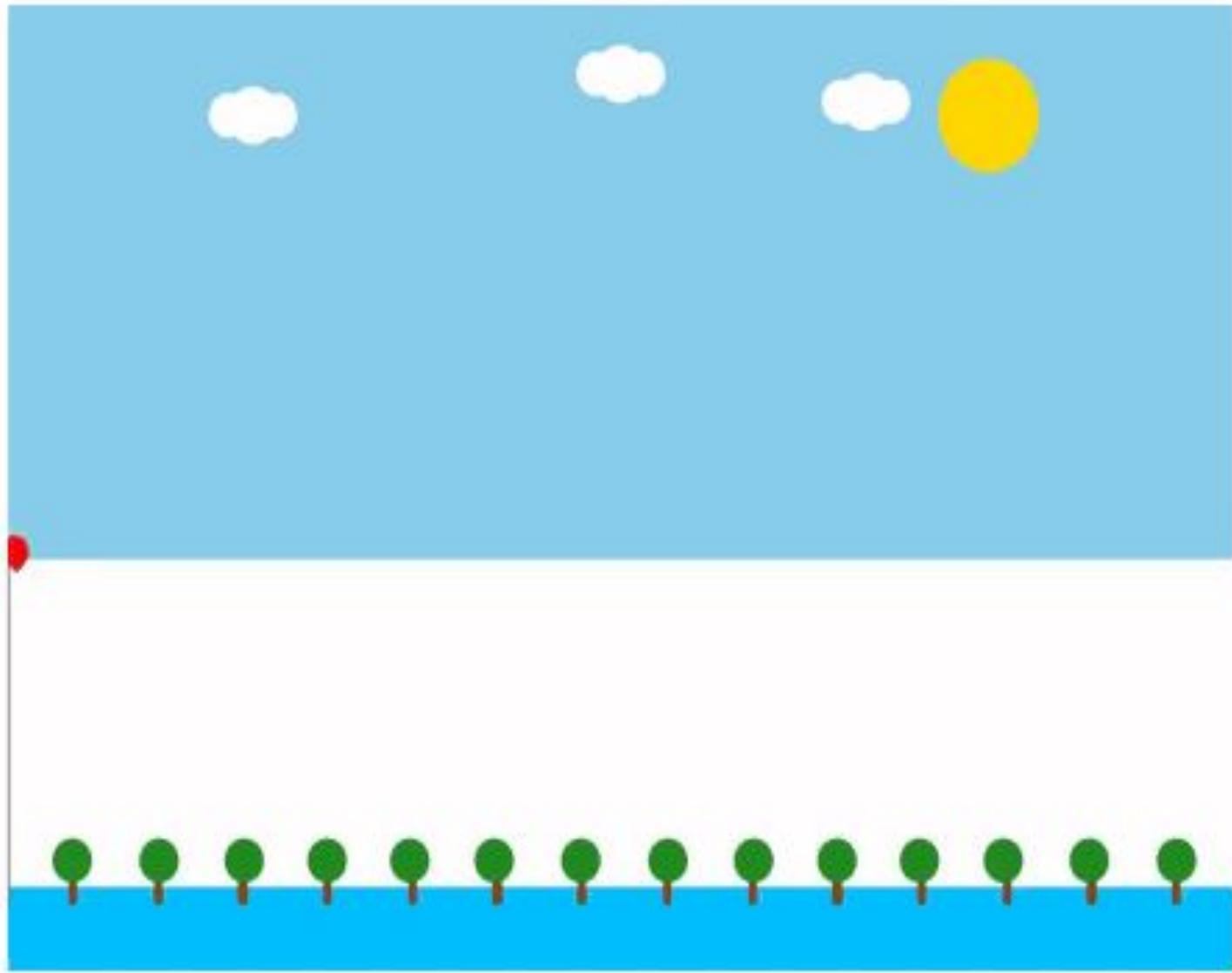
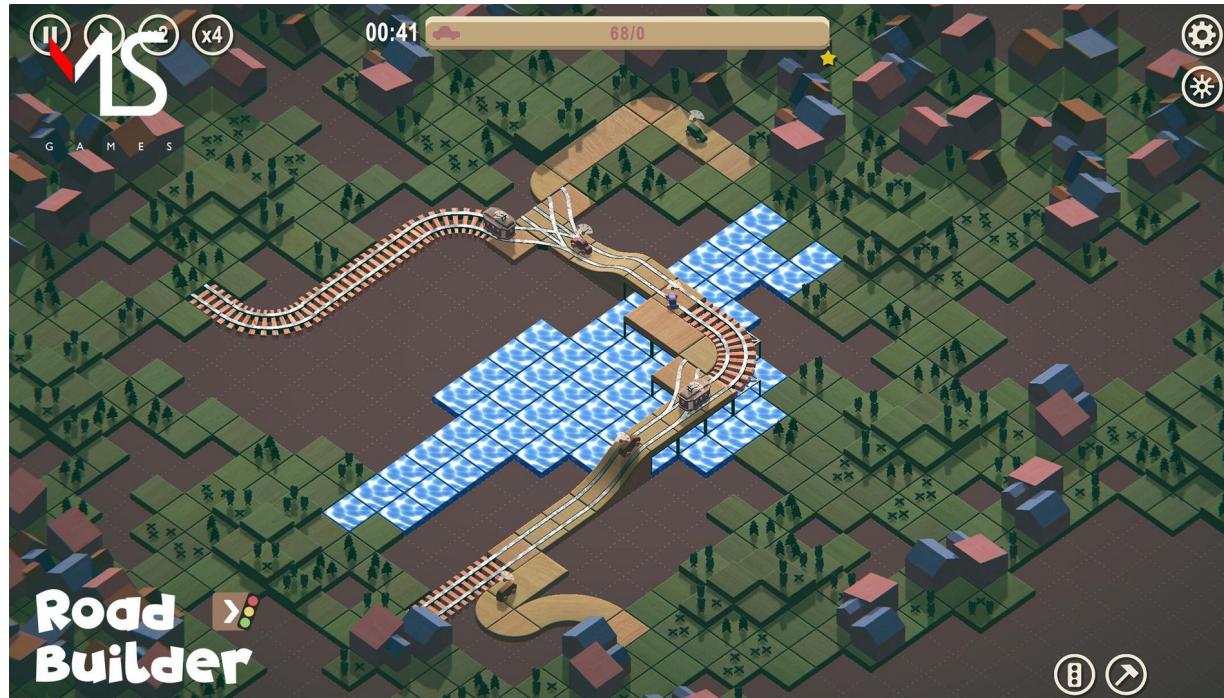


curves







Curve

- Intuition: A set of points drawn with a pen
- Mapping from time to place
- $f(t) = (x, y)$ for $t \in [0,1]$



Curve Representations (Types)

1. Explicit Representation
2. Implicit Representation
3. Parametric Representation
4. Subdivision Representation
5. Procedural Representation



Curve Representations (Overview)

- There are several ways to represent a curve

- **Explicit:** $y = f(x)$

$$y = mx + b$$

$$y = x^2$$

- (-) Must be a single valued function

- (-) Vertical lines, say $x = d$? No way to represent using single valued function

- **Implicit:** $f(x,y) = 0$

$$x^2 + y^2 - r^2 = 0$$

- (+) y can be multiple valued function of x

- (-) Continuity hard to detect

- **Parametric:** $(x, y) = (x(t), y(t))$

$$(x, y) = (\cos t, \sin t)$$

- (+) Easy to specify, modify and control

- (-) Extra hidden variable t , the parameter, non intuitive



Parametric Representation

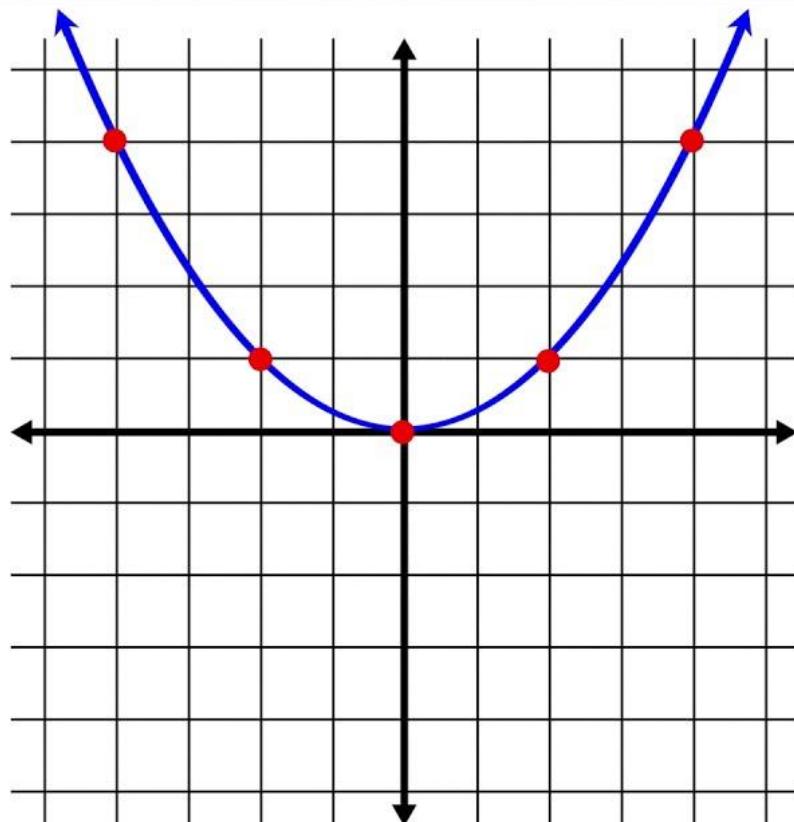
- Curves: single parameter t (e.g. time)
 - $x = x(t), y = y(t), z = z(t)$
- Circle:
 - $x = \cos(t), y = \sin(t), z = 0$
- Tangent described by derivative

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \quad \frac{dp(t)}{dt} = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{bmatrix}$$

- Magnitude is “velocity”

$$f(t) = (2t, t^2)$$

Understanding Parametric Equations



$x = 2t$	$y = t^2$
----------	-----------

t	x	y
-2	-4	4
-1	-2	1
0	0	0
1	2	1
2	4	4

Same curve, different parameterization

- $f(t) = (t, 0)$

$t = 0$

$t = 1$

- $t \in [0, 1]$

- $f(t) = (1 - t, 0)$

$t = 1$

$t = 0$

- $t \in [0, 1]$



Parametric Curve Visualization

- 2D Curve Visualization

- <https://www.geogebra.org/m/cAsHbXEU>

- 3D Curve Visualization

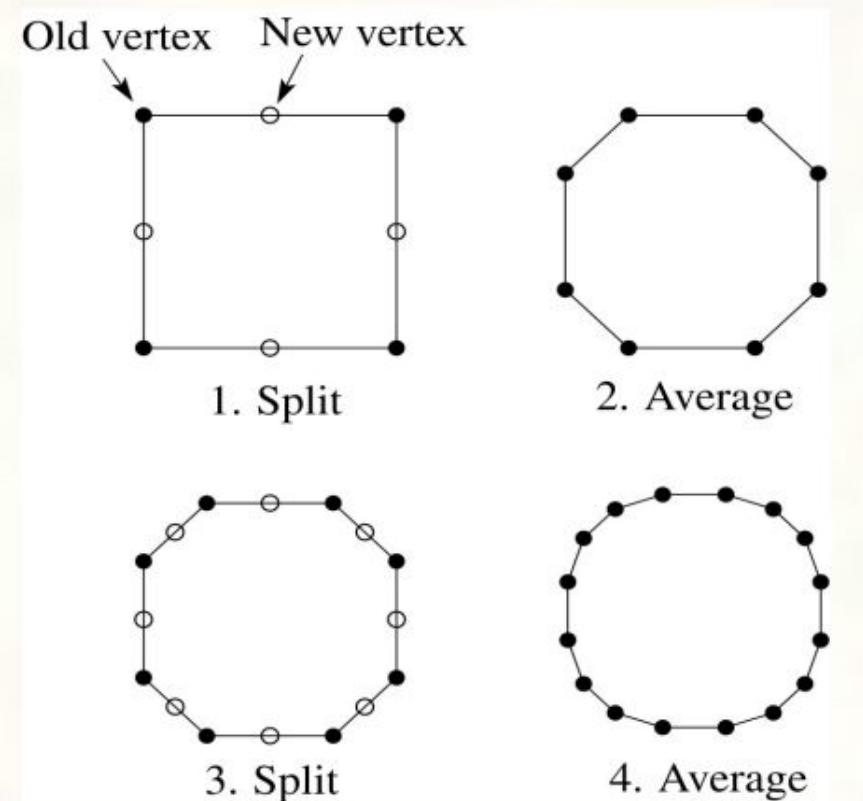
- https://christopherchudzicki.github.io/MathBox-Demos/parametric_curves_3D.html

Subdivision Representation

- Start with a set of points
- Have a rule that adds points [possibly removing old ones]
- Repeat the rule
- Each subdivision makes the set of points closer to the intended curve



- Start with a piecewise linear curve
- Insert new vertices at the midpoints (the **splitting step**)
- Average each vertex with the “next” (clockwise) neighbor (the **averaging step**)
- Go to the splitting step



Example

Lagrange Polynomial

- Given $n+1$ points $(x_0, y_0), (x_1, y_1) \dots \dots (x_n, y_n)$
- To construct a curve that passes through these points we can use Lagrange polynomial defined as follows:.

$$y = f(x) = \sum_{k=0}^n y_k L_{n,k}$$

$$L_{n,k} = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

Lagrange Polynomial (Example)

- Let's consider the following data points:

$$(x_0, y_0) = (0, 2)$$

$$(x_1, y_1) = (1, 3)$$

$$(x_2, y_2) = (2, 5)$$

$$(x_3, y_3) = (3, 10)$$

$$(x_4, y_4) = (4, 20)$$

Step 1: Define Each Basis Polynomial $L_j(x)$

Each basis polynomial $L_j(x)$ is defined as:

$$L_j(x) = \prod_{\substack{0 \leq m \leq 4 \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$$

$$L_0(x)$$

$$\begin{aligned}
 L_0(x) &= \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{(0 - 1)(0 - 2)(0 - 3)(0 - 4)} \\
 &= \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{(-1)(-2)(-3)(-4)} = \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{24}
 \end{aligned}$$



$L_1(x)$

$$L_1(x) = \frac{(x - 0)(x - 2)(x - 3)(x - 4)}{(1 - 0)(1 - 2)(1 - 3)(1 - 4)}$$

$$= \frac{x(x - 2)(x - 3)(x - 4)}{(1)(-1)(-2)(-3)} = \frac{x(x - 2)(x - 3)(x - 4)}{-6}$$

$L_2(x)$

$$L_2(x) = \frac{(x - 0)(x - 1)(x - 3)(x - 4)}{(2 - 0)(2 - 1)(2 - 3)(2 - 4)}$$

$$= \frac{x(x - 1)(x - 3)(x - 4)}{(2)(1)(-1)(-2)} = \frac{x(x - 1)(x - 3)(x - 4)}{4}$$



$$L_3(x)$$

$$\begin{aligned}
 L_3(x) &= \frac{(x-0)(x-1)(x-2)(x-4)}{(3-0)(3-1)(3-2)(3-4)} \\
 &= \frac{x(x-1)(x-2)(x-4)}{(3)(2)(1)(-1)} = \frac{-x(x-1)(x-2)(x-4)}{6}
 \end{aligned}$$

$$L_4(x)$$

$$\begin{aligned}
 L_4(x) &= \frac{(x-0)(x-1)(x-2)(x-3)}{(4-0)(4-1)(4-2)(4-3)} \\
 &= \frac{x(x-1)(x-2)(x-3)}{(4)(3)(2)(1)} = \frac{x(x-1)(x-2)(x-3)}{24}
 \end{aligned}$$



Step 2: Multiply Each $L_j(x)$ by y_j

Now build the full polynomial:

$$P(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x) + y_3L_3(x) + y_4L_4(x)$$

Plug in the y_j values:

$$P(x) = 2 \cdot L_0(x) + 3 \cdot L_1(x) + 5 \cdot L_2(x) + 10 \cdot L_3(x) + 20 \cdot L_4(x)$$

That is:

$$\begin{aligned} P(x) &= 2 \cdot \frac{(x-1)(x-2)(x-3)(x-4)}{24} + 3 \cdot \frac{x(x-2)(x-3)(x-4)}{-6} + 5 \cdot \frac{x(x-1)(x-3)(x-4)}{4} \\ &\quad + 10 \cdot \frac{-x(x-1)(x-2)(x-4)}{6} + 20 \cdot \frac{x(x-1)(x-2)(x-3)}{24} \end{aligned}$$



Lagrange Polynomial (Problems)

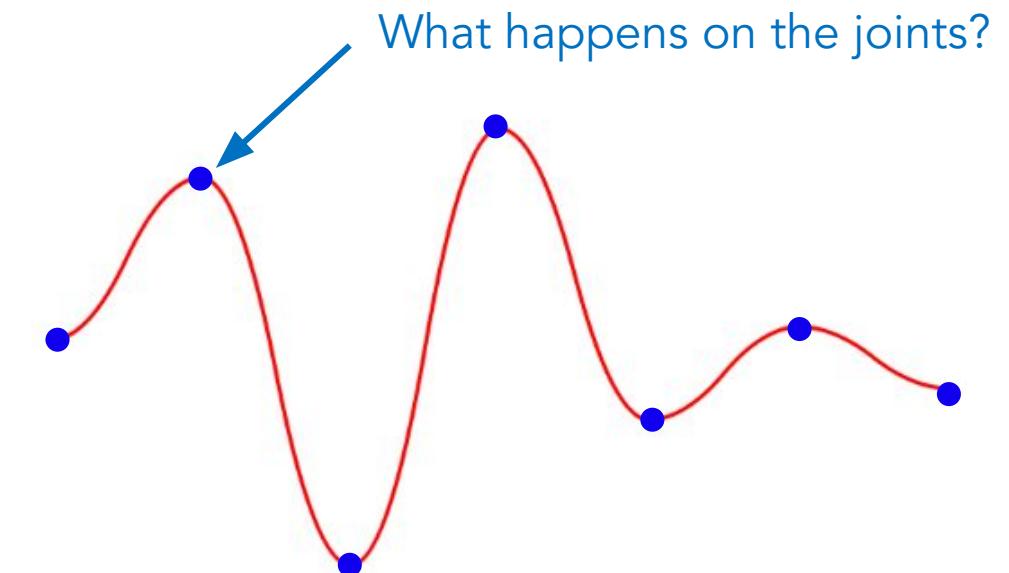
Problems:

- $y=f(x)$, no multiple values
- Higher order functions tend to oscillate
- No local control (change any (x_i, y_i) changes the whole curve)
- Computationally expensive due to high degree.



How to draw complex curves

- Break them into “manageable” smaller curves
- These smaller curves are called **splines**



Solution: Piecewise Linear Polynomial

- To overcome the problems with Lagrange polynomial
 - Divide given points into overlap sequences of 4 points
 - Construct **3rd degree** polynomial that passes through these points, p_0, p_1, p_2, p_3 then p_3, p_4, p_5, p_6 etc.
 - Then glue the curves so that they appear **sufficiently smooth** at joint points.

Piecewise Linear Polynomial

Questions:

- Why 3rd Degree curves used?
- How to measure smoothness at joint point?



Why Cubic Curves

Cubic polynomials are most often used for piecewise because:

- (1) Lower-degree polynomials offer too little flexibility in controlling the shape of the curve.
- (2) Higher-degree polynomials can introduce unwanted wiggles and also require more computation.



Why Cubic Curves

- (3) No lower-degree representation allows a curve segment to be defined by two given endpoints with given derivative at each endpoints.

- (4) No lower-degree curves are non planar in 3D.



Piecewise Linear Polynomial

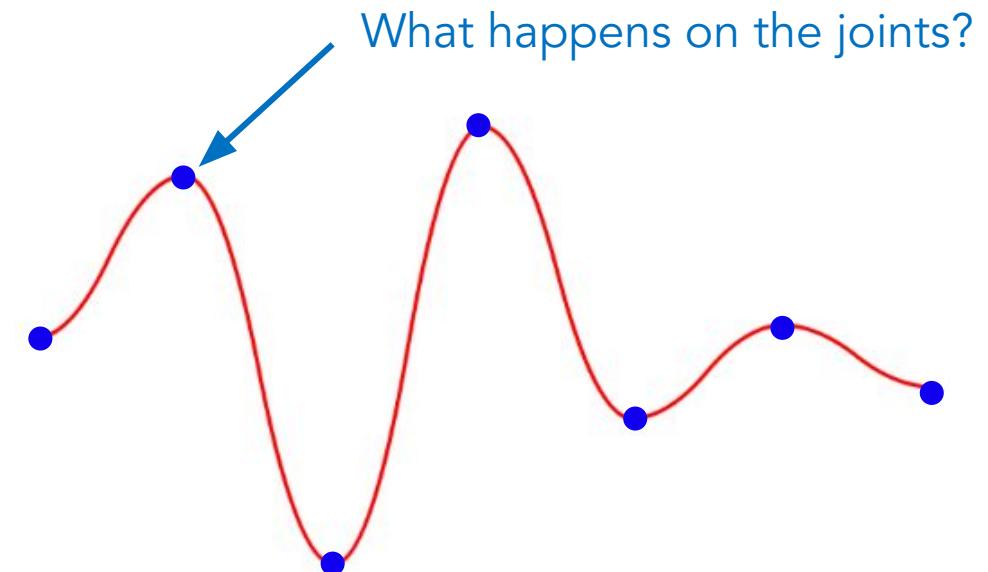
Questions:

- Why 3rd Degree curves used?
- How to measure smoothness at joint point?



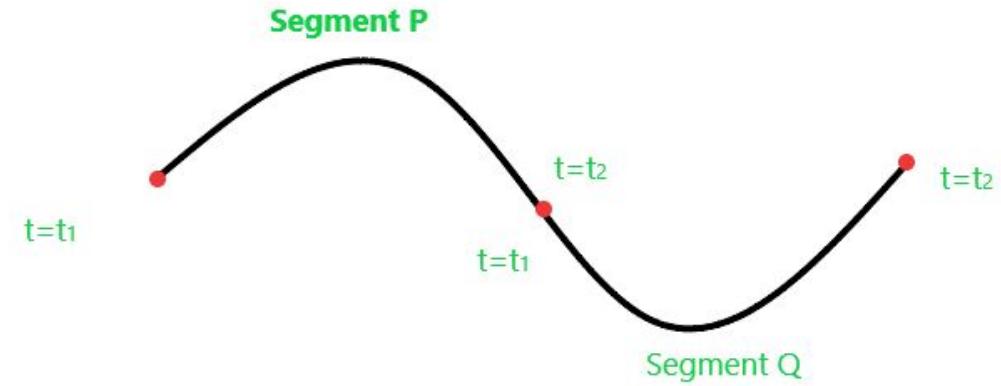
Introducing Continuity

- AKA “Smoothness”
- Is there any gaps, sudden turns?
- Not same as “not wiggly”
- A curve can be wiggly but still smooth.

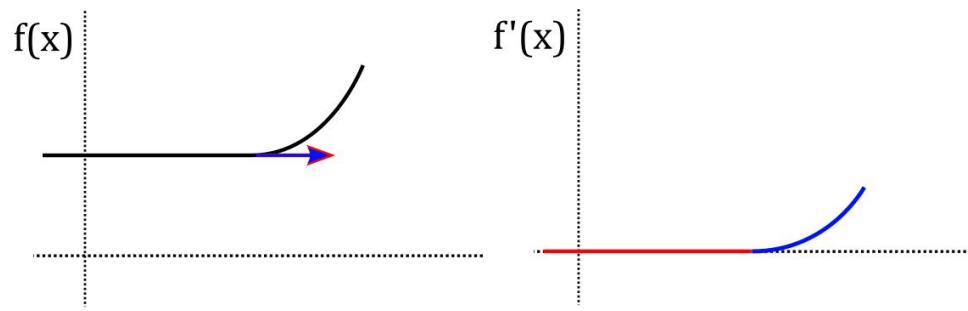


Continuity

- Are the points next to each other?
- Can we draw without lifting the pen?
- Approach a point from left and right, do we get the same point?

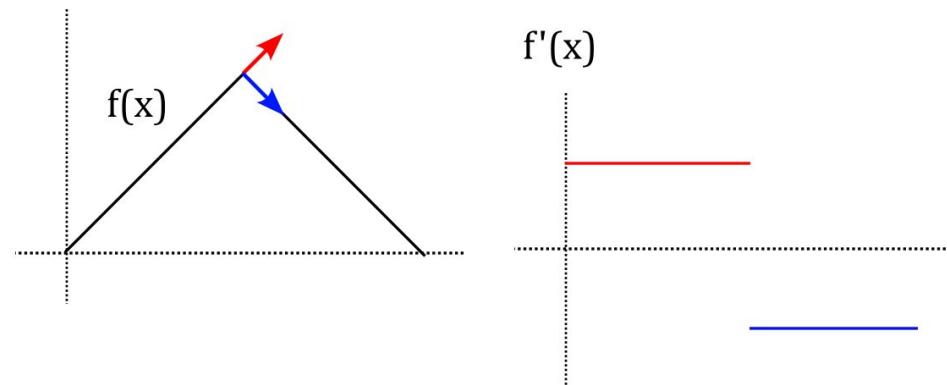


Continuity in Direction



First derivative same at joint

First derivative changes abruptly at joint



Derivative of a Curve

◦

$$f(t) = (f_x(t), f_y(t))$$

$$f'(t) = \frac{\partial}{\partial t} f(t) = \left(\frac{\partial}{\partial t} f_x(t), \frac{\partial}{\partial t} f_y(t) \right)$$

◦ A velocity vector, tangent to the curve



Discontinuity Example

Piecewise line segments:

```
f(u) = if u<.5 then (u,0) else (u,1)  
or  
f(u) = (u<.5) ? (u,0) : (u,1)
```

Position discontinuity at $u=.5$



Discontinuity Example

Piecewise line segments:

$$f(u) = \text{if } u < .5 \text{ then } (u, u) \text{ else } (u, .5)$$

Tangent (first derivative) discontinuity at $u = .5$

Note: discontinuities happen when we switch

Parametric Continuity, $C(n)$

We say a curve is $C(n)$ continuous

If all its derivatives up to (and including) n are continuous

$C(0)$ - position

$C(1)$ - position and tangent (1^{st} derivative)

$C(2)$ - position, tangent and 2^{nd} derivative



How much continuity do we need

$C(0)$ - no gaps

$C(1)$ - no corners

$C(2)$ - looks smooth

Higher...

Important for airflow (airplane, car, boat design)

Important for reflections



Geometric Continuity, G(n)

Speed Matters?

```
f(u) = if u<0.5 then (u,0) else (2u-0.5,0)
```

It's a horizontal line

The pen doesn't change direction

It does change "speed" at the point



C and G continuity

$C(n)$ continuity - all derivatives up to n match

$G(n)$ continuity - direction of all derivatives up to n match



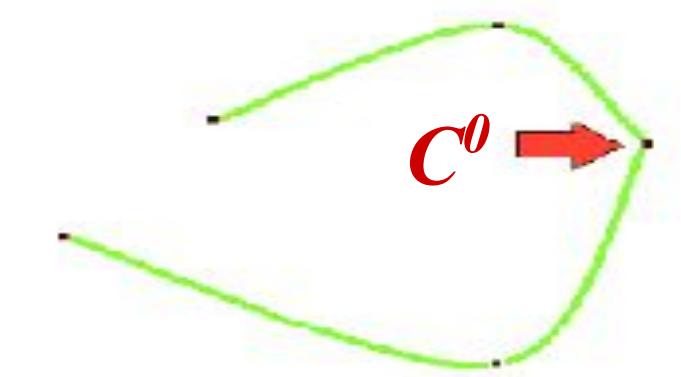
Measure of Smoothness

G^0 Geometric Continuity \Leftrightarrow C^0 Parametric Continuity

If two curve segments join together.

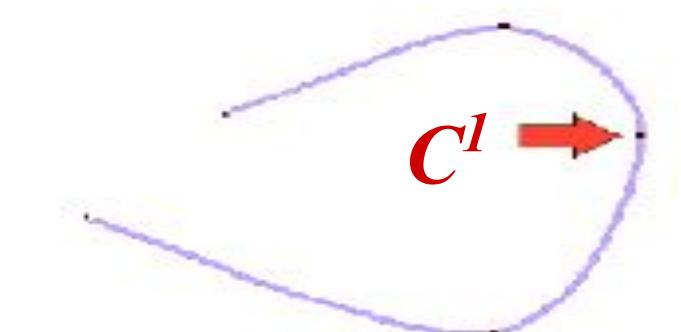
G^1 Geometric Continuity

If the **directions** (but not necessarily the magnitudes) of the two segments' tangent vectors are equal at a join point.



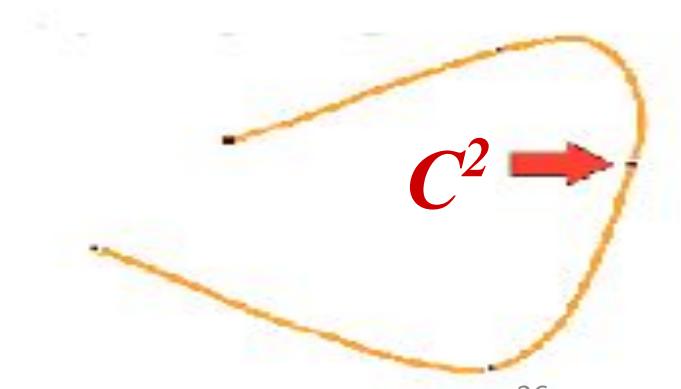
C^1 Parametric Continuity

If the **directions and magnitudes** of the two segments' tangent vectors are equal at a join point.



C^2 Parametric Continuity

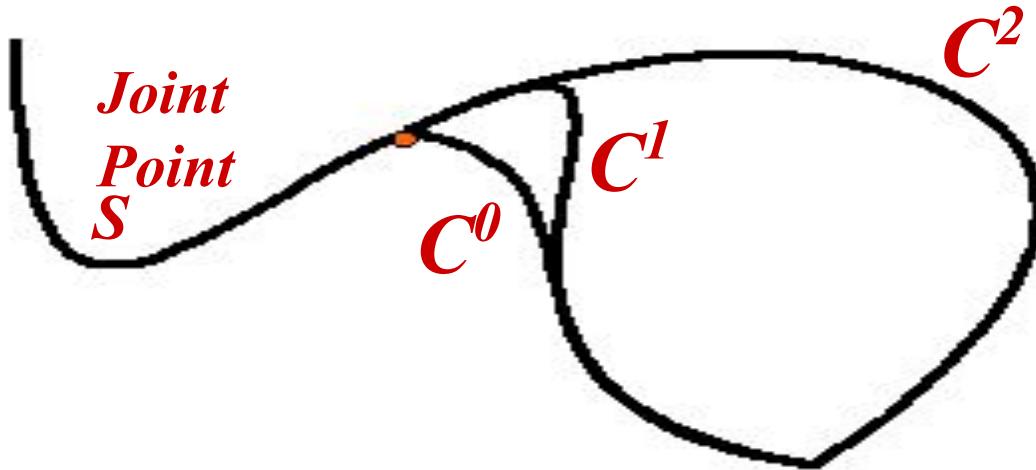
If the direction and magnitude of $Q^2(t)$ (curvature or **acceleration**) are equal at the join point.



C^n Parametric Continuity

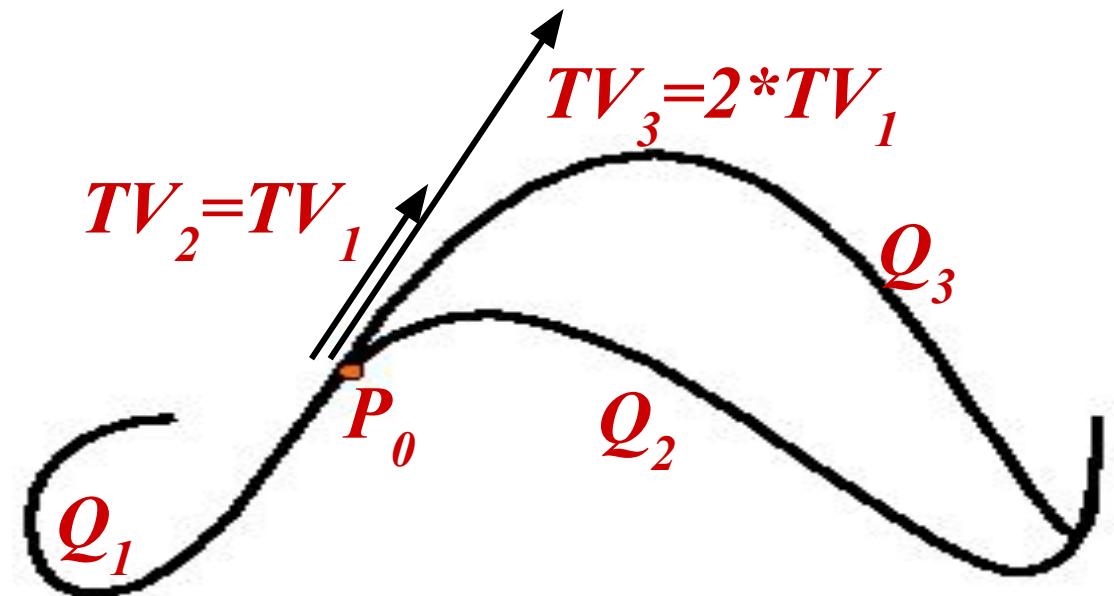
If the direction and magnitude of $Q^n(t)$ through the n th derivative are equal at the join point.

Measure of Smoothness



- By increasing parametric continuity we can increase smoothness of the curve.

- Q_1 & Q_2 are C^1 and G^1 continuous
- Q_1 & Q_3 are G^1 continuous only as Tangent vectors have different magnitude.
- Observe the effect of increasing in magnitude of TV



Line Segment Parameterization

- $f(t) = P_0 + t(P_1 - P_0)$
- $f(t) = \mathbf{a}_0 + \mathbf{a}_1 t$

- Two info to draw a line segment
 - Start point
 - End point / Velocity Vector



Linear Interpolation (or, Lerp)

- $$\text{lerp}(P_0, P_1, t) = P_0 + t(P_1 - P_0)$$
- Will be useful later



Cubic Curves

- $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
- Convenient enough to draw, complex enough to draw a wiggly shape
- Four info to draw a curve
 - Starting point, $f(0) = a_0$
 - Ending point, $f(1) = a_0 + a_1 + a_2 + a_3$
 - Direction at starting point, $f'(0) = a_1$
 - Direction at ending point, $f'(1) = a_1 + 2a_2 + 3a_3$



Desirable Properties of a Curve

- Simple control
 - lines need only two points
 - curves will need more (but not significantly more)
- Intuitive control
 - Physically meaningful quantities like position, tangent, curvature etc.
- Global Vs. Local Control
 - Portion of curve affected by a control point.
- General Parameterization
 - Handle multi-valued x-y mapping





BÉZIER CURVE

History

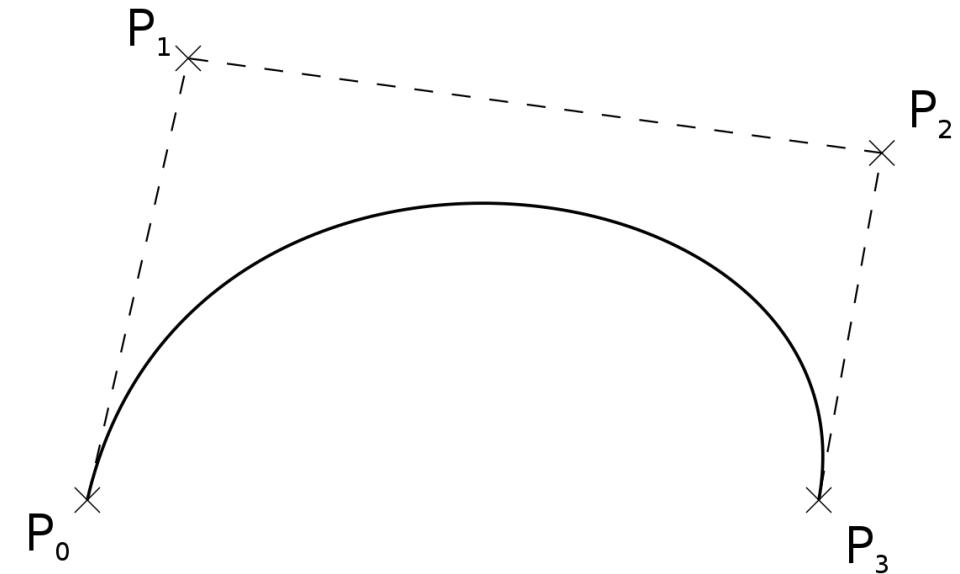
- Named after French engineer Pierre Bézier (1910–1999),
- First used it in the 1960s for designing curves for the bodywork of Renault cars.

Pierre Étienne Bézier was a French engineer and one of the founders of the fields of solid, geometric and physical modelling as well as in the field of representing curves, especially in computer-aided design and manufacturing systems. As an engineer at Renault, he became

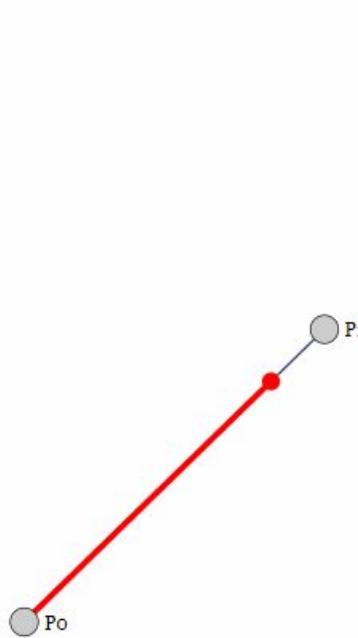


Bézier Curve

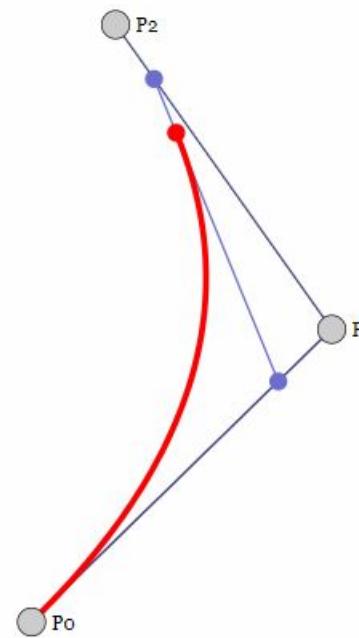
- Will have some control points
- Move around control points to change shape of the curve
- For example: 4 control points to build a cubic Bézier curve



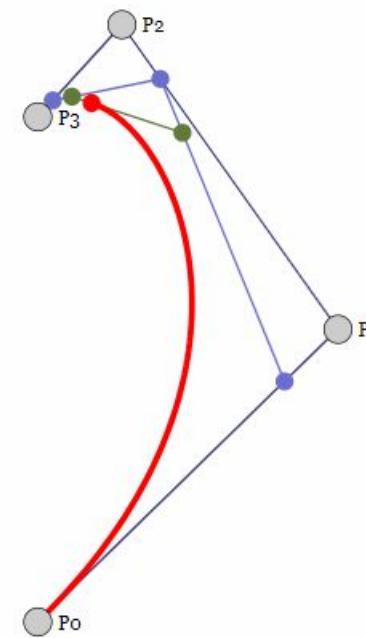
Family of Bézier Curves



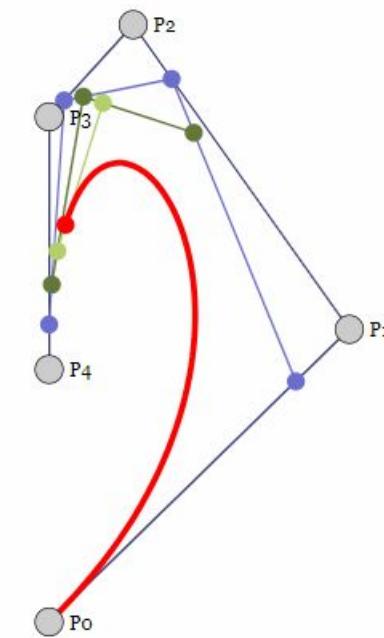
Linear



Quadratic



Cubic



Quartic

Bézier Curve Demonstration

- Single Curve Graph
 - <https://www.desmos.com/calculator/d1ofwre0fr>
- Family of Bezier Curve Animation
 - <https://www.jasondavies.com/animated-bezier/>



How to draw Bézier Curves

- Cubic Bezier Curve Code
- <https://editor.p5js.org/shonku/sketches/vonrqpOhd>



Draw Cubic Bézier Curves

Given 4 control points p_1, p_2, p_3, p_4

For $t = 0$ to 1 in proper interval:

$$q_1 = \text{lerp}(p_1, p_2, t)$$

$$q_2 = \text{lerp}(p_2, p_3, t)$$

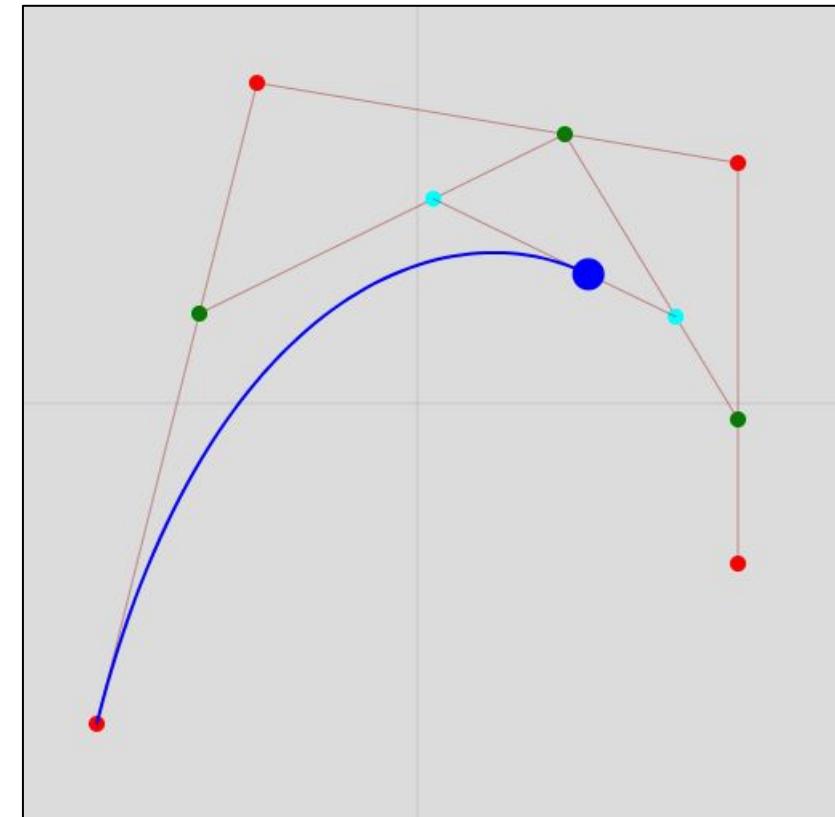
$$q_3 = \text{lerp}(p_3, p_4, t)$$

$$r_1 = \text{lerp}(q_1, q_2, t)$$

$$r_2 = \text{lerp}(q_2, q_3, t)$$

$$s = \text{lerp}(r_1, r_2, t)$$

Draw(s)



Draw Bézier curves

- Process of recursively finding a point on a Bézier curve has a name
 - “De Casteljau’s Algorithm”



Equation of a Cubic Bézier Curve

$$\begin{aligned}
 s &= r_1 + (r_2 - r_1)t \\
 r_1 &= q_1 + (q_2 - q_1)t \\
 r_2 &= q_2 + (q_3 - q_2)t
 \end{aligned}
 \quad \rightarrow \quad
 \begin{aligned}
 s &= r_1 + (r_2 - r_1)t \\
 &= q_1 + (q_2 - q_1)t + (q_2 + (q_3 - q_2)t - q_1 - (q_2 - q_1)t)t \\
 &= q_1 + tq_2 - tq_1 + (q_2 + tq_3 - tq_2 - q_1 - tq_2 + tq_1)t \\
 &= q_1 + tq_2 - tq_1 + tq_2 + t^2q_3 - t^2q_2 - tq_1 - t^2q_2 + t^2q_1 \\
 &= q_1(1 - 2t + t^2) + q_2(2t - 2t^2) + q_3(t^2)
 \end{aligned}$$

$$\begin{aligned}
 q_1 &= p_1 + (p_2 - p_1)t \\
 q_2 &= p_2 + (p_3 - p_2)t \\
 q_3 &= p_3 + (p_4 - p_3)t
 \end{aligned}
 \quad \rightarrow \quad
 \begin{aligned}
 &= (p_1 + (p_2 - p_1)t)(1 - 2t + t^2) + (p_2 + (p_3 - p_2)t)(2t - 2t^2) + (p_3 + (p_4 - p_3)t)(t^2) \\
 &= (p_1 + tp_2 - tp_1)(1 - 2t + t^2) + (p_2 + tp_3 - tp_2)(2t - 2t^2) + (p_3 + tp_4 - tp_3)(t^2) \\
 &= (p_1 - 2tp_1 + t^2p_1 + tp_2 - 2t^2p_2 + t^3p_2 - tp_1 + 2t^2p_1 - t^3p_1) + (2tp_2 - 2t^2p_2 \\
 &+ 2t^2p_3 - 2t^3p_3 - 2t^2p_2 + 2t^3p_2) + (t^2p_3 + t^3p_4 - t^3p_3) \\
 &= (1 - 3t + 3t^2 - t^3)p_1 + (3t - 6t^2 + 3t^3)p_2 + (3t^2 - 3t^3)p_3 + t^3p_4 \\
 &= (1 - t)^3 p_1 + 3t(1 - t)^2 p_2 + 3t^2(1 - t) p_3 + t^3 p_4
 \end{aligned}$$



Matrix Eqⁿ of a Cubic Bézier Curve

Dimensions

$$Q(t) = (1-3t+3t^2-t^3)p_1 + (3t-6t^2+3t^3)p_2 + (3t^2-3t^3)p_3 + t^3p_4 \quad [1]$$

$$Q(t) = [(1 - 3t + 3t^2 - t^3) \quad (3t - 6t^2 + 3t^3) \quad (3t^2 - 3t^3) \quad t^3] \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \quad [1 \times 4] [4 \times 1]$$

$$Q(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \quad [1 \times 4][4 \times 4][4 \times 1]$$

$$Q(t) = \textcolor{red}{T} \cdot \textcolor{green}{M}_B \cdot \textcolor{blue}{G}_B$$

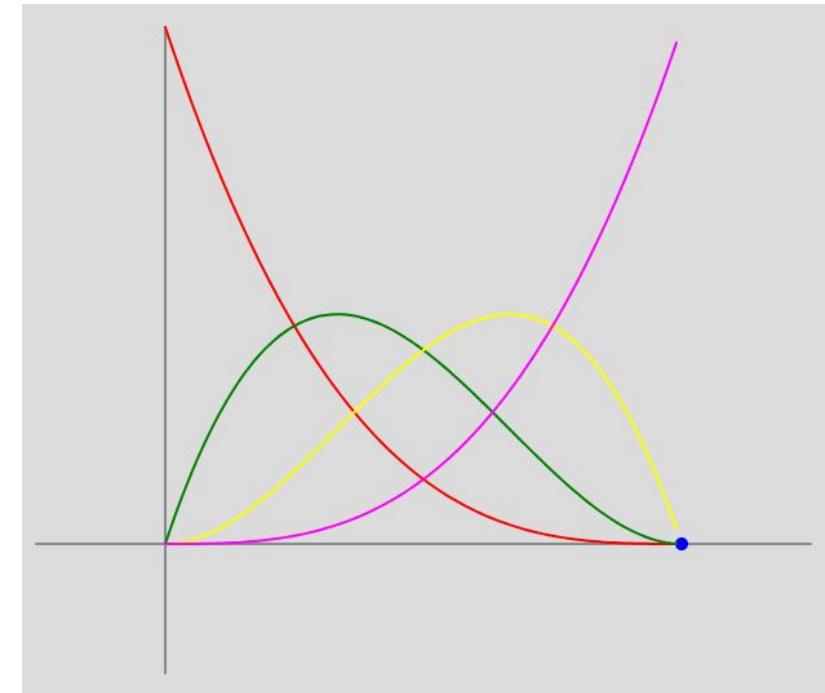
M_B : The basis Matrix of Bézier Curve

G_B : Geometric properties of Bézier Curve

The Bernstein Polynomials

- How much a point contributes to a certain position?

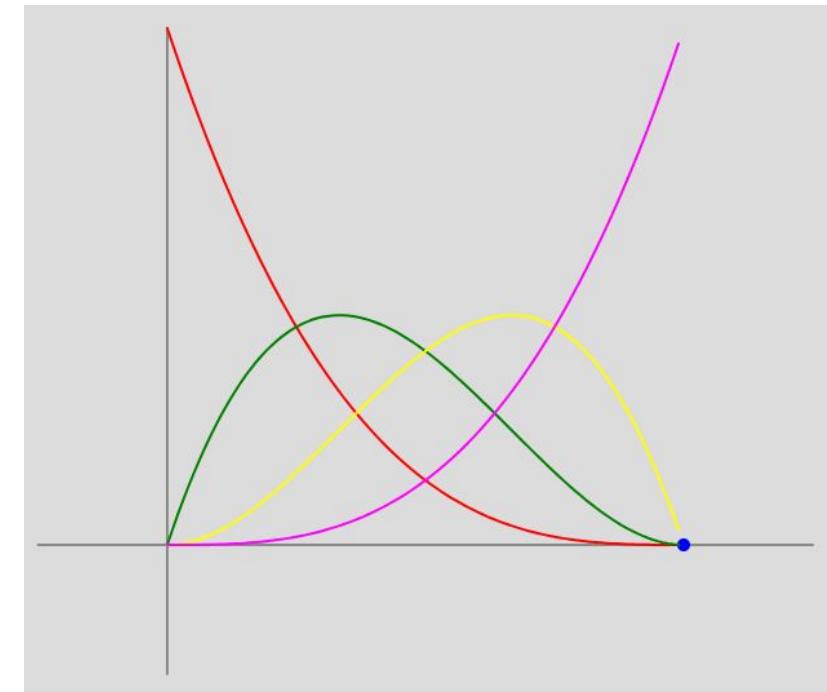
- $Q(t) = (1-3t+3t^2-t^3)p_1 + (3t-6t^2+3t^3)p_2 + (3t^2-3t^3)p_3 + t^3p_4$



The Bernstein Polynomials

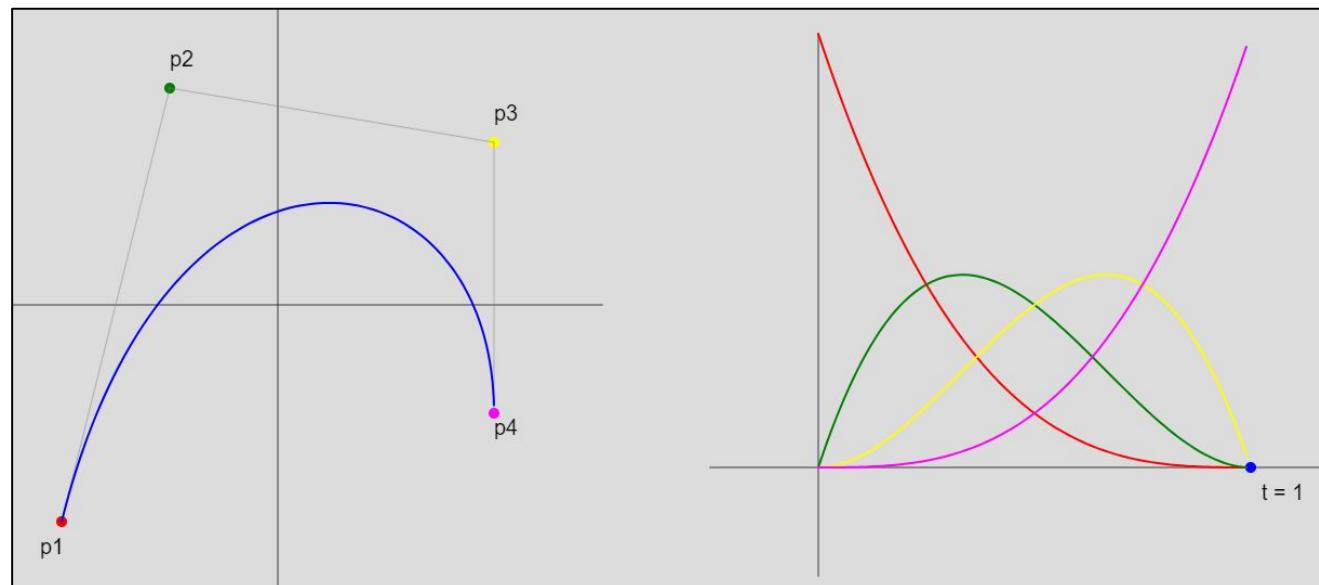
- $Q(t) = (1-3t+3t^2-t^3)p_1 + (3t-6t^2+3t^3)p_2 + (3t^2-3t^3)p_3 + t^3p_4$

- Sum of polynomials = 1
- So, a point on the curve is a ***convex combination*** of the control points



The Bernstein Polynomials

- Different Bernstein polynomial values for different values of t :
- <https://editor.p5js.org/shonku/sketches/9dlzE16jW>



Convex Hull Property

Convex combination of n points

$$P_1\lambda_1 + P_2\lambda_2 + \cdots + P_n\lambda_n$$

Where $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$

Any convex combination will lie inside the convex hull of the n points.

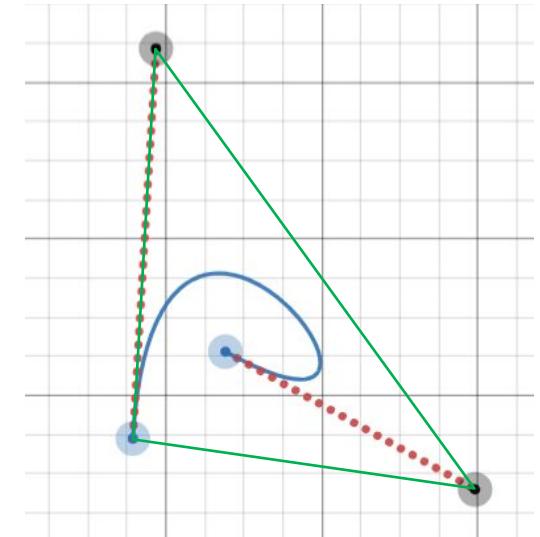
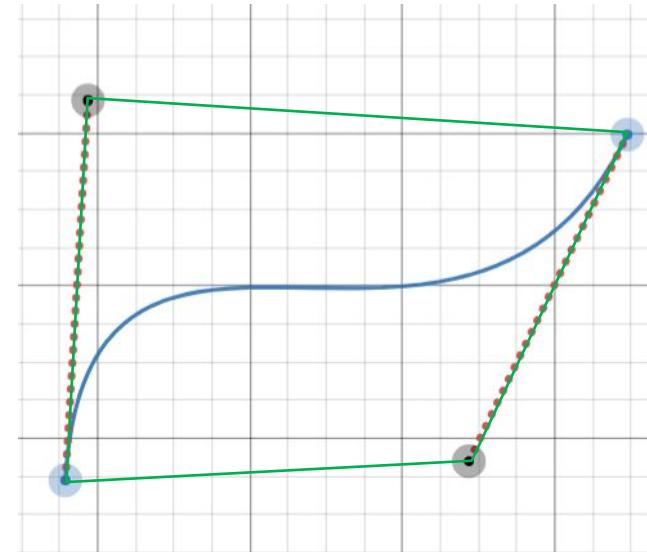
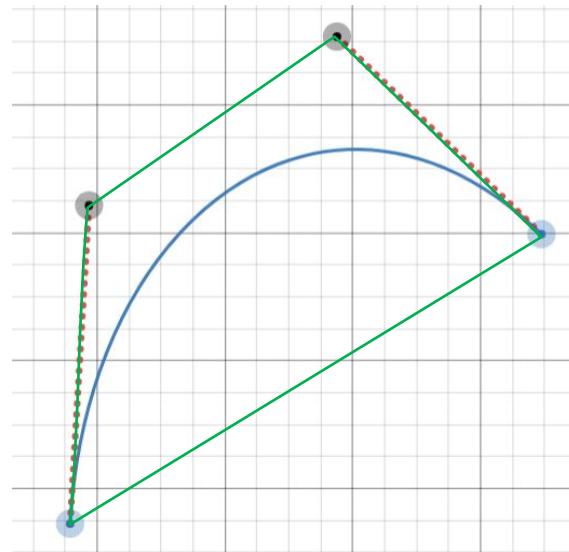
Not need for this course:

<https://math.stackexchange.com/questions/229354/proof-that-the-convex-hull-of-a-finite-set-is-equal-to-all-convex-combinations>

Convex Hull Property

As, $Q(t)$ is a convex combination of P_1, P_2, P_3 and P_4

Cubic Bézier curve must be bounded by convex hull of P_1, P_2, P_3 and P_4



Derivatives

Position: $Q(t) = (1 - 3t + 3t^2 - t^3)P_1 + (3t - 6t^2 + 3t^3)P_2 + (3t^2 - 3t^3)P_3 + t^3P_4$

Velocity $Q'(t) = (-3 + 6t - 3t^2)P_1 + (3 - 12t + 9t^2)P_2 + (6t - 9t^2)P_3 + 3t^2P_4$

Acceleration: $Q''(t) = (6 - 6t)P_1 + (-12 + 18t)P_2 + (6 - 18t)P_3 + 6tP_4$

Jolt: $Q'''(t) = (-6)P_1 + (18)P_2 + (-18)P_3 + 6P_4$

<https://editor.p5js.org/shonku/sketches/XFv3T4ita>

Derivatives

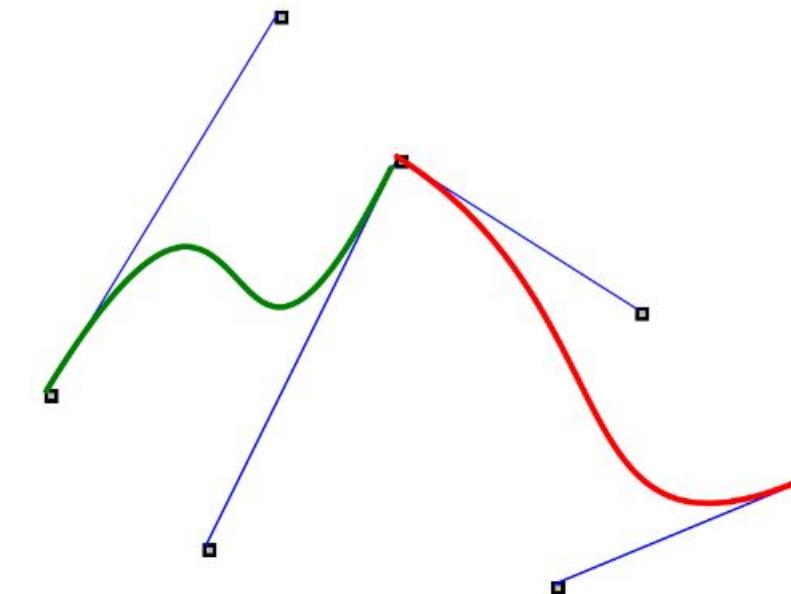
Velocity at $t = 0$ and $t = 1$

$$Q'(0) = (-3)P_1 + (3)P_2 = 3(P_2 - P_1)$$

$$Q'(1) = (-3 + 6 - 3)P_1 + (3 - 12 + 9)P_2 + (6 - 9)P_3 + 3P_4 = 3(P_4 - P_3)$$

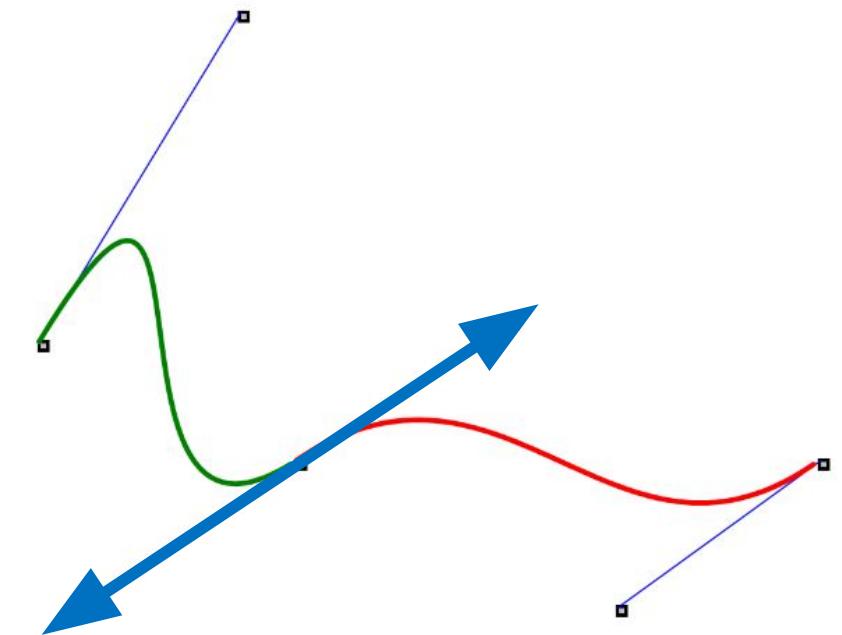
Join Two Cubic Bézier Curve

- C(0) continuous
- Not enough



Join Two Cubic Bézier Curve

- How to ensure $C(1)$?
- See the smoothness in action:
- <https://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html>
- Coding Example:
- <https://editor.p5js.org/shonku/sketches/1wWKxEu0j>



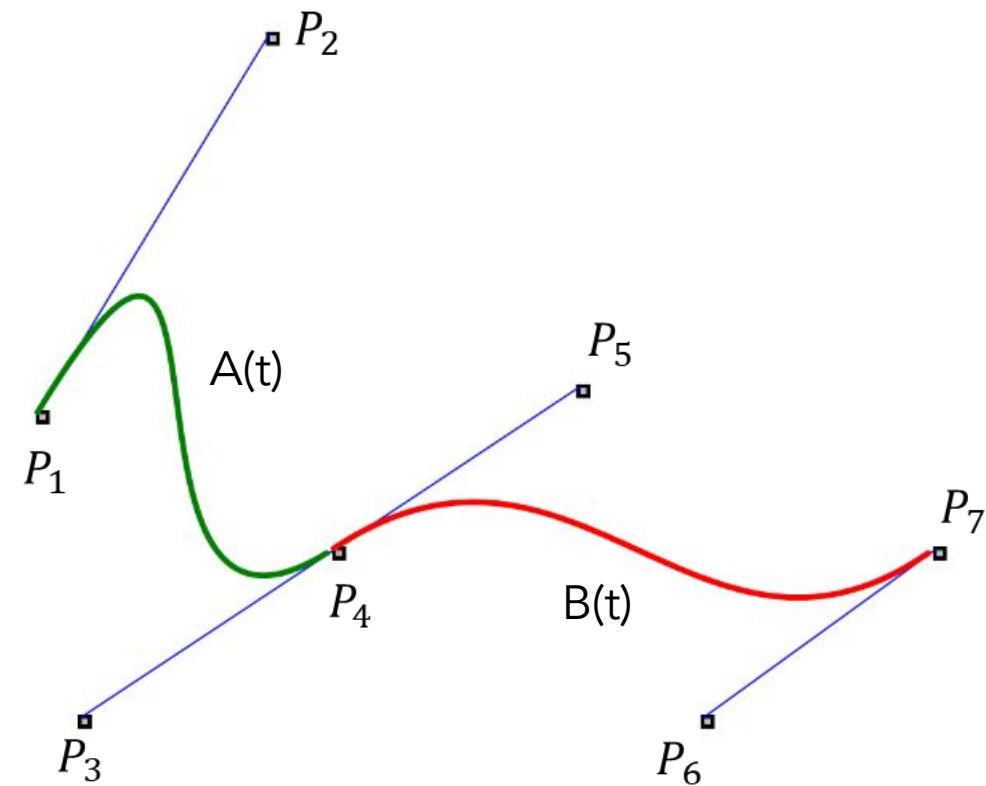
Join Two Cubic Bézier Curve

- For $C(1)$ continuity at point P_4
- Velocity of two curves at the joint must be equal

$$\text{So, } A'(1) = B'(0)$$

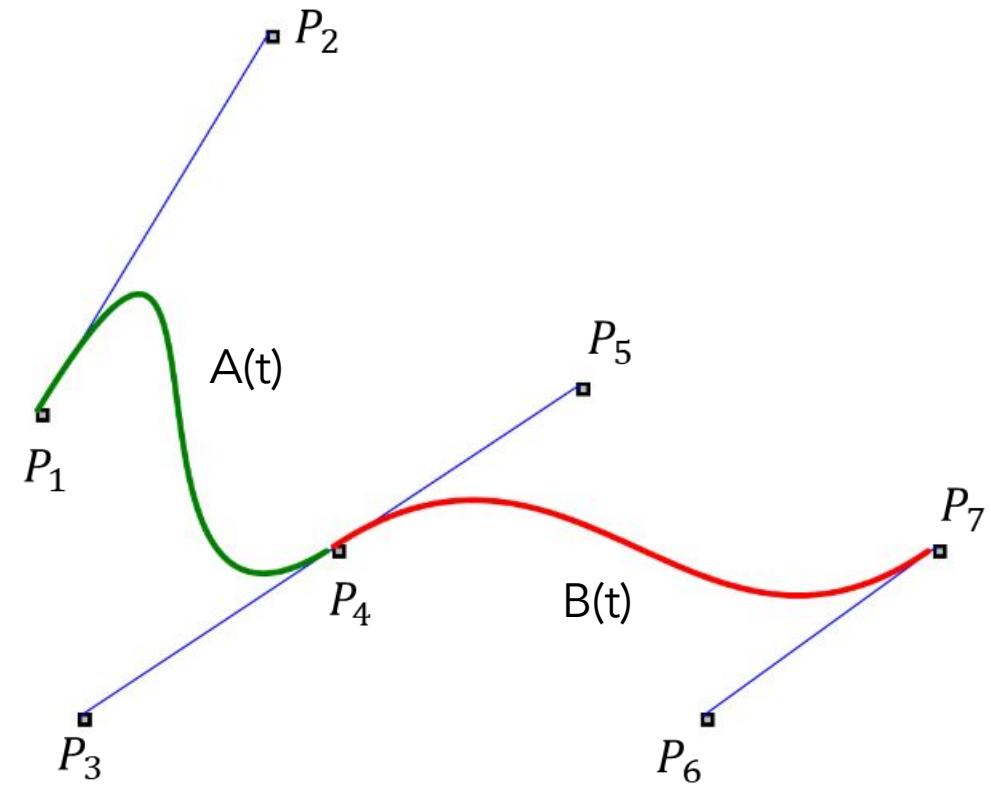
$$\Rightarrow 3(P_4 - P_3) = 3(P_5 - P_4)$$

$$\Rightarrow P_5 = 2P_4 - P_3$$



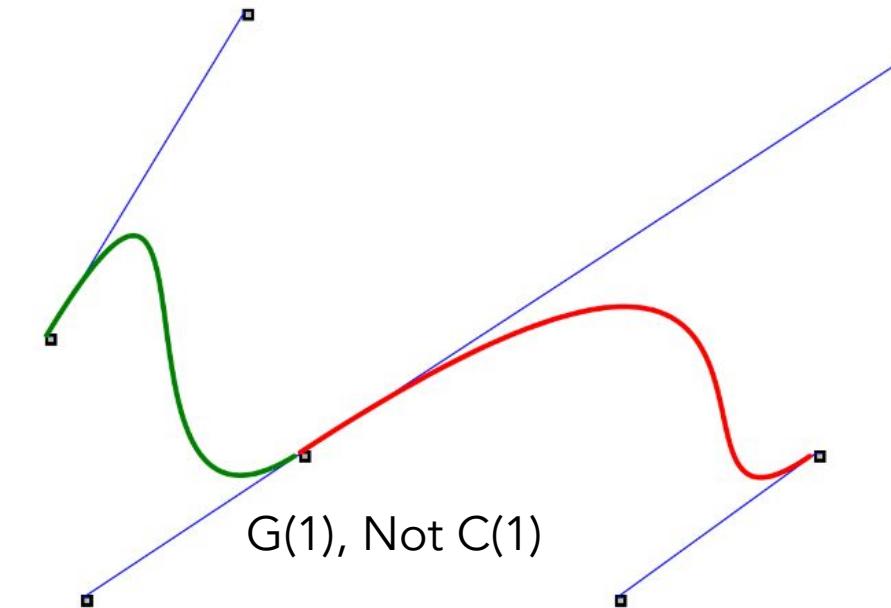
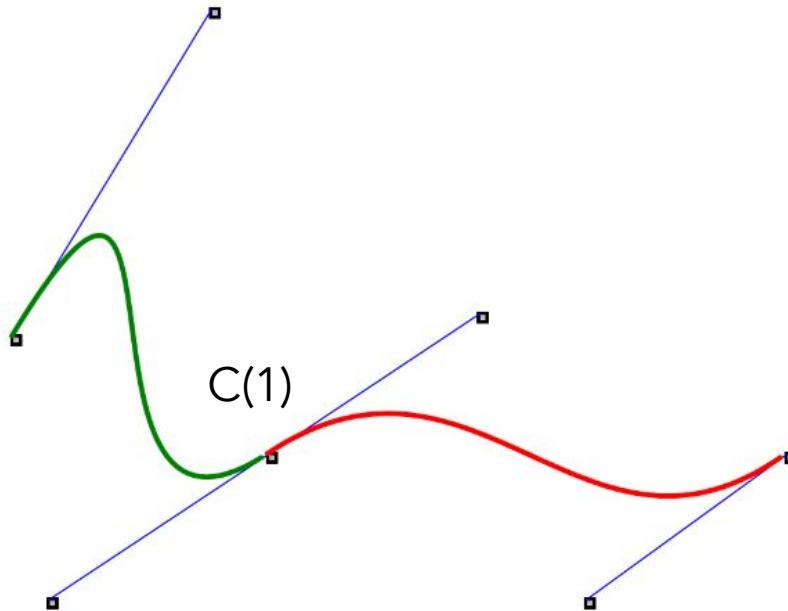
Join Two Cubic Bézier Curve

- $P_5 = 2P_4 - P_3$
- This condition must hold to be C(1) continuous
- Notice that we can no longer control P_5 . It is decided by P_4 and P_3



Join Two Cubic Bézier Curve

- Difference of C and G continuity



That's all

Thanks to...

- Prof. Michael Gleicher for his CS599 course

<https://pages.graphics.cs.wisc.edu/559-sp22/>

- https://www.cs.utexas.edu/users/fussell/courses/cs384g-fall2011/lectures/lecture17-Subdivision_curves.pdf

Thanks to...

- Freya Holmér
- “The Beauty of Bézier Curves”
- <https://www.youtube.com/watch?v=aVwxzDHniEw>
- “The Continuity of Splines”
- <https://www.youtube.com/watch?v=jvPPXbo87ds>

