# *Geometric Transformation*

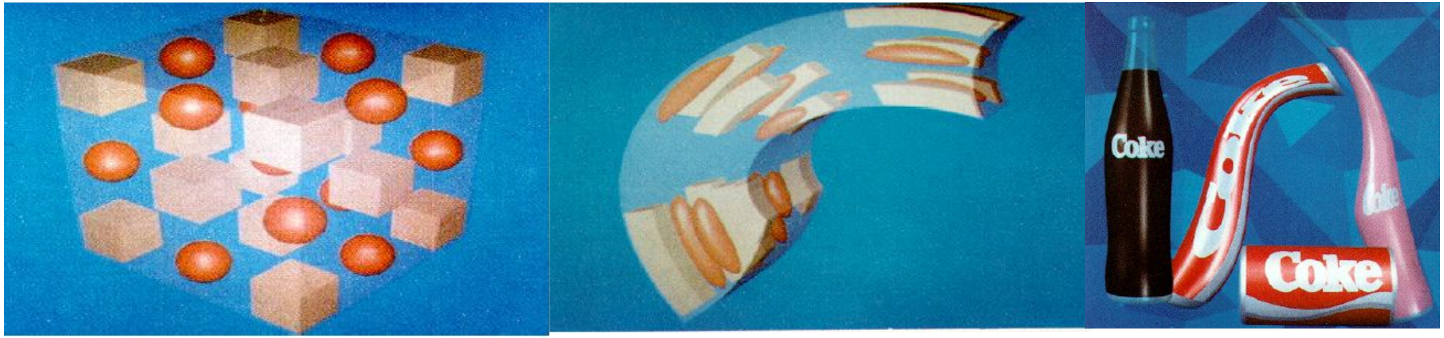
# *What is a Transformation*

- Transformation:
  - An operation that changes one configuration into another
- For images, shapes, *etc.*
  - A geometric transformation maps positions that define the object to other positions
  - Linear transformation means the transformation is defined by a linear function... which is what matrices are good for.

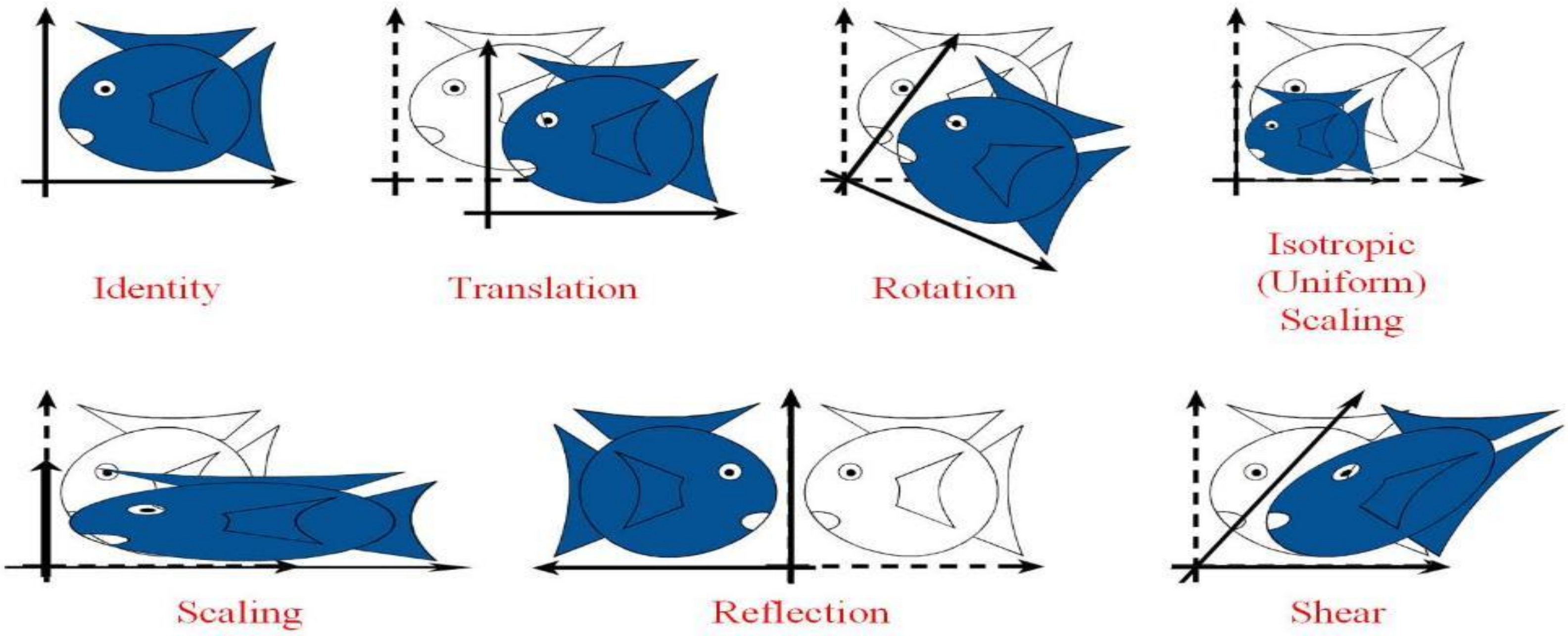
# *What is a Transformation?*

- A function that maps points $x$ to points $x'$:

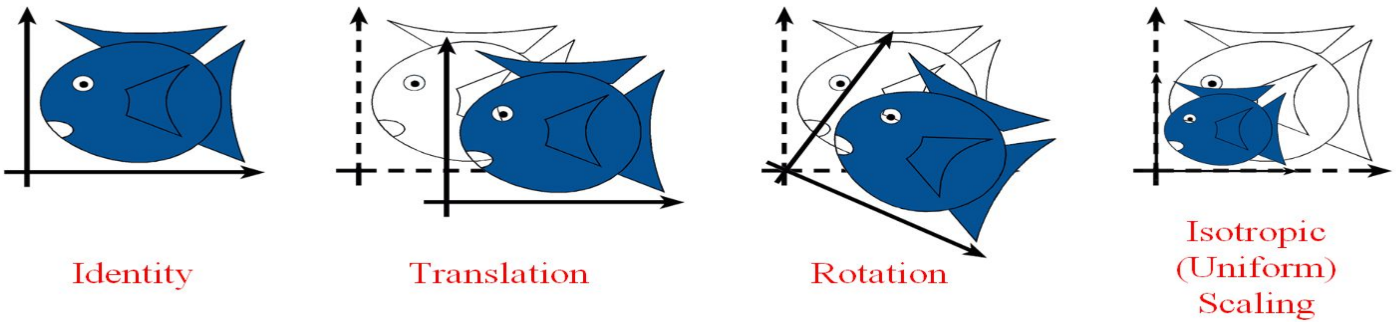
  Applications: animation, deformation, viewing, projection, real-time shadows, …



# *Examples of Transformation*



# *Simple Transformations*




- Can be combined
- Are these operations invertible?

  *Yes, except scale = 0*

# Rigid-Body / Euclidean Transforms

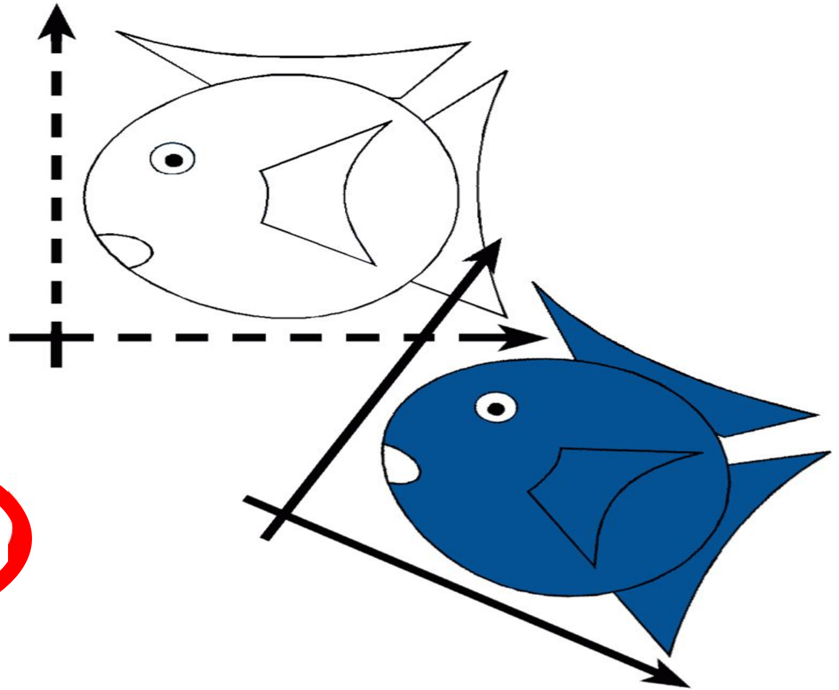
- Preserves distances
- Preserves angles

***Rigid / Euclidean***

Translation

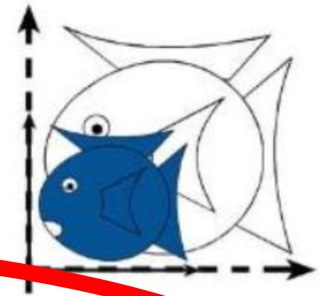
Identity

Rotation



# *Similitudes / Similarity Transforms*

- Preserves angles



***Similitudes***

***Rigid / Euclidean***

Translation

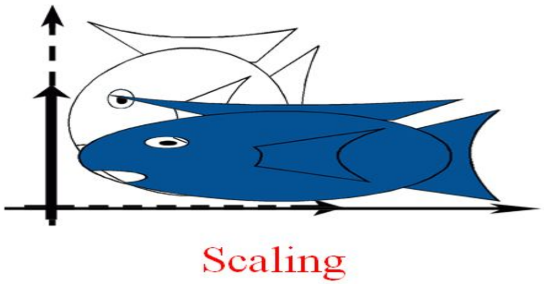
Identity

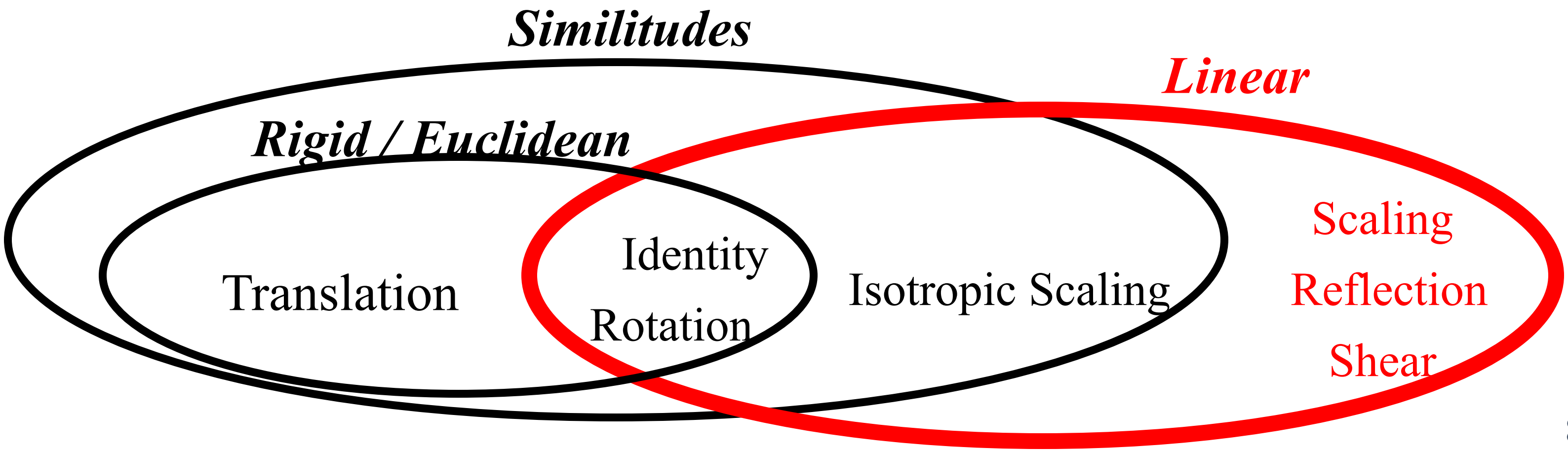
Rotation

Isotropic Scaling

Isotropic (Uniform) Scaling

# *Linear Transformations*

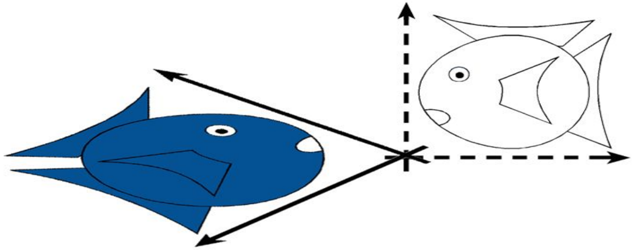


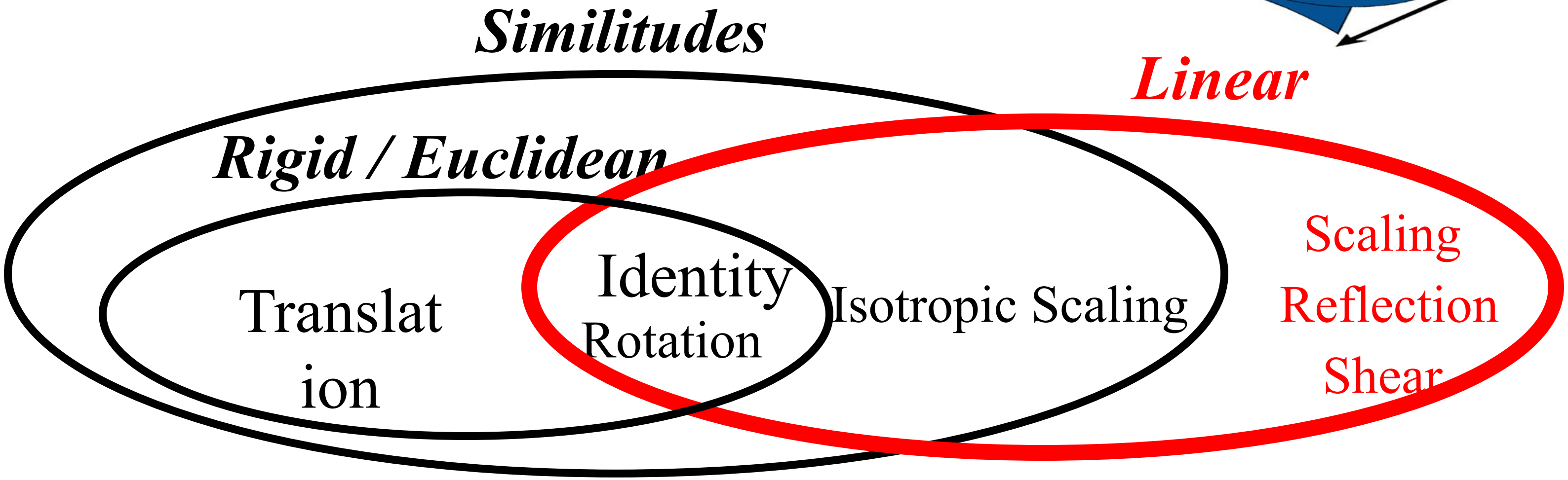



# *Linear Transformations*

***Additivity: L(p + q) = L(p) + L(q)***

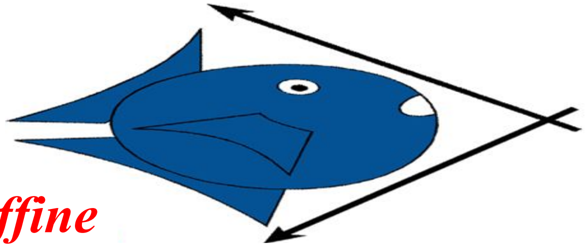
***Homogeneity: L(ap) = a L(p)***

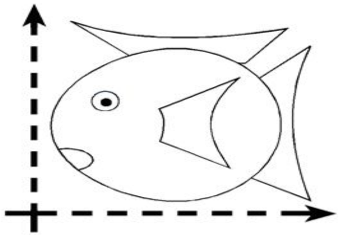


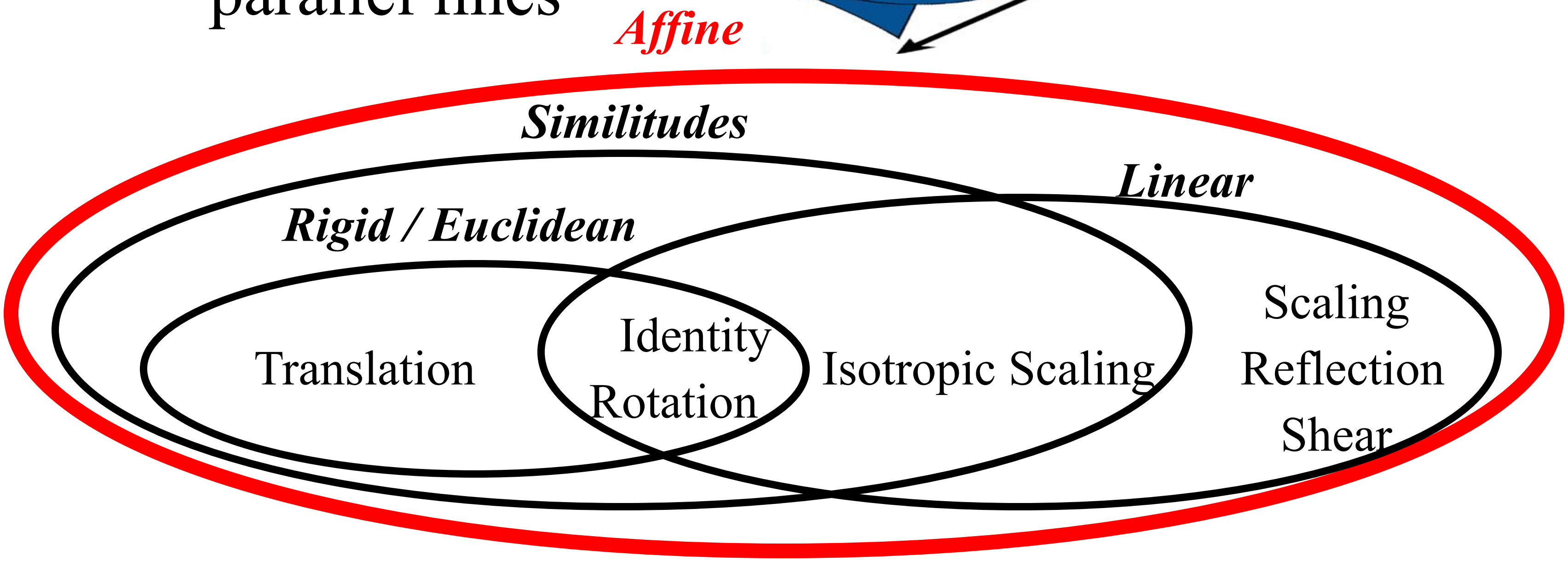


# *Affine Transformations*

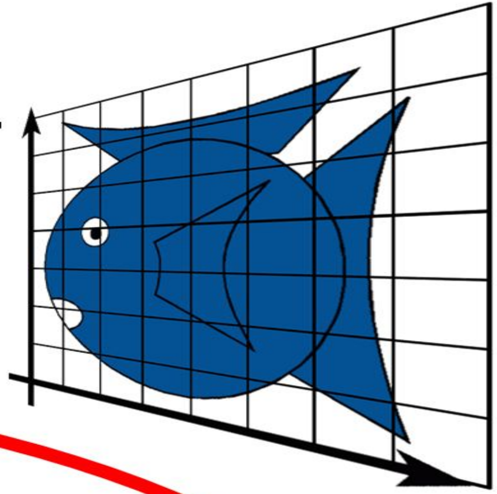
- preserves parallel lines



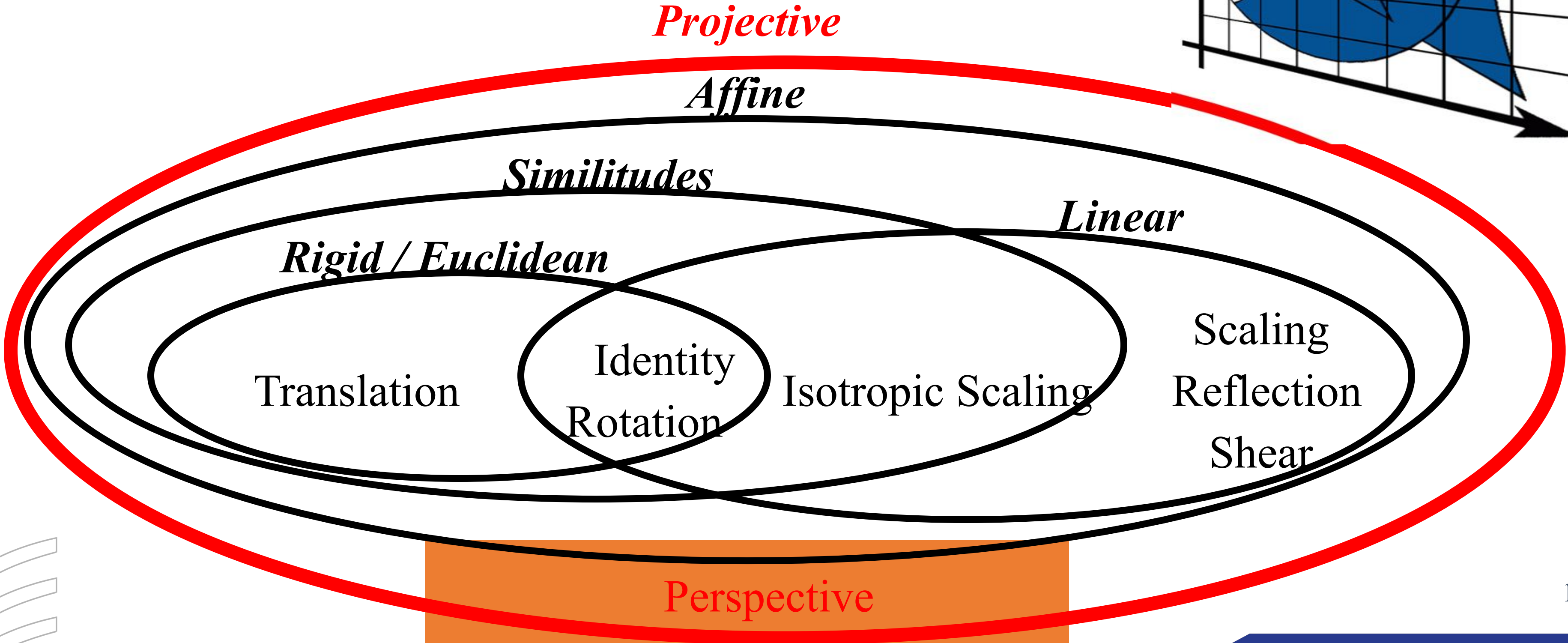




# *Projective Transformations*



- preserves lines



# How are Transforms Represented?

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = Mp + t$$

# Translation in homogeneous coordinates

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

Cartesian formulation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = M p + t$$

Homogeneous formulation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = M p$$

# Homogeneous Coordinates

- Translation, scaling and rotation are expressed (non-homogeneously) as:
  - translation: $P' = P + T$
  - Scale: $P' = S \cdot P$
  - Rotate: $P' = R \cdot P$
- Composition is difficult to express, since translation not expressed as a matrix multiplication
- Homogeneous coordinates allow all three to be expressed homogeneously, using multiplication by $3 \times 3$ matrices
- W is 1 for affine transformations in graphics

# *Homogeneous Coordinates*

- Add an extra dimension
  - in 2D, we use 3 x 3 matrices
  - In 3D, we use 4 x 4 matrices
- Each point has an extra value, *w*

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$p' = \color{red}{M}\, p$$

# Homogeneous Coordinates

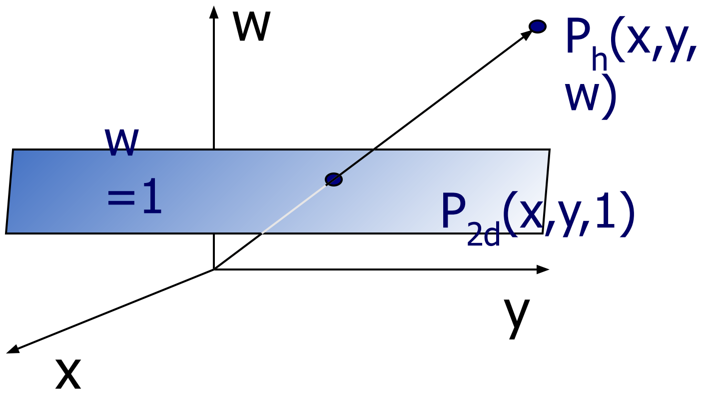
- Most of the time w = 1, and we can ignore it

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- If we multiply a homogeneous coordinate by an *affine matrix*, w is unchanged

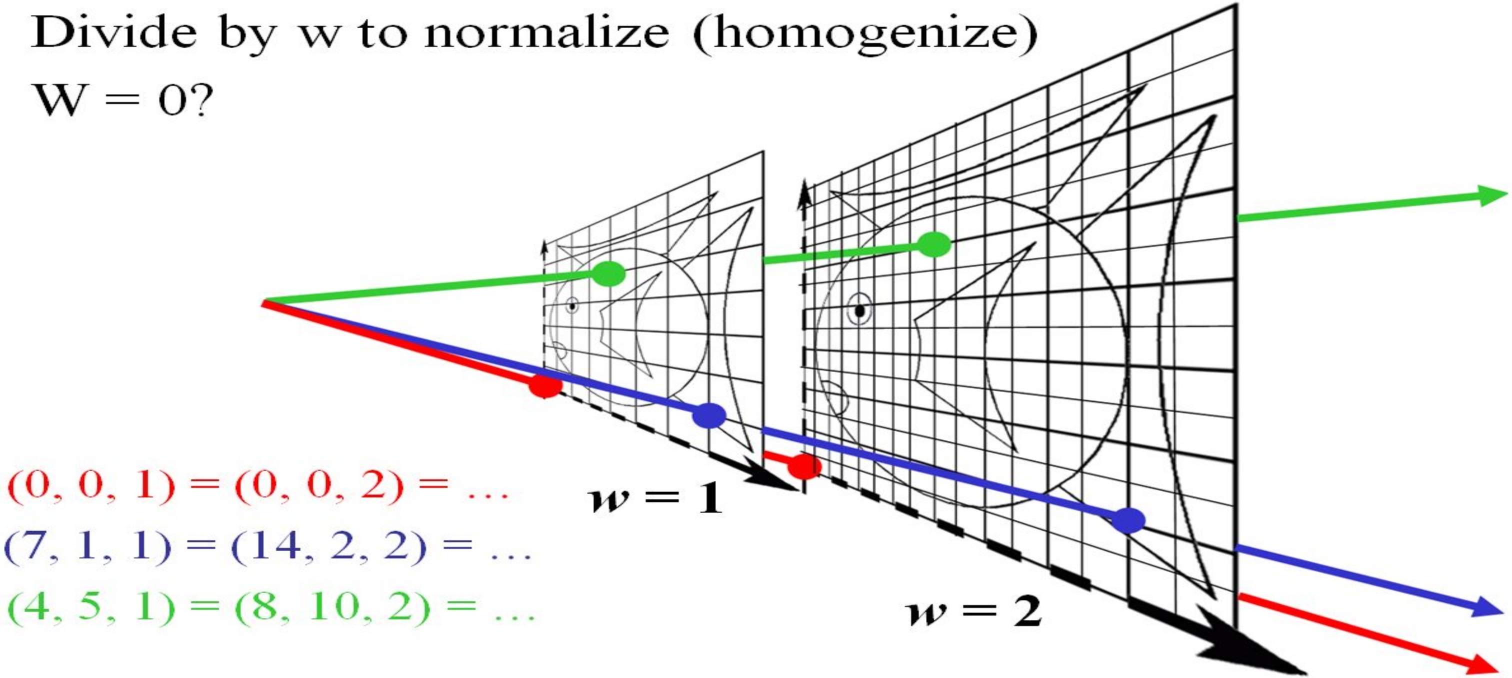
# *Homogeneous Coordinates*

- $P_{2d}$ is a projection of $P_h$ onto the $w = 1$ plane
- So an infinite number of points correspond to : they constitute the whole line (tx, ty, tw)



# *Homogeneous Visualization*

- Divide by w to normalize (homogenize)
- W = 0?



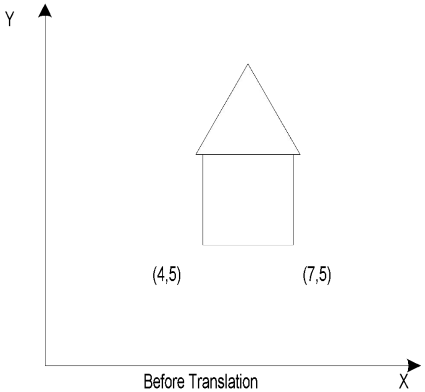
# *Mechanics of Rigid Transformations*

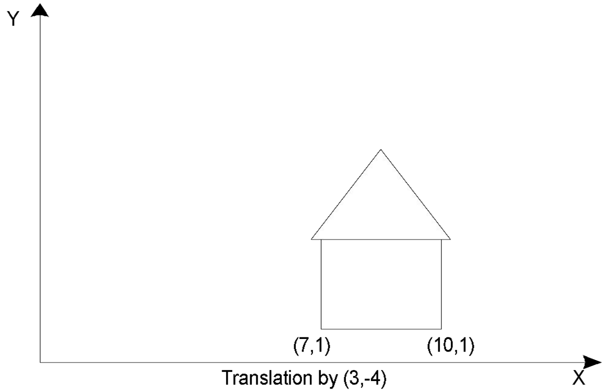
Translate

Rotate

Scale

# *Translation – 2D*



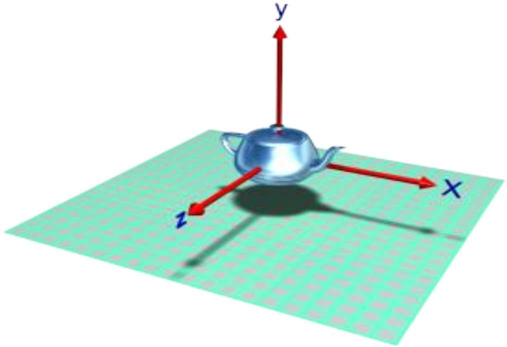

$$x' = x + d_x$$
$$y' = y + d_y$$

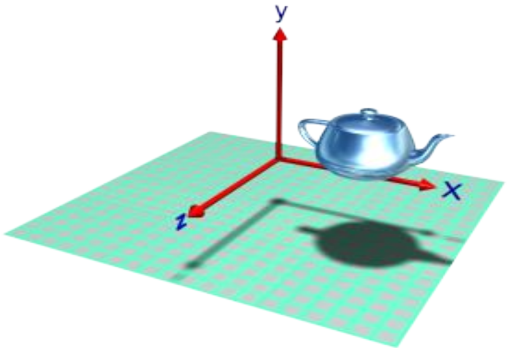
$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad T = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \qquad P' = P + T$$

*Homogeniou s Form*

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Translation – 3D



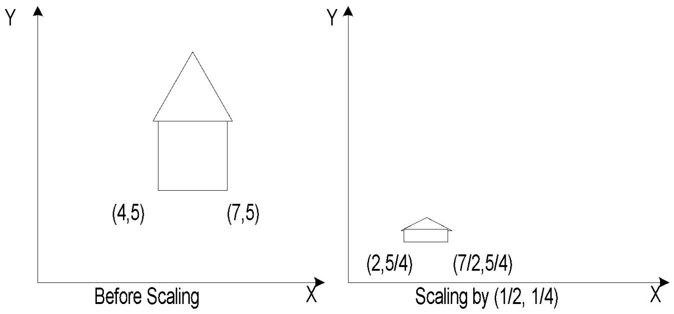

$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$

$$\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{bmatrix}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$T(d_x, d_y, d_z) \; * \; P \; = \; P'$$

# Scaling – 2D




Types of Scaling:

- Differential ( $s_x$ != $s_y$ )
- Uniform ( $s_x$ = $s_y$ )

$$x' = s_x * x$$

$$y' = s_y * y$$

$$S \quad * \quad P = P'$$

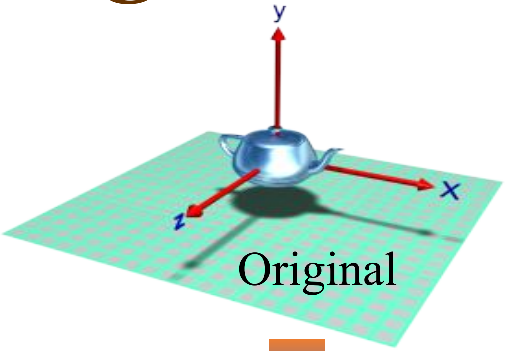
$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

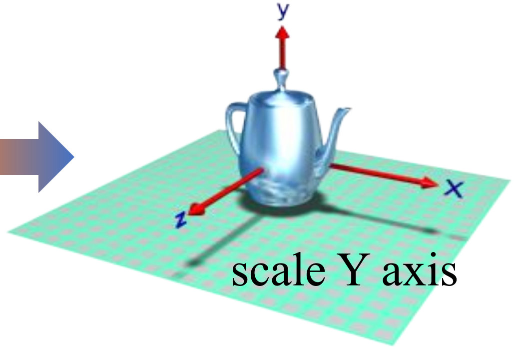
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x * s_x \\ y * s_y \end{bmatrix}$$

Homogenious Form

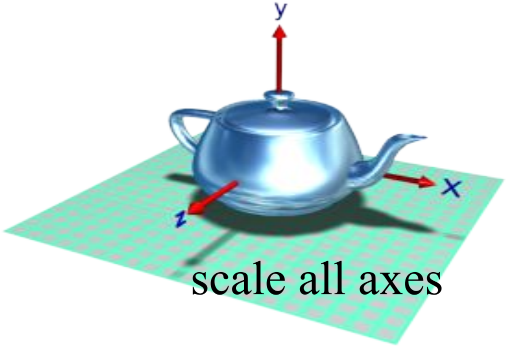
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# *Scaling – 3D*




$$x' = s_x * x$$
$$y' = s_y * y$$
$$z' = s_z * z$$




$$S(s_x, s_y, s_z) \quad * \quad P \quad = \quad P'$$

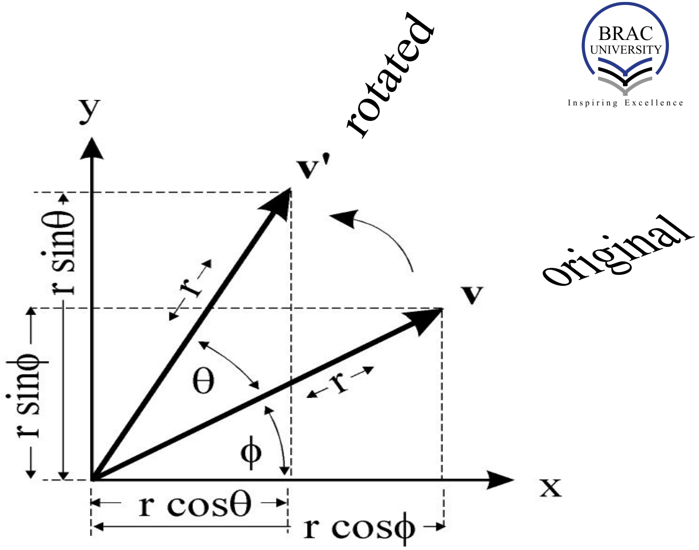
$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x * s_x \\ y * s_y \\ z * s_z \\ 1 \end{bmatrix}$$

# *Rotation – 2D*

$$\mathbf{v} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$$

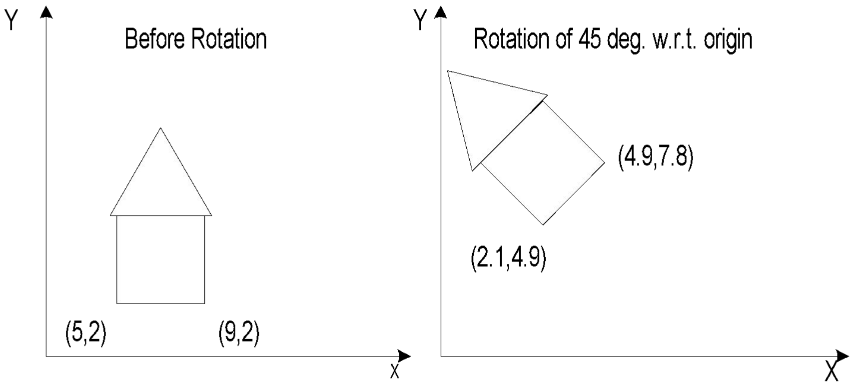
$$\mathbf{v}' = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$$




$$\text{expand}\,(\phi+\theta) \Rightarrow \begin{cases} x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\text{but} \quad \begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix} \Rightarrow \begin{matrix} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{matrix}$$

# Rotation – 2D




$$x * \cos\theta - y * \sin\theta = x'$$

$$x * \sin\theta + y * \cos\theta = y'$$

$$R \quad * \quad P = \quad P'$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x * \cos\theta - y * \sin\theta \\ x * \sin\theta + y * \cos\theta \end{bmatrix}$$

Homogenious Form

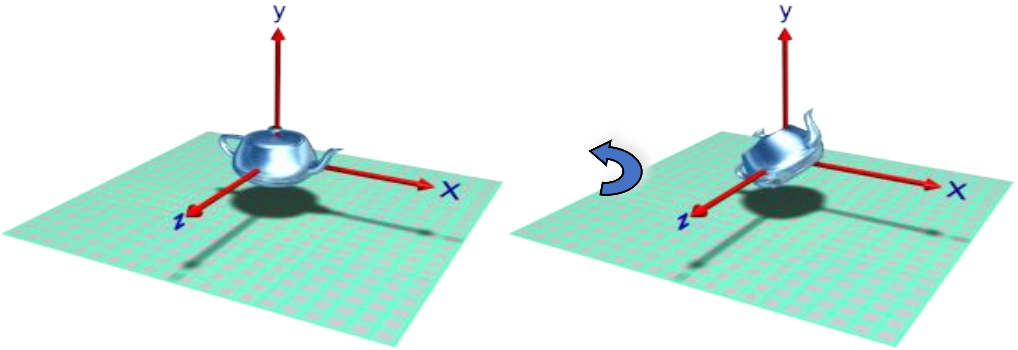
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Rotation – 3D

For 3D-Rotation 2 parameters are needed

- Angle of rotation
- Axis of rotation

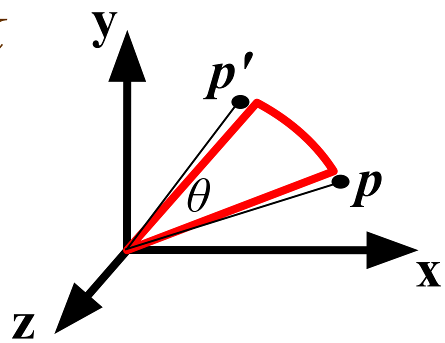
Rotation about z-axis:




$$\begin{array}{cccc} R_{\theta,k} & * & P & = & P' \\ \Downarrow & & \Downarrow & & \Downarrow \end{array}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x * \cos\theta - y * \sin\theta \\ x * \sin\theta + y * \cos\theta \\ z \\ 1 \end{bmatrix}$$

# *Rotation about Z axis: $R_{\theta,k}$*




- About z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# Rotation about Y-axis & X-axis

About y-axis

$$\begin{array}{ccc} R_{\theta,j} & * \; P \; = & P' \\ \Downarrow & \Downarrow & \Downarrow \end{array}$$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x*\cos\theta + z*\sin\theta \\ y \\ -x*\sin\theta + z*\cos\theta \\ 1 \end{bmatrix}$$

About x-axis

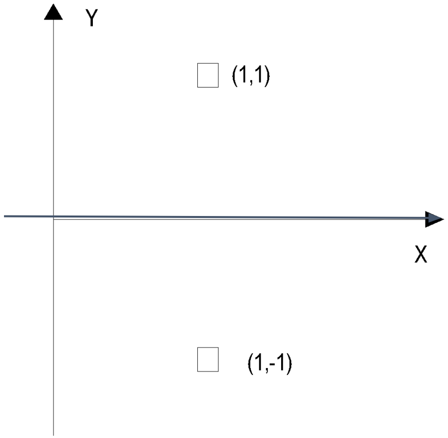
$$\begin{array}{ccc} R_{\theta,i} & * \; P \; = & P' \\ \Downarrow & \Downarrow & \Downarrow \end{array}$$

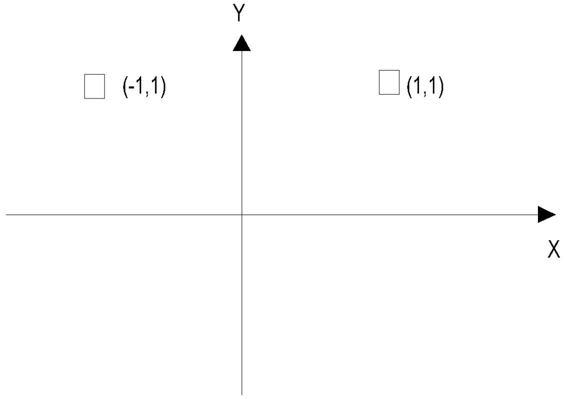
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y*\cos\theta - z*\sin\theta \\ y*\sin\theta + z*\cos\theta \\ 1 \end{bmatrix}$$

# *Properties of rotation matrix*

- The columns of rotation matrix are unit vectors perpendicular to each other
- The column vectors indicate where the unit vectors along the principal axes are transformed
- The rows of rotation matrix are unit vectors perpendicular to each other
- The row vectors indicate the vectors that are transformed into the unit vectors along the principal axes
- The inverse of rotation matrix is its transpose

# *Mirror Reflection*




Reflection about X - axis

$$x' = x \quad y' = -y$$

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about Y - axis

$$x' = -x \quad y' = y$$

$$M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation$(d_x, d_y)$

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Scale$(s_x, s_y)$

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation(θ)

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection$(x\ axis)$

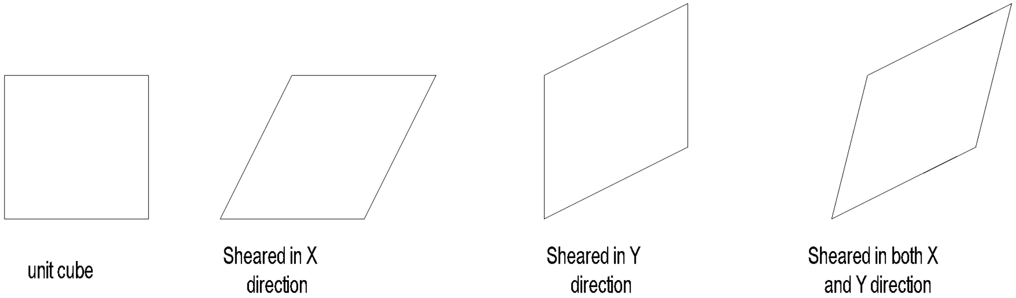
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection$(y\ axis)$

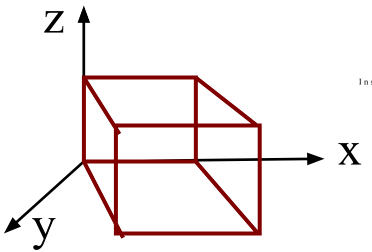
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# *Shearing Transformation*

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_{xy} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Shear along Z-axis

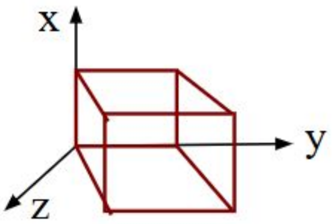



$$SH_{xy}(sh_x, sh_y) \quad * \quad P \quad = \quad P'$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + z * sh_x \\ y + z * sh_y \\ z \\ 1 \end{bmatrix}$$

# *Shear along X-axis*

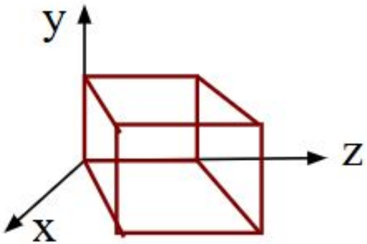



$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ Sh_y & 1 & 0 & 0 \\ Sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

**3D Shearing Matrix**

(In X axis)

# *Shear along Y-axis*




$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix

(In Y axis)

# *Inverse Transforms*

- In general: $\mathbf{A}^{-1}$ undoes effect of $\mathbf{A}$
- Special cases:
  - Translation: negate $t_x$ and $t_y$
  - Rotation: transpose
  - Scale: invert diagonal (axis-aligned scales)
- Others:
  - Invert matrix
  - Invert SVD matrices

# *Inverse Transformations*

Translaiton : $T^{-1}_{(dx,dy)} = T_{(-dx,-dy)}$

Rotation : $R^{-1}_{(\theta)} = R_{(-\theta)} = R^{T}_{(\theta)}$

Sclaing : $S^{-1}_{(sx,sy)} = S_{(1/sx,\,1/sy)}$

Mirror Ref : $M^{-1}_{x} = M_{x}$

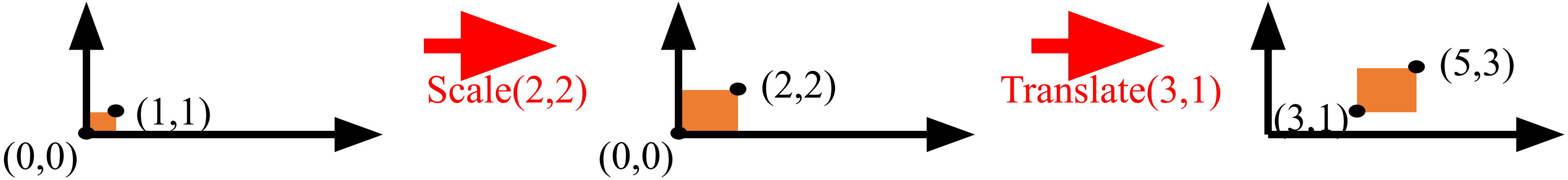
$M^{-1}_{y} = M_{y}$

Shear : $Sh^{-1}_{x} = Sh_{-x}$

# *Composing Transformations*

# *How are transforms combined?*

Scale then Translate



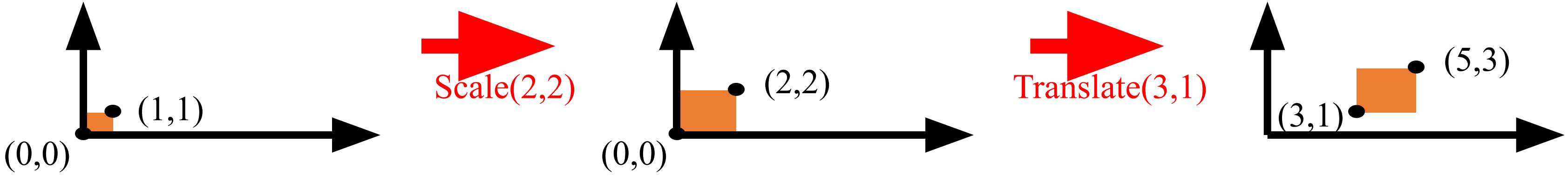

Use matrix multiplication: p' = T ( S p ) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

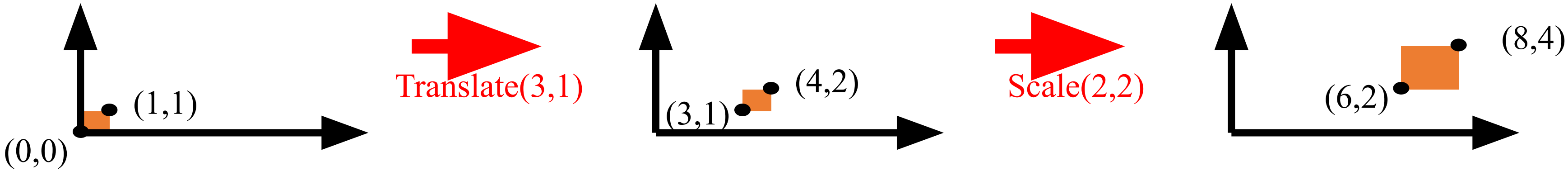
Caution: matrix multiplication is NOT commutative!

# *Non-commutative Composition*

Scale then Translate: p' = T ( S p ) = TS p




Translate then Scale: p' = S ( T p ) = ST p



# *Non-commutative Composition*

Scale then Translate: $p' = T(Sp) = TSp$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Translate then Scale: $p' = S(Tp) = STp$

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

# *Combining Translations, Rotations*

- Order matters!! TR is not the same as RT (demo)
- General form for rigid body transforms
- We show rotation first, then translation (commonly used to position objects) on next slide. Slide after that works it out the other way

# Rotate then Translate

$$P' = (TR)P = MP = RP + T$$

$$M = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & T_x \\ R_{21} & R_{22} & R_{23} & T_y \\ R_{31} & R_{32} & R_{33} & T_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$$

# Translate then Rotate

$$P' = (RT)P = MP = R(P+T) = RP + RT$$

$$M = \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{3\times3} & R_{3\times3}T_{3\times1} \\ 0_{1\times3} & 1 \end{pmatrix}$$

# *Associativity of Matrix Multiplication*

Create new affine transformations by multiplying sequences of the above basic transformations.

$$\mathbf{q} = \mathbf{CBAp}$$

$$\mathbf{q} = ((\mathbf{CB})\,\mathbf{A})\,\mathbf{p} = (\mathbf{C}\,(\mathbf{B}\,\mathbf{A}))\mathbf{p} = \mathbf{C}\,(\mathbf{B}\,(\mathbf{Ap}))\text{ etc.}$$

matrix multiplication is associative.

To transform just a point, better to do $\mathbf{q} = \mathbf{C}(\mathbf{B}(\mathbf{Ap}))$

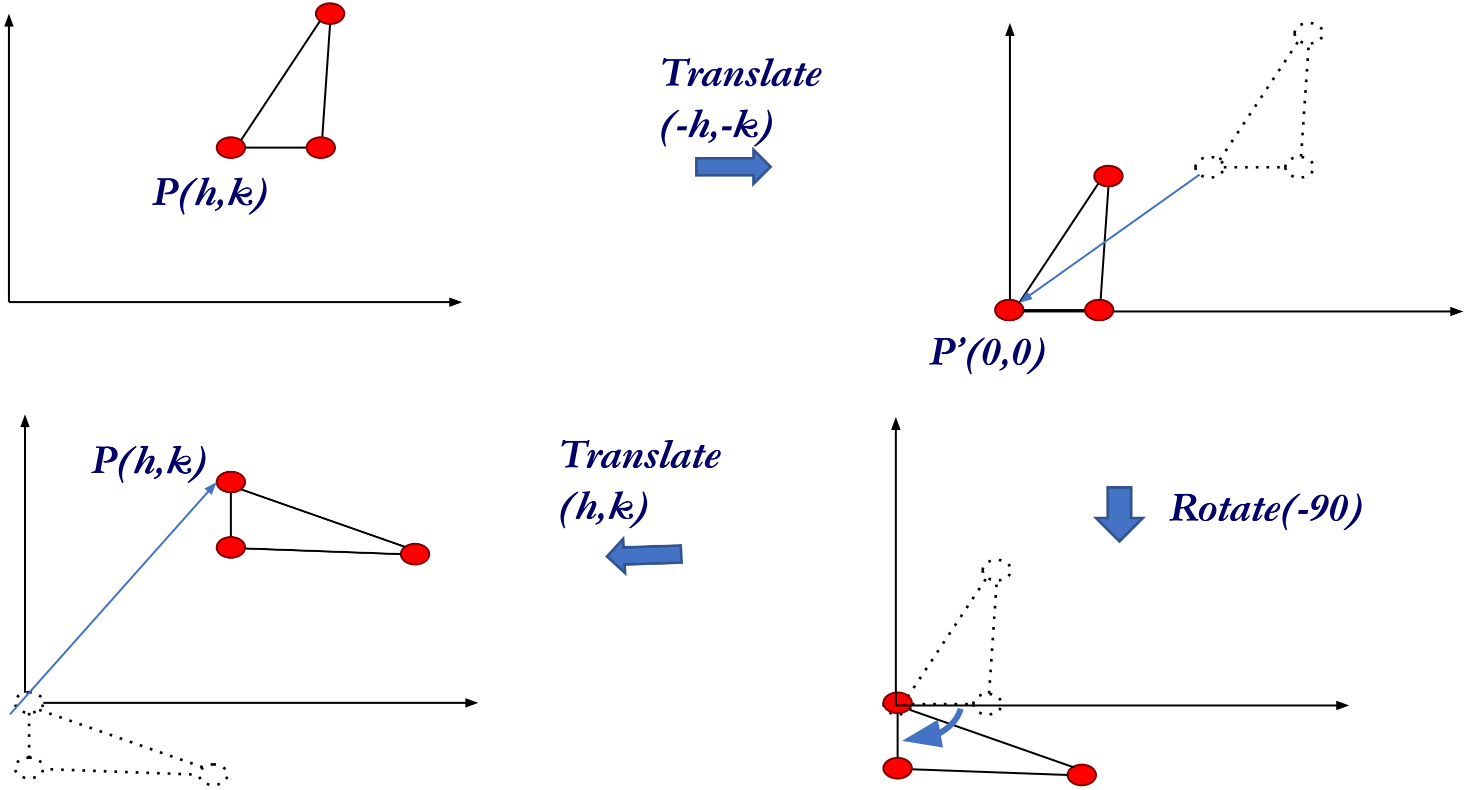
But to transform many points, best to do

$$\mathbf{M} = \mathbf{CBA}$$

then do $\mathbf{q} = \mathbf{Mp}$ for any point **p** to be rendered.

For geometric pipeline transformation, define **M** and set it up with the model-view matrix and apply it to any vertex subsequently defined to its setting.

# *Example Composite Transforms*


P(h,k)
Translate
(-h,-k)
P'(0,0)
Rotate(-90)
Translate
(h,k)
P(h,k)

# *Rotation of $\theta$ about P(h,k): $R_{\theta,P}$*

**Step 1:** Translate P(-h,-k) to origin

**Step 2:** Rotate θ w.r.t to origin

**Step 3:** Translate (0,0) to P(h,k)

$\mathbf{R_{\theta,P} = T(h\,,k) * R_{\theta} * \;T(-h\,,-k)}$

$$P' = T_{(h,k)} \times R_{(\theta)} \times T_{(-h,-k)} \times P$$

$$Composite\ matrix, M = T_{(h,k)} \times R_{(\theta)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as rotating 90 degree counterclockwise with respect to point (5, 5). Calculate the composite transformation matrix in homogeneous form. Then, find the new coordinates of the point (10, 10) after transformation.

$$Composite\ matrix, M = T_{(5,5)} \times R_{(90)} \times T_{(-5,-5)}$$

$$Composite\ matrix, M = T_{(5,5)} \times R_{(90)} \times T_{(-5,-5)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

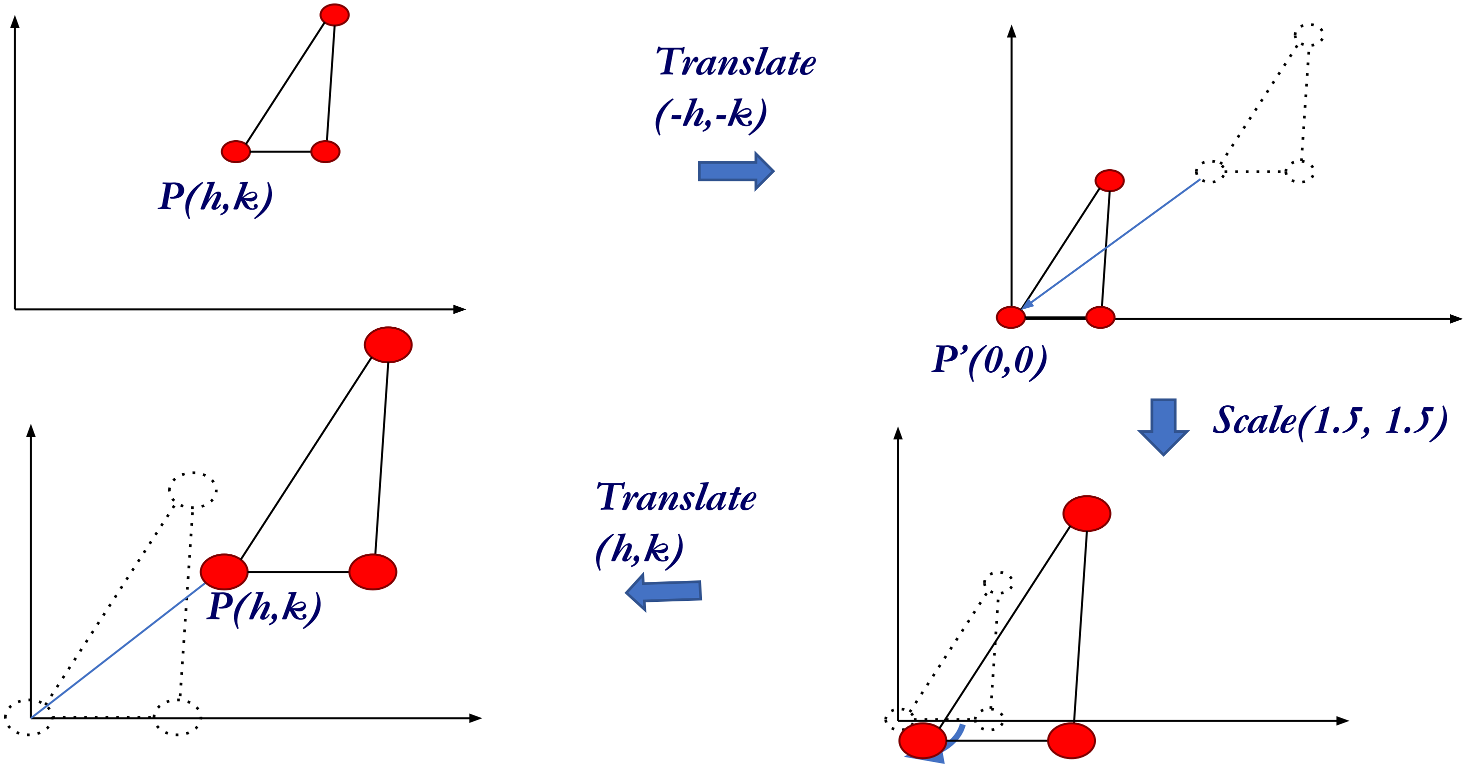
$$= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = M \times P$$

$$= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$$

So, the new coordinate of P is (0, 10)


P(h,k)
Translate
(-h,-k)
P'(0,0)
Scale(1.5, 1.5)
Translate
(h,k)
P(h,k)

# *Scaling of $S_{a,b}$ about $P(h,k)$: $R_{S,P}$*

**Step 1:** Translate P(-h,-k) to origin

**Step 2:** Scale $S_{a,b}$ w.r.t to origin

**Step 3:** Translate (0,0) to P(h,k)

$$\mathbf{R_{S,P} = T(h\ ,k) * S(a,b) * T(-h\ ,-k)}$$

$$P' = T_{(h,k)} \times S_{(a,b)} \times T_{(-h,-k)} \times P$$

$$Composite\ matrix, M = T_{(h,k)} \times S_{(a,b)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as scaling 2 times in both axis with respect to point (5, 5). Calculate the composite transformation matrix in homogeneous form. Then, find the new coordinates of the point (10, 10) after transformation.

$$Composite\ matrix, M = T_{(5,5)} \times S_{(2,2)} \times T_{(-5,-5)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

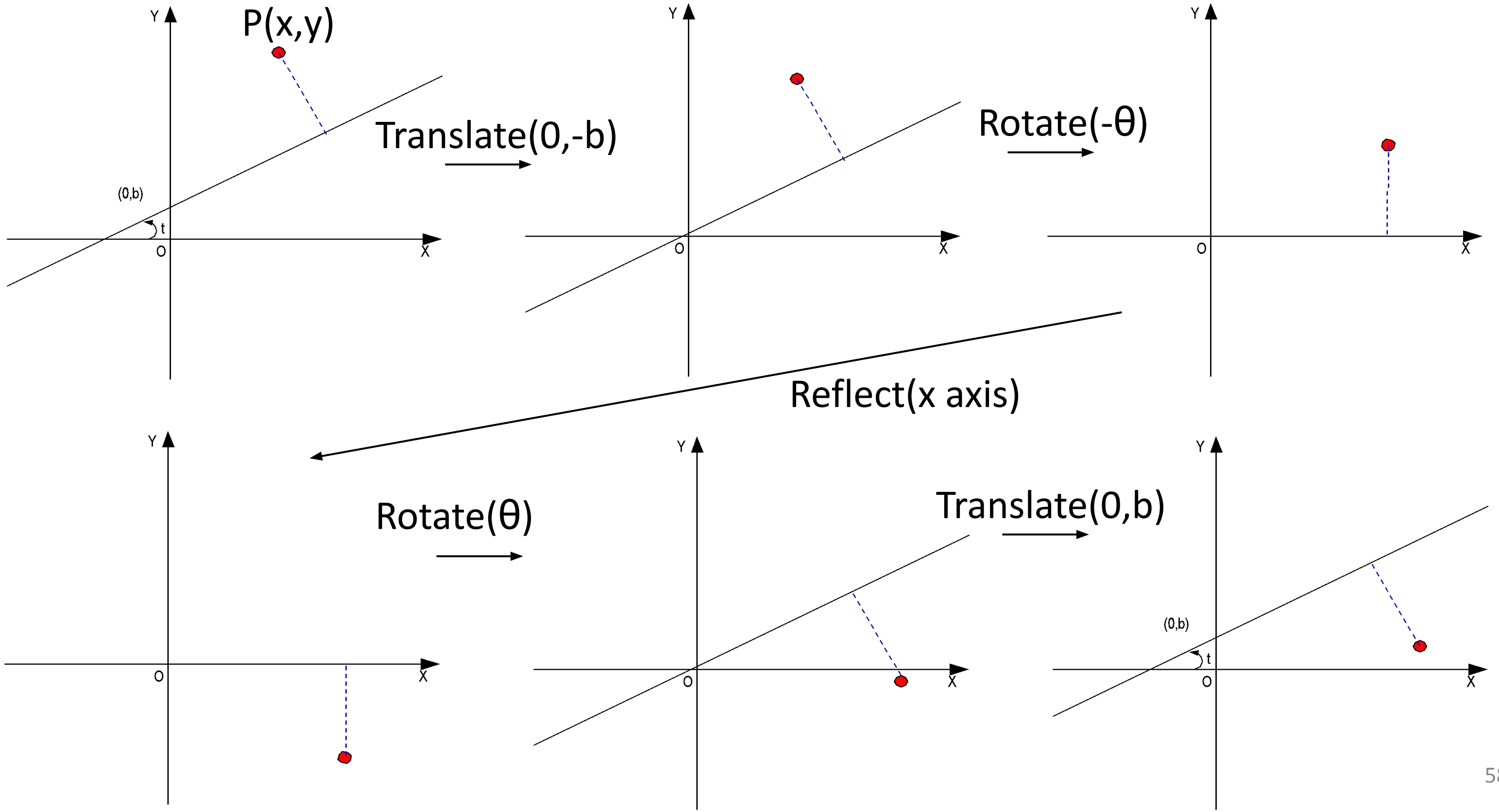
$$P' = M \times P$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 15 \\ 1 \end{bmatrix}$$

So, the new coordinate of P is (15, 15)

# Reflection about line L, $M_L$



# Reflection about line L, $M_L$

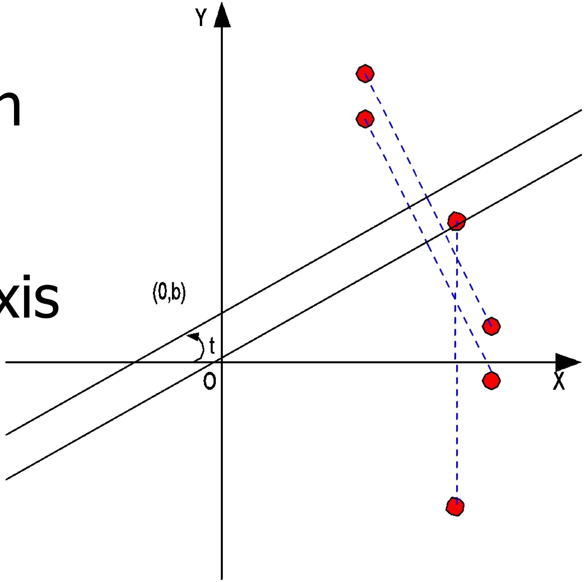
**Step 1:** Translate (0,-b) to origin

**Step 2:** Rotate -θ degrees

**Step 3:** Mirror reflect about X-axis

**Step 4:** Rotate θ degrees

**Step 5:** Translate origin to (0,b)




$$M_L = T(0,b) * R(\theta) * M_x * R(-\theta) * T(0,-b)$$

# Reflect the point (10, 5) with respect to the line y=x+2.

$$b = 2, m = 1, \theta = \tan^{-1} 1 = 45$$

$$M_{composite} = T_{(0,2)} \times R_{(45)} \times Refl_{(x\ axis)} \times R_{(-45)} \times T_{(0,-2)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix}$$

New coordinate (3, 12)

# Exercise

**Check this link for the sample problems:**

https://docs.google.com/document/d/1FGYfEREicKold6MJ6hzQoVzP5hbnD-2lu23xl5vNYSQ/edit?usp=sharing

# References


- Chapter 4 and Chapter 6, Schaum's Outline of Computer Graphics (2nd Edition) by Zhigang Xiang, Roy A. Plastock
- Chapter 5, Computer Graphics: Principles and Practice in C (2nd Edition) by James D. Foley, Andries van Dam, Steven K. Feiner, John F. Hughes
- Chapter 6, Fundamentals of Computer Graphics, by Peter Shirley