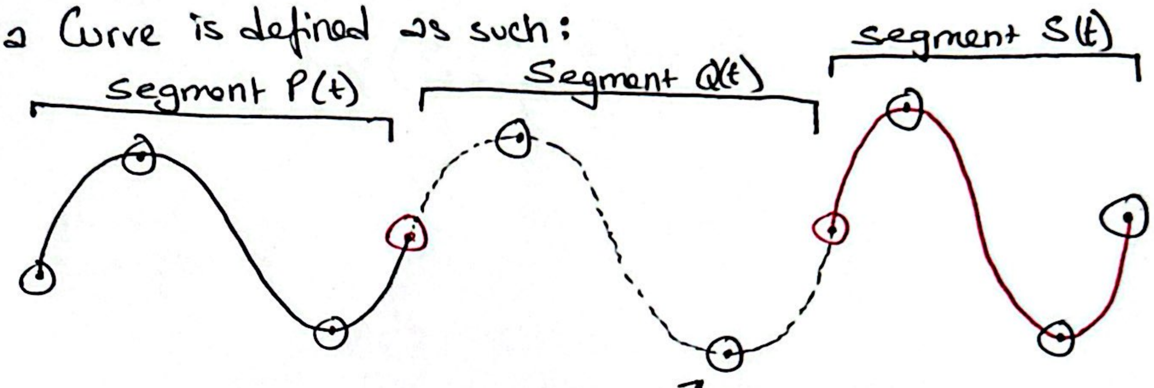
CURVES ~ (PART 2)

Recap of Previous Note:

→ we decided to divide the higher degree polynomial curve into smaller segments of (cubic polynomials).
The reason was : Computational complexity.

→ There's another reason (modification)

∅ Say a Curve is defined as such:



Now that this curve of degree 9 has been divided into 3 segments (P(t), Q(t), S(t)).
Ex: Changing a point in the curve Q(t) won't affect the rest of the curve segments (P(t) OR S(t))

→ Hence, it is easier to modify & control.

∅ if this curve was not defined using smaller cubic polynomial curves. Then changing this point would've created an undesired effect on the whole curve itself.

Linear Interpolation (Parametric)

∅ if a line / two points are defined as P_0 & P_1

it can represented as such:

$$P(t) = P_0 + t(P_1 - P_0)$$

∅ Instead, the notation that we're going to use from now on can be stated as such:

$$\underbrace{lerp}_{\text{linear interpolation}}(\underbrace{P_0}_{\text{start}}, \underbrace{P_1}_{\text{end}}, \underbrace{t}_{t \in \{0, 1\}}) = P_0 + t(P_1 - P_0)$$

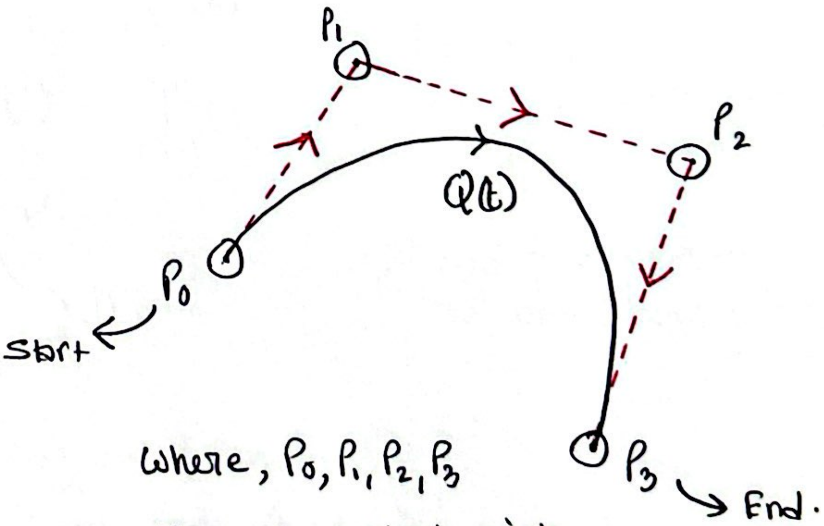
∅ BÉZIER CURVE

∅ We will have some control points. Moving them around would change the shape of the curve.

4 points → Cubic Bezier Curve
3 " → Quadratic " "
2 " → Linear " "

We can increase the number of points as well.

Example: 4 points (Cubic Bezier Curve)



Where, P_0, P_1, P_2, P_3 are our control points.

We are going to be using linear interpolation to determine the trajectory of the curve from t=0/1

∅ For demonstration on how the algorithm works: See the link provided in the slide.

Ø For 2 points (Line)

We know,

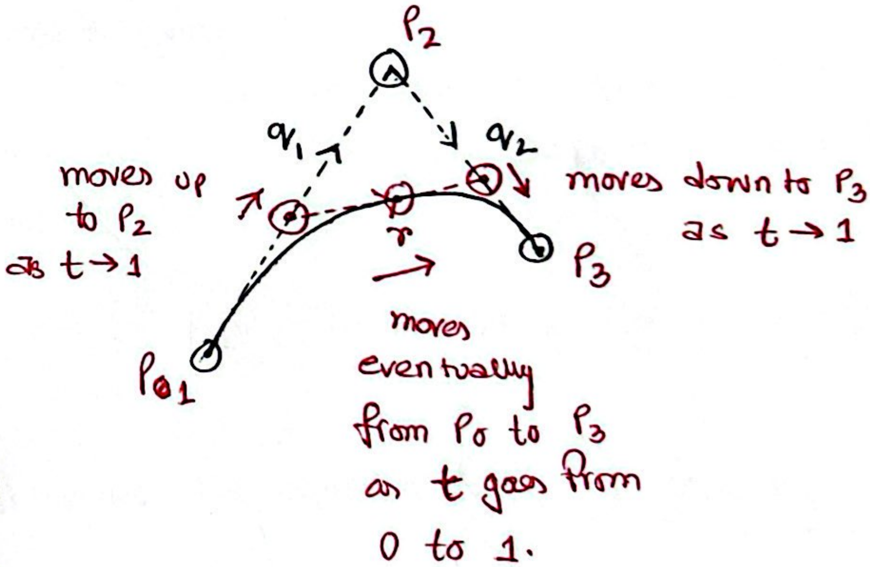
$$lerp(p1, p2, t) = P_1 + t(P_2 - P_1)$$

Ø For 3 points (Quadratic Bezier Curve)

We'll have three control points (P_1, P_2, P_3)

→ Then we'll run linear interpolation from $(P_1$ to $P_2)$ & $(P_2$ to $P_3)$ simultaneously resulting in q_1 & q_2

→ Then, simultaneously we'll also be running a linear interpolation from q_1 to q_2 where, ($t = 0$ goes to $t = 1$). This will result in r. And the respective (x, y, z) values of r at t interval will result in drawing the curve itself.



So,

$$q_1 = lerp(P_1, P_2, t)$$

$$q_2 = lerp(P_2, P_3, t)$$

$$r = lerp(q_1, q_2, t)$$

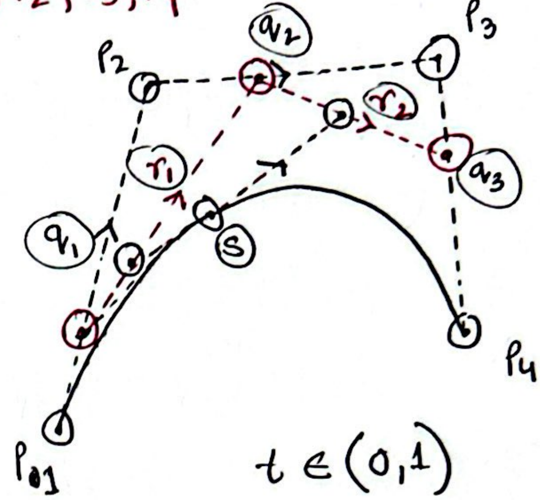
→ We will be drawing the points along r as t goes from 0 to 1.

Ø See the animation provided to understand better: Check Slide.

Ø for 4 points (Cubic Bezier Curve)

We'll have 4 control points P_1, P_2, P_3, P_4

Same concept applied.



Ø $q_1 = lerp(P_1, P_2, t)$

$q_2 = lerp(P_2, P_3, t)$

$q_3 = lerp(P_3, P_4, t)$

$t \in (0,1)$

→ See the animation again in the slide.

Ø $r_1 = lerp(q_1, q_2, t)$

$r_2 = lerp(q_2, q_3, t)$

Ø $s = lerp(r_1, r_2, t)$

→ We'll be plotting the point in s as t goes from 0 to 1

→ The concept here is going to be:

→ we'll start off from S and recursively make our way up to P_1, P_2, P_3, P_4 terms.

This is called "De Castlejau's Algo"

Ø Now we'll derive the equation for S recursively.

$$\Rightarrow S = \text{lerp}(r_1, r_2, t)$$

Converting to parametric form of function.

$$\emptyset \Rightarrow S = r_1 + (r_2 - r_1)t$$

$$= \underbrace{q_1 + (q_2 - q_1)t}_{r_1} + \{\underbrace{q_2 + (q_3 - q_2)t}_{r_2} - \underbrace{q_1 - (q_2 - q_1)t}_{r_1}\}t$$

$$= q_1 + tq_2 - tq_1 + \{q_2 + tq_3 - tq_2 - q_1 - tq_2 + tq_1\}t$$

$$= q_1 + tq_2 - tq_1 + tq_2 + t^2q_3 - t^2q_2 - tq_1 - t^2q_2 + t^2q_1$$

$$= q_1(1 - t - t + t^2) + q_2(t + t - t^2 - t^2) + q_3(t^2)$$

$$= q_1(1 - 2t + t^2) + q_2(2t - 2t^2) + q_3(t^2)$$

$$= (P_1 + (P_2 - P_1)t)\cdot(1 - 2t + t^2) + \cancel{q}(P_2 + (P_3 - P_2)t)\cdot(2t - 2t^2) + \dashrightarrow$$

$$\dashrightarrow + (P_3 + (P_4 - P_3)t)(t^2)$$

$$= (P_1 + tP_2 - tP_1)\cdot(1 - 2t + t^2) + (P_2 + tP_3 - tP_2)(2t - 2t^2) + (P_3 + tP_4 - tP_3)\cdot(t^2)$$

↓ after further expansion we'll take P_1, P_2, P_3, P_4 as common. Resulting in the following derivation.

$$\cancel{(1 - t^3)\cdot P_1 +}$$

$$S = \underbrace{(1 - 3t + 3t^2 - t^3)}_{\text{coefficient of } P_1} P_1 + \underbrace{(3t - 6t^2 + 3t^3)}_{\text{coeff of } P_2} P_2 + \underbrace{(3t^2 - 3t^3)}_{\text{coeff of } P_3}\cdot P_3 + \underbrace{(t^3)}_{\text{coeff of } P_4}\cdot P_4$$

Also ↓

$Q(t) \rightarrow$ equation for the curve in terms of t.

Matrix Eq^n of a Cubic Bezier Curve

$$Q(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

T matrix

M_B: The Basis Matrix of Bezier Curve

G_B: Geometric properties matrix of the curve → control points.

So,

$$Q(t) = T \cdot M_B \cdot G_B$$

→ So for a cubic Bezier Curve we can calculate values of (x, y, z) at different intervals of t where, $t \in (0,1)$.

Ø if say we want to calculate the value of x at $t = 0.5$

→ The t matrix will be : $\begin{bmatrix} 0.5^3 & 0.5^2 & 0.5 & 1 \end{bmatrix}$

→ The Basis matrix will be as it is for Cubic curve (Bezier).

→ The G_B matrix will only have the x coordinate values of P_4, P_3, P_2, P_1 :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

x coordinate values of the control points.

Ø in case of 3D → we'll have to find the value across z axis as well. The z values of control points will be given.

Q) Suppose a Cubic Bezier Curve is defined by the control points : $P_1(0,0)$, $P_2(2,2)$, $P_3(4,-2)$, $P_4(6,0)$ find the x & y coordinate values at $t = 0.75$.

We Know,

$$Q(t) = [t^3 \; t^2 \; t \; 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

So,

$$\underset{\text{t value}}{x(0.75)} = [27/64 \;\; 9/16 \;\; 3/4 \;\; 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$$

$$= [1/64 \;\; 9/64 \;\; 27/64 \;\; 27/64]\begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} = 9/2 \quad \text{Ans:}$$

same thing for y.

$$y(0.75) = [\underbrace{1/64 \;\; 9/64 \;\; 27/64}_{\text{this won't change}} \;\; 27/64]\begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix} = -9/16 \quad \text{Ans:}$$

So,

$$Q(0.75) = (4.5, -0.5625) \quad \text{Ans:}$$