Projection

Introduction

Projection

Is a type of Transformation

Transforming points in a coordinate of dimension n into a coordinate system of dimension m where m < n

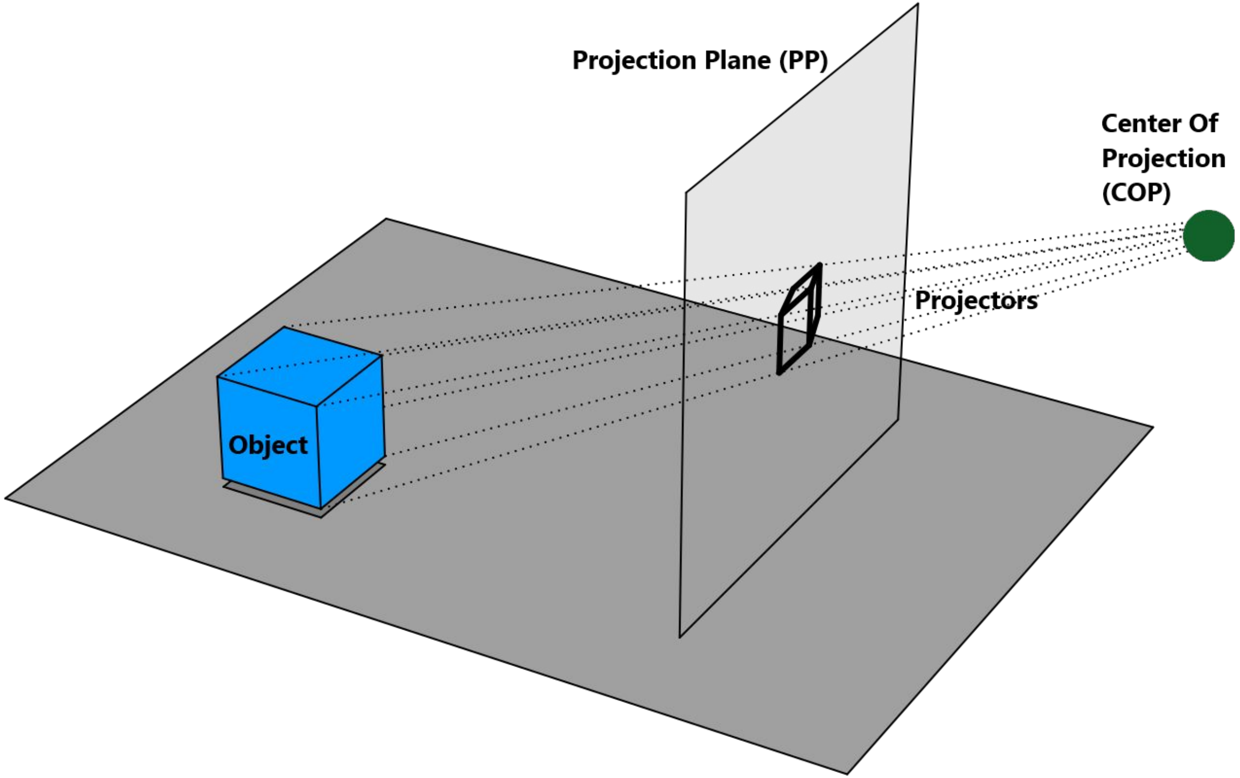
Planar Geometric Projection: Projecting an object on to a flat plane

- Turning 3D shapes to its 2D plane (this chapter)
- From 2D to 1D also possible

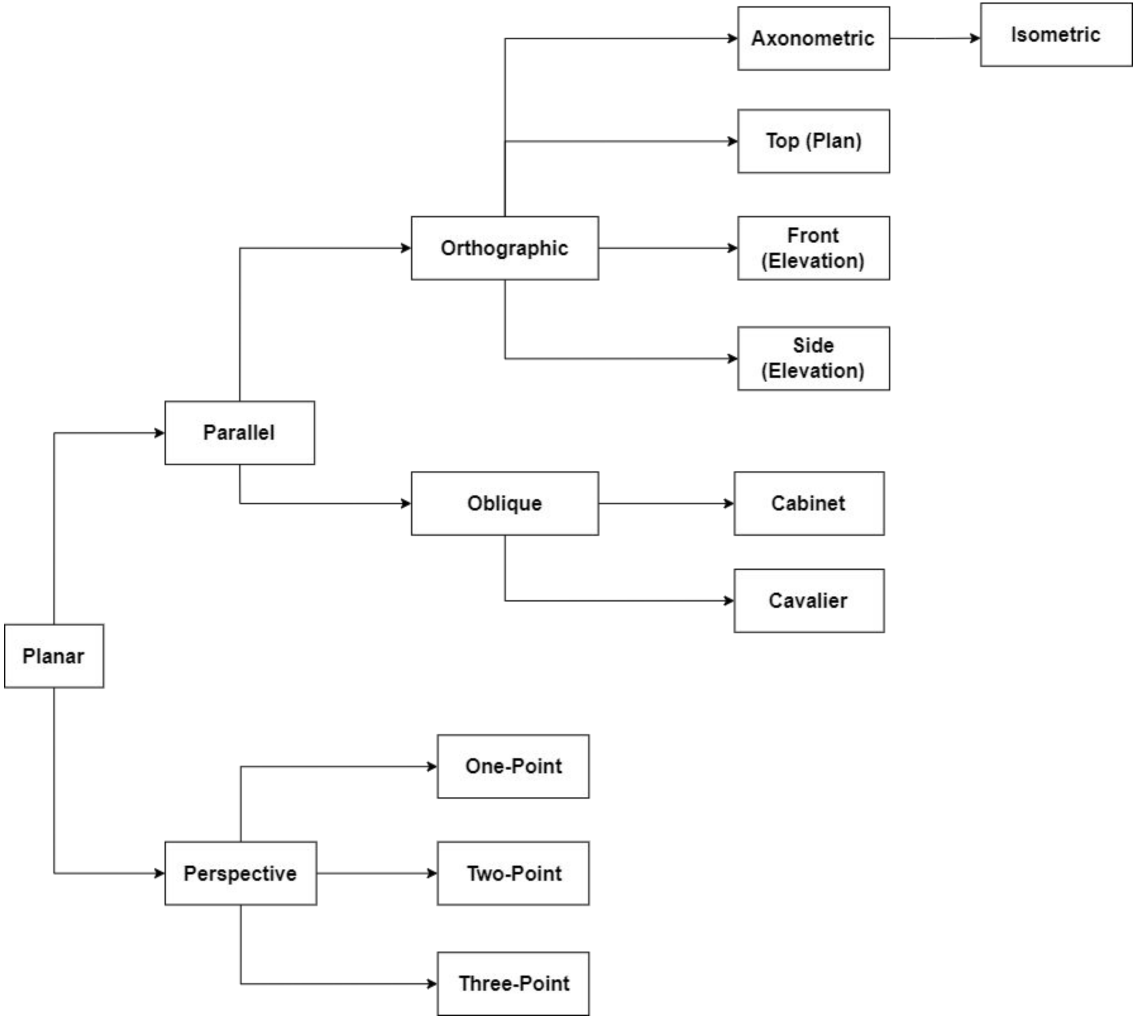
Projection (Terminologies)

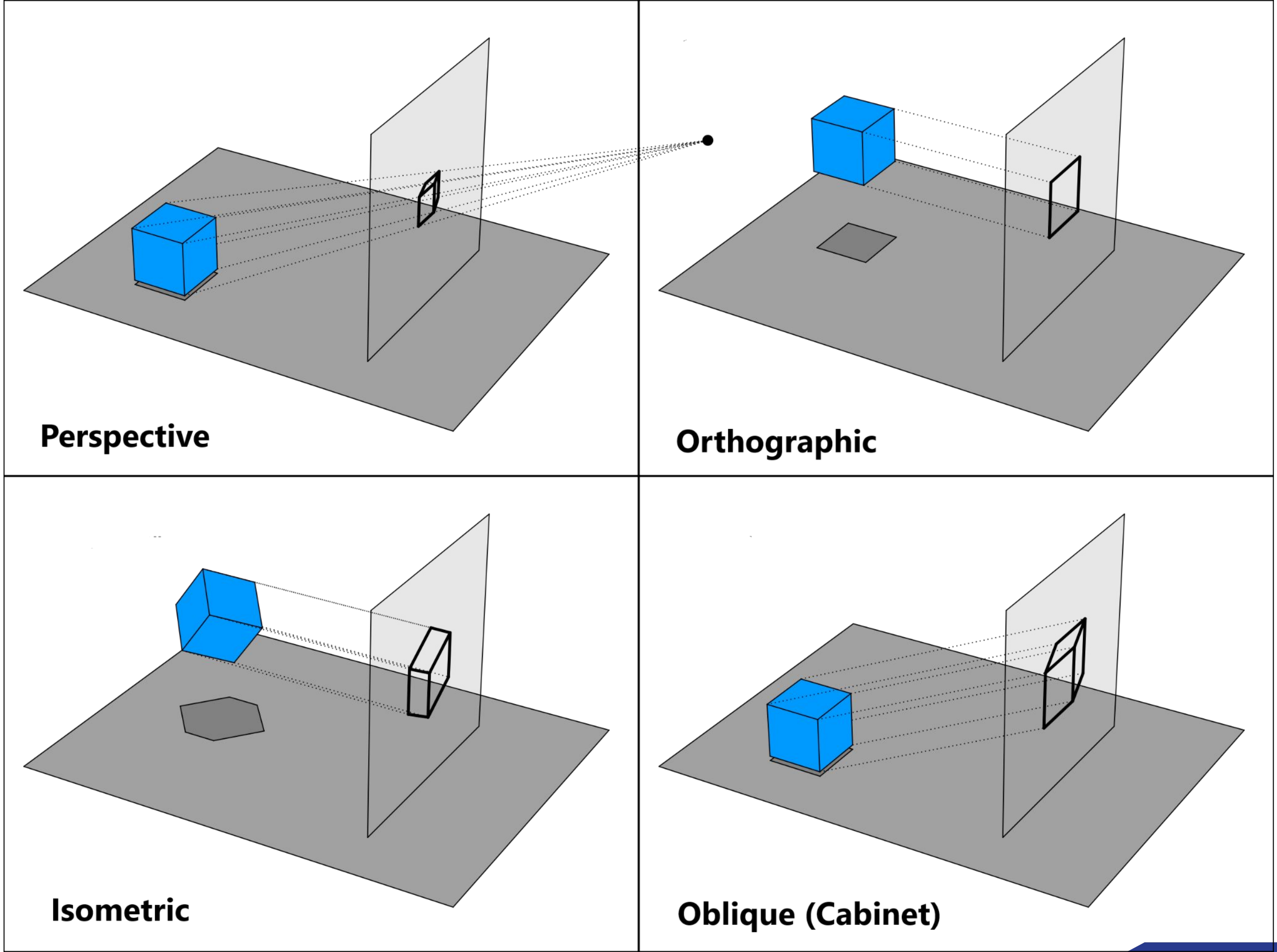
- **Object/Target Point (P)**: A single point on an object/geometry we're trying to project (calculate projection of)
- **Projected Point (P′)**: The projected point after a projection transformation is applied for a given point P′. Projected points together constitutes the projected image
- **Projectors**: The "rays" responsible for projection. They'll intersect the projection plane.
- **Projection Plane (PP)**: The plane where the projected image of the object will appear on. This plane holds all of the P′ points. Also known as view plane.
- **Center Of Projection (COP)**: The singular point where all the projectors/projection rays converge (only applicable to perspective projection).
- **Direction Of Projection (DOP)**: The direction towards which the parallel projection rays go towards (only applicable to parallel projection).

Projection (Anatomy)

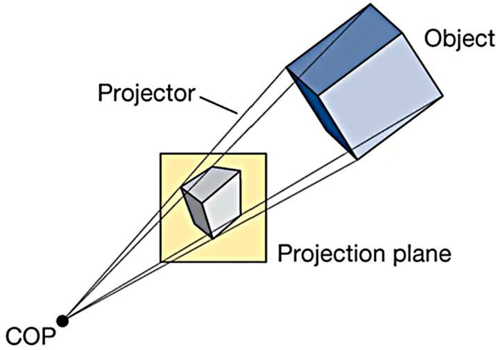


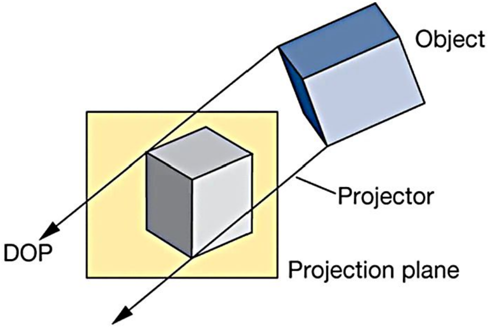
Projection (Types)




Perspective
Orthographic
Isometric
Oblique (Cabinet)

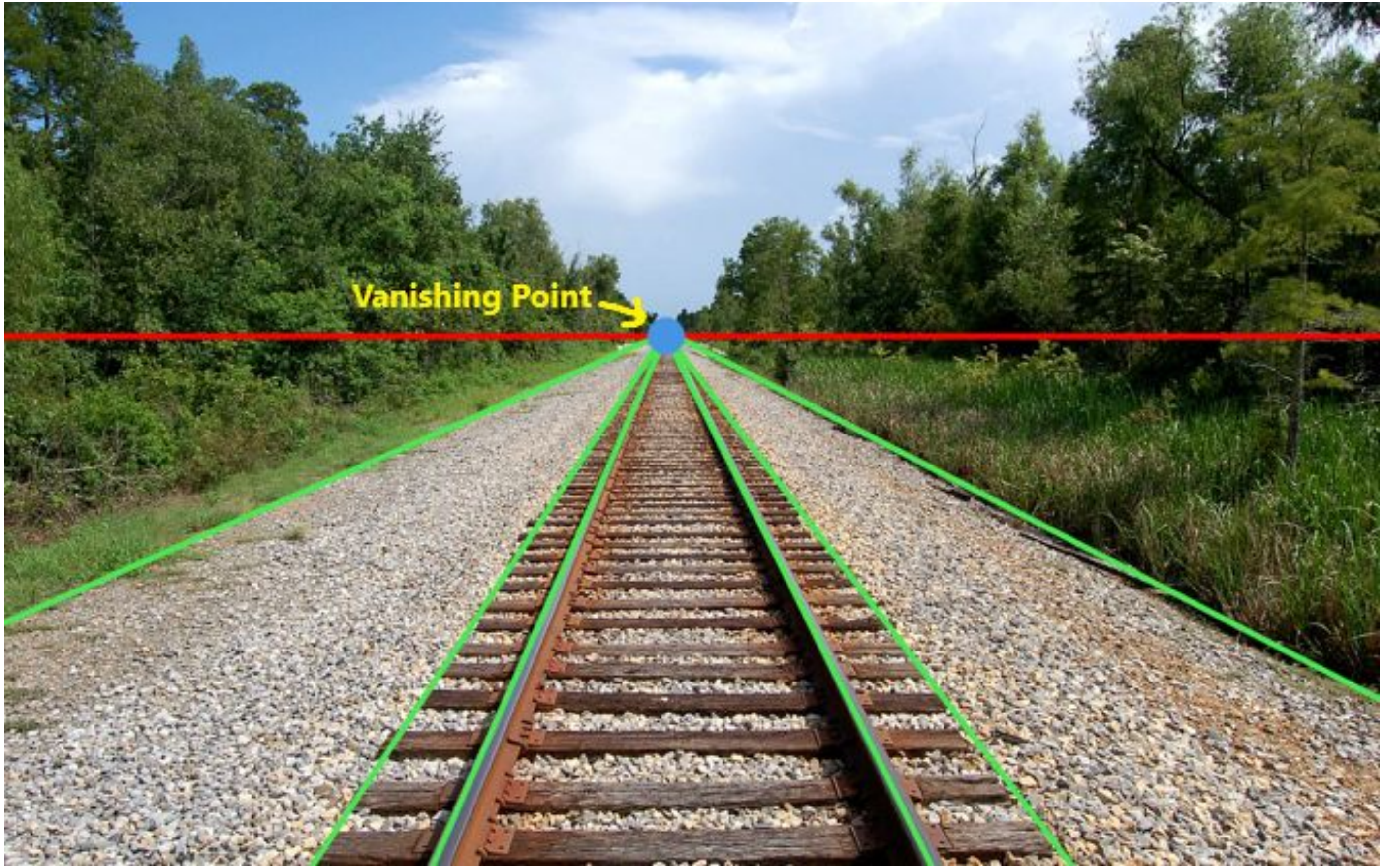
Perspective vs Parallel





	Perspective	Parallel
Nature of Projectors	Projectors aren't parallel, and will converge at a point called COP.	The projectors are always parallel, will never converge.
Configured By	The Center of Projection (COP) and the distance between COP and the projection plane (PP)	The angles those eventually control the Direction of Projection (DOP)
Foreshortening	Yes, so more familiar to human eye	No, appears less realistic
Examples	1-point, 2-point, 3-point	Cabinet, Cavalier, Orthographic

Perspective Projection



Perspective Projection (Attributes)

Vanishing Point: All lines that are parallel to each other (in 3D space) appear to converge into a single point (in projected space)

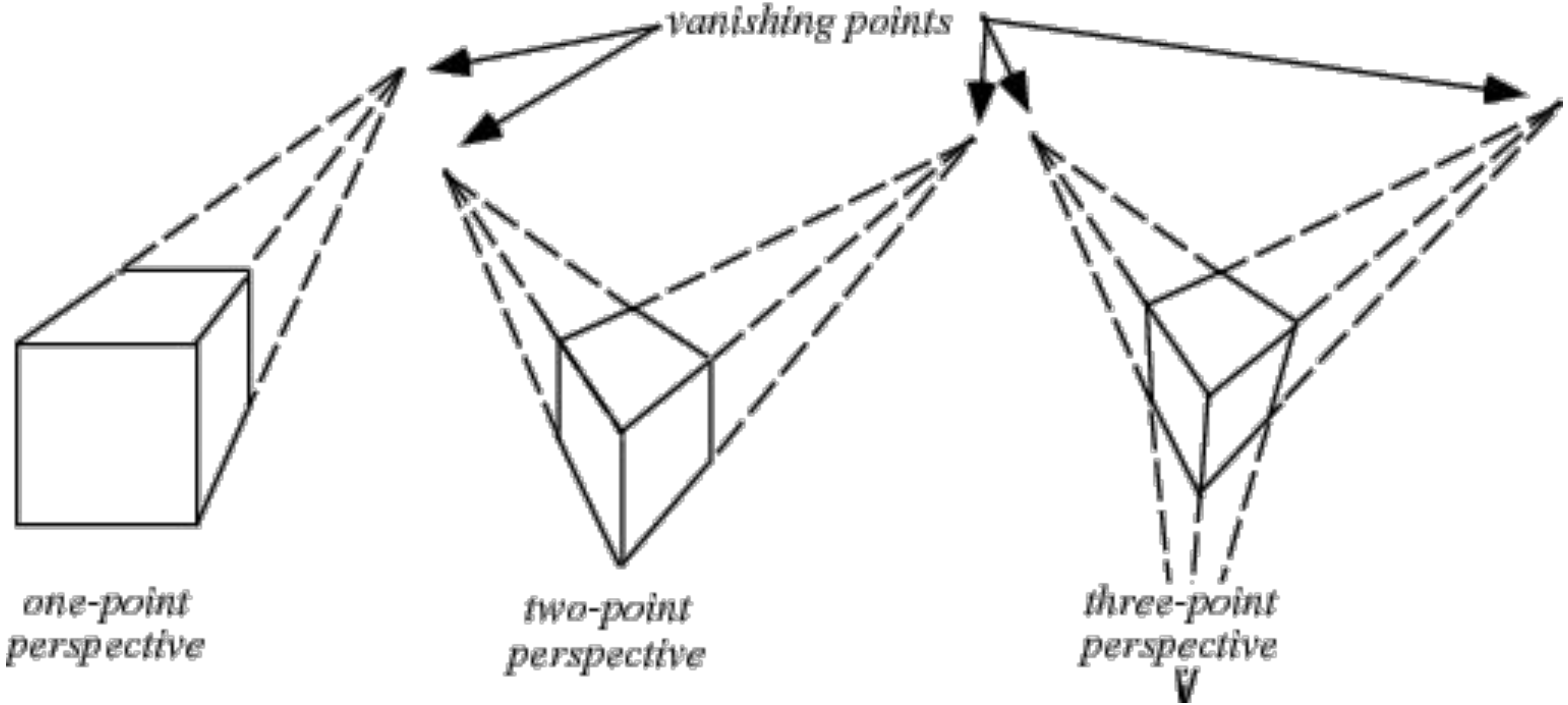
- One-point Perspective Projection: 1 vanishing point
- Two-point Perspective Projection: 2 vanishing points
- Three-point Perspective Projection: 3 vanishing points

In reality, there are infinite number of vanishing points.

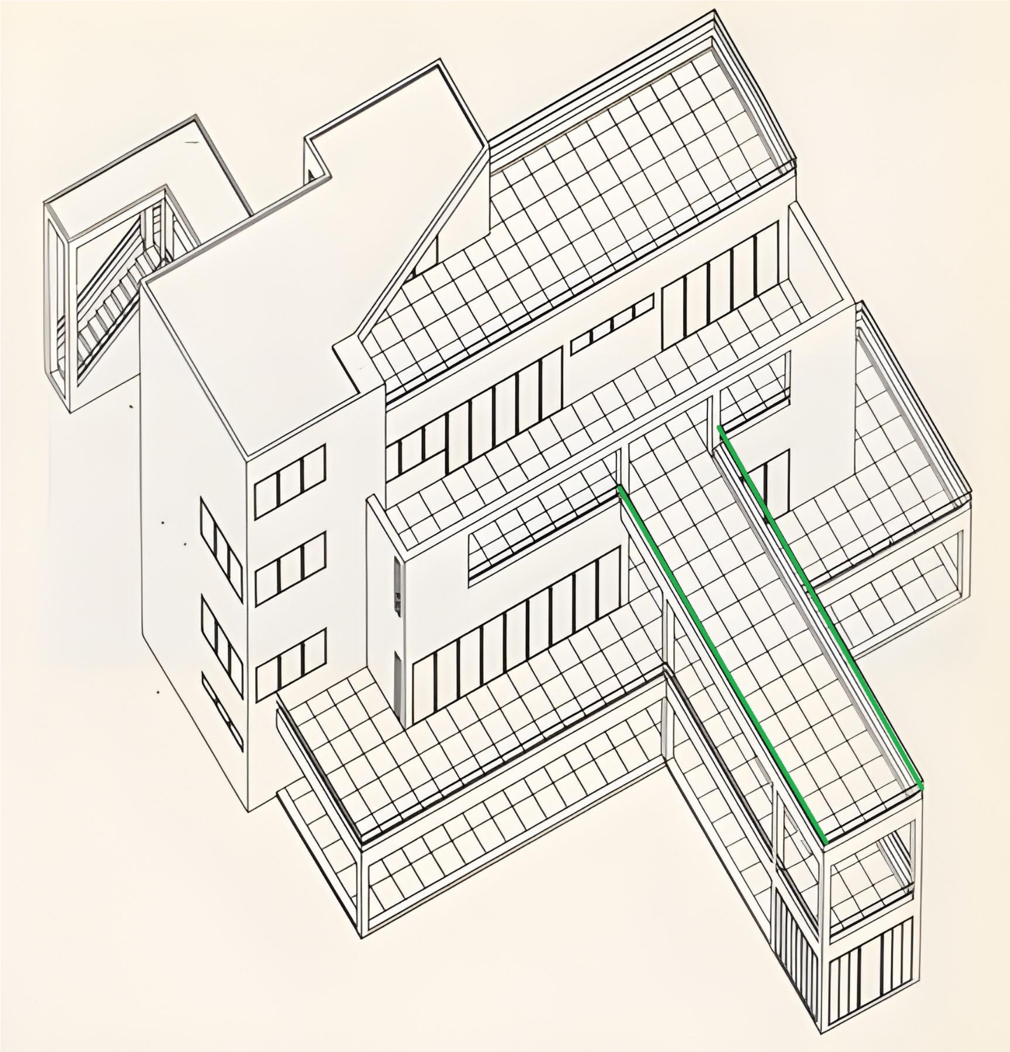
Foreshortening: Basically the phenomenon of far-away objects appearing smaller than the closer ones (despite having equal size).

More realistic since resembles human vision. Used in video games, animation, VR

Perspective Projection (Types)



Parallel Projection



Parallel Projection (Attributes)

Parallel Lines: Lines that are parallel to each other (in 3D space) even still appear parallel to each other (in projected space)

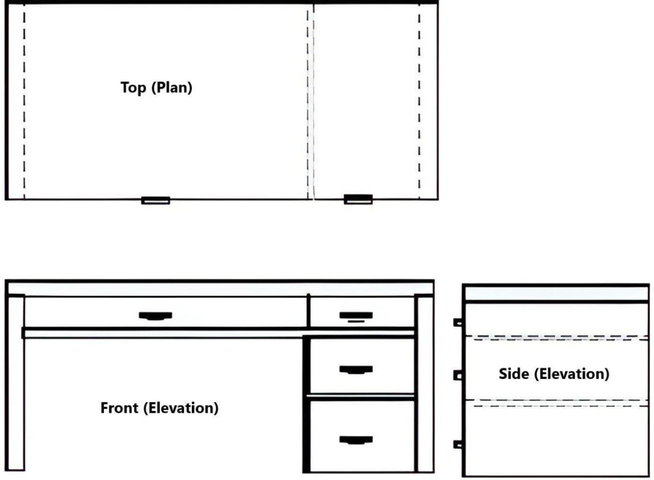
No Convergence: Since the lines remain parallel there are no single point the projectors converge into (in other words, they converge at infinity). So there are no COP, but there is a Direction Of Projection (DOP).

No Foreshortening: Objects that are far-away appear the same size as the closer ones.

For Orthographic the DOP is equal to the normal vector of the plane, whereas they are not equal in case of Oblique.

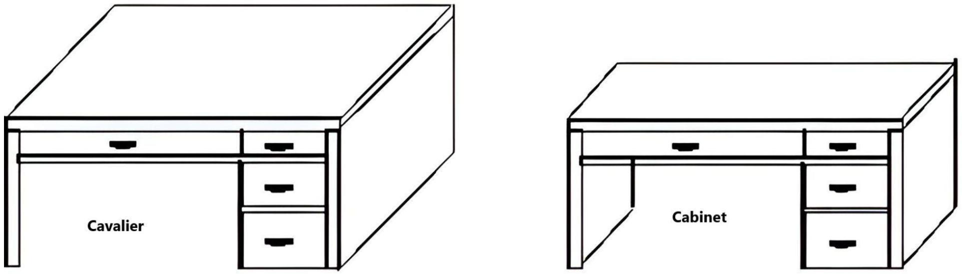
Parallel Projected Images are less realistic, but they are good for architectural planning and design.
(Some video games use isometric projection which is a type of parallel projection)

Parallel Projection (Types)

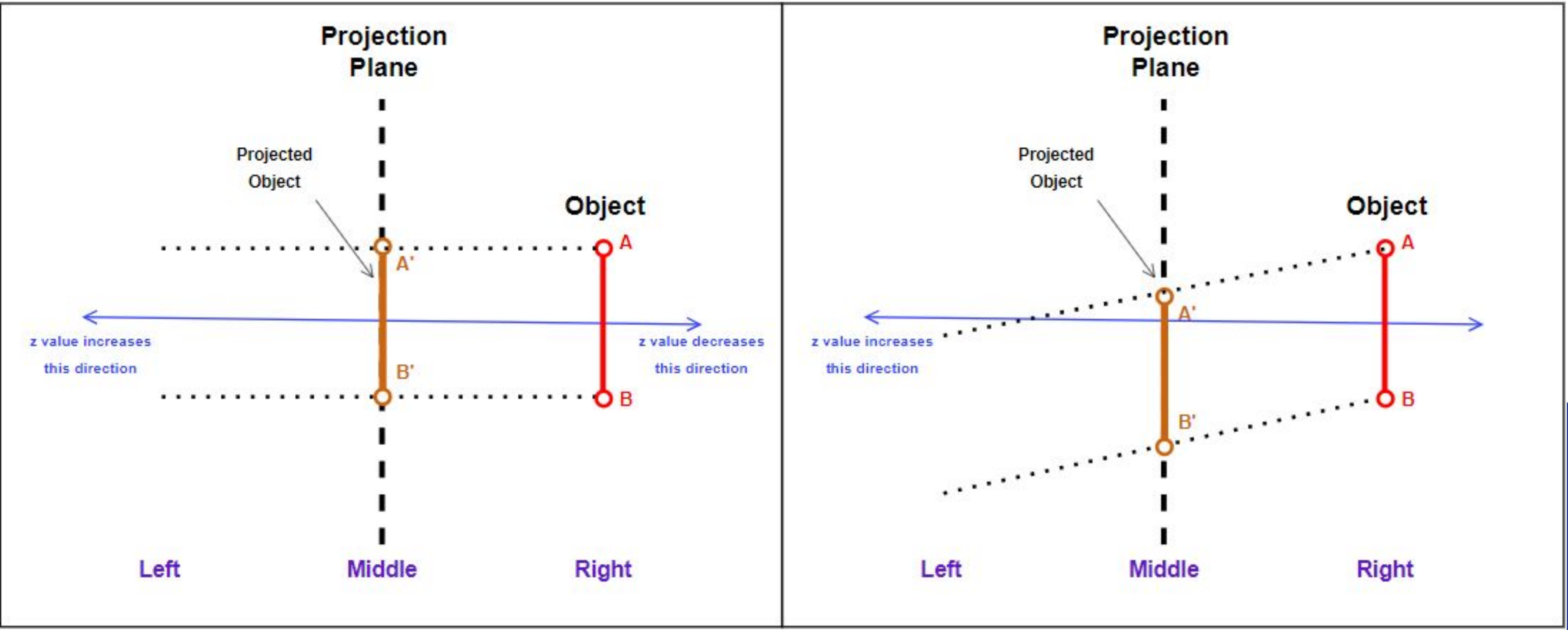


Orthographic

Oblique



Parallel Projection (Geometric Setup)



Orthographic

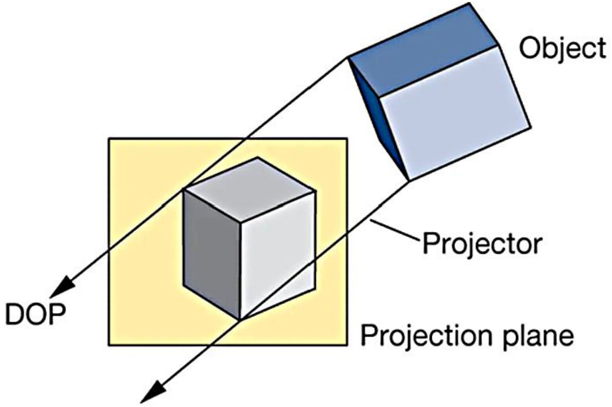
Oblique

Performing Projection Math

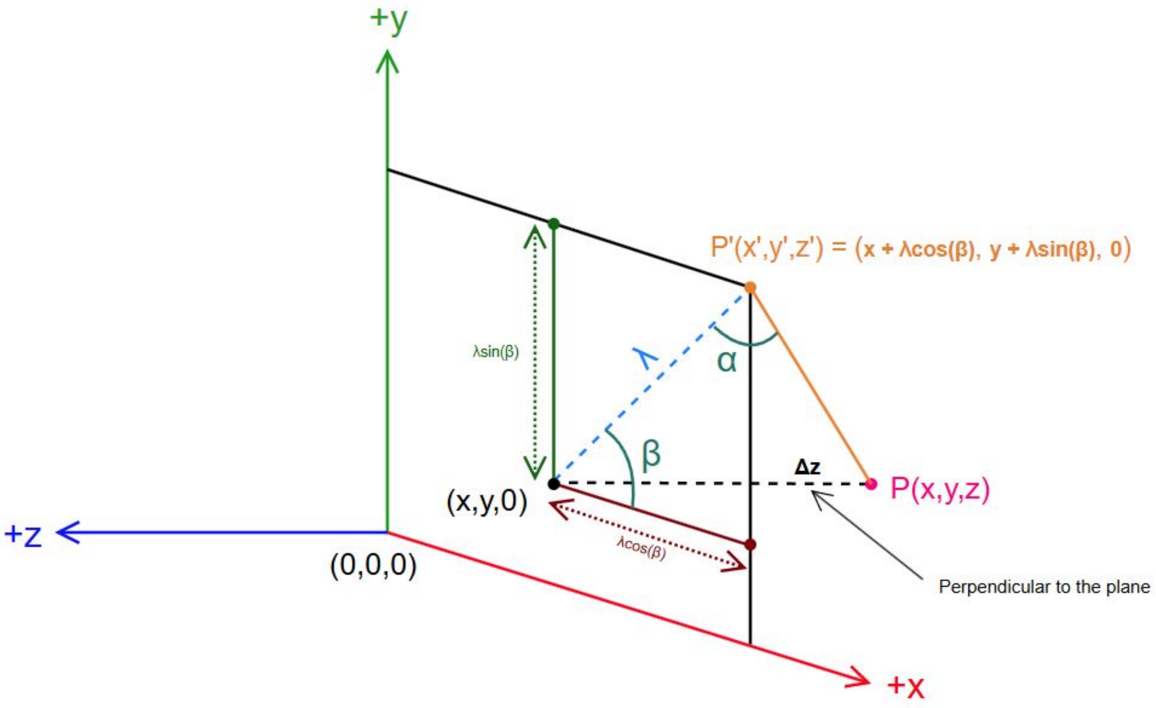
- Depending on the scenario, we need to use a specific projection matrix. A 4x4 homogeneous coordinate matrix since we're in 3D.
- The point to be projected P(x,y,z) will need to be converted into a 4x1 column-matrix, the w/last value will be 1.
- Multiplying the projection matrix and the point column-matrix will give a 4x1 result containing the transformation point.
- The w/last value of result won't necessarily be 1. In this case, we need to divide all values in the result by w to get the actual result called P′(x′,y′,z′).

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} \longrightarrow \begin{bmatrix} \left(\frac{x'.w}{w}\right) \\ \left(\frac{y'.w}{w}\right) \\ \left(\frac{z'.w}{w}\right) \\ \left(\frac{w}{w}\right) \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Deriving Parallel Projection Matrix for XY Plane



- The projection rays don't converge, so there are no Center Of Projection (COP)
- There is however, a Direction Of Projection (DOP)
- This DOP is steered by some angles/scaling factors, which control the ultimate projection transformation
- Different angle/scaling factor configuration for Cabinet, Cavalier, Orthographic



P′ is the projected point on plane for P

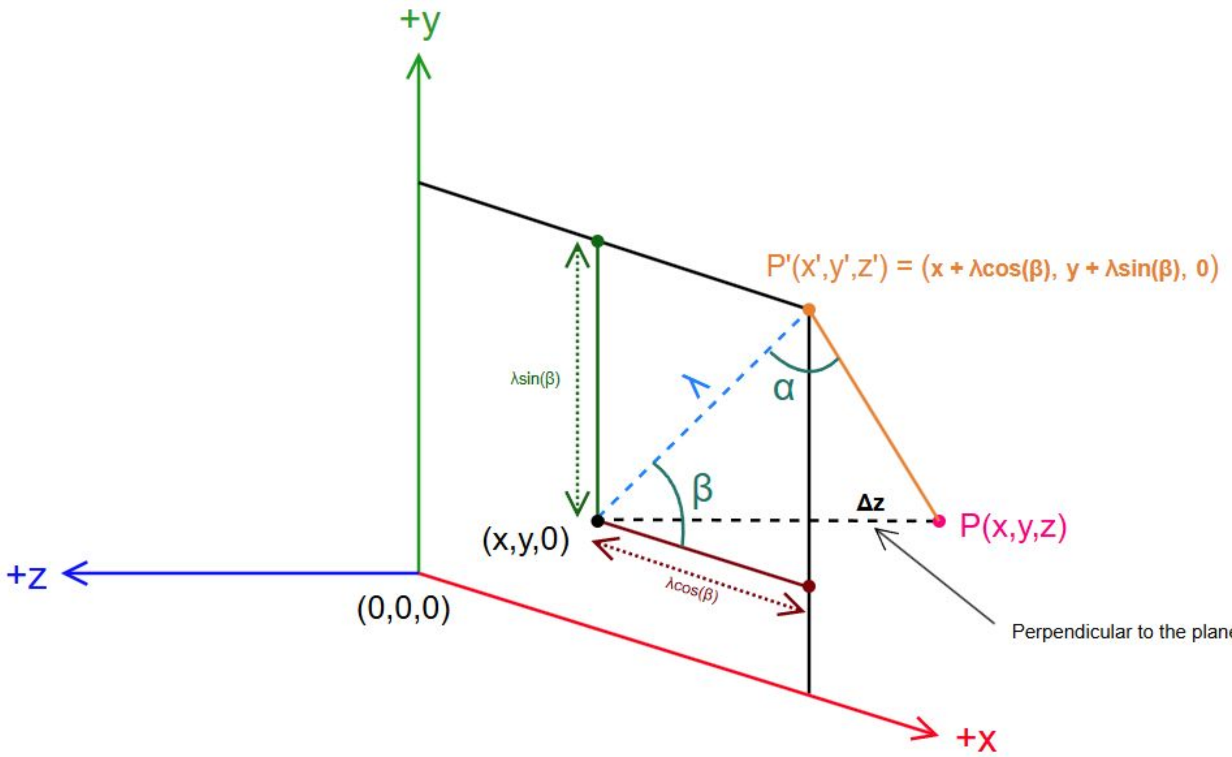
The distance between the plane and P is Δz

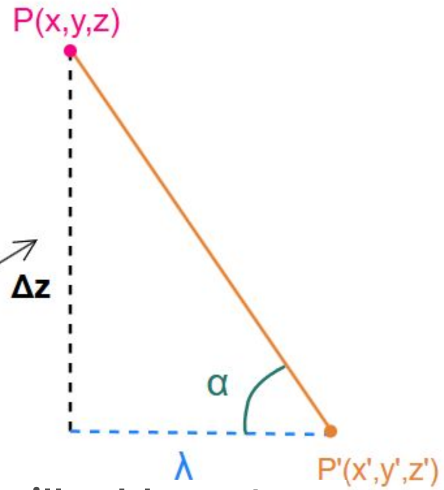
(x, y, 0) is an equivalent point for P(x, y, z) put on the projection plane

β is inclination angle

$x' = x + \lambda \cos(\beta)$ ....1(a)

$y' = y + \lambda \sin(\beta)$ .....1(b) α is the angle between the projection plane and projection ray, this is the angle of skewness.





$$tan(\alpha) = \frac{\Delta z}{\lambda} \quad \Rightarrow \quad \lambda = \frac{\Delta z}{tan(\alpha)}$$

We will address Δz as just z since:

$\Delta z = (z - 0) = z$

From equations 1(a) and 1(b), we can write

$$x' = x + \left(\frac{cos\beta}{tan\alpha}\right)z$$

$$y' = y + \left(\frac{sin\beta}{tan\alpha}\right)z$$

Since on the projection plane, all z values are 0 (for this particular scenario)

$$z' = 0$$

Rewriting with color coded coefficients

$$x' = (1)x + \left(\frac{cos\beta}{tan\alpha}\right)z$$

$$y' = (1)y + \left(\frac{sin\beta}{tan\alpha}\right)z$$

$$z' = 0$$

Coefficients Color Coding:
(x = red, y = green, z = blue, leftover = purple)

Strategy: Converting x′, y′, z′ equations to a 4x4 matrix

$$
\begin{aligned}
x'.w &= a_1x + b_1y + c_1z + d_1 \\
y'.w &= a_2x + b_2y + c_2z + d_2 \\
z'.w &= a_3x + b_3y + c_3z + d_3 \\
w &= a_4x + b_4y + c_4z + d_4
\end{aligned}
$$

We don't have any w equation.

We will consider w = 1 (since parallel projection)

$$
\begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

$$
\boxed{\begin{aligned}
x'.w &= x' \\
y'.w &= y' \\
z'.w &= z' \\
&\text{(Since w = 1)}
\end{aligned}}
$$

$$x' = (1)x + \left(\frac{cos\beta}{tan\alpha}\right)z$$

$$y' = (1)y + \left(\frac{sin\beta}{tan\alpha}\right)z$$

$$z' = 0$$

$$w = 1$$

$$\begin{bmatrix} 1 & 0 & \left(\frac{cos\beta}{tan\alpha}\right) & 0 \\ 0 & 1 & \left(\frac{sin\beta}{tan\alpha}\right) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix}$$

Since, w = 1 (only for this scenario)

$$\begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Considering Δz = z as 1, we can get this relation:

$$tan(\alpha) = \frac{1}{\lambda} \quad \Rightarrow \quad \lambda = \frac{1}{tan(\alpha)}$$

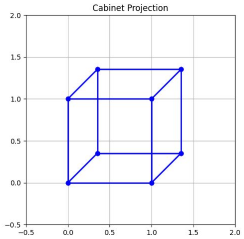
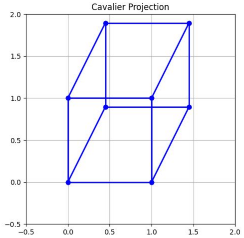
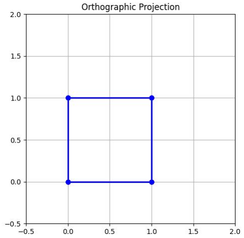
By which, we can get an alternate matrix:

$$\begin{bmatrix} 1 & 0 & \lambda cos\beta & 0 \\ 0 & 1 & \lambda sin\beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix}$$

For the math,

We can either build the matrix using
(α, β) or (λ, β)

Configurations for different types of parallel projections

Type	Oblique?	λ	α (deg)	β (deg)	Preview
Cabinet	Yes	0.5	63.4	0-360	
Cavalier	Yes	1	45	0-360	
Orthographic	No	0	90	0-360	

Example: For a point P (10, 20, -40), calculate the projected point P′ using Cavalier Projection, orientation angle is 30° for projection plane of XY.

For Cavalier, $\alpha = 45°$. And it is given that, $\beta = 30°$
The whole calculated matrix with the point multiplied is:

$$\begin{bmatrix} 1 & 0 & 0.866 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ -40 \\ 1 \end{bmatrix} = \begin{bmatrix} -24.64 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We don't need to divide values by w since w is already 1

P′ is (-24.64, 0)

Example: Derive the matrix for orthographic projection

Here, the projection rays will be perpendicular to the projection plane, α will be 90 degrees, making λ as 0, so the value of β won't even be considered.
Putting these values into the formula matrix gives:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Calculate Orthographic Projection on XY Plane for the point P (d, e, f)

We don't need to know any other value since the orthographic projection matrix is constant, we simply multiply our provided point P (as a homogenous column-matrix) with the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ e \\ 0 \\ 1 \end{bmatrix}$$

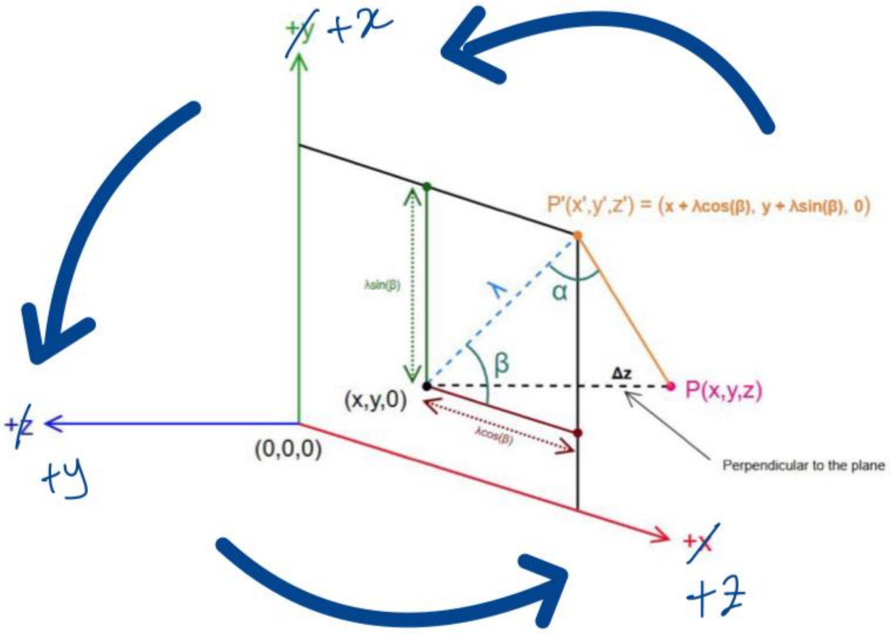
In fact, applying orthographic projection on XY Plane to a point is equivalent to copying the point's x and y values, while setting its z value to 0. [(x, y, z) becomes (x, y, 0)]

But how can we find the Parallel projection matrix for different projection plane other than XY?

Same way

- Rotate the coordinate system clockwise to get to YZ plane.
- Rotate the coordinate system counterclockwise to get to ZX plane.
- Modify the matrix accordingly.

Projection Plane: ZX / Y =0



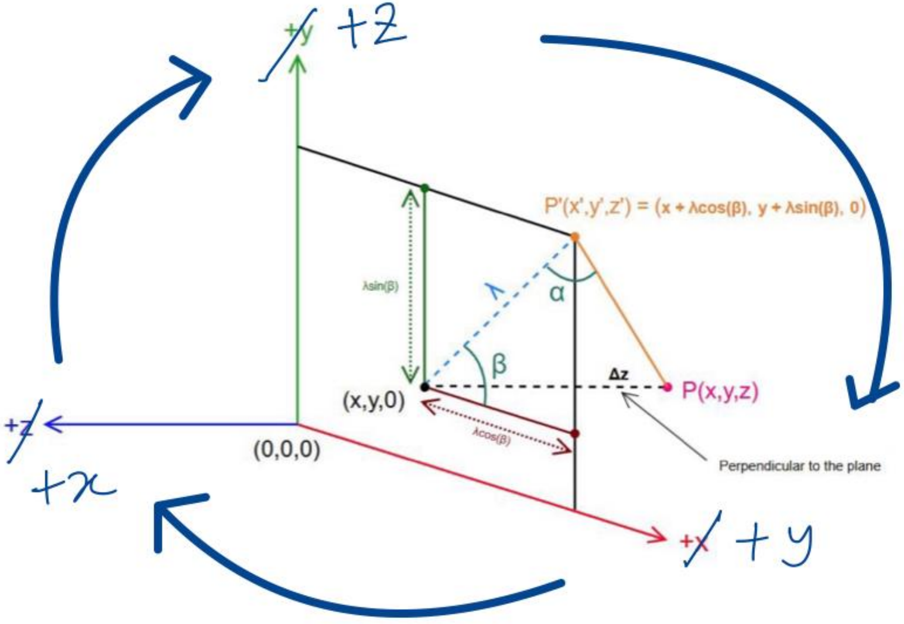
$$z' = z + \lambda \cos\beta \times y$$

$$x' = x + \lambda \sin\beta \times y$$

$$\begin{vmatrix} x' \\ y' \\ z' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & \lambda\sin\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda\cos\beta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

here, $\lambda = \dfrac{1}{\tan\alpha}$

Actually, $\lambda = \dfrac{\Delta y}{\tan\alpha}$



Projection Plane: YZ / X =0

$$y' = y + \lambda\cos\beta \times x$$

$$z' = z + \lambda\sin\beta \times x$$

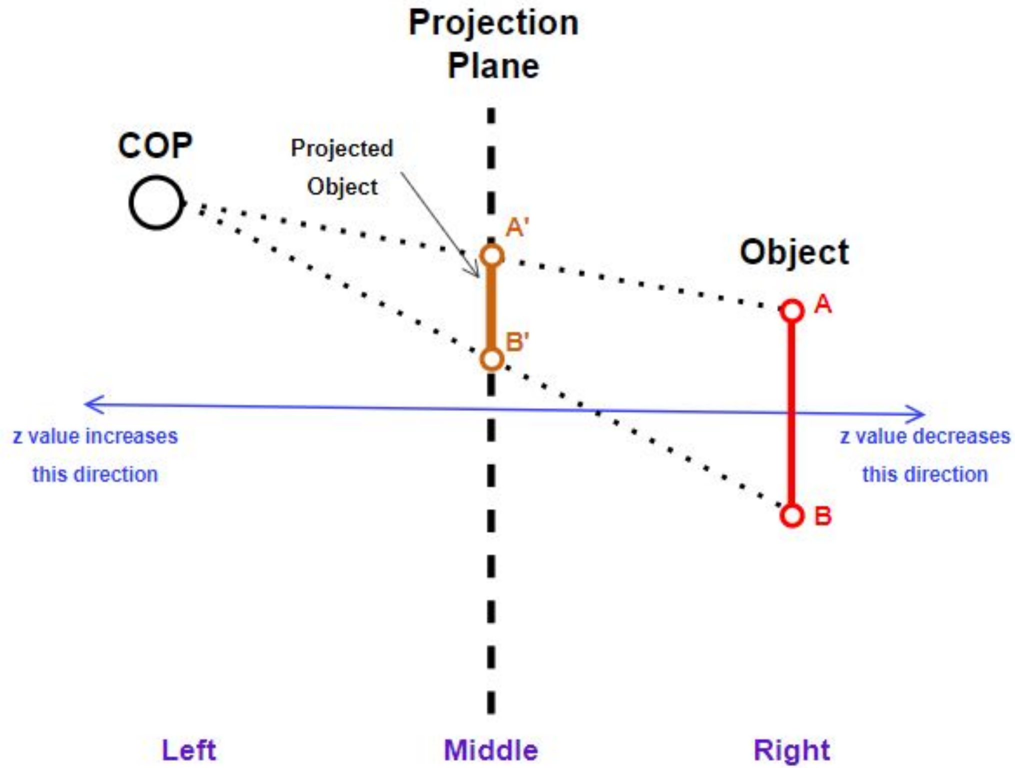
$$\begin{vmatrix} x' \\ y' \\ z' \\ 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ \lambda\cos\beta & 1 & 0 & 0 \\ \lambda\sin\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

here, $\lambda = \dfrac{1}{\tan\alpha}$

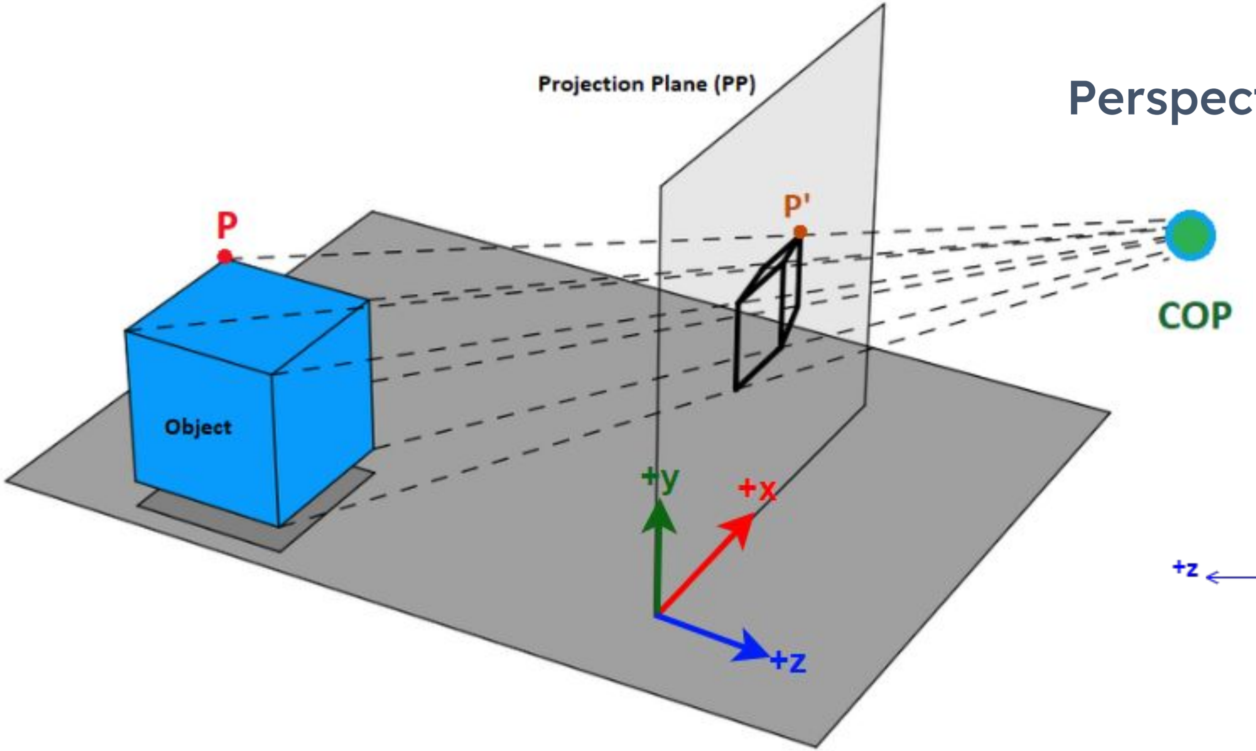
Actually, $\lambda = \dfrac{\Delta x}{\tan\alpha}$

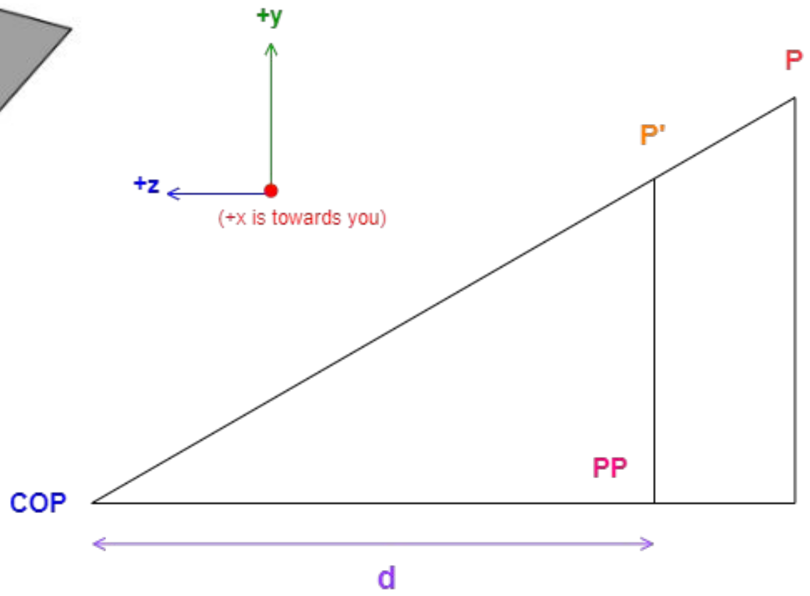
Perspective Projection
(Simple Purpose)

Perspective Projection (Geometric Setup)

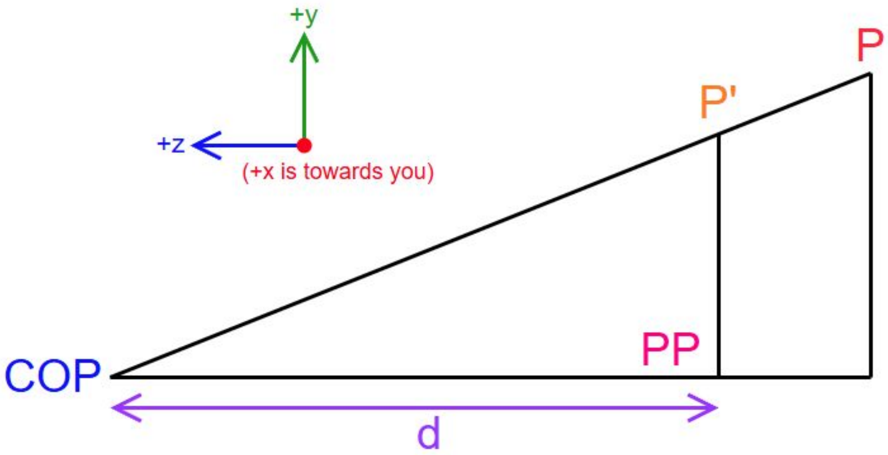


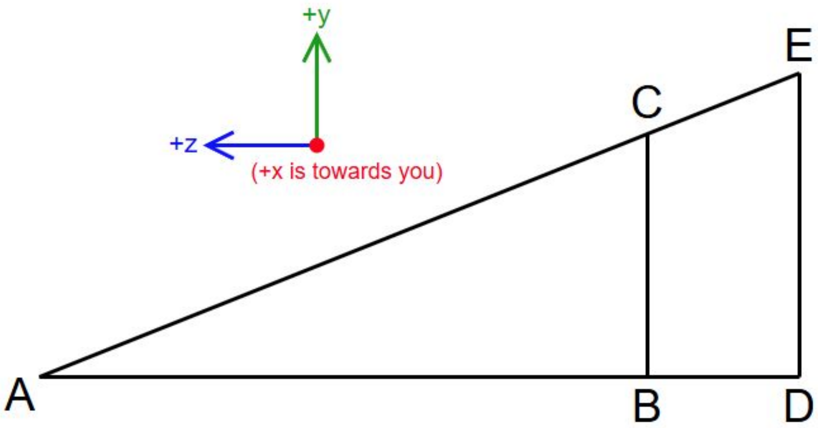
Perspective Projection Overview





Objective: We need to calculate $P'(x', y', z')$ from $P(x, y, z)$, so we need to derive the equations for x', y', z'



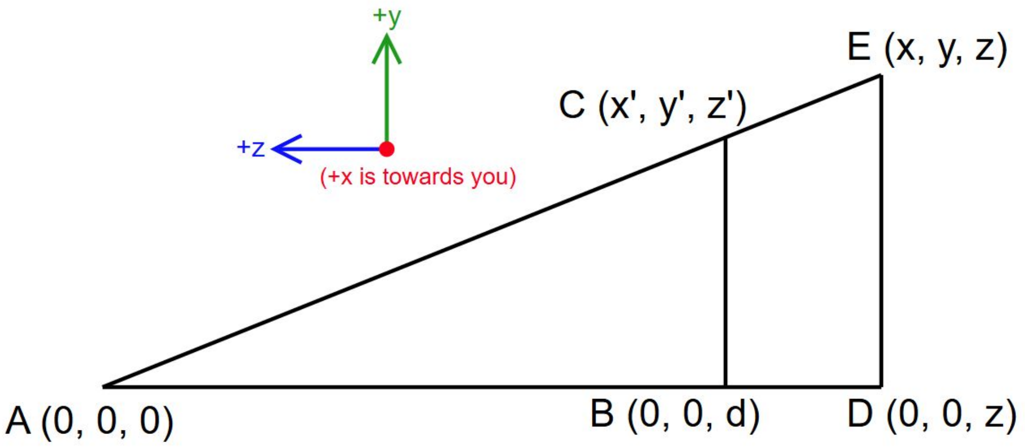


We know that for such triangle, $\frac{BC}{AB} = \frac{DE}{AD}$

Scenario: COP is at origin (0, 0, 0)

Note: **value of d is inherently negative** in this scenario

Using the Relation $\frac{BC}{AB} = \frac{DE}{AD}$ We perform for each axis



$$\frac{x'}{d} = \frac{x}{z} \Rightarrow x' = \frac{x}{\left(\frac{z}{d}\right)}$$

$$\frac{y'}{d} = \frac{y}{z} \Rightarrow y' = \frac{y}{\left(\frac{z}{d}\right)}$$

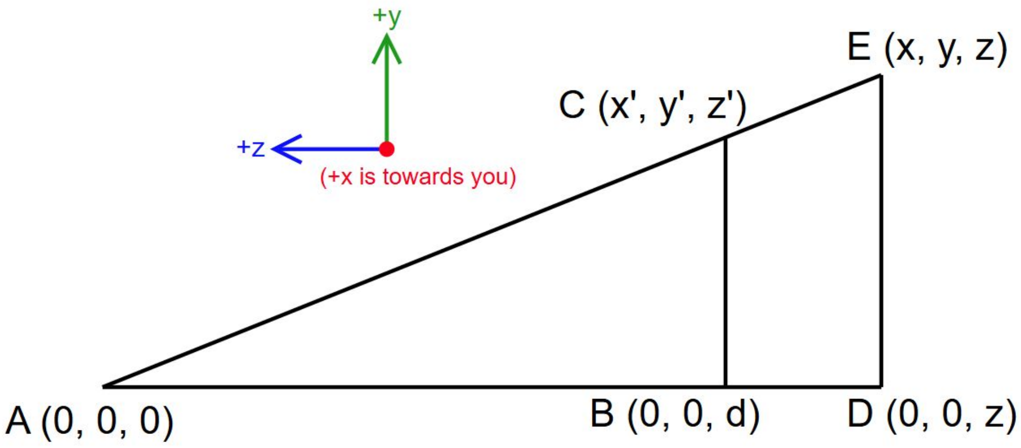
$$z' = \frac{z}{\left(\frac{z}{d}\right)}$$

We have got the equations for x′, y′, z′. Notice how their denominators are same **(z/d)**. We can declare this denominator as "w"

$$w = \frac{z}{d}$$

Scenario: COP is at origin (0, 0, 0)

Rearranging (and color coding coefficients for your convenience)
Moving the denominator a.k.a "w" to the L.H.S.



$$x'.w = (1)x$$
$$y'.w = (1)y$$
$$z'.w = (1)z$$
$$w = (\frac{1}{d})z$$

Now we can build the matrix!

Coefficients Color Coding:
(x = red, y = green, z = blue, leftover = purple)

Strategy: Converting x′, y′, z′ equations to a 4x4 matrix

$$x'.w = a_1x + b_1y + c_1z + d_1$$
$$y'.w = a_2x + b_2y + c_2z + d_2$$
$$z'.w = a_3x + b_3y + c_3z + d_3$$
$$w = a_4x + b_4y + c_4z + d_4$$

$$\Downarrow$$

$$\begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scenario: COP is at origin (0, 0, 0)

Building the actual matrix from set of equations

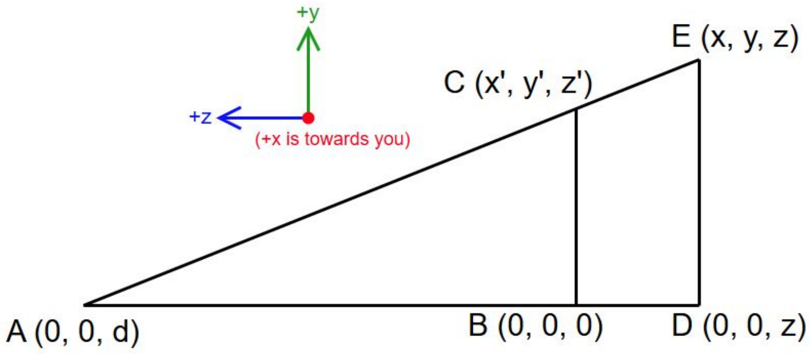
$$x'.w = (1)x$$
$$y'.w = (1)y$$
$$z'.w = (1)z$$
$$w = (\frac{1}{d})z$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}$$

Scenario: Projection Plane is at origin (0, 0, 0)

Note: value of d is inherently **positive** in this scenario

Using the Relation $\frac{BC}{AB} = \frac{DE}{AD}$ We perform for each axis



$$\frac{x'}{d} = \frac{x}{d-z} \Rightarrow x' = \frac{x}{\frac{d-z}{d}} = \frac{x}{(1-\frac{z}{d})}$$

$$\frac{y'}{d} = \frac{y}{d-z} \Rightarrow y' = \frac{y}{\frac{d-z}{d}} = \frac{y}{(1-\frac{z}{d})}$$

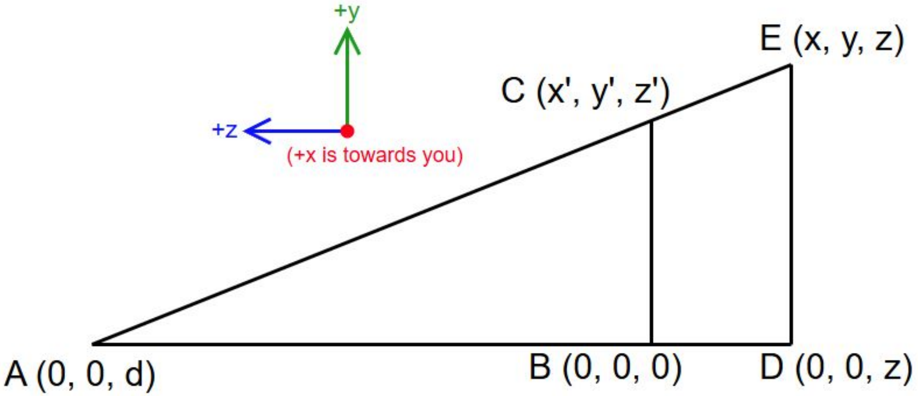
$$z' = 0$$

z′ is 0 because it is on the projection plane, which itself is at origin.
We have got the equations for x′, y′, z′. Notice how their denominators are same **(1 - z/d)**. We can declare this denominator as "w"

$$w = 1 - \frac{z}{d}$$

Scenario: Projection Plane is at origin (0, 0, 0)

Rearranging (and color coding coefficients for your convenience)
Moving the denominator a.k.a "w" to the L.H.S.



$$x'.w = (1)x$$
$$y'.w = (1)y$$
$$z'.w = 0$$
$$w = (\frac{-1}{d})z + 1$$

Now we can build the matrix!

Coefficients Color Coding:
(x = red, y = green, z = blue, leftover = purple)

Recall the Strategy: Converting x′, y′, z′ equations to a 4x4 matrix

$$x'.w = a_1x + b_1y + c_1z + d_1$$
$$y'.w = a_2x + b_2y + c_2z + d_2$$
$$z'.w = a_3x + b_3y + c_3z + d_3$$
$$w = a_4x + b_4y + c_4z + d_4$$

$$\Downarrow$$

$$\begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

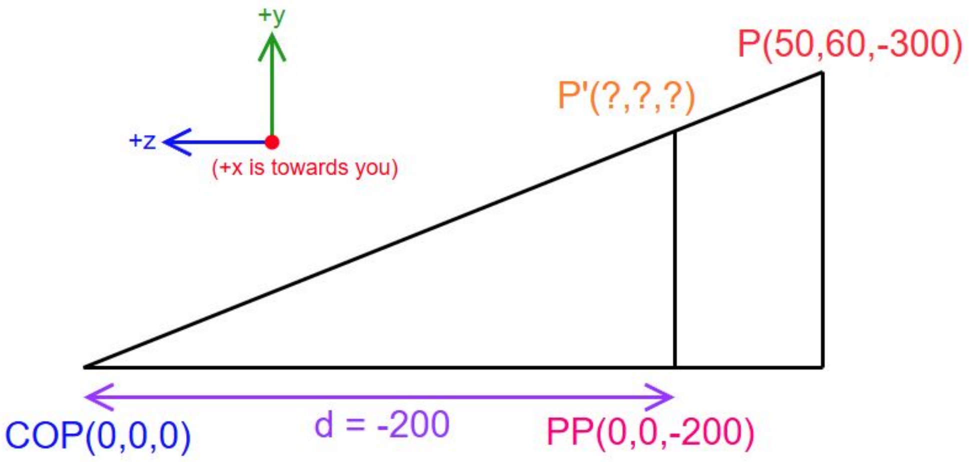
Scenario: Projection Plane is at origin (0, 0, 0)

Building the actual matrix from set of equations

$$x'.w = (1)x$$
$$y'.w = (1)y$$
$$z'.w = 0$$
$$w = (\frac{-1}{d})z + 1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{d} & 1 \end{bmatrix}$$

Example: For COP at origin, calculate the projected point P′ for a given point P(50, 60, -300), if the plane is 200 units in Z axis away from the COP.



{ d = z value of PP }

Since the Projection Plane (PP) lies on the **negative side** of the z-axis, all points on that plane will have a negative z value, which is why **d is negative**

Now' using our matrix for "COP at Origin", we place the d value there, and multiply it by our point P (as a column vector)

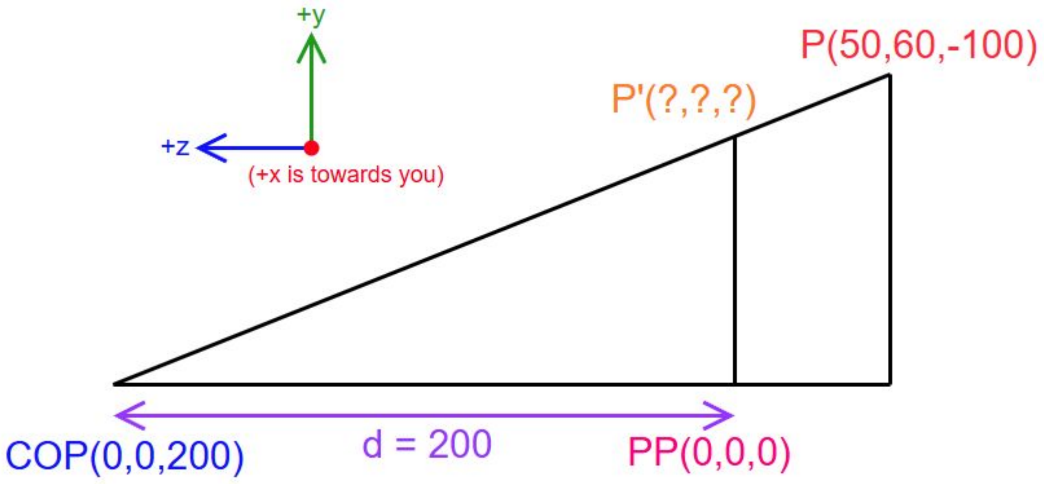
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{-200} & 0 \end{bmatrix} \times \begin{bmatrix} 50 \\ 60 \\ -300 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 60 \\ -300 \\ \frac{3}{2} \end{bmatrix}$$

Now, divide all the values by w = 3/2 to get the final answer

$$\begin{bmatrix} 50 \\ 60 \\ -300 \\ \frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 33.33 \\ 40 \\ -200 \\ 1 \end{bmatrix}$$

P' = (33.33, 40, -200)

Example: For PP at origin, calculate the projected point P′ for a given point P(50, 60, -100), if the plane is 200 units in Z axis away from the COP.



{ d = z value of COP }

Since the COP lies on the **positive side of the z-axis**, its z values will be positive, which makes the **value of d positive**

Now' using our matrix for "PP at Origin", we place the d value there, and multiply it by our point P (as a column vector)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{200} & 1 \end{bmatrix} \times \begin{bmatrix} 50 \\ 60 \\ -100 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 60 \\ 0 \\ \frac{3}{2} \end{bmatrix}$$

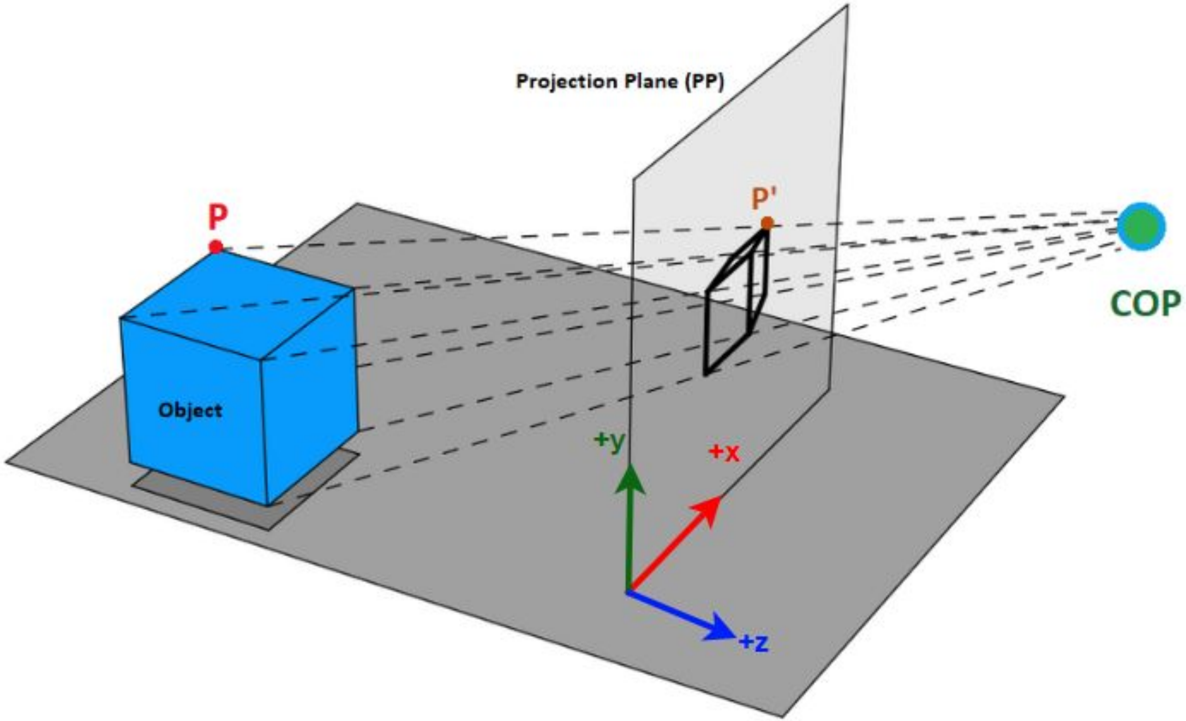
Now, divide all the values by w = 3/2 to get the final answer

$$\begin{bmatrix} 50 \\ 60 \\ 0 \\ \frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 33.33 \\ 40 \\ 0 \\ 1 \end{bmatrix}$$

P′ = (33.33, 40)

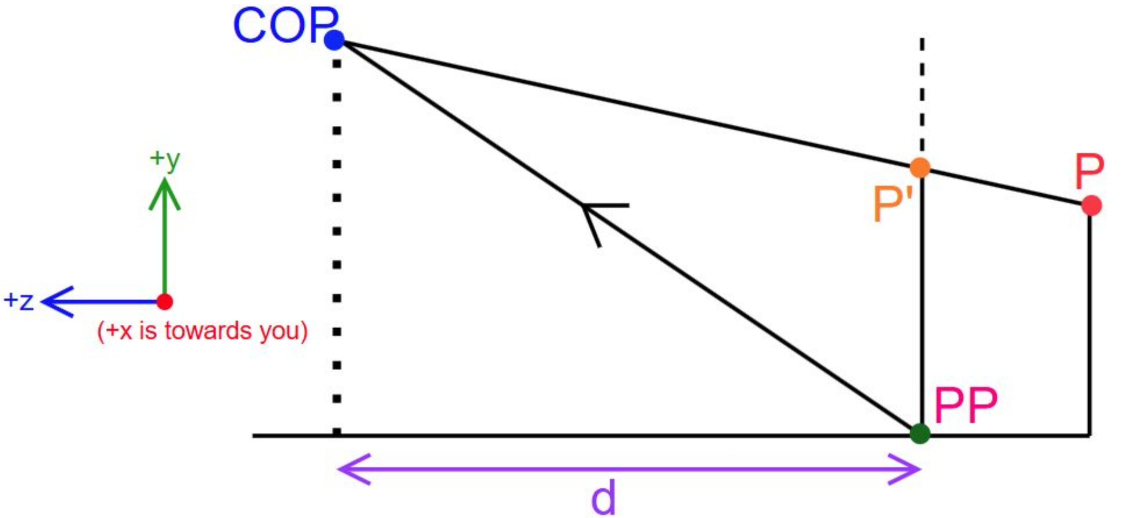
Perspective Projection
(General Purpose)

Perspective Projection Overview



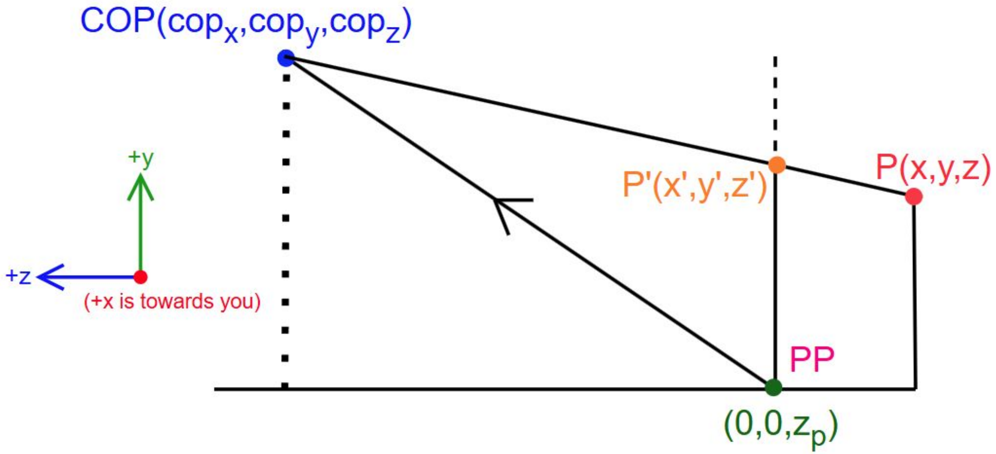
In case of General Purpose Perspective Projection, the COP is not necessarily on the z-axis

Scenario



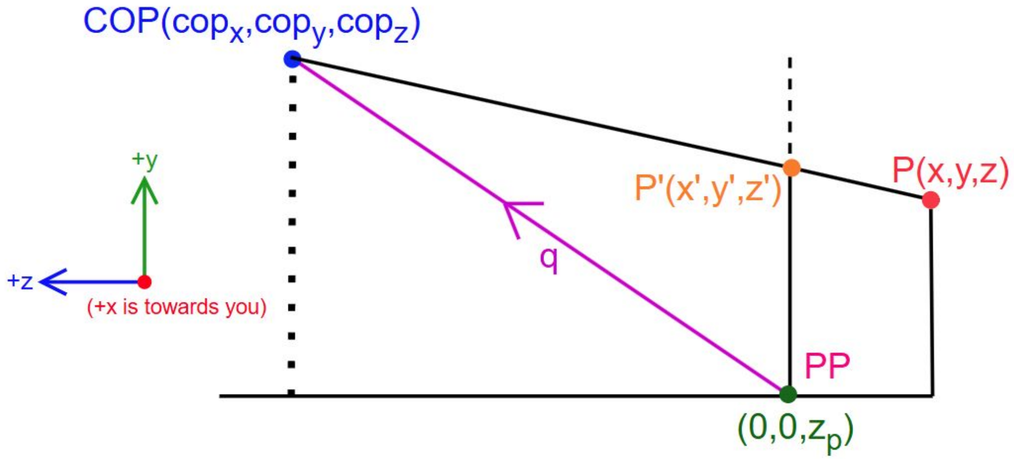
- COP is at an arbitrary point, not necessarily on the z-axis
- We don't care if COP or PP is at origin or not.
- The distance between COP and PP (expressed as d) is the z-distance between them.

Scenario



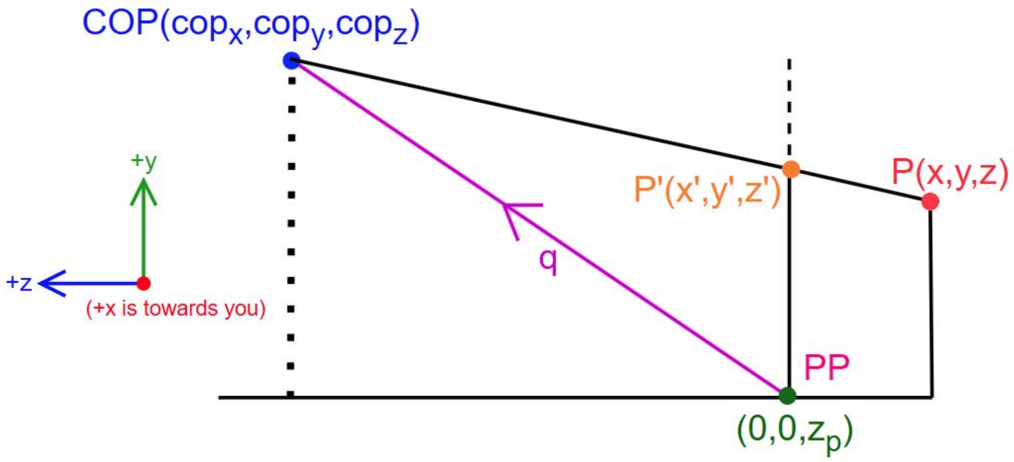
- COP is expressed component-wise as (cop_x, cop_y, cop_z)
- The z-axis pierces through the PP, the piercing point is (0, 0, z_p)
- The difference between cop_z and z_p is equals to d

Scenario



- We consider a vector called q that goes from $(0, 0, z_p)$ to COP
- So, $q = (qx, qy, qz) = (cop_x, cop_y, cop_z) - (0, 0, z_p) = (cop_x, cop_y, cop_z - z_p)$
- (qx, qy, qz) is sometimes expressed as (Qdx, Qdy, Qdz)

Scenario



$q_x = cop_x$

$q_y = cop_y$

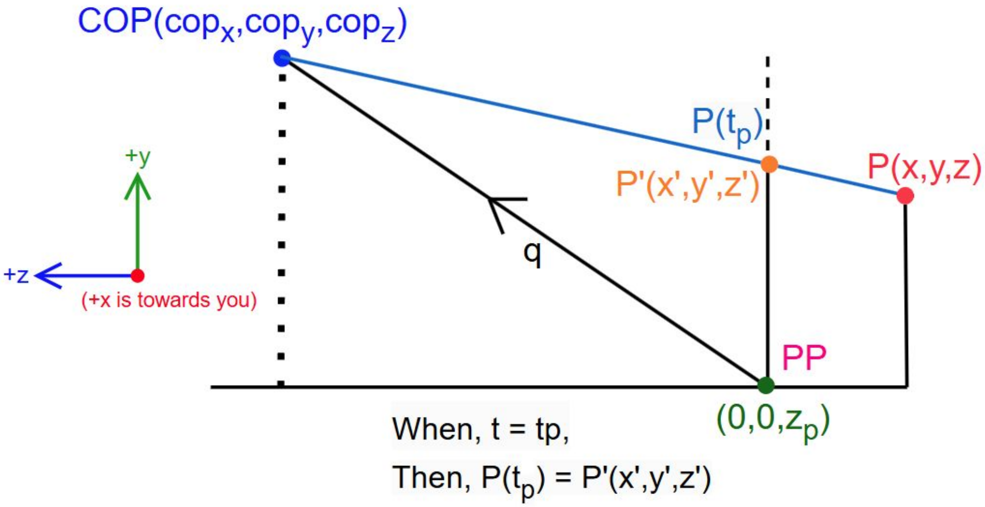
$q_z = cop_z - z_p$

$cop_x = q_x$ ....1(a)

$cop_y = q_y$ ....1(b)

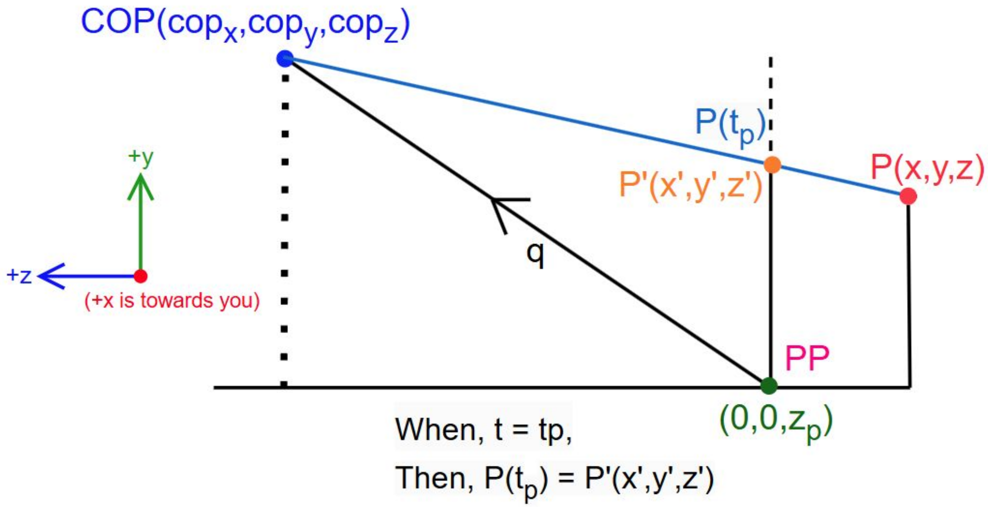
$cop_z = z_p + q_z$ ....1(c)

Scenario



We consider a parametric line segment that goes between COP and P
This line segment is basically our projection ray

Scenario



$$P(t) = P_0 + t(P_1 - P_0)$$

Here we consider,

$$P_0 = COP$$
$$P_1 = P$$

So,

$$P(t) = COP + t(P - COP)$$

- At some point, for a certain value of t, P(t) will be the same as P′
- We call that t value t_p

$$P' = P(t_p) = COP + t_p(P - COP) \quad \text{........(2)}$$

Breaking the equation the last equation into x, y, z components

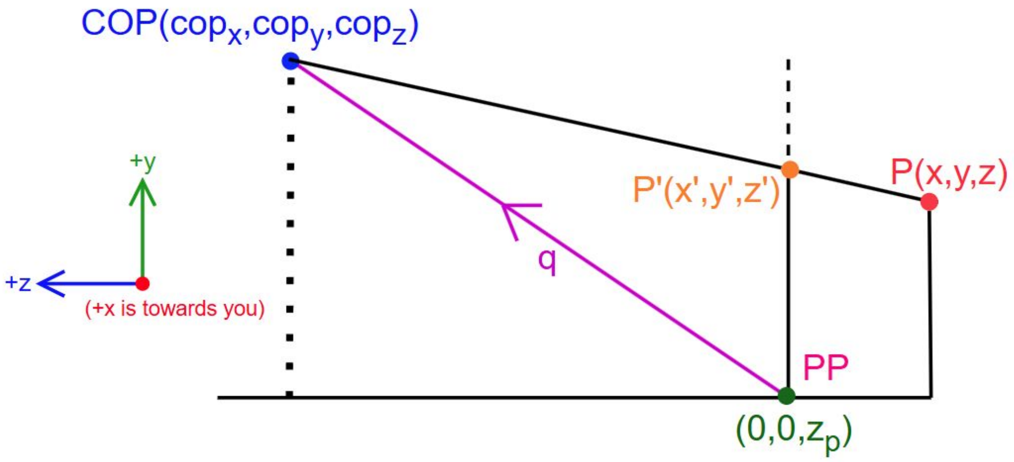
$$x' = q_x + t_p(x - q_x) \quad \text{........3(a)}$$

$$y' = q_y + t_p(y - q_y) \quad \text{........3(b)}$$

$$z' = (q_z + z_p) + t_p(z - q_z - z_p) \quad \text{........3(c)}$$

We need to remove t_p from these equations, since we don't know the value of t_p
Using the equation of z′ we can write:

$$t_p = \frac{z' - q_z - z_p}{z - q_z - z_p}$$



Since **P′(x′,y′,z′)** and **(0,0,z_p)** both reside on the same plane (the PP), their z value are the same.
which means:

$z' = z_p$ ....(4)

(Remember this equation)

$$t_p = \frac{z' - q_z - z_p}{z - q_z - z_p} = \frac{-q_z}{z - q_z - z_p} = \frac{1}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

Now we substitute the value of t_p in the equations of x′ and y′ from before

For x′

$$x' = q_x + \left(\frac{1}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}\right)(x - q_x)$$

$$x' = \frac{-\frac{z}{q_z}q_x + q_x + z_p\frac{q_x}{q_z} + x - q_x}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

$$x' = \frac{x + (\frac{-q_x}{q_z})z + (z_p.\frac{q_x}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})} \qquad \text{........5(a)}$$

For y′, following the same way

$$y' = \frac{y + (\frac{-q_y}{q_z})z + (z_p.\frac{q_y}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})} \qquad \text{........5(b)}$$

For x′

$$x' = q_x + \left(\frac{1}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}\right)(x - q_x)$$

$$x' = \frac{-\frac{z}{q_z}q_x + q_x + z_p\frac{q_x}{q_z} + x - q_x}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

$$x' = \frac{x + (\frac{-q_x}{q_z})z + (z_p . \frac{q_x}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})} \quad \text{........5(a)}$$

Notice how they have the same denominator

(Remember this)

For y′, following the same way

$$y' = \frac{y + (\frac{-q_y}{q_z})z + (z_p . \frac{q_y}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})} \quad \text{........5(b)}$$

For z′

$$z' = z_p$$

[Ok, but it lacks the common denominator]

$$z' = z_p . \left(\frac{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})} \right)$$

[multiply by denominator/denominator]
[multiply the above part with z_p]
[keep the bottom part intact]

$$z' = \frac{(\frac{-z_p}{q_z})z + (z_p + \frac{z_p^2}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

........5(c)

We've brought the common denominator present in x′ and y′ (Remember this)

All in one place

$$x' = \frac{x + (\frac{-q_x}{q_z})z + (z_p.\frac{q_x}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

$$y' = \frac{y + (\frac{-q_y}{q_z})z + (z_p.\frac{q_y}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

$$z' = \frac{(\frac{-z_p}{q_z})z + (z_p + \frac{z_p^2}{q_z})}{(\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})}$$

We can call the common denominator "w"

$$x' = \frac{x + (\frac{-q_x}{q_z})z + (z_p.\frac{q_x}{q_z})}{w}$$

$$y' = \frac{y + (\frac{-q_y}{q_z})z + (z_p.\frac{q_y}{q_z})}{w}$$

$$z' = \frac{(\frac{-z_p}{q_z})z + (z_p + \frac{z_p^2}{q_z})}{w}$$

$$w = (\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})$$

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Moving "w" to the L.H.S

$$x'.w = x + (\frac{-q_x}{q_z})z + (z_p.\frac{q_x}{q_z})$$

$$y'.w = y + (\frac{-q_y}{q_z})z + (z_p.\frac{q_y}{q_z})$$

$$z'.w = (\frac{-z_p}{q_z})z + (z_p + \frac{z_p^2}{q_z})$$

$$w = (\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})$$

Same thing, but with color coded coefficients

$$x'.w = (1)x + (\frac{-q_x}{q_z})z + (z_p.\frac{q_x}{q_z})$$

$$y'.w = (1)y + (\frac{-q_y}{q_z})z + (z_p.\frac{q_y}{q_z})$$

$$z'.w = (\frac{-z_p}{q_z})z + (z_p + \frac{z_p^2}{q_z})$$

$$w = (\frac{-1}{q_z})z + (1 + \frac{z_p}{q_z})$$

Coefficients Color Coding:
(x = red, y = green, z = blue, leftover = purple)

Strategy: Converting x′, y′, z′ equations to a 4x4 matrix

$$
\begin{aligned}
x'.w &= a_1x + b_1y + c_1z + d_1 \\
y'.w &= a_2x + b_2y + c_2z + d_2 \\
z'.w &= a_3x + b_3y + c_3z + d_3 \\
w &= a_4x + b_4y + c_4z + d_4
\end{aligned}
$$

$$\Downarrow$$

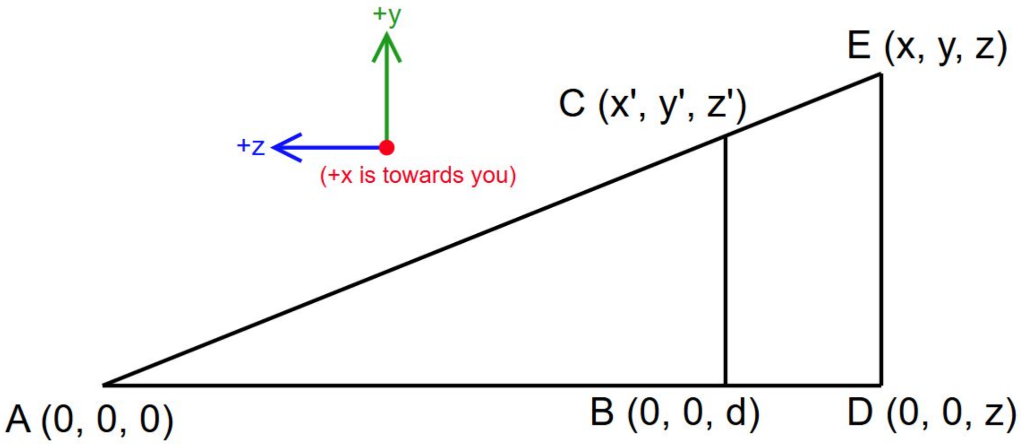
$$
\begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

Ultimate Matrix System

$$\begin{bmatrix} x'.w \\ y'.w \\ z'.w \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & \left(\frac{-q_x}{q_z}\right) & \left(z_p.\frac{q_x}{q_z}\right) \\ 0 & 1 & \left(\frac{-q_y}{q_z}\right) & \left(z_p.\frac{q_y}{q_z}\right) \\ 0 & 0 & \left(\frac{-z_p}{q_z}\right) & \left(z_p + \frac{z_p^2}{q_z}\right) \\ 0 & 0 & \left(\frac{-1}{q_z}\right) & \left(1 + \frac{z_p}{q_z}\right) \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

We only need to know the constants q_x, q_y, q_z, z_p to build the matrix

Getting Simple Perspective Projection Matrix from the General One (COP at Origin)



COP = A = (0, 0, 0)

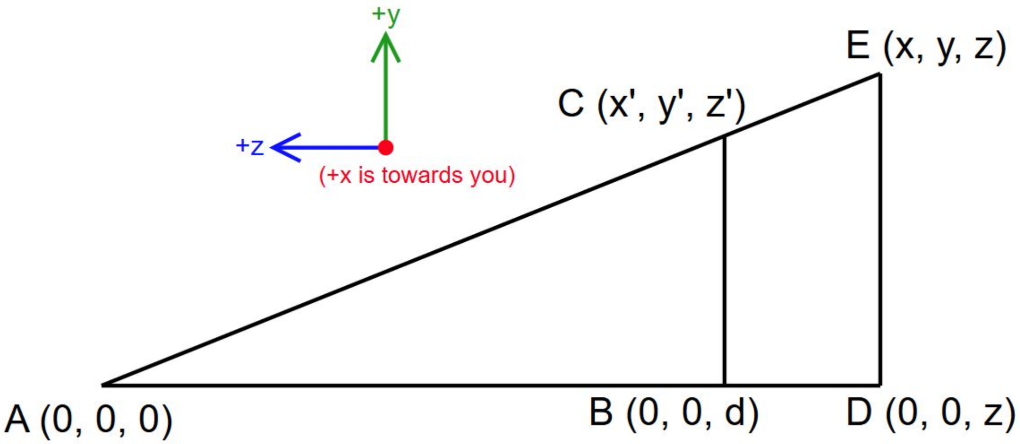
$(0, 0, z_p)$ = B = (0, 0, d)

So, $z_p = d$

Vector q(qx, qy, qx) = COP - $(0, 0, z_p)$ = (0, 0, 0) - (0, 0, d) = (0, 0, -d)

So, $q_x = 0$ and $q_y = 0$ and $q_z = -d$

Getting Simple Perspective Projection Matrix from the General One (COP at Origin)



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}$$

COP = A = (0, 0, 0)

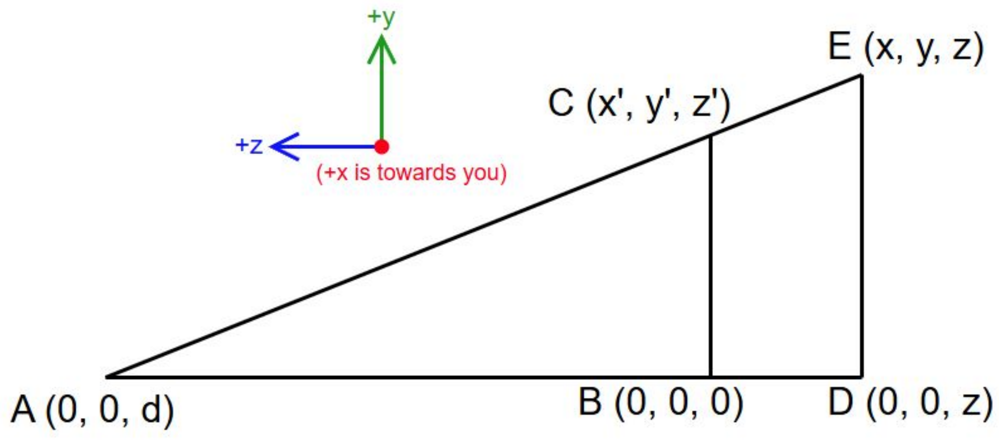
(0, 0, z_p) = B = (0, 0, d)

So, $z_p = d$

Vector q(qx, qy, qx) = COP - (0, 0, z_p) = (0, 0, 0) - (0, 0, d) = (0, 0, -d)

So, $q_x = 0$ and $q_y = 0$ and $q_z = -d$

Getting Simple Perspective Projection Matrix from the General One (PP at Origin)



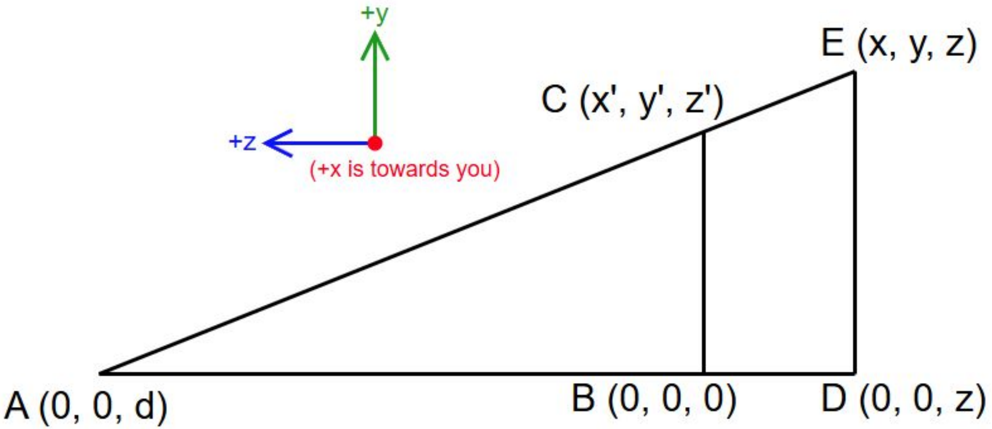
COP = A = (0, 0, d)
$(0, 0, z_p)$ = B = (0, 0, 0)

So, $z_p = 0$

Vector q(qx, qy, qx) = COP - $(0, 0, z_p)$ = (0, 0, d) - (0, 0, 0) = (0, 0, d)

So, $q_x = 0$ and $q_y = 0$ and $q_z = d$

Getting Simple Perspective Projection Matrix from the General One (PP at Origin)



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{d} & 1 \end{bmatrix}$$

COP = A = (0, 0, d)

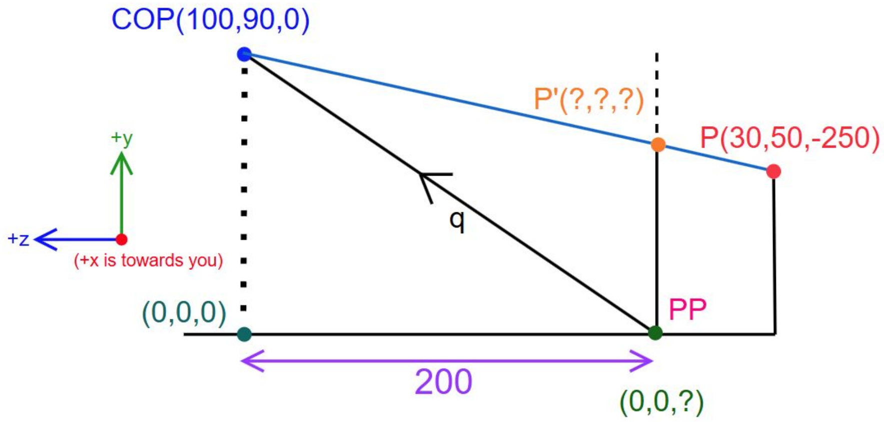
$(0, 0, z_p)$ = B = (0, 0, 0)

So, $z_p = 0$

Vector q(qx, qy, qx) = COP - $(0, 0, z_p)$ = (0, 0, d) - (0, 0, 0) = (0, 0, d)

So, $q_x = 0$ and $q_y = 0$ and $q_z = d$

Example: For COP at (100, 90, 0), calculate the projected point P′ for a point P(30, 50, -250) where the Projection Plane (PP) is 200 units away from COP

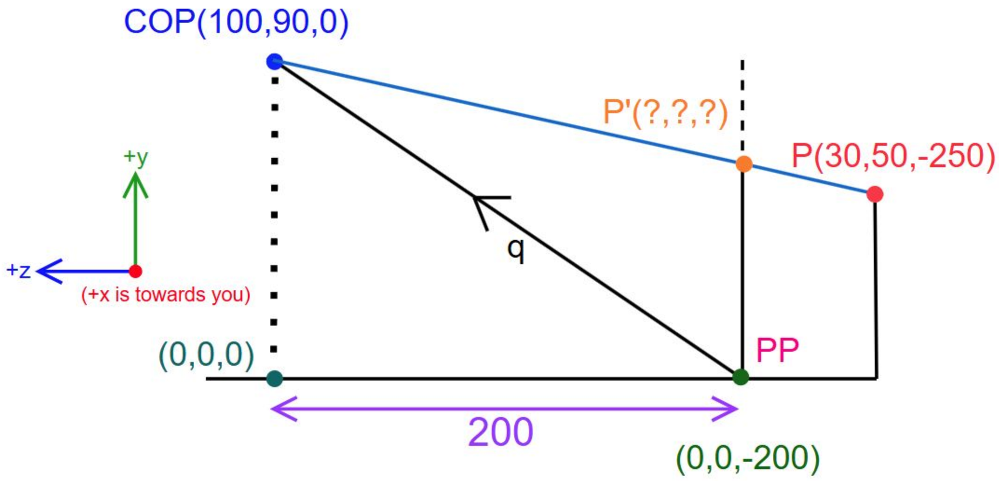


The z level of COP is 0, while the z level of P is -250.
Since there's a projection on the plane, the plane is likely to be between these two points.
The plane itself will have a z value that's between 0 and -250 (which is a negative value).

It's already said the distance between COP and PP is 200 units. This distance is across z-axis.
Thus, the z value of PP is (0-200) = -200. This is the z_p value.

So, z_p = -200

Example: Continued…



$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} & -100 \\ 0 & 1 & \frac{-9}{20} & -90 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{200} & 0 \end{bmatrix}$$

Since we've got the z_p, we can now calculate the q vector.

q = COP - $(0,0,z_p)$ = (100,90,0) - (0,0,-200) = (100,90,200)

So, q_x = 100 and q_y = 90 and q_z = 200 (and z_p = -200 from previously)

We now but these values into the general purpose perspective projection matrix, we will get the matrix from the upper-right.

Example: Continued...

Now, simply multiply the point P as a column vector.

$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} & -100 \\ 0 & 1 & \frac{-9}{20} & -90 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{200} & 0 \end{bmatrix} \times \begin{bmatrix} 30 \\ 50 \\ -250 \\ 1 \end{bmatrix} = \begin{bmatrix} 55 \\ 72.5 \\ -250 \\ 1.25 \end{bmatrix}$$

Almost done, all we need to do is divide each value in the result by "w" which is 1.25

$$\begin{bmatrix} 55 \\ 72.5 \\ -250 \\ 1.25 \end{bmatrix} \longrightarrow \begin{bmatrix} \left(\frac{55}{1.25}\right) \\ \left(\frac{72.5}{1.25}\right) \\ \left(\frac{-250}{1.25}\right) \\ \left(\frac{1.25}{1.25}\right) \end{bmatrix} = \begin{bmatrix} 44 \\ 85 \\ -200 \\ 1 \end{bmatrix}$$