

## Exercise 2.11 Solution

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Let  $C_{BR}$  be the true direction cosine matrix of the attitude with respect to the reference frame and  $C_{B'R}$  be the estimated direction cosine matrix of the attitude. Then, the error direction cosine matrix,  $C_{BB'}$ , is given by

$$C_{BB'} = C_{BR} C_{RB'} = C_{BR} C_{B'R}^T$$

As the direction matrix is a function of the quaternion, see (2.37) in the book, the same operation with the corresponding quaternions is given by

$$\delta \mathbf{q}_{BB'} = \mathbf{q}_{BR} \otimes \mathbf{q}_{RB'} = \mathbf{q}_{BR} \otimes \mathbf{q}_{B'R}^{-1} \quad (1)$$

where  $\otimes$  is the quaternion product defined by

$$\mathbf{q}^{(1)} \otimes \mathbf{q}^{(2)} = \begin{bmatrix} q_4^{(1)} \mathbf{q}_{13}^{(2)} + q_4^{(2)} \mathbf{q}_{13}^{(1)} - [\mathbf{q}_{13}^{(1)} \times] \mathbf{q}_{13}^{(2)} \\ q_4^{(1)} q_4^{(2)} - \left( \mathbf{q}_{13}^{(1)} \right)^T \mathbf{q}_{13}^{(2)} \end{bmatrix}$$

$$\mathbf{q}^{(i)} = \begin{bmatrix} \mathbf{q}_{13}^{(i)} \\ q_4^{(i)} \end{bmatrix} = \begin{bmatrix} q_1^{(i)} & q_2^{(i)} & q_3^{(i)} & q_4^{(i)} \end{bmatrix}^T$$

for  $i$  be the positive integers, and the inverse quaternion is given by

$$\mathbf{q}^{-1} = \begin{bmatrix} -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix}^T$$

i.e., the same angle rotation with the opposite direction to the one of  $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$ .

Take the time derivative of (1)

$$\delta \dot{\mathbf{q}}_{BB'} = \dot{\mathbf{q}}_{BR} \otimes \mathbf{q}_{B'R}^{-1} + \mathbf{q}_{BR} \otimes \dot{\mathbf{q}}_{B'R}^{-1} \quad (2)$$

where

$$\dot{\mathbf{q}}_{BR} = \frac{1}{2} \Omega(\omega) \mathbf{q}_{BR}$$

To obtain the time derivative of  $\mathbf{q}_{B'R}^{-1}$  in (1), take the time derivative of the following equation:

$$\mathbf{q}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} = [0 \ 0 \ 0 \ 1]^T$$

as follows:

$$\dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} + \mathbf{q}_{B'R} \otimes \dot{\mathbf{q}}_{B'R}^{-1} = \mathbf{0}$$

where  $\mathbf{0}$  is the  $4 \times 1$  zero vector. Apply  $\mathbf{q}_{RB'} \otimes$  both sides

$$\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} + \mathbf{q}_{RB'} \otimes \mathbf{q}_{B'R} \otimes \dot{\mathbf{q}}_{B'R}^{-1} = \mathbf{0}$$

and it becomes

$$\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} + \dot{\mathbf{q}}_{B'R}^{-1} = \mathbf{0} \Rightarrow \dot{\mathbf{q}}_{B'R}^{-1} = -\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1}$$

where

$$\dot{\mathbf{q}}_{B'R} = \frac{1}{2} \Omega(\hat{\omega}) \mathbf{q}_{B'R} \quad (3)$$

and

$$\hat{\omega} = \omega - \delta\omega$$

Obtain the expression of the right-hand side of

$$\dot{\mathbf{q}}_{B'R}^{-1} = -\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1}$$

using the symbolic calculations in MATLAB or python (See matlab\_ or python\_) as follows:

$$\begin{aligned} \dot{\mathbf{q}}_{B'R}^{-1} &= \frac{1}{2} \begin{bmatrix} -q_2\hat{\omega}_3 + q_3\hat{\omega}_2 - q_4\hat{\omega}_1 \\ -q_3\hat{\omega}_1 + q_1\hat{\omega}_3 - q_4\hat{\omega}_2 \\ -q_1\hat{\omega}_2 + q_2\hat{\omega}_1 - q_4\hat{\omega}_3 \\ -q_1\hat{\omega}_1 - q_2\hat{\omega}_2 - q_3\hat{\omega}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \hat{\omega}_3 & -\hat{\omega}_2 & -\hat{\omega}_1 \\ -\hat{\omega}_3 & 0 & \hat{\omega}_1 & -\hat{\omega}_2 \\ \hat{\omega}_2 & -\hat{\omega}_1 & 0 & -\hat{\omega}_3 \\ \hat{\omega}_1 & \hat{\omega}_2 & \hat{\omega}_3 & 0 \end{bmatrix} \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ q_4 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -[\hat{\omega} \times] & -\hat{\omega} \\ \hat{\omega}^T & 0 \end{bmatrix} \mathbf{q}_{B'R}^{-1} = \frac{1}{2} \mathcal{U}(\hat{\omega}) \mathbf{q}_{B'R}^{-1} \end{aligned}$$

where the subscript  $B'R$  is omitted for the notational simplicity, or it can be written as

$$\begin{aligned} \dot{\mathbf{q}}_{B'R}^{-1} &= \frac{1}{2} \begin{bmatrix} -[\hat{\omega} \times] & -\hat{\omega} \\ \hat{\omega}^T & 0 \end{bmatrix} \mathbf{q}_{B'R}^{-1} = \frac{1}{2} \begin{bmatrix} [(\mathbf{q}_{B'R}^{-1})_{13} \times] \hat{\omega} - \hat{\omega} (\mathbf{q}_{B'R}^{-1})_4 \\ \hat{\omega}^T (\mathbf{q}_{B'R}^{-1})_{13} \end{bmatrix} \\ &= -\frac{1}{2} \mathbf{q}_{B'R}^{-1} \otimes \begin{bmatrix} \hat{\omega} \\ 0 \end{bmatrix} \quad (4) \end{aligned}$$

Substitute (3) and (4) into (2)

$$\begin{aligned}\delta\dot{\mathbf{q}}_{BB'} &= \left[ \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}_{BR} \right] \otimes \mathbf{q}_{B'R}^{-1} - \frac{1}{2}\mathbf{q}_{BR} \otimes \left\{ \mathbf{q}_{B'R}^{-1} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \right\} \\ &= \frac{1}{2}\Omega(\boldsymbol{\omega}) [\mathbf{q}_{BR} \otimes \mathbf{q}_{B'R}^{-1}] - \frac{1}{2} [\mathbf{q}_{BR} \otimes \mathbf{q}_{B'R}^{-1}] \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix}\end{aligned}\quad (5)$$

where the following two equalities

$$\begin{aligned}\left[ \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}_{BR} \right] \otimes \mathbf{q}_{B'R}^{-1} &= \frac{1}{2}\Omega(\boldsymbol{\omega}) [\mathbf{q}_{BR} \otimes \mathbf{q}_{B'R}^{-1}] \\ \mathbf{q}_{BR} \otimes \left\{ \mathbf{q}_{B'R}^{-1} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \right\} &= [\mathbf{q}_{BR} \otimes \mathbf{q}_{B'R}^{-1}] \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix}\end{aligned}$$

are proved in matlab\_ or python\_.

Equation (5) is further expanded as follows:

$$\begin{aligned}\delta\dot{\mathbf{q}}_{BB'} &= \frac{1}{2}\Omega(\boldsymbol{\omega})\delta\mathbf{q}_{BB'} - \frac{1}{2}\delta\mathbf{q}_{BB'} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} = \frac{1}{2}\Omega(\boldsymbol{\omega})\delta\mathbf{q}_{BB'} + \frac{1}{2}\mathcal{U}(\hat{\boldsymbol{\omega}})\mathbf{q}_{BB'} \\ &= \frac{1}{2} [\Omega(\hat{\boldsymbol{\omega}} + \delta\boldsymbol{\omega}) + \mathcal{U}(\hat{\boldsymbol{\omega}})] \mathbf{q}_{BB'} \\ &= \frac{1}{2} \left\{ \begin{bmatrix} -[\hat{\boldsymbol{\omega}} \times] - [\delta\boldsymbol{\omega} \times] & \hat{\boldsymbol{\omega}} + \delta\boldsymbol{\omega} \\ -\hat{\boldsymbol{\omega}}^T - \delta\boldsymbol{\omega}^T & 0 \end{bmatrix} + \begin{bmatrix} -[\hat{\boldsymbol{\omega}} \times] & -\hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\omega}}^T & 0 \end{bmatrix} \right\} \mathbf{q}_{BB'}$$

Therefore,

$$\delta\dot{\mathbf{q}}_{BB'} = \frac{1}{2} \begin{bmatrix} -2[\hat{\boldsymbol{\omega}} \times] - [\delta\boldsymbol{\omega} \times] & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix} \mathbf{q}_{BB'}$$

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The proof follows the paper by Bani Younes A, Mortari D. *Derivation of All Attitude Error Governing Equations for Attitude Filtering and Control*, Sensors (Basel). 2019;19(21):4682. Published 2019 Oct 28. doi:10.3390/s19214682.