# Outage Probability in OFDMA Protocol

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## Introduction

This project investigates the outage probability in Orthogonal Frequency Division Multiple Access (OFDMA) systems, where users compete for limited subcarrier resources. We model the user distribution as a Poisson process and develop computational methods to determine the probability that the total subcarrier demand exceeds the available supply. The analysis combines stochastic geometry with wireless communication theory, implemented through Python simulations to optimize resource allocation. OFDMA is a fundamental multiple-access technique in modern wireless networks (4G/5G) that allocates subcarriers to users in both the time and frequency domains. Our work focuses on calculating the **outage probability** - the likelihood that the cumulative demand for subcarriers exceeds available resources. Key challenges include:

- Modeling user distribution as a Poisson process
- Calculating subcarrier requirements based on distance and SNR
- Implementing compound Poisson distributions

## 1 System Model

The system consists of:

- Circular cell with radius $R = 300$m
- Users distributed via Poisson process with intensity $\lambda = 0.01$ m$^{-2}$
- Active user probability $p = 0.01$
- Subcarrier requirements follow:

$$N(x) = \left\lceil \frac{C}{W \log_2(1 + \frac{K}{\|x\|^\gamma})} \right\rceil \tag{1}$$

where $C = 200$ kb/s, $W = 250$ kHz, $K = 10^6$, $\gamma = 2.8$

# 2 Theory

## Question 1

The active user process is derived through probabilistic thinning. Consider the activation function:

$$f : R^2 \to [0,1], \quad x \mapsto p$$

Given that

- The original user distribution forms a Poisson process with intensity $\lambda dx$
- Each user activates independently with probability $p$

Active users constitute a thinned Poisson process with intensity measure:

$$\mu_p(A) = \int_A f(x)\lambda dx = \lambda p \int_A dx$$

This confirms that the active process maintains Poisson statistics with reduced intensity $\lambda p$.

## Question 2

The number of active customers $N_{\text{active}}(t)$ follows a Poisson distribution with a parameter equal to the integrated intensity measure on the cell area. For a circular cell $C$ of radius $R$, we compute:

$$E[N_{\text{active}}] = \int_C \lambda p \, dx = \lambda p \cdot \text{Area}(C) = \lambda p \pi R^2$$

Thus, the mean number of active customers is:

$$\boxed{\pi R^2 \lambda p}$$

where:

- $\lambda$ is the user density (users/m$^2$)
- $p$ is the activation probability
- $R$ is the cell radius

## Question 3

Let us fix an integer $k \in \{1, \ldots, N_{\max}\}$. We aim to describe the region within the cell where all users require exactly $k$ subcarriers.

This occurs when the function $N(x)$ satisfies:

$$N(x) = k \quad \Longleftrightarrow \quad k - 1 < \frac{C}{W \log_2 \left(1 + \frac{K}{\|x\|^\gamma}\right)} \leq k$$

Rewriting the inequality:

$$\frac{C}{k} \leq W \log_2\left(1 + \frac{K}{\|x\|^\gamma}\right) < \frac{C}{k-1}$$

Exponentiating both sides:

$$2^{\frac{C}{Wk}} \leq 1 + \frac{K}{\|x\|^\gamma} < 2^{\frac{C}{W(k-1)}}$$

Solving for $\|x\|$, we obtain the following annular region:

$$\left(\frac{K}{2^{\frac{C}{W(k-1)}} - 1}\right)^{1/\gamma} < \|x\| \leq \left(\frac{K}{2^{\frac{C}{Wk}} - 1}\right)^{1/\gamma}$$

To simplify notation, we define:

$$R_{k-1} = \left(\frac{K}{2^{\frac{C}{W(k-1)}} - 1}\right)^{1/\gamma}, \quad R_k = \left(\frac{K}{2^{\frac{C}{Wk}} - 1}\right)^{1/\gamma}$$

Therefore, users requiring exactly $k$ subcarriers are located within the annular region bounded by the radii $R_{k-1}$ and $R_k$ from the origin. This region forms a ring-shaped area inside the circular cell.

## Question 4

We aim to determine the distribution of $A_3$, which represents the number of users requiring exactly 3 subcarriers.

To proceed, we consider the collection of random variables $(\zeta_k)_{k=1}^{N_{\max}}$, where each $\zeta_k$ denotes the number of users requiring $k$ subcarriers. This process is derived from a spatial Poisson process of users with intensity measure $\lambda\, dx$, combined with a spatial probability function $q_k(x)$ defined as:

$$q_k : R^2 \to [0, 1], \quad q_k(x) = P[N(x) = k]$$

From this, it follows that each $\zeta_k$ is a Poisson random variable with intensity measure:

$$\mu_k(B) = \lambda \int_B q_k(x)\, dx$$

In particular, for $k = 3$, the number of users requiring 3 subcarriers is:

$$A_3 = \zeta_3 \sim \text{Poisson}(\mu_3)$$

Geometrically, the set of points where $N(x) = 3$ corresponds to an annular region $B_3$ centered at the origin, bounded by radiuses $R_2$ and $R_3$, which are derived from the subcarrier condition established in Question 3.

Thus, the intensity becomes:

$$\mu_3 = \lambda p \int_{B_3} dx = \lambda p \pi (R_3^2 - R_2^2)$$

Hence, $A_3$ follows a Poisson distribution with parameter $\lambda p \pi (R_3^2 - R_2^2)$.

## Question 5

To corroborate our theoretical results through simulation, we first generated random user locations uniformly distributed within a circular cell of radius $R$. For each user, we calculated their required number of subcarriers using the given formula $N(x, y)$. We then filtered these users to isolate those requiring exactly 3 subcarriers ($A_3$), repeating this process 100 times to build an empirical distribution.

Next, we performed a Kolmogorov-Smirnov test to compare this empirical distribution with the theoretically predicted Poisson distribution. This statistical test quantifies how well our simulation matches the analytical model by measuring the maximum difference between their cumulative distribution functions and providing a $p$-value for significance testing.

## Question 6

We want to show that the total number of required subcarriers can be written as:

$$F(\varphi) = \sum_{k=1}^{N_{\max}} k \zeta_k$$

Since $N(x) = 0$ for $\|x\| > R_{N_{\max}}$, we have:

$$F(\varphi) = \sum_{\substack{x \in \varphi \\ \|x\| \leq R_{N_{\max}}}} N(x) = \sum_{k=1}^{N_{\max}} \sum_{\substack{x \in \varphi \\ R_{k-1} \leq \|x\| \leq R_k}} N(x) = \sum_{k=1}^{N_{\max}} k \zeta_k$$

For each $k \in \{1, \ldots, N_{\max}\}$, the inner sum runs over all users located at positions $x \in \varphi$ such that $R_{k-1} \leq \|x\| \leq R_k$, where the number of required subcarriers is constant and equal to $k$. We define $\zeta_k$ as the number of such users.

Since the annular regions $\{x \in \varphi \mid R_{k-1} \leq \|x\| \leq R_k\}$ are disjoint for different values of $k$, the random variables $\zeta_k$ are independent. Each $\zeta_k$ follows a Poisson distribution with parameter:

$$\lambda p \pi (R_k^2 - R_{k-1}^2)$$

Therefore, $F(\varphi)$ is a compound Poisson random variable.

## Question 7

Let $X_\mu \sim \text{Poisson}(\mu)$. We compute:

$$E[e^{\theta X_\mu}] = \sum_{i=0}^{\infty} e^{\theta i} \frac{\mu^i e^{-\mu}}{i!} = e^{-\mu} \sum_{i=0}^{\infty} \frac{(\mu e^\theta)^i}{i!} = e^{-\mu} e^{\mu e^\theta} = e^{\mu(e^\theta - 1)}$$

Then:

$$e^{-K\mu\theta} E[e^{\theta X_\mu}] = e^{-K\mu\theta + \mu(e^\theta - 1)}$$

The minimum is reached at $\theta = \ln K$, and we get:

$$\min_{\theta > 0} e^{-K\mu\theta} E[e^{\theta X_\mu}] = e^{-\mu(K(\ln K - 1) + 1)}$$

**Justification.** Define the function:

$$g(\theta) = -K\mu\theta + \mu(e^\theta - 1)$$

To find the minimizer, we compute the derivative:

$$g'(\theta) = -K\mu + \mu e^\theta = \mu(e^\theta - K)$$

We solve $g'(\theta) = 0$:

$$e^\theta = K \quad \Longrightarrow \quad \theta = \ln K$$

Since $g''(\theta) = \mu e^\theta > 0$, this critical point is a minimum. Therefore, the minimum of the original expression is reached at $\theta = \ln K$.

## Question 8

Using the fact that

$$P[X_\mu \geq K\mu] \leq e^{-K\mu\theta} E\left[e^{\theta X_\mu}\right],$$

we get:

$$P[X_\mu \geq K\mu] \leq \min_{\theta > 0} e^{-K\mu\theta} E\left[e^{\theta X_\mu}\right] = K^{-K\mu} e^{\mu(K-1)}.$$

Therefore, to ensure the probability is at most $10^{-4}$, it suffices to find $K_\mu$ satisfying:

$$K^{-\mu K_\mu} e^{\mu(K_\mu - 1)} \leq 10^{-4}.$$

Taking logarithms on both sides:

$$-\mu K_\mu \ln K_\mu + \mu(K_\mu - 1) \leq -4 \ln 10$$

Dividing by $\mu$:

$$K_\mu(1 - \ln K_\mu) \geq 1 - \frac{4\ln 10}{\mu}$$

Thus, we need to find a solution $K_\mu > 1$ to the equation:

$$K_\mu(1 - \ln K_\mu) = 1 - \frac{4\ln 10}{\mu}$$

**Existence of the solution.** Define the function $f(K) = K(1 - \ln K)$ on $K > 0$. This function is continuous and strictly decreasing on $(1, +\infty)$, with:

$$\lim_{K \to 1^+} f(K) = 1, \quad \lim_{K \to +\infty} f(K) = -\infty$$

Since the right-hand side $1 - \frac{4\ln 10}{\mu}$ is less than 1 for any $\mu > 0$, the equation $f(K_\mu) = 1 - \frac{4\ln 10}{\mu}$ has a unique solution $K_\mu > 1$ by the intermediate value theorem.