MA 221 Differential Equations Recitation F

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Section 2.1: Problem 28

Question
Find the critical points and phase portrait of the given autonomous first-order differential equation. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the xy-plane determined by the graphs of the equilibrium solutions.

$$\frac{dy}{dx} = \frac{ye^y - 9y}{e^y}$$

Solution:
Critical points are the solution of the equation:

$$\begin{aligned} f(y) &= ye^y - 9y = 0 \\ \frac{ye^y - 9y}{e^y} &= 0 \\ y(e^y - 9) &= 0 \\ y = 0 &\text{ or } e^y - 9 = 0 \\ y = 0 &\text{ or } y = \ln 9 = 2\ln 3 \end{aligned}$$

We have the critical points at $y = 0$ and $y = 2\ln 3$. To find out where our arrows in the phase portrait point, use:

$$\begin{aligned} f(-1) &= 9e - 1 > 0 \\ f(1) &= 1 - \frac{9}{e} < 0 \\ f(3) &= 3 - \frac{27}{e^3} > 0 \end{aligned}$$

Since both arrows are pointing towards the critical point at $y = 0$, it is a stable critical point. The critical point at $y = 2\ln 3$ is unstable.

Section 2.2: Problem 33

Question:
Find an explicit solution of the given initial value problem. Determine the exact interval I of definition by analytical methods. Use a graphing utility to plot the graph of the solution.

$$e^y dx - e^{-x} dy = 0$$

Solution:
We rewrite the equation as:

$$e^y dx = e^{-x} dy$$

Multiplying both sides by $e^x e^{-y}$, we get:

$$e^x dx = e^{-y} dy$$

Integrating both sides, we obtain:

$$\begin{aligned} \int e^x dx &= \int e^{-y} dy \\ e^x &= -e^{-y} + C \\ e^{-y} &= -e^x + C \\ y &= -\ln(-e^x + C) \end{aligned}$$

Using the initial condition $y(0) = 0$, we find:

$$\begin{aligned} 0 &= -\ln(-e^0 + C) \\ 0 &= \ln(C - 1) \\ C &= 2 \end{aligned}$$

Thus, the solution to the initial value problem is:

$$y = -\ln(-e^x + 2)$$

The solution is only defined when $-e^x + 2 > 0$, which gives:

$$\begin{aligned} e^x &< 2 \\ x &< \ln 2 \end{aligned}$$

Therefore, the solution is defined on the interval $I = (-\infty, \ln 2)$.

Section 2.3: Problem 16

Question:
Find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

$$ydx = (ye^y - 2x)dy$$

Solution:
We begin by rewriting the equation:

$$\frac{dx}{dy} + \frac{2x}{y} = e^y$$

The integrating factor is:

$$\mu(y) = e^{\int \frac{2}{y} dy} = y^2$$

Multiplying both sides by y^2, we get:

$$y^2 dx + 2xy^2 dy = y^2 e^y dy$$

This simplifies to:

$$\frac{d}{dy}(xy^2) = y^2 e^y$$

Integrating both sides:

$$xy^2 = e^y(y^2 - 2y + 2) + C$$

Thus, the general solution is:

$$x = e^y \frac{y^2 - 2y + 2}{y^2} + \frac{C}{y^2}$$

Section 2.4: Problem 8

Question:
Determine whether the given differential equation is exact. If it is exact, solve it.

$$\left(1 + \ln x + \frac{y}{x}\right) dx = (1 - \ln x)dy$$

Solution:
Given:

$$M(x, y) = \left(1 + \ln x + \frac{y}{x}\right), \quad N(x, y) = -(1 - \ln x)$$

We find:

$$\frac{\partial M}{\partial y} = \frac{1}{x}, \quad \frac{\partial N}{\partial x} = \frac{1}{x}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. We find a function $f(x, y)$ such that:

$$\frac{\partial f}{\partial x} = 1 + \ln x + \frac{y}{x}$$

Integrating with respect to x:

$$f(x, y) = x(1 + \ln x) + y \ln x - y = C$$

Thus, the general solution is:

$$x(1 + \ln x) + y(\ln x - 1) = C$$

Section 2.5: Problem 26

Question:
Solve the given differential equation using an appropriate substitution.

$$\frac{dy}{dx} = \sin(x+y)$$

Solution:
Let $u = x + y$. Then:

$$\frac{du}{dx} = 1 + \frac{dy}{dx}, \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

Substituting:

$$\sin u = \frac{du}{dx} - 1$$

$$\frac{du}{dx} = \sin u + 1$$

Separating variables:

$$\int \frac{1}{\sin u + 1} du = \int dx$$

Solving the integral:

$$\int \frac{1 - \sin u}{\cos^2 u} du = \int dx$$

$$\tan u - \sec u = x + C$$

Substituting back $u = x + y$, we get:

$$\tan(x+y) - \sec(x+y) = x + C$$

Thus, the solution to the differential equation is:

$$\tan(x+y) - \sec(x+y) = x + C$$