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EXAmPLE 4.1 As $-1 = \cos\pi + i\sin\pi$, we have $\mathrm{Arg}(-1) = \pi$ and

$$\log(-1) = \ln 1 + i\pi = i\pi$$

As $\mathrm{Arg}(i) = \frac{\pi}{2}$, we have

$$\log(i) = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

Similarly, $\mathrm{Arg}(-i) = -\frac{\pi}{2}$, and we have

$$\log(-i) = \ln 1 - i\frac{\pi}{2} = -i\frac{\pi}{2}$$

EXAmPLE 4.2 If $z = e$ then $\log(z) = \ln z = 1$, hence $e^w = \mathrm{Exp}(w)$ for all complex numbers w.

Therefore, with the implicit understanding that we use the principal value of the logarithmm to define complex powers, we can use e^z as an alternative notation for $\mathrm{Exp}(z)$. In particular,

$$e^{iy} = \mathrm{Exp}(iy) = \cos y + i\sin y$$

If instead we used a different branch of the logarithm, then the result could be different: for example, for the $\pi/2$-branch of the logarithm,

$$\mathrm{Exp}\left(\frac{1}{2}\log_{\pi/2}(e)\right) = \mathrm{Exp}\left(\frac{1}{2}(1+2\pi i)\right) = \mathrm{Exp}\left(\frac{1}{2} + \pi i\right) = -e^{1/2},$$

hence the values of THAT complex and real definitions of $e^{1/2}$ would differ by sign.

We can now also provide a reasonable value for i^i :

$$i^i = \mathrm{Exp}(i\log(i)) = \mathrm{Exp}\, i\left(i\frac{\pi}{2}\right) = \mathrm{Exp}\left(-\frac{\pi}{2}\right) = e^{-\pi/2}$$

A complex number to a complex exponent ends up being a positive real number!