Separable equations and first-order linear equations

MA221, Lecture 3

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Separable Equations

A **separable ODE** is any equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} \overset{\star}{=} f(x)g(y).$$

These are solvable, provided we know the anti-derivatives of f and $1/g$ (call them F and G, respectively):

$$\star \Rightarrow \frac{1}{g(y)}\frac{dy}{dx} = f(x) \Rightarrow \int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \underbrace{\int f(x)\,dx}_{= F(x)+C}$$

$$\Rightarrow F(x) + C = \underbrace{\int \frac{1}{g(y)}\frac{dy}{dx}\,dx}$$

$$\Rightarrow \quad u = y = y(x)$$

$$du = y'(x)\,dx = \frac{dy}{dx}\,dx$$

$$\Rightarrow \boxed{F(x)+C = G(y)}\ \star \quad = \int \frac{1}{g(u)} \cdot du = G(u) = G(y)$$

Example 1: $1 - (y^3 - y^2 + y - 1)xy' \overset{*}{=} 0$

$$* \Rightarrow (y^3 - y^2 + y - 1)\, x \frac{dy}{dx} = 1 \Rightarrow (y^3 - y^2 + y - 1)\, dy = \frac{1}{x} dx$$

$$\Rightarrow \int (y^3 - y^2 + y - 1)\, dy = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2} - y + C = \ln |x|$$

Example 2: $y' \overset{\star}{=} 3t^2 y$ subject to $y(0) = 1$

$$\star \Rightarrow \frac{dy}{dt} = 3t^2 y \Rightarrow \frac{1}{y}\,dy = 3t^2\,dt \Rightarrow \int \frac{1}{y}\,dy = \int 3t^2\,dt$$

$$\Rightarrow \ln|y| + C = t^3$$

$$y(0)=1 \Rightarrow \begin{matrix} y=1 \\ t=0 \end{matrix} : \underbrace{\ln|1|}_{=0} + C = 0^3 \Rightarrow C = 0.$$

$$\Rightarrow \ln|y| = t^3 \Rightarrow e^{\ln|y|} = e^{t^3}$$

$$\Rightarrow |y| = e^{t^3} \Rightarrow y = \pm e^{t^3} \Rightarrow \boxed{y = e^{t^3}}$$

rule out $-e^{t^3}$: $y(0)=1$

Example 3: $\frac{dy}{dx} \overset{\star}{=} x\sqrt{y}$

$$\star \Rightarrow \frac{1}{\sqrt{y}}dy = x\,dx \Rightarrow y^{-\frac{1}{2}}dy = x\,dx$$

$$\Rightarrow \int y^{-\frac{1}{2}}dy = \int x\,dx$$

$$\Rightarrow 2\sqrt{y} = \frac{y^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = \int y^{-\frac{1}{2}}dy = \int x\,dx = \frac{x^2}{2} + C$$

$\Rightarrow$ Can this implicit solution be made explicit? how?

Other examples to try on your own:

- . $\frac{dy}{dt} = 4t^3y - y$
- . $y'(\tan y)(t^2+1) = t$
- . $\ln y \cdot \sec(\sin x)\frac{dy}{dx} = 3\cos x$

$$y'(\tan y)(t^2+1) \overset{*}{=} t$$

$$* \Rightarrow \frac{dy}{dt} \cdot \tan y \cdot (t^2+1) = t$$

$$\Rightarrow \tan y \, dy = \frac{t}{t^2+1} dt \Rightarrow \underbrace{\int \tan y \, dy}_{A} = \underbrace{\int \frac{t}{t^2+1} dt}_{B}$$

$$A = \int \tan y \, dy = \int \frac{\sin y}{\cos y} dy = -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos y| + C$$

$u = \cos y$

$du = -\sin y \, dy$

$$B = \int \frac{t}{t^2+1} dt = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln(t^2+1)$$

$u = t^2+1$

$du = 2t\,dt$

$$A = B \Rightarrow \boxed{-\ln|\cos y| + C = \frac{1}{2}\ln(t^2+1)}$$