math426_math_626assignment_2_samir_banjara

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Question1: Let $Q \in \mathbb{C}^{m \times m}$. Show that the following statements are equivalent

1. Q is an orthogonal matrix.

Suppose, Q is an orthogonal matrix, then the pairwise elements of any m mutual othronormal vectors of Q, $\vec{n_1}, \vec{n_2}, \cdots, \vec{n_m} \in \mathbb{C}^m$ are orthogonal.

$$\langle n_j, n_k \rangle = \delta_{jk}$$

then,

$$Q = \begin{bmatrix} \vec{n_1} & \vec{n_2} & \cdots & \vec{n_m} \end{bmatrix}$$

and,

$$Q^* = \begin{bmatrix} \vec{n_1^*} \\ \vec{n_2^*} \\ \vdots \\ \vec{n_m^*} \end{bmatrix}$$

thus,

$$QQ^* = \begin{bmatrix} \vec{n_1} & \vec{n_2} & \cdots & \vec{n_m} \end{bmatrix} \begin{bmatrix} \vec{n_1^*} \\ \vec{n_2^*} \\ \vdots \\ \vec{n_m^*} \end{bmatrix} =$$

$$= \begin{bmatrix} \vec{n_1}\,\vec{n_1^*} & \cdots & \vec{n_m}\,\vec{n_1^*} \\ \vdots & \ddots & \vdots \\ \vec{n_1}\,\vec{n_m^*} & \cdots & \vec{n_m}\,\vec{n_m^*} \end{bmatrix} =$$

$$= \begin{bmatrix} \langle \vec{n_j}, \vec{n_k} \rangle \end{bmatrix} = \begin{bmatrix} \delta_{jk} \end{bmatrix} = I_m \qquad \text{for } 1 \leq j,\, k \leq m$$

Hence the magnitude of two vectors Q and Q^* is equal to the identity vector I_m and orthonormal.

2. $||Q\mathbf{x}|| = ||\mathbf{x}||$.

From our previous work on Question 1.1, we assumed Q to be orthogonal and hence the product of matrix Q and its complex conjugate Q^* is equal to be the identity vector.

$$QQ^* = I_m$$

Let A be an $n \times n$ matrix. Then, A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Then B must be the inverse of A, denoted A^{-1}.

We know that, a orthogonal matrix is a square $m \times m$ matrix that satisfies $M^* M = I_m$ and the inverse must satisfy, $A\,A^{-1} = I_n$,

thus if an inverse of an matrix is equal to its adjoint, then it is orthogonal.

$$\|\mathbf{x}\| = \|Q\mathbf{x}\|$$

$$\|\mathbf{x}\| = \|Q\,Q^{-1}|\ \|\mathbf{x}\|$$

$$\text{or} \quad \|\mathbf{x}\| = |Q\,Q^*|\ \|\mathbf{x}\|$$

$$\|\mathbf{x}\| = I_m \cdot\ \|\mathbf{x}\|$$

$$\|x\| = \|x|$$

3. $(Q\mathbf{x})^*(Q\mathbf{y}) = x^* y$

Using the property, $(\alpha\mathbf{x})^*(\beta\mathbf{y}) = \alpha^*\,\beta\,\mathbf{x}^*\,\mathbf{y}$

$$(Q\mathbf{x})^*\,(Q\mathbf{y}) = \mathbf{x}^*\,\mathbf{y}\,Q\,Q^* = \mathbf{x}^*\,\mathbf{y}\,I_n = \mathbf{x}^*\,\mathbf{y}$$

We have thus proved all statements to be true are equivalent.

Question2: Let $A \in \mathbb{C}^{m\times m}$ be Hermitian. Show that the following statements are true:

1. All eigenvalues of A are real.
2. If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to distinct eigenvalues, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Let λ be an arbitrary eigenvalue of an Hermitian matrix A and let x be an eigenvector corresponding to the eigenvalue λ

Then we have $Ax = \lambda x$

Multiply both sides by x^{-T} where,$x^{-T} = x^*$ thus,

$$x^*(Ax) = x^*(\lambda x) = \lambda x^* x = \lambda\|x\|$$

or

$$x^*(Ax) = (Ax)^*\ \overline{x} = x^*\ A^*\,\overline{x}$$

Dot product is communicative, let $u = \overline{x},\ v = Ax$

Thus, $u \cdot v = u^* v = v^* u = v \cdot u$ or $x^* A^*\,\overline{x} = \lambda\,\|x\|$

taking the complex conjugate of this equality we have,

$$x^* A^* = \overline{\lambda}\, \|x\|$$

since, matrix A is Hermitian, we have $A^* = A$

Which results in,

$$\overline{\lambda}\, \|x\| = x^* A x = x^* \lambda x = \lambda \|x\|$$

Because x is an eigenvector, $x \neq 0$ and $\|x\| \neq 0$, $\lambda = \overline{\lambda}$

thus, eigenvalue λ is a real number.

Additionally, from the equality,

$$x^*(Ax) = x^*(\lambda x) = \lambda x^* x = \lambda \|x\|$$

the we see that $xAx^* = \lambda\, x\, x^*$ is a complex number

but $A^* = A$, $v \neq 0$ and given $xA = \lambda x$

both xAx^* and $x\, x^*$ are positive real numbers, it follows that λ is also real.

Since we have x and y as eigenvectors with distinct real eigenvalues, which we just showed true for every Hermitian matrix, because we assumed λ was arbitrary.

Now lets supposed for x and y as eigenvectors we have λ and μ as distinct eigenvalues. we have from work above

because we showed $y\,(Ax) = (\lambda x)\, y = (yA)\, x = (\mu y)\, x = (\mu x)\, y$

and so, $(\lambda x)\, y = (\mu x)\, y$

then $(\lambda - \mu)\, x \cdot y = 0$ Since $\lambda - \mu \neq 0$, then $x \cdot y = 0$ which satisfies the condition for two vectors to be orthogonal it's dot product must be 0.

$$x \perp y$$

Thus, eigenvectors corresponding to distinct real eigenvalues of the Hermitian matrix $A \in \mathbb{C}^{m \times m}$ are orthogonal.