MA225 - Sequences and Series

Pedro Vilanova

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Motivation for series

Key motivation: Approximate functions. Non-trivial functions, including those that can't be easily integrated or differentiated, can be expressed as power series. This allow us to approximate functions with arbitrary precision.

Example: Gaussian Function:

$$f(x) = e^{-x^2}.$$

Why it's challenging:

- The integral does not have an elementary antiderivative, meaning it cannot be expressed in terms of functions like polynomials, exponentials, logarithms, trigonometric functions, etc.
- Differentiating e^{-x^2} repeatedly results in increasingly complicated expressions.

Motivation for series

We can express e^{-x^2} as a power series centered at $x = 0$ (a Maclaurin series):

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \,.$$

This series converges for all x, and it allows us to approximate e^{-x^2} to any desired degree of accuracy by truncating the series at a certain number of terms.

By using the power series, we can work with this function in contexts where direct integration or differentiation would be intractable.

Applications:

- Probability Theory: The Gaussian (or normal) distribution, a cornerstone of statistics, is modeled by the function e^{-x^2}.
- Physics: It appears in the study of heat distribution (the heat equation) and quantum mechanics (the wave functions of particles).

Introduction

Series applications:

- is a powerful tool to express functions as infinite polynomials. (series whose terms involve powers of a variable).
- help to evaluate complicated functions.
- approximate definite integrals.
- solve differential equations that model physical behavior.

https://www.wolframcloud.com/env/pguerra0/MA225-L1.nb

Sequences are essential for series. This is the next topic.

Sequence

Definition

A sequence $\{a_n\}$ is an infinite ordered list of numbers of the form

$$a_1, a_2, \dots, a_n, \dots$$

- *The subscript n is called the index variable of the sequence.*
- *Each number a_n is a term of the sequence.*
- *Sequences are defined by explicit formulas, in which case $a_n = f(n)$ for some function $f(n)$ defined over the natural numbers. Example: $a_n = \frac{1}{n}$.*
- *In other cases, sequences are defined by using a recurrence relation (not emphasized in this course).*

Formally a sequence is a function from $\mathbb{N}_+ \to \mathbb{R}$. Here $\mathbb{N}_+$ denotes the set $\{1, 2, 3, ...\}$.

Example:

$$2, 4, 8, 16, 32, \ldots$$

This is a sequence $a_1 = 2, a_2 = 4$, and $a_3 = 8$, etc.

This sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4 \text{ and } a_5 = 2^5\,.$$

Assuming the pattern continues we obtain the formula:

$$a_n = 2^n\,.$$

Equivalent notation: $\{2^n\}_{n\geq 1}$.

Example 2:

$$a_n = \frac{n + (-1)^n}{n}\,, \quad n \geq 1\,.$$

Example 3:

$$\frac{1}{2}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{8}, \frac{1}{10}, \ldots$$

Sequence

Note 1: A sequence can be described using a recurrence relation: by expressing the n-th term a_n in terms of the previous a_{n-1} plus a basis case. In the previous example,

$$\begin{aligned} a_1 &= 2 \\ a_n &= 2a_{n-1}\,, \qquad n \geq 2\,. \end{aligned}$$

Note 2: The starting index does not have to be one. For example,

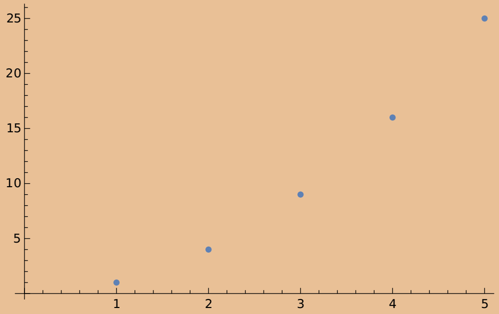
$$a_0, a_1, a_2, \ldots$$

We will see the starting index is not essential. By default is 1.

Sequence

Since a sequence is a function, we can plot its graph:

$$\text{all points } (n, a_n) \text{ for } n \in \mathbb{N}_+.$$



$$a_n = 2^n$$

Example: Sequences without a defining formula.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) Let a_n be the digit in the n-th decimal place of the number e. Then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots\}.$$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1\,, \quad f_2 = 1\,, \quad f_n = f_{n-1} + f_{n-2}\,, \quad n \geq 3\,.$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}.$$

This sequence first appeared when the 13th-century Italian mathematician Fibonacci solved a problem concerning the breeding of pairs of rabbits.

Two types of sequences

Two types of sequences occur often: arithmetic sequences and geometric sequences.

In an **arithmetic** sequence, the difference between every pair of consecutive terms is the same. For example,

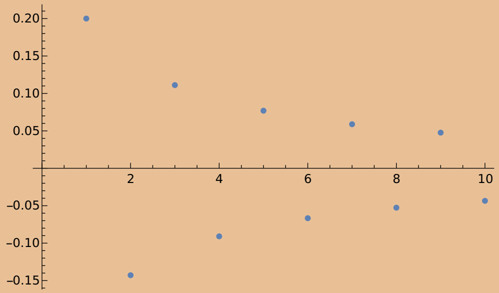
$$3, 7, 11, 15, 19, \dots$$

Formula?

Example: Find an explicit formula for the n^{th} term of the sequence

$$\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \dots.$$

Two types of sequences

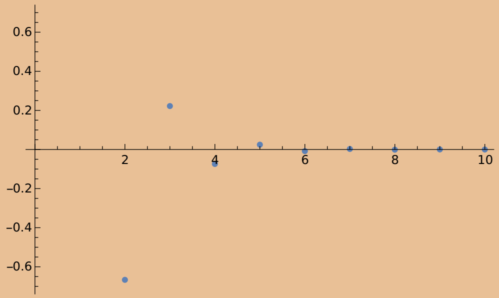


Two types of sequences

In a **geometric** sequence, the ratio of every pair of consecutive terms is the same. For example,

$$2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \ldots$$

We see that the ratio between a_n and a_{n-1} for $n \geq 2$ is $-1/3$. Since it starts at 2 and $a_n = -\frac{1}{3}a_{n-1}$ we can deduce the formula.



Two types of sequences

In summary, for constants $a, d, r \in \mathbb{R}$:

Arithmetic:

$$a,\ (a+d),\ (a+2d), \ldots$$

Geometric:

$$a,\ ar,\ ar^2, \ldots$$

Examples

Examples: Find an explicit formula for the n^{th} term of the sequence.

1. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$
2. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots$

Examples - Solution

1. Alternating sequence with "arithmetic/arithmetic". The odd terms in the sequence are negative, and the even terms are positive. Therefore, the n^{th} term includes a factor of $(-1)^n$. Next, consider the sequence of numerators even terms are positive. Therefore, the n^n term includes a factor of $(-1)^n$.

Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$ We can see that both of these sequences are arithmetic sequences. The n^{th} term in the sequence of numerators is n, and the n^{th} term in the sequence of denominators is $n+1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1}\,.$$

Examples - Solution

2. "geometric/arithmetic". The sequence of numerators 3, 9, 27, 81, 243, ... is a geometric sequence. The numerator of the n^{th} term is 3^n The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the n^{th} term is $4 + 3(n-1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n+1}$.

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger, that is $n \to \infty$.

Examples:

1. $\{1 + 3n\} = \{4, 7, 10, 13, \ldots\}$.

   The terms become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$

2. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} \ldots\right\}$.

   The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.

3. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$.

   The terms alternate but do not approach one single value as $n \to \infty$.

4. $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

   The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

Definition (convergent and divergent sequences)

Given a sequence $\{a_n\}$, if the terms can become arbitrarily close to a number L as n becomes sufficiently large, we say $\{a_n\}$ is a convergent sequence and L is the limit of the sequence. In this case, we write

$$\lim_{n\to\infty} a_n = L\,, \qquad \text{or} \quad a_n \to L\,.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a divergent sequence.

Limit of a Sequence

Example: the sequences $\{(-1)^n\}$ and $\{1+3n\}$ diverge in different ways.

The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1 (oscillates).

On the other hand, the sequence $\{1+3n\}$ diverges because the terms $1+3n \to \infty$ as $n \to \infty$. We say the sequence $\{1+3n\}$ diverges to infinity and write $\lim_{n\to\infty}(1+3n) = \infty$.

A sequence can also diverge to negative infinity. For example, the sequence $\{-5n+2\}$.

Note: Infinity (∞) is not a real number; it's a concept that represents an unbounded quantity.

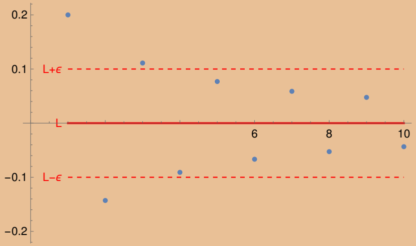
Formal definition

Definition

A sequence $\{a_n\}_{n\geq 1}$ converges to a real number L if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}_+$ such that for all $n \geq N$ we have

$$|a_n - L| < \epsilon\,.$$

Note the number N depends on ϵ. The number L is the limit of the sequence. The notation is the same as before.



$$a_n = \frac{(-1)^{n+1}}{3+2n}$$

Limit of a Sequence Defined by a Function

Theorem

Consider a sequence $\{a_n\}_{n\geq 1}$ *such that* $a_n = f(n)$. *If there exists a real number* L *such that*

$$\lim_{x\to\infty} f(x) = L\,,$$

then $\{a_n\}$ *converges and*

$$\lim_{n\to\infty} a_n = L\,.$$

Example 1: Consider the sequence $\{1/n\}$ and the related function $f(x) = 1/x$. Since f is defined for $x > 0$ and satisfies $\lim_{x\to\infty} 1/x = 0$ then $\lim_{n\to\infty} \{1/n\} = 0$.

Example 2: Consider the sequence $\{(1/2)^n\}$ and the related function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty}(1/2)^x = 0$, then $\lim_{n\to\infty} \{(1/2)^n\} = 0$.

Limit of a Sequence Defined by a Function

Example 3: For any real number r such that $0 \leq r < 1$, we know $\lim_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges.

Moreover:

- If $r = 1$, then $r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1.
- If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply the theorem. However, since the terms r^n in the sequence become arbitrarily large as $n \to \infty$, we can conclude that the sequence $\{r^n\}$ diverges.

In summary,

$$r^n \to 0 \text{ if } r \in (0,1)$$
$$r^n \to 1 \text{ if } r = 1$$
$$r^n \to \infty \text{ if } r > 1.$$

Later we will see the case when $r < 0$.

Limit of a Sequence Defined by a Function

Remark: The converse of the theorem is not necessarily true: knowing that the sequence $\{a_n\}$ converges to some limit L does not imply that the function $f(x)$ converges to L as x tends to infinity.

Consider the following counterexample:

- Define the function $f(x) = \sin(\pi x) + \frac{1}{x}$.
- The sequence $\{a_n\}$ is given by $a_n = f(n) = \sin(\pi n) + \frac{1}{n}$. Since $\sin(\pi n) = 0$ for all integers n, we have $a_n = \frac{1}{n}$.
- The sequence $\{a_n\}$ clearly converges to 0 as $n \to \infty$.

However, the function $f(x) = \sin(\pi x) + \frac{1}{x}$ does **not** converge to 0 as $x \to \infty$ because $\sin(\pi x)$ oscillates between -1 and 1 for non-integer values of x, and the $\frac{1}{x}$ term tends to 0, but the sum does not approach any single value.

Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$

ii. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$

iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$

iv. $\lim_{n\to\infty} (a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n) = A \cdot B$

v. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Remark: Laws apply when a_n and b_n converge to a limit.

Example: Consider $a_n = \frac{1}{n^2}$. As shown earlier, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Using rule iv., for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

Exercise: For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

1. $\{(2/3)^n + (1/4)^n\}$. Hint: Use previous geometric sequence result.
2. $\{5 - \frac{3}{n^2}\}$. Hint: Use previous example.
3. $\{\frac{n^2+1}{n^3+2n^2+1}\}$.
4. $\{\frac{3n^4-7n^2+5}{6-4n^4}\}$.
5. $\{\frac{2^n}{n^2}\}$ Hint: Use L'Hopital's rule.

Algebraic Limit Laws Exercise solution

5. Consider the related function $f(x) = \frac{2^x}{x^2}$ for $x > 0$. Since $2^x \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, apply L'Hôpital's rule and take the derivatives of the numerator and denominator

$$\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{2^x \ln 2}{2x} =$$

(take derivatives again)

$$= \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} = \infty\,.$$

(See `https://en.wikipedia.org/wiki/L'H%C3%B4pital's_rule`)

Continuous function

Recall that if f is a continuous function at L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences.

Proposition

Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$.

This property often enables us to find limits for complicated sequences.

Example: Consider the sequence $a_n = 5 - \frac{3}{n^2}$. It is very simple to see that $5 - \frac{3}{n^2} \to 5$.

Now consider the sequence $b_n = \sqrt{5 - \frac{3}{n^2}}$.

Since $\sqrt{x}$ is a continuous function at $x = 5$,

$$\lim_{n\to\infty} \sqrt{5 - \frac{3}{n^2}} = \sqrt{\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right)} = \sqrt{5}.$$

Basically, we can interchange limit and the continuous function.

Continuous function

Example 2: Determine if the sequence $\left\{\sin(\frac{\pi n}{n+1})\right\}$ converges. If it converges, find its limit. Consider $b_n = \frac{\pi n}{n+1}$. Since $\lim_{n\to\infty} b_n = \pi$ and $\sin(x)$ is continuous everywhere, $\lim_{n\to\infty} \sin\left(\frac{\pi n}{n+1}\right) = \sin(\pi) = 0$. In other words we can say

$$\lim_{n\to\infty} \sin\left(\frac{\pi n}{n+1}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi n}{n+1}\right).$$

In other words, lim "commutes" with a continuous function.

Exercise: Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

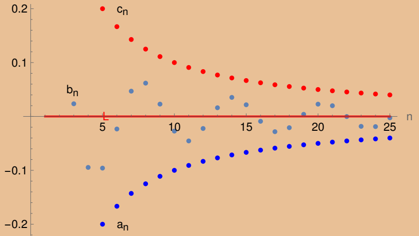
Counterexample: Let $f(x) = 1/x$. In this case we cannot apply the property of the continuous function. Consider the sequence $a_n = 1/n^2$. We know already that $a_n \to 0$. Since $f(x)$ is not continuous at 0, we cannot apply the property. Indeed, we have $\lim_{n\to\infty} f(a_n) = \frac{1}{1/n^2} = n^2$ diverges.

Squeeze Theorem

Consider sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$. Suppose there exists an $N \in \mathbb{N}_+$ such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N.$$

If there exists a real number L such that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$.



Each term b_n satisfies $a_n \leq b_n \leq c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Squeeze Theorem

Example: Consider $\left\{\left(-\frac{1}{2}\right)^n\right\}$. Since

$$-\frac{1}{2^n} \le \left(-\frac{1}{2}\right)^n \le \frac{1}{2^n}, \qquad n \ge 1,$$

and $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Exercise: Find $\lim_{n\to\infty} \frac{2n-\sin n}{n}$. Hint: Use the fact that $-1 \le \sin(n) \le 1$.

Exercise: (useful to understand the next one). Show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. First observe we have the upper bounding sequence ("c_n" in the theorem's notation)

$$\frac{2^n}{n!} = \frac{\overbrace{2\cdot 2\cdot 2\cdots 2}^{n \text{ copies}}}{1\cdot 2\cdot 3\cdots n} \le \frac{2\cdot 2}{1\cdot 2}\left(\frac{2}{3}\right)^{n-2}, \quad \text{for } n > 2.$$

Finally using S.T.,

$$\lim_{n\to\infty} \frac{2\cdot 2}{1\cdot 2}\left(\frac{2}{3}\right)^{n-2} = 0$$

as we wanted to prove. Note the "a_n" sequence in the theorem's notation is the sequence $a_n = 0$.

What if we want to prove the limit of the sequence $\frac{3^n}{n!}$ is 0? We use the same idea

$$\frac{3^n}{n!} = \frac{\overbrace{3\cdot 3\cdot 3\cdots 3}^{n \text{ copies}}}{1\cdot 2\cdot 3\cdots n} \leq \frac{3\cdot 3\cdot 3}{1\cdot 2\cdot 3}\left(\frac{3}{4}\right)^{n-3}, \quad \text{for } n > 3\,.$$

What if we want to prove the limit of the sequence $\frac{r^n}{n!}$ for $r \in \mathbb{R}$, $r > 0$ is 0? This is a bit more complicated because r may not be a natural number. But we can choose a natural number N such that $N > r$. Then, for $n > N$ we have

$$\frac{r^n}{n!} = \frac{\overbrace{r \cdot r \cdot r \cdots r}^{n \text{ copies}}}{1 \cdot 2 \cdot 3 \cdots n} \leq \frac{N \cdot N \cdot N \cdots N}{1 \cdot 2 \cdot 3 \cdots N} \frac{N \cdots N}{N{+}1 \cdots n}, \quad \text{for } n > N > r\,.$$

Note that $\frac{N \cdot N \cdot N \cdots N}{1 \cdot 2 \cdot 3 \cdots N}$ does not depend on n so it's a constant term in the long run. As $n \to \infty$ we have that

$$\lim_{n\to\infty} \frac{N \cdots N}{N{+}1 \cdots n} = 0$$

and thus

$$\lim_{n\to\infty} \frac{r^n}{n!} = 0\,.$$

Exercise: (this exercise is a generalization of the previous one). Show that $\lim_{n\to\infty} \frac{r^n}{n!} = 0$ for fixed $r \in \mathbb{R}$. If $r > 0$ we are in the previous example, if $r = 0$ the sequence is zero. So we focus on the case when $r < 0$. So we can denote $r = -q$, where $q \in \mathbb{R}$ and $q > 0$. Then we have

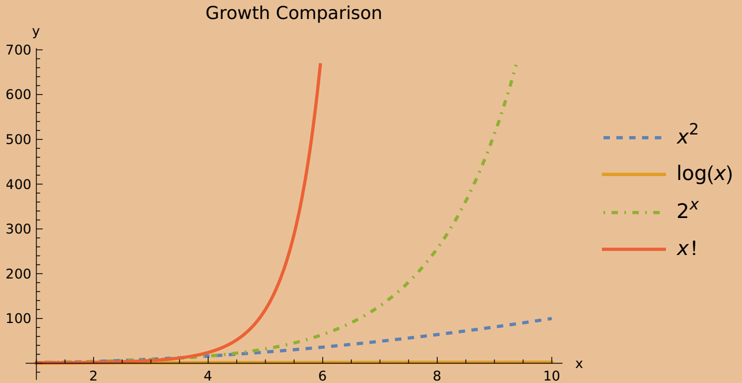
$$\frac{r^n}{n!} = \frac{(-q)^n}{n!} = \frac{(-1)^n}{n!}\frac{q^n}{n!}\,.$$

We know from the previous exercise that $\lim \frac{q^n}{n!} = 0$ because $q > 0$ and also we know $\lim_{n\to\infty} \frac{(-1)^n}{n!}$ since upstairs is bounded and downstairs grows to infinity. The limit of the product of two converging sequences is converging and converges to the product of the limits (algebraic limit law iv). So we have proved the exercise.

Some hints to find the bounding sequences

- Exponential functions (like e^x) and power functions (like x^n) for $n > 1$ can often serve as bounds for other slower-growing functions. For instance, $\ln(x)$ grows slower than any power function. In turn, $\sqrt{x}$ and $x^{0.3}$ grow slower than x but faster than $\ln(x)$. Functions like $x!$ (factorial) grow faster than any exponential function a^x (for any positive base a).
- Analog reasoning can be done considering inverses. For example $1/x!$, they represent a function that decays faster than any exponential decay a^{-x}, and so on.
- Sometimes factoring the dominant term helps. Example $a_n = \frac{n^2+3n+5}{n^3+2n^2+7}$ factor out the dominant term from both numerator and denominator, (x^2 and x^3) which helps in finding a simpler equivalent sequence that's easier to bound.

Some hints to find the bounding sequences



Some hints to find the bounding sequences

- If your sequence involves trigonometric functions (like $\sin(x)$ or $\cos(x)$), remember their outputs are always bounded between -1 and 1.
- Utilize properties of absolute values, logarithms, and other functions to create inequalities (see examples we solved in the lecture).
- Sometimes plotting the terms of a sequence can provide insight into its behavior and potential bounding sequences.
- Practice: As with many mathematical techniques, practice is key. The more problems you solve involving the Squeeze theorem, the more patterns you'll recognize, and the quicker you'll become at identifying potential bounding sequences.

Bounded Sequence

Definition

A sequence $\{a_n\}$ is bounded above if there exists a real number M such that

$$a_n \leq M$$

for all $n \in \mathbb{N}_+$.

A sequence $\{a_n\}$ is bounded below if there exists a real number m such that

$$m \leq a_n$$

for all $n \in \mathbb{N}_+$.

A sequence $\{a_n\}$ is a bounded sequence if it is bounded above and bounded below.

If a sequence is not bounded, it is an unbounded sequence.

Bounded Sequence

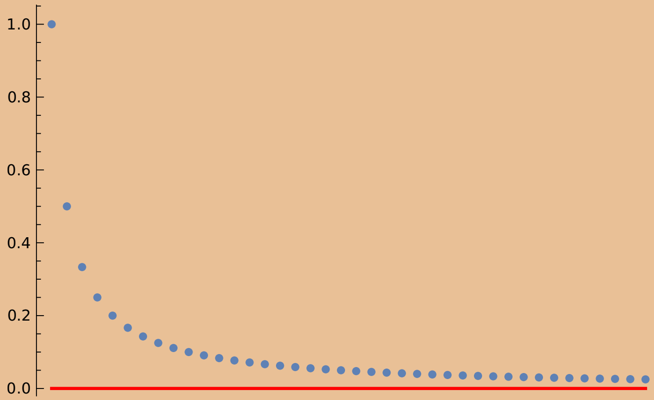
Example: $\{1/n\}$ is bounded above because $1/n \leq 1$ for all $n \in \mathbb{N}_+$. It is also bounded below because $1/n \geq 0$ for all $n \in \mathbb{N}_+$. Therefore, $\{1/n\}$ is a bounded sequence.

Example 2: Consider $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

Relationship between boundedness and convergence?

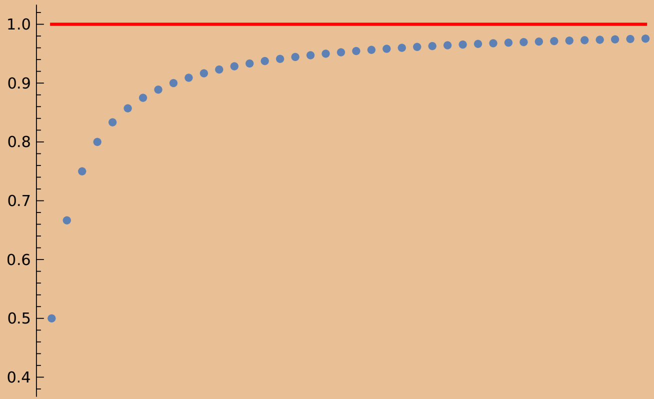
A sequence being bounded is not a sufficient condition for convergence. Example: $\{(-1)^n\}$ is bounded, but diverges since oscillates between 1 and -1 and never approaches a finite number.

Bounded Sequence



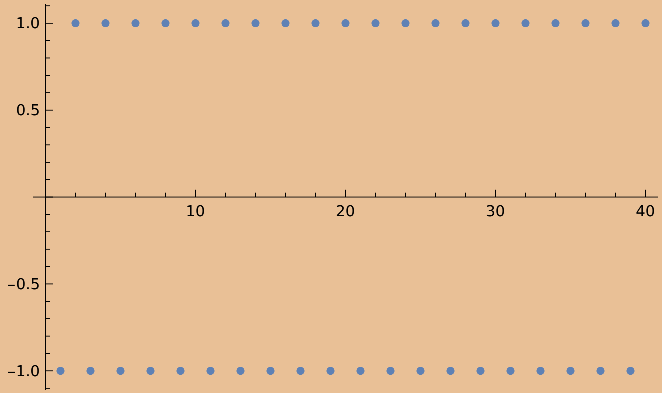
Boudned and decreasing sequence and thus converges, $a_n = 1/n$. (red line is the limit).

Bounded Sequence



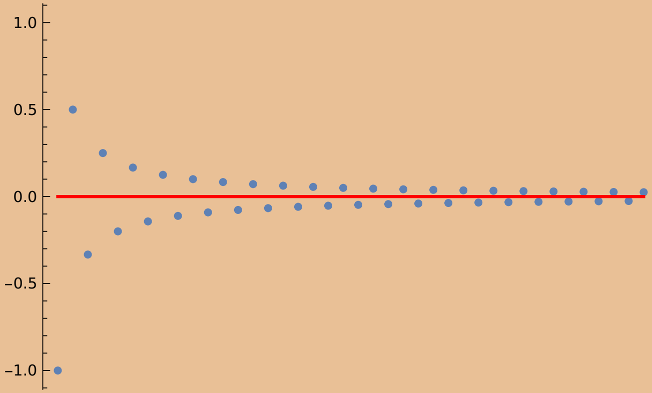
Bounded and increasing sequence and thus converges, $a_n = n/n + 1$. (red line is the limit).

Bounded Sequence



Bounded sequence not increasing nor decreasing, and thus convergence cannot be guaranteed. In this case, this sequence diverges: $a_n = (-1)^n$.

Bounded Sequence



Bounded sequence not increasing nor decreasing, and thus convergence cannot be guaranteed. In this case, this sequence converges: $a_n = (-1)^n/n$. (red line is the limit).

Bounded Sequence

Proposition

If a sequence $\{a_n\}$ converges, then it is bounded.

We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Also suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \dots$.

Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities: the sequence converge or diverge to infinity. Since the sequence is bounded, if cannot diverge to infinity. We conclude $\{a_n\}$ converges.

Note: If the sequence is decreasing we can also conclude that the sequence converges.

Bounded Sequence

Example: Consider

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}.$$

This sequence is increasing and bounded above thus it converges.

Example 2: Consider

$$\left\{-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\right\}.$$

We see that $-1/2 < -1/3 < -1/4 < \cdots$ so the sequence is increasing.

Example 3: Consider

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\right\}.$$

Even though the sequence is not increasing for all values of n starting with the 8-th term the sequence is increasing. In this case, we say the sequence is eventually increasing. Since the sequence is bounded above, it converges.

Sequences - Summary

We introduced sequences and we determined when a sequence has a limit L. Then we presented several "tools" to aid in the task of finding if a sequence has a limit. To mention:

- Algebraic limit laws.
- Use the fact that the limit can be switched with a continuous function.
- Squeeze theorem also allows to find limits of sequences by finding the limit of easier sequences.
- Bounded + increasing (or decreasing) sequence means convergence.

Series

A sequence is an infinite ordered list of numbers. If you add these numbers together, you get a series. Series will allow us to write certain functions as polynomials with an infinite number of terms.

Definition

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

For each $k \in \mathbb{N}_+$, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the k^{th} partial sum of the series.

Key observation: The partial sums form a sequence:

$$S_1, S_2, \ldots, S_k, \ldots .$$

If the sequence of partial sums converges to a real number S, the series converges.

If the series converges, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S .$$

If the sequence of partial sums diverges, we have the series diverge.

Note: the index for a series can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \text{ can also be written } \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Note 2: Changing a finite number of terms ina series does not affect its convergence or divergence. If converges, only affects the value of the sum.

Example: For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
2. $\sum_{n=1}^{\infty} (-1)^n$
3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Exercise: Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint: Look at the sequence of partial sums.

Answer:

Harmonic Series

This series is interesting because it diverges, but it diverges very slowly.

Harmonic series has the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

k	10	100	1000	10,00	100,000	$1,000,000$
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after 1,000,000 terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges.

Fortunately, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

Consider even terms: $S_2 = 1 + 1/2$, $S_4 = 1 + 1/2 + 1/3 + 1/4$. Notice the last two terms in S_4:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

Therefore,

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right) \end{aligned}$$

From the pattern $S_2 = 1 + 1/2$, $S_4 > 1 + 2(1/2)$, and $S_8 > 1 + 3(1/2)$ we infer that $S_{2^j} > 1 + j(1/2)$.

Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

1. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule)
2. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)
3. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Note: The product and quotient rule are not valid as in the sequences case.

Example: $\sum_{n=0}^{\infty} \frac{(-1)^n + 2^{n+1}}{3^{n+2}}$. Converges or diverges?

Geometric Series

Geometric series has the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

The number r is called the ratio, and the number a is the initial term.

Example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is a geometric series with initial term $a = 1$ and ratio $r = 1/2$.

Geometric Series

When does a geometric series converge? Consider

$$\sum_{n=1}^{\infty} ar^{n-1}$$

for $a > 0$. Its sequence of partial sums S_k is given by

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}.$$

Consider $r = 1$:

$$S_k = a + a(1) + a(1)^2 + \cdots + a(1)^{k-1} = ak.$$

Since $a > 0$, $ak \to \infty$.

Geometric Series

Consider $r \neq 1$: Multiply Equation by $1 - r$:

$$\begin{aligned}(1-r)S_k &= a(1-r)\left(1 + r + r^2 + r^3 + \cdots + r^{k-1}\right) \\ &= a\left(\left(1 + r + r^2 + r^3 + \cdots + r^{k-1}\right) - \left(r + r^2 + r^3 + \cdots + r^k\right)\right) \\ &= a\left(1 - r^k\right).\end{aligned}$$

All the other terms cancel out. Therefore,

$$S_k = \frac{a\left(1 - r^k\right)}{1 - r}, \qquad \text{for } r \neq 1.$$

We know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$.

Assuming $|r| < 1$ we have

$$\lim_{k\to\infty} S_k = \frac{a}{1-r}\,.$$

In summary we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1\,. \end{cases}$$

Note: If we need to evaluate $\sum_{n=1}^{\infty} r^n$, recall

$$\sum_{n=1}^{\infty} r^n = r \sum_{n=1}^{\infty} r^{n-1}\,.$$

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

1. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$. Hint: What is a and r?
2. $\sum_{n=1}^{\infty} e^{2n}$

Solution (2): Writing this series as

$$e^2 \sum_{n=1}^{\infty} \left(e^2\right)^{n-1}$$

we can see that this is a geometric series where $a = r = e^2 > 1$. Therefore, the series diverges.

Example: Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.
Hint: $r = -2/5$. Answer: $5/7$.

Telescoping series

Definition

A telescoping series is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty}(b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \dots$$

so the general term of the partial sums satisfy

$$S_k = b_1 - b_{k+1}$$

because all intermediate terms cancel.

Writing repeating decimals as fractions of integers

Application of geometric series to write $3.\overline{26} = 3.262626\ldots$ as a fraction of integers.

Write

$$\begin{aligned} 3.262626\ldots &= 3 + \frac{26}{100} + \frac{26}{1000} + \frac{26}{100,000} + \cdots \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots \end{aligned}$$

Ignoring the 3, the rest is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Using the formula for geometric series we have

$$\frac{26/10^2}{1-(1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}$$

Thus,

$$3.262626\ldots = 3 + \frac{26}{99} = \frac{323}{99}.$$

Exercise: Write $5.2\overline{7}$ as a fraction of integers. In this case $a = 7/100$ and $r = 1/10$. Answer: $475/90$.

Appendix: Necessity and sufficiency

In mathematics, necessity and sufficiency are used to describe a conditional or implicational relationship between two predicates (statement or a declarative sentence that is true or false.) P and Q.

"If P then Q" ($P \Rightarrow Q$ or "Q if P" or "P only if Q"), means

- Q is necessary for P, because the truth of Q is guaranteed by the truth of P (equivalently, it is impossible to have P without Q).
- P is sufficient for Q, because P being true always implies that Q is true, but P not being true does not always imply that Q is not true.

Simultaneous necessity and sufficiency: P is necessary and sufficient for Q means

- P is necessary for Q, $P \Leftarrow Q$
- P is sufficient for Q, $P \Rightarrow Q$

In other words: P and Q is necessary for the other, $P \Rightarrow Q \wedge Q \Rightarrow P$ (each is sufficient for or implies the other).

In summary: "P iff Q" ($P \Leftrightarrow Q \equiv Q \Leftrightarrow P$ or "P if and only if Q")