2-Dimensional Kinematics

Vectors:

$\boxed{\text{A vector is a quantity with both \underline{magnitude} and \underline{direction}, that obeys certain rules for addition and multiplication.}}$

- Notation: $\vec{A}$ ← ("vector A", indicated by "harpoon" on top)
- Representation:

← (arrow • point shows direction • length shows magnitude)

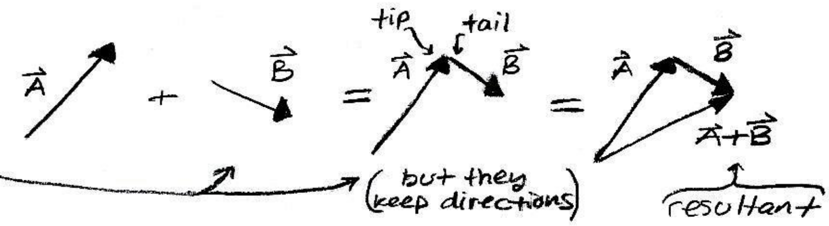
Scalars:

A scalar is a number with no direction
- but can have any sign
- can even be imaginary

- ex: $5, -1, 0, i \equiv \sqrt{-1}, 3-4i, \infty, -\infty, \pi$ are all scalars
- CAUTION:
  - All magnitudes are scalars
  - But not all scalars are magnitudes
- magnitude $\equiv$ $\boxed{\text{Any \underline{real} scalar } m \text{ such that } m \geq 0.}$ (All nonnegative (real) scalars can be magnitudes)

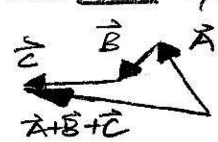
Vector Addition:

- vectors add "tip to tail":
- vectors can slide
- the result $\vec{A}+\vec{B}$ is the "resultant"



- To draw a resultant: (after tip-to-tail sliding, draw new vector, with its tail at the tail of "first" vector, and its head at the head of "last" vector slid)
- ex:
  - Given:

- $\vec{A}+\vec{B}+\vec{C}$ is:



← (The resultant is drawn "tail to tip")

- Vector Addition is Commutative:

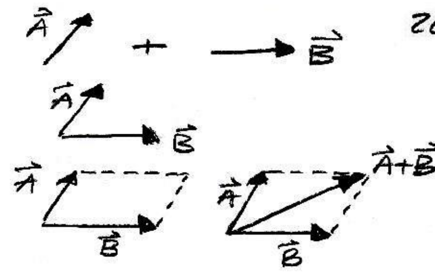
$$\boxed{\vec{A}+\vec{B} = \vec{B}+\vec{A}}$$

← (The order of vectors added doesn't matter; the resultant is the same.)

- Vector Addition is Associative:

$$\boxed{(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})} \qquad (1)$$

- *Parallelogram Method*: (for 2 vectors):
  - Slide both vectors until <u>tails touch</u>
  - Form parallelogram with $\vec{A}$, $\vec{B}$ a sides
  - Draw resultant from <u>touching tails</u> to <u>unused corner of parallelogram</u>
  - Is same as tail to tip method, but only works on two per step.



Multiplication of Vectors by Scalars:

- given a scalar c: $-\infty \le c \le \infty$ (limiting ourselves to real-valued c for now)

  a "scaled vector" is

$$\boxed{\vec{B} \equiv c\vec{A} = \vec{A}c} \tag{2}$$

- $\vec{B}$ is also a vector, and:

$$\boxed{\begin{array}{ll} \bullet \text{ if } |c|>1, & \vec{B} \text{ is } \underline{\text{longer than}}\ \vec{A} \\ \bullet \text{ if } |c|=1, & \vec{B} \text{ has } \underline{\text{same length}} \text{ as } \vec{A} \\ \bullet \text{ if } |c|<1, & \vec{B} \text{ is } \underline{\text{shorter than}}\ \vec{A} \end{array}} \tag{3}$$

  - so c "scales" $\vec{A}$, thus the name "scalar"

- The <u>sign</u> of a scalar <u>Affects the direction</u> of a <u>vecter</u>:

  <u>ex</u>: if $c=-2$, then $\vec{A}$ and $c\vec{A}$ are:

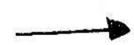
(c = −2 flipped the direction of $\vec{A}$, due to sign(c) = −1, and doubled its length, since $|c| = 2$)

- <u>Scalar Multiplication is Distributive</u>:

$$\boxed{\begin{array}{l} c(\vec{A}+\vec{B}) = c\vec{A} + c\vec{B} \quad (4) \\ c\vec{A} + d\vec{A} = (c+d)\vec{A} \quad (5) \end{array}}$$

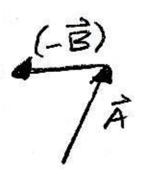
- <u>ex</u>: {The <u>Difference of Two Vectors</u> is just adding a vector scaled by −1:}

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

  - so if $\vec{A} \equiv$  and $\vec{B} \equiv$ 

  - then $-\vec{B} =$ 

  - so <u>add</u> $\vec{A}$ and $(-\vec{B})$:

$\vec{A} + (-\vec{B}) =$ 

$=$  $\vec{A} - \vec{B}$

Magnitude (Length) of Vectors:

$$\boxed{|\vec{A}| \equiv \text{the magnitude of } \vec{A}} \qquad (6)$$

- $|\vec{A}|$ is scalar (it's just a length)
- $|\vec{A}| \geq 0$ (length is nonnegative)
- CAUTION:
  - $A \equiv |\vec{A}|$ often used as abbreviation, so then $A \geq 0$
  - But it's good to state this to avoid confusion with more general scalars

Unit Vectors:

$$\equiv \boxed{\begin{array}{l}\text{vectors of length 1} \\ \hat{A}\ ,\ \text{s.t. } |\hat{A}| = 1\end{array}} \leftarrow \left(\text{we use a "hat" } \hat{A} \text{ to distinguish these}\right) \qquad (7)$$

- Can point in any direction
- used to keep track of direction
- To get $\hat{A}$ from $\vec{A}$:

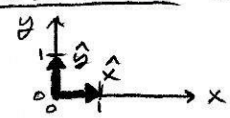
($\hat{A}$ is the unit vector in the direction of $\vec{A}$)

$$\boxed{\hat{A} \equiv \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}} \ ; \left(\text{if } |\vec{A}| \neq 0, \text{ more on this soon}\right) \qquad (8)$$

Component View of Vectors:

- Define two unit vectors in perpendicular directions:

$$\hat{x}\ ,\ \hat{y} \qquad (9)$$

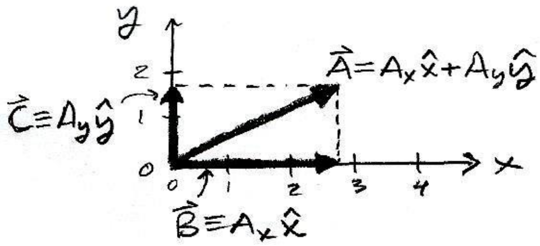


- Define new vectors by scaling these as: $\vec{B} \equiv A_x \hat{x}$, $\vec{C} \equiv A_y \hat{y}$ $\qquad A_x \in [-\infty, \infty]$, $A_y \in [-\infty, \infty]$ (10)
- Add the vectors:

$$\vec{A} \equiv \vec{B} + \vec{C} \qquad (11)$$

$$\boxed{\vec{A} = A_x \hat{x} + A_y \hat{y}} \leftarrow \left(\text{Component-view of } \vec{A} \text{ in Cartesian Coordinates}\right) \qquad (12)$$

- Visually:



(We can add perpendicular vectors because we can add any vectors by definition $\vec{A} = \vec{B} + \vec{C}$)

- Meaning: (A_x is the component of $\vec{A}$ in the $\hat{x}$ direction; A_y is the component of $\vec{A}$ in the $\hat{y}$ direction) (13)
- CAUTION:
  - if $\vec{A} \equiv A\hat{u}$, A could be negative!
  - here A is a general scalar unless otherwise specified
  - if you want to restrict the sign of A, simply include "$A \equiv |\vec{A}|$" or "$A \geq 0$" in its definition (just be clear!)

(see "CAUTION" at top of page)

Magnitude of Vectors in Component View:

- If components A_x, A_y belong to perpendicular ("orthogonal") unit vectors, as in:

$$\vec{A} \equiv A_x\hat{x} + A_y\hat{y} \qquad (14)$$

- then magnitude follows Pythagorean Theorem:

$$\boxed{|\vec{A}| = \sqrt{A_x^2 + A_y^2}} \qquad (15)$$

(In 3D, we have $\vec{A} \equiv A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$, $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$)

- if $A \equiv |\vec{A}|$, then

$$A = \sqrt{A_x^2 + A_y^2} \qquad (16)$$

Unit Vectors are Just Special-Case Vectors:

- Given most general 2D vector in (14),
- Set $A_x = 0$ and $A_y = 1$:

$$\vec{A} = 0\hat{x} + 1\hat{y} = \hat{y} \qquad (17)$$

(when components are zero, it's okay to omit their unit vector)

- so then $\vec{A} = \hat{y}$

$$\boxed{\text{unit vectors along principal axes are when one component is 1, and all other components are 0.}} \qquad (18)$$

Addition of Vectors by Components:

- let

$$\vec{A} \equiv A_x\hat{x} + A_y\hat{y} \quad \text{and} \quad \vec{B} \equiv B_x\hat{x} + B_y\hat{y} \qquad (19)$$

- add:

$$\vec{C} \equiv \vec{A} + \vec{B}$$

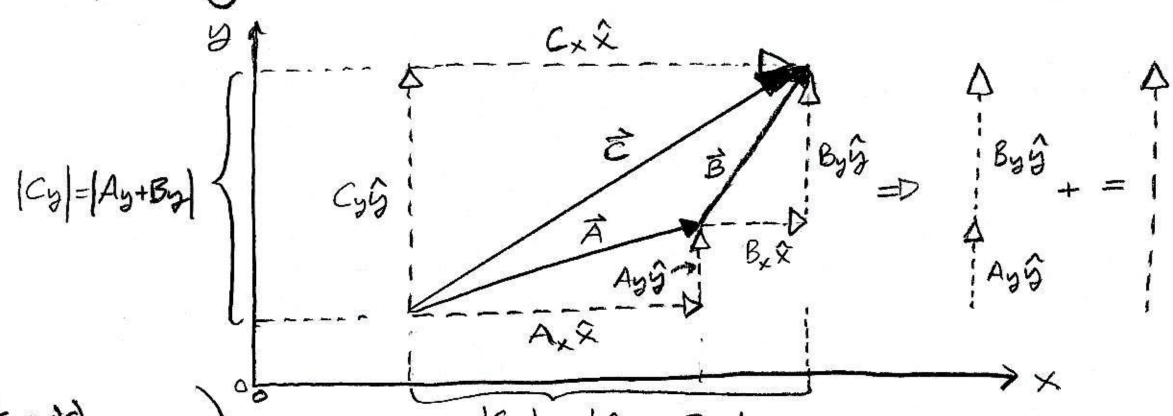
$$\vec{C} = A_x\hat{x} + A_y\hat{y} + B_x\hat{x} + B_y\hat{y}$$

(Group by unit vectors)

$$\vec{C} = A_x\hat{x} + B_x\hat{x} + A_y\hat{y} + B_y\hat{y}$$

$$\boxed{\begin{aligned} \vec{C} &= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} \\ \vec{C} &= \quad C_x\hat{x} \quad + \quad C_y\hat{y} \end{aligned}} \quad \text{so} \quad \boxed{\begin{aligned} C_x &= A_x + B_x \\ C_y &= A_y + B_y \end{aligned}} \qquad (20)$$

- Graphically:



$$C_y\hat{y} = A_y\hat{y} + B_y\hat{y} = (A_y + B_y)\hat{y}$$

(just add components along each perpendicular direction separately!)

$\Downarrow$

$$A_x\hat{x} + B_x\hat{x}$$

$$= \quad C_x\hat{x} = A_x\hat{x} + B_x\hat{x} = (A_x + B_x)\hat{x}$$

• ex ① : { Given $\vec{A} = 3\hat{x} - 4\hat{y}$, what is the component of $\vec{A}$ in the direction of $\hat{y}$? }

• Since a general 2D vector is $\vec{A} = A_x\hat{x} + A_y\hat{y}$, where A_y is the compont of $\vec{A}$ in the $\hat{y}$ direction, then:

$$\boxed{\text{The "}\hat{y}\text{ component" of } \vec{A} = 3\hat{x} - 4\hat{y} \text{ is } A_y = -4}$$

• ex ② : { If $\vec{A} \equiv 3\hat{x} + 4\hat{y}$ and $\vec{B} \equiv 1\hat{x} - 3\hat{y}$, what is $\vec{A}+\vec{B}$? }

$$\boxed{\begin{aligned} \vec{A}+\vec{B} &= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} \\ &= (3+1)\hat{x} + (4-3)\hat{y} \\ &= 4\hat{x} + \hat{y} \end{aligned}}$$

• ex ③ : { In ②, what is the component of $\vec{A}+\vec{B}$ in the $\hat{x}$ direction? }

$$\boxed{\text{The } \hat{x} \text{ component of } \vec{A}+\vec{B} \text{ is } A_x + B_x = 3+1 = 4}$$

• ex ④ : { Given $\vec{C} \equiv -\hat{x} + 2\hat{y}$, what is the magnitude of $\vec{C}$? }

$$\boxed{|\vec{C}| = \sqrt{C_x^2 + C_y^2} = \sqrt{(-1)^2 + (2)^2} = \sqrt{1+4} = \sqrt{5}}$$

• ex ⑤ : { In ②, what is the magnitude of $\vec{A}+\vec{B}$? }

• let $\vec{C} \equiv \vec{A} + \vec{B}$ (new $\vec{c}$ here) (1)

• then $C_x = A_x + B_x$ and $C_y = A_y + B_y$ (2)

• so since $|\vec{C}| = \sqrt{C_x^2 + C_y^2}$ (3)

• then plugging in to (3) from (1) and (2):

$$\boxed{\begin{aligned} |\vec{A}+\vec{B}| &= \sqrt{(A_x+B_x)^2 + (A_y+B_y)^2} \\ &= \sqrt{(3+1)^2 + (4-3)^2} \\ &= \sqrt{4^2 + 1^2} \\ &= \sqrt{17} \end{aligned}} \qquad (4)$$

• ex ⑥ : { What is the unit vector in the direction of $\vec{B} \equiv a\hat{x} + b\hat{y}$? }

$$\boxed{\hat{B} \equiv \frac{\vec{B}}{|\vec{B}|} = \frac{\vec{B}}{\sqrt{B_x^2 + B_y^2}} = \frac{a\hat{x} + b\hat{y}}{\sqrt{a^2 + b^2}}} \quad (\text{if } |\vec{B}| \neq 0)$$

• ex ⑦ : { ⓐ what are the $\hat{x}$ and $\hat{y}$ components of $\hat{B}$ in ⑥? ⓑ what is the length of $\hat{B}$? }

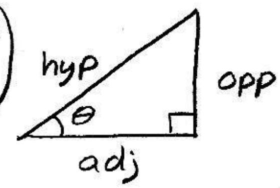
ⓐ $\boxed{(\hat{B})_x = \frac{a}{\sqrt{a^2+b^2}}, \ (\hat{B})_y = \frac{b}{\sqrt{a^2+b^2}}}$

ⓑ $\boxed{|\hat{B}| = \sqrt{(\hat{B})_x^2 + (\hat{B})_y^2} = \sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1}$

Trigonometric View of Vector Components:

- "trig" functions $\equiv$ ratios of sides of a right triangle:

(hyp $\equiv$ hypotenuse (side across from 90°))



(opp $\equiv$ opposite (side opposite from θ))

(adj $\equiv$ adjacent (side next to θ, that's not hyp))

- $\sin(\theta) \equiv \dfrac{opp}{hyp}$ ("sine of theta") (21)
- $\cos(\theta) \equiv \dfrac{adj}{hyp}$ ("cosine of theta") (22)
- $\tan(\theta) \equiv \dfrac{opp}{adj}$ ("tangent of theta") (23)

- Note: $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{opp}{hyp} \cdot \dfrac{1}{\left(\frac{adj}{hyp}\right)} = \dfrac{opp}{\cancel{hyp}} \cdot \dfrac{\cancel{hyp}}{adj} = \dfrac{opp}{adj}$ ✓ (24)

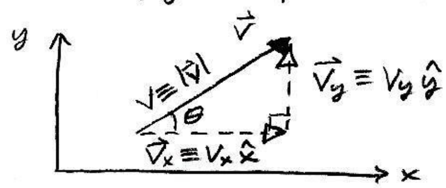
- memonic:

$$\boxed{\text{sohcahtoa}} \quad (25)$$

- Fact: these ratios are the same for a given θ, no matter how big the right triangle is!

Application To Vectors:

- given $\vec{V}$, to get its x, y components:
  - Form a right triangle:



- let $\vec{V}$ be hypotenuse
- draw nonhyp. sides parallel to coord. axes (axes must be perpendicular)
- write in θ, $V \equiv |\vec{V}|$, and component vectors in proper direction in relation to $\vec{V}$

- Use trig definitions:

$$\cos(\theta) \equiv \frac{adj}{hyp} = \frac{V_x}{V} \quad ; \quad \sin(\theta) \equiv \frac{opp}{hyp} = \frac{V_y}{V} \quad (26)$$

- Solve for components:

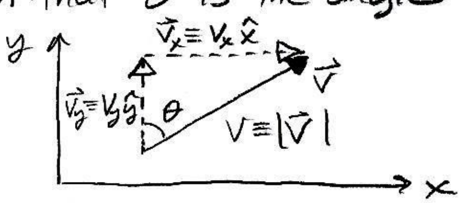
(cos(θ) and sin(θ) scale-down V by the proper amounts) →

$$\boxed{\begin{aligned} V_x &= V\cos(\theta) \\ V_y &= V\sin(\theta) \end{aligned}} \quad (27)$$

← (Given θ and $V \equiv |\vec{V}|$, we get V_x and V_y)

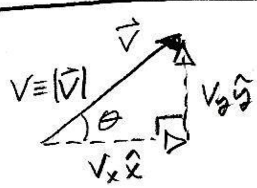
- CAUTION: $\boxed{\text{definition of } \theta \text{ affects equations for components}}$ (28)

ex: if we're given that θ is the angle between $\vec{V}$ and the y-axis, then:



$$\Rightarrow \boxed{\begin{aligned} V_x &= V\sin(\theta) \\ V_y &= V\cos(\theta) \end{aligned}} \quad (29)$$

- Given Components, We Get θ:



$$\Rightarrow \tan(\theta) \equiv \frac{opp}{adj} = \frac{V_y}{V_x} \quad (30)$$

$$\boxed{\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)} \quad (31)$$

($\tan^{-1}(x) \equiv$ "inverse tangent of x", not $\dfrac{1}{\tan(x)}$)

- CAUTION:
  - components are not magnitudes or lengths
  - trig functions are ratios of components that sometimes involve lengths $\Rightarrow$ $\boxed{\text{these ratios can be negative}}$

Two Ways To Specify a Vector:

- **Cartesian** $\equiv$ $\boxed{\text{Give its components, } V_x \text{ and } V_y}$

(if you know V_x, V_y, you know $\vec{V}$)

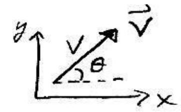
$$\vec{V} = V_x \hat{x} + V_y \hat{y} \tag{32}$$

- **Polar** $\equiv$ $\boxed{\text{Give its magnitude } V \equiv |\vec{V}| \text{ and direction } \theta \text{ (relative to coordinate system)}}$

(if you know V, θ, you know $\vec{V}$)

$$\vec{V} = V\cos(\theta)\,\hat{x} + V\sin(\theta)\,\hat{y} \tag{33}$$

where θ is the angle between $\vec{V}$ and $\hat{x}$



Conversions of Vector Descriptions:

- Cartesian to Polar:
  - Given V_x, V_y,

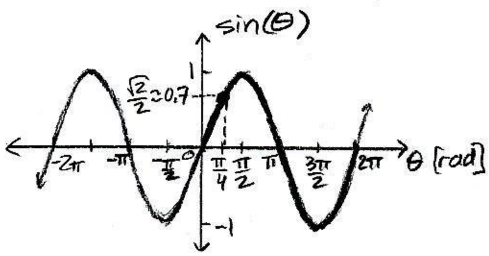
$$\boxed{V = \sqrt{V_x^2 + V_y^2}\,, \quad \theta = \tan^{-1}\left(\frac{V_y}{V_x}\right) \quad \text{where } \theta \text{ is angle between } \vec{V} \text{ and } \hat{x}} \tag{34}$$

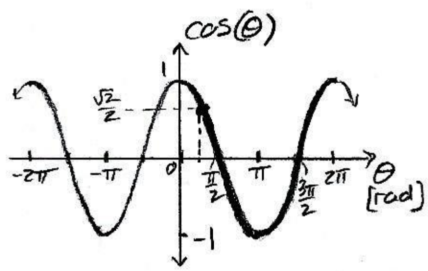
- Polar to Cartesian:
  - Given V, θ (betw. $\vec{V}$ and $\hat{x}$),

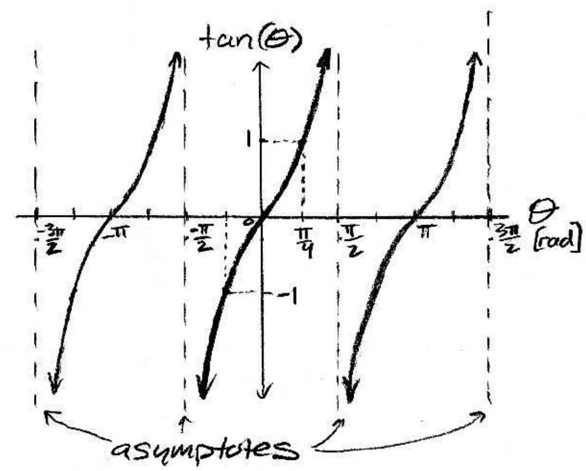
$$\boxed{V_x = V\cos(\theta)\,, \quad V_y = V\sin(\theta)} \tag{35}$$

- Stating (34) or (35) is equivalent to stating (32) or (33).

Review of Trig Functions:







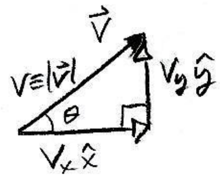
- Converting radians to degrees:

$$\theta\,[\cancel{\text{rad}}] \cdot \left(\frac{180^\circ}{\pi\,[\cancel{\text{rad}}]}\right) = \theta^\circ$$

- The Pythagorean Identity:

$$\boxed{\cos^2(\theta) + \sin^2(\theta) = 1 \quad \forall\, \theta \text{ ("for all" theta)}} \tag{36}$$

ex: • Given



- then:

$$V = \sqrt{V_x^2 + V_y^2} \Rightarrow V_x^2 + V_y^2 = V^2 \tag{37}$$

- components in polar form:

$$V_x = V\cos(\theta)\,, \quad V_y = V\sin(\theta) \tag{38}$$

- put (38) into (37):

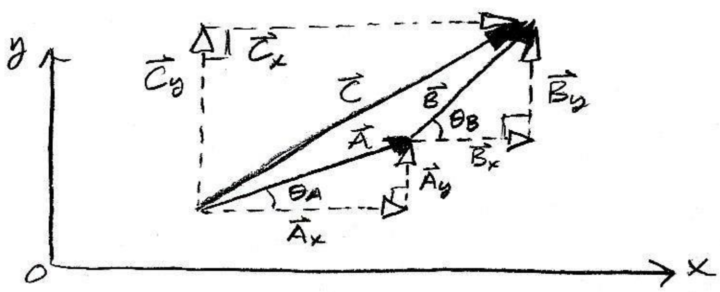
$$\cancel{V^2}\cos^2(\theta) + \cancel{V^2}\sin^2(\theta) = \cancel{V^2}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \quad ✓ \tag{39}$$

- gives back (36)

Addition of Vectors in Polar Form:

- Recall from (20):



- where

$$\vec{C} \equiv C_x \hat{x} + C_y \hat{y} \qquad (40)$$
$$\vec{C}_x \equiv C_x \hat{x} \qquad (41)$$
$$\vec{C}_y \equiv C_y \hat{y} \qquad (42)$$

- and

$$C_x = A_x + B_x \qquad (43)$$
$$C_y = A_y + B_y \qquad (44)$$

- Polar form of component vectors:

(where $A \equiv |\vec{A}|$, $B \equiv |\vec{B}|$)

$$\vec{A} \equiv A_x \hat{x} + A_y \hat{y} = A\cos(\theta_A)\hat{x} + A\sin(\theta_A)\hat{y} \qquad (45)$$
$$\vec{B} \equiv B_x \hat{x} + B_y \hat{y} = B\cos(\theta_B)\hat{x} + B\sin(\theta_B)\hat{y} \qquad (46)$$

since

$$A_x = A\cos(\theta_A) \quad \text{and} \quad B_x = B\cos(\theta_B)$$
$$A_y = A\sin(\theta_A) \qquad\qquad B_y = B\sin(\theta_B) \qquad (47)$$

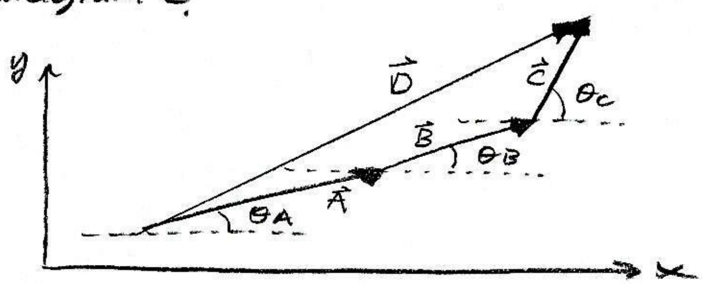
- So putting (47) into (43) and (44):

$$\boxed{\begin{aligned} C_x &= A_x + B_x = A\cos(\theta_A) + B\cos(\theta_B) \\ C_y &= A_y + B_y = A\sin(\theta_A) + B\sin(\theta_B) \end{aligned}} \qquad \begin{aligned}(48)\\(49)\end{aligned}$$

- Thus, in full:

$$\boxed{\vec{C} = \left[A\cos(\theta_A) + B\cos(\theta_B)\right]\hat{x} + \left[A\sin(\theta_A) + B\sin(\theta_B)\right]\hat{y}} \qquad (50)$$

- ex. {What is $\vec{D}$ in terms of θ_A, θ_B, θ_C, $A \equiv |\vec{A}|$, $B \equiv |\vec{B}|$, $C \equiv |\vec{C}|$ in the following diagram?}



- "Generic" form of $\vec{D}$:

$$\vec{D} = D_x \hat{x} + D_y \hat{y} \qquad (1)$$

- components:

$$D_x = A_x + B_x + C_x \qquad (2)$$
$$D_y = A_y + B_y + C_y \qquad (3)$$

- polar form of $\vec{A}, \vec{B}, \vec{C}$ components:

$$\left.\begin{aligned} A_x &= A\cos(\theta_A) & B_x &= B\cos(\theta_B) & C_x &= C\cos(\theta_C) \\ A_y &= A\sin(\theta_A) & B_y &= B\sin(\theta_B) & C_y &= C\sin(\theta_C) \end{aligned}\right\} \qquad (4)$$

- plug (4) into (2):

$$D_x = A\cos(\theta_A) + B\cos(\theta_B) + C\cos(\theta_C) \qquad (5)$$
$$D_y = A\sin(\theta_A) + B\sin(\theta_B) + C\sin(\theta_C) \qquad (6)$$

- put (5,6) into (1):

((5) and (6) were enough, but this is okay too)

$$\boxed{\vec{D} = \left[A\cos(\theta_A) + B\cos(\theta_B) + C\cos(\theta_C)\right]\hat{x} + \left[A\sin(\theta_A) + B\sin(\theta_B) + C\sin(\theta_C)\right]\hat{y}} \qquad (7)$$