The 69th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2008

A1 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g: \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

A2 Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A3 Start with a finite sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

A4 Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

A5 Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

A6 Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \log |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4, 4, 2$ is a subsequence of $2, 4, 6, 4, 2$, but $2, 2, 4$ is not.)

B1 What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

B2 Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n \to \infty} \frac{n! F_n(1)}{\ln n}.$$

B3 What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

B4 Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \dots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \dots, h(p^3 - 1)$ are distinct modulo p^3.

B5 Find all continuously differentiable functions $f: \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a,b) = 1$.) (Note: gcd means greatest common divisor.)

B6 Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \dots, n\}$ is *k-limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \dots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.