First-order linear equations and variation of parameter

MA221, Lecture 4

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Updated office hours

M : 2pm – 3pm NB316, 3pm–4pm (zoom)

W : 2pm – 4pm NB316

Th : 1pm – 2pm (zoom)

F : 2pm – 3pm NB316

Linear First-Order ODEs (or LFO ODEs)

A LFO ODE is an ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y \overset{\star}{=} Q(x).$$

These are solvable by way of a "trick" that makes use of the product rule for derivatives. But first, we start as follows:

Define $\rho(x) = e^{\int P(x)\mathrm{d}x}$. This function is known as an integrating factor for the LFO ODE.

What is $\rho'(x)$? $\rho'(x) = \left[e^{\int P(x)dx}\right]' = e^{\int P(x)dx} \cdot P(x) = \rho(x)P(x)$

$\star \Rightarrow \underline{Q(x)\rho(x)} = \rho(x)\frac{dy}{dx} + \rho(x)P(x)y = \rho(x)\frac{dy}{dx} + \rho'(x)y$

$= [\underline{\rho(x)y}]'$

Linear First-Order ODEs (or LFO ODEs)

Now we can compute an *explicit* solution to the equation.

$$[\rho(x)y]' = Q(x)\rho(x)$$

$$\implies \rho(x)y = \int [\rho(x)y]'\,dx = \int Q(x)\rho(x)\,dx$$

$$\implies \underbrace{y = \frac{1}{\rho(x)} \int Q(x)\rho(x)\,dx}_{\text{explicit solution to } \star}$$

Linear First-Order ODEs (or LFO ODEs)

Example 1: $y' + 2xy \overset{\star}{=} x^3$ ... $P(x) = 2x$ ... $Q(x) = x^3$

we didn't cover this in class!

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2+C} = e^{x^2}$$

choose $C=0$

$$\star \Rightarrow x^2 e^{x^2} = y' e^{x^2} + 2xy\, e^{x^2} = [y e^{x^2}]'$$

$$\Rightarrow y e^{x^2} = \int [y e^{x^2}]'\,dx = \int x^3 e^{x^2}\,dx$$

$$\Rightarrow y = e^{-x^2}\left(\int x^3 e^{x^2}\,dx\right)$$

$$= e^{-x^2}\left(\int x^2 \cdot x e^{x^2}\,dx\right)$$

$$u = x^2 \Rightarrow du = 2x\,dx$$

$$= e^{-x^2}\left(\frac{1}{2}\int u\, e^{u}\,du\right)$$

$$A = u \qquad\qquad dA = du$$

$$dB = e^{u}\,du \quad\Rightarrow\quad B = e^{u}$$

$$= e^{-x^2}\left(\frac{1}{2}\left[u e^{u} - \int e^{u}\,du\right]\right)$$

$$y = e^{-x^2}\left(\frac{1}{2}\left[x^2 e^{x^2} - e^{x^2} + C\right]\right)$$

Linear First-Order ODEs (or LFO ODEs)

Example 2: $\frac{\mathrm{d}y}{\mathrm{d}t} \overset{\star}{=} 1 + t + y + ty$

$$\frac{dy}{dt} + P(t)y = Q(t)$$

$$\star \Rightarrow \frac{dy}{dt} = 1+t + (1+t)y \Longrightarrow \frac{dy}{dt} + (-1-t)y \overset{\star'}{=} 1+t$$

$$P(t) = -1-t$$

$$Q(t) = 1+t$$

$$\rho(t) = e^{\int P(t)\,dt} = e^{\int(-1-t)\,dt} = e^{-t-\frac{t^2}{2}+c}$$

$$\underset{c=0}{\overset{\text{choose}}{=}} e^{-t-\frac{t^2}{2}}$$

$$\circledast' \Rightarrow (1+t)e^{-t-\frac{t^2}{2}} = e^{-t-\frac{t^2}{2}}\frac{dy}{dt} + e^{-t-\frac{t^2}{2}}(-1-t)y$$

$$= \left[e^{-t-\frac{t^2}{2}}y\right]'$$

$$\Rightarrow e^{-t-\frac{t^2}{2}}y = \int \left[e^{-t-\frac{t^2}{2}}y\right]' dt = \int \underbrace{(1+t)e^{-t-\frac{t^2}{2}}dt}$$

$u = -t - \frac{t^2}{2}$

$du = (-1-t)\,dt$

$= -(1+t)\,dt$

$$= -\int e^u du$$

$$= -e^u + C$$

$$= -e^{-t-\frac{t^2}{2}} + C$$

$$\Rightarrow e^{-t-\frac{t^2}{2}}y = -e^{-t-\frac{t^2}{2}} + C$$

$$\Rightarrow y = e^{t+\frac{t^2}{2}}\left(-e^{-t-\frac{t^2}{2}} + C\right)$$

$$\Rightarrow \boxed{y = -1 + Ce^{t+\frac{t^2}{2}}}$$

Example 3: $y' + \dfrac{2y}{x} = 0$ ★ ... $P(x) = \dfrac{2}{x}$ ... $Q(x) = 0$

"homogeneous linear equation"

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x| + C}$$

$$\underset{C=0}{\overset{\text{choose}}{=}} e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$$

$$★ \Rightarrow 0 = x^2 y' + \left(\frac{2y}{x}\right)x^2 = x^2 y' + 2xy$$

$$= [x^2 y]'$$

$$\Rightarrow x^2 y = \int [x^2 y]'\,dx = \int 0\,dx = C$$

$$\Rightarrow \boxed{y = \frac{C}{x^2}}, \quad x \neq 0$$

Variation of parameter

A linear differential equation is **homogeneous** if all (non-zero) expressions in the equation include a factor of the solution function or any of its derivatives. Equations that aren't homogeneous are **inhomogeneous**.

Solutions to inhomogeneous equations can be obtained from solutions to the corresponding homogeneous equation with a technique called **variation of parameter**:

Homogeneous: $\frac{dy}{dx} + P(x)y = 0$

Inhomogeneous: $\frac{dy}{dx} + P(x)y = Q(x)$

Variation of parameter

Suppose $u(x)$ is a solution to $\frac{dy}{dx} + P(x)y = 0$

$C \cdot u(x)$ is also a solution to

(verify!)

VoP: Suppose $y(x) = C(x)u(x)$ is a solution to $\frac{dy}{dx} + P(x)y = Q(x)$. What is $C(x)$?

Linear First-Order ODEs (or LFO ODEs)

Example 3: $y' + \dfrac{2y}{x} \overset{\bigstar}{=} \dfrac{\cos x}{x^2}$

we solved $y' + \dfrac{2y}{x} = 0$: $y = \dfrac{C}{x^2}$

VoP $\Rightarrow$ suppose $y(x) = \dfrac{C(x)}{x^2}$ is a solution to $\bigstar$

$$y'(x) = \frac{x^2 C'(x) - 2x\,C(x)}{x^4}$$

$$\bigstar \Rightarrow \frac{x^2 C'(x) - 2x\,C(x)}{x^4} + \frac{2C(x)}{x^3} = \frac{\cos x}{x^2}$$

$$\Rightarrow x^2 C'(x) - 2x\,C(x) + 2x\,C(x) = x^2 \cos x$$

$$\Rightarrow x^2 C'(x) = x^2 \cos x$$

$$\Rightarrow C'(x) = \cos x$$

$$\Rightarrow C(x) = \int C'(x)\,dx = \int \cos x\,dx = \sin x + C$$

$$\Rightarrow y = \frac{C(x)}{x^2} = \frac{\sin x + C}{x^2}$$