Algebra Review

Properties of Real Numbers:

- Commutative:

$$a+b=b+a \tag{1}$$
$$ab=ba \tag{2}$$

- Associative:

$$(a+b)+c=a+(b+c)=b+(a+c) \tag{3}$$
$$(ab)c=a(bc)=b(ac) \tag{4}$$

- Distributive:

$$a(b+c)=ab+ac \tag{5}$$
$$(a+b)c=ac+bc \tag{6}$$

(Note: if we work these equations the other way, it's called "factoring": $ac+bc=(a+b)c$)

- note: if $a=-1$ in (5),

$$-(b+c)=-b-c \tag{7}$$

- Identity:
  - Additive identity: $$a+0=a \tag{8}$$
  - Multiplicative identity: $$a\cdot 1=a \tag{9}$$
- Inverse:
  - Additive inverse: $$a+(-a)=0 \quad \text{and} \quad (-a)+a=0 \tag{10}$$
  - mult. inverse: $$a\cdot\left(\frac{1}{a}\right)=1 \quad \text{and} \quad \left(\frac{1}{a}\right)\cdot a=1\ ;\ (a\neq 0) \tag{11}$$

Order of Operations:

1. please	P	parentheses		
2. excuse	E	exponents		
3. my	M	multiplication	} left to right	(12)
4. dear	D	division		
5. Aunt	A	addition	} left to right	
6. Sally	S	subtraction		

Examples:

- two binomials:

$$(a+b)(c+d)=\big(a(c+d)+b(c+d)\big)$$
$$=\big(ac+ad+bc+bd\big)=ac+ad+bc+bd \tag{13}$$

- squared binomial:

$$(a+b)^2=(a+b)(a+b)=a(a+b)+b(a+b)$$
$$=a^2+ab+ba+b^2$$
$$=a^2+ab+ab+b^2$$

$$\boxed{(a+b)^2=a^2+2ab+b^2} \tag{14}$$

- also:

$$(a-b)^2=(a-b)(a-b)=a(a-b)-b(a-b)$$
$$=a^2-ab-ba+b^2$$

$$\boxed{(a-b)^2=a^2-2ab+b^2} \tag{15}$$

- WARNING:

$$(a\pm b)^2\neq a^2\pm b^2 \tag{16}$$

(exponents do <u>not</u> distribute!)

- FOIL Method:

1. firsts F
2. outers O
3. inners I
4. lasts L

$$(a+b)(c+d)=\overset{F}{ac}+\overset{O}{ad}+\overset{I}{bc}+\overset{L}{bd} \tag{17}$$

(but only works for two binomials; best to use distribution law directly)

Fractions:

- Definition of Division:

$$\frac{a}{b} \equiv a \cdot \frac{1}{b} \quad ; \; b \neq 0 \qquad \left(\begin{array}{l}\text{Note: "}\equiv\text{" means} \\ \text{"is defined as"}\end{array}\right) \tag{18}$$

- parts of a fraction:

$$\frac{a}{b} \equiv \frac{\text{numerator}}{\text{denominator}} \leftarrow \left(\begin{array}{l}\text{think:} \\ \text{"d" for "downstairs"}\end{array}\right) \tag{19}$$

- implicit parentheses:

$$\frac{a+b}{c+d} = \frac{(a+b)}{(c+d)} \tag{20}$$

- distribution over numerator:

$$\frac{a+b}{c} = \frac{(a+b)}{c} = \frac{1}{c} \cdot (a+b) = \frac{1}{c} \cdot a + \frac{1}{c} \cdot b$$

$$\boxed{\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}} \qquad \left(\begin{array}{l}\text{works in} \\ \text{both directions}\end{array}\right) \tag{21}$$

- WARNING:

$$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c} \tag{22}$$

  - because:

$$\frac{a}{b+c} = \frac{1}{b+c} \cdot a \tag{23}$$

- adding fractions with different denominators:
  - need to get a common denominator to get form like right side of (21)
  - so multiply by 1 in a clever way

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \underbrace{1} + \frac{c}{d} \cdot \underbrace{1} \rightarrow \left(\begin{array}{l}\text{Note:} \\ \frac{x}{x} = 1 \\ \text{due to (11)}\end{array}\right)$$

$$= \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b}$$

$$= \frac{ad}{bd} + \frac{cb}{db} \leftarrow \left(\begin{array}{l}\text{now denominators} \\ \text{are the same,} \\ \text{so use (21) in} \\ \text{reverse}\end{array}\right)$$

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}} \tag{24}$$

- note that we also used: (more general form here) →

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}} \tag{25}$$

- inverse of inverse:
  - recall:

$$\left(\frac{1}{a}\right) \cdot a = 1 \tag{26}$$

  - now let $a \equiv \frac{1}{b}$, so:

$$\frac{1}{\left(\frac{1}{b}\right)} \cdot \frac{1}{b} = 1 \tag{27}$$

  - mult. both sides by b:

$$\frac{1}{\left(\frac{1}{b}\right)} \cdot \underbrace{\frac{1}{b} \cdot b} = \underbrace{1 \cdot b} \tag{28}$$

$$\frac{1}{\left(\frac{1}{b}\right)} \cdot \underbrace{\frac{b}{b}} = b$$

$$\frac{1}{\left(\frac{1}{b}\right)} \cdot 1 = b$$

(notice smaller length of fraction bar in denominator indicates that it's <u>inside</u> the denominator)

$$\boxed{\frac{1}{\left(\frac{1}{b}\right)} = b} \quad \text{or} \quad \boxed{\frac{1}{\;\frac{1}{b}\;} = b} \tag{29}$$

- alternative method: $\boxed{\text{division by a fraction is multiplication by its reciprocal}}$ (30)

- so:

$$\boxed{\frac{1}{\left(\frac{1}{b}\right)} = 1 \cdot \left(\frac{b}{1}\right) = b} \qquad (31)$$

- more generally:

$$\boxed{\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}} \qquad (32)$$

- recalling $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$, we can do this:

$$\boxed{\frac{a}{b+c} = \frac{a}{b+c} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{1}{\frac{b}{a} + \frac{c}{a}}} \quad (a \neq 0) \qquad (33)$$

Exponents:

- if $a > 0$, and n is a positive integer $(1, 2, \ldots)$:

$$\boxed{a^0 = 1} \quad \boxed{a^1 = a} \qquad (34)$$

$$\boxed{a^n = \prod_{k=1}^{n} a = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}} \qquad (35)$$

$$\boxed{a^{-n} = \frac{1}{a^n}} \qquad (36)$$

(note, by (36), $1^{-1} = 1$, $(-1)^{-1} = -1$, $-\frac{a}{b} = -\frac{a}{b} = \frac{a}{-b}$)

$$\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}} \qquad (37)$$

(is the "nth root of a", meaning that if $x = a^{\frac{1}{n}}$, then $x^n = (a^{\frac{1}{n}})^n = a^1 = a$)

$$\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m} \quad \text{for any integer } m \qquad (38)$$

- for $a, b > 0$ and x, y any rational numbers (ratios of integers):

$$\boxed{a^x a^y = a^{x+y}} \qquad (39)$$

$$\boxed{\frac{a^x}{a^y} = a^x a^{-y} = a^{x-y}} \qquad (40)$$

$$\boxed{(a^x)^y = \underset{1}{a^x} \cdot \underset{2}{a^x} \cdot \underset{\cdots}{\cdots} \cdot \underset{y}{a^x} = a^{xy}} \qquad (41)$$

$$\boxed{(ab)^x = a^x b^x} \quad \boxed{\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}} \; b \neq 0 \qquad (42)$$

Square Roots:

$$\boxed{\sqrt{a} \equiv \sqrt[2]{a}} \implies \sqrt{a} \cdot \sqrt{a} = a \qquad (43)$$

$$\boxed{\sqrt{ab} = \sqrt{a}\sqrt{b}} \quad \boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}} \quad \text{(works for nth roots too)} \qquad (44)$$

- WARNING:

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b} \qquad (45)$$

- useful fact:

$$\boxed{|a| \equiv \sqrt{a^2}} \qquad (46)$$

- also:

$$\boxed{\text{if } a^2 = b, \text{ then } a = \pm\sqrt{b} \equiv \begin{cases} a = +\sqrt{b} \\ \text{or} \\ a = -\sqrt{b} \end{cases}} \qquad (47)$$

Absolute Values:

- for real a,b:

$$\boxed{|a| \geq 0} \qquad (48)$$

$$\boxed{a = \mathrm{sgn}(a)\,|a|}\;;\; \mathrm{sgn}(a) \equiv \begin{cases} +1\,; & a>0 \\ 0\,; & a=0 \\ -1\,; & a<0 \end{cases} \qquad (49)$$

- again:

$$\boxed{|a| \equiv \sqrt{a^2}} \qquad (50)$$

$$\boxed{|ab| = |a|\cdot|b|} \quad \boxed{\left|\frac{a}{b}\right| = \frac{|a|}{|b|}} \rightarrow b \neq 0 \qquad (51)$$

$$\boxed{|a^n| = |a|^n} \quad \text{for integer } n \qquad (52)$$

- inequalities:
  - for $a > 0$:

$$\boxed{|x| = a \iff x = \pm a} \qquad (53)$$

$$\boxed{|x| < a \iff -a < x < a} \qquad (54)$$

$$\boxed{|x| > a \iff (x < -a) \text{ or } (x > a)} \qquad (55)$$

Factoring:

- some special cases:

$$\boxed{a^2 - b^2 = (a+b)(a-b)} \qquad (56)$$

$$\boxed{a^3 - b^3 = (a-b)(a^2+ab+b^2)} \qquad (57)$$

$$\boxed{a^3 + b^3 = (a+b)(a^2-ab+b^2)} \qquad (58)$$

- general product of binomials in x:

$$(x+g)(x+h) = x^2 + (g+h)x + gh \qquad (59)$$

  - so given:

$$x^2 + ax + b \qquad (60)$$

factoring means finding a and b s.t.

$$\boxed{\begin{aligned} a &= g+h \\ b &= gh \end{aligned}} \qquad (61)$$

- more generally, we have the quadratic formula:

($x_\pm$ are the "roots" of the quadratic ax^2+bx+c) →

$$\boxed{\begin{aligned} &\text{if } ax^2+bx+c = 0 \\ &\text{then } x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \end{aligned}} \qquad (62)$$

or

$$\boxed{x_\pm = \frac{-b}{2a} \pm \sqrt{\left(\frac{-b}{2a}\right)^2 - \frac{c}{a}}} \qquad (63)$$

which means:

$$ax^2+bx+c = (x - x_+)(x - x_-) = 0 \qquad (64)$$

since whenever $x = x_+$ or $x = x_-$ then one of the factors of ax^2+bx+c will be 0.

Logarithms:

$$p = \log_b(a) \equiv \boxed{\text{the power that base } b \text{ must be raised to to get } a\text{, so } b^p = a} \tag{65}$$

$$\boxed{\log_b(x) + \log_b(y) = \log_b(xy)} \tag{66}$$

$$\boxed{\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)} \tag{67}$$

$$\boxed{\log_b(a^x) = x\log_b(a)} \tag{68}$$

$$\boxed{\log_b(1) = 0} \tag{69}$$

$$\boxed{\log_b(b) = 1} \tag{70}$$

$$\boxed{b^{\log_b(x)} = x} \tag{71}$$

$$\boxed{\log_b(b^x) = x} \tag{72}$$

$$\boxed{x^{\log_b(y)} = y^{\log_b(x)}} \tag{73}$$

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}} \tag{74}$$

- natural logarithm:

$$\boxed{\ln(a) \equiv \log_e(a)} \tag{75}$$

- so, by (74):

$$\boxed{\log_b(a) = \frac{\ln(a)}{\ln(b)}} \tag{76}$$

- also, from (74):

$$\boxed{\log_b(a) = \frac{1}{\log_a(b)}} \tag{77}$$

$$\boxed{\log_{b^n}(a) = \frac{1}{n}\log_b(a)} \tag{78}$$

$$\boxed{\log_{\frac{1}{b}}(a) = \log_b\left(\frac{1}{a}\right) = -\log_b(a)} \tag{79}$$

$$\boxed{b^{\log_c(a)} = a^{\log_c(b)}} \tag{80}$$

- and many others...

Sigma Notation:

- Abbreviation of a sum:

(Σ ≡ Greek letter Sigma for "sum") →

$$\boxed{\sum_{i=1}^{n} c_i \equiv c_1 + c_2 + \cdots + c_n} \tag{81}$$

- i ≡ index for counting
- index bounds:

$$\sum_{i=a}^{b} c_i \tag{82}$$

b (upper/ending value)

a (lower/starting value)

  - this is an abbreviation of:

$$\sum_{i=a}^{i=b} c_i \tag{83}$$

  - index behavior:
    - i starts on $\underline{a}$
    - then it increments by 1 as $\underline{a+1}$
    - then it increments by 1 to $\underline{a+2}$

    ⋮

    - lastly, it increments by 1 to $\underline{a+b-a}=b$
  - sum behavior:
    - each version of the $\underline{\text{summand}}$ c_i is added cumulatively to the others

- ex:

$$\sum_{i=1}^{3} c_i = c_1 + c_2 + c_3 \tag{84}$$

- ex: summand could just be a number, independent of index:

$$\sum_{i=1}^{4} 2 = 2+2+2+2 = 8 \tag{85}$$

  - so also:

$$\boxed{\sum_{i=1}^{n} 1 = \underbrace{1+1+\cdots+1}_{n \text{ terms}} = n} \tag{86}$$

  - and:

$$\boxed{\sum_{i=1}^{n} c = c\sum_{i=1}^{n} 1 = cn} \tag{87}$$

- ex: summand can depend on index explicitly:

$$\boxed{\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n)} \tag{88}$$

  - such as

$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^\infty}{\infty!}} \tag{89}$$

- informal usage:

$$\boxed{\Sigma F \text{ is often used to mean } \sum_{k=1}^{n} F_k = F_1 + \cdots + F_n} \tag{90}$$

- Dataset Usage:
  - summand indices act as $\underline{\text{labels}}$ to distinguish different data points which may or may not have same values:

data point labels	data values [m]
x_1	3.10
x_2	−1.64
x_3	3.10

- mean (average):

$$\begin{aligned} \bar{x} &= \frac{1}{3}\sum_{j=1}^{3} x_j \\ &= \frac{1}{3}(3.10 - 1.64 + 3.10) \\ &= 1.52 \end{aligned} \tag{91}$$