Assignment 4

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Question1: Write an `R` program to perform the following steps.

1. Simulate a noisy data set from a line with intercept $a = 2.1$ and slope $b = 3.4$. Let x range in $[-10, 10]$ with step-size 0.1 and add gaussian noise with mean 0 and variance 0.5.
2. Generate a `data.frame` containing simulated x and y. Center the data using `scale` function.
3. Plot the data using `ggplot2`
4. Perform a `SVD` decomposition and create the change of coordinate matrix P from the `SVD` calculations. What do the rows of P represent?
    1. P is the matrix that transforms X into Y
        - Because P(change if coordinate matrix) is a rotation and a stretch that transforms X and Y
        - Hence, the rows of P, $\{p_1, \ldots, p_m\}$ are the set of new basis vectors to express the columns of X
        - They, are also the Principle Components
5. Generate the rotated data set Y by projecting onto the newly calculated PCA coordinates. Plot the rotated data.
6. What is the variance in each `PCA` direction? Plot the results using `ggplot2` bar graphs.
7. Data de-nosing: On the diagonal part of the `SVD`, set the entries corresponding to low variance directions to zero. Reconstruct the data matrix using the new `SVD`. Plot the original data and the de-noised data on the same coordinates.=

Question2: Suppose $A \in \mathbb{C}^{m \times m}$ has an SVD $A = U\Sigma V^*$. Find the spectral decomposition of

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

The given square symmetric(Hermetrian) matrix, it can be factorized into two matrices U and D.

Matrix U is an orthogonal matrix.

- $UT = U^{-1}$

Matrix D is a diagonal matrix.

Spectral Decomposition is matrix factorization because we can multiply the matrices to get back the original matrix $M = UDU^*$. Another way of saying this is that, we are diagonalizing the matrix M

Solution :

Since the SVD of A is given by $A = U\Sigma V^*$,

$$Mx = \lambda x \implies \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{bmatrix} \begin{array}{l} \implies A^*\mathrm{x}_2 = \lambda \mathrm{x}_1 \\ \implies A\mathrm{x}_1 = \lambda \mathrm{x}_2 \end{array}$$

Hence,

$$AV = U\Sigma A^*U = V\Sigma$$

Thus,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} V\Sigma \\ U\Sigma \end{bmatrix}$$

Also,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V \\ -U \end{bmatrix} = \begin{bmatrix} -V\Sigma \\ U\Sigma \end{bmatrix}$$

Putting these together:

$$\begin{aligned} \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V & V \\ U & -U \end{bmatrix} &= \begin{bmatrix} V\Sigma & -V\Sigma \\ U\Sigma & U\Sigma \end{bmatrix} \\ &= \begin{bmatrix} V & -V \\ U & U \end{bmatrix} \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix} \\ &= \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & -\Sigma \end{bmatrix} \end{aligned}$$

Multiply both sides by,

$$\begin{bmatrix} V & V \\ U & -U \end{bmatrix}^{-1}$$

to get,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & -\Sigma \end{bmatrix} \begin{bmatrix} V & V \\ U & -U \end{bmatrix}^{-1}$$

Find eigenvalues :

$$\det\left(\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} -\lambda & A^* \\ A & -\lambda \end{pmatrix} = (-\lambda \cdot -\lambda) - (A^*A) = \lambda^2 - I$$

Thus, the eigenvalues are: $\lambda_1 = 1, \lambda_2 = -1$

and the diagonal matrix D is composed of the eigenvalues:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvector for the given matrix: $\lambda_1 = 1$

$$(A - \lambda I)) : \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & A^* \\ A & -1 \end{pmatrix}$$

Reduce the matrix $\begin{pmatrix} -1 & A^* \\ A & -1 \end{pmatrix}$,

$(-1)R_1 \to R_1$

$$\begin{pmatrix} 1 & -A^* \\ A & -1 \end{pmatrix}$$

$(-A)R_1 + R_2$

$$\begin{pmatrix} 1 & -A^* \\ 0 & 0 \end{pmatrix}$$

The system associated with the eigenvalue $\lambda = 1$

$$(A - 1I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -A^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this reduces to the equation

$$x - A^* y = 0$$

Isolate

$$x = A^* y$$

Plug into $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} A^*(y) \\ y \end{pmatrix} \quad y \neq 0$$

let $y = A$

$$\begin{pmatrix} I \\ A \end{pmatrix}$$

The eigenvector for the given matrix: $\lambda_2 = -1$

$$(A - \lambda I) : \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & A^* \\ A & 1 \end{pmatrix}$$

Solve, reduce the matrix:

$$\begin{pmatrix} 1 & A^* \\ A & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & A^* \\ 0 & 0 \end{pmatrix}$$

$$(A + 1I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & A^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this reduces to the equation

$$x + A^* y = 0$$

Isolate

$$x = -A^* y$$

Plug into $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} -A^* y \\ y \end{pmatrix} \quad y \neq 0$$

let $y = A$

$$\begin{pmatrix} -1 \\ A \end{pmatrix}$$

Eigenvectors for

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

$$= \begin{pmatrix} 1 \\ A \end{pmatrix}, \begin{pmatrix} -1 \\ A \end{pmatrix}$$

The Eigenvectors corresponding to the Eigenvalues in ‘D compose the columns of P

$$P = \begin{pmatrix} 1 & -1 \\ A & A \end{pmatrix}$$

and P^{-1} is

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ A & A \end{pmatrix}^{-1}$$

Thus,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} U & -U \\ V & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \begin{bmatrix} U & -U \\ V & V \end{bmatrix}^{-1}$$