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Chapter 2.3

Problem 13

$$x^2y' + x(x+2)y = e^x$$

Divide all terms by x^2 to get the normal form,

$$y' + \frac{(x+2)y}{x} = \frac{e^x}{x^2}$$

Integrating factor is given by,

$$\begin{aligned}\mu(x) &= \exp\left[\int 1 + \frac{2}{x}\,dx\right]\\ &= \exp\left[x + 2\ln|x|\right]\\ &= e^x e^{x^2}\\ &= e^x x^2\end{aligned}$$

multiply the integrating factor,

$$e^x x^2 + e^x x(x+2)y = e^x$$

take the integral of the LHS,

$$\begin{aligned}e^x x^2 y &= \int e^x\,dx\\ &= e^x + C\end{aligned}$$

solve for y,

$$y = \frac{1}{x^2} + C$$

Because we have a x in the denominator, $x \neq 0$ and $x = 0$ is a singularity, Thus interval is $x \in (-\infty, 0) \cup (0, \infty)$

Additionally, $\frac{C}{x^2 e^x}$ is a transient term

Problem 23

Solve the IVP

$$y\frac{dx}{dy} - x = 2y^2\ ;\ y(1) = 5$$

1. Rearrange the equation

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

2. Identify the integrating factor,

$$\begin{aligned}\mu(x) &= \exp\left[\int -\frac{1}{y}\,dx\right] \\ &= \exp\left[-\ln(y)\right] \\ &= \frac{1}{y}\end{aligned}$$

3. Multiply by the Integrating factor

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y}\frac{x}{y} = \frac{1}{y}2y$$

    1. simplify

$$\frac{d}{dy}\left(\frac{x}{y}\right) = 2$$

4. Integrate with respect to y,

$$\frac{x}{y} = \int 2\,dy = 2y + C$$

5. Solve for x,

$$x = 2y^2 + C$$

6. Plug in $x = 1, y = 5$

$$1 = 2(5)^2 + C$$

7. Final solution

$$x = 2y^2 - 49$$

Chapter 2.4

Problem 4

$$(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$$

$$\begin{aligned} M(x,y) &= \sin y - y\sin x \\ N(x,y) &= \cos x + x\cos y - y \end{aligned}$$

Calculate $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin y) - \frac{\partial}{\partial y}(y\sin x) \\ &= \cos y - \sin x \end{aligned}$$

And $\frac{\partial N}{\partial x}$,

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos x) + \frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial x}(y) \\ &= -\sin x + \cos y \end{aligned}$$

Thus we have shown, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the given equation is an exact equation

Then to solve it we integrate $M(x,y)$ with respect to x

$$\int \sin y - y\sin x \, dx = x\sin y + y\cos x + g(x)$$

then differentiate with respect to y

$$\frac{d}{dx}(x\sin y + y\cos x + g(x)) = x\cos(y) + \cos x + g'(x)$$

then we set $x\cos(y) + \cos x + g'(x)$ to $N(x,y)$

$$\begin{aligned} x\cos(y) + \cos x + g'(x) &= \cos x + x\cos y - y \\ g'(x) &= -y \\ &\text{or} \\ g(x) &= \frac{-y^2}{2} \end{aligned}$$

We have shown that,

$$f(x,y) = x\sin y + y\cos x - \frac{y^2}{y}$$

and the solution we see is,

$$x \sin y + y \cos x - \frac{y^2}{y} = C_1$$

Problem 29

Verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor and verify that the new equation is exact. Solve.

$$(-xy \sin x + 2y \cos x)dx + (2x \cos x)dy = 0\ ;\ \mu(x, y) = xy$$

we identify,

$$\begin{aligned} M(x,y) &= -xy \sin x + 2y \cos x \\ N(x,y) &= 2x \cos x \end{aligned}$$

verify that $M(x,y) \neq N(x,y)$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 2 \cos x - x \sin x \\ \frac{\partial N}{\partial x} &= 2 \cos x - 2x \sin x \end{aligned}$$

Thus not exact. Lets multiply by the given integrating factor.

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

$$\begin{aligned} (xy)(-xy \sin x + 2y \cos x)dx + (xy)(2x \cos x)dy = 0 \\ (-x^2y^2 \sin x + 2xy^2 \cos x)dx + (2x^2y \cos x)dy = 0 \end{aligned}$$

Testing for exactness,

$$\begin{aligned} \frac{\partial M}{\partial y}\left(-x^2y^2 \sin x + 2xy^2 \cos x\right) &= -2x^2y \sin x + 4xy \cos x \\ \frac{\partial N}{\partial x}\left(2x^2y \cos x\right) &= 4xy \cos x - 2x^2y \sin x \end{aligned}$$

They are now exact. Thus there is function $f(x,y)$

Now solve it

$$(-x^2y^2 \sin x + 2xy^2 \cos x)dx + (2x^2y \cos x)dy = 0$$

Integrate $(-x^2y^2 \sin x + 2xy^2 \cos x)$ with respect to x

$$\int \left(-x^2y^2 \sin x + 2xy^2 \cos x\right) dx = x^2y^2 \cos x + g(x)$$

Differentiate with respect to y and set the result equal to $N(x, y)$, we obtain,

$$\frac{d}{dx}\left(x^2y^2\cos x + g(x)\right) = 2x^2y\cos(x) + g'(x)$$

$$\begin{aligned} 2x^2y\cos x + g'(x) &= 2x^2y\cos x \\ g'(x) &= 0 \end{aligned}$$

thus we find,

$$f(x, y) = x^2y^2\cos(x)$$

Chapter 2.5

Problem 18

Solve the given differential equation (Bernoulli equation) by using an appropriate substitution.

$$x\frac{dy}{dx} - (1 + x)y = xy^2$$