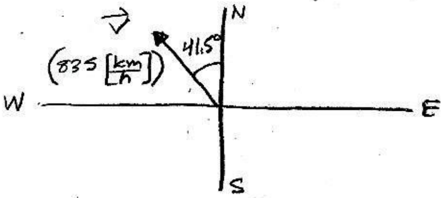
Solutions

Homework #[01b]

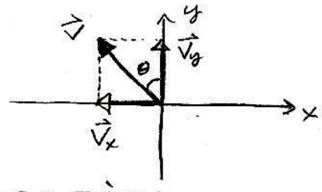
① An airplane is traveling 835 [km/h] in a direction 41.5° west of north.

ⓐ Find the components of the velocity vector in the northerly and westerly directions

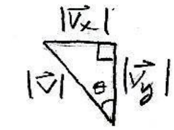
ⓑ How far north and how far west has the plane traveled after 1.75 [h] ?



- Diagram w/ coord. sys. :



$\Longrightarrow$



(shifted $\vec{v}_x$ up, since vectors can slide)

(won't mention this again)

- since $|\vec{v}_x|$ is opposite θ, use sine:

$$|\vec{v}_x| = |\vec{v}|\sin(\theta) \quad (1)$$

- since $|\vec{v}_y|$ is adjacent to θ, cosine:

$$|\vec{v}_y| = |\vec{v}|\cos(\theta) \quad (2)$$

- As a generic vector,

$$\vec{v} \equiv v_x\hat{x} + v_y\hat{y} \quad (3)$$

where the components are:

$$v_x \text{ and } v_y \quad (4)$$

- To deal with sign, look at the diagram and insert the sign appropriate to the direction in front of the absolute value of the component vector

$$v_x = -|\vec{v}_x| \quad (5)$$

$$v_y = +|\vec{v}_y| \quad (6)$$

- put (1) and (2) into (5), (6):

$$\begin{cases} v_x = -|\vec{v}|\sin(\theta) & (7) \\ v_y = +|\vec{v}|\cos(\theta) & (8) \end{cases}$$

- These are the scalar components of $\vec{v}$ as defined in (3)
- However, the problem specifies in the "northerly and westerly directions", and west is the $-y$ direction.
- Therefore, if we are going to verbally say "in the westerly direction" in the answer, then that takes care of the sign part of v_x, so then we only need its magnitude with that verbal direction:

ⓐ

$$\boxed{\begin{aligned} v_{WEST} &\equiv -v_x = |\vec{v}|\sin(\theta) = 553 \left[\tfrac{km}{h}\right] \text{ in the westerly direction} \\ v_{NORTH} &\equiv +v_y = |\vec{v}|\cos(\theta) = 625 \left[\tfrac{km}{h}\right] \text{ in the northerly direction} \end{aligned}} \quad (9)$$

- Note: In a usual problem, we'd just want $\vec{v}$, so $\vec{v} = v_x\hat{x} + v_y\hat{y}$

$$= -|\vec{v}|\sin(\theta)\,\hat{x} + |\vec{v}|\cos(\theta)\,\hat{y}$$

$$= \left(-553\left[\tfrac{km}{h}\right]\right)\hat{x} + \left(625\left[\tfrac{km}{h}\right]\right)\hat{y} \quad (10)$$

- Steady velocity in each direction, so use

$x = x_0 + v_x t \quad , \quad y = y_0 + v_y t$

$\Delta x = v_x t \quad , \quad \Delta y = v_y t$

- Adopt to directions desired:

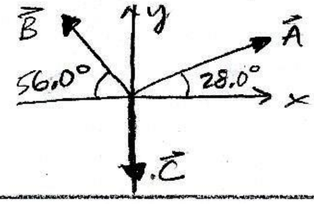
$\Delta x_{WEST} = v_{WEST}\,t \; ; \; \Delta y_{NORTH} = v_{NORTH}\,t$

$t = 1.75\,[h]$

ⓑ

$$\boxed{\begin{aligned} \Delta x_{WEST} &= |\vec{v}|\sin(\theta)\,t = 968\,[km] \text{ westwards} \\ \Delta y_{NORTH} &= |\vec{v}|\cos(\theta)\,t = 1090\,[km] \text{ northwards} \end{aligned}} \quad (11)$$

(2) Three vectors are shown in the diagram. Their magnitudes are given in arbitrary units. Determine the sum of the three vectors. Give the resultant in terms of (a) components (b) magnitude and angle with the +x axis:



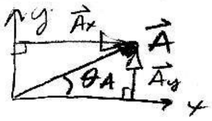
$A \equiv |\vec{A}| = 44.0$

$B \equiv |\vec{B}| = 26.5$

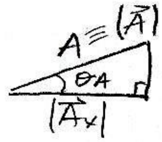
$C \equiv |\vec{C}| = 31.0$

- Break each vector into components in the coordinate system:

$\vec{A}$:



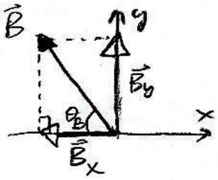
$\Rightarrow$



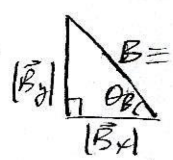
$\Rightarrow$

$$A_x = +|\vec{A}_x| = +A\cos(\theta_A) \qquad (1)$$
$$A_y = +|\vec{A}_y| = +A\sin(\theta_A) \qquad (2)$$

$\vec{B}$:



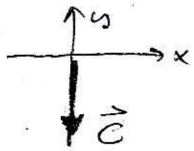
$\Rightarrow$



$\Rightarrow$

$$B_x = -|\vec{B}_x| = -B\cos(\theta_B) \qquad (3)$$
$$B_y = +|\vec{B}_y| = +B\sin(\theta_B) \qquad (4)$$

$\vec{C}$:



$\Rightarrow$ $\vec{C} = \underbrace{C_x}_{0}\hat{x} + C_y\hat{y}$ $\Rightarrow$ $C_x = 0 \qquad (5)$

$= C_y\hat{y} = \vec{C}_y$ $\qquad C_y = -|\vec{C}_y| = -|\vec{C}| = -C \qquad (6)$

- Add vectors by component:
  - Resultant: $\vec{D} \equiv \vec{A} + \vec{B} + \vec{C} \qquad (7)$
  - then if $\vec{D} \equiv D_x\hat{x} + D_y\hat{y} \qquad (8)$

  we have:

$$D_x = A_x + B_x + C_x \qquad (9)$$
$$D_y = A_y + B_y + C_y \qquad (10)$$

- put (1–6) into (9,10):

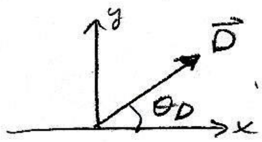
$\theta_A = 28.0°$

$\theta_B = 56.0°$

(a) The resultant is $\vec{D} = D_x\hat{x} + D_y\hat{y}$ where,

$$D_x = A\cos(\theta_A) - B\cos(\theta_B) + 0 = 24.0$$
$$D_y = A\sin(\theta_A) + B\sin(\theta_B) - C = 11.6 \qquad (11)$$

(b) Defining θ_D as the angle between $\vec{D}$ and the +x axis:



then $\theta_D = \tan^{-1}\left(\frac{D_y}{D_x}\right)$

$\theta_D = 25.8°$

and the magnitude of $\vec{D}$ is

$$D \equiv |\vec{D}| = \sqrt{D_x^2 + D_y^2} = 26.7$$

where we used D_x and D_y from (11) in both D and θ_D

(3) Estimate by what factor a person can jump farther on the Moon as compared to the Earth if the takeoff speed and angle are the same. The acceleration due to gravity on the Moon is one-sixth what it is on Earth.

- A jumper on flat ground is a typical projectile problem and identical to the cannonball problem from the lecture.
- Therefore, the general solution is the same (work it out yourself!) for the distance or "range":

$$\Delta x \equiv x - x_0 = \frac{v_0^2 \sin(2\theta_0)}{g} \quad ; \quad \begin{aligned} \theta_0 &\equiv \text{takeoff angle} \\ v_0 &\equiv \text{takeoff speed} \\ g &\equiv \text{acceleration magnitude due to local gravity} \end{aligned} \tag{1}$$

- Adapt to each planet:

(keep v_0 and θ_0 same since problem says so)

$$\Delta x_{\oplus} = \frac{v_0^2 \sin(2\theta_0)}{g_{\oplus}} \tag{2}$$

$$\Delta x_{\text{☾}} = \frac{v_0^2 \sin(2\theta_0)}{g_{\text{☾}}} \tag{3}$$

- Ratio of lunar range to earth range:

$$\frac{\Delta x_{\text{☾}}}{\Delta x_{\oplus}} = \frac{\cancel{v_0^2 \sin(2\theta_0)}}{g_{\text{☾}}} \cdot \frac{1}{\left(\frac{\cancel{v_0^2 \sin(2\theta_0)}}{g_{\oplus}}\right)} = \frac{g_{\oplus}}{g_{\text{☾}}} \tag{4}$$

- given fact:

$$g_{\text{☾}} = \frac{g_{\oplus}}{6} \tag{5}$$

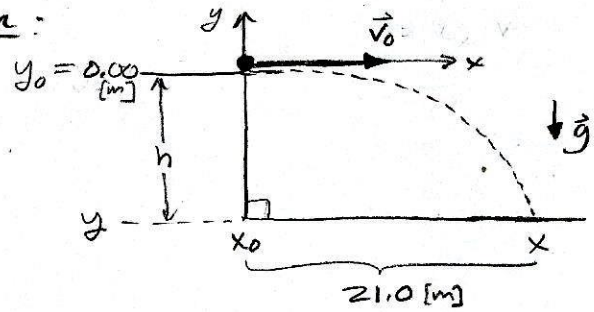
- put (5) into (4):

$$\boxed{\frac{\Delta x_{\text{☾}}}{\Delta x_{\oplus}} = \frac{\cancel{g_{\oplus}}}{\left(\frac{\cancel{g_{\oplus}}}{6}\right)} = 6} \rightarrow$$

(A person can jump 6 times farther in range on the Moon than on Earth with same launch angle and launch speed.)

(4) {A ball thrown horizontally at 12.2 [m/s] from the roof of a building lands 21.0 [m] from the base of the building. How high is the building?}

• Diagram:



x givens	y givens
$x_0 = 0.00$ [m]	$y_0 = 0.00$ [m]
$x = 21.0$ [m]	$v_{y0} = 0$ [m/s]
$v_{x0} = 12.2$ [m/s]	$a_y = -9.80$ [m/s²]
$a_x = 0$ [m/s²]	

(1)

• want y to get $h \equiv |y - y_0|$. (2)

• x kinematic eqn:

$$x = x_0 + v_{x0} t \quad \text{, (now need } t \text{ too)} \tag{3}$$

• y kinematic eqn: have y_0, v_{y0}, a_y, want y, t, so use (706) from lecture:

$$y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2 \tag{4}$$

• solve (3) for t:

$$t = \frac{x - x_0}{v_{x0}} \tag{5}$$

• put (5) into (4) to elim. t:

$$y = y_0 + v_{y0}\left(\frac{x - x_0}{v_{x0}}\right) + \frac{1}{2} a_y \left(\frac{x - x_0}{v_{x0}}\right)^2 \tag{6}$$

• and since $y_0 = 0$, $v_{y0} = 0$, $x_0 = 0$, this simplifies to

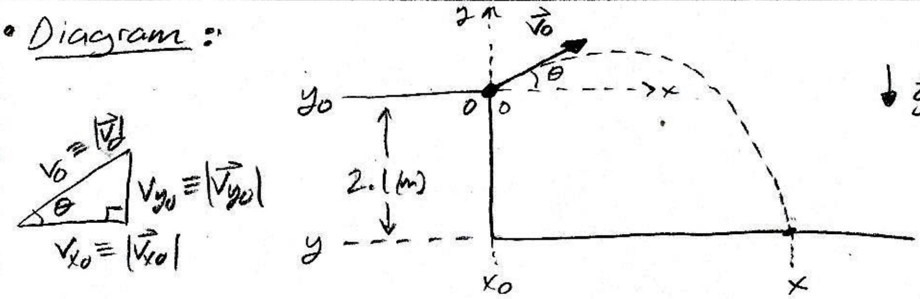
$$y = \frac{a_y}{2}\left(\frac{x}{v_{x0}}\right)^2 \tag{7}$$

• so the height of the building is

$$\boxed{h \equiv |y - y_0| = \left| \frac{a_y}{2}\left(\frac{x}{v_{x0}}\right)^2 \right| = 14.5 \text{ [m]}} \tag{8}$$

(5) A shot-putter throws the "shot" (mass 7.3 [kg]) with an initial speed of 14.4 [m/s] at a 34.0° angle to the horizontal. Calculate the horizontal distance traveled by the shot if it leaves the athlete's hand at a height of 2.10 [m] above the ground.

- Diagram:



$\downarrow \vec{g}$

x givens	y givens
$x_0 = 0.00$ (m)	$y_0 = 0.00$
$x = ?$	$y = -2.1$ [m]
$v_{x_0} = v_0 \cos(\theta)$	$v_{y_0} = v_0 \sin(\theta)$
$a_x = 0$	$a_y = -g = -9.80 \left[\frac{m}{s^2}\right]$

(1)

- want x, and also given $v_0 = 14.4 \left[\frac{m}{s}\right]$, $\theta = 34.0°$ (2)

- x kinematic eqn:

$$x = x_0 + v_{x_0} t \quad \text{; (now need } t \text{ too)} \qquad (3)$$

- y kinematic eqn: have y_0, y, v_{y_0}, a_y, want t, so use (7ob):

$$y = y_0 + v_{y_0} t + \frac{1}{2} a_y t^2 \qquad (4)$$

- since we want x, then (3) is our answer but we must solve (4) to get t; so rewrite it for quadratic eqn:

$$\underbrace{\left[\tfrac{1}{2} a_y\right]}_{a} t^2 + \underbrace{\left[v_{y_0}\right]}_{b} t + \underbrace{\left[y_0 - y\right]}_{c} = 0 \qquad (5)$$

$$t_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (6)$$

$$t_\pm = \frac{-v_{y_0} \pm \sqrt{v_{y_0}^2 - 4 \cdot \frac{1}{2} a_y (y_0 - y)}}{2 \cdot \frac{1}{2} a_y}$$

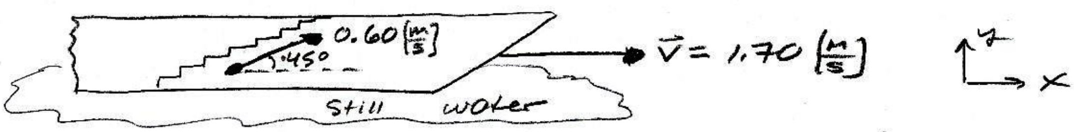
$$t_\pm = \frac{-v_0 \sin(\theta) \pm \sqrt{v_0^2 \sin^2(\theta) + 2g(-y)}}{-g} \qquad (7)$$

$$t_+ = -0.229 \text{ [s]}, \quad t_- = 1.87 \text{ [s]} \quad \text{since } t_- > 0 \qquad (8)$$

- so plug $t = t_-$ into (3):

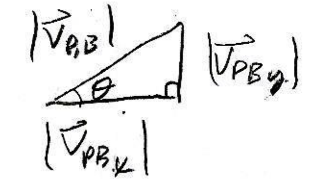
$$\boxed{x = x_0 + v_0 \cos(\theta) t_- = 22.3 \text{ [m]}} \qquad (9)$$

(6) A passenger on a boat moving at 1.70 [m/s] on a still lake walks up a flight of stairs at a speed of 0.60 [m/s], as in the diagram. The stairs are angled at 45° above the direction of motion. What is the velocity of the passenger relative to the water?



• Define variables:

$$\vec{V}_{B,W} = V_{B,W_x}\,\hat{x} = \left(1.70\left[\tfrac{m}{s}\right]\right)\hat{x} \equiv \begin{pmatrix}\text{velocity of Boat}\\ \text{relative to Water}\end{pmatrix} \quad (1)$$



$\theta = 45°$

$|\vec{V}_{P,B}| = 0.60\left[\frac{m}{s}\right]$

$$\vec{V}_{P,B} = V_{P,B_x}\,\hat{x} + V_{P,B_y}\,\hat{y} \equiv \begin{pmatrix}\text{velocity of Passenger}\\ \text{relative to Boat}\end{pmatrix} \quad (2)$$

where

$$V_{P,B_x} = +|\vec{V}_{PB_x}| = +|\vec{V}_{P,B}|\cos(\theta) \quad (3)$$

$$V_{P,B_y} = +|\vec{V}_{PB_y}| = +|\vec{V}_{P,B}|\sin(\theta) \quad (4)$$

$$\vec{V}_{P,W} = V_{P,W_x}\,\hat{x} + V_{P,W_y}\,\hat{y} = ? \begin{pmatrix}\text{velocity of}\\ \text{passenger}\\ \text{relative to}\\ \text{water}\end{pmatrix} \quad (5)$$

• Use relative velocity eqn:

$$\vec{V}_{P,W} = \vec{V}_{P,\underbrace{B + \vec{V}_{B}}_{\text{same index}},W} \quad (6)$$

same order as left side

• put (1) and (2) into (6):

$$\vec{V}_{P,W} = \left(V_{PB_x}\,\hat{x} + V_{P,B_y}\,\hat{y}\right) + \left(V_{B,W_x}\,\hat{x}\right) \quad (7)$$

$$= \left(V_{P,B_x} + V_{B,W_x}\right)\hat{x} + V_{P,B_y}\,\hat{y} \quad (8)$$

← (group by unit vector)

• put (3, 4) into (8):

$$\vec{V}_{P,W} = \left(|\vec{V}_{P,B}|\cos(\theta) + V_{B,W_x}\right)\hat{x} + \left(|\vec{V}_{P,B}|\sin(\theta)\right)\hat{y}$$

$$= \left[\left(0.60\left[\tfrac{m}{s}\right]\right)\cos(45°) + \left(1.70\left[\tfrac{m}{s}\right]\right)\right]\hat{x} + \left[\left(0.60\left[\tfrac{m}{s}\right]\right)\sin(45°)\right]\hat{y}$$

$$= \left(2.12\left[\tfrac{m}{s}\right]\right)\hat{x} + \left(0.424\left[\tfrac{m}{s}\right]\right)\hat{y}$$

or

$$|\vec{V}_{P,W}| = \sqrt{V_{P,W_x}^2 + V_{P,W_y}^2} = 2.17\left[\tfrac{m}{s}\right],$$

$$\theta_{P,W} = \tan^{-1}\left(\frac{V_{P,W_y}}{V_{P,W_x}}\right) \simeq 11° \text{ above } +x\text{-direction}$$

(7) Given the vector $\vec{v} \equiv a\hat{x} + b\hat{y} + c\hat{z}$, answer the following:

(a) what are the components v_x, v_y, v_z of $\vec{v}$?

(b) what is the magnitude of $\vec{v}$? (That is, find $|\vec{v}|$.)

(c) what is the unit vector in the direction of $\vec{v}$?

(a) Scalar components are the scalars that multiply the unit vector for their particular direction. So:

$$V_x = a \quad , \quad V_y = b \quad , \quad V_z = c \tag{1}$$

since then

$$\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z} = a\hat{x} + b\hat{y} + c\hat{z} = \vec{v} \checkmark \tag{2}$$

(b) Magnitude of a vector in Cartesian form is given by the Pythagorean theorem:

(b)
$$\begin{aligned} |\vec{V}| &= \sqrt{V_x^2 + V_y^2 + V_z^2} \\ &= \sqrt{a^2 + b^2 + c^2} \end{aligned} \tag{3}$$

(c): The definition of the unit vector in the direction of $\vec{v}$ is:

$$\hat{v} \equiv \frac{\vec{v}}{|\vec{v}|} \tag{4}$$

- so put $\vec{V}$ from the problem statement into the numerator of (4),
- and put $|\vec{v}|$ from (3) into the denominator of (4):

(c)
$$\hat{v} = \frac{a\hat{x} + b\hat{y} + c\hat{z}}{\sqrt{a^2 + b^2 + c^2}} \tag{5}$$

$$= \frac{a}{\sqrt{a^2+b^2+c^2}}\hat{x} + \frac{b}{\sqrt{a^2+b^2+c^2}}\hat{y} + \frac{c}{\sqrt{a^2+b^2+c^2}}\hat{z} \tag{6}$$

(8) Given the two vectors:

$$\vec{A} \equiv a\hat{x} - b\hat{y} \quad \text{and} \quad \vec{B} \equiv c\hat{x} + d\hat{y}$$ (i)

(a) Find $\vec{A}+\vec{B}$

(b) Find $\vec{A}-\vec{B}$

(c) Find $|\vec{A}|$, $|\vec{B}|$, and $|\vec{A}+\vec{B}|$

(d) What is the scalar y-component of $\vec{C} \equiv \vec{A}-\vec{B}$?

(a) • vectors add "component-wise":

$$\vec{A}+\vec{B} = a\hat{x} - b\hat{y} + c\hat{x} + d\hat{y}$$ (collect terms by unit vector)

$$= a\hat{x} + c\hat{x} - b\hat{y} + d\hat{y}$$ (and pull out common unit vectors to "group" them by unit vector)

$$= (a+c)\hat{x} + (-b+d)\hat{y}$$ (1)

so:

(a) $$\boxed{\vec{A}+\vec{B} = (a+c)\hat{x} + (-b+d)\hat{y}}$$ (2)

(b):

$$\vec{A}-\vec{B} = \vec{A}+(-\vec{B}) = a\hat{x} - b\hat{y} + (-[c\hat{x}+d\hat{y}])$$

$$= a\hat{x} - b\hat{y} + (-c\hat{x} - d\hat{y})$$

$$= a\hat{x} - b\hat{y} - c\hat{x} - d\hat{y}$$

$$= (a-c)\hat{x} + (-b-d)\hat{y}$$

so: (3)

(b) $$\boxed{\vec{A}-\vec{B} = (a-c)\hat{x} + (-b-d)\hat{y}}$$

(c):

(c) $$\boxed{|\vec{A}| = \sqrt{a^2+b^2}, \quad |\vec{B}| = \sqrt{c^2+d^2} \\ |\vec{A}+\vec{B}| = \sqrt{(a+c)^2 + (-b+d)^2}}$$ (4)

(if this is confusing, define $\vec{C} \equiv \vec{A}+\vec{B}$ and note that $|\vec{C}| = \sqrt{C_x^2 + C_y^2}$)

(d)

• if $\vec{C} \equiv \vec{A}-\vec{B}$,

• then from (3):

$$\vec{C} = (a-c)\hat{x} + (-b-d)\hat{y} \equiv C_x\hat{x} + C_y\hat{y}$$ (5)

• so (d) $$\boxed{C_y = -b-d}$$ (6)

(a) In the 2D kinematic equations for projectiles where $a_x = 0$ and a_y is constant, rewrite them without time.
That is:

(a) solve the x equation for t

(b) plug the solution for t from (a) into the y-direction kinematic equations

(c) what x-direction parameters have the ability to affect y-direction variables such as y and v_y?

- Solving the x eqn. for t:

$$x = x_0 + v_{x_0} t \qquad (1)$$

$$\text{(a)} \quad \boxed{t = \frac{x - x_0}{v_{x_0}}} \qquad (2)$$

- Plug (2) into the y-direction kinematic eqns:

(b)

$$v_y = v_{y_0} + a_y \frac{x - x_0}{v_{x_0}} \qquad (3)$$

$$y = y_0 + v_{y_0} \frac{x - x_0}{v_{x_0}} + \frac{1}{2} a_y \left(\frac{x - x_0}{v_{x_0}} \right)^2 \qquad (4)$$

$$v_y^2 = v_{y_0}^2 + 2a_y (y - y_0) \qquad (5)$$

$$y = y_0 + \frac{1}{2}(v_{y_0} + v_y)\left(\frac{x - x_0}{v_{x_0}} \right) \qquad (6)$$

$$y = y_0 + v_y \frac{x - x_0}{v_{x_0}} - \frac{1}{2} a_y \left(\frac{x - x_0}{v_{x_0}} \right)^2 \qquad (7)$$

- In (b), we see that v_y and y depend on x, x_0, and v_{x_0}, so:

(c) x, x_0, and v_{x_0} have the ability to affect y-direction variables such as y and v_y. This is because the same time t passes as the object moves in both directions. (8)

* Note: this is not exactly true in Einstein's Special Relativity because in that case, time passes differently at different velocities. However, that is only a significant effect at velocities near the speed of light, so at "usual" velocities, the above result is correct.