

# Introduction to differential equations, initial value problems, non-uniqueness

MA221, Lecture 1

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# What is a differential equation?

- Numerical equations are equations with numerical solutions  
 $\hookrightarrow x^2 - 1 = 0 \dots x = \pm 1$
- Functional equations are equations with function solutions  
 $\hookrightarrow (f(x))^2 - 1 = 0 \dots f(x) = \pm 1$  & infinitely many other solutions
- Informally, a DiffEq is a functional equation involving derivatives of the solution functions

# What is a differential equation?

Ex:  $f''(x) + f'(x) = 0$  ODE  
2nd order  
linear

$f(x)f'(x) + [f''(x)]^5 = f^{(4)}(x)$  ODE  
4th order  
non-linear

$\hookrightarrow yy' + [y'']^5 = y^{(4)}$

$\hookrightarrow y \frac{dy}{dx} + \left[ \frac{d^2 y}{dx^2} \right]^5 = \frac{d^4 y}{dx^4}$

# Classification of differential equations

- Classify by “type”: ODE or PDE?

“ordinary”  
solutions are  
univariate

“partial”  
solutions are multivariate

- Classify by “order”: highest order derivative is...?

- Classify by “linearity”: linear or non-linear?

# What are some examples of differential equations?

Example:  $y' = 0$  has solution  $y = C$ ;  $C \in \mathbb{R}$

Ex:  $y'' = 0$  has solution  $y = ax + b$ ;  $a, b \in \mathbb{R}$

# What are some examples of differential equations?

Ex:  $y'' + y = 0$ .  $y = A \sin x + B \cos x$

how to verify?

$$y' = A \cos x - B \sin x$$

$$y'' = -A \sin x - B \cos x$$

$$\Rightarrow y'' + y = 0. \checkmark$$

# What are some examples of differential equations?

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# Verifying (explicit) solutions

Example:  $\frac{dy}{dx} = x\sqrt{y}$  has an explicit solution  $y = \frac{1}{16}x^4$

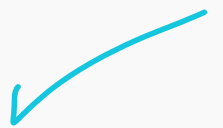
$$(LHS) = \frac{1}{4}x^3$$

$$\frac{dy}{dx} = \frac{1}{4}x^3$$

$$(RHS) = x\sqrt{y} = x\sqrt{\frac{1}{16}x^4} = x\left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3$$

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$$\text{General solution: } y = \left(\frac{x^2}{4} + C\right)^2$$





# Verifying (implicit) solutions

Example:  $\frac{dy}{dx} = -\frac{x}{y}$  has an implicit solution  $x^2 + y^2 \stackrel{\star}{=} 1$

Use implicit differentiation on this 

$$(\text{RHS of } \star)' = 0$$

$\parallel$

$$(\text{LHS of } \star)' = 2x + 2y \frac{dy}{dx}$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

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general solution:  $x^2 + y^2 = C$



# Initial value problems

Example:  $y'' + y \neq 0$  subject to  $y' \left( \frac{\pi}{2} \right) = y \left( \frac{\pi}{2} \right) = 1$  is an IVP. What is a solution to this IVP?

recall, general solution to ~~A~~ is  $y = A \sin x + B \cos x$

$$1 = y \left( \frac{\pi}{2} \right) = A \underbrace{\sin \left( \frac{\pi}{2} \right)}_{=1} + B \underbrace{\cos \left( \frac{\pi}{2} \right)}_{=0} = A$$

Since  $y' = A \cos x - B \sin x$ ,

$$1 = y' \left( \frac{\pi}{2} \right) = A \underbrace{\cos \left( \frac{\pi}{2} \right)}_{=0} - B \underbrace{\sin \left( \frac{\pi}{2} \right)}_{=1} = -B$$

$$\Rightarrow y = \sin x - \cos x$$

# Non-Uniqueness

While we expect a given differential equation to have multiple solutions (“a family of solutions”), it is also possible for an IVP to have more than one solution.

$$\frac{dy}{dx} = x\sqrt{y} \quad \text{subject to} \quad y(0) = 0$$