3

EXAMPLE 3.1. Let $(z_n)_n$ be the sequence given by $z_n = \left(\frac{1+i}{2}\right)^n$ for $n \geqslant 1$. Since $\left|\frac{1+i}{2}\right| = \frac{1}{\sqrt{2}}$, we have $|z_n| = \left|\frac{1+i}{2}\right|^n = \left(\frac{1}{\sqrt{2}}\right)^n$. As $0 \leqslant \frac{1}{\sqrt{2}} < 1$, it follows that $\left(\frac{1}{\sqrt{2}}\right)^n \to 0$ as $n \to \infty$, and therefore $|z_n| \to 0$ as $n \to \infty$. As a consequence, $Z = 0$ is the limit of the sequence z_n,

$$\lim_{n\to\infty}\left(\frac{1+i}{2}\right)^n = 0$$

and the sequence $(z_n)_n$ is convergent to 0 . To reach that conclusion we have used only that $0 \leqslant \left|\frac{1+i}{2}\right| < 1$, hence for all complex numbers ζ such that $0 \leqslant |\zeta| < 1$ we have

$$\lim_{n\to\infty}\zeta^n = 0$$

EXAMPLE 3.2 Let $(z_n)_{n\geqslant 1}$ be the sequence given by

$$z_n = \left(1 + \frac{i}{n}\right)^n$$

If we write

then

$$1 + \frac{i}{n} = r_n\left(\cos\theta_n + i\sin\theta_n\right)$$
$$z_n = r_n^n\left(\cos\left(n\theta_n\right) + i\sin n\theta_n\right))$$

We start with the modulus:

$$|z_n| = \left|1 + \frac{i}{n}\right|^n = \left(1 + \frac{1}{n^2}\right)^{n/2} = \left[\left(1 + \frac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{2n}}$$

We wrote the sequence that way to use the fact that if $(x_n)_n$ is a sequence of real numbers such that $x_n \to \infty$, then

$$\lim_{n\to\infty}\left(1 + \frac{1}{x_n}\right)^{x_n} = e$$

Using that property for $x_n = n^2$, we conclude that

$$\lim_{n\to\infty}|z_n| = \left[\lim_{n\to\infty}\left(1 + \frac{1}{n^2}\right)^{n^2}\right]^{\lim_{n\to\infty}\frac{1}{2n}} = e^0 = 1.$$

For the argument, we have

$$\cos\theta_n = \frac{1}{\sqrt{1 + 1/n^2}} \to 1, \quad \sin\theta_n = \frac{1/n}{\sqrt{1 + 1/n^2}} \to 0$$

as $n \to \infty$, hence we can choose θ_n such that $\theta_n \to 0$; in particular, for large enough n, we have $\theta_n \in (-\pi/2, \pi/2)$, meaning that we can choose $\theta_n = \arctan\frac{1}{n}$. Then, using l'Hopital's rule,

$$\begin{aligned}\lim_{n\to\infty} n\theta_n &= \lim_{n\to\infty} n\arctan(1/n) = \lim_{n\to\infty}\frac{\arctan(1/n)}{1/n} \\ &= \lim_{x\to 0}\frac{\arctan x}{x} = \lim_{x\to 0}\frac{1/\left(1 + x^2\right)}{1} = 1.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\left(1 + \frac{i}{n}\right)^n = \cos 1 + i\sin 1$$

The same method can be used to show that for $z = x + iy$,

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^x(\cos y + i\sin y).$$

Make a note of this result; we will see it again soon.

ExAmple 3.3 Let $\zeta \in \mathbb{C}$ such that $|\zeta| > 1$. Then $0 < \left|\frac{1}{\zeta}\right| < 1$ and therefore

$$\lim_{n\to\infty}\left(\frac{1}{\zeta}\right)^n = 0 \Longleftrightarrow \lim_{n\to\infty}\frac{1}{\zeta^n} = 0 \Longleftrightarrow \lim_{n\to\infty}\zeta^n = \infty.$$

Combining with the result of Example 3.1, we conclude that

$$\lim_{n\to\infty} z^n = \begin{cases} 0 & , \text{ if } |z| < 1 \\ \infty & , \text{ if } |z| > 1 \end{cases}$$

What happens when $|z| = 1$?

Suppose $z = \cos(2\pi\alpha) + i\sin(2\pi\alpha)$, with $0 \leqslant \alpha < 1$; then

$$z^n = \cos(2n\pi\alpha) + i\sin(2n\pi\alpha).$$

When $\alpha = 0$, corresponding to $z = 1$, the sequence $(z^n)_n$ is a constant sequence, with all terms equal to 1 , and therefore it is a convergent sequence, with limit equal to 1.

When $\alpha \in (0,1)$ is a rational number, the sequence $(n\alpha(\bmod 1))_n$ is a non-constant periodic sequence, and therefore the sequence $(z^n)_n$ is also a non-constant periodic sequence, hence a sequence without a limit.

When $\alpha \in (0,1)$ is an irrational number, the sequence $(n\alpha(\bmod 1))_n$ is dense in $[0,1]$: for every value x in $[0,1]$, and whatever close to x we look, we will find at least one term (in fact, infinitely many terms) of the sequence. Therefore in that case the sequence $(z^n)_n$ does not have a limit. Hence

$$\lim_{n\to\infty} z^n = \begin{cases} 0 & , \text{ if } |z| < 1 \\ \infty & , \text{ if } |z| > 1 \\ 1 & , \text{ if } z = 1 \\ \text{DNE} & , \text{ if } |z| = 1 \text{ and } z \neq 1 \end{cases}.$$

Bottom line: the complex world is sometimes simpler than the real one! (... and all it takes is allowing imaginary entities...) But don't be fooled and let your guard down - complexity is lurking ahead.

EXAMPLE 3.4 Let $z = \frac{i}{2}$ and consider the geometric series

$$\sum_{n\geqslant 0} z^n = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots$$

with ratio $z \in \mathbb{C}$ and first term $a_0 = 1$. What would be a reasonable value to assign to the evaluation of the that infinite sum?

The partial sums are given by

$$s_n = 1 + z + z^2 + \cdots + z^n = \frac{z^{n+1}-1}{z-1} = \frac{(i/2)^{n+1}-1}{(i/2)-1}$$

Since $|i/2| = 1/2 < 1$, the values of $(i/2)^{n+1}$ get close and stay close to 0 as n gets large, and therefore

$$\lim_{n\to\infty} s_n = \frac{0-1}{i/2-1} = \frac{4}{5} + \frac{2}{5}i$$

and it makes sense to assign that value to the infinite sum. More general, if $|z| < 1$, then

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots = \frac{1}{1-z}.$$

A few comments here: we use $\sum_{n\geqslant 0} z^n$, to refer to a series (abstract concept), but we change the notation to $\sum_{n=0}^{\infty} z^n$ to refer to/denote the complex number that we assign as the sum of the series. Then, while the right hand side above $\frac{1}{1-z}$ makes sense for all $z \neq 1$, it represents the sum of the geometric series only when $|z|<1$.

If the first term a_0 has a non-zero value, then the partial sums are simply multiplied by that value and the nature of the series doesn't change; its sum, however, becomes

$$\sum_{n=0}^{\infty} a_0 z^n = a_0\left(1+z+z^2+z^3+\cdots+z^n+\cdots\right) = \frac{a_0}{1-z}.$$

ExAmPle 3.5 Consider the series $\sum_{n\geqslant 0} \frac{1}{n!}$. The square of that series (the Cauchy product of two copies of the series) is the series

$$\begin{aligned}\left(\sum_{n\geqslant 0}\frac{1}{n!}\right)^2 &= \left(\sum_{n\geqslant 0}\frac{1}{n!}\right)\left(\sum_{n\geqslant 0}\frac{1}{n!}\right) = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{(n-k)!}\right) = \\ &= \sum_{n\geqslant 0}\frac{1}{n!}\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\right) = \sum_{n\geqslant 0}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} = \sum_{n\geqslant 0}\frac{2^n}{n!}\end{aligned}$$

More general, consider the series $\sum_{n\geqslant 0}\frac{1}{n!}z^n$ and $\sum_{n\geqslant 0}\frac{1}{n!}w^n$, where z and w are complex numbers. The product of those series is

$$\begin{aligned}\left(\sum_{n\geqslant 0}\frac{1}{n!}z^n\right)\left(\sum_{n\geqslant 0}\frac{1}{n!}w^n\right) &= \sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{(n-k)!}z^k w^{n-k}\right) = \\ &= \sum_{n\geqslant 0}\frac{1}{n!}\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}z^k w^{n-k}\right) = \\ &= \sum_{n\geqslant 0}\frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}z^k w^{n-k}\right) = \sum_{n\geqslant 0}\frac{1}{n!}(z+w)^n.\end{aligned}$$

With sums and products we can compute algebraic expressions involving series.

Example 3.6 Consider the series

$$\begin{aligned}\sum_{n\geqslant 0}\frac{(-1)^n z^{2n+1}}{(2n+1)!} &= z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots \\ \sum_{n\geqslant 0}\frac{(-1)^n z^{2n}}{(2n)!} &= 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\end{aligned}$$

We compute the sum of the squares of these series,

$$\left(\sum_{n\geqslant 0}\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right)^2 + \left(\sum_{n\geqslant 0}\frac{(-1)^n z^{2n}}{(2n)!}\right)^2.$$

We write down the first few terms of the squares, determined as if we were squaring polynomials:

$$\begin{aligned}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)^2 &= z^2-2\frac{z^4}{3!}+\frac{z^6}{(3!)^2}+2\frac{z^6}{5!}+\cdots \\ \left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)^2 &= 1-2\frac{z^2}{2!}+\frac{z^4}{(2!)^2}+2\frac{z^4}{4!}-2\frac{z^6}{6!}-2\frac{z^6}{2!4!}+\cdots\end{aligned}$$

Since the first series involves only odd powers of z, the terms of the square of the series will involve only products of odd degree terms, hence will be even degree terms. Same for the second series, where all the initial terms have even degree. A quick check shows that the first term is 1 , and then when adding the coefficients of z^2-terms from both series, the sum is 0 , and the same happens for the z^4-terms, and z^6-terms, ... Indeed, if $n \geqslant 1$, then the coefficient of z^{2n} is (with indices $a, b \geqslant 0$)

$$
\begin{aligned}
&\sum_{a+b=n-1} \frac{(-1)^{a+b}}{(2a+1)!(2b+1)!}+\sum_{a+b=n} \frac{(-1)^{a+b}}{(2a)!(2b)!} \\
&\quad=\frac{1}{(2n)!}\left[(-1)^{n-1} \sum_{k \text{ odd}}\binom{2n}{k}+(-1)^{n} \sum_{k \text{ even}}\binom{2n}{k}\right] \\
&\quad=\frac{(-1)^{n}}{(2n)!} \sum_{k=0}^{2n}(-1)^{k}\binom{2n}{k}=\frac{(-1)^{n}}{(2n)!}(1+(-1))^{2n}=0 .
\end{aligned}
$$

Therefore, as series,

$$
\left(\sum_{n \geqslant 0} \frac{(-1)^{n} z^{2n+1}}{(2n+1)!}\right)^{2}+\left(\sum_{n \geqslant 0} \frac{(-1)^{n} z^{2n}}{(2n)!}\right)^{2}=\sum_{n \geqslant 0} \delta_{n, 0},
$$

where $\delta_{n, 0}=1$ when $n=0$ and $\delta_{n, 0}=0$ when $n \neq 0$.

EXAMPLE 3.7 We return to the constant sequence $\left(a_{n}=1\right)_{n \geqslant 0}$ and the corresponding power series $\sum_{n \geqslant 0} z^{n}$.

If $z=1$, then the sequence of partial sums becomes

$$
s_{n}(1)=1+1^{1}+1^{2}+\cdots+1^{n}=n+1
$$

for all $n \geqslant 0$; this sequence does not have a finite limit, hence it is not convergent. Therefore the power series $\sum_{n \geqslant 0} z^{n}$ diverges for $z=1$.

If $z \neq 1$, then the sequence of partial sums becomes

$$
s_{n}(z)=1+z+z^{2}+\cdots+z^{n}=\frac{z^{n+1}-1}{z-1}=\frac{1}{1-z}\left(1-z^{n+1}\right)
$$

for all $n \geqslant 0$. Now recall that the sequence $\left(z^{n+1}\right)_{n \geqslant 0}$ converges if and only if $|z|<1$ (when the limit is 0) or $z=1$ (when the limit is 1). As $z=1$ is not included in the current case, it follows that for $z \neq 1$, the sequence $\left(s_{n}(z)\right)_{n \geqslant 0}$ converges if and only if $|z|<1$, and if that is the case, then the sequence of partial sums converges to $\frac{1}{1-z}$.

Therefore the region of convergence of the complex power series $\sum_{n \geqslant 0} z^{n}$ is the open unit disk centered at 0 , and

$$
\sum_{n=0}^{\infty} z^{n}=\frac{1}{1-z} \quad \text { if }|z|<1 .
$$

More Examples. We consider below several examples that will show what situations may occur when studying the region of convergence of a power series.

ExAmple 3.8 Consider the power series

$$
\sum_{n \geqslant 1} \frac{1}{n} z^{n}
$$

The series of moduli is

$$
\sum_{n \geqslant 1}\left|\frac{1}{n} z^{n}\right|=\sum_{n \geqslant 1} \frac{1}{n}|z|^{n} .
$$

The series converges for $z=0$ and for $z \neq 0$, to apply the Ratio Test we compute the long-term behavior of the ratio of consecutive terms

$$
\lim _{n \rightarrow \infty} \frac{\frac{1}{n+1}|z|^{n+1}}{\frac{1}{n}|z|^{n}}=\lim _{n \rightarrow \infty} \frac{n}{n+1}|z|=|z| .
$$

Therefore, if $|z|<1$, then the Ratio Test tells us that the series

$$
\sum_{n \geqslant 1} \frac{1}{n} z^{n}
$$

is absolutely convergent, hence convergent, and if $|z|>1$, then the series is divergent.

What happens if $|z|=1$? It depends: the real values of z with $|z|=1$ already provide some insight. For $z=1$ we get the divergent harmonic series $\sum_{n \geqslant n} \frac{1}{n}$, and for $z=-1$ we get the convergent alternate harmonic series $\sum_{n \geqslant 1} \frac{(-1)^n}{n}$. For $z=i$ we get the series

$$\sum_{n \geqslant 1} \frac{i^n}{n}=i-\frac{1}{2}-\frac{1}{3} i+\frac{1}{4}+\frac{1}{5} i-\frac{1}{6}-\frac{1}{7} i+\frac{1}{8}-\cdots$$

The terms of the series do converge to 0 , but that does not guarantee the convergence of the series. It does, however, allow us to group the terms four-at-a-time, changing the order within each group, but without changing the order of the groups:

$$\begin{aligned}
\sum_{n \geqslant 1} \frac{i^n}{n} & =\left(-\frac{1}{2}+\frac{1}{4}+i-\frac{1}{3} i\right)+\left(-\frac{1}{6}+\frac{1}{8}+\frac{1}{5} i-\frac{1}{7} i\right)+\cdots \\
& =\sum_{n \geqslant 0}\left(-\frac{1}{4 n+2}+\frac{1}{4 n+4}\right)+\left(\frac{1}{4 n+1}-\frac{1}{4 n+3}\right) i \\
& =\sum_{n \geqslant 0}-\frac{2}{(4 n+2)(4 n+4)}+\frac{1}{(4 n+1)(4 n+5)} i
\end{aligned}$$

Both the real and the imaginary part compare with the p-series $\sum \frac{1}{n^p}$ with $p=2>1$, hence both the real and the imaginary part are convergent series of real numbers. The conclusion is that the power series $\sum_{n \geqslant 1} \frac{1}{n} z^n$ is convergent for $z=i$. Note that we figured out only the nature of the series for $z=i$ (convergent series), but we haven't yet made any progress in finding the sum of the series for $z=i$.

EXAMPLE 3.9 Let $\sum_{n \geqslant 0} \frac{z^n}{1+2^n}$. The series of moduli is

$$\sum_{n \geqslant 0}\left|\frac{z^n}{1+2^n}\right|=\sum_{n \geqslant 0}\left|\frac{z}{2}\right|^n \frac{2^n}{1+2^n}=\sum_{n \geqslant 0}\left|\frac{z}{2}\right|^n \frac{1}{1+2^{-n}} .$$

To use the Root Test (the Ratio Test would be fine, too) we compute

$$\lim _{n \rightarrow \infty} \sqrt[n]{\left|\frac{z}{2}\right|^n \frac{1}{1+2^{-n}}}=\lim _{n \rightarrow \infty} \frac{|z|}{2} \sqrt[n]{\frac{1}{1+2^{-n}}}=\frac{|z|}{2}$$

Therefore the series $\sum_{n \geqslant 0} \frac{z^n}{1+2^n}$ is absolutely convergent, hence convergent, for $|z|<2$, and the series is divergent if $|z|>2$.

For $|z|=2$ we get

$$\lim _{n \rightarrow \infty}\left|\frac{z^n}{1+2^n}\right|=\lim _{n \rightarrow \infty}\left|\frac{z}{2}\right|^n \frac{1}{1+2^{-n}}=\lim _{n \rightarrow \infty} \frac{1}{1+2^{-n}}=1,$$

hence the limit of the terms of the series is not 0. Again, the Test for Divergence implies that the series $\sum_{n \geqslant 0} \frac{z^n}{1+2^n}$ is divergent for $|z|=2$.

Therefore the region of convergence of the series $\sum_{n \geqslant 0} \frac{z^n}{1+2^n}$ is the open disk $|z|<2$.

Example 3.10 Consider the power series

$$\sum_{n \geqslant 0} \frac{1}{n!} z^n$$

For $z=0$, the series has only one non-zero term, hence it is convergent, and in fact it is absolutely convergent. For $z \neq 0$, to apply the Ratio Test for the series of moduli we compute the limit of the ratio

$$\lim _{n \rightarrow \infty}\left|\frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n}\right|=\lim _{n \rightarrow \infty}\left|\frac{z}{n+1}\right|=0$$

and we conclude that the limit is $0<1$ no matter what non-zero value z takes. Therefore the series $\sum_{n \geqslant 0} \frac{1}{n!} z^n$ is absoultely convergent, hence convergent, for all $z \in \mathbb{C}$.

EXAMPLE 3.11

Let $\sum_{n \geqslant 0} n! z^n$. For $z=0$, the series has only one non-zero term, hence it is convergent, and in fact it is absolutely convergent. For $z \neq 0$, to apply the Ratio Test for the series of moduli we compute the limit of the ratio

$$\lim_{n \rightarrow \infty}\left|\frac{(n+1)! z^{n+1}}{n! z^n}\right|=\lim_{n \rightarrow \infty}(n+1)|z|=\infty$$

and we conclude that the limit is greater than 1, no matter what nonzero value z takes. Therefore the series $\sum_{n \geqslant 0} n! z^n$ is not absoultely convergent for $z \neq 0$.

In fact, for $z \neq 0$, the fact that $\lim_{n \rightarrow \infty}(n+1)|z|=\infty$ implies that, eventually, the modulus of $n! z^n$ will increase at each term by a factor of at least 2 , and therefore $\lim_{n \rightarrow \infty}\left|n! z^n\right|=\infty$. The Test for Divergence then implies that the power series $\sum_{n \geqslant 0} n! z^n$ is divergent for all $z \neq 0$. The region of convergence then consists of just the value $z=0$.

Example 3.12 Consider the power series

$$\sum_{n \geqslant 0} \frac{1}{(2 n)!}(-1)^n z^{2 n}=1-\frac{1}{2} z^2+\frac{1}{24} z^4-\frac{1}{720} z^6+\cdots$$

The odd terms are missing (coefficients equal to 0), and the coefficients of the even terms are real and alternating in sign, as reciprocals of factorials of even non-negative integers. As it stands, the Ratio Test can't be applied, since every other term of ratios would include division by 0 . However, we can change the variable and consider a new variable $w=z^2$. The power series becomes

$$\sum_{n \geqslant 0} \frac{1}{(2 n)!}(-1)^n w^n=1-\frac{1}{2} w+\frac{1}{24} w^2-\frac{1}{720} w^3+\cdots$$

and now we can compute the limit of the ratios of moduli of consecutive terms:

$$\lim_{n \rightarrow \infty}\left|\frac{(-1)^n}{(2 n)!} \cdot \frac{(2 n+2)!}{(-1)^{n+1}}\right|=\lim_{n \rightarrow \infty}(2 n+1)(2 n+2)=\infty$$

hence the radius of convergence of the w-series is $R=\infty$. The series is absolutely convergent for all $w \in \mathbb{C}$, hence for all $z \in \mathbb{C}$. The radius of convergence of the original z-series is also $R=\infty$.