# Graphical Approach To Linear Motion:

- Recall definition of a line:

$$y = mx + b \quad ; \quad \begin{cases} m \equiv \frac{\Delta y}{\Delta x} \equiv \frac{y_2 - x_1}{x_2 - x_1} \equiv \text{slope} \\ b \equiv y\text{-intercept} \end{cases} \tag{1}$$

- Now rewrite with $x \to t$, $b \to y_0$

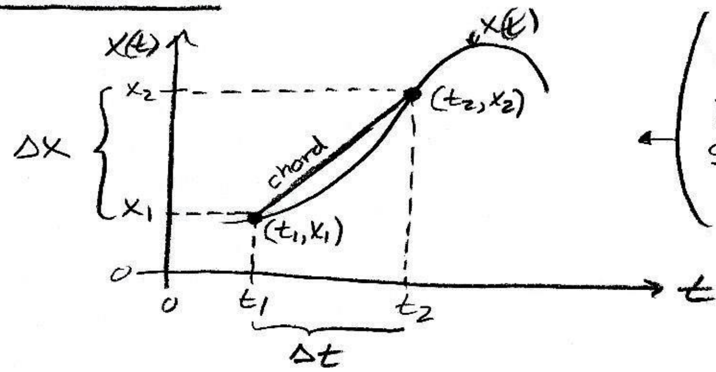
$$y = \frac{\Delta y}{\Delta t} t + y_0 \tag{2}$$

- Now put $y \to x$:

$$x = x_0 + \bar{v} t \quad \leftarrow \left( \begin{array}{l} \bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0} = \frac{x - x_0}{t} \\ \text{so} \quad x = x_0 + \bar{v} t \end{array} \right) \tag{3}$$

- This is (8) from earlier:
  - $x_0$ is the intercept of dependent variable $x(t)$
  - $\bar{v}$ is the slope of the line $x(t)$

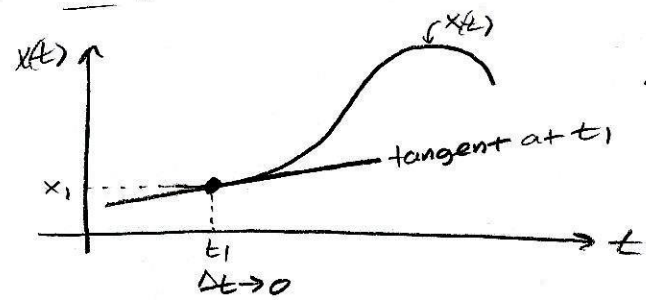
- What if $v(t)$ is not a line?




$\leftarrow \left( \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \right.$ is the slope of the chord joining points $(t_1, x_1)$ and $(t_2, x_2)$ on the position curve $\left. \right)$

- Now, as $\Delta t \to 0$, $t_2$ gets closer and closer to $t_1$ until they touch:

  - tangent line is a straight line that only touches the curve at one point, without crossing the curve

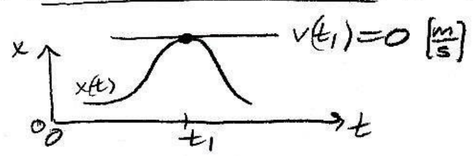


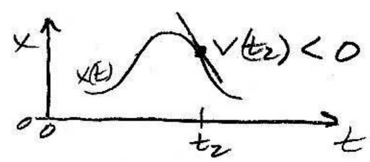

$\leftarrow \left( \lim_{\Delta t \to 0} \left( \frac{\Delta x}{\Delta t} \right) = v \right.$ is the slope of the tangent line at the point on the $x(t)$ curve $\left. \right)$

- Thus, $\boxed{\text{instantaneous velocity is the slope of the position curve at a point (slope of the tangent of position)}}$

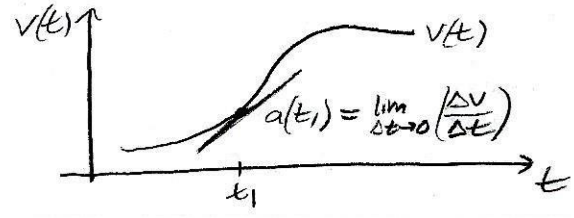
- Since the slope of the tangent is different at different points on $x(t)$, then $v(t)$ is not constant when $x(t)$ is not a line.

- Note: this instantaneous slope can also be 0 or negative:




- Acceleration is the instantaneous Slope of Velocity:




  - if $v$ is a line, $a =$ constant
  - if $v$ is not a line, $a$ depends on time (is different tangent slope at different times)

# Derivatives:

$$\boxed{\text{the derivative of } f(t) \text{ is the slope of the tangent line to } f(t) \text{ at } t.} \quad (1)$$

- definition:

$$\boxed{\frac{df(t)}{dt} \equiv \lim_{\Delta t \to 0}\left(\frac{\Delta f(t)}{\Delta t}\right)} = \lim_{\Delta t \to 0}\left(\frac{f(t_2) - f(t_1)}{t_2 - t_1}\right) \quad (2)$$

- to get a more useful form, let

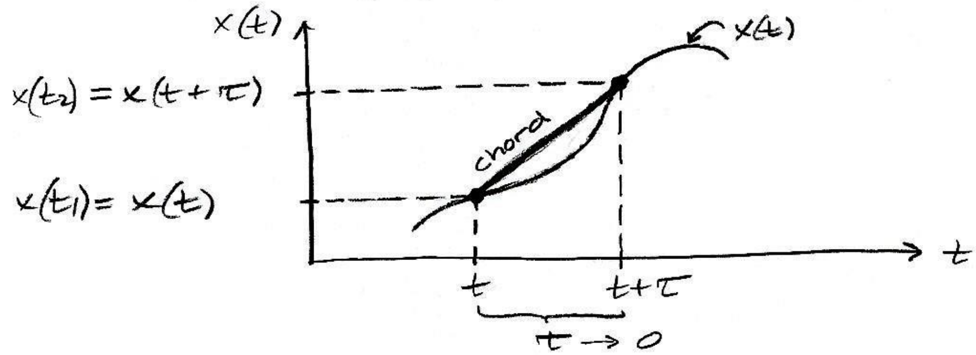
$$t_1 \equiv t \quad (3)$$

$$t_2 \equiv t + \tau \quad (4)$$

- so (2) becomes

$$\frac{df(t)}{dt} = \lim_{([t+\tau]-t)\to 0} \frac{f(t+\tau) - f(t)}{(t+\tau) - t}$$

$$\boxed{\frac{df(t)}{dt} = \lim_{\tau \to 0} \frac{f(t+\tau) - f(t)}{\tau}} \quad (5)$$




(6)

- key points:
  - as $\tau \to 0$, it becomes $\tau = dt$ an "infinitesimal" (7)
  - since limit means only to approach zero, we say $\lim_{\tau \to 0} \tau = dt \equiv$ a "nonzero infinitesimal" (8)
  - so $\frac{1}{dt} \neq \infty$, and we can use $dt$ algebraically (9)

- How to Compute Derivatives:
  - let $\lim_{\tau\to 0} \tau = dt \neq 0$ in $\frac{df(t)}{dt}$ (10)
  - simplify as much as possible (11)
  - discard any terms left with $dt$ (take the "standard part") (12)

- ex:
  - Suppose $x(t) \equiv ct^2$
  - then

$$\frac{dx(t)}{dt} = \lim_{\tau \to 0}\left(\frac{x(t+\tau) - x(t)}{\tau}\right) = sp\left(\frac{x(t+dt) - x(t)}{dt}\right)$$

$$= sp\left(\frac{c(t+dt)^2 - c(t)^2}{dt}\right) = sp\left(\frac{c\left(t^2 + 2tdt + (dt)^2 - t^2\right)}{dt}\right)$$

(taking the standard part means finding the nearest real number that has no infinitesimal part)

$$= sp\left(\frac{2ctdt}{dt} + \frac{c(dt)^2}{dt}\right) = sp(2ct + cdt)$$

$$\boxed{\frac{dx(t)}{dt} = 2ct} \quad (13)$$

- In practice: (We try to break-down a problem to standard quantities whose derivatives we've already computed.) (14)

- ex: $\frac{d}{dt} cf(t) = c\frac{df(t)}{dt}$ ; $\frac{d}{dt} t^n = nt^{n-1}$ ; $\frac{d}{dt} e^{ct} = ce^{ct}$ ; $\frac{d}{dt} f(t)g(t) = \left(\frac{df}{dt}\right)g + f\left(\frac{dg}{dt}\right)$ (15)

# Deriving Kinematic Eqns. from Taylor Series:

- Recall Taylor series:

(Expansion of $f(t)$ about the point $t=t_0$) → $$f(t)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf(t')}{dt'^n}\bigg|_{t'=t_0}(t-t_0)^n \leftarrow \left(\begin{array}{c}(t-t_0)^0\equiv 1\\ 0!\equiv 1\\ \frac{d^0}{dt^0}f(t)\equiv f(t)\end{array}\right) \quad (1)$$

- so for <u>position</u>: $$x(t)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nx(t')}{dt'^n}\bigg|_{t'=t_0}(t-t_0)^n \quad (2)$$
  - if acceleration is constant: $$a=\frac{dv}{dt}=\frac{d}{dt}\frac{dx}{dt}=\frac{d^2x}{dt^2}=C \leftarrow \left(\text{where } \frac{d}{dt}C=0\right) \quad (3)$$
  - so its derivative is zero: $$\frac{da}{dt}=\frac{d^3x}{dt^3}=\frac{dC}{dt}=0 \quad (4)$$
  - as are all higher derivatives: $$\frac{d^{n\geq 3}x(t)}{dt^n}=0 \quad (5)$$
  - so (5) into (2) gives:

$$x(t)=\sum_{n=0}^{2}\frac{1}{n!}\frac{d^nx(t')}{dt'^n}\bigg|_{t'=t_0}(t-t_0)^n$$
$$=\frac{1}{0!}\frac{d^0}{dt^0}x(t')\bigg|_{t'=t_0}(t-t_0)^0+\frac{1}{1!}\frac{d^1}{dt'^1}x(t')\bigg|_{t'=t_0}(t-t_0)^1+\frac{1}{2!}\frac{d^2}{dt'^2}x(t')\bigg|_{t'=t_0}(t-t_0)^2$$
$$=x(t')\big|_{t'=t_0}+\frac{dx(t')}{dt'}\bigg|_{t'=t_0}(t-t_0)+\frac{1}{2}\frac{d^2x(t')}{dt'^2}\bigg|_{t'=t_0}(t-t_0)^2$$
$$=x(t_0)+v(t_0)(t-t_0)+\frac{1}{2}a(t_0)(t-t_0)^2$$

(• use: $x_0\equiv x(t_0)$, $v_0\equiv v(t_0)$ • and $t_0\equiv 0$ • and $a(t_0)=a(t)\equiv a$ since $a\equiv$ const.) →

$$x(t)=x_0+v_0t+\frac{1}{2}at^2 \leftarrow [\text{is } (17b)] \quad (6)$$

$$\frac{dx(t)}{dt}=0+v_0+\frac{1}{2}a2t$$

- then differentiate: $$v(t)=v_0+at \leftarrow [\text{is } (17a)] \quad (7)$$
- solve (7) for $t$: $$t=\frac{v-v_0}{a} \quad (8)$$
- put (8) into (6), and solve for $v^2$:

$$x=x_0+v_0\left(\frac{v-v_0}{a}\right)+\frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2$$
$$x-x_0=\frac{v_0v}{a}-\frac{v_0^2}{a}+\frac{1}{2a}(v^2-2vv_0+v_0^2)$$
$$2a(x-x_0)=\cancel{2v_0v}-2v_0^2+v^2-\cancel{2vv_0}+v_0^2$$
$$2a(x-x_0)=v^2-v_0^2$$
$$v^2=v_0^2+2a(x-x_0) \leftarrow [\text{is } (17c)] \quad (9)$$

- To eliminate $a$, use fact that $a=$ const., then $v$ is a line: $$\bar{v}=\frac{x-x_0}{t-t_0}=\frac{x-x_0}{t} \quad (10)$$
  - solve (10) for $x$: $$x=x_0+\bar{v}t \quad (11)$$
  - then use alternative formula for $\bar{v}$: $$\bar{v}=\frac{1}{2}(v_0+v) \quad (12)$$
  - (12) into (11): $$x=x_0+\frac{1}{2}(v_0+v)t \leftarrow [\text{is } (17d)] \quad (13)$$

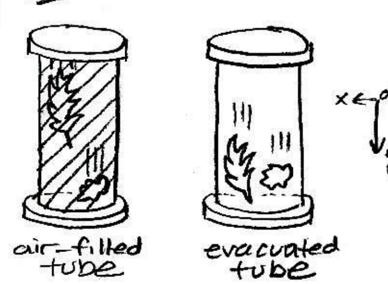
- <u>Higher-Order Kinematics</u>:
  - if 3rd deriv. of $x$ is constant, $j\equiv\frac{d^3x}{dt^3}=$ const. (14)
    - we get a new term: $$x(t)=x_0+v_0(t-t_0)+\frac{1}{2}a(t-t_0)^2+\frac{1}{6}j(t-t_0)^3 \quad (15)$$
    - but we will get <u>more</u> kinematic eqns.
- what if $x(t)$ has <u>infinitely</u> many nonzero time derivatives?
  - then: (there would be an <u>infinite</u> number of kinematic equations!) (16)
  - and, as with all Taylor series: $x(t)$ is only exactly correct <u>at $t_0$</u> (17)
- so: (we need something more powerful in general when $a\neq$ constant...) (18)

# Freely Falling Objects:

- Galileo Galilei: (1564 – 1642)

$$\boxed{\begin{array}{l}\bullet \text{ The speed of a falling object is } \underline{\text{independent}} \text{ of its mass} \\ \bullet \text{ All objects fall with the same constant acceleration in the abscence of air or other resistance}\end{array}}$$

ex:




- Starting from rest, an object's displacement (vertically to ground) is proportional to elapsed time squared:

$$\boxed{\Delta y \propto t^2} \quad (1)$$

- From (17b) (with $x \to y$):

$$y = y_0 + v_0 t + \tfrac{1}{2} a t^2 \quad (2)$$

- if $v_0 = 0$ (starts from rest),

$$\underbrace{y - y_0} = \underbrace{\tfrac{1}{2}}_{\text{constant}} a t^2 \quad (3)$$

$$\Delta y \propto t^2 \quad \checkmark \quad (4)$$

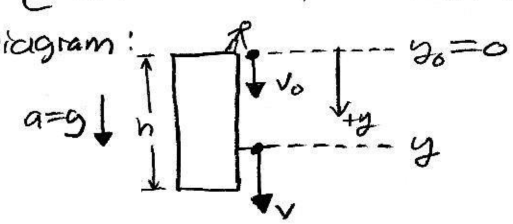
- Acceleration Due To Gravity:
  - The constant $a$ in (24-26) is $|a| = g$ (5)

where

$$\left(\begin{array}{l} g \text{ itself} \\ \text{is always} \\ \text{positive} \end{array}\right)^{*} \rightarrow \boxed{g \equiv 9.80 \left[\tfrac{m}{s^2}\right]} \leftarrow \left(\begin{array}{l}\text{acceleration} \\ \text{due to gravity} \\ \text{at Earth's surface}\end{array}\right) \quad (6)$$

  - $g$ is actually not constant over space (gets weaker farther up), but Earth is so big compared to us, that $g \approx$ constant in most of our problems. (Difference of $g$ from top to bottom of a tall building is negligible, for example)

- ex: {Suppose a ball is thrown downwards from a tower at height $h$, with initial speed $v_0$. What is its velocity at time $t$?}
  - Diagram:




  - want $v$, have $t, v_0, a$, since

$$a = g \quad (1)$$

  - so use (17a) with $x \to y$:

$$\boxed{\begin{array}{c} v = v_0 + at = v_0 + gt \\ \text{in the +y direction as shown in diagram}\end{array}} \quad (2)$$

- Notice not all information was needed (didn't use $y_0, y,$ or $h$)
- We chose $+y$ in direction of motion

* Since $+y$ defined downward, and $a$ due to gravity points downward, then:

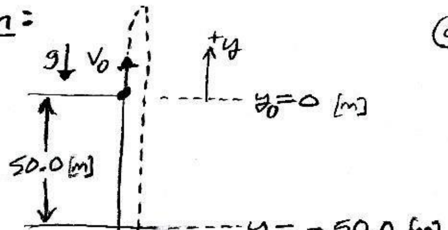
$$a \equiv +|a| = +g \quad (3)$$

  - if we had chosen $+y$ upwards, then we'd need $a \equiv -|a| = -g$, and we'd expect the value of $v$ to be negative

- Always specify direction for vectors
  - the diagram does this automatically
  - but it's good to be clear and state it in the answer

ex: A person on the edge of a cliff tosses a rock upward at $15.0\,[\frac{m}{s}]$. Supposing that the ball moves straight upward before falling back down, and then falls all the way to the base of the $50.0\,[m]$ cliff, ⓐ how long does it take for the rock to reach the base of the cliff? ⓑ what total distance does the rock travel? (Ignore air resistance and person's height).

• Diagram:




ⓐ • we have:

$$y_0 = 0\,[m] \tag{1}$$
$$y = -50.0\,[m] \tag{2}$$
$$a = -g = -9.80\,\left[\frac{m}{s^2}\right] \tag{3}$$
$$V_0 = +15.0\,\left[\frac{m}{s}\right] \tag{4}$$

• we want:

$$t = ? \tag{5}$$

• So, since (17b) has these, solve it for $t$:
$(x \to y)$

$$y = y_0 + V_0 t + \tfrac{1}{2}at^2 = y_0 + V_0 t - \tfrac{1}{2}gt^2 \tag{6}$$
$$-\tfrac{1}{2}gt^2 + V_0 t + y_0 - y = 0 \tag{7}$$

• is quadratic in $t$:
  • given $\quad At^2 + Bt + C = 0 \quad (8)$
  • solutions are:

(quadratic formula) →
$$t_\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = -\left(\frac{B}{2A}\right) \pm \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \tag{9}$$

  • so in (7);
$$A = -\tfrac{1}{2}g\,,\quad B = V_0\,,\quad C = y_0 - y \tag{10}$$

  • so putting (10) into (9):

$$t_\pm = \frac{-V_0 \pm \sqrt{V_0^2 - 4(-\frac{1}{2}g)(y_0 - y)}}{2(-\frac{1}{2}g)}$$

ⓐ
$$\boxed{t_\pm = \frac{-1}{g}\left(-V_0 \pm \sqrt{V_0^2 + 2g(y_0 - y)}\right)} \tag{11}$$

• and plugging in from (1–4) gives

$$t_+ = -2.01\,[s]\,,\ \text{and}\ t_- = 5.07\,[s] \tag{12}$$

• By causality, we are only interested in times after $t_0 = 0\,[s]$, so $t_-$ is the physical solution:
$$\boxed{t_- = 5.07\,[s]} \tag{13}$$

• In ⓑ, first, the rock's trip to the peak of its trajectory has parameters:

$$y_0 = 0\,[m],\ y = ?\,,\ a = -g\,,\ V_0 = +15.0\,\left[\frac{m}{s}\right],\ V = 0\,\left[\frac{m}{s}\right]\ \left(\begin{array}{c}\text{momentarily}\\ \text{at rest}\end{array}\right) \tag{14}$$

• So solve (17c) for $y$:
$$V^2 = V_0^2 - 2g(y - y_0)$$
$$y = \frac{V_0^2 - V^2}{2g} + y_0 \tag{15}$$

• Then, since it also travels this distance back to its launch elevation, and then travels the cliff height $h = 50.0\,[m]$, the total distance traveled is

ⓑ
$$\boxed{D = |y| + |y| + |h| = 2y + h = \frac{V_0^2 - V^2}{g} + 2y_0 + h} \tag{16}$$

• using numbers in (14):
$$\boxed{= 73.0\,[m]}$$