11 Written Assignment

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Question 1

Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the $xy-$plane, and below the cone $z = \sqrt{x^2 + y^2}$

Solution :

Rewrite the sphere as $p^2 = 4 \Rightarrow p = 2$
Thus, the sphere is centered at the origin with radius 2.

Cone $z = \sqrt{x^2 + y^2}$ is the same as $\phi = \frac{\pi}{4}$

Define solid using spherical coordinates,

$$E = \{(\rho, \theta, \phi) \mid o \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi \quad \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}\}$$

Calculate the volume of the solid as the integral,

$$\begin{aligned}
\iiint_E dV &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho \\
&= [-\cos\phi] \left|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot [\theta]_0^{2\pi} \cdot \left[\frac{1}{3}\rho^3\right]_0^2 \right. \\
&= \left[\frac{1}{\sqrt{2}}\right] \cdot [2\pi] \cdot \left[\frac{8}{3}\right] \\
&= \frac{8\sqrt{2}}{3}\pi
\end{aligned}$$

Question 2

Evaluate the integral $\iint_R (x+y)e^{x^2-y^2}$

Solution :

$u = x + y$ and $v = x - y$
thus,

$$x = \frac{u+v}{2}$$
$$y = \frac{u-v}{2}$$

R is defined $0 \leq u \leq 3$ and $0 \leq v \leq 2$

The Jacobian is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \frac{-1}{2}$$

Thus, the integral is

$$\begin{aligned}
\iint_R (x+y)e^{x^2-y^2} dA &= \int_0^3 \int_0^2 ue^{uv} \mid -\frac{1}{2} \mid \; dv\, du \\
&= \frac{1}{2} \int_0^3 e^{uv} \Big]_0^2 du \\
&= \frac{1}{2} \int_0^3 e^{2u} - 1\, du \\
&= \frac{1}{2} \cdot \left[\frac{1}{2} e^{2u} - u\right]_0^3 \\
&= \frac{1}{2} \cdot \left[\frac{1}{2} e^6 - 3 - \frac{1}{2} + 0\right] \\
&= \frac{1}{4}(e^{6-7})
\end{aligned}$$