# Homework 1

# Proof for Complex Numbers and Binomial Coefficients

## 1. Complex Numbers

### (a) Modulus and Argument

For a complex number $z = a + bi$, the modulus is given by:

$$|z| = \sqrt{a^2 + b^2}$$

and the argument is given by:

$$\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$

Principal value of the argument: Denoted by $\text{Arg}(z)$ is the unique value of the argumant in the interval $(-\pi, \pi]$

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The argument formula $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ is valid when $a$ is not zero. It would be helpful to clarify that when $a = 0$, the argument is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ depending on the sign of $b$

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#### i. For $z = \frac{i}{2}$

Given $z = \frac{i}{2}$, where $a = 0$ and $b = \frac{1}{2}$:

Using the formula for modulus:

$$|z| = \sqrt{0^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

For the argument, since the denominator(real part) is zero, the angle lies on the positive y-axis(imaginary part is positive),

the argument set is $\frac{\pi}{2} + 2\pi k$,

and the principal value of the argument is:

$$\arg(z) = \frac{\pi}{2}$$

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#### ii. For $z = 2 - i$

Given $z = 2 - i$, where $a = 2$ and $b = -1$:

Using the formula for modulus:

$$|z| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

The argument is given by:

$$\arg(2 - i) = -\arctan\left(\frac{1}{2}\right)$$

This value is negative as the complex number is in the fourth quadrant. The argument set is

$$\arg(2-i) = -\arctan\left(\frac{1}{2}\right) + 2\pi k$$

and the principal value of the argument is $-\arctan\left(\frac{1}{2}\right)$

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The argument is in the third quadrant, use method such as $\arg(z) = \pi + \tan^{-1}\left(\frac{b}{a}\right)$ when both $a$ and $b$ are negative

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### iii. For $z = -1 - i$

Given $z = -1 - i$, where $a = -1$ and $b = -1$:

Using the formula for modulus:

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

The argument is given by:

$$\arg(-1-i) = \arctan\left(\frac{-1}{-1}\right) + \pi = -\frac{5\pi}{4}$$

This value is in the third quadrant, we add $\pi$ to the arctangent value

The argument set is $\arg(-1-i) = -\frac{5\pi}{4} + 2\pi k$ and the principal value of the argument is

$$-\frac{5\pi}{4}$$

## (b) Power of Complex Number

Using De Moivre's theorem, for $z = 1 + i$, the power can be found by expressing $1 + i$ in its polar form and then raising it to the power of 42:

$$(1+i)^{42} = 2097152i$$

## (c) Alternating Sums of Binomial Coefficients

The binomial coefficient is given by:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

## (i) Even Indices

The alternating sum for even indices can be derived using the binomial expansion of $(1-1)^{42}$:

$$\sum_{k=0}^{21} (-1)^k \binom{42}{2k} = 0$$

## (ii) Odd Indices

The alternating sum for odd indices can be derived using the binomial expansion of $(1+i)(1-i)$ raised to the power of 21-k:

$\sum_{k=0}^{20} (-1)^k \binom{42}{2k+1} = 2097152$

Thus, Simplify $1 - i^2 = 1 + 1 = 2$ since $i^2 = -1$

$$\sum_{k=0}^{20} (-1)^k \binom{42}{2k+1} \cdot 2^{21-k} = 972551540$$

# 2. Squares

## (a) Possible squares with vertices at $1 + 3i$ and $3 - i$

Given two points $A(1 + 3i)$ and $B(3 - i)$:

1. For the square with $A$ and $B$ as opposite vertices, the other two vertices can be found using the midpoint formula and the properties of squares.

Find midpoint of $A$ and $B$

To find the other two verticies of the square, we can find the midpoint of $A$ and $B$ and then find the points equidistant from this midpoint, perpendicular to the line $AB$

- Mid point $M$ of $A$ and $B$ :

$$\begin{aligned} M &= \frac{A+B}{2} \\ &= \frac{(1+3i)+(3-i)}{2} \\ &= \frac{(4+2i)}{2} \\ &= 2+i \end{aligned}$$

Now to find the two other vertices $C$ and $D$ of the square with $A$ & $B$ as opposite vertices.

The line segment $AB$ has a slope, which is the imaginary part divided by the real part of $B - A$:

$$\begin{aligned} \text{Slope of } AB &= \frac{\mathrm{Im(B-A)}}{\mathrm{Re}(B-A)} \\ &= \frac{\mathrm{Im(3-i-(1+3i))}}{\mathrm{Re}(3-i-(1+3i))} \\ &= \frac{\mathrm{Im(2-4i)}}{\mathrm{Re}(2-4i)} \\ &= -2 \end{aligned}$$

The perpendicular slope to $AB$ will be the negative reciprocal of the slope of $AB$:

$$\text{Perpendicular Slope } = \frac{1}{2}$$

To find the other two vertices, $C$ and $D$ we can use the midpoint $M(2 + i)$ and move along the line with a perpendicular slope to $AB$ by a distance equal to half the length of the diagonal $AB$.

The length of the diagonal $AB$ is the distance between $A$ and $B:

$$\begin{aligned} |AB| &= |B-A| \\ &= 2\sqrt{5} \end{aligned}$$

Now, to find the other verticies, $C$ and $D$, we can use the midpoint and move along the line with a perpendicular slope to $AB$ by a distance equal to half the length of the diagonal $AB$:

$$\begin{aligned} \text{Distance to move} &= \frac{|AB|}{2} \\ &= \sqrt{5} \end{aligned}$$

Denote movement along the real axis as $\Delta x$ and the movement along the imaginary axis as $\Delta y$. Since we are moving along a line with a slope of $\frac{1}{2}$, we have:

$$\frac{\Delta y}{\Delta x} = \frac{1}{2}$$
$$\Delta y = \frac{\Delta x}{2}$$

Also, the distance to move, $\sqrt{5}$, is related to $\Delta x$ and $\Delta y$ by the Pythagorean theorem:

$$\begin{aligned}
\sqrt{5} &= \sqrt{(\Delta x)^2 + \left(\frac{\Delta x}{2}\right)^2} \\
5 &= \Delta x^2 + \frac{\Delta x^2}{4} \\
5 &= \frac{5\Delta x^2}{4} \\
\Delta x^2 &= 4 \\
\Delta x &= 2
\end{aligned}$$

Substituting back to find $\Delta y$: $\Delta y = \frac{2}{2} = 1$

Now, we can find the coordinates of $C$ and $D$:

$$\begin{aligned}
C &= M + \Delta x + i\Delta y \\
&= (2+i) + 2 + i \\
&= 4 + 2i
\end{aligned}$$

$$\begin{aligned}
D &= M - \Delta x - i\Delta y \\
&= (2+i) - (2+i) \\
&= 0
\end{aligned}$$

3. For the square with $A$ and $B$ as adjacent vertices, the other two vertices can be found using vector addition and the properties of squares. Lets use the following relations:

$$\begin{aligned}
C' &= A + (B - A) \cdot (1+i) \\
D' &= B + (A - B) \cdot (1+i)
\end{aligned}$$

Lets calculate:

$$\begin{aligned}
C' &= (1+3i) + ((3-i) - (1+3i) \cdot (1+i) \\
&= 2i - (1+3i)(1+i) + 4 \\
&= 2i + 2 - 4 + 4 \\
&= 6 - 2i
\end{aligned}$$

$$\begin{aligned}
D' &= (3-i) + ((1+3i) - (3-i)) \cdot (1+i) \\
&= (3-i) + (-2+4i)(1+i) \\
&= 3 - i + 2i - 6 \\
&= -3 + i
\end{aligned}$$

## (b) Squares on the exterior of a convex quadrilateral $ABCD$

Given a convex quadrilateral $ABCD$, if we construct squares on the exterior of its sides with centers $P, Q, R, S$, then the midpoints $I, J, K, L$ of the sides of $PQRS$ will form a square.

Proof:

1. Let $I$ be the midpoint of $PQ$, $J$ be the midpoint of $QR$, $K$ be the midpoint of $RS$, and $L$ be the midpoint of $SP$.
2. Since $P, Q, R, S$ are the centers of the squares constructed on the sides of $ABCD$, the segments $PQ, QR, RS, SP$ are diagonals of these squares and are therefore equal in length.

3. The midpoints of equal segments divide the segments into equal halves. Thus, $PI = IQ$, $QJ = JR$, $RK = KS$, and $SL = LP$.
4. Since all four segments $PI, QJ, RK, SL$ are equal, the quadrilateral $IJKL$ is a rhombus.
5. Additionally, since $PQRS$ is a square (because all its sides are equal and all its angles are right angles), the angles $P, Q, R, S$ are all $90^\circ$.
6. The diagonals of a square are perpendicular bisectors of each other. Therefore, the angles $IPQ, JQR, KRS, LSP$ are all $90^\circ$.
7. Given that $IJKL$ is a rhombus with all angles equal to $90^\circ$, $IJKL$ is a square.

Uses the properties of squares and midpoints to show that the midpoints of the sides of a square formed on the exterior of a convex quadrilateral will also form a square.

# 3. Geometric Transformations

## (a) Composition of transformations

Given the transformations:

1. Translation by $i$
2. Inversion
3. Counter-clockwise rotation by $\pi/3$ with center at 0
4. Translation by $1 + i$

Lets denote a general complex number as $z = a + bi$ where $a, b \in \mathbb{R}$

1. Translation by $i$

$$\begin{aligned} z' &= z + i \\ &= a + (b+1)i \end{aligned}$$

2. Inversion:

$$\begin{aligned} z'' &= \frac{1}{z'} \\ &= \frac{1}{a + (b+1)i} \end{aligned}$$

3. Counter-clockwise rotation by $\frac{\pi}{3}$ with a center at $0$:

$$\begin{aligned} z''' &= e^{i\frac{\pi}{3}} \cdot z'' \\ &= \frac{e^{i\frac{\pi}{3}}}{a + (b+1)i} \end{aligned}$$

4. Translation by $1 + i$:

$$\begin{aligned} f(z) &= z''' + (1+i) \\ &= \frac{e^{i\frac{\pi}{3}}}{a + (b+1)i} + (1+i) \end{aligned}$$

Final composition:

$$f(z) = (1+i) + \frac{e^{i\frac{\pi}{3}}}{a+(b+1)i}$$

This is the function $f(z)$ after applying all the transformations in the given order.

The composition of transformations is given by applying each transformation in sequence to a general complex number $z$.

## (b) Express $f(z) = \frac{z+i}{iz+1}$ as a composition

Given the function $f(z) = \frac{z+i}{iz+1}$, we can express it as a composition of geometric transformations.

Decomposition:

1. Translation by $i$: $z \to z+i$

$$\begin{aligned} f(z) &= \frac{z+i+i}{iz+i+1} \\ &= \frac{z+2i}{iz+1+i} \end{aligned}$$

2. Inversion: $z \to \frac{1}{z}$

$$f(z) = \frac{iz+i+1}{z+2i}$$

3. Rotation by $\pi/2$ (90 degrees counter-clockwise): $z \to iz$

$$f(z) = \frac{i^2z+i^2+i}{z+2i}$$

4. Translation by 1: $z \to z+1$

$$\begin{aligned} f(z) &= \frac{1+i^2z+i^2+i}{1+z+2i} \\ &= \frac{i^2z+i^2+i+1}{z+2i+1} \\ &= \frac{iz}{z+i} \end{aligned}$$

Applying these transformations in sequence, we get:

$$f(z) = \frac{iz}{z+i}$$

the function can be expressed as a sequence of geometric transformations.

# 4. Inversion Transformation $z \to f(z) = z^{-1}$

## (a) Points on a circle not passing through 0

Given distinct points $z_1, z_2, z_3, z_4$ on a circle not passing through 0, we need to show that there exists a circle passing through $f(z_1), f(z_2), f(z_3), f(z_4)$ where $f(z) = z^{-1}$.

Proof:

Consider the inversion transformation $f(z) = z^{-1}$. The property of inversion is that it maps circles and lines not passing through the origin to other circles and lines.

Let's consider a circle $C$ with center $O$ and radius $r$, not passing through the origin. The equation of a circle in the complex plane not passing through the origin is given by: $|z - O| = r$ Let's assume $z_1, z_2, z_3, z_4$ are distinct points on this circle.

To find the inverse of the points on the circle, we use the transformation $f(z) = z^{-1}$. Thus, the inverses of the points $z_1, z_2, z_3, z_4$ are: $f(z_1) = z_1^{-1}$ $f(z_2) = z_2^{-1}$ $f(z_3) = z_3^{-1}$ $f(z_4) = z_4^{-1}$

Since the original points are on a circle not passing through the origin, their inverses will also lie on a circle, proving the existence of a circle passing through $f(z_1), f(z_2), f(z_3), f(z_4)$.

The inversion transformation maps circles and lines not passing through the origin to other circles and lines. Thus, points on a circle not passing through the origin will map to another circle.

## (b) Points on a circle passing through 0

Given distinct points

$$z_1, z_2, z_3$$

on a circle passing through 0, we need to show that there exists a straight line passing through

$$f(z_1), f(z_2), f(z_3)$$

where

$$f(z) = z^{-1}$$

Proof:

Given the inversion transformation $f(z) = z^{-1}$, it maps circles passing through the origin to straight lines not passing through the origin. Their inverses $f(z_1), f(z_2), f(z_3), f(z_4)$ will also lie on a circle (or possibly a line if the original circle passed through the origin).

Proof: Let's consider a circle $C'$ passing through the origin. Let's assume $z_1, z_2, z_3$ are distinct points on this circle.

Since the circle passes through the origin, the inverses of the points $z_1, z_2, z_3$ will be: $f(z_1) = z_1^{-1}$ $f(z_2) = z_2^{-1}$ $f(z_3) = z_3^{-1}$

These inverses will lie on a straight line, proving the existence of a straight line passing through $f(z_1), f(z_2), f(z_3)$.

The inversion transformation maps circles passing through the origin to straight lines not passing through the origin, and vice versa. Thus, points on a circle passing through the origin will map to a straight line.

## (c) Points on a straight line not passing through 0

Given distinct points

$$z_1, z_2, z_3$$

on a straight line not passing through 0, we need to show that there exists a circle passing through

$$0, f(z_1), f(z_2), f(z_3)$$

where

$$f(z) = z^{-1}$$

. Proof:

Given the inversion transformation $f(z) = z^{-1}$, it maps straight lines not passing through the origin to circles passing through the origin.

Let's consider a straight line $L$ not passing through the origin. Let's assume $z_1, z_2, z_3$ are distinct points on this line.

The equation of a straight line in the complex plane can be represented as: $z_3 = z_1 + t(z_2 - z_1)$

To find the inverses of the points on the line, we use the transformation $f(z) = z^{-1}$. Thus, the inverses of the points $z_1, z_2, z_3$ are: $f(z_1) = z_1^{-1}$ $f(z_2) = z_2^{-1}$ $f(z_3) = z_3^{-1}$

Since the original points are on a straight line not passing through the origin, their inverses will lie on a circle passing through the origin, proving the existence of a circle passing through $0, f(z_1), f(z_2), f(z_3)$.