W02. Invertibility One Step at a Time

Question 1

Discuss and critique the statement below. Go beyond ascertaining if the statement is true or false and provide a full discussion.

> Let M be a 2×2 matrix which is not invertible. Is it possible to change just one entry of M so that the resulting matrix is invertible?

Solution :
Suppose M is the singular matrix $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Matrix M is not invertible because all entries of the bottom row and second column are zero's.
By the (Invertible Matrix Theorem)(https://www.wikiwand.com/en/Invertible_matrix | Invertible Matrix Theorem), an square matrix A is Invertible or *non-singular* if there exists a matrix B such that,

$$AB = BA = I_n$$

If B exists, it is unique and is called the Inverse matrix of A, denoted A^{-1}.

Let's list all the equivalent statements given by the Invertible Matrix Theorem:

- There is an $n \times n$ matrix B such that $AB = I_n = BA$
- A is invertible, that is A has an inverse, is nonsingular, and is nondegenerate
- A is row-equivalent to the $n \times n$ Identity matrix I_n
- A is column-equivalent to the $n \times n$ Identity matrix I_n
- A has n pivot positions
- A has full rank; that is $\operatorname{rank}(A) = n$
- Based on the $\operatorname{rank}(A) = n$, the equation $Ax = 0$ has only the trivial solution $x = 0$ and the equation $Ax = b$ has exactly one solution for each b in K^n
- The kernel if A is trivial, that is, it contains only the null vector as an element, $\ker(A) = \{0\}$
- The columns of A are linearly independent
- The columns of A span K^n
- $\operatorname{Col}(A) = K^n$
- The columns of A form a basis of K^n

- The linear transformation mapping x to Ax is a bijection from K^n to K^n
- The number zero is not an Eigenvalue of A
- The transpose A^T is an invertible matrix (hence rows of A are linearly independent, span K^n, and form a basis of K^n)
- The matrix A can be expressed as a finite product of elementary matrices

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Conversion of the singular matrix M to become a non-singular matrix can be done by changing just one entry. Lets change a property of the matrix M that aligns with any one of the statement above in the Invertible Matrix Theorem. We know that the $\dim(M) = 2$ and the $\text{rank}(M) = 1$. Let's change the entry $M_{2,2}$ from its current entry 0 to 1.
Lets view the updated matrix M

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

By changing one entry of matrix M, we have also changed the $\text{rank}(M)$ from 1 to 2. Thus, because the $\dim(A) = 2$ and $\text{rank}(M) = 2$, the matrix M now is of *full rank*, and therefore is invertible (non-singular).
But if we have the zero matrix

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

denoted by Z.
We can not change one entry of the matrix Z to transform it into a invertible matrix. This is because $\text{rank}(Z) = 0$ while $\dim(Z) = 2$. We would need 2 entry changes to transform it into an invertible matrix.

Thus we can conclude given a non invertible (singular) square matrix D, it takes a $r = \dim(D) - \text{rank}(D)$ entry changes to convert a invertible (singular) matrix to an invertible (non-singular) matrix.

Question 2

Let W_3 be the matrix below. Ascertain is the matrix is invertible and explain your reasoning. If not, is it possible to change just one entry in W_3 so that the resulting matrix is invertible? What is the minimum number of entry changes needed to attain invertibility?

$$W_3 = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}$$

Solution :
To determine if an matrix is invertible or not, we can compare the properties of matrix W_3 to the properties listed above in the Invertible Matrix Theorem. $\text{rank}(W_3) = 2$, while the $\dim(W_3) = 3$. Thus from the conclusion of *Question 1*,

Conclusion from Solution of Question 1

given a non invertible (singular) square matrix D, it takes a $r = \dim(D) - \operatorname{rank}(D)$ entry changes to convert a invertible (singular) matrix to an invertible (non-singular) matrix.

We can, say that it takes

$$\begin{aligned} r &= \dim(W_3) - \operatorname{rank}(w_3) \\ 1 &= 3 - 2 \\ r &= 1 \end{aligned}$$

$r = 1$ entry changes to convert the noninvertible matrix W_3 to an invertible matrix.
Note: We need to change one entry in the linearly dependent column of W_3 to convert it an linearly independent column vector. So we get an $\operatorname{rank}(w_3) = 3$ instead of $\operatorname{rank}(w_3) = 3$.

The 3^{rd} column of W_3 is a linearly dependent vector. This is shown by applying the Gauss Jordan method and examining the reduced matrix of W_3. Shown below is the reduced matrix of W_3.

$$W_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \text{ reduced matrix of } W_3$$

Thus, the 3^{rd} column of W_3, can be described as the following linear combination,

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Furthermore, by examining the reduced matrix of W_3, it is shown that W_3 has $n = 2$ pivot positions (analogues to the number of linearly independent column vectors). Additionally, because the reduced matrix of W_3 is not an Identity matrix I_3, by the Invertible Matrix Theorem;

Invertible Matrix Theorem

An square matrix A is Invertible or *non-singular* if there exists a matrix B such that,

$$\begin{aligned} AB &= BA \\ &= I_n \end{aligned}$$

If B exists, it is unique and is called the Inverse matrix of A, denoted A^{-1}.

since a series of elementary row operations on the matrix W_3, denoted by the elementary matrix $B = e_1 \cdot e_2 \cdot \ \ldots \ \cdot e_k$ did not result in an identity matrix I_3, the inverse $B = A^{-1}$ does not exist.

Since the 3^{rd} column vector of matrix W_3 is linearly dependent, changing one entry of this column vector to get an linearly independent vector, will result in a invertible matrix W_3.
Lets change entry $W_{3,3} = 6$ to $W_{3,3} = 7$, the updated matrix is shown below.

$$W_3 = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 7 \end{pmatrix}$$

We now have $n = 3$ linearly independent vectors or $n = 3$ pivot positions or $\text{rank}(W_3) = 3 = \dim(W_3)$. Thus, updating one entry of the noninvertible matrix W_3 from $W_{3,3} = 6$ to $W_{3,3} = 7$, transform the matrix into an invertible non-singular matrix with the inverse denoted $(W_3)^{-1}$ shown below:

$$(W_3)^{-1} = \begin{pmatrix} \frac{-11}{6} & \frac{-1}{6} & 1 \\ \frac{13}{6} & \frac{5}{6} & -2 \\ \frac{-1}{2} & \frac{-1}{2} & 1 \end{pmatrix}$$

Question 3

Let W_4 be the matrix below. Ascertain if the matrix is invertible and explain your reasoning. If it is not, is it possible to change just one entry in W_4 so that the resulting matrix is invertible? What is the minimum number of entry changes needed to attain invertibility?

$$W_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

Solution :

Lets dissect the given matrix W_4.
Using the
Gauss Jordan Method (or Gaussian Elimination) we can obtain the $\text{rref}(W_4)$ (reduced row echelon form), as done for W_3 in *Question 2*. Examining the rref of W_4 allows us to determine the rank of the matrix, anaglous to pivot positions (or number of linearly independent column vectors). Shown below is the $\text{rref}(W_4)$

$$\text{rref}(W_4) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, after examination of $\text{rref}(W_4)$,

$$\begin{aligned} \text{rank}(W_4) &= 2 \\ \dim(W_4) &= 4 \\ 2 = 4 - 2 &\quad \text{by conclusion of Question 1} \end{aligned}$$

The given matrix W_4 without any entry changes is not invertible by the Invertible Matrix Theorem, since $\text{rank}(W_4) \neq \dim(W_4)$ and the $\text{rref}(W_4) \neq I_4$.
It is not possible to change just one entry in the matrix, there need to be 2 entry changes to convert the given noninvertible matrix W_4 to am invertible nonsingular matrix.
We can do so by changing the one entry of the 3^{rd} and one entry 4^{th} dependent column vectors.

The 3^{rd} and 4^{th} columns of W_4 are dependent vectors, because they can be expressed as linear combinations of the 1^{st} and the 2^{nd} columns of W_4 as shown below;

$$\begin{pmatrix} 3 \\ 7 \\ 11 \\ 15 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 5 \\ 9 \\ 13 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 6 \\ 10 \\ 14 \end{pmatrix} \qquad ; 3^{\text{rd}} \text{ column linear combo}$$

$$\begin{pmatrix} 4 \\ 8 \\ 12 \\ 16 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 5 \\ 9 \\ 13 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 6 \\ 10 \\ 14 \end{pmatrix} \qquad ; 4^{\text{th}} \text{ column linear combo}$$

Thus if changing the entries $W_{3,3}$ and $W_{4,4}$ from 11 and 16 to 1 and 1 respectively, will result in a invertible matrix. The updated matrix with the changes is shown below;

$$W_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 1 & 12 \\ 13 & 14 & 15 & 1 \end{pmatrix}$$

Taking the rref of this updated matrix results in an identity matrix, as shown below,

$$\text{rref}(W_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To conclude, the change of entries $W_{3,3}$ and $W_{4,4}$ from 11 and 16 to 1 and 1 respectively and taking the rref of the updated matrix results in an identity matrix, satisfying the Invertible Matrix Theorem. Therefore the updated matrix is invertible with the inverse denoted $(W_4)^{-1}$ is shown below;

$$(W_4)^{-1} = \begin{pmatrix} \frac{-28}{15} & \frac{11}{10} & \frac{-1}{10} & \frac{-2}{10} \\ \frac{37}{20} & \frac{-5}{4} & \frac{1}{5} & \frac{1}{5} \\ \\ \frac{-1}{10} & \frac{1}{5} & \frac{-1}{10} & 0 \\ \frac{-2}{15} & \frac{1}{5} & 0 & \frac{-1}{5} \end{pmatrix}$$