Ma221 typical problems fitting to the outcomes

1. **Derivation of solutions to first-order equations classified as linear, separable, or exact.**

1a

Consider the following autonomous first-order differential equation.

$$\frac{dy}{dx} = y^2 - 4y$$

Find the critical points and phase portrait of the given differential equation.

1b

Solve the given differential equation by separation of variables.

$$e^x y \frac{dy}{dx} = e^{-y} + e^{-4x - y}$$

1c

Find the general solution of the given differential equation.

$$y' = 2y + x^2 + 9$$

1d

Find the general solution of the given differential equation.

$$x^2 y' + x(x + 2)y = e^x$$

1e

Solve equation

$$\left(1 + \ln(x) + \frac{y}{x}\right) dx = (3 - \ln(x))\, dy$$

2. **Modeling some processes and phenomena**

2a Assume that in the absence of immigration and emigration, the growth of a country's population

$P(t)$ satisfies $dP/dt = kP$ for some constant $k > 0$. Determine a differential equation governing the growing population $P(t)$ of the country when individuals are allowed to immigrate into the country at a constant rate $r > 0$.

2b Suppose a student carrying a flu virus returns to an isolated college campus of N students. Determine a differential equation governing the number of students $x(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interactions between students with the flu and students who have not yet contracted it. Use $k > 0$ for the constant of proportionality.

2c Suppose that a large mixing tank initially holds 200 gallons of water in which 70 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 2 gal/min, and when the solution is well stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $A(t)$ in the tank at time $t > 0$.

2d In the theory of learning, the rate at which a subject is memorized is assumed to be proportional to the amount that is left to be memorized. Suppose M denotes the total amount of a subject to be memorized and $A(t)$ is the amount memorized in time $t > 0$. Determine a differential equation for the amount $A(t)$. Assume the constant of proportionality is $k > 0$.

2e A drug is infused into a patient's bloodstream at a constant rate of r grams per second. Simultaneously, the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at time t. Determine a differential equation for the amount $x(t)$. Use $k > 0$ for the constant of proportionality.

3. **Solving homogeneous linear ordinary differential equations using auxiliary algebraic equations**

3a Find the general solution of the given second-order differential equation $y'' - y' - 6y = 0$.
3b Solve differential equation $x^2y'' + 7xy' + 5y = 0$
3c Find the eigenvalues λ_n and eigenfunctions $y_n(x)$ for boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

4. **Solving non-homogeneous linear ordinary differential equations**

4a
Solve the given differential equation by undetermined coefficients.

$$y'' + 2y' = 2x + 3 - e^{-2x}$$

4b Solve differential equation $y'' + y = \csc(x)$.
4c Solve differential equation $xy'' - 5y' = x^5$

4d Solve differential equation $x^2y'' + xy' - y = \ln(x)$

5. **Derivation of the Fourier coefficients**

5a Find the Fourier series of f on the interval

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$$

5b Expand the given function in an appropriate cosine or sine series

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

5c Find sine and cosine Fourier series on interval (0,1) for f(x)=x(1- x/2) satisfying boundary conditions f(0)=0, f'(1)=0.

6. **Separation of Variables to solve the examples of linear second order partial differential equations (heat and wave equations)**

6a A rod of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the temperature $u(x, t)$. The ends are insulated, and there is heat transfer from the lateral surface of the rod into the surrounding medium held at temperature 50°. The initial temperature is 100° throughout.

$$k\frac{\partial^2 u}{\partial x^2} - h(u - 50) = \frac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0, \quad h > 0 \text{ a constant}$$

6b

Solve the wave equation $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \ 0 < x < L,\ t > 0$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$$
$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(L - x), \quad 0 < x < L$$

7. **Apply Laplace transform methods to differential equations and their systems**

7a Solve the initial-value problem

$$y'' - 9y' + 20y = 3t\,\mathcal{U}(t - 1), \quad y(0) = 0,\ y'(0) = 1$$

7b Solve the initial-value problem.

$$y' - 3y = 2t\,\delta(t - 4), \quad y(0) = 0$$

8. **Find the series solution to ODEs**
   8a. Solve Bessel equation $xy'' + y' + xy = 0$

9. **Solve homogeneous linear systems of ODEs**
   9a.

   Find the general solution of the given system.

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 6x + 5y$$