MA225 - Series Part 2

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Convergence tests - Divergence test

We determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums S_k. In practice, explicitly calculating this limit can be difficult or impossible.

Luckily, several tests exist that allow us to determine convergence or divergence for many types of series.

Definition (The divergence test)

If $\lim_{n\to\infty} a_n = L \neq 0$ *or* $\lim_{n\to\infty} a_n$ *diverges, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Why? The term a_k can be written $a_k = S_k - S_{k-1}$. Then

$$\begin{aligned}\lim_{k\to\infty} a_k &= \lim_{k\to\infty} (S_k - S_{k-1}) \\ &= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} \\ &= S - S = 0.\end{aligned}$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the n^{th} term $a_n \to 0$ as $n \to \infty$.

Convergence tests - Divergence test

Important note: What if $\lim_{n\to\infty} a_n = 0$? We cannot take a conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$ using the divergence test.

Example: $\lim_{n\to 0} \frac{1}{n} = 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Examples: For each of the following series, apply the divergence test.

1. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
3. $\sum_{n=1}^{\infty} e^{1/n^2}$

Convergence tests - Integral test

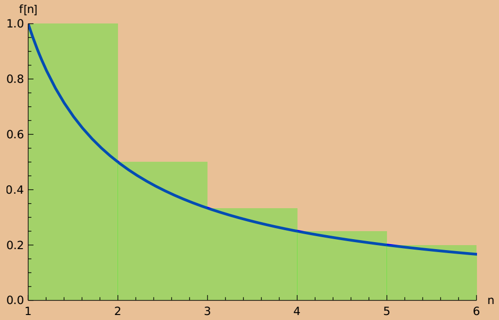
Motivation for the integral test:

Suppose we want to show that the harmonic series diverges:

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_1^{k+1} \frac{1}{x} dx &= \ln(x)\Big|_1^{k+1} \\ &= \ln(k+1) - \ln(1) \\ &= \ln(k+1). \end{aligned}$$

Since $\lim_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums S_k is unbounded and, consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

Convergence tests - Integral test



Each rectangle represents one term in the series: $a_1, a_2, .., a_5$. Function $f(x) = 1/x$ goes from 1 to 6.

$$\sum_{n=1}^{5} \frac{1}{n} > \int_1^6 \frac{1}{x} dx$$

Convergence tests - Integral test

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Assume $k \geq 2$. Then

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2} dx &= 1 - \left.\frac{1}{x}\right|_1^k \\ &= 1 - \frac{1}{k} + 1 \\ &= 2 - \frac{1}{k} < 2. \end{aligned}$$

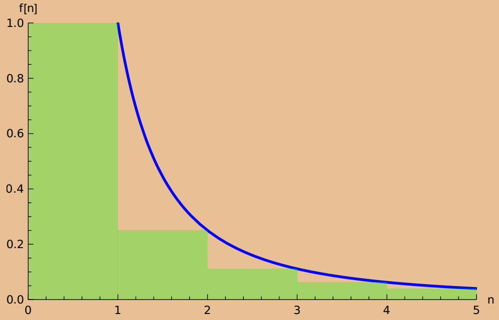
We conclude that the sequence of partial sums S_k is bounded. We also see that S_k is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2}\,, \quad k \geq 2\,.$$

Since S_k is increasing and bounded, it converges (recall week 1).

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Convergence tests - Integral test



The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area below $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

Convergence tests - Integral test

Property (The Integral Test)

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a $N \in \mathbb{N}$ such that the following conditions are satisfied:

1. *f is continuous,*
2. *f is decreasing, and*
3. *$f(n) = a_n$ for all $n \geq N$.*

Then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_N^{\infty} f(x)dx$$

both converge or both diverge.

Convergence tests - Integral test

Note: Convergence of $\int_N^\infty f(x)dx$ implies convergence of the related series $\sum_{n=1}^\infty a_n$, but it does not imply that the values are equal.

Example: For each of the following series, use the integral test to determine whether the series converges or diverges.

1. $\sum_{n=1}^\infty \frac{1}{n^3}$
2. $\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$

Example: Use the integral test to determine whether the series $\sum_{n=1}^\infty \frac{n}{3n^2+1}$ converges or diverges.

Convergence tests - Integral test

Solution (2): Compare

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \quad \text{vs.} \quad \int_1^{\infty} \frac{1}{\sqrt{2x-1}}\, dx.$$

Since

$$\int_1^{\infty} \frac{1}{\sqrt{2x-1}}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{\sqrt{2x-1}}\, dx$$

which evaluates to:

$$\lim_{b\to\infty} \sqrt{2x-1}\Big|_1^b = \lim_{b\to\infty} \left(\sqrt{2b-1} - 1\right) = \infty,$$

the integral diverges. Therefore, the series also diverges.

To evaluate the integral recall the change of variable $u = 2x + 1$.

p-series

A generalization of the Harmonic series.

Definition

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a p-series.

We already know the p-series converges if $p = 2$ and diverges if $p = 1$.

What about other values of p? Use Divergence or Integral tests.

Case $p \leq 0$:

- If $p < 0$, then $1/n^p \to \infty$.
- If $p = 0$, then $1/n^p \to 1$.

Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\,, \qquad \text{diverges if } p \leq 0\,.$$

Case $p > 0$:

$f(x) = 1/x^p$, $x \geq 1$ is a positive, continuous, decreasing function.
Therefore, for $p > 0$, we use the integral test.

If $p > 0$ we have

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty}\frac{1}{1-p}x^{1-p}\bigg|_1^b = \lim_{b\to\infty}\frac{1}{1-p}(b^{1-p}-1).$$

The second equality assumes $p \neq 1$ (this case we already know the series diverge).

Since

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that

$$\int_1^\infty \frac{1}{x^p}dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } p < 1\,. \end{cases}$$

By the integral test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \\ \text{diverges} & \text{if } p \leq 1 \,. \end{cases}$$

Digression: Remainder Estimate

Allows to estimate the sum of a series (which can be otherwise difficult).

Property (Remainder Estimate from the Integral Test)

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

1. *f is continuous,*
2. *f is decreasing, and*
3. *$f(n) = a_n$ for all integers $n \geq 1$.*

Let S_N be the N-th partial sum of $\sum_{n=1}^{\infty} a_n$.

Then, for all $N \in \mathbb{N}$,

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)dx\,.$$

Digression: Remainder Estimate

In other words, the remainder $R_N := \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the estimate:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx \,.$$

(this reads: the integral is either an overestimate or an underestimate of the error).

Example: Using the remainder property for $a_n = 1/n^2$ we immediately get for $N = 1$

$$1 + \int_2^{\infty} \frac{1}{x^2} dx < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx \,.$$

Digression: Remainder Estimate

Exercise: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

a. Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error. Hint: Using a calculator we can get $S_{10} \approx 1.1975$.

By remainder we have $R_N < \int_N^{\infty} \frac{1}{x^3} dx$, where $N = 10$. We have

$$\int_N^{\infty} \frac{1}{x^3}\, dx = \lim_{b\to\infty} \int_N^{b} \frac{1}{x^3}\, dx = \lim_{b\to\infty} -\frac{1}{2x^2}\Big|_N^b = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2N^2}\right) = \frac{1}{2N^2}.$$

Therefore, the error is given by

$$R_{10} < \frac{1}{2(10^2)} = 0.005.$$

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001 .

Comparison test

A method that allows to prove convergence or divergence of a suitable series by using our knowledge about other series, for example geometric or *p*-series.

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} .$$

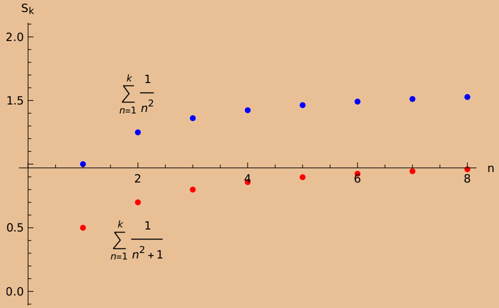
This series looks similar to the convergent p series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} .$$

The sequence of partial sums S_k for each series is increasing.

Comparison test

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274



Each of the partial sums for the given series is less than the corresponding partial sum for the converging p-series

Comparison test

Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all $n \in \mathbb{N}_+$, we have

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the series on the right converges, the sequence S_k is <u>bounded</u> above. Also, is easy to see that the sequence S_k is <u>increasing</u>. Since S_k is increasing and bounded above, S_k converges.

Comparison test

Now consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n - 1/2}$$

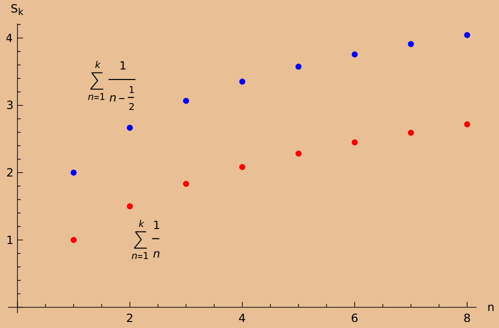
This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

The sequence of partial sums for each series is increasing.

Comparison test

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179



Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

Comparison test

Furthermore

$$\frac{1}{n-1/2} > \frac{1}{n} > 0$$

for every $n \in \mathbb{N}_+$. Therefore, we have

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n}$ is unbounded. Consequently, S_k is an unbounded sequence, and therefore diverges.

Comparison test

Property (Comparison Test)

1. *($C \Longrightarrow C$) Suppose there exists $N \in \mathbb{N}_+$ such that $0 \leq a_n \leq b_n$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *($D \Longrightarrow D$) Suppose there exists exists $N \in \mathbb{N}_+$ such that $a_n \geq b_n \geq 0$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Note: To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a bounding series. Since we know the convergence properties of geometric series and p-series, these series are often used.

Note 2:

In (1), we need to find an upper bound that converges, to determine convergence.

In (2), we need to find a lower bound that diverges, to determine divergence.

Comparison test

Example: For each of the following series, use the comparison test to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$
2. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$
3. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

Solution (2) Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$, it converges.
Also,

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

for every positive integer n. Therefore, $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

Comparison test tips

Most of the tips for choosing bounding sequences for the Squeeze Theorem also apply here.

- Ensure the terms a_n and b_n are non-negative.
- Common used series to compare are geometric series and p-series

Limit comparison test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult.

Motivation: Consider

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}\,.$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}\,.$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2 - 1} > \frac{1}{n^2}$$

for all $n \geq 2$.

Limit comparison test

We can instead compute

$$\lim_{n\to\infty} \frac{1/\left(n^2-1\right)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 \quad \text{“same order”}.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \quad \text{converges}.$$

This test only requires that the ratio of terms approaches a non-zero, finite limit.

This can be easier in situations where it's difficult to establish clear inequalities between terms but the general behavior of the series is understood.

Limit comparison test

Property (Limit comparison test)

Let $a_n \geq 0$, $b_n > 0$ for all $n \geq 1$.

1. *Same order of growth: If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.*
2. *a_n grows significantly slower: If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*
3. *b_n grows significantly slower: If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Note: Here b_n is not required to be $b_n \geq a_n$ nor $b_n \leq a_n$ like in the comparison test.

Note 2: if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information.

Limit comparison test

Example: For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
2. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$

Limit comparison test

Solution (1): Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

Solution (2): Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. We see that

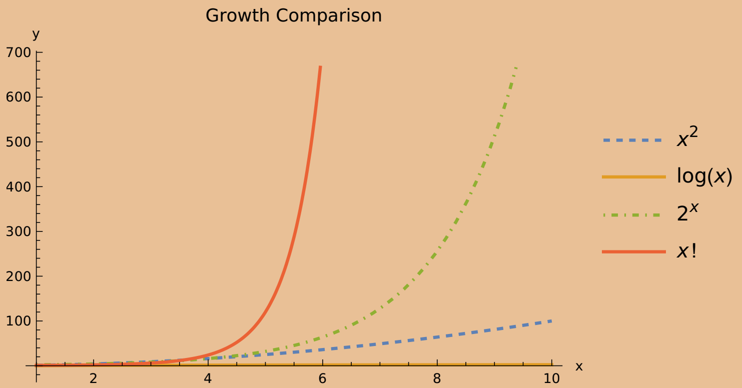
$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1 + \left(\frac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

Recall growth comparison



Alternating test

Definition (Alternating Series)

Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

$$\sum_{n-1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \dots$$

where $b_n \geq 0$ *for* $n \geq 1$.

Alternating test

Property (Alternating test)

An alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n$$

converges if

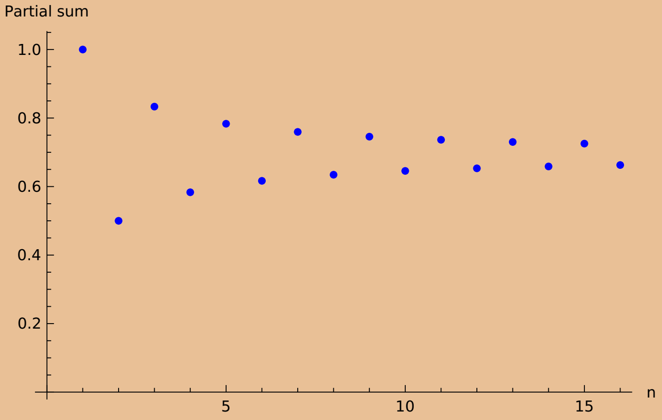
1. $0 \leq b_{n+1} \leq b_n$ *for all* $n \geq 1$ *(non-increasing) and*
2. $\lim_{n\to\infty} b_n = 0$.

Why? $\{S_{2k}\}$ is a decreasing sequence that is bounded below, so converges. Similarly, the even terms $\{S_{2k+1}\}$ form an increasing sequence that is bounded above so converges.

Alternating test

Example: The alternating harmonic series is a classical example of converging alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots .$$



Alternating test

Examples: For each of the following alternating series, determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$
3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$

Alternating test

Solution (1): Since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and $\frac{1}{n^2} \to 0$, the series converges.

Solution (2): Since $n/(n+1) \nrightarrow 0$ as $n \to \infty$, we cannot apply the alternating series test. Instead, we can use the divergence test. Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, the series diverges.

Solution (3): Is $\left\{\frac{n}{2^n}\right\}$ decreasing? What is $\lim_{n\to\infty} \frac{n}{2^n}$?

Ratio test

This test will be especially useful in the discussion of power series.

Consider a series $\sum_{n=1}^{\infty} a_n$.

We know that $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ vs. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

We now introduce the ratio test, which provides a way of measuring how fast the terms of a series approach zero.

Property (Ratio Test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

1. *If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely and thus converges.*
2. *If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $\rho = 1$, the test does not provide any information.*

Note: This test do not require us to find a comparable series b_n, like the comparison tests.

Note 2: The ratio test is particularly useful for series whose terms contain factorials or exponential, where the ratio of terms simplifies the expression.

Ratio test

Examples: For each of the following series, use the ratio test to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Exercise: Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges.

Solution (1): From the ratio test, we can see that

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}.$$

Since $(n+1)! = (n+1)\cdot n!$,

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

Since $\rho < 1$, the series converges.

Solution (2): We can see that

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.\end{aligned}$$

Since $\rho > 1$, the series diverges.

Absolute and Conditional Convergence

We explore the relationship between the convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$.

For example, consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. The series whose terms are the absolute value of these terms is the harmonic series, since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$.

Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits <u>conditional convergence</u>.

Now consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ exhibits <u>absolute convergence</u>.

Absolute and Conditional Convergence

Definition

A series $\sum_{n=1}^{\infty} a_n$ exhibits absolute convergence if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits conditional convergence if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Property (Absolute Convergence Implies Convergence)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Why? Let $a_n^+ := \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$, and $a_n^- := \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}$.

Then is immediate that $a_n^+ \leq |a_n|$ and $a_n^- \leq |a_n|$. Since $|a_n| = a_n^+ + a_n^-$ and $a_n = a_n^+ - a_n^-$ we have the result.

Absolute and Conditional Convergence

Examples: For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$. Hint: To check whether the series converges absolutely, we need to consider the series $\sum_{n=1}^{\infty} |\frac{(-1)^{n+1}}{3n+1}|$. If that one converges, then we also know that the original series converge. If not, we need to check if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ converges. So in total we may need to solve two series problems.
2. $\sum_{n=1}^{n=1} \frac{\cos(n)}{n^2}$

Absolute and Conditional Convergence

Solution (1): We can see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,

$$\lim_{n\to\infty} \frac{1/(3n+1)}{1/n} = \frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since decreasing

$$\frac{1}{3(n+1)+1} < \frac{1}{3n+1} \text{ and } \frac{1}{3n+1} \to 0,$$

the series converges by the alternating test. We can conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ converges conditionally.

Absolute and Conditional Convergence

Solution (2): Noting that $|\cos(n)| \leq 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right| \quad \text{vs.} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and

$$\frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$$

by the comparison test, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ converges, and therefore $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely and thus converges.

Quiz problem 3 sample

For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$
3. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$
4. $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Quiz problem 3 sample

To determine if $f(x) = \frac{1}{x\ln(x)}$ is decreasing over the interval $[2, \infty)$, we can examine the sign of its derivative.
First, let's find $f'(x)$. Given:

$$f(x) = \frac{1}{x\ln(x)}$$

The derivative can be found using the quotient rule:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Where:

$$u(x) = 1, \text{ thus } u'(x) = 0$$

$$v(x) = x\ln(x)$$

Using the product rule for $v'(x)$:

$$v'(x) = \ln(x) + x\frac{1}{x} = \ln(x) + 1$$

Quiz problem 3 sample

Using the above values in the quotient rule:

$$f'(x) = \frac{0 - \ln(x) - 1}{(x\ln(x))^2} = -\frac{\ln(x) + 1}{(x\ln(x))^2}$$

We can see that for $x \geq 2$:

- $\ln(x) > 0$
- $(x\ln(x))^2 > 0$ (square of a positive number)

Therefore, $f'(x)$ is negative for all $x \geq 2$ and thus the function $f(x) = \frac{1}{x\ln(x)}$ is <u>decreasing</u> on $[2, \infty)$.