math426_math_626_assignment_3_samir_banjara

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Question1: Calculate the SVD of the following matrix manually. Verify our answer by calculating the SVD in R.

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

- **Solution**:

First we compute the singular values σ_i by finding the eigenvalues of AA^T and A^TA,

$$AA^T = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$$

characteristic polynomial is $\det(AA^T - \lambda I) = (\lambda - 50)(\lambda - 200)$

and so eigenvalues are of A^TA and AA^T are

$$\lambda_1 = 200$$
$$\lambda_2 = 50$$
$$\lambda_3 = 0$$

and

$$\sigma_1 = 10\sqrt{2}$$
$$\sigma_2 = 5\sqrt{2}$$

and so,

$$\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$$

Note: singular values in Σ are square roots of eigenvalues of $AA^T or A^TA$ in descending order

We know that A^TA is symmetric, and so the eigenvectors will be orthogonal.

Now we find the right singular vectors (orthonormal set of eigenvectors that make up columns of V)

- for $\lambda = 200$ we have,

$$A^TA - 200I = \begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

A unit length vector in the null space is

$$v_1 = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix}$$

- for $\lambda = 50$ we have

$$A^T A - 50I = \begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

A unit length vector in the null space is

$$v_2 = \begin{bmatrix} -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix}$$

compute V by $\sigma v_i = A u_i$ or $v_i = \frac{1}{\sigma} A u_i$

$$v_1 = \frac{1}{10\sqrt{2}} \cdot \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$v_2 = \frac{1}{5\sqrt{2}} \cdot \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

Therfore,

$$V = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

The columns of the matrix U are the normalized unit length vectors (left singular vectors) - which is obtained by dividing each coordinate of the given vector its the magnitude.

thus,

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Resulting in the final SVD of: $A = U\Sigma V^*$

$$A = U\Sigma V^* = \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

Question2: Let A be a full ranked $m \times m$ matrix and let B be an $m \times n$ matrix. Show that $rank(AB) = rank(B)$.

- **Solution**

$range(A) = column(A) = span\{a_1, \cdots, a_n\}$

since, given $y \in span(AB)$

we can choose $x \in F$

and then, we have $y = (AB)x = A(bx) \in span(A)$

- or $AB = \{Ab_1, \cdots, Ab_n\}$ and $Ab_j \in span(A)$
- So, each of the columns of AB is contained in $span(A)$

That is, $rank(AB) \leq rank(A)$

Hence, $span(AB) \subseteq span(A)$

Because $span(AB) \subseteq span(A)$,

any basis for $span(AB)$ can be extended to a basis for $span(A)$

and so $dim(span(AB)) \leq dim(span(A))$

note that $null(B) \subseteq null(AB)$

since given $x \in null(AB)$

we have $(AB)x = A(Bx) = A0 = 0$

so that $x \in null(AB)$

Since $null(B) \subseteq null(AB)$

we have, $dim(null(B) \leq dim(null(AB)$

that is, $nullity(B) \leq nullity(AB)$

thus,

$$\begin{aligned} rank(AB) &= n - nullity(AB) \leq n - nullity(B) \\ &= rank(B) \end{aligned}$$

Different Solution

Because the multiplication of $A^T B$ results in a each of its column as linear combination of the rows of B and columns of A^T

$$\begin{aligned} rank(AB) &= rank((AB)^T) \\ &= rank(A^T B^T) \leq rank(B^T) \\ &= rank(B) \end{aligned}$$

Question3: Two matrices A and B in $\mathbb{C}^{m \times m}$ are unitarily similar if $A = QBQ^*$.

Show that A and B are unitarily similar then they have the same singular values.

- **solution**

By Theorem (5.4)

The nonzero singular values of A are square roots of the nonzero eigenvalues of AA^* (these matrices have the same nonzero eigenvalues)

Because A and B are unitarily similar,

for $A, B \in \mathbb{C}^{m \times m}$ we have,

$$A^* = A^-1 \qquad B^* = B^-1$$
$$AA^* = I \qquad BB^* = I$$

and

$$A = QBQ^*$$
$$B = QAQ^*$$

Now suppose $B = U\Sigma V^*$ is a SVD for B, with U and V unitary and Σ a diagonal matrix wiht decresing σ_i values

becasue, $A = QBQ^*$ and $B = QAQ^*$ with Q unitary, then

$$\begin{aligned} A &= (U\Sigma V^*)^* (U\Sigma V^*) \\ &= V\Sigma^* U^* U\Sigma V^* \\ &= VV^*(\Sigma^*\Sigma) \\ or &= (QU)\Sigma(V^*Q^*) \end{aligned}$$

And

$$\begin{aligned} B &= (U\Sigma V^*)^* (U\Sigma V^*) \\ &= V\Sigma^* U^* U\Sigma V^* \\ &= VV^*(\Sigma^*\Sigma) \\ or &= (QU)\Sigma(V^*Q^*) \end{aligned}$$

we are factorizing A using the the singular values of B The product of unitary matrices is unitary thus, these are SVD's for A and B

We see that AA^* is similar to $\Sigma\Sigma^*$ & hence has the same n eigenvalues, also BB^* is similar to $\Sigma\Sigma^*$ & Hence has same n eigenvalues, AA^* and BB^* have the same eigenvalues(singular values).

Unitarily similar implies same eigenvalues or singular values.

- Show that the converse is not true by providing a counter example (Hint: Start with the matrix Σ and construct A and B, one of which is symmetric and the other is not).

Show that A and B are unitarily similar then they have the same singular values. converse: If A and B have the same singular values then A B are unitarily similar.

We need to show that A and B are not unitarily similar.

Let B non-symmetric (non square matrix), In the reduced SVD of B,

$$B = \hat{U}\hat{\Sigma}V^*$$

The singular values are in a square diagonal matrix $\hat{\Sigma}$.

Construct a square matrix A with the same singular values as B by multiplying $\hat{\Sigma}$ by unitary U and V. (Factorization of A using B)

However no unitary matrix Q exists such that $A = QBQ^*$ because A and Q are square and B is not square.