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Example 2.1 If $P=X^2+1$, then $P=(X-i)(X+i)$. The zeroes/roots are $z_1=i$ and $z_2=-i$, and both have multiplicity 1
.

If $P=\left(X^2+1\right)^2=X^4+2X^2+1$, then $P=(X-i)^2(X+i)^2$. The roots are $z_1=i$ and $z_2=-i$, and both have multiplicity 2
.

If $P=X(X+1)(X-i)^2$, then the roots are $z_1=0, z_2=-1$, and $z_3=i$. The multiplicities are $m_1=m_2=1$ and $m_3=2$, and the sum of the multiplicities is $1+1+2=4$, the degree of the polynomial.

A rational function is a quotient of two polynomial functions:

$$f(z)=\frac{P(z)}{Q(z)}.$$

If z_0 is a zero of Q of multiplicity m_Q and a zero of P of multiplicity m_P, then $P(z)=(z-z_0)^{m_P}P_1(z), Q(z)=(z-z_0)^{m_Q}Q_1(z)$
, and

$$f(z)=\frac{(z-z_0)^{m_P}P_1(z)}{(z-z_0)^{m_Q}Q_1(z)}=(z-z_0)^{m_P-m_Q}\frac{P_1(z)}{Q_1(z)}.$$

Since $P_1(z_0)\neq 0\neq Q_1(z_0)$, it follows that, after simplifying the common powers of $z-z_0$ from P and Q, the function f can be defined at z_0 if and only if $m_P\geqslant m_Q$.

Let $f(z)=\frac{P(z)}{Q(z)}$ be a rational function, such that P and Q do not have any common zeroes - we will call such a rational function reduced and unless explicitly specified otherwise, all the rational functions we will consider are reduced. A zero z_0 of P is called a zero of f, and the multiplicity of z_0 as a zero of f is the multiplicity of z_0 as a zero of P. A zero z_0 of Q is called a pole of f, and the multiplicity of z_0 as a zero of Q is called the order of the pole z_0 of f. The domain of $f=P/Q$ is $\mathbb{C}$ minus the set of all its poles - the entire plane with a finite number reduced rational function zero pole order of a pole of points removed.

EXAMPLE 2.2. Let

$$f(z)=\frac{z(z+1)(z-i)^2}{(z+i)^2(z-1)}$$

The zeroes of f are $0,-1$, and i, with multiplicities 1,1 , and 2 , respectively. The poles of f are $-i$ and 1 , with orders 2 and 1 , respectively. The domain of f is $\mathbb{C}\backslash\{1,-i\}$.

EXAMPLE 2.3 Let $f_1(z)=z+a, f_2(z)=bz+c$, and $f_3(z)=1/z$. Then

$$\begin{aligned}(f_2\circ f_3\circ f_1)(z)&=(f_2\circ f_3)(z+a)=f_2\left(\frac{1}{z+a}\right)=\frac{b}{z+a}+c\\&=\frac{cz+(ac+b)}{z+a}\\(f_1\circ f_3\circ f_2)(z)&=(f_1\circ f_3)(bz+c)=f_1\left(\frac{1}{bz+c}\right)=\frac{1}{bz+c}+a\\&=\frac{(ab)z+(ac+1)}{bz+c}.\end{aligned}$$

In both cases the end result is a rational function, with polynomials of degree 1 as numerators and denominators.

EXAMPLE 2.4 The unique Möbius transformation that sends the triple $(-1,0,1)$ to $(0,1,i)$ is the solution of the equation

$$\begin{aligned}&\frac{f(z)-1}{f(z)-i}:\frac{0-1}{0-i}=\frac{z-0}{z-1}:\frac{-1-0}{-1-1}\Longleftrightarrow\\&\frac{f(z)-1}{f(z)-i}\cdot i=\frac{z}{z-1}\cdot 2\Longleftrightarrow f(z)=\frac{-z-1}{(2i+1)z-1}.\end{aligned}$$

EXAMPLE 2.5 To find the unique Möbius transformation that sends the triple $(0,1,i)$ to $(i,1,\infty)$, we take advantage of the presence of ∞ in the second triple and use the equality of cross-ratios

$$(f(z),\infty;i,1)=(z,i;0,1)$$

Then $f(z)$ is the solution of the equation

$$\begin{aligned}&\frac{f(z)-i}{f(z)-1}:\frac{\infty-i}{\infty-1}=\frac{z-0}{z-1}:\frac{i-0}{i-1}\Longleftrightarrow\\&\frac{f(z)-i}{f(z)-1}=\frac{(i-1)z}{i(z-1)}\Longleftrightarrow f(z)=\frac{-iz+1}{z-i}\end{aligned}$$

To understand the geometric transformations that build f, we use

$$\begin{aligned}f(z)&=\frac{-iz+1}{z-i}=(-i)\frac{z+i}{z-i}=(-i)\left(1+\frac{2i}{z-i}\right)=-i+\frac{2}{z-i}\\&=2\left(\frac{1}{z-i}-i\right)+i.\end{aligned}$$

Therefore f is the translation by $-i$, followed by an inversion, and then by a homothety of ratio 2 with center i.

EXAMPLE 2.6 Let $z=1$ and $n=5$; we want to find the complex numbers ζ such that $\zeta^5=1$. Since de Moivre's formula (1.4) provides a nice way of computing powers of complex numbers in trigonometric form, we will work with trigonometric forms of complex numbers. We therefore look for $\zeta=\rho(\cos\alpha+i\sin\alpha)$ such that

$$1=\cos 0+i\sin 0=(\rho(\cos\alpha+i\sin\alpha))^5=\rho^5(\cos(5\alpha)+i\sin(5\alpha)).$$

Equating the moduli and the argument sets we obtain

$$\rho^5=1\quad\text{and}\quad 5\alpha=2k\pi,\ \text{for some } k\in\mathbb{Z},$$

from which we conclude that

$$\rho=1\quad\text{and}\quad\alpha=\frac{2k\pi}{5},\ \text{for some } k\in\mathbb{Z}.$$

The modulus is therefore equal to 1 for all solutions of $\zeta^5=1$. Different values k_1 and k_2 of k will result in the same value for ζ if and only if the corresponding values of α differ by an integer multiple of 2π, and that is the case if and only if k_1 and k_2 differ by a multiple of 5 . As a consequence, there are exactly five different values for ζ, corresponding to $k=0,1,2,3,4$. The solutions of $\zeta^5=1$ are therefore given by

$$\begin{aligned}\zeta_0&=\cos 0+i\sin 0=1\\\zeta_1&=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}:=\omega\\\zeta_2&=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}=\omega^2\\\zeta_3&=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=\omega^3\\\zeta_4&=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=\omega^4\end{aligned}$$

All five are on the unit circle and the angles between consecutive ones are all equal to $\frac{2\pi}{5}$; therefore $\zeta_0=1,\zeta_1=\omega,\zeta_2=\omega^2,\zeta_3=\omega^3$ and $\zeta_4=\omega^4$ are the vertices of a regular pentagon.

EXAMPLE 2.7 The 4^{th} -roots of unity are

$$\begin{aligned}1&=\cos\left(0\cdot\frac{2\pi}{4}\right)+i\sin\left(0\cdot\frac{2\pi}{4}\right)=\cos 0+i\sin 0\\\omega&=\cos\left(1\cdot\frac{2\pi}{4}\right)+i\sin\left(1\cdot\frac{2\pi}{4}\right)=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i\\\omega^2&=\cos\left(2\cdot\frac{2\pi}{4}\right)+i\sin\left(2\cdot\frac{2\pi}{4}\right)=\cos\pi+i\sin\pi=-1\\\omega^3&=\cos\left(3\cdot\frac{2\pi}{4}\right)+i\sin\left(3\cdot\frac{2\pi}{4}\right)=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i\end{aligned}$$

and they are the roots of

$$X^4-1=\left(X^2-1\right)\left(X^2+1\right)=(X-1)(X+1)(X-i)(X+i).$$

EXAMPLE 2.8 We can now find the roots of X^4+1; those are the solutions of $\zeta^4=-1=\cos\pi+i\sin\pi$; we will work with the trigonometric form and look for $\zeta=\rho(\cos\alpha+i\sin\alpha)$ such that

$$\cos\pi+i\sin\pi=(\rho(\cos\alpha+i\sin\alpha))^4=\rho^4(\cos(4\alpha)+i\sin(4\alpha)).$$

Equating the moduli and the argument sets we obtain

$$\rho^4=1 \quad \text{and} \quad 4\alpha=\pi+2k\pi, \text{ for some } k\in\mathbb{Z},$$

from which we conclude that

$$\rho=1 \quad \text{and} \quad \alpha=\frac{\pi}{4}+\frac{2k\pi}{4}, \text{ for some } k\in\mathbb{Z}.$$

The values of α for $k=0,1,2,3,4$ are given by

$$\frac{\pi}{4}, \frac{\pi}{4}+\frac{\pi}{2}, \frac{\pi}{4}+\pi, \frac{\pi}{4}+\frac{3\pi}{2}, \frac{\pi}{4}+2\pi$$

and we note that, in general, for every integer k, the values for k and for $k+4$ differ by 2π, hence they correspond to the same argument set, hence to the same complex number ζ. There are exactly 4 complex numbers ζ such that $\zeta^4=-1$; those four numbers have the same modulus, 1 , and their argument sets are spaced in increments of $\pi/2$. Let

$$\zeta_0=\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)$$

be the first of those solutions and

$$\omega=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)=i.$$

Then the four solutions of $\zeta^4=-1$ are

$$\zeta_0, \zeta_0\omega, \zeta_0\omega^2, \zeta_0\omega^3$$

they correspond to ζ_0 and the images of successive counterclockwise rotations of angle $\frac{\pi}{2}$ about the origin, until, after four such rotations, we get back to ζ_0. Therefore the solutions of $\zeta^4=-1$ are the vertices of a square centered at the origin. That square is obtained from the square corresponding to 4^{th}-roots of unity, rotated counterclockwise by an angle $\frac{\pi}{4}=\frac{1}{4}\operatorname{Arg}(-1)$.

EXAMPLE 2.9 Let $z=1+i$; we want to find the complex numbers ζ such that $\zeta^6=z$. We will work with trigonometric forms of complex numbers. Then

$$z=1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

and we look for $\zeta=\rho(\cos\alpha+i\sin\alpha)$ such that

$$\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=(\rho(\cos\alpha+i\sin\alpha))^6=\rho^6(\cos(6\alpha)+i\sin(6\alpha)).$$

Equating the moduli and the argument sets we obtain

$$\rho^6=\sqrt{2} \quad \text{and} \quad 6\alpha=\frac{\pi}{4}+2k\pi, \text{ for some } k\in\mathbb{Z},$$

from which we conclude that

$$\rho=(\sqrt{2})^{1/6} \quad \text{and} \quad \alpha=\frac{\pi}{24}+\frac{2k\pi}{6}, \text{ for some } k\in\mathbb{Z}.$$

The values of α for $k=0,1,2,3,4,5,6$ are given by

$$\frac{\pi}{24}, \frac{\pi}{24}+\frac{\pi}{3}, \frac{\pi}{24}+\frac{2\pi}{3}, \frac{\pi}{24}+\pi, \frac{\pi}{24}+\frac{4\pi}{3}, \frac{\pi}{24}+\frac{5\pi}{3}, \frac{\pi}{24}+2\pi$$

and we note that, in general, for every integer k, the values for k and for $k+6$ differ by 2π, hence they correspond to the same argument set, hence to the same complex number ζ. There are exactly 6 complex numbers ζ such that $\zeta^6 = 1+i$; those six numbers have the same modulus, $(\sqrt{2})^{1/6}$, and their argument sets are spaced in increments of $\pi/3$. Let

$$\zeta_0 = (\sqrt{2})^{1/6}\left(\cos\left(\frac{\pi}{24}\right) + i\sin\left(\frac{\pi}{24}\right)\right)$$

be the first of those solutions and

$$\omega = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

Then the six solutions of $\zeta^6 = 1+i$ are

$$\zeta_0, \zeta_0\omega, \zeta_0\omega^2, \zeta_0\omega^3, \zeta_0\omega^4, \text{ and } \zeta_0\omega^5$$

they correspond to ζ_0 and the images of successive counterclockwise rotations of angle $\frac{\pi}{3}$ about the origin, until, after six such rotations, we get back to ζ_0. Therefore the solutions of $\zeta^6 = 1+i$ are the vertices of a regular hexagon centered at the origin. That hexagon is obtained from the regular hexagon corresponding to 6^{th}-roots of unity, rotated counterclockwise by an angle $\frac{\pi}{24} = \frac{1}{6}\mathrm{Arg}(1+i)$ and rescaled by a factor of $(\sqrt{2})^{1/6} = |1+i|^{1/6}$.

EXAMPLE 2.10 For $w = z^2$ we have

$$u + iv = (x+iy)^2 = x^2 - y^2 + i(2xy),$$

hence $u(x,y) = x^2 + y^2$ and $v(x,y) = 2xy$.

For $w = \bar{z}$ we have

$$u + iv = \overline{x+iy} = x - iy,$$

hence $u(x,y) = x$ and $v(x,y) = -y$.

For $w = |z|$ we have

$$u + iv = |x+iy| = \sqrt{x^2+y^2},$$

hence $u(x,y) = \sqrt{x^2+y^2}$ and $v(x,y) = 0$.