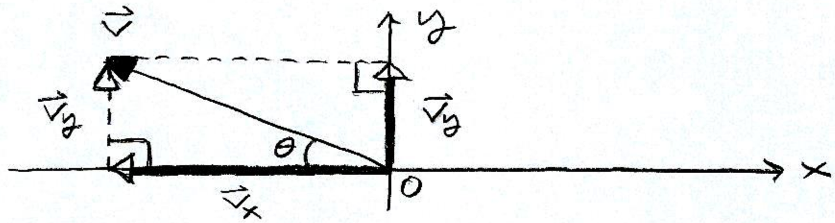
Dealing with Vector Components in Negative Directions:

- Given $\vec{v}$:



- $\vec{v}$ has generic Cartesian form:

$$\vec{v} \equiv v_x \hat{x} + v_y \hat{y} \quad \longleftarrow \quad \left(\text{notice no minus signs here; scalar components } v_x, v_y \text{ contain that info!}\right) \tag{51}$$

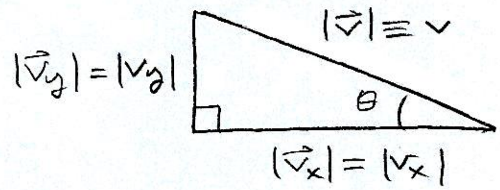
- with vector components:

$$\vec{v}_x \equiv v_x \hat{x} \tag{52}$$

$$\vec{v}_y \equiv v_y \hat{y} \tag{53}$$

$$\left(\text{so } \vec{v} = \vec{v}_x + \vec{v}_y\right)$$

- make a magnitude triangle with $|\vec{v}|$ as hypotenuse:



$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} \tag{54}$$

$$|\vec{v}_x| = |v_x \hat{x}| = |v_x|\,|\underbrace{\hat{x}}_{1}| = |v_x| \tag{55}$$

$$|\vec{v}_y| = |v_y \hat{y}| = |v_y|\,|\underbrace{\hat{y}}_{1}| = |v_y| \tag{56}$$

- Fact: for any real scalar c:

$$\boxed{c = \mathrm{sgn}(c)\,|c| = \pm|c|} \qquad \mathrm{sgn}(c) \equiv \begin{cases} +1; & c>0 \\ 0; & c=0 \\ -1; & c<0 \end{cases} \tag{57}$$

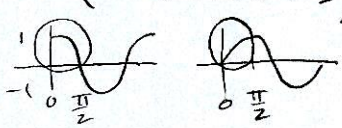
- So write v_x, v_y like $c = \pm|c|$, using vector component arrows to get sign!

$$\left.\begin{aligned} v_x &= -|v_x| \quad \left(\text{because } \longleftarrow \vec{v}_x \text{ points in } -x \text{ direction}\right) \\ v_y &= +|v_y| \quad \left(\text{because } \uparrow \vec{v}_y \text{ points in } +y \text{ direction}\right) \end{aligned}\right\} \tag{58}$$

- To get $|v_x|, |v_y|$, use magnitude triangle:

(don't have to worry about sign here: θ is acute $\theta \in [0, 90°]$, and all sides are magnitudes, so positive) →

$$\left.\begin{aligned} |v_x| &= v\cos(\theta) \\ |v_y| &= v\sin(\theta) \end{aligned}\right\} \quad \left(\text{for acute } \theta,\ \cos(\theta) \in [0,1],\ \sin(\theta) \in [0,1]\right) \tag{59}$$



- plug (59) into (58):

$$\left.\begin{aligned} v_x &= -v\cos(\theta) \\ v_y &= +v\sin(\theta) \end{aligned}\right\} \tag{60}$$

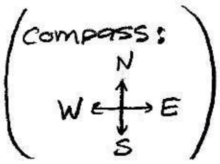
- For full vector answer, plug (60) into (51):

$$\vec{v} = v_x \hat{x} + v_y \hat{y}$$

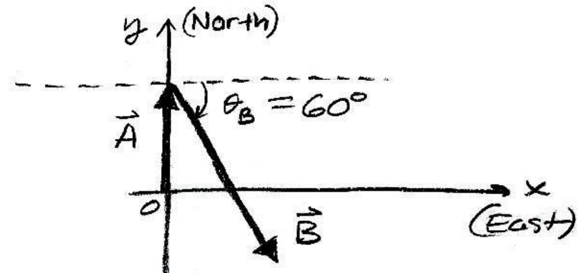
$$\boxed{\begin{aligned} \vec{v} &= -v\cos(\theta)\,\hat{x} + v\sin(\theta)\,\hat{y} \\ &= v\left(-\cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\right) \end{aligned}} \tag{61}$$

• **ex** {You drive 22.0 [km] north, then you suddenly turn and drive in a direction 60.0° south of east for 47.0 [km]. What is your total displacement?}

• Diagram:

(compass:


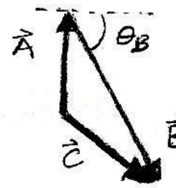
)



• Interpret:
"60° south of east"
means
"60° below the +x axis" (1)

• Asks for total displacement, so use definition of vector addition:

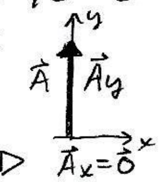
$$\vec{C} \equiv \vec{A} + \vec{B} \qquad (2)$$



• And write each vector generically:

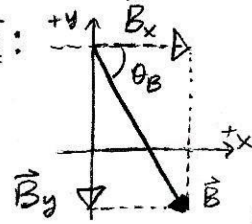
$$\vec{A} = A_x \hat{x} + A_y \hat{y} \qquad (3)$$
$$\vec{B} = B_x \hat{x} + B_y \hat{y} \qquad (4)$$
$$\vec{C} = C_x \hat{x} + C_y \hat{y} \qquad (5)$$

• Analyze each separately:

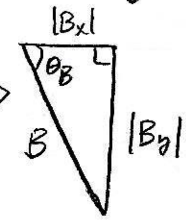
$\vec{A}$:


$\Rightarrow$ $\vec{A}_x = \vec{0}$

$A_x = 0$
$A_y = +|A_y|$

$\vec{B}$:


$B_x = +|B_x|$
$B_y = -|B_y|$ $\Rightarrow$



$|B_x| = B\cos(\theta_B)$
$|B_y| = B\sin(\theta_B)$
$B \equiv |\vec{B}|$ (6)

(notice: we simplified the trig problem to include lengths only)

• Use component addition:

(• CAUTION:
• Do not confuse B_y for $|B_y|$
• Note the steps)

$$C_x = A_x + B_x = 0 + B\cos(\theta_B) = B\cos(\theta_B) \qquad (7)$$
$$C_y = A_y + B_y = |A_y| - |B_y| = |A_y| - B\sin(\theta_B) \qquad (8)$$

• Plug (7,8) into (5) for symbolic answers:

• Cartesian Method:
• Polar Method:

$$\boxed{\begin{aligned} &\vec{C} = B\cos(\theta_B)\,\hat{x} + \left[|A_y| - B\sin(\theta_B)\right]\hat{y} \\ &\text{OR} \\ &C \equiv |\vec{C}| = \sqrt{C_x^2 + C_y^2} = \sqrt{B^2\cos^2(\theta_B) + \left[|A_y| - B\sin(\theta_B)\right]^2} \\ &\theta_C = \tan^{-1}\left(\frac{C_y}{C_x}\right) = \tan^{-1}\left(\frac{|A_y| - B\sin(\theta_B)}{B\cos(\theta_B)}\right) \quad \text{where } \theta_C \text{ is between } \vec{C} \text{ and the } +x \text{ axis.} \end{aligned}} \qquad (9)$$

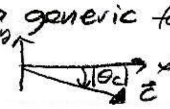
• For numerical answers state the givens: $A \equiv |\vec{A}| = 22.0$ [km], $B \equiv |\vec{B}| = 47.0$ [km], $\theta_B = 60.0°$ (10)
$|A_y| = 22.0$ [km]

• plug them into symbolic formulas:

(Cartesian): $\boxed{\vec{C} = C_x\hat{x} + C_y\hat{y} \quad \text{where} \begin{cases} C_x = B\cos(\theta_B) = 23.5 \text{ [km]} \\ C_y = |A_y| - B\sin(\theta_B) = -18.7 \text{ [km]} \end{cases}}$

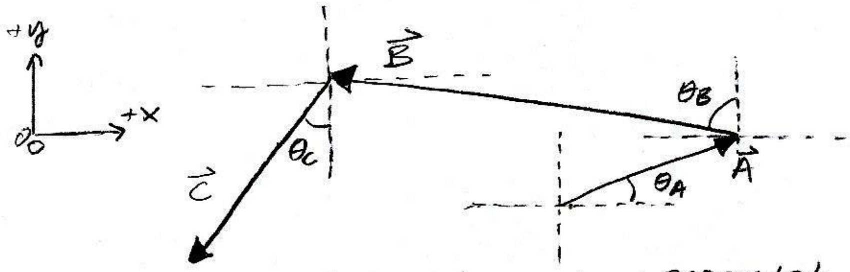
(Polar): $\boxed{\begin{aligned} &C \equiv |\vec{C}| = \sqrt{C_x^2 + C_y^2} = 30.0 \text{ [km]} \\ &\theta_C = \tan^{-1}\left(\frac{C_y}{C_x}\right) = -38.5° = +38.5° \text{ below the } +x \text{ axis (south of east)} \end{aligned}}$

• Note: • use of the formula $\theta_C = \tan^{-1}\left(\frac{C_y}{C_x}\right)$ assumes a generic form

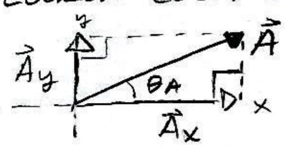

(11)
• so then if θ_C comes out negative, that means:


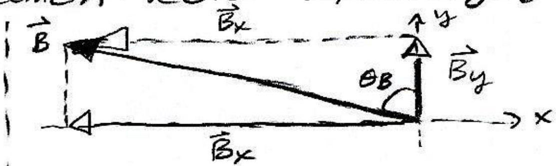
• both figures are correct

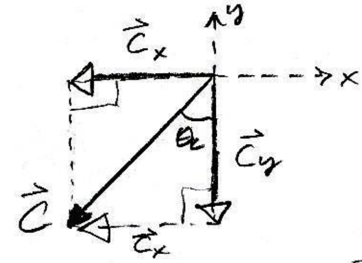
• ex: { Given the three displacements $\vec{A}, \vec{B}, \vec{C}$ and angles $\theta_A, \theta_B, \theta_C$ shown below, what is the total displacement $\vec{D}$? }



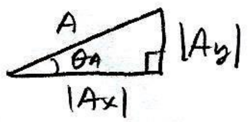
• Look at each displacement vector separately:

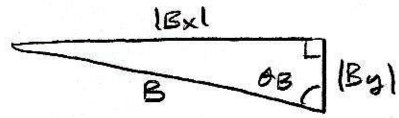


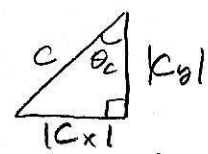




$$A_x = +|A_x| \qquad B_x = -|B_x| \qquad C_x = -|C_x|$$
$$A_y = +|A_y| \qquad B_y = +|B_y| \qquad C_y = -|C_y| \qquad (1)$$







$$|A_x| = A\cos(\theta_A) \qquad |B_x| = B\sin(\theta_B) \qquad |C_x| = C\sin(\theta_C)$$
$$|A_y| = A\sin(\theta_A) \qquad |B_y| = B\cos(\theta_B) \qquad |C_y| = C\cos(\theta_C) \qquad (2)$$

• Put (2) into (1):

$$A_x = +A\cos(\theta_A) \qquad B_x = -B\sin(\theta_B) \qquad C_x = -C\sin(\theta_C)$$
$$A_y = +A\sin(\theta_A) \qquad B_y = +B\cos(\theta_B) \qquad C_y = -C\cos(\theta_C) \qquad (3)$$

• State answer generically by vector addition definition:

$$\vec{D} \equiv D_x\hat{x} + D_y\hat{y} \text{, where } \begin{cases} D_x = A_x + B_x + C_x \\ D_y = A_y + B_y + C_y \end{cases} \qquad (4)$$

• Plug (3) into (4):

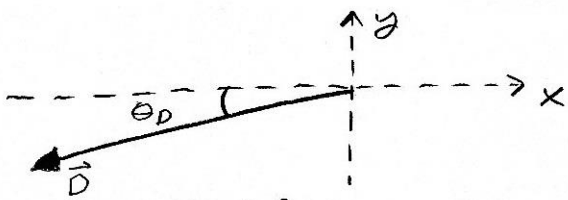
$$\vec{D} = D_x\hat{x} + D_y\hat{y} \text{ where } \begin{cases} D_x = A\cos(\theta_A) - B\sin(\theta_B) - C\sin(\theta_C) \\ D_y = A\sin(\theta_A) + B\cos(\theta_B) - C\cos(\theta_C) \end{cases} \qquad (5)$$

$A \equiv |\vec{A}|$, $B \equiv |\vec{B}|$, $C \equiv |\vec{C}|$

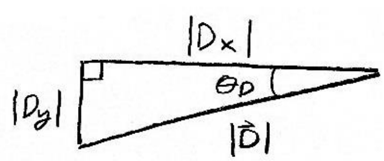
and in polar form:

$$D \equiv \sqrt{D_x^2 + D_y^2} \text{ where , and } \theta_D = \tan^{-1}\left(\frac{|D_y|}{|D_x|}\right) \text{ also where} \qquad (6)$$

where θ_D is: angle between $\vec{D}$ and $-x$ axis.



← resultant from given diagram

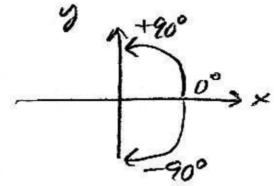


Warning About $\tan^{-1}(x)$:

$\boxed{\text{It doesn't really work!}}$

- The range is:

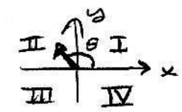
$$-90^\circ < \tan^{-1}(x) < 90^\circ \qquad (62)$$



- But that only works for half of all angles!
- That's why books just say:

$$\boxed{\tan(\theta) = \frac{A_y}{A_x}} \leftarrow \text{(not wrong!)} \qquad (63)$$

  - From sign information in A_y and A_x, we can then determine the proper quadrant of θ



- The solution is a two-argument function: (for output in radians):

$$\boxed{\begin{aligned} \operatorname{atan2}(y,x) \equiv &\begin{cases} \operatorname{atan}\left(\frac{y}{x}\right) & \text{if } x>0 \\ \operatorname{atan}\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \text{ and } y \geq 0 \\ \operatorname{atan}\left(\frac{y}{x}\right)-\pi & \text{if } x<0 \text{ and } y<0 \\ +\frac{\pi}{2} & \text{if } x=0 \text{ and } y>0 \\ -\frac{\pi}{2} & \text{if } x=0 \text{ and } y<0 \\ \text{undefined} & \text{if } x=0 \text{ and } y=0 \end{cases} \end{aligned}} \qquad (64)$$

where $\operatorname{atan}\left(\frac{y}{x}\right) \equiv \arctan\left(\frac{y}{x}\right) \equiv \tan^{-1}\left(\frac{y}{x}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the typical inverse tangent function you'd find on a calculator, and

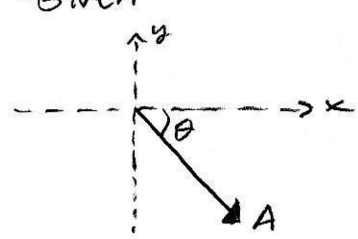
$$\operatorname{atan2}(y,x) \in (-\pi, \pi] \qquad (65)$$

  - to get output on $[0, 2\pi)$, just add 2π to negative results:

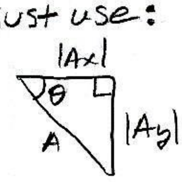
$$\boxed{\operatorname{atan2}_{2\pi}(y,x) \equiv \begin{cases} \operatorname{atan2}(y,x) & \text{if } \operatorname{atan2}(y,x) \geq 0 \\ \operatorname{atan2}(y,x)+2\pi & \text{if } \operatorname{atan2}(y,x)<0 \end{cases}} \qquad (66)$$

- But we <u>won't use</u> (64) or (66)
- Instead, just use (63) and the usual $\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$ aided by inspection of signs of vector components

ex: • Given



• just use:



$$\Rightarrow \boxed{\underbrace{\theta = \tan^{-1}\left(\frac{|A_y|}{|A_x|}\right)}_{\text{guarantees } 0^\circ \leq \theta < 90^\circ}}$$

• And specify quadrant:

"In quadrant IV"

or

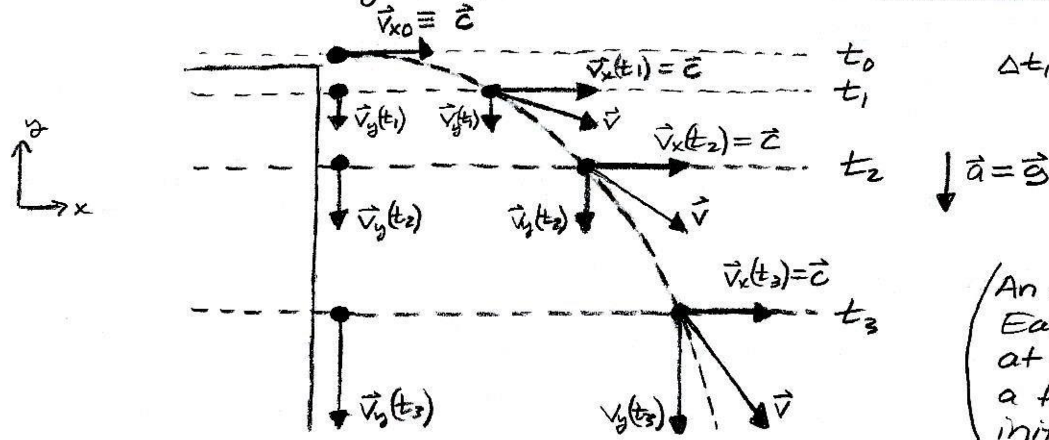
"below the +x axis" (best)

because $\operatorname{sgn}(A_x) = +1$

$\operatorname{sgn}(A_y) = -1$

Projectile Motion in Two Dimensions:

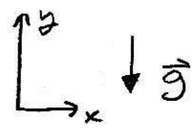
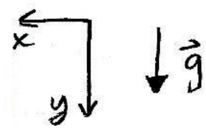
- x-component and y-component motion are <u>independent</u>



$\Delta t_{1,0} = \Delta t_{2,1} = \Delta t_{3,2}$

(An object dropped near Earth's surface falls at the same rate as a falling object with an initial horizontal velocity)

- Since no horizontal $\vec{a}$ here, then $\underline{\vec{v}_{x0} = \vec{c} \text{ stays constant}}$, unaffected by vertical acceleration

- <u>Ways to write $\vec{g}$</u>:
  - if  then $\boxed{\vec{g} = -g\hat{y}}$
  - if  then $\boxed{\vec{g} = +g\hat{y}}$
  - Either way, g is defined positive: $\boxed{g \equiv |\vec{g}| \approx 9.80 \left[\frac{m}{s^2}\right]}$

(67)

Kinematic Equations for Constant Acceleration in 2D:

(for $t_0 \equiv 0$)

x component	y component		
$v_x = v_{x0} + a_x t$	$v_y = v_{y0} + a_y t$	(68a)	
$x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2$	$y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2$	(68b)	(68)
$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$	$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$	(68c)	
$x = x_0 + \frac{1}{2}(v_{x0} + v_x)t$	$y = y_0 + \frac{1}{2}(v_{y0} + v_y)t$	(68d)	
$x = x_0 + v_x t - \frac{1}{2} a_x t^2$	$y = y_0 + v_y t - \frac{1}{2} a_y t^2$	(68e)	

- Unified as:

$$\vec{v} = \vec{v}_0 + \vec{a}t$$
$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

(• for $j = \{x, y\}$ where $r_x \equiv x$, $r_{x0} \equiv x_0$, $r_y \equiv y$, $r_{y0} \equiv y_0$) $\rightarrow$

$$v_j^2 = v_{j0}^2 + 2a_j(r_j - r_{j0})$$
$$\vec{r} = \vec{r}_0 + \frac{1}{2}(\vec{v}_0 + \vec{v})t$$
$$\vec{r} = \vec{r}_0 + \vec{v}t - \frac{1}{2}\vec{a}t^2$$

where $\vec{r} \equiv x\hat{x} + y\hat{y}$, $\vec{v} \equiv v_x\hat{x} + v_y\hat{y}$

and $\vec{r}_0 \equiv x_0\hat{x} + y_0\hat{y}$, $\vec{v}_0 \equiv v_{x0}\hat{x} + v_{y0}\hat{y}$, $\vec{a} \equiv a_x\hat{x} + a_y\hat{y}$

(69)

Projectiles with $a_x = 0$:

- And choose a_y as appropriate for coordinate system:
  - $\uparrow y \ \downarrow \vec{g}$: $a_y = -g$ or $\downarrow y \ \downarrow \vec{g}$: $a_y = +g$

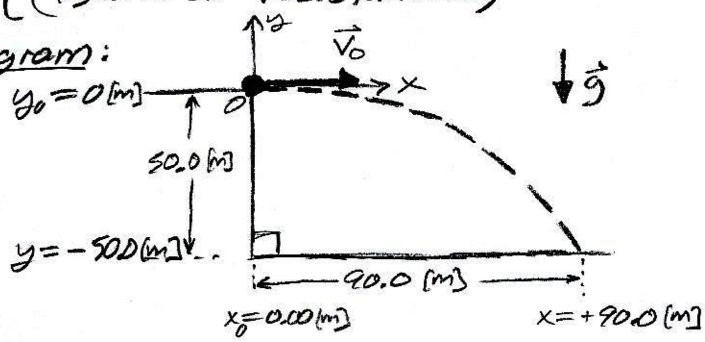
$v_x = v_{x0}$	$v_y = v_{y0} + a_y t$	(70a)	
$x = x_0 + v_{x0} t$	$y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2$	(70b)	
	$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$	(70c)	(70)
	$y = y_0 + \frac{1}{2}(v_{y0} + v_y)t$	(70d)	
	$y = y_0 + v_y t - \frac{1}{2} a_y t^2$	(70e)	

Strategy for Solving Projectile Problems in 2D:

- We only consider the path
  - ignore the starting and stopping mechanisms
  - use only initial velocity
  - and velocity just before object "stops" (or up to some time of interest)
- List givens, and desired variables
- Draw a diagram, with coordinate system
- State times (time interval is same for both x and y motion)
- Treat x and y motion separately at first, then relate by time

- ex: { A stunt driver on a motorcycle speeds horizontally off a 50.0 [m]-high cliff. How fast must the motorcycle leave the cliff top to land on level ground below, 90.0 [m] from the base of the cliff? (ignore air resistance) }

- Diagram:



- x givens:
  $x_0 = 0.00$ [m]
  $x = 90.0$ [m]
  $a_x = 0$ $[\frac{m}{s^2}]$
- y givens:
  $y_0 = 0.00$ [m]
  $y = -50.0$ [m]
  $V_{y0} = 0$ $[\frac{m}{s}]$
  $a_y = -g = -9.80$ $[\frac{m}{s^2}]$ (1)
- want:
  V_{x0} (2)

- x part: ($70b_x$) is:

$$x = x_0 + V_{x0} t \tag{3}$$

So solve for V_{x0}:

$$V_{x0} = \frac{x - x_0}{t} \tag{4}$$

- But we don't know t, so now, we also want t.
- y part: We have y_0, y, V_{y0}, a_y and want t, so solve ($70b_y$) for t:

$$y = y_0 + V_{y0} t + \tfrac{1}{2} a_y t^2 \tag{5}$$

$$\tfrac{1}{2} a_y t^2 + V_{y0} t + y_0 - y = 0$$

$$t_\pm = \frac{-V_{y0} \pm \sqrt{V_{y0}^2 - 2a_y(y_0 - y)}}{a_y} = \begin{matrix} t_+ = -3.194 \text{ [s]} \\ t_- = +3.194 \text{ [s]} \end{matrix} \tag{6}$$

- by causality, we want t_- since $t_- > 0$ [s], so

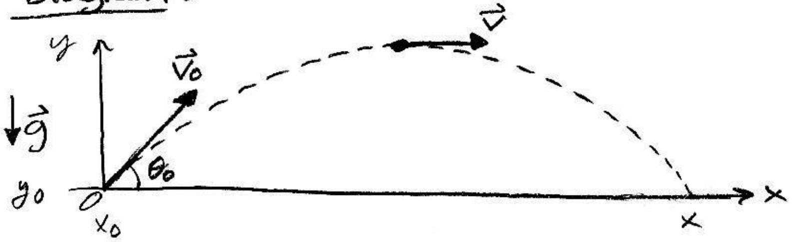
$$t = t_- = +3.194 \text{ [s]} \tag{7}$$

- Put (7) into (4):

$$\boxed{\begin{aligned} V_{x0} &= \frac{x - x_0}{t} = \frac{(x - x_0) a_y}{-V_{y0} - \sqrt{V_{y0}^2 - 2a_y(y_0 - y)}} \\ &\approx \frac{90.0 \text{ [m]} - 0.00 \text{ [m]}}{3.194 \text{ [s]}} \\ &\approx 28.2 \left[\tfrac{m}{s}\right] \end{aligned}} \tag{8}$$

ex: A cannon points at an angle θ_0 above the horizontal, and fires a cannon ball with a muzzle speed of v_0. If the cannon is in a ditch, so its muzzle-end is at ground level, and if the surrounding ground is level, find ⓐ the maximum height of the cannon ball ⓑ the time of travel of the cannon ball in the air ⓒ how far away the cannon ball hits the ground. Ignore air resistance.

• **Diagram:**



• Instead of breaking this problem into two halves, treat it as a whole:

• <u>x givens</u>: x_0, v_{x0}, $a_x = 0$ <u>y givens</u>: y_0, v_{y0}, $a_y = -g$ (1)

• <u>Note</u>: we are given $v_0 \equiv |\vec{v}_0|$ and θ_0, which completely specify $\vec{v}_0$ and thus v_{x0} and v_{y0}:

(which is why we stated v_{x0}, v_{y0} as givens) $\quad \vec{v}_0 \equiv v_{x0}\hat{x} + v_{y0}\hat{y}$; $\begin{cases} v_{x0} = v_0\cos(\theta_0) \\ v_{y0} = v_0\sin(\theta_0) \end{cases}$ (2)

• We <u>want</u> $\{x, y, t\}$ and we <u>have</u> $\{x_0, v_{x0}\}$ and $\{y_0, v_{y0}, a_y\}$

• so look for eqns that involve the "wants" <u>and</u> "haves"

• <u>x-part</u>: $\begin{pmatrix} \text{have } x_0, v_{x0} \\ \text{want } x, t \end{pmatrix} \to (70b_x) \to \quad x = x_0 + v_{x0}t$ (3)

• <u>y-part</u>: $\begin{pmatrix} \text{have } y_0, v_{y0}, a_y \\ \text{want } y, t \end{pmatrix} \to (70b) \to \quad y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$ (4)

• <u>Note</u>: together, (3) and (4) give the <u>position</u> $\vec{r} \equiv x\hat{x} + y\hat{y}$ <u>at any time t</u>.

• **ⓐ**: • let coordinates of highest point be: at "t_1":

$x_1 \equiv x(t_1)$, $y_1 \equiv y(t_1)$, so: $v_y(t_1) = 0 \leftarrow$ (peak height condition)   (5)

• <u>have</u> v_y (at t_1), v_{y0}, a_y, y_0, <u>want</u> y (at t_1)

• so use (70c): $\quad v_y^2 = v_{y0}^2 + 2a_y(y - y_0) \leftarrow$ (at t_1, so $v_y \to v_y(t_1)$, $y \to y_1$)   (6)

• so solve (6) for $y(t_1)$:

(used (5), (2), (1)) $\to$ ⓐ $\boxed{y(t_1) = \dfrac{v_y^2(t_1) - v_{y0}^2}{2a_y} + y_0 = \dfrac{0^2 - v_0^2\sin^2(\theta_0)}{-2g} = \dfrac{v_0^2\sin^2(\theta_0)}{2g}}$ (7)

height of cannonball at peak of trajectory

• **ⓑ**: • let coordinates of endpoint be just x, y, t, where $y = 0$ (landing height at t)   (8)

• <u>have</u> $y = 0$, y_0, v_{y0}, a_y, <u>want</u> t, so solve (4) for t w/ $y = 0$:

$\frac{1}{2}a_y t^2 + v_{y0}t + 0 = 0$ (used $y_0 = 0$, $y = 0$)   (9)

$t\left(\frac{1}{2}a_y t + v_{y0}\right) = 0 \Rightarrow t = 0$ (launch time) or $t = -\dfrac{v_{y0}}{\frac{1}{2}a_y}$ (10)

• so

ⓑ $\boxed{t = \dfrac{-v_{y0}}{\frac{1}{2}a_y} = \dfrac{-2v_{y0}}{-g} = \dfrac{2v_0\sin(\theta_0)}{g} \quad \text{time in flight}}$ (11)

• **ⓒ**: • total distance on ground is x at total time t: (put (11) into (3)) $\to$ ⓒ $\boxed{x = x_0 + v_0\cos(\theta_0)\dfrac{2v_0\sin(\theta_0)}{g} = \dfrac{v_0^2\sin(2\theta_0)}{g}}$ (12)