Exercises 1.2

In Problems 1 and 2, $y = 1/(1 + c_1 e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\dfrac{1}{3}$
2. $y(-1) = 2$

In Problems 3, 4, 5, and 6, $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval I over which the solution is defined.

3. $y(2) = \dfrac{1}{3}$
4. $y(-2) = \dfrac{1}{2}$
5. $y(0) = 1$
6. $y\left(\dfrac{1}{2}\right) = -4$

In Problems 7, 8, 9, and 10, $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

7. $x(0) = -1$, $x'(0) = 8$

8. $x(\pi/2) = 0$, $x'(\pi/2) = 1$

9. $x(\pi/6) = \frac{1}{2}$, $x'(\pi/6) = 0$

10. $x(\pi/4) = \sqrt{2}$, $x'(\pi/4) = 2\sqrt{2}$

In Problems 11, 12, 13, and 14, $y = c_1 e^x + c_2 e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

11. $y(0) = 1$, $y'(0) = 2$

12. $y(1) = 0$, $y'(1) = e$

13. $y(-1) = 5$, $y'(-1) = -5$

14. $y(0) = 0$, $y'(0) = 0$

In Problems 15 and 16 determine by inspection at least two solutions of the given first-order IVP.

15. $y' = 3y^{2/3}$, $y(0) = 0$

16. $xy' = 2y$, $y(0) = 0$

In Problems 17, 18, 19, 20, 21, 22, 23, and 24 determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

17. $\frac{dy}{dx} = y^{2/3}$

18. $\frac{dy}{dx} = \sqrt{xy}$

19. $x\frac{dy}{dx} = y$

20. $\frac{dy}{dx} - y = x$

21. $(4-y^2)y' = x^2$

22. $(1+y^3)y' = x^2$

23. $(x^2+y^2)y' = y^2$

24. $(y-x)y' = y+x$

In Problems 25, 26, 27, and 28 determine whether Theorem 1.2.1 guarantees that the differential equation $y' = \sqrt{y^2-9}$ possesses a unique solution through the given point.

25. $(1, 4)$

26. $(5, 3)$

27. $(2, -3)$

28. $(-1, 1)$

29.

(a) By inspection find a one-parameter family of solutions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial-value problem $xy' = y$, $y(0) = 0$.

(b) Explain part (a) by determining a region R in the xy-plane for which the differential equation $xy' = y$ would have a unique solution through a point (x_0, y_0) in R.

(c) Verify that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part (a).

30.

(a) Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

(b) Since $f(x, y) = 1 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Use the family of solutions in part (a) to find an explicit solution of the first-order initial-value problem $y' = 1 + y^2$, $y(0) = 0$. Even though $x_0 = 0$ is in the interval $(-2, 2)$, explain why the solution is not defined on this interval.

(c) Determine the largest interval I of definition for the solution of the initial-value problem in part (b).

31.

(a) Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

(c) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

32.

(a) Show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(1) = 1$, is $y = 1/(2 - x)$.

(b) Then show that a solution from the family in part (a) of

Problem 31 that satisfies $y' = y^2$, $y(3) = -1$, is $y = 1/(2 - x)$.

(c) Are the solutions in parts (a) and (b) the same?

33.

(a) Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\,dy/dx = 3x$.

(b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval I of definition of each explicit solution.

(c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions in part (b) satisfies $y(-2) = 3$?

34.

(a) Use the family of solutions in part (a) of Problem 33 to find an implicit solution of the initial-value problem $y\,dy/dx = 3x$, $y(2) = -4$. Then, by hand, sketch the graph of the explicit solution of this problem and give its interval I of definition.

(b) Are there any explicit solutions of $y\,dy/dx = 3x$ that pass through the origin?

In Problems 35, 36, 37, and 38 the graph of a member of a family of solutions of a second-order differential equation $d^2y/dx^2 = f(x, y, y')$ is given. Match the solution curve with at least one pair of the following initial conditions.

(a) $y(1) = 1$, $y'(1) = -2$

(b) $y(-1) = 0$, $y'(-1) = -4$

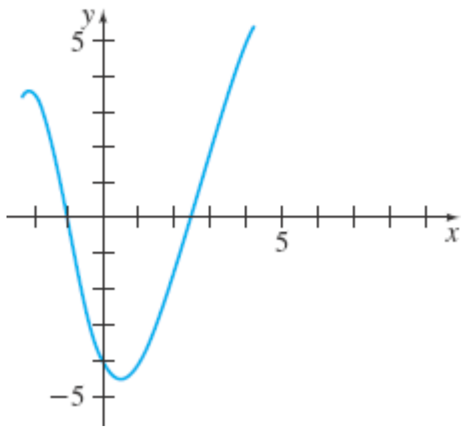
(c) $y(1) = 1$, $y'(1) = 2$

(d) $y(0) = -1, y'(0) = 2$

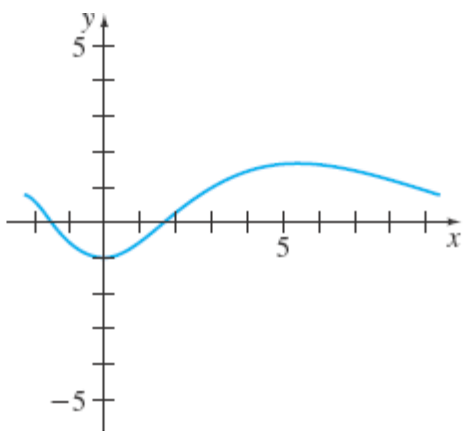
(e) $y(0) = -1, y'(0) = 0$

(f) $y(0) = -4, y'(0) = -2$

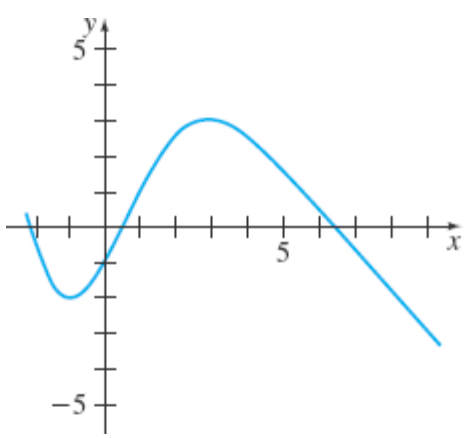
35. **Figure 1.2.7**



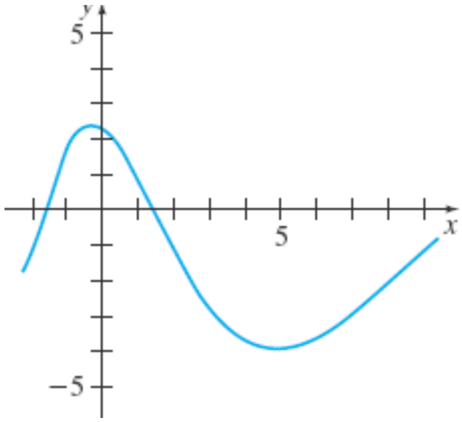
36. **Figure 1.2.8**



37. **Figure 1.2.9**



38. **Figure 1.2.10**



In Problems 39, 40, 41, 42, 43, and 44, $y = c_1 \cos 2x + c_2 \sin 2x$ is a two-parameter family of solutions of the second-order DE $y'' + 4y = 0$. If possible, find a solution of the differential equation that satisfies the given side conditions. The conditions specified at two different points are called boundary conditions.

39. $y(0) = 0$, $y(\pi/4) = 3$

40. $y(0) = 0$, $y(\pi) = 0$

41. $y'(0) = 0$, $y'(\pi/6) = 0$

42. $y(0) = 1$, $y'(\pi) = 5$

43. $y(0) = 0$, $y(\pi) = 2$

44. $y'(\pi/2) = 1$, $y'(\pi) = 0$

Discussion Problems

In Problems 45 and 46 use Problem 55 in Exercises 1.1 and (2) and (3) of this section.

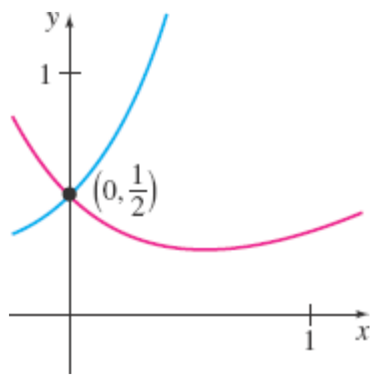
45. Find a function whose graph at each point (x, y) has the slope given by $8e^{2x} + 6x$ and has the y-intercept $(0, 9)$.

46. Find a function whose second derivative is $y'' = 12x - 2$ at each point (x, y) on its graph and $y = -x + 5$ is tangent to the graph at the point corresponding to $x = 1$.

47. Consider the initial-value problem $y' = x - 2y$, $y(0) = \frac{1}{2}$.

Determine which of the two curves shown in Figure 1.2.11 is the only plausible solution curve. Explain your reasoning.

Figure 1.2.11



48. Show that

$$x = \int_0^y \frac{1}{\sqrt{t^3 + 1}}\, dt$$

is an implicit solution of the initial-value problem

$$2\frac{d^2y}{dx^2} - 3y^2 = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Assume that $y \geq 0$. [*Hint*: The integral is nonelementary. See (ii) in the Remarks at the end of Section 1.1.]

49. Determine a plausible value of x_0 for which the graph of the solution of the initial-value problem $y' + 2y = 3x - 6$, $y(x_0) = 0$ is tangent to the x-axis at $(x_0, 0)$. Explain your reasoning.

50. Suppose that the first-order differential equation $dy/dx = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

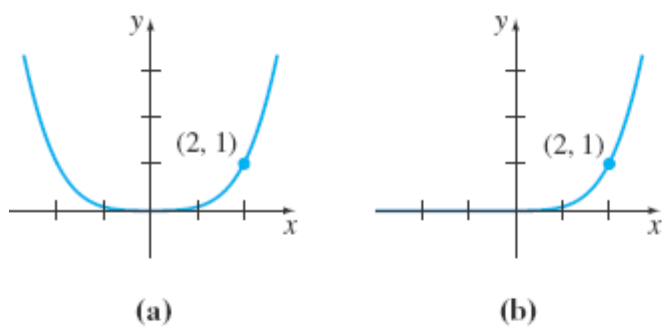
51. The functions $y(x) = \frac{1}{16}x^4$, $-\infty < x < \infty$ and

$$y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

have the same domain but are clearly different. See Figures 1.2.12(a) and 1.2.12(b), respectively. Show that both functions are solutions of the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ on the interval $(-\infty, \infty)$. Resolve the apparent contradiction between this fact and the last sentence in Example 5.

Figure 1.2.12

Two solutions of the IVP



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