Direction fields, autonomous ODEs, phase portraits

MA221, Lecture 2

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First order ODEs of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$

Despite looking simple, such equations may not be readily solvable.
However, you will still be able to *visualize* how solutions to these equations behave.

Some examples:

- $\dfrac{dy}{dx} = x - y^2$ (non-linear)
- $\dfrac{dy}{dx} = x^2 - y$ (linear)
- $\dfrac{dy}{dx} = x^2 y^3$
- $\dfrac{dy}{dx} = \cos x$
- $\dfrac{dy}{dx} = \cos y$

Special examples of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$

- What if f depends only on x? $\frac{dy}{dx} = f(x)$

  Example: $\frac{dy}{dx} = x\cos(3x^2)$

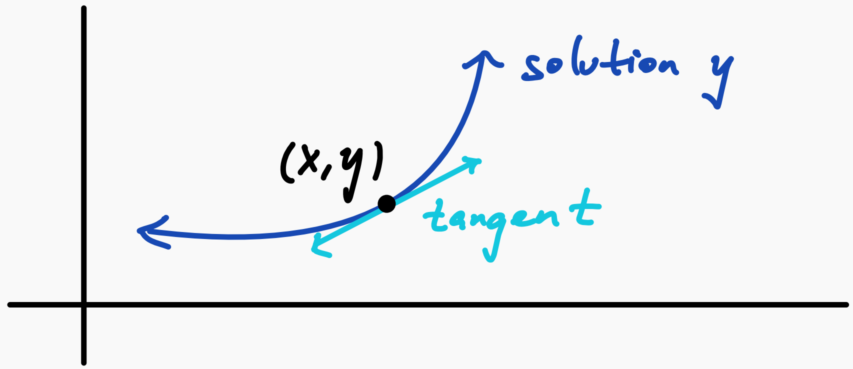
  $$y = \int \frac{dy}{dx}\,dx = \int x\cos(3x^2)\,dx = \ldots = \frac{1}{6}\sin(3x^2) + C$$

  $$\begin{bmatrix} u = 3x^2 \\ du = 6x\,dx \end{bmatrix}$$

- What if f depends only on y? $\underbrace{\frac{dy}{dx} = f(y)}_{\text{"Autonomous ODEs"}}$

Direction fields

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$. For each point (x, y) in $\mathbb{R}^2$, $f(x, y)$ represents the slope of the tangent line to the solution y at (x, y).



The corresponding **direction field** (or **slope field**) for this differential equation is a grid containing small segments of these tangent lines.

Direction fields

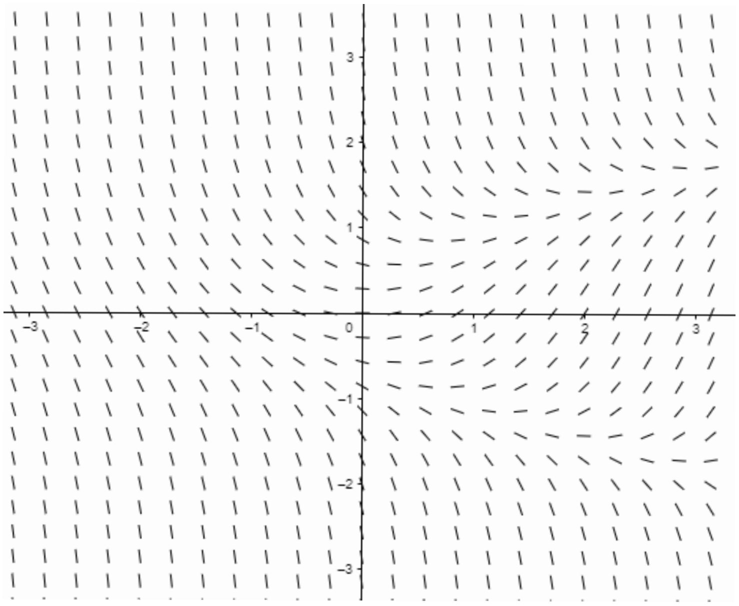
Doing this by hand is very tedious!

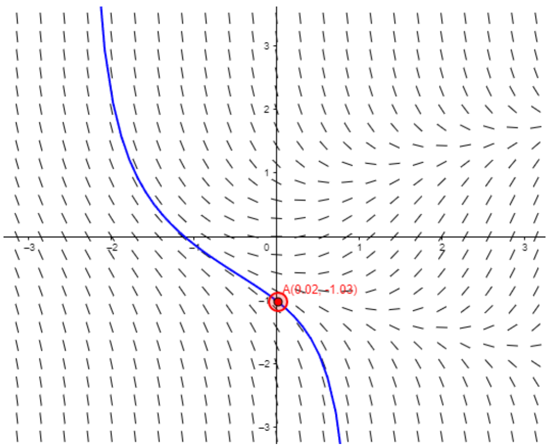
Sample grid for $\frac{dy}{dx} = x - y^2$:

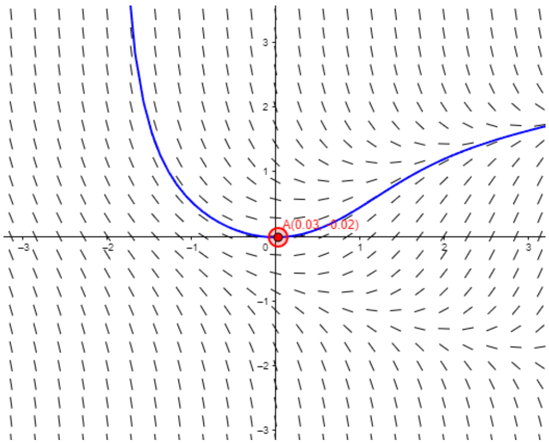
$y \backslash x$	-3	-2	-1	0	1	2	3
-3	-12	-11	-10	-9	-8	-7	-6
-2	-7	-6	-5	-4	-3	-2	-1
-1	-4	-3	-2	-1	0	1	2
0	-3	-2	-1	0	1	2	3
1	-4	-3	-2	-1	0	1	2
2	-7	-6	-5	-4	-3	-2	-1
3	-12	-11	-10	-9	-8	-7	-6

Each grid entry is $\frac{dy}{dx}$ evaluated at the appropriate x,y- values!

Direction field for $\frac{dy}{dx} = x - y^2$







Autonomous ODEs

An **autonomous ODE** is an ODE of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$. Since f (and therefore $\frac{\mathrm{d}y}{\mathrm{d}x}$) depends only on y, the slope field for an autonomous ODE "stays constant" along any ~~vertical~~ horizontal line:

This makes visualizing solutions to autonomous equations quite easy to do by hand; provided you can identify the roots of f!

Phase lines

The **phase line** of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$ can be obtained as follows:

1) Find the roots/zeroes of f and plot them on a vertical line. The roots c are called "critical points" of the equation.

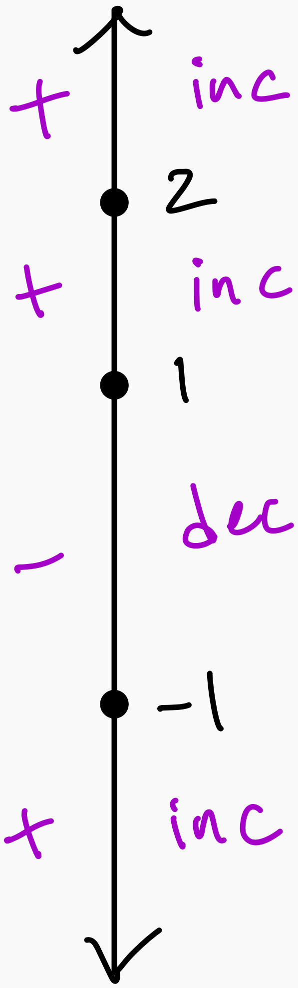
2) What happens between critical points? Is f positive or negative on these intervals?

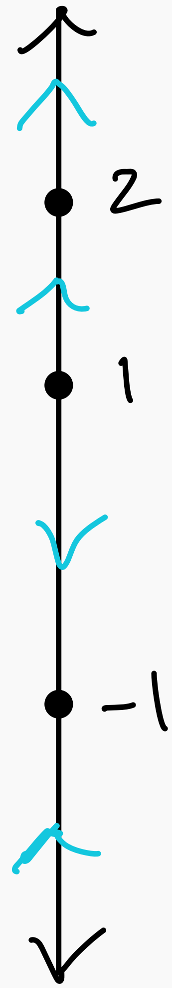
$\downarrow$ y increases

$\downarrow$ y decreases

Phase lines

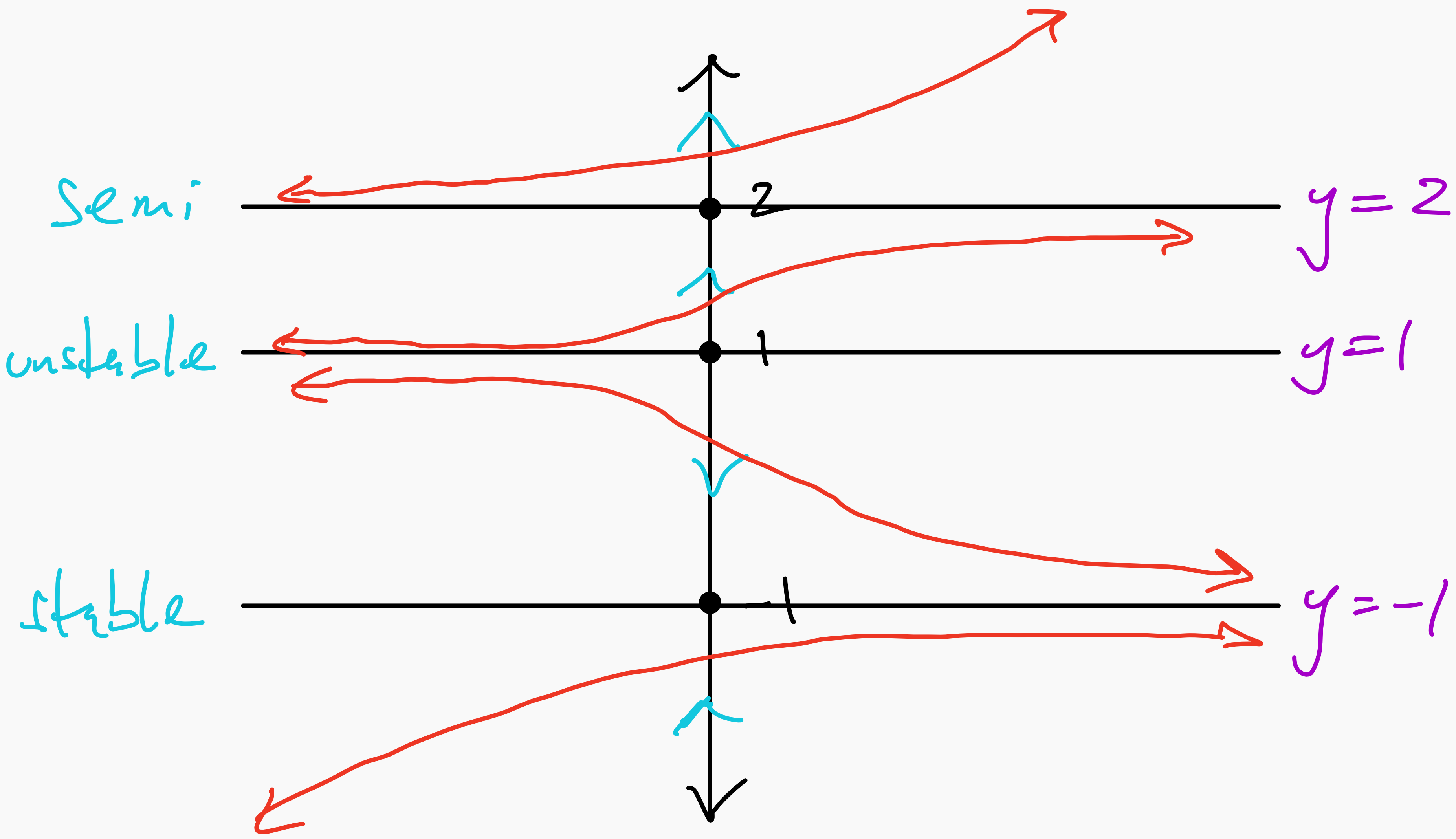
Example: Compute the **phase line** of $\frac{dy}{dx} = (y^2 - 1)(y - 2)^2$





Phase portraits

A phase line can be extrapolated to a **phase portrait**:



Stability of critical points

Critical points fall into one of three categories, depending on the behavior of the solution y around those points: *stable*, *semi-stable*, or *unstable*.