10 HW

$$R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Calculate Transitive Closure

a can communicate to b and c
both b and c can communicate to d
assume communication is one way,

this can be described by the relation
$R = \{(a,b),(a,c),(b,d),(c,d)\}$

we want to change the system so a can communicate with d and still maintain the previous system.

- Described as
  - Find the smallest relation R^+ which contains R as a sub set $R \subset R^+$
  - and is transitive
- Thus, $R^+ = \{(a,b),(a,c),(b,d),(c,d),(a,d)\}$

Hence the **transitive closure** of R,

- denoted by R^+
  - is the smallest transitive relation that contains R as a subset

It follows that is $(a,b) \in S$ and $(b,c) \in S$ then, $(a,c) \in S^+$

$A = \{1,2,3,4\}$
$S = \{(1,2),(2,3),(3,4)\}$ be the relation on set A

- This is called the successor relation on A
  - since each element is related to its successor.

$S^+ = S$

$S = \{(1,2),(2,3),(3,4)\}$
$SS = S^2 = \{(1,3),(2,4)\}$
$S^+ = S \cup S^2 = \{(1,2)(1,3),(2,3),(2,4),(3,4)\}$
$S^2 S = S^3 = \{(1,4)\}$

- because $(1,3) \in S^2$ and $(3,4) \in S$
  this shows that $S^3 \subseteq S^+$
  Thus,
  $S^+ = S \cup S^2 \cup S^3 = \{(1,2)(1,3),(1,4)(2,3),(2,4),(3,4)\}$

Let r be a relation on the set $A = \{\text{Kira}, \text{Frank}, \text{Lola}, \text{Sarah}, \text{Judy}, \text{Reuben}\}$

with the relation matrix,

$$R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix of the transitive closure R^+,

- can be computed by the equation $R^+ = R + R^2 + R^3 + \cdots + R^n$.
  - By using the polynomial evaluation methods,
    - we can compute R^+ with $n-1$ matrix multiplications:
      - $R^+ = R(I + R(I + (\cdots R(I + R)\cdots)))$
        - for example, if n = 3
        - $R^+ = R(I + R(I + R))$

We can make use of the fact that if T is a relation matrix $T + T = T$ due to the fact that $1 + 1 = 1$ in boolean arithmetic.

- let, $S_k = R + R^2 + \cdots + R^k$
  - Then,
    - $$\begin{aligned} R &= S_1 \\ S_1(I + S_1) &= R(I + R) = R + R^2 = S_2 \\ S_2(I + S_2) &= (R + R^2)(I + R + R^2) \\ &= (R + R^2) + (R^2 + R^3) + (R^3 + R^4) \\ &= R + R^2 + R^3 + R^4 \\ &= S_4 \end{aligned}$$
    - Similarly,
    - $$S_4(I + S_4) = S_8$$
    - and by induction we can prove
    - $$S_{2^k}(I + S_{2^k}) = S_{2^{k+1}}$$
    - Each matrix multiplication doubles the number of terms that have been added to the sum that is currently computed.

For our problem we have

Let r be a relation on the set $A = \{\text{Kira}, \text{Frank}, \text{Lola}, \text{Sarah}, \text{Judy}, \text{Reuben}\}$
$n = 6$
with the relation matrix,

$$R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix of the transitive closure R^+,

- can be computed by the equation $R^+ = R + R^2 + R^3 + \cdots + R^n$.

$$R^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $R^+ = R^1 + R^2 + R^3 + R^4 + R^5 + R^6$

$$R^+ = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q1.2

In set notion,

$$R^+ = \{(2,1),(3,1),(4,1),(4,2),(5,1),(5,2),(5,4),(6,1),(6,2),(6,4)\}$$

Q1.3

the **transitive closure** of

$$S$$

,

- denoted by

  $$S^+$$

  or

  $$S^*$$

> - is the smallest transitive relation that contains
>
> $$S$$
>
>   as a subset.

For our specific, problem,
relation

$$R$$

is the successor relation on set

$$A$$

.

The **Transitive Closure** specifies an ancestral family tree,

- if there is a edge between two vertices, then they are
  > - either a parent and child,
  > - grandparent, and child,
  > - or great grandparent and child.

Q2.1

$$A = \{c, d, e\}$$

so total # of equivalence relations on the set is 5

$$R_1 = \{(a,a),(b,b),(c,c)\}$$

$$R_2 = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$$

$$R_3 = \{(a,a),(b,b),(c,c),(b,c),(c,b)\}$$

$$R_4 = \{(a,a),(b,b),(c,c),(a,c),(c,a)\}$$

$$R_5 = \{(a,a),(b,b),(c,c),(a,b),(b,a),(c,a),(b,c),(b,c),(c,b)\}$$

Each equivalent relation on a set yields a partition of that set in disjoint equivalence classes for a finite set.
The number of equivalence relations is the number of partitions.

Q 2.2

How many different partial ordering can we dine on the set

$$A = \{\backslash\mathbf{x}, y\}$$

?

$$A \times A = \{(x, x), (y, y), (x, y), (y, x)\}$$

partial order is reflexive, lets list the reflexive relations on

$$A$$

$$R_1 = \{(x, x), (y, y)\}$$

$$R_2 = \{(x, x), (x, y), (y, y)\}$$

$$R_3 = \{(x, x), (y, z), (y, y)\}$$

$$R_4 = \{(x, x), (x, y), (y, x)(y, y)\}$$

- Due to antisymmetric property, we cant use

$$R_4$$

so, there exist 3 different partial orderings for set

$$A$$

Q 2.3

How many different total orderings on set

$$A = \{p, q\}$$

?

$$A \times A = \{(p, p), (p, q), (q, p), (q, q)\}$$

Total ordering is

- partial ordering and two elements of

$$A$$

  are comparable.

so, 2 total orderings,

$$\{p < q,\ q < p\}$$