Bernoulli equations

MA221, Lecture 6

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this week: NB319

Office hours

M : 2pm - 3pm NB316, 3pm-4pm (zoom)

W : 2pm - 4pm NB 316

Th : 1pm - 2pm (zoom)

F : 2pm - 3pm NB316

Recall that a **linear first-order equation** is an ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x).$$

We have two methods for solving these equations: (1) via integrating factors or (2) via variation of parameter.

Example 1: Solve $xy' + 2y = x^3 - x$.

$$\Rightarrow y' + \underbrace{\boxed{\frac{2}{x}}}_{P(x)} \cdot y \overset{\star}{=} \underbrace{x^2 - 1}_{Q(x)}.$$

Look at homogeneous version of $\star$: $\star_H$

$$\boxed{P(x) = \frac{2}{x}}$$

$$\star_H: \quad y' + \frac{2}{x} y = 0$$

$$\rho(x) = e^{\int P(x)\,dx} = e^{2\ln|x|}$$

$$= e^{\ln|x|^2} = x^2$$

multiply by ρ $\Longrightarrow$

$$0 = x^2 y' + \frac{2}{x} x^2 y$$
$$= x^2 y' + 2xy$$
$$= [x^2 y]'$$

$$\Rightarrow \quad C = \int 0\,dx = \int [x^2 y]'\,dx = x^2 y$$

$$\Rightarrow \quad y = \frac{C}{x^2} \rightsquigarrow \text{solution to } (\star_H).$$

VoP: Define $y = \dfrac{C(x)}{x^2}$ and suppose y is a solution to $\star$: what is $C(x)$?

$$y' + \frac{2}{x}\cdot y \overset{\star}{=} x^2 - 1.$$

$$y = \frac{C(x)}{x^2} \Rightarrow y' = \frac{x^2 C'(x) - 2x\,C(x)}{x^4}$$

$$\star \Rightarrow \frac{x^2 C'(x) - 2x\,C(x)}{x^4} + \frac{2}{x}\cdot\frac{C(x)}{x^2} = x^2 - 1$$

$$\Rightarrow x^2 C'(x) \underbrace{- 2x\,C(x)} + \underbrace{2x\,C(x)} = x^6 - x^4$$

$$\Rightarrow x^2 C'(x) = x^6 - x^4 \Rightarrow C'(x) = x^4 - x^2$$

$$\Rightarrow C(x) = \int C'(x)\,dx = \int (x^4 - x^2)\,dx = \frac{x^5}{5} - \frac{x^3}{3} + C$$

$$\Rightarrow y = \frac{C(x)}{x^2} = \frac{\frac{x^5}{5} - \frac{x^3}{3} + C}{x^2} = \underline{\frac{x^3}{5} - \frac{x}{3} + \frac{C}{x^2}}$$

is our solution to $\star$.

- "largest interval over which general solution is defined":
  - Identify $P(x)$: $P(x) = \frac{2}{x}$
  - Domain of $P(x)$: $(-\infty, 0) \cup (0, \infty)$

    pick either

- "Transient terms": Expressions in your solution that approach 0 as $x \to \infty$.

$$y = \underbrace{\frac{x^3}{5}}_{\text{not}} - \underbrace{\frac{x}{3}}_{\text{not}} + \underbrace{\frac{C}{x^2}}_{\text{transient}}$$

Bernoulli Equations

A **Bernoulli equation** is a first order ODE of the form

$$\frac{dy}{dx} + P(x)y \overset{\star}{=} Q(x)y^r.$$

The $r = 0$ and $r = 1$ cases aren't the most "interesting" given what we've seen so far: · if $r=0$: $\star$ is $\boxed{\text{LFO}}$

· if $r=1$: $\star$ is $\frac{dy}{dx} + P(x)y = Q(x)y \Rightarrow \frac{dy}{dx} = y(Q(x)-P(x))$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (Q(x)-P(x)) \Rightarrow \frac{1}{y}dy = (Q(x)-P(x))dx$$

$\boxed{\text{separable}}$

But if $r \neq 0, 1$, then we have something more interesting and we solve $(\star)$ by transforming it using an appropriately chosen substitution.

$$u = y^{1-r} \Rightarrow \frac{du}{dx} = (1-r)y^{-r}\frac{dy}{dx}$$

Bernoulli Equations

Example 2: $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^2 + 3y^2$

$$\frac{dy}{dx} + P(x)y = Q(x)y^r$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = \frac{2x}{y} + \underbrace{\frac{3y}{2x}}$$

$$\Rightarrow \frac{dy}{dx} + \underbrace{\left[-\frac{3}{2x}\right]}_{P(x)} y = \underbrace{2x}_{Q(x)} y^{-1}$$

$\left(\begin{array}{l} r = -1 \\ \text{Bernoulli} \end{array} \right)$

$$u = y^{1-r} = y^{1-(-1)} = y^2 \Rightarrow \frac{du}{dx} = 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}\frac{du}{dx}$$

$$\Rightarrow \frac{1}{2y}\frac{du}{dx} + \left[\frac{-3}{2x}\right]y = 2xy^{-1}$$

$$\Rightarrow \frac{du}{dx} + 2y\left[\frac{-3}{2x}\right]y = 2y\left(2xy^{-1}\right)$$

$$\Rightarrow \frac{du}{dx} + \left[\frac{-3}{x}\right]y^2 = 4x$$

$$\Rightarrow \underbrace{\frac{du}{dx} + \left[\frac{-3}{x}\right]u = 4x}$$

LFO in u. Solve for u.

solution will invoke u not y,

but you can replace u with y^2.

Bernoulli equations

Example 3: $y(6y^2 - t - 1) + 2ty' = 0$

Didn't cover this one in class!

$y' + P(t)y = Q(t)y^r$

$\frac{y(6y^2 - t - 1)}{2t} + y' = 0 \Rightarrow \frac{6y^3 - ty - y}{2t} + y' = 0$

$\Rightarrow \frac{3y^3}{t} - \frac{y}{2} - \frac{y}{2t} + y' = 0$

$\Rightarrow y' + \underbrace{\left[\frac{-t-1}{2t}\right]}_{P(t)} y = \underbrace{-\frac{3}{t}}_{Q(t)} y^3$

($r = 3$ Bernoulli eqn)

$$\boxed{u} = y^{1-r} = y^{1-3} = \boxed{y^{-2}}$$

$$\Rightarrow \frac{du}{dt} = -2y^{-3}\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = -\frac{y^3}{2}\frac{du}{dt}$$

$$\star \Rightarrow -\frac{y^3}{2}\cdot\frac{du}{dt} + \left[\frac{-t-1}{2t}\right]y = -\frac{3}{t}y^3$$

solve for $\frac{du}{dt}$

$$\Rightarrow \frac{du}{dt} + \left[-\frac{2}{y^3}\right]\left[\frac{-t-1}{2t}\right]y = -\frac{3}{t}y^3\left[-\frac{2}{y^3}\right]$$

$$\Rightarrow \frac{du}{dt} + \left[\frac{t+1}{t}\right]y^{-2} = \frac{6}{t}$$

$$\Rightarrow \underbrace{\frac{du}{dt} + \left[\frac{t+1}{t}\right]u = \frac{6}{t}}$$

LFO in u. Solve!

solution will be in terms of u.

replace u with y^{-2} at the end!

Bernoulli equations

Example 4: $3\sqrt{y}\frac{\mathrm{d}y}{\mathrm{d}x} + y^{\frac{3}{2}} = e^{-x}$

Didn't cover this one in class!

$$\Rightarrow \frac{dy}{dx} + \frac{y^{\frac{3}{2}}}{3\sqrt{y}} = \frac{e^{-x}}{3\sqrt{y}}$$

$$\Rightarrow \frac{dy}{dx} + \underbrace{\frac{1}{3}}_{P(x)} y = \underbrace{\frac{1}{3}e^{-x}}_{Q(x)} y^{-\frac{1}{2}} \quad \left(\begin{array}{c} r = -\frac{1}{2} \\ \text{Bernoulli} \\ \text{eqn} \end{array} \right)$$

$$u = y^{1-r} = y^{1-(-\frac{1}{2})} = y^{\frac{3}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{3}{2} y^{\frac{1}{2}} \frac{dy}{dx} = \frac{3}{2}\sqrt{y}\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{3} \cdot \frac{1}{\sqrt{y}} \frac{du}{dx}$$

$$\Rightarrow \frac{2}{3} \cdot \frac{1}{\sqrt{y}} \frac{du}{dx} + \frac{1}{3} y = \frac{1}{3} e^{-x} y^{-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} + \frac{1}{3} y \left[\frac{3\sqrt{y}}{2}\right] = \frac{1}{3} e^{-x} y^{-\frac{1}{2}} \left[\frac{3\sqrt{y}}{2}\right]$$

$$\Rightarrow \quad \frac{du}{dx} + \frac{1}{2} y^{\frac{3}{2}} = \frac{1}{3} e^{-x} \cdot \frac{3}{2}$$

$$\Rightarrow \quad \underbrace{\frac{du}{dx} + \frac{1}{2} u = \frac{1}{2} e^{-x}}_{\text{LFO.}}$$