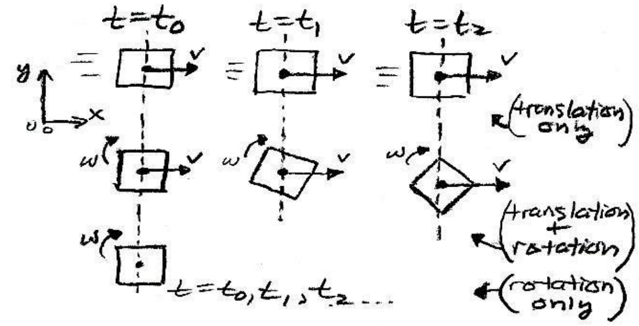
Motion: Kinematics

Kinds of Motion:

- Translational (linear)
- Rotational



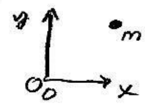
- Each kind is <u>independent</u> of the other:
- Each kind can happen alone or with the other
- Each kind has <u>3 independent directions</u>

(we'll stick to 1-dimensional translational motion for now)

Object Idealizations:

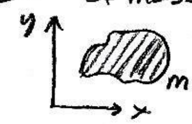
- <u>particle</u> ≡ a single <u>point</u>
  - no spatial extent
  - can <u>only translate</u>
  - many <u>nonpoint</u> objects can be <u>treated as points</u>

<u>ex</u>: particle of mass m:



- <u>rigid body</u> ≡ a <u>continuous solid</u>
  - no internal motion
  - ideal for rotations
  - ideal for collisions
  - translates like a particle

<u>ex</u>: rigid body of mass m:



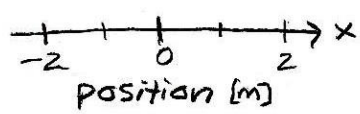
Coordinate Systems: ≡

a rigid set of <u>axes</u>, each having <u>coordinates</u> marking location along them, and a definite <u>point of origin</u>

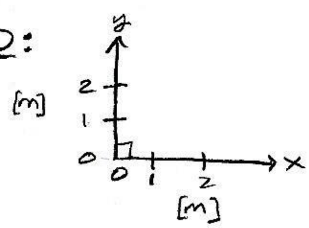
- <u>ex</u>

Cartesian Coordinates:

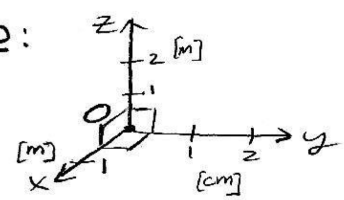
- <u>1 Dimension (1D)</u>:



- <u>2D</u>:



- <u>3D</u>:



- <u>Notes</u>:
  - Always <u>label each axis</u> with variables
  - Always mark a few relevant coordinates
  - Show zero whenever possible
  - Usually good to use <u>word</u> axis labels
    - but a variable (x, y, z) is sufficient
    - but <u>at least</u> specify <u>units and variable</u>
  - Can omit portions not needed (like 2D and 3D above)

- <u>Reference Frame</u> ≡

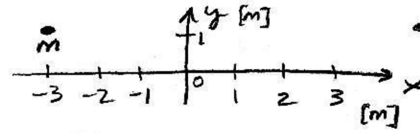
a coordinate system attached to something

- <u>Coordinate System Freedom</u>
  - we can pick any coordinate system we want! → (But we must be consistent!)
  - "up" doesn't have to be positive
  - often nice to make <u>the direction of motion positive</u> (but not necessary)
  - sometimes, we can break a problem into stages, and use different coordinates in each stage... but then coordinates need <u>careful conversion</u>

Position: $\equiv$ $\boxed{\text{the spatial coordinates of an object}}$

- can change with time
- can have any sign

ex:



(position of the particle is (in meters): $(x,y)=(-3,1)$)

Displacement: $\equiv$ $\boxed{\text{the change in position of an object}}$

- Definition:

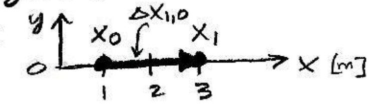
$$\boxed{\Delta x \equiv x_f - x_i} \quad ; \quad x_f \equiv \text{final position}, \quad x_i \equiv \text{initial position}$$

- can have any sign
- is not generally equal to distance traveled
- useful notation:

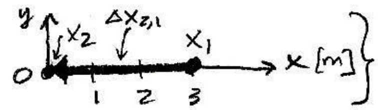
$$\boxed{\Delta x_{f,i} \equiv x_f - x_i}$$

(good for problems with many displacements)

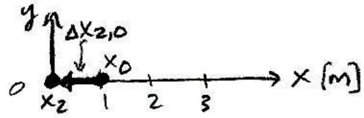
- ex: {If you walk 2 [m] forward, and then walk 3 [m] backward, what is your total displacement? what is the total distance traveled?}
  - State "known" quantities first:
    - two separate displacements: $\Delta x_{1,0}$ and $\Delta x_{2,1}$
    - draw diagrams:
      - part 1:



      - part 2:



      - total:



- By coordinate system freedom, we can choose it so that x_0 is anywhere!
- so $x_0 = 1\,[m]$, $x_1 = 3\,[m]$ (1)

and $\Delta x_{1,0} = x_1 - x_0 = 2\,[m]$ ✓ (2)

- Note: in part 2, the initial point is the final point of part 1
- so $x_1 = 3\,[m]$, $x_2 = 0\,[m]$ (3)

and
$$\Delta x_{2,1} = x_2 - x_1 = 0[m] - 3[m] = -3\,[m] \quad (4)$$

- Total displacement is the sum of consecutive displacements:

$$\Delta x_{2,0} = \Delta x_{1,0} + \Delta x_{2,1} \quad (5)$$

- put (2) and (4) into (5):

$$\boxed{\Delta x_{2,0} = (2[m]) + (-3[m]) = -1\,[m]} \quad (6)$$

- check:
  - By definition, displacement only depends on endpoints, so we could have just said:

$$\Delta x_{2,0} \equiv x_2 - x_0 \quad (7)$$

  - then, from (1) and (3), $x_0 = 1\,[m]$ and $x_2 = 0\,[m]$, so (7) is

$$\Delta x_{2,0} = (0[m]) - (1[m]) = -1\,[m] \;✓ \quad (8)$$

- Note, only endpoints matter because:

$$\Delta x_{2,0} = \Delta x_{1,0} + \Delta x_{2,1} = \cancel{x_1} - x_0 + x_2 - \cancel{x_1} = x_2 - x_0 \quad (9)$$

- Total distance is:

$$\boxed{D \equiv |\Delta x_{1,0}| + |\Delta x_{2,1}| = |2[m]| + |-3[m]| = 5\,[m]}$$

so . . .

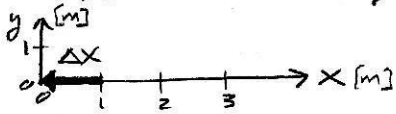
Distance Traveled $\equiv$

$\boxed{\text{The total distance traveled in one dimension is the sum of magnitudes of each distinct one-directional displacement}}$ → $\boxed{D \equiv \sum_{a=1}^{n} |\Delta x_{a,a-1}|}$ (10)

- for n distinct displacements
- starting at x_0, ending at x_n

Vectors: $\equiv$ $\boxed{\text{quantities with both } \underline{\text{magnitude}} \text{ and } \underline{\text{direction}}}$ (more details coming soon)

- direction represented by an arrow
- direction also indicated by sign relative to coordinate system
- ex: displacement is a vector: $\Delta x = -1\,[m]$



Average Speed:

- is not velocity
- is not a vector
- is a scalar (has no direction)

$$\boxed{\bar{s} \equiv \frac{D}{\Delta t}} \rightarrow \left(\text{average speed} \equiv \frac{\text{distance traveled}}{\text{time elapsed}}\right)$$

- where $\begin{cases} D \equiv \text{distance traveled} \\ \Delta t \equiv t_f - t_i \equiv \text{time elapsed}; \end{cases} \begin{cases} t_f \equiv \text{final time} \\ t_i \equiv \text{initial time} \end{cases}$

Average Velocity:

- is a vector
- in 1-D, we can omit subscripts

$$\boxed{\bar{v} \equiv \bar{v}_x \equiv \frac{\Delta x}{\Delta t}} \rightarrow \left(\text{average velocity} \equiv \frac{\text{displacement}}{\text{time elapsed}}\right)$$

- detail: $\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x(t_f) - x(t_i)}{t_f - t_i}$ (positions are functions of time)
- notice that the enpoints in Δx match the times in Δt

- ex: {In the previous example, if your whole trip from start to finish took 2.0 [s], what was your average speed and average velocity?}
  - Given:

$$\Delta t \equiv t_f - t_i = 2.0\,[s] \qquad (1)$$
$$D = 5\,[m] \qquad (2)$$
$$\Delta x \equiv x_f - x_i \equiv x_2 - x_0 = \Delta x_{2,0} = -1\,[m] \qquad (3)$$

  - then average speed is:

$$\text{a} \quad \boxed{\bar{s} \equiv \frac{D}{\Delta t} = \frac{5\,[m]}{2\,[s]} = 2.5\left[\frac{m}{s}\right]} \qquad (4)$$

  - and average velocity is: b

$$\bar{v}_x \equiv \frac{\Delta x}{\Delta t} = \frac{-1\,[m]}{2\,[s]} = -0.5\left[\frac{m}{s}\right] \quad \text{(numerical direction indication)}$$

or

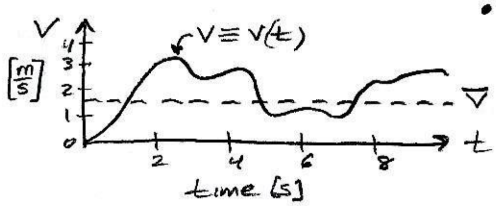
$$\bar{v}_x = \underbrace{0.5\left[\frac{m}{s}\right]}_{\text{is } |\bar{v}_x| \text{ when this is used}} \text{ in the } -x \text{ direction} \quad \text{(verbal direction indication)} \qquad (5)$$

Instantaneous Velocity:
(or just "velocity")

$$\boxed{v \equiv \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}} \leftarrow \left(\text{the value of the ratio } \frac{\Delta x}{\Delta t} \text{ as } \Delta t \text{ approaches zero}\right)$$

- v is a definite value
- is not some number over 0

- Instantaneous velocity is generally different from average velocity:



← (but v can be the same as $\bar{v}$ if v is constant)

Instantaneous Speed:

$$\boxed{s \equiv \lim_{\Delta t \to 0}\left(\frac{D}{\Delta t}\right) = |v|} \leftarrow \left(\text{because as } \Delta t \to 0, \ D \text{ approaches } |\Delta x| \text{ in } \left|\lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right)\right| = |v|\right)$$

$$\left(\text{instantaneous speed} = \text{magnitude of instantaneous velocity}\right)$$

Acceleration: $\equiv$ $\boxed{\text{change in velocity over time}}$

- average acceleration:
  - is a vector

$$\boxed{\bar{a} \equiv \frac{\Delta v}{\Delta t}} \leftarrow \left(\frac{\text{change in velocity}}{\text{time elapsed}}\right)$$

$$\bar{a} \equiv \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} = \frac{v(t_f) - v(t_i)}{t_f - t_i}$$

- instantaneous acceleration:
  - (or just "acceleration")
  - is a vector

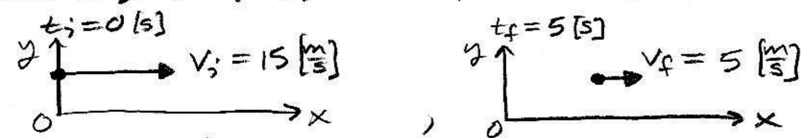
$$\boxed{a \equiv \lim_{\Delta t \to 0}\left(\frac{\Delta v}{\Delta t}\right) = \frac{dv}{dt}}$$

- deceleration $\equiv$ $\boxed{\text{whenever acceleration opposes velocity}}$ (a and v pointing in opposite directions)
  - is just a form of acceleration
  - so we usually don't use this term

- ex: {A car has initial velocity of $v_i = 15\ [\frac{m}{s}]$, and suddenly hits the breaks and slows down to a final velocity of $v_f = 5\ [\frac{m}{s}]$ over a time of $5\ [s]$. What was the car's average acceleration?}
  - Givens:

$$v_i = 15\ \left[\tfrac{m}{s}\right] \quad (1)$$
$$v_f = 5\ \left[\tfrac{m}{s}\right] \quad (2)$$
$$\Delta t = 5\ [s] \quad (3)$$

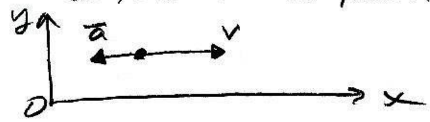
  - since no direction change specified, then signs of v_i and v_f are the same (4)
  - Diagram:



  - put (1-3) into definition of $\bar{a}$:

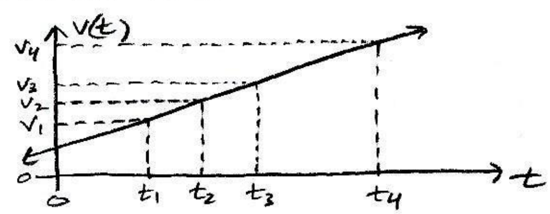
$$\boxed{\bar{a} \equiv \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t} = \frac{5\left[\frac{m}{s}\right] - 15\left[\frac{m}{s}\right]}{5\ [s]} = \frac{-10\left[\frac{m}{s}\right]}{5\ [s]} = -2\left[\frac{m}{s^2}\right]} \quad (5)$$

  - since car was moving in the +x direction, but its velocity decreased, then it was decelerating, so its $\bar{a}$ points in the −x direction:



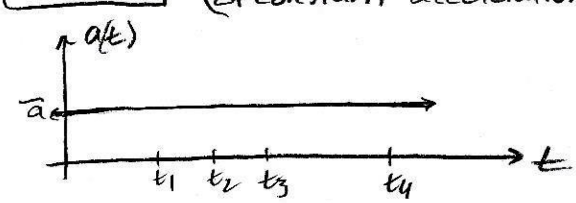
Constant Acceleration: $\boxed{\text{For constant } a\text{, changes in } v \text{ are the same between any two times}}$

- this is true even as $\Delta t \to 0$
- and $\bar{a}$ is the slope of the line $v(t)$



$$\frac{\Delta v_{2,1}}{\Delta t_{2,1}} = \frac{\Delta v_{4,3}}{\Delta t_{4,3}} = \bar{a}$$

- so when $v(t)$ is a line: $\boxed{a = \bar{a}}$ (special case of constant acceleration)



$\leftarrow$ (a has value $\bar{a}$ at all times because the slope of $v(t)$ above is constant)

- often, constat a is a good approximation

Motion With Constant Acceleration:

- Simplifications:
  - since a is constant, $$\bar{a} = a \tag{1}$$
  - by coord.-sys. freedom, $$t_i \equiv 0\,[\mathrm{s}] \equiv t_0 \tag{2}$$
  - let t_f be simply "t": $$t_f \equiv t \tag{3}$$
- So average velocity is:

$$\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x(t) - x(0)}{t - 0} \equiv \frac{x - x_0}{t} \tag{4}$$

where we abbreviated:

$$x \equiv x(t) \quad \text{and} \quad x_0 \equiv x(0) \tag{5}$$

- similarly $\bar{a}$ is:

$$\bar{a} \equiv \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t} = a \quad \leftarrow (\bar{a} = a \text{ from } (1)) \tag{6}$$

with abbreviations

$$v \equiv v(t) \quad \text{and} \quad v_0 \equiv v(0) \tag{7}$$

- Solve (4) for x:

$$x = x_0 + \bar{v}t \quad \leftarrow \left(\begin{aligned} \text{From } \bar{v} &= \frac{x - x_0}{t} \\ \bar{v}t &= x - x_0 \\ x_0 + \bar{v}t &= x \end{aligned} \right) \leftarrow \left(\begin{array}{l} \text{Write steps out} \\ \text{whenever you} \\ \text{need to -- it helps!} \end{array} \right) \tag{8}$$

- Solve (6) for v:

$$v = v_0 + at \qquad (\text{no } (x - x_0)) \tag{9}$$

- Since a is constant, $v \equiv v(t)$ is a <u>line</u>, so $\bar{v}$ is the <u>average</u> of initial and final velocities:

$$\bar{v} = \tfrac{1}{2}(v_0 + v) \qquad (\text{no } a, t, (x - x_0)) \tag{10}$$

- Put (10) into (8):

$$x = x_0 + \tfrac{1}{2}(v_0 + v)t \qquad (\text{no } a) \tag{11}$$

- Put (9) into (11):

$$x = x_0 + \tfrac{1}{2}(v_0 + [v_0 + at])t$$

$$x = x_0 + \tfrac{1}{2}(2v_0 + at)t = x_0 + (v_0 + \tfrac{1}{2}at)t$$

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \qquad (\text{no } v) \tag{12}$$

- To eliminate t, solve for t in (9):

$$t = \frac{v - v_0}{a} \tag{13}$$

- Put (13) into (12):

$$x = x_0 + v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 = x_0 + \frac{v_0 v}{a} - \frac{v_0^2}{a} + \frac{1}{2}\frac{a}{a^2}(v^2 - 2vv_0 + v_0^2)$$

$$x = x_0 + \cancel{\frac{v_0 v}{a}} - \frac{v_0^2}{a} + \frac{v^2}{2a} - \cancel{\frac{v v_0}{a}} + \frac{v_0^2}{2a} = x_0 - \frac{v_0^2}{2a} + \frac{v^2}{2a}$$

$$x - x_0 = \frac{1}{2a}(v^2 - v_0^2)$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad (\text{no } t) \tag{14}$$

- To eliminate v_0, solve (9) for v_0:

$$v_0 = v - at \tag{15}$$

- Put (15) into (12):

$$x = x_0 + (v - at)t + \tfrac{1}{2}at^2 = x_0 + vt - at^2 + \tfrac{1}{2}at^2$$

$$x = x_0 + vt - \tfrac{1}{2}at^2 \qquad (\text{no } v_0) \tag{16}$$

- Summary:

(Kinematic Equations for 1-D motion with Constant acceleration and $t_0 \equiv 0$) →

Equation	missing quantity		
$v = v_0 + at$	$x - x_0$	(17a)	(17)
$x = x_0 + v_0 t + \tfrac{1}{2}at^2$	v	(17b)	
$v^2 = v_0^2 + 2a(x - x_0)$	t	(17c)	
$x = x_0 + \tfrac{1}{2}(v_0 + v)t$	a	(17d)	
$x = x_0 + vt - \tfrac{1}{2}at^2$	v_0	(17e)	

- Supplementary: $\quad \bar{v} = \tfrac{1}{2}(v_0 + v) \qquad a, t, x - x_0 \qquad (18)$

How To Bring Back t_0:

- notice equations with t are <u>functions</u> of t (also from our definitions in (5) and (7))
- <u>ex</u>:

$$x(t) = x_0 + v_0 t + \tfrac{1}{2} a t^2 \tag{19}$$

- so now replace argument t with $t - t_0$:

$$x(t-t_0) = x_0 + v_0(t-t_0) + \tfrac{1}{2}a(t-t_0)^2 \tag{20}$$

- Then make new abbrev.:

$$x \equiv x(t) \equiv x(\underset{(\text{old})}{t} - t_0) \tag{21}$$

so

$$x = x_0 + v_0(t-t_0) + \tfrac{1}{2}a(t-t_0)^2 \tag{22}$$

- This can be useful in multi-part problems if we don't want to reset the time in each part. (But we still need to label different t_0's: t_{0_1}, t_{0_2}, etc.)

- <u>ex</u>: You're driving at 100 $\left[\frac{km}{h}\right]$ and notice a parked police car, causing you to brake over a distance of 88.0 [m] until your speed is 80.0 $\left[\frac{km}{h}\right]$.
  - ⓐ What was your acceleration during braking? (assume it was constant)
  - ⓑ How much time did it take you to decrease your speed?
  - ⓒ If the police officer were slow to react and only measured your final speed, but knew the time it took you to pass the visible distance over which you braked, could the officer determine your initial speed?

- Helps to pick positive direction in direction of motion
- Givens:

$$\Delta x = x - x_0 = 88.0\,[m]\,;\ \begin{cases} x \equiv 88.0\,[m] \\ x_0 \equiv 0\,[m] \end{cases} \tag{1}$$

$$v_0 = 100\left[\tfrac{km}{h}\right]\cdot\left(\frac{1000\,[m]}{1\,[km]}\right)\cdot\left(\frac{1\,[h]}{3600\,[s]}\right) = 27.78\left[\tfrac{m}{s}\right] \tag{2}$$

$$v = 80.0\left[\tfrac{km}{h}\right]\cdot\left(\frac{1000\,[m]}{1\,[km]}\right)\cdot\left(\frac{1\,[h]}{3600\,[s]}\right) = 22.22\left[\tfrac{m}{s}\right] \tag{3}$$

- we <u>want a</u>, so the equation needs "a" in it!
  - we have x, x_0, v_0, v and want a, and (17c) has these, so solve it for a:

$$v^2 = v_0^2 + 2a(x - x_0)$$

$$v^2 - v_0^2 = 2a(x-x_0)$$

ⓐ
$$\boxed{a = \frac{1}{2}\frac{v^2 - v_0^2}{x - x_0} = \frac{1}{2}\frac{\left(22.22\left[\tfrac{m}{s}\right]\right)^2 - \left(27.78\left[\tfrac{m}{s}\right]\right)^2}{88.0\,[m]} \approx -1.58\left[\tfrac{m}{s^2}\right]} \tag{4}$$

- In ⓑ, we <u>want t</u> (really $\Delta t = t - t_0 = t - 0 = t$), but now we have a, so we have a, v, v_0 and want t, so all we need is (17a), so solve for t:

$$v = v_0 + at$$

$$v - v_0 = at$$

ⓑ
$$\boxed{t = \frac{v - v_0}{a} = \frac{22.22\left[\tfrac{m}{s}\right] - 27.78\left[\tfrac{m}{s}\right]}{-1.58\left[\tfrac{m}{s^2}\right]} \approx 3.52\,[s]} \tag{5}$$

- For ⓒ, the officer only knows $x - x_0$, v, t, and wants v_0, so since (17d) has these, solve it for v_0:

$$x = x_0 + \tfrac{1}{2}(v + v_0)t$$

$$2(x - x_0) = (v_0 + v)t \tag{6}$$

ⓒ
$$\boxed{v_0 = \frac{2(x-x_0)}{t} - v = \frac{2(88.0\,[m])}{3.52\,[s]} - 22.22\left[\tfrac{m}{s}\right] \approx 27.78\left[\tfrac{m}{s}\right]}$$

which is correct,
so <u>yes</u>, the officer can determine your initial speed but had to assume constant acceleration by this method. (Note: actual methods are more sophisticated)