Section 3.1 – 3.3

Discrete Random Variables and Probability Distribution

Objectives

- Calculate the expected value of a probability distribution.
- Interpret the variance and the standard deviation of a probability distribution.

Random Variables

- A **random variable** is a variable whose numeric value is determined by the outcome of a probability experiment.
- A random variable is called **discrete** if it has either a finite or a countable number of possible values.
- A random variable is called **continuous** if its possible values contain a whole interval of numbers.

Experiment	**Number *X***	**Possible Values of *X***

Probability Distribution

- A **probability distribution of a discrete random variable *X*** is a table or formula that gives the probabilities for every value of the random variable *X*, where $0 \leq P(X = x) \leq 1$ and $\sum P(X = x_i) = 1$.

Properties of a Probability Distribution

1. All of the probabilities are between 0 and 1, inclusive. That is,

$$0 \leq P(X = x) \leq 1.$$

2. The sum of the probabilities is 1. That is, $\sum P(X = x_i) = 1$.

Example: Creating a Discrete Probability Distribution

A fair coin is tossed twice. Let X be the number of heads that are observed.

a. Construct the probability distribution of X.

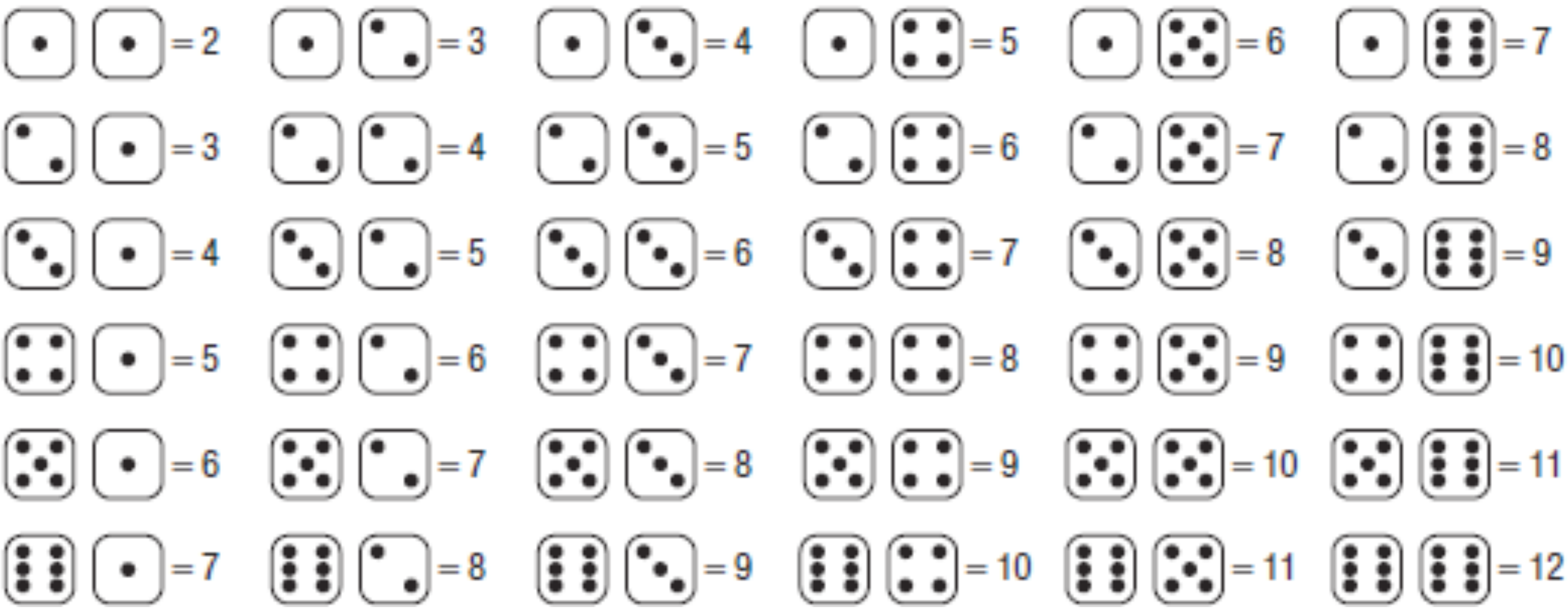
b. Find the probability that at least one head is observed.

Example: Creating a Discrete Probability Distribution

Create a discrete probability distribution for *X*, the sum of two rolled dice. Then find:

a. $P(X \geq 9)$

b. Find the probability that *X* takes an even value.



= 2 = 3 = 4 = 5 = 6 = 7
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Example: Discrete Probability Distribution

Consider the following tables.

x	2	5	6	8
P(X = x)	-1	0.5	0.7	0.8

x	2	5	6	8
P(X = x)	0	0.5	0.3	0.4

Can they be a discrete probability distribution?

Mean (Expected Value)

The **mean (expected value)** for a discrete random variable X is equal to the mean of the probability distribution of X and is given by

$$E(X) = \mu = \sum \left[x_i \cdot P(X = x_i) \right]$$

where x_i is the i^{th} value of the random variable X.

Example: Calculating Expected Values

Suppose that Randall and Blake decide to make a friendly wager on the football game they are watching one afternoon. For every kick the kicker makes, Blake has to pay Randall \$30.00. For every kick the kicker misses, Randall has to pay Blake \$40.00. Prior to this game, the kicker has made 18 of his past 23 kicks this season.

a. Construct the probability distribution of Randall's bet.

b. What is the expected value of Randall's bet for one kick?

c. Suppose that the kicker attempts four kicks during the game. How much should Randall expect to win in total?

Example: Calculating Expected Values

Peyton is trying to decide between two different investment opportunities. The two plans are summarized in the table below. The left column for each plan gives the potential earnings, and the right columns give their respective probabilities. Which plan should he choose?

Investment Plans			
Plan A		**Plan B**	
Earnings	**Probability**	**Earnings**	**Probability**
\$1200	0.1	\$1500	0.3
\$950	0.2	\$800	0.1
\$130	0.4	–\$100	0.2
–\$575	0.1	–\$250	0.2
–\$1400	0.2	–\$690	0.2

Variance and Standard Deviation for a Discrete Probability Distribution

- The **variance** for a discrete probability distribution of a random variable *X* is given by

$$Var(X) = E\left[\left(X - E(X)\right)^2\right] = E[(X-\mu)^2] = \sum(x_i - \mu)^2\, p(x_i)$$

➢ Another formula:

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \sum x_i^2\, p(x_i) - \mu^2$$

where x_i is the i^{th} value of the random variable *X* and μ = E(X) is the mean of the probability distribution.

- The **standard deviation for a discrete probability distribution** of a random variable *X* is the square root of the variance, given by the following formulas.

$$\sigma_X = \sqrt{Var(X)}$$

Example: Find variance and standard deviation of a discrete random variable

A discrete random variable X has the following probability distribution:

x	-1	0	1	4
$P(X = x)$	0.2	0.5	α	0.1

Compute each of the following quantities.

a) α
b) The mean $\mu = E(X)$ of X
c) The variance of X
d) The standard deviation of X

Solution:

Example: Find variance and standard deviation of a discrete random variable

x	-1	0	1	4
$P(X = x)$	0.2	0.5	α	0.1

a) $\alpha = 1 - (0.2 + 0.5 + 0.1) = 0.2$

b) $\mu = E(X) = (-1)(0.2) + (0)(0.5) + (1)(0.2) + (4)(0.1) = 0.4$

c) Way 1: Compute using the definition of variance.

$$Var(X) = E[(X - \mu)^2] = \sum (x_i - \mu)^2 \, p(x_i)$$

x	-1	0	1	4
$x_i - \mu$	-1.4	-0.4	0.6	3.6
$(x_i - \mu)^2$	1.96	0.16	0.36	12.96
$(x_i - \mu)^2 p(x_i)$	0.392	0.08	0.072	1.296

$$Var(X) = 0.392 + 0.08 + 0.072 + 1.296 = 1.84$$

Example: Find variance and standard deviation of a discrete random variable

x	-1	0	1	4
P(X = x)	0.2	0.5	0.2	0.1

Way 2: Compute using the shortcut formula.

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \sum x_i^2\, p(x_i) - \mu^2$$

x	-1	0	1	4
x_i^2	1	0	1	16
$x_i^2\, p(x_i)$	0.2	0	0.2	1.6
$\sum x_i^2\, p(x_i)$	2			

$$Var(X) = 2 - (0.4)^2 = 1.84$$

d) $\sigma_X = \sqrt{1.84} = 1.3565$

Properties of The Expected Value and Variance

1) The expected value of any function *h(X)* is computed by

$$E[h(X)] = \sum h(x)\, p(x)$$

In particular,

$$E(aX + b) = a\,E(X) + b$$

2) The variance of any function *h(X)* is computed by

$$Var[h(X)] = E\left[\left(h(X) - E\big(h(X)\big)\right)^2\right] = \sum \left(h(x) - E\big(h(X)\big)\right)^2 \cdot p(x)$$

In particular,

$$Var(aX + b) = a^2 \cdot Var(X)$$

$$\sigma_{aX+b} = |a| \cdot \sigma_X$$

Example:

A computer store has purchased three computers of a certain type at $500 apiece. It will sell them for $1000 apiece. The manufacturer has agreed to repurchase any computers still unsold after a specified period at $200 apiece. Let X denote the number of computers sold, and suppose that p(0) = 0.1, p(1) = 0.2, p(2) = 0.3, and p(3) = 0.4. Let *h(X)* denote the profit associated with selling X units.

a) What is the expected profit?

b) What is the standard deviation of the profit?

Solution:

a) The probability distribution of X is

x	0	1	2	3
P(X=x)	0.1	0.2	0.3	0.4

$$E(X) = (0)(0.1) + (1)(0.2) + (2)(0.3) + (3)(0.4) = 2$$

The profit function is $h(X) = 1000X - 1500 + 200(3 - X) = 800X - 900.$

$$E\left(h(X)\right) = 800\, E(X) - 900 = 800(2) - 900 = 700$$

The expected profit is $700.

Example:

b)

x	0	1	2	3
x_i^2	0	1	4	9
$x_i^2\, p(x_i)$	0	0.2	1.2	3.6
$\sum x_i^2\, p(x_i)$	5			

$$Var(X) = 5 - (2)^2 = 1$$

$$\sigma_X = \sqrt{Var(X)} = 1$$

$$\sigma_{h(X)} = \sigma_{800X-900} = 800\sigma_X = 800$$

The standard deviation of the profit is $800.