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Lecture 4 Notes

These notes correspond to Section 11.3 in Stewart and Sections 2.3 and 3.1 in Marsden and Tromba.

Partial Derivatives

Now that we have become acquainted with functions of several variables, and what it means for such functions to have limits and be continuous, we are ready to analyze their behavior by computing their instantaneous rates of change, as we know how to do for functions of a single variable. However, in contrast to the single-variable case, the instantaneous rate of change of a function of several variables cannot be described by a single number that represents the slope of a tangent line. Instead, such a slope can only describe how *one* of the function's dependent variables (outputs) varies as *one* of its independent variables (inputs) changes. This leads to the concept of what is known as a *partial derivative*.

Terminology and Notation

- Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a scalar-valued function of a single variable. Recall that the *derivative of $f(x)$ with respect to x at x_0* is defined to be

$$\frac{df}{dx}(x_0) = f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- Let $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a scalar-valued function of two variables, and let $(x_0, y_0) \in D$. The *partial derivative of $f(x, y)$ with respect to x at (x_0, y_0)* is defined to be

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Note that only values of $f(x, y)$ for which $y = y_0$ influence the value of the partial derivative with respect to x . Similarly, the *partial derivative of $f(x, y)$ with respect to y at (x_0, y_0)* is defined to be

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Note the two methods of denoting partial derivatives used above: $\partial f / \partial x$ or f_x for the partial derivative with respect to x . There are other notations, but these are the ones that we will use.

Example Let $f(x, y) = x^2y$, and let $(x_0, y_0) = (2, -1)$. Then

$$f_x(2, -1) = \lim_{h \rightarrow 0} \frac{(2+h)^2(-1) - 2^2(-1)}{h} = \lim_{h \rightarrow 0} \frac{-(4+4h+h^2)+4}{h} = \lim_{h \rightarrow 0} \frac{-4h-h^2}{h} = -4,$$

$$f_y(2, -1) = \lim_{h \rightarrow 0} \frac{2^2(-1+h) - 2^2(-1)}{h} = \lim_{h \rightarrow 0} \frac{4(h-1)+4}{h} = \lim_{h \rightarrow 0} \frac{4h}{h} = 4.$$

□

In the preceding example, the value $f_x(2, -1) = -4$ can be interpreted as the slope of the line that is tangent to the graph of $f(x, -1) = -x^2$ at $x = 2$. That is, we consider the *restriction* of f to the portion of its domain where $y = -1$, and thus obtain a function of the single variable x , $g(x) = f(x, -1) = -x^2$. Note that if we apply differentiation rules from single-variable calculus to g , we obtain $g'(x) = -2x$, and $g'(2) = -4$, which is the value we obtained for $f_x(2, -1)$.

Similarly, if we consider $f_y(2, -1) = 4$, this can be interpreted as the slope of a line that is tangent to the graph of $p(y) = f(2, y) = 4y$ at $y = -1$. Note that if we differentiate p , we obtain $p'(y) = 4$, which, again, shows that the partial derivative of a function of several variables can be obtained by “freezing” the values of all variables except the one with respect to which we are differentiating, and then applying differentiation rules to the resulting function of one variable.

It follows from this relationship between partial derivatives of a function of several variables and the derivative of a function of a single variable that other interpretations of the derivative are also applicable to partial derivatives. In particular, if $f_x(x_0, y_0) > 0$, which is equivalent to $g'(x_0) > 0$ where $g(x) = f(x, y_0)$, we can conclude that f is *increasing* as x varies from x_0 , along the line $y = y_0$. Similarly, if $f_y(x_0, y_0) < 0$, which is equivalent to $p'(y_0) < 0$ where $p(y) = f(x_0, y)$, we can conclude that f is *decreasing* as y varies from y_0 along the line $x = x_0$.

- Let the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{R}^n$ be defined as follows: for each $i = 1, 2, \dots, n$, \mathbf{e}_i has components that are all equal to zero, except the i th component, which is equal to 1. Then these vectors are called the *standard basis vectors* of \mathbb{R}^n .

Example If $n = 3$, then

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We also have that $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$.

- Let $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar-valued function of n variables x_1, x_2, \dots, x_n . Then, the *partial derivative* of f with respect to x_i at $\mathbf{x}_0 \in \mathbb{R}^n$, where $1 \leq i \leq n$, is defined to be

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = f_{x_i}(\mathbf{x}_0)$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}.
\end{aligned}$$

Example Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ be defined by $f(\mathbf{x}) = (\mathbf{c} \cdot \mathbf{x})^2$, where $\mathbf{c} \in \mathbb{R}^4$ is the vector $\mathbf{c} = \langle 4, -3, 2, -1 \rangle$. Let $\mathbf{x}_0 \in \mathbb{R}^4$ be the point $\mathbf{x}_0 = \langle 1, 3, 2, 4 \rangle$. Then, the partial derivative of f with respect to x_2 at \mathbf{x}_0 is given by

$$\begin{aligned}
f_{x_2}(\mathbf{x}_0) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_2) - f(\mathbf{x}_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(\mathbf{c} \cdot (\mathbf{x}_0 + h\mathbf{e}_2))^2 - (\mathbf{c} \cdot \mathbf{x}_0)^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{(\mathbf{c} \cdot \mathbf{x}_0 + h\mathbf{c} \cdot \mathbf{e}_2)^2 - (\mathbf{c} \cdot \mathbf{x}_0)^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{(\mathbf{c} \cdot \mathbf{x}_0)^2 + 2(\mathbf{c} \cdot \mathbf{x}_0)(h\mathbf{c} \cdot \mathbf{e}_2) + (h\mathbf{c} \cdot \mathbf{e}_2)^2 - (\mathbf{c} \cdot \mathbf{x}_0)^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{2h(\mathbf{c} \cdot \mathbf{x}_0)(\mathbf{c} \cdot \mathbf{e}_2) + h^2(\mathbf{c} \cdot \mathbf{e}_2)^2}{h} \\
&= \lim_{h \rightarrow 0} 2(\mathbf{c} \cdot \mathbf{x}_0)(\mathbf{c} \cdot \mathbf{e}_2) + h(\mathbf{c} \cdot \mathbf{e}_2)^2 \\
&= 2(\mathbf{c} \cdot \mathbf{x}_0)(\mathbf{c} \cdot \mathbf{e}_2) \\
&= 2(\langle 4, -3, 2, -1 \rangle \cdot \langle 1, 3, 2, 4 \rangle)(\langle 4, -3, 2, -1 \rangle \cdot \langle 0, 1, 0, 0 \rangle) \\
&= 2[4(1) - 3(3) + 2(2) - 1(4)](-3) \\
&= 2(-5)(-3) \\
&= 30.
\end{aligned}$$

This shows that f is increasing sharply as a function of x_2 at the point \mathbf{x}_0 . Note that the same result can be obtained by defining

$$g(x_2) = f(1, x_2, 2, 4) = (\mathbf{c} \cdot \langle 1, x_2, 2, 4 \rangle)^2 = (\langle 4, -3, 2, -1 \rangle \cdot \langle 1, x_2, 2, 4 \rangle)^2 = (4 - 3x_2)^2,$$

differentiating this function of x_2 to obtain $g'(x_2) = 2(4 - 3x_2)(-3)$, and then evaluating this derivative at $x_2 = 3$ to obtain $g'(3) = 2(4 - 3(3))(-3) = 30$.

- Just as functions of a single variable can have second derivatives, third derivatives, or derivatives of any order, functions of several variables can have higher-order partial derivatives. To that end, let $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar-valued function of n variables x_1, x_2, \dots, x_n . Then, the *second partial derivative of f with respect to x_i and x_j* at $\mathbf{x}_0 \in D$ is defined to be

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = f_{x_i x_j}(\mathbf{x}_0)$$

$$\begin{aligned}
&= \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right) (\mathbf{x}_0) \\
&= \lim_{h_i \rightarrow 0} \frac{f_{x_j}(\mathbf{x}_0 + h_i \mathbf{e}_i) - f_{x_j}(\mathbf{x}_0)}{h_i} \\
&= \lim_{(h_i, h_j) \rightarrow (0,0)} \frac{f(\mathbf{x}_0 + h_i \mathbf{e}_i + h_j \mathbf{e}_j) - f(\mathbf{x}_0 + h_i \mathbf{e}_i) - f(\mathbf{x}_0 + h_j \mathbf{e}_j) + f(\mathbf{x}_0)}{h_i h_j}.
\end{aligned}$$

The second line of the above definition is the most helpful, in terms of describing how to compute a second partial derivative with respect to x_i and x_j : first, compute the partial derivative with respect to x_j . Then, compute the partial derivative of the result with respect to x_i , and finally, evaluate at the point \mathbf{x}_0 . That is, the second partial derivative, or a partial derivative of higher order, can be viewed as an *iterated* partial derivative.

- A commonly used method of indicating that a function is evaluated at a given point, especially if the formula for the function is complicated or otherwise does not lend itself naturally to the usual notation for evaluation at a point, is to follow the function with a vertical bar, and indicate the evaluation point as a subscript to the bar. For example, given a function $f(x)$, we can write

$$f'(4) = \left. \frac{df}{dx} \right|_{x=4} = \left. \frac{df}{dx} \right|_4$$

or, given a function $f(x, y)$, we can write

$$f_x(2, 3) = \left. \frac{\partial f}{\partial x} \right|_{x=2, y=3} = \left. \frac{\partial f}{\partial x} \right|_{(2,3)}.$$

This notation is similar to the use of the vertical bar in the evaluation of definite integrals, to indicate that an antiderivative is to be evaluated at the limits of integration.

Results

The following theorem is very useful for reducing the amount of work necessary to compute all of the higher-order partial derivatives of a function.

- *Clairaut's Theorem:* Let $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, and let $\mathbf{x}_0 \in D$. If the second partial derivatives f_{xy} and f_{yx} are continuous on D , then they are equal:

$$f_{xy}(\mathbf{x}_0) = f_{yx}(\mathbf{x}_0).$$

Example Let $f(x, y) = \sin(2x) \cos^2(4y)$. Then

$$f_x = 2 \cos^2(4y) \cos(2x), \quad f_y = -8 \sin(2x) \cos(4y) \sin(4y),$$

which yields

$$f_{xy} = (2 \cos^2(4y) \cos(2x))_y = -16 \cos(2x) \cos(4y) \sin(4y)$$

and

$$f_{yx} = (-8 \sin(2x) \cos(4y) \sin(4y))_x = -16 \cos(2x) \cos(4y) \sin(4y),$$

and we conclude that these *mixed partial derivatives* are equal. \square

Techniques

We now describe the most practical techniques for computing partial derivatives.

- As mentioned previously, computing the partial derivative of a function with respect to a given variable, at a given point, is equivalent to “freezing” the values of all *other* variables at that point, and then computing the derivative of the resulting function of one variable at that point.

However, generally it is most practical to compute the partial derivative as a *function* of all of the independent variables, which can then be evaluated at any point at which we wish to know the *value* of the partial derivative, just as when we have a function $f(x)$, we normally compute its derivative as a function $f'(x)$, and then evaluate that function at any point x_0 where we want to know the rate of change.

Therefore, the most practical approach to computing a partial derivative of a function f with respect to x_i is to apply differentiation rules from single-variable calculus to differentiate f with respect to x_i , while *treating all other variables as constants*. The result of this process is a *function* that represents $\partial f / \partial x_i(x_1, x_2, \dots, x_n)$, and then values can be substituted for the independent variables x_1, x_2, \dots, x_n .

Example To compute $f_x(\pi/2, \pi)$ of $f(x, y) = e^{-(x^2+y^2)} \sin 3x \cos 4y$, we treat y as a constant, since we are differentiating with respect to x . Using the Product Rule and the Chain Rule from single-variable calculus, as well as the rules for differentiating exponential and trigonometric functions, we obtain

$$\begin{aligned} f_x(\pi/2, \pi) &= \left. \frac{\partial}{\partial x} [e^{-(x^2+y^2)} \sin 3x \cos 4y] \right|_{x=\pi/2, y=\pi} \\ &= \cos 4y \left. \frac{\partial}{\partial x} [e^{-(x^2+y^2)} \sin 3x] \right|_{x=\pi/2, y=\pi} \\ &= \cos 4y \left[\sin 3x \frac{\partial}{\partial x} [e^{-(x^2+y^2)}] + e^{-(x^2+y^2)} \frac{\partial}{\partial x} [\sin 3x] \right] \Big|_{x=\pi/2, y=\pi} \\ &= \cos 4y \left[e^{-(x^2+y^2)} \sin 3x \frac{\partial}{\partial x} [-(x^2 + y^2)] + 3e^{-(x^2+y^2)} \cos 3x \right] \Big|_{x=\pi/2, y=\pi} \end{aligned}$$

$$\begin{aligned}
&= \cos 4y \left[-2xe^{-(x^2+y^2)} \sin 3x + 3e^{-(x^2+y^2)} \cos 3x \right] \Big|_{x=\pi/2, y=\pi} \\
&= \cos 4\pi \left[-2(\pi/2)e^{-((\pi/2)^2+\pi^2)} \sin(3\pi/2) + 3e^{-((\pi/2)^2+\pi^2)} \cos 3(\pi/2) \right] \\
&= \pi e^{-5\pi^2/4}.
\end{aligned}$$

Similarly, to compute $f_y(\pi/2, \pi)$, we treat x as a constant, and apply these differentiation rules to differentiate with respect to y . Finally, we substitute $x = \pi/2$ and $y = \pi$ into the resulting derivative.

This approach to differentiation can also be applied to compute higher-order partial derivatives, *as long as any substitution of values for the variables is deferred to the end*.

Example To evaluate the second partial derivatives of $f(x, y) = \ln |x + y^2|$ at $x = 1$, $y = 2$, we first compute the first partial derivatives of f :

$$\begin{aligned}
f_x &= \frac{1}{x + y^2} \frac{\partial}{\partial x} [x + y^2] = \frac{1}{x + y^2}, \\
f_y &= \frac{1}{x + y^2} \frac{\partial}{\partial y} [x + y^2] = \frac{2y}{x + y^2}.
\end{aligned}$$

Next, we differentiate each of these partial derivatives with respect to both x and y to obtain

$$\begin{aligned}
f_{xx} &= (f_x)_x \\
&= \left(\frac{1}{x + y^2} \right)_x \\
&= -\frac{1}{(x + y^2)^2} \frac{\partial}{\partial x} [x + y^2] \\
&= -\frac{1}{(x + y^2)^2}, \\
f_{xy} &= (f_x)_y \\
&= \left(\frac{1}{x + y^2} \right)_y \\
&= -\frac{1}{(x + y^2)^2} \frac{\partial}{\partial y} [x + y^2] \\
&= -\frac{2y}{(x + y^2)^2}, \\
f_{yx} &= f_{xy} \\
&= -\frac{2y}{(x + y^2)^2}, \\
f_{yy} &= (f_y)_y
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{2y}{x+y^2} \right)_y \\
&= \frac{(x+y^2)(2y)_y - 2y(x+y^2)_y}{(x+y^2)^2} \\
&= \frac{2(x+y^2) - 4y^2}{(x+y^2)^2} \\
&= \frac{2(x-y^2)}{(x+y^2)^2}.
\end{aligned}$$

Finally, we can evaluate these second partial derivatives at $x = 1$ and $y = 2$ to obtain

$$f_{xx}(1, 2) = -\frac{1}{25}, \quad f_{xy}(1, 2) = f_{yx}(1, 2) = -\frac{4}{25}, \quad f_{yy}(1, 2) = -\frac{6}{25}.$$

□

Example Let $f(x, y, z) = x^2y^4z^3$. We will compute the second partial derivatives of this function at the point $(x_0, y_0, z_0) = (-1, 2, 3)$ by repeated computation of first partial derivatives. First, we compute

$$f_x = (x^2y^4z^3)_x = (x^2)_x y^4 z^3 = 2xy^4z^3,$$

by treating y and z as constants, then

$$f_y = (x^2y^4z^3)_y = (y^4)_y x^2 z^3 = 4x^2y^3z^3,$$

by treating x and z as constants, and then

$$f_z = (x^2y^4z^3)_z = x^2y^4(z^3)_z = 3x^2y^4z^2.$$

We then differentiate each of these with respect to x , y and z to obtain the second partial derivatives:

$$\begin{aligned}
f_{xx} &= (f_x)_x = (2xy^4z^3)_x = 2y^4z^3, \\
f_{xy} &= (f_x)_y = (2xy^4z^3)_y = (2x)(4y^3)(z^3) = 8xy^3z^3, \\
f_{xz} &= (f_x)_z = (2xy^4z^3)_z = (2xy^4)(3z^2) = 6xy^4z^2, \\
f_{yx} &= (f_y)_x = (4x^2y^3z^3)_x = 8xy^3z^3, \\
f_{yy} &= (f_y)_y = (4x^2y^3z^3)_y = (4x^2)(3y^2)(z^3) = 12x^2y^2z^3, \\
f_{yz} &= (f_y)_z = (4x^2y^3z^3)_z = (4x^2y^3)(3z^2) = 12x^2y^3z^2, \\
f_{zx} &= (f_z)_x = (3x^2y^4z^2)_x = 6xy^4z^2, \\
f_{zy} &= (f_z)_y = (3x^2y^4z^2)_y = (3x^2)(4y^3)(z^2) = 12x^2y^3z^2, \\
f_{zz} &= (f_z)_z = (3x^2y^4z^2)_z = (3x^2y^4)(2z) = 6x^2y^4z.
\end{aligned}$$

Then, these can be evaluated at (x_0, y_0, z_0) by substituting $x = -1$, $y = 2$, and $z = 3$ to obtain

$$\begin{aligned} f_{xx}(-1, 2, 3) &= 864, & f_{xy}(-1, 2, 3) &= -1728, & f_{xz}(-1, 2, 3) &= -864, \\ f_{yx}(-1, 2, 3) &= -1728, & f_{yy}(-1, 2, 3) &= 1296, & f_{yz}(-1, 2, 3) &= 864, \\ f_{zx}(-1, 2, 3) &= -864, & f_{zy}(-1, 2, 3) &= 864, & f_{zz}(-1, 2, 3) &= 288. \end{aligned}$$

Note that the order in which partial differentiation operations occur does not appear to matter; that is, $f_{xy} = f_{yx}$, for example. That is, Clairaut's Theorem applies for any number of variables. It also applies to any order of partial derivative. For example,

$$\begin{aligned} f_{xyy} &= (f_{xy})_y = (8xy^3z^3)_y = 24xy^2z^3, \\ f_{yyx} &= (f_{yy})_x = (12x^2y^2z^3)_x = 24xy^2z^3. \end{aligned}$$

□

- In single-variable calculus, *implicit differentiation* is applied to an equation that *implicitly* describes y as a function of x , in order to compute dy/dx . The same approach can be applied to an equation that implicitly describes any number of dependent variables in terms of any number of independent variables. The approach is the same as in the single-variable case: differentiate both sides of the equation with respect to the independent variable, leaving derivatives of dependent variables in the equation as unknowns. The resulting equation can then be solved for the unknown partial derivatives.

Example Consider the equation

$$x^2z + y^2z + z^2 = 1.$$

If we view this equation as one that implicitly describes z as a function of x and y , we can compute z_x and z_y using implicit differentiation with respect to x and y , respectively. Applying the Product Rule yields the equations

$$2xz + x^2z_x + y^2z_x + 2zz_x = 0,$$

$$x^2z_y + 2yz + y^2z_y + 2zz_y = 0,$$

which can then be solved for the partial derivatives to obtain

$$z_x = -\frac{2xz}{x^2 + y^2 + 2z}, \quad z_y = -\frac{2yz}{x^2 + y^2 + 2z}.$$

Practice Problems

1. For each of the following functions, compute all first partial derivatives.

(a)

$$f(x, y) = \sec x \tan y + \ln |\sec x + \tan y|$$

(b)

$$f(u, v) = \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \cos u$$

(c)

$$f(x, y, z) = 2\sqrt{x^2 + y^2 + z^2}$$

2. For each of the following functions, determine whether the function is increasing as a function of x , and as a function of y , when $x = -1$ and $y = 1$.

(a)

$$f(x, y) = xy + y^2$$

(b)

$$f(x, y) = \frac{\sin(\pi x) \cos(\pi y)}{\sqrt{x^2 + y^2}}$$

(c)

$$f(x, y) = \frac{e^x + e^y}{e^x - e^y}$$

3. For each of the functions from Problem 2, compute all second partial derivatives.
4. Given a function $z = f(x, y)$, what can the second partial derivatives tell us about the graph of a function?
5. Use implicit differentiation to compute w_x , w_y and w_z , where $w = f(x, y, z)$ is implicitly described by the equation

$$\sin(wx) \cos(wy) = \ln(w^2 + z^2).$$

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Additional Practice Problems

Additional practice problems from the recommended textbooks are:

- Stewart: Section 11.3, Exercises 1-31 odd, 37-55 odd
- Marsden/Tromba: Section 2.3, Exercises 1, 3; Section 3.1, Exercises 1-9 odd