Introduction to differential equations, initial value problems, non-uniqueness

MA221, Lecture 1

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What is a differential equation?

- Numerical equations are equations with numerical solutions

  $\hookrightarrow x^2 - 1 = 0 \quad \ldots \quad x = \pm 1$

- Functional equations are equations with function solutions

  $\hookrightarrow (f(x))^2 - 1 = 0 \quad \ldots \quad f(x) = \pm 1$ & infinitely many other solutions

- Informally, a DiffEq is a functional equation involving derivatives of the solution functions

What is a differential equation?

Ex: $f''(x) + f'(x) = 0$ ODE 2nd order linear

$f(x)f'(x) + [f''(x)]^5 = f^{(4)}(x)$ ODE 4th order non-linear

$\hookrightarrow yy' + [y'']^5 = y^{(4)}$

$\hookrightarrow y\frac{dy}{dx} + \left[\frac{d^2y}{dx^2}\right]^5 = \frac{d^4y}{dx^4}$

Classification of differential equations

- Classify by "type": ODE or PDE?

  "ordinary" solutions are univariate

  "partial" solutions are multivariate

- Classify by "order": highest order derivative is...?

- Classify by "linearity": linear or non-linear?

What are some examples of differential equations?

Example: $y' = 0$ has solution $y = C$; $C \in \mathbb{R}$

Ex: $y'' = 0$ has solution $y = mx + b$; $m, b \in \mathbb{R}$

What are some examples of differential equations?

Ex: $y'' + y = 0$. $\underbrace{y = A \sin x + B \cos x}_{\text{how to verify?}}$

$y' = A\cos x - B\sin x$

$y'' = -A\sin x - B\cos x$

$\Rightarrow y'' + y = 0$. ✓

What are some examples of differential equations?

Verifying (explicit) solutions

Example: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{y}$ has an explicit solution $\underbrace{y = \dfrac{1}{16}x^4}$

$\dfrac{dy}{dx} = \dfrac{1}{4}x^3$

$(LHS) = \dfrac{1}{4}x^3$

$(RHS) = x\sqrt{y} = x\sqrt{\dfrac{1}{16}x^4} = x\left(\dfrac{1}{4}x^2\right) = \dfrac{1}{4}x^3$ ✓

general solution: $y = \left(\dfrac{x^2}{4} + C\right)^2$

Verifying (implicit) solutions

Example: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x}{y}$ has an implicit solution $x^2+y^2 \overset{\star}{=} 1$

Use implicit differentiation on this

$$(\text{RHS of } \star)' = 0$$

$$\|$$

$$(\text{LHS of } \star)' = 2x + 2y\,\frac{dy}{dx}$$

$$2x + 2y\,\frac{dy}{dx} = 0$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

✓

general solution: $x^2+y^2 = C$

Initial value problems

Example: $y'' + y \overset{*}{=} 0$ subject to $y'\left(\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right) = 1$ is an IVP. What is a solution to this IVP?

recall, general solution to $*$ is $y = A\sin x + B\cos x$

$$1 = y\left(\tfrac{\pi}{2}\right) = A\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} + B\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} = A$$

Since $y' = A\cos x - B\sin x$,

$$1 = y'\left(\tfrac{\pi}{2}\right) = A\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} - B\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} = -B$$

$$\Longrightarrow y = \sin x - \cos x$$

Non-Uniqueness

While we expect a given differential equation to have multiple solutions ("a family of solutions"), it is also possible for an IVP to have more than one solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{y} \quad \text{subject to} \quad y(0) = 0$$