Exact equations and integrating factors

MA221, Lecture 5

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Exact Equations

Suppose that you have an equation of the form

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} \overset{\star}{=} 0.$$

This equation is **exact** if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y).$$

To solve exact equations, we make use of two facts:

- **Fact 1 (the exactness test):** If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $\star$ is exact.
- **Fact 2:** The solution to $(\star)$ is given *implicitly* by the equation $F(x,y) = C$, for any constant C, and where F is the function from the definition of an "exact equation."

The exactness test

If the equation is exact...

$$
\begin{tikzcd}
 & M \arrow[r, "\frac{\partial}{\partial y}"] & \frac{\partial M}{\partial y} \arrow[dd, equal] \\
F \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & \\
 & N \arrow[r, "\frac{\partial}{\partial x}"'] & \frac{\partial N}{\partial x}
\end{tikzcd}
$$

equality implies exactness

Example 1: $\dfrac{x+y}{1+y^2}\dfrac{\mathrm{d}y}{\mathrm{d}x} = -x - \arctan y$ ... $M + N\dfrac{dy}{dx} = 0$

$$\underbrace{x + \arctan y}_{M} + \underbrace{\frac{x+y}{1+y^2}}_{N}\frac{dy}{dx} = 0$$

exactness test:

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial N}{\partial x} = \left[\frac{x+y}{1+y^2}\right]_x = \left[\frac{x}{1+y^2} + \frac{y}{1+y^2}\right]_x = \frac{1}{1+y^2}$$

$$\implies \frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

$$\frac{\partial F}{\partial x} = M \Rightarrow F = \int \frac{\partial F}{\partial x}\,dx = \int M\,dx = \int (x+\arctan y)\,dx$$

$$= \frac{x^2}{2} + x\arctan y + C(y)$$

$$\Rightarrow C(y) = F - \frac{x^2}{2} - x\arctan y$$

$$\Rightarrow C'(y) = \frac{\partial F}{\partial y} - 0 - \frac{x}{1+y^2} = N - \frac{x}{1+y^2}$$

$$= \frac{x+y}{1+y^2} - \frac{x}{1+y^2} = \frac{y}{1+y^2}$$

$$\Rightarrow C(y) = \int C'(y)\,dy = \int \frac{y}{1+y^2}\,dy \quad \left[\begin{array}{l} u = 1+y^2 \\ du = 2y\,dy \end{array}\right]$$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+y^2) + C$$

$$\Rightarrow F = \frac{x^2}{2} + x\arctan y + C(y)$$

$$\Rightarrow F = \frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2) + C$$

$$\Rightarrow \frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2) + C = D$$

$$\Rightarrow \boxed{\frac{x^2}{2} + x\arctan y + \frac{1}{2}\ln(1+y^2) = C}$$

Exact equations

Example 2: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}$ subject to $y(0) = 2$

Rewrite as $\underbrace{(\cos x \sin x - xy^2)}_{M} + \underbrace{y(1-x^2)}_{N} \dfrac{dy}{dx} = 0$

$$\underbrace{\frac{\partial M}{\partial y} = -2xy \quad \dots \quad \frac{\partial N}{\partial x} = [y(1-x^2)]_x = [y - x^2 y]_x = -2xy}_{\text{equation is exact! There exists } F(x,y) \text{ such that}}$$

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \underbrace{\frac{\partial F}{\partial y} = N}$$

previous example we started with $\dfrac{\partial F}{\partial x} = M$;

in this example, we'll use $\dfrac{\partial F}{\partial y} = N$.

$$F = \int \frac{\partial F}{\partial y}\,dy = \int N\,dy = \int y(1-x^2)\,dy = \frac{y^2(1-x^2)}{2} + C(x)$$

$$\Rightarrow C(x) = F - \frac{y^2}{2} + \frac{y^2x^2}{2}$$

$$\Rightarrow C'(x) = \frac{\partial F}{\partial x} - 0 + xy^2 = N + xy^2$$

$$= \cos x \sin x - xy^2 + xy^2 = \cos x \sin x$$

$$\Rightarrow C(x) = \int \cos x \sin x\,dx = \int u\,du = \frac{u^2}{2} + C$$

$u = \sin x$

$du = \cos x\,dx$

$$= \frac{\sin^2 x}{2} + C$$

$$\Rightarrow F = \frac{y^2(1-x^2)}{2} + \frac{\sin^2 x}{2} + C$$

$$\Rightarrow \text{solution: } \frac{y^2(1-x^2)}{2} + \frac{\sin^2 x}{2} = C$$

Now, we account for the initial value $y(0)=2$:

$$C = \frac{2^2(1-0^2)}{2} + \frac{\sin^2 0}{2} = 2 + 0 = 2.$$

So $$\frac{y^2(1-x^2)}{2} + \frac{\sin^2 x}{2} = 2$$

or $$\boxed{y^2(1-x^2) + \sin^2 x = 4}$$

"Troubleshooting" non-exact equations

If our equation isn't exact, what can we do?

- If $P = \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x, then

  define $\rho(x) = e^{\int P(x)\,dx}$

- If $P = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y, then

  define $\rho(y) = e^{\int P(y)\,dy}$

Example 3: $\underbrace{xy}_{M}\,\mathrm{d}x + \underbrace{(2x^2 + 3y^2 - 20)}_{N}\,\mathrm{d}y \overset{*}{=} 0 \ldots$ $M + N\frac{dy}{dx} = 0$

$$\Updownarrow$$

$$M\,dx + N\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= x \\ \frac{\partial N}{\partial x} &= 4x \end{aligned}\right] \text{not exact!}$$

$$P = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

$$\rho(y) = e^{\int P(y)\,dy} = e^{\int \frac{3}{y}\,dy} = e^{3\ln|y|} = e^{\ln|y|^3}$$

$$= |y|^3 = y^3$$

under constraint $y > 0$

multiply both sides of ☆ by $\check{\rho}(y) = y^3$:

$$☆ \implies \underbrace{xy^4}_{\tilde{M}}dx + \underbrace{y^3(2x^2+3y^2-20)}_{\tilde{N} = 2x^2y^3+3y^5-20y^3}dy = 0 \quad ☆'$$

$$\left.\begin{aligned} \frac{\partial \tilde{M}}{\partial y} &= 4xy^3 \\ \frac{\partial \tilde{N}}{\partial x} &= 4xy^3 \end{aligned}\right] \quad \begin{aligned} &☆' \text{ is exact,} \\ &\text{even though } ☆ \text{ is not.} \end{aligned}$$