math426_math626_assignment_1_samir_banjara

Samir Banjara

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Question 1: State the fundamental theorem of invertible matrices. Use *Lists* to format the equivalent statements.

Let $\mathbf{A}$ be an $n \times n$ matrix. Then, $\mathbf{A}$ is invertible if there exists an $n \times n$ matrix $\mathbf{B}$ such that $AB = BA = I$.

- **Equivalent statements**
  - A has an inverse of A^{-1},
  - $\text{rank}(A) = m$,
  - $\text{range}(A) = \mathbb{C}^m$
  - $\text{null}(A) = \{0\}$
  - $0$is not an eigenvalue of$A$,
  - $0$is not a singular value of$A$,
  - $\det(A) \neq 0$.

Consider the $n \times 2n$ augmented matrix $C = (A \mid I_n)$.

Then,

$$A^{-1}C = (A^{-1}A \mid A^{-1}I_n) = (I_n \mid A^{-1}) \tag{1}$$

Because A^{-1} is the product of elementary matrices, $A^{-1} = E_p E_{p-1} \dots E_1$ thus, equation1 becomes

$$E_p E_{p-1} \cdots E_1 (A \mid I_n) = A^{-1}C = (I_n \mid A^{-1})$$

Then,

$$E_p E_{p-1} \dots E_1 (A \mid I_n) = (I_n \mid B) \tag{2}$$

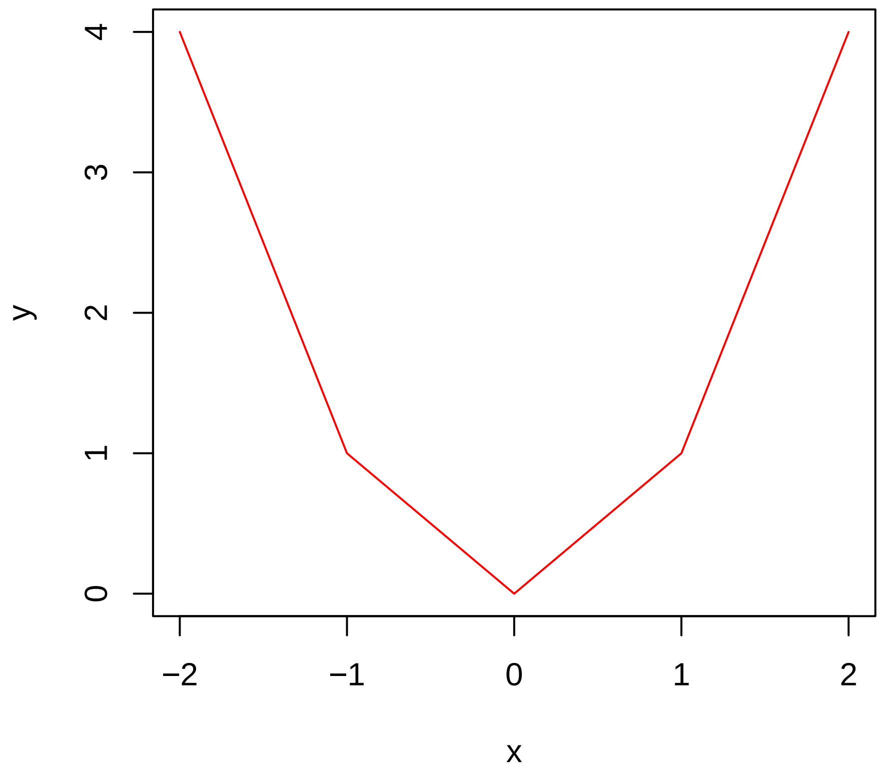
Letting $M = E_p E_{p-1} \dots E_1$, we have from 2, that

$$(MA \mid M) = M(A \mid I_n) = (I_n \mid B)$$

Hence,$MA = I_n$ and $M = B$. It follow that $M = A^{-1}$.

Question 2: Generate a code block and plot the function $y = x^2$ is red from -2 to 2. Make sure the code as well as the output are displayed in the pdf.

```r
myFunction = function(x, a){
  result = x^a
  return (result)
}
x = -2:2
y = myFunction(x,2)
plot(x, y, col = "red", type = 'l', xlab="x", ylab="y")
```



Question 3: Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix.

1. Show that $Range(A)$ is the space spanned by the columns of$A$.

Proof: Let V and W be a vector space, and let transformation representing matrix A be linear $(T : V \to W)$. If $\beta = v_1, v_2, \cdots, v_n$ is a basis for V then,

$$R(T) = \text{span}\left(\{T(v_1), T(v_2), \cdots, T(v_n)\}\right.$$

Clearly $T(v_i) \in R(T), \quad$ for each i. Because $R(T)$ is a subspace, $R(T)$ contains

$$\text{span}\left(\{T(v_1), T(v_2), \cdots, T(v_n)\} = \text{span}\left(T(B)\right)\right. \tag{3}$$

Suppose $w \in R(T)$. Then, $w = T(v) \quad \exists v \in V$. Because β is a basis for V we have,

$$v = \sum_{i=1}^{n} a_i v_i \quad \exists a_1, a_2, \cdots, a_n \in F \tag{4}$$

Since T is linear, $w = T(V) = \sum_{i=1}^{n} a_i T(v_i) \in \text{span}(T(\beta))$.

So, $R(T)$ is contained in span $(T(\beta))$

2. Show that $dim\,(Null(A)) + dim\,(Range(A)) = n$. This is referred to as *Rank Theorem.*

Proof: Let V and W be vector space and let the linear transformation $T : V \to W$ represent a matrix, if V is finite dimensional then, $dim(R(T)) + dim(N(T)) = dim(V)$.

If W is a subspace of a finite dimension vector space V. Then any basis for W can be extended to a basis for V. Thus, we claim $S = \{T(v_{k+1}), T(v_{v+2}), \cdots, T(v)\}$ is a basis for $R(t)$.

We first need to prove that S generates $R\,(T\,)$. Suppose,dim$(V) = n,\ \dim(N(T)) = k,\ and\ \{v_1, v_2, \cdots, v_k\}$ is a basis for $N(T)$ using the fact that $T\,(v_i\,) = 0$ for $1 \le i \le k$ we have,

$$\begin{aligned}R(T) &= \operatorname{span}\left(\{T\,(v_1\,), T\,(v_2\,), \cdots, T\,(v_n\,)\}\right. \\ &= \operatorname{span}\left(\{T\,(v_{k+1}\,), T\,(v_{k+2}\,), \cdots, T\,(v_n\,)\}\right. \\ &= \operatorname{span}\,(S\,)\end{aligned}$$

Now we prove that S is linearly independent. Suppose that,

$$\sum_{i\,=\,k+1}^{n} b_i\,T\,(v_i\,) = 0 \quad \text{for } b_{k+1}, b_{k+2}, \cdots, b_n \in F$$

using the fact that T is linear we have,

$$T\left(\sum_{i\,=\,k+1}^{n} b_i\,v_i\right) = 0$$

so,

$$\sum_{i\,=\,k+1}^{n} b_i\,v_i \in N\,(T\,)$$

Hence, there exists $c_1, c_2, \cdots, c_k \in F$ such that,

$$\sum_{i\,=\,k+1}^{n} b_i\,v_i = \sum_{i\,=\,1}^{k} c_i\,v_i \quad \text{or} \quad \sum_{i=1}^{k} (-c_i\,)\,v_i + \sum_{i\,=\,k+1}^{n} b_i\,v_i = 0$$

Since, β is a basis for \$V\$, we have $b_i = 0$ for all i.

Hence, S is linearly independent and $T\,(v_{k+1}\,), T\,(v_{k+2}\,), \cdots, T\,(v_n\,)$ are distinct;

$$\begin{aligned}\therefore\ \operatorname{rank}\,(T\,) &= n - k \\ dim(R(T)) &= dim(V) - dim(N(T)) \\ dim(R(T)) + dim(N(T)) &= dim(V)\end{aligned}$$