Hw_1_Infinite_Series_Samir_Banjara

Pledge: I pledge my honor that I have abided by the Stevens Honor System. Signature: Samir Banjara

Question 1

Determine whether each sequence is convergent or divergent. If convergent find the limit.

a)

$$\lim_{n \to \infty} ne^{-n}$$

Solution: First lets convert it,

$$\lim_{n \to \infty} ne^{-n} \implies \lim_{n \to \infty} \frac{n}{e^n}$$

Taking the limit give us the indeterminate form,

$$\lim_{n \to \infty} \frac{n}{e^n} = \frac{\infty}{\infty}$$

We can use L' Hopital's rule and take the derivative of both the numerator and denominator, which give us, $\lim_{n \to \infty} \frac{1}{e^n}$, and then we can take the limit again.

$$\lim_{n \to \infty} \frac{n}{e^n} := \lim_{n \to \infty} \frac{1}{e^n} = \frac{1}{e^\infty} = 0$$

b)

$$\lim_{n \to \infty} \frac{\sin^2 n}{n+3}$$

Solution: We can use the squeeze theorem to derive the limit.

- Setting up bounds. - For all $\forall$ real $\mathbb{R}$ number n, $\forall\, n \in \mathbb{R}$,

$$-1 \leq \sin n \leq 1$$

  - Squaring both sides

$$0 \leq \sin^2 n \leq 1$$

  This is because $\sin n$ oscillates between -1 and 1, so $\sin^2 n$ oscillates between 0 and 1.

- Divide the inequality by $n+3$ - Since $n+3$ is positive for all $n \geq 1$, we can divide the inequality without changing the direction of the inequality:

$$0 \leq \frac{\sin^2 n}{n+3} \leq \frac{1}{n+3}$$

  Consider the Sequence for the Lower and Upper Bounds
- Lets us define
  - Lower bound sequence: $a_n = 0$
  - Upper bound sequence: $b_n = \frac{1}{n+3}$
  - Given sequence: $c_n = \frac{\sin^2 n}{n+3}$

So we have:

$$a_n \leq c_n \leq b_n \text{ for all } n \geq 1$$

- Computing the limits of the Lower and Upper bound as $n \to \infty$: - Lower bound:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$$

  - Upper bound:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+3} = 0$$

Application of Squeeze Theorem (Sandwich Theorem)': The Squeeze Theorem states that if $a_n \leq c_n \leq b_n$ for all n beyond some index N, and if:

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

then,

$$\lim_{n\to\infty} c_n = L$$

Since both the Lower and Upper bounds converge to 0 it follows that,

$$\lim_{n\to\infty} \frac{\sin^2 n}{n+3} = 0$$

c)

$$\lim_{n\to\infty} \frac{5n^2+2}{3n^2+3}$$

Solution: First, lets divide all terms in the numerator and denominator by n^2, which is the greatest power in the denominator.

$$\lim_{n\to\infty} \frac{\frac{5n^2}{n^2} + \frac{2}{n^2}}{\frac{3n^2}{n^2} + \frac{3}{n^2}} \implies \lim_{n\to\infty} \frac{5 + \frac{2}{n^2}}{3 + \frac{3}{n^2}}$$

Then we can take the limit,

$$\lim_{n\to\infty} \frac{5 + \frac{2}{n^2}}{3 + \frac{3}{n^2}} = \frac{5 + \frac{2}{\infty^2}}{3 + \frac{3}{\infty^2}} = \frac{5+0}{3+0} = \frac{5}{3}$$

Question 2

a)

Express the repeating decimal $0.45454545\ldots$ as a geometric series.

Solution:

$$.454545\ldots = 0.45 + .0045 + .000045 + \cdots = \sum_{n=1}^{\infty} 0.45 \cdot (0.01)^{n-1}$$

b)

Use the sum formula for a convergent geometric series to express this decimal as a rational number, e.g. as a quotient of two integers.

Solution: This is a geometric series with common ratio $r = 0.01$, and initial term $a = 0.45$.

Since $|r| < 1$, this series converges.

$$0.454545\ldots = \text{sum} = \frac{a}{1-r} = \frac{0.45}{1-(0.01)} = \frac{0.45}{0.99} = \frac{45}{99}$$

Question 3

a)

Prove that if $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty}(a_n + b_n)$ diverges.

Hint: To derive a contradiction assume $\sum_{n=0}^{\infty}(a_n + b_n)$ converges and consider $\sum_{n=0}^{\infty}(a_n + b_n) - \sum_{n=0}^{\infty} a_n$

Solution: We know through through the Algebraic Property of Convergent Series (Linearity of Series) that if two series $\sum a_n$ & $\sum b_n$ are convergent, then the following linear combination of series is also convergent.

$$\sum(a_n + b_n) = \sum a_n + \sum b_n$$
$$\sum(a_n - b_n) = \sum a_n - \sum b_n$$

Proof:

Assume $\sum(a_n + b_n)$ is convergent.

Given that $\sum a_n$ is convergent, by the Linearity of Series, we can say that $\sum(a_n + b_n) - a_n = \sum b_n$ converges.

This contradicts the given information $\sum b_n$ is divergent.

Conclusion: Therefore, by contradiction the initial assumption was incorrect and $\sum(a_n + b_n)$ is convergent.

b)

Is the series $\sum_{n=0}^{\infty} \dfrac{n + (-1)^n}{n^2}$ convergent or divergent? Explain!

Solution:

Proof:

First, we observe that for all $n \geq 1$:

$$\sum_{n=0}^{\infty} \frac{n + (-1)^n}{n^2} = \frac{n}{n^2} + \frac{(-1)^n}{n^2}$$

The first series $\sum_{n=0}^{\infty} \frac{n}{n^2}$ is equivalent to the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent.

The second series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$, is a convergent alternating series.

We can rewrite it as,

$$S = \sum_{n=0}^{\infty} \frac{n + (-1)^n}{n^2} = \frac{n}{n^2} + \frac{(-1)^n}{n^2} = \frac{1}{n} + \frac{(-1)^n}{n^2}$$

Thus the series can be expressed as the sum of two series by the Linearity of Series. The partial sums of the original series is shown bellow,

$$
\begin{aligned}
S_N &= \sum_{n=0}^{\infty} \frac{n+(-1)^n}{n^2} \\
&= \sum_{n=0}^{\infty} \frac{n}{n^2} + \frac{(-1)^n}{n^2} \\
&= \sum_{n=0}^{\infty} \frac{1}{n} + \frac{(-1)^n}{n^2} \\
&= \sum_{n=0}^{\infty} \frac{1}{n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}
\end{aligned}
$$

1. Divergence of the first partial sum $H = \sum_{n=0}^{\infty} \frac{1}{n}$:

The series $H = \sum_{n=0}^{\infty} \frac{1}{n}$ is a harmonic series that diverges to infinity as N approaches infinity.

This is easily seen by the integral test.

$$
\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t \to \infty} \ln t - \ln 1 = \infty
$$

2. Convergence of $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n^2}$:

Conditions for the alternating series test $A = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n^2}$:

- Let $b_n = \dfrac{1}{n^2}$
- $b_{n+1} > 0$ for all $n \geq 1$
- $b_{n+1} = \dfrac{1}{(n+1)^2} < \dfrac{1}{n^2} = b_n$
    - Thus $\{b_n\}$ is a decreasing sequence.
- $\lim_{n \to \infty} b_n = 0$

By the Alternating Series Test, the series A converges.

- Moreover, since the series $\sum_{n=1}^{\infty} b_n$ converges, which is a $p\text{-series}$, with $p = 2 > 1$, the series A converges absolutely.

Lets say that it converges to some finite value, say L

$\textbf{Conclusion:}$ The sum of a divergent series and a convergent series is divergent.

Because for any $M > 0$, there exists N such that,

$$
\sum_{n=0}^{N} \frac{n}{n^2} > M - L
$$

Adding the convergent alternating series to both sides, we get,

$$S_N = \sum_{n=0}^{N} \frac{n + (-1)^n}{n^2} > M$$

Which shows that the partial sums of the original series can be made arbitrarily large, implying that the series $S = H + A$ is divergent.

Thus, the series $\sum_{n=0}^{\infty} \dfrac{n + (-1)^n}{n^2}$ is divergent.