hw3

(1)

(a) To determine whether the series converges for $z = -i$, we can use the ratio test.

The ratio of consecutive terms is given by

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+1}(-i)^{n+1}}{\frac{1}{n}(-i)^n}\right| = \left|\frac{n}{n+1}\right| = \frac{n}{n+1}.$$

Taking the limit as n approaches infinity, we have

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since the limit is less than 1, the series converges for $z = -i$.

(b) To determine whether the series converges for $z = \frac{1}{2}(-1 + i\sqrt{3})$, we can again use the ratio test.

The ratio of consecutive terms is given by

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n}\left(\frac{1}{2}(-1+i\sqrt{3})\right)^{n+1}}{\frac{1}{n}\left(\frac{1}{2}(-1+i\sqrt{3})\right)^{n}}\right| = \left|\frac{\frac{1}{2}(-1+i\sqrt{3})}{1}\right| = \frac{1}{2}\sqrt{3}.$$

Since the ratio is a constant value less than 1, the series converges for $z = \frac{1}{2}(-1 + i\sqrt{3})$.

(2)

Since $\sum_{n\geq 0} a_n(z-2)^n$ is convergent for $z = -1$, it converges within a certain radius of convergence. Let's call this radius R.

For $z = -2 - 2i$, we can calculate the distance from -1 using the distance formula:

$$\sqrt{((-2)-(-1))^2 + ((-2)-0)^2} = \sqrt{2+4} = \sqrt{6}.$$

Since $\sqrt{6} < R$, the series $\sum_{n\geq 2023} a_n(-2-2i)^n$ is absolutely convergent.

(3)

(a)

To compute $\mathrm{Exp}(z)$, we can use the formula $\mathrm{Exp}(z) = \sum_{n\geq 0} \frac{z^n}{n!}$.

$$\mathrm{Exp}(z) = \sum_{n\geq 0} \frac{(-1-2i)^n}{n!} = 1 + (-1-2i) + \frac{(-1-2i)^2}{2} + \frac{(-1-2i)^3}{3!} + \cdots$$

To compute $\log(z)$, we can use $\log(z) = \log|z| + i\arg(z)$, where $\arg(z)$ is the principal value of the argument of z.

$$\log(z) = \log|-1-2i| + i\arg(-1-2i) = \log\sqrt{5} + i\left(\frac{3\pi}{2}\right)$$

To compute z^i, where $i = \sqrt{-1}$,

$$z^i = (-1-2i)^i = \exp(i\log(-1-2i))$$

(b)

To compute $\mathrm{Exp}(z)$, we can again use the formula $\mathrm{Exp}(z) = \sum_{n\geq 0} \frac{z^n}{n!}$.

$$\mathrm{Exp}(z)=\sum_{n\geq 0}\frac{(-1+ti)^n}{n!}=1+(-1+ti)+\frac{(-1+ti)^2}{2}+\frac{(-1+ti)^3}{3!}+\cdots$$

To compute $\log(z)$, we can use $\log(z)=\log|z|+i\arg(z)$, where $\arg(z)$ is the principal value of the argument of z.

$$\log(z)=\log|-1+ti|+i\arg(-1+ti)=\log\sqrt{1+t^2}+i\left(\frac{\pi+\arctan(t)}{2}\right)$$

To compute z^i, where $i=\sqrt{-1}$,

$$z^i=(-1+ti)^i=\exp(i\log(-1+ti))$$

(4)

(a)

We can represent n^{-2+i} as $e^{(i\ln n-2\ln n)}$.

Using Euler's formula, $e^{ix}=\cos x+i\sin x$, we have

$$e^{(i\ln n-2\ln n)}=e^{-2\ln n}e^{i\ln n}=\frac{e^{i\ln n}}{n^2}=\frac{\cos(\ln n)+i\sin(\ln n)}{n^2}.$$

(b)

To show that the series $\sum_{n\geq 1}n^{-2+i}$ is absolutely convergent, we can use the comparison test.

Let's compare the series to the convergent series $\sum_{n\geq 1}\frac{1}{n^2}$.

Using the inequality $|\cos(\ln n)|\leq 1$ and $|\sin(\ln n)|\leq 1$, we have

$$\left|\frac{\cos(\ln n)+i\sin(\ln n)}{n^2}\right|\leq\frac{1}{n^2}.$$

Since the series $\sum_{n\geq 1}\frac{1}{n^2}$ converges, and we have a constant upper bound for the absolute value of each term in the series $\sum_{n\geq 1}n^{-2+i}$, we can conclude that the series $\sum_{n\geq 1}n^{-2+i}$ is absolutely convergent.

(c)

To determine whether the series $\sum_{n\geq 1}\frac{1}{n^z}$ is convergent or divergent for $z=x+iy$, where $x>1$, we can use the comparison test again.

Using the same inequality as in part (b), we have

$$\left|\frac{\cos(\ln n)+i\sin(\ln n)}{n^z}\right|\leq\frac{1}{n^x}.$$

Since $x>1$, the series $\sum_{n\geq 1}\frac{1}{n^x}$ converges.

Therefore, by the comparison test, we can conclude that the series $\sum_{n\geq 1}\frac{1}{n^z}$ is convergent for $z=x+iy$, where $x>1$.

(5)

(a)

To determine the radius of convergence of the power series $\sum_{n\geq 0}F_nz^n$, we can use the ratio test.

The ratio of consecutive terms is given by

$$\left|\frac{F_{n+1}}{F_n}\right|=\left|\frac{\frac{1}{\sqrt{5}}\left(\phi^{n+1}-(-\phi)^{-n-1}\right)}{\frac{1}{\sqrt{5}}(\phi^n-(-\phi)^{-n})}\right|=\left|\frac{\phi^{n+1}-(-\phi)^{-n-1}}{\phi^n-(-\phi)^{-n}}\right|=\left|\frac{\phi^{n+1}(-\phi)^n+(-\phi)^{n+1}\phi^{-n-1}}{\phi^n(-\phi)^n+(-\phi)^n\phi^{-n}}\right|=\left|\frac{\phi+(-1)\phi^{-n-1}}{1+(-1)\phi^{-2n}}\right|.$$

Taking the limit as n approaches infinity, we have

$$\lim_{n\to\infty}\left|\frac{\phi+(-1)\phi^{-n-1}}{1+(-1)\phi^{-2n}}\right| = \left|\frac{\phi}{1}\right| = |\phi|,$$

where $|\phi|$ is the modulus of ϕ.

Therefore, the radius of convergence of the power series is $|\phi|$.

(b)

For z in the disk of convergence, the sum of the series $\sum_{n\geq 0} F_n z^n$ is given by

$$\sum_{n\geq 0} F_n z^n = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots.$$

Since the Fibonacci sequence has the general term $F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n})$, we can substitute this expression into the series.

$$\sum_{n\geq 0} F_n z^n = \sum_{n\geq 0} \frac{1}{\sqrt{5}}\left(\phi^n - (-\phi)^{-n}\right) z^n = \frac{1}{\sqrt{5}}\left(\sum_{n\geq 0}\phi^n z^n - \sum_{n\geq 0}(-\phi)^{-n} z^n\right).$$

Using the formula for the sum of a geometric series, we can simplify this expression as

$$\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1+\phi^{-1}z}\right).$$

Therefore, the sum of the series is $\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1+\phi^{-1}z}\right)$.