Name:

1. Problem 3 from PS2.

2. In this problem we study a result that allows to estimate the sum of a series (which can be otherwise difficult).

Estimating the Sum of a Series.

Suppose we use the Integral Test to show that a series $\sum a_n$ is convergent. It would be nice if we could also find an approximation to the sum S of the series. We know that any partial sum S_n is an approximation to S because

$$\lim_{n\to\infty} S_n = S.$$

But we would really like some measure of the accuracy of the approximation. To measure accuracy, we need to estimate the magnitude of

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

The remainder R_n is the error made when S_n, the sum of the first n terms, is used as an approximation to the total sum.

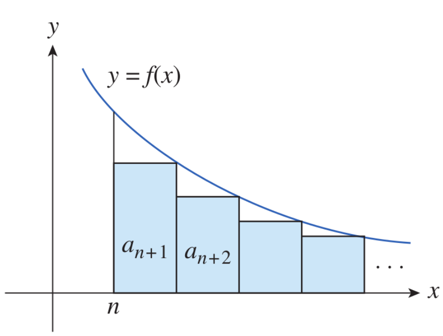
Assume that the associated function f is decreasing on $[n, \infty)$ and use the same notation and concepts as in the Integral Test. Compare the areas of the rectangles with the area under the graph of $y = f(x)$ for $x > n$ in the Figure (left).

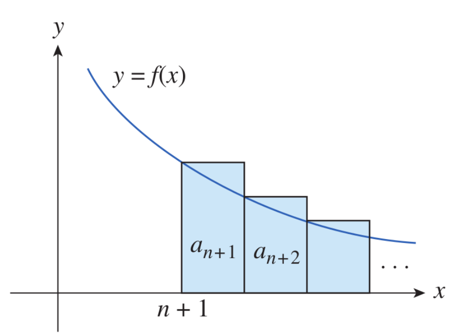
This suggests that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x)\,dx\,.$$

Similarly, Figure (right) suggests that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^\infty f(x)\,dx$$





This geometric argument leads to the following error estimate.

Proposition 1 (Remainder Estimate for the Integral Test). *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent with sum S and sequence of partial sums $\{S_n\}$. If $R_n = S - S_n$, then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx\,.$$

(a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first ten terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

3. a) Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_{1}^{\infty} \frac{1}{x^{1.3}}\,dx$$

What can you conclude about the series?

b) Suppose f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order.

$$\int_{1}^{6} f(x)\,dx \quad \sum_{i=1}^{5} a_i \quad \sum_{i=2}^{6} a_i$$

4. It is important to distinguish between

$$\sum_{n=1}^{\infty} n^b \quad \text{and} \quad \sum_{n=1}^{\infty} b^n$$

What is the first series called? The second? For what values of b does the first series converge? For what values of b does the second series converge?

5. Use the Integral Test to determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}\,.$$

6. Use the Comparison Test to determine whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$.

b) $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$.