General theory of linear ordinary differential equations

MA221, Lecture 8

- Reminder: Exam 1 on W, 9/25
- Exam 1 review: Sunday, 9/22, 8pm–10pm Kidde 228

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major exam 1 topics

- phase portraits
- seperable equations
- LFO equations → think of VoP
- exact equations
- Bernoulli equations

Higher Order Linear Equations

Recall that an n-th order linear equation is an equation of the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \ldots + a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y \overset{\star}{=} g(x).$$

In general, this is an **inhomogeneous** equation. However, if $g(x)$ is identically 0, we get the **homogeneous** equation

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \ldots + a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y \overset{\star_H}{=} 0.$$

Recall that when computing solutions to LFOs, we were able to go from a solution to $(\star_H)$ to a solution to $(\star)$. We will adopt this general philosophy to solve higher order linear equations.

Moving forward, the coefficient functions a_n, a_{n-1}, ... a_1, a_0, and $g(x)$ for an n-th order linear equation will be assumed to be continuous on a common interval I. Moreover, we assume a_n is non-zero on the interval I.

Superposition principle for homogeneous equations

Fact: (*Theorem 4.1.2 in Zill*). Let y_1, y_2, ..., y_k be solutions to a given *homogeneous* equation

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \ldots + a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y \overset{\star_H}{=} 0$$

over some interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x)$$

is also a solution to the given equation.

Other good things to know:

- Any scalar multiple of a solution to $(\star_H)$ is also a solution to $(\star_H)$
- $y = 0$ is always a solution to $(\star_H)$

In linear algebra terms: solutions to $\star_H$ generate a vector space

Linear dependence and independence

Consider a set consisting of the functions $f_1(x), f_2(x), \ldots, f_n(x)$. These functions are said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

for every x in I. Otherwise, the set of functions is **linearly independent**.

In other words, $\{f_1, f_2, \ldots, f_n\}$ is linearly independent if

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

implies

$$c_1 = c_2 = c_3 = \ldots = c_n = 0,$$

for x in I.

Linear dependence and independence

Example 1: The set $\{\underbrace{10}_{=f_1}, \underbrace{\cos^2 x}_{=f_2}, \underbrace{\sin^2 x}_{=f_3}\}$ is linearly dependent on $(-\infty, \infty)$.

$$c_1 \cdot 10 + c_2 \cdot \cos^2 x + c_3 \cdot \sin^2 x \overset{?}{=} 0$$

possible if not all c_1, c_2, c_3 are 0?

recall: $\cos^2 x + \sin^2 x = 1$

what if $c_1 = 1$? try $c_2 = c_3 = -10$

$$1 \cdot 10 + (-10)\cos^2 x + (-10)\sin^2 x$$

$$= 10 + [-10]\underbrace{(\cos^2 x + \sin^2 x)}_{=1}$$

$$= 10 - 10 = 0$$

$$\boxed{\begin{aligned} c_1 &= 1 \\ c_2 &= -10 \\ c_3 &= -10 \end{aligned}}$$

$\Longrightarrow$ lin. dependent.

Example 2: The set $\{x+1, x-1, x\}$ is linearly dependent on $(-\infty, \infty)$.

f_1 f_2 f_3

$$c_1(x+1) + c_2(x-1) + c_3 x \overset{?}{=} 0$$

↳ is this possible if not all of c_1, c_2, c_3 are zero?

what if $c_1 = c_2 = 1$.

$$\underbrace{1\cdot(x+1) + 1\cdot(x-1)}_{=2x} + c_3 x$$

→ choose $c_3 = -2$

$$= 2x + (-2)x = 0$$

$\Rightarrow$ linearly dependent!

$c_1 = 1$

$c_2 = 1$

$c_3 = -2$

Linear dependence and independence

Example 3: The set $\{\underbrace{x+1}_{f_1}, \underbrace{x}_{f_2}, \underbrace{x^2}_{f_3}, \underbrace{x^3 - 2x^2}_{f_4}\}$ is linearly independent on $(-\infty, \infty)$.

assume true.

$$0 = c_1(x+1) + c_2 x + c_3 x^2 + c_4(x^3 - 2x^2)$$

$$= c_1 x + c_1 + c_2 x + c_3 x^2 + c_4 x^3 - 2c_4 x^2$$

$$= c_1 + (c_1 + c_2)x + (c_3 - 2c_4)x^2 + c_4 x^3$$

$$\begin{cases} c_1 = 0 \\ c_1 + c_2 = 0 \\ c_3 - 2c_4 = 0 \\ c_4 = 0 \end{cases}$$

$\Longrightarrow c_2 = -c_1 = 0$

$\Longrightarrow c_3 = 2c_4 = 2 \cdot 0 = 0$

$\Longrightarrow c_1 = c_2 = c_3 = c_4 = 0$

$\Longrightarrow$ linear indep.

The Wronskian

For a set $\{f_1, f_2, \ldots, f_n\}$ of functions each possessing derivatives of order *at least* $n-1$, we define the **Wronskian** to be the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{bmatrix}$$

We will largely focus on the situations when $n = 2$ and $n = 3$ so make sure to review the definition of the determinant in these situations!

Fact: (*Theorem 4.1.3 in Zill*). Let $y_1, y_2, \ldots, y_n$ be solutions to

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \ldots + a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y \overset{\star_H}{=} 0$$

on some interval I. Then the following are equivalent:

- $\{y_1, y_2, \ldots, y_n\}$ is linearly independent on I
- $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for all x in I

Fundamental sets of solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \ldots + a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y \overset{\star_H}{=} 0$$

on an interval I is said to be a **fundamental set** of solutions on I.

Given a set of n solutions to an nth-order linear equation, we can use the Wronskian to determine if the set is linearly independent (and therefore form a fundamental set). See the example in the next slide!

Important Fact: Every nth-order linear equation has a fundamental set!

homogeneous

(consisting of n functions.)

general solution to an n-th order homogeneous linear equation describes a vector space with dimension n.

$\Rightarrow$ fundamental set = basis of this space.

Fundamental sets of solutions

Example 4: Consider the differential equation

order 2

$$x^2y'' - 4xy' + 8y = 0$$

2 solutions

on the interval $(0, \infty)$. The functions x^2 and x^4 are solutions to this equation (**verify!**). Do these solutions constitute a fundamental set?

Is $\{x^2, x^4\}$ a fundamental set?

Just check lin. ind. of $\underline{\underline{x^2}}$ and $\underline{\underline{x^4}}$:

f_1 f_2

$$W(f_1, f_2)(x) = \det \begin{bmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{bmatrix}$$

$$= x^2 \cdot 4x^3 - x^4 \cdot 2x$$

$$= 4x^5 - 2x^5 = 2x^5 > 0$$

on $(0, \infty)$.

$\Rightarrow \{x^2; x^4\}$ is fundamental.