A Complete Solution Guide to Real and Complex Analysis I

by Kit-Wing Yu, PhD

kitwing@hotmail.com

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About the author

Dr. Kit-Wing Yu received his B.Sc. (1st Hons), M.Phil. and Ph.D. degrees in Math. at the HKUST, PGDE (Mathematics) at the CUHK. After his graduation, he has joined United Christian College to serve as a mathematics teacher for at least seventeen years. He has also taken the responsibility of the mathematics panel since 2002. Furthermore, he was appointed as a part-time tutor (2002 – 2005) and then a part-time course coordinator (2006 – 2010) of the Department of Mathematics at the OUHK.

Besides teaching, Dr. Yu has been appointed to be a marker of the HKAL Pure Mathematics and HKDSE Mathematics (Core Part) for over thirteen years. Between 2012 and 2014, Dr. Yu was invited to be a Judge Member by the World Olympic Mathematics Competition (China). In the research aspect, he has published over twelve research papers in international mathematical journals, including some well-known journals such as J. Reine Angew. Math., Proc. Roy. Soc. Edinburgh Sect. A and Kodai Math. J.. His research interests are inequalities, special functions and Nevanlinna's value distribution theory.

Preface

Professor Walter Rudin[a] is the author of the classical and famous textbooks: *Principles of Mathematical Analysis*, *Real and Complex Analysis*, and *Functional Analysis*. (People commonly call them "*Baby Rudin*", "*Papa Rudin*" and "*Grandpa Rudin*" respectively.) Undoubtedly, they have produced important and extensive impacts to the study of mathematical analysis at university level since their publications.

As far as you know, *Real and Complex Analysis* keeps the features of *Principles of Mathematical Analysis* which are well-organized and expositions of theorems are clear, precise and well-written. Therefore, *Real and Complex Analysis* is always one of the main textbooks or references of graduate real analysis course in many universities. Actually, some universities will request their Ph.D. students to study this book for their qualifying examinations.

After the publication of the book *A Complete Solution Guide to Principles of Mathematical Analysis*, some purchasers have suggested me to write a solution book of *Real and Complex Analysis*. (To the best of the author's knowledge, *Papa Rudin* has **no** solution manual.) I was afraid of doing so at the beginning because the exercises are at graduate level and they are much more difficult than those in *Baby Rudin*. Fortunately, I was used to keeping solutions of mathematics exercises done by me in my undergraduate and graduate study. In fact, I have kept at least 25% of the exercises in the first six chapters of *Papa Rudin*. Therefore, after thorough consideration, I decided to start the next project of a solution book to Rudin's *Real and Complex Analysis*.

Since *Papa Rudin* consists of two components, I plan to write the solutions for "Real Analysis" part first. In fact, the present book *A Complete Solution Guide to Real and Complex Analysis I* covers all the exercises of Chapters 1 to 9 and its primary aim is to help every mathematics student and instructor to understand the ideas and applications of the theorems in Rudin's book. To accomplish this goal, I have adopted the way I wrote the book *A Complete Solution Guide to Principles of Mathematical Analysis*. In other words, I intend writing the solutions as comprehensive as I can so that you can understand *every* detailed part of a proof easily. Apart from this, I also keep reminding you what theorems or results I have applied by quoting them *repeatedly* in the proofs. By doing this, I believe that you will become fully aware of the meaning and applications of each theorem.

Before you read this book, I have two gentle reminders for you. Firstly, as a mathematics instructor at a college, I understand that the growth of a mathematics student depends largely on how hard he/she does exercises. When your instructor asks you to do some exercises from Rudin, you are not suggested to read my solutions unless you have tried your best to prove them yourselves. Secondly, when I prepared this book, I found that some exercises require knowledge that Rudin did not cover in his book. To fill this gap, I refer to some other analysis or topology

[a] `https://en.wikipedia.org/wiki/Walter_Rudin`.

books such as [9], [10], [22], [42] and [47]. Other useful references are [1], [12], [24], [28], [29], [59], [64], [66] and [67]. Of course, it is not a surprise that we will regard the exercises in *Baby Rudin* as some known facts and if you want to read proofs of them, you are strongly advised to read my book [63].

The features of this book are as follows:

- It covers all the 176 exercises from Chapters 1 to 9 with *detailed* and *complete* solutions. As a matter of fact, my solutions show every detail, every step and every theorem that I applied.
- There are 11 illustrations for explaining the mathematical concepts or ideas used behind the questions or theorems.
- Sections in each chapter are added so as to increase the readability of the exercises.
- Different colors are used frequently in order to highlight or explain problems, lemmas, remarks, main points/formulas involved, or show the steps of manipulation in some complicated proofs. (ebook only)
- Necessary lemmas with proofs are provided because some questions require additional mathematical concepts which are not covered by Rudin.
- Many useful or relevant references are provided to some questions for your future research.

Since the solutions are written solely by me, you may find typos or mistakes. If you really find such a mistake, please send your valuable comments or opinions to

`kitwing@hotmail.com`.

Then I will post the updated errata on my website

`https://sites.google.com/view/yukitwing/`

irregularly.

Kit Wing Yu
April 2019

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CHAPTER 1

Abstract Integration

1.1 Problems on σ-algebras and Measurable Functions

Problem 1.1

Rudin Chapter 1 Exercise 1.

Proof. Let X be a set. Assume that $\mathfrak{M}$ was an infinite σ-algebra in X which has only countably many members. Then we have $\mathfrak{M} = \{\varnothing, X, A_1, A_2, \ldots\}$. Let $x \in X$ be fixed, we define the set

$$S_x = \bigcap_{x \in A} A, \tag{1.1}$$

where $x \in A \in \mathfrak{M}$. (By this definition (1.1), it is clear that $x \in S_x$.) Since $\mathfrak{M}$ is countable, the intersection in (1.1) is actually at most countable. Thus it follows from Comment 1.6(c) that $S_x \in \mathfrak{M}$. Next we define the set

$$\mathfrak{S} = \{S_x \,|\, x \in X\}$$

so that $\mathfrak{S} \subseteq \mathfrak{M}$. We need a lemma about the set $\mathfrak{S}$:

Lemma 1.1

If $S_x, S_y \in \mathfrak{S}$ and $S_x \cap S_y \neq \varnothing$, then $S_x = S_y$.

Proof of Lemma 1.1. Let $z \in S_x \cap S_y$. We want to show that $S_z = S_x$. On the one hand, since $z \in S_x$, we know from the definition (1.1) that $z \in A$ *for all* A containing x. Now if $w \in S_x$, then $w \in A$ for all A containing x. By the previous observation, such A must contain z, so we get $w \in S_z$ and then

$$S_x \subseteq S_z. \tag{1.2}$$

On the other hand, if $w \in S_z$, then $w \in A$ *for all* A containing z. Thus the hypothesis $z \in S_x \cap S_y$ implies that $w \in S_x \cap S_y$, i.e., $S_z \subseteq S_x \cap S_y$. By this result and the definition (1.1), we obtain

$$S_z \subseteq S_x \cap S_y \subseteq S_x. \tag{1.3}$$

Combining the observations (1.2) and (1.3), we have the desired result that $S_z = S_x$. Similarly, we can show that $S_z = S_y$. Hence we conclude that $S_x = S_y$. ∎

Now we return to the proof of the problem. By Lemma 1.1, we may assume that all elements in $\mathfrak{S}$ are distinct. If $B \in \mathfrak{M}$, then the definition (1.1) certainly gives $y \in S_y$ for every $y \in B$ so that

$$B \subseteq \bigcup_{y \in B} S_y. \tag{1.4}$$

Therefore, if the cardinality of $\mathfrak{S}$ is N, then the cardinality of its power set $2^{\mathfrak{S}}$ is 2^N and so there are at most 2^N elements in $\mathfrak{M}$ by the inclusion (1.4), a contradiction. Thus the set $\mathfrak{S}$ must be infinite.

Recall that $\mathfrak{S} \subseteq \mathfrak{M}$ and $\mathfrak{M}$ is countable, $\mathfrak{S}$ is also countable by [49, Theorem 2.8, p. 26]. Since $\mathfrak{M}$ is a σ-algebra, it must contain the power set of $\mathfrak{S}$. However, it is well-known that the power set of an infinite countable set is uncountable [47, Problem 22, p. 16]. We conclude from these facts that $\mathfrak{M}$ is uncountable, a contradiction. This completes the proof of the problem. ■

Problem 1.2

Rudin Chapter 1 Exercise 2.

Proof. Put $\mathbf{f}(x) = (f_1(x), \ldots, f_n(x))$. Since $f_1, \ldots, f_n : X \to \mathbb{R}$, $\mathbf{f}$ maps the measurable space X into $\mathbb{R}^n$. Let Y be a topological space and $\Phi : \mathbb{R}^n \to Y$ be continuous. Define

$$h(x) = \Phi\big((f_1(x), \ldots, f_n(x))\big).$$

Since $h = \Phi \circ \mathbf{f}$, Theorem 1.7(b) shows that it suffices to prove the measurability of $\mathbf{f}$.

To this end, we first consider $\mathbf{f}^{-1}(R)$ for an open rectangle R in $\mathbb{R}^n$. By the definition, $R = I_1 \times \cdots \times I_n$, where I_i is an open interval in $\mathbb{R}$ for $1 \leq i \leq n$. Now we know from [42, Exercise 3, p. 21] that

$$\begin{aligned}
\mathbf{f}^{-1}(R) &= \{x \in X \,|\, \mathbf{f}(x) \in R\} \\
&= \{x \in X \,|\, f_i(x) \in I_i \text{ for } 1 \leq i \leq n\} \\
&= \{x \in X \,|\, f_1(x) \in I_1\} \cap \cdots \cap \{x \in X \,|\, f_n(x) \in I_n\} \\
&= f_1^{-1}(I_1) \cap \cdots \cap f_n^{-1}(I_n).
\end{aligned}$$

Since each f_i is measurable and each I_i is open in $\mathbb{R}$, the set $f_i^{-1}(I_i)$ is measurable in X by Definition 1.3(c). Thus this implies that $\mathbf{f}^{-1}(R)$ is measurable in X by Comment 1.6(c).

Our proof will be complete if we can show that *every* open set V in $\mathbb{R}^n$ can be written as a countable union of open rectangles R_j in $\mathbb{R}^n$. We need some topology. By [42, Exercise 8(a), p. 83], the countable collection

$$\mathscr{B} = \{I = (a, b) \,|\, a < b \text{ and } a, b \in \mathbb{Q}\}$$

is a *basis* that generates the *standard topology* on $\mathbb{R}$. Then we follow from [42, Theorem 15.1, p. 86] that the collection

$$\mathscr{C} = \{I_1 \times \cdots \times I_n \,|\, I_i = (a_i, b_i),\ a_i, b_i \in \mathbb{Q},\ 1 \leq i \leq n\} \tag{1.5}$$

is a basis for the (product) topology of $\mathbb{R}^n$.[a] Since elements in the collection (1.5) are open rectangles and it is countable by [49, Theorem 2.13, p. 29], every open set V in $\mathbb{R}^n$ is, in fact, a countable union of open rectangles R_j, i.e,

$$\mathbf{f}^{-1}(V) = \mathbf{f}^{-1}\Big(\bigcup_{j=1}^{\infty} R_j\Big) = \bigcup_{j=1}^{\infty} \mathbf{f}^{-1}(R_j).$$

[a] For details, please read [42, §13 and §15].

Hence $\mathbf{f}^{-1}(V)$ is also a measurable set in X by Definition 1.3(a)(iii). This completes the proof of the problem. ∎

Problem 1.3

Rudin Chapter 1 Exercise 3.

Proof. Let $f : X \to \mathbb{R} \subset [-\infty, \infty]$ and $\mathfrak{M}$ be the σ-algebra of X. Let α be real. By [49, Theorem 1.20(b), p. 9], we can find a sequence $\{r_n\}$ of rational numbers such that $\alpha < r_n$ for all $n \in \mathbb{N}$, $r_n \to \alpha$ as $n \to \infty$. In other words, we have

$$(\alpha, \infty] = \bigcup_{n=1}^{\infty} [r_n, \infty]$$

which implies that

$$f^{-1}((\alpha, \infty]) = f^{-1}\Big(\bigcup_{n=1}^{\infty} [r_n, \infty]\Big) = \bigcup_{n=1}^{\infty} f^{-1}([r_n, \infty]).$$

Recall that

$$f^{-1}([r_n, \infty]) = \{x \mid f(x) \geq r_n\}$$

is assumed to be measurable for each n, so we have $f^{-1}([r_n, \infty]) \in \mathfrak{M}$ and then we follow from Definition 1.3(a)(iii) that $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$. Since α is arbitrary, we conclude from Theorem 1.12(c) that f is measurable. This finishes the proof of the problem. ∎

Problem 1.4

Rudin Chapter 1 Exercise 4.

Proof.

(a) Let $\alpha_k = \sup\{-a_k, -a_{k+1}, \ldots\}$ for $k = 1, 2, \ldots$. By the definition, we have $\alpha_k \geq -a_n$ for all $n \geq k$ and if $\alpha \geq -a_n$ for all $n \geq k$, then $\alpha \geq \alpha_k$. Note that this is equivalent to the fact that $-\alpha_k \leq a_n$ for all $n \geq k$ and if $-\alpha \leq a_n$ for all $n \geq k$, then $-\alpha \leq -\alpha_k$. In other words, we have

$$-\alpha_k = \inf\{a_k, a_{k+1}, \ldots, \}$$

for $k = 1, 2, \ldots$. Thus this implies that

$$\sup_{n\geq k}(-a_n) = \sup\{-a_k, -a_{k+1}, \ldots\} = -\inf\{a_k, a_{k+1}, \ldots\} = -\inf_{n\geq k}(a_n). \tag{1.6}$$

Similarly, we have

$$\inf_{n\geq k}(-c_n) = -\sup_{n\geq k}(c_n) \tag{1.7}$$

for a sequence $\{c_n\}$ in $[-\infty, \infty]$. By applying the equality (1.6) and then the equality (1.7), we achieve that

$$\limsup_{n\to\infty}(-a_n) = \inf_{k\geq 1}\Big\{\sup_{n\geq k}(-a_n)\Big\} = \inf_{k\geq 1}\Big\{-\underbrace{\inf_{n\geq k}(a_n)}_{c_n}\Big\} = -\sup_{k\geq 1}\Big\{\inf_{n\geq k}(a_n)\Big\} = -\liminf_{n\to\infty}(a_n)$$

which is our desired result.

(b) This part is proven in [63, Problem 3.5, pp. 32, 33].

(c) Since $a_n \leq b_n$ for all $n = 1, 2, \ldots$, we must have

$$\alpha_k = \inf_{n\geq k}(a_n) \leq \inf_{n\geq k}(b_n) = \beta_k \tag{1.8}$$

for all $k = 1, 2, \ldots$. Thus $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[-\infty, \infty]$ such that $\alpha_n \leq \beta_n$ for all $n = 1, 2, \ldots$. By a similar argument, we also have

$$\sup_{k\geq m} \alpha_k \leq \sup_{k\geq m} \beta_k \tag{1.9}$$

for all $m = 1, 2, \ldots$. Combining the inequalities (1.8) and (1.9), we have

$$\sup_{k\geq m}\Big\{\inf_{n\geq k}(a_n)\Big\} \leq \sup_{k\geq m}\Big\{\inf_{n\geq k}(b_n)\Big\}$$

for all $m = 1, 2, \ldots$. By Definition 1.13, we have the desired result.

For a counterexample to part (b), we consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ for all $n = 1, 2, \ldots$. On the one hand, we have $a_n + b_n = 0$ for all $n = 1, 2, \ldots$ so that

$$\limsup_{n\to\infty}(a_n + b_n) = 0. \tag{1.10}$$

On the other hand, we have

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k} = 1 \quad \text{and} \quad \limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{2k+1} = 1$$

which imply that

$$\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = 2. \tag{1.11}$$

Hence we obtain from the results (1.10) and (1.11) that the strict inequality can hold in part (b). This completes the proof of the problem. ■

Problem 1.5

Rudin Chapter 1 Exercise 5.

Proof.

(a) Let $\mathfrak{M}_X$ be a σ-algebra of X. Further, suppose that $S_f(\pm\infty) = \{x \in X \mid f(x) = \pm\infty\}$, and $S_g(\pm\infty) = \{x \in X \mid g(x) = \pm\infty\}$. We see that

$$S_f(\infty) = \bigcap_{n=1}^{\infty}\{x \in X \mid f(x) > n\}.$$

Since f is measurable and $(n, \infty]$ is open[b] in $[-\infty, \infty]$, $f^{-1}((n, \infty]) = \{x \mid f(x) > n\} \in \mathfrak{M}_X$ by Theorem 1.12(b) for each $n \in \mathbb{N}$. Thus $S_f(\infty) \in \mathfrak{M}_X$ by Comment 1.6(c). Similarly, all the other sets $S_f(-\infty), S_g(\infty)$ and $S_g(-\infty)$ belong to $\mathfrak{M}_X$ too.

Next, we let $X' = X \setminus (S_f(\infty) \cup S_f(-\infty) \cup S_g(\infty) \cup S_g(-\infty))$. Since $S_f(\infty), S_f(-\infty), S_g(\infty)$ and $S_g(-\infty)$ are measurable, $X' \in \mathfrak{M}$ by Definition 1.3(a)(ii) and Comment 1.6(b). By

[b]See the proof of Theorem 1.12(c) in [51, p. 13]

the comment following Proposition 1.24, we know that X' is itself a measure space and if we let $\mathfrak{M}_{X'}$ be a σ-algebra of X', then

$$\mathfrak{M}_{X'} \subseteq \mathfrak{M}_X. \tag{1.12}$$

Furthermore, the restricted mappings $f_{X'} : X' \to \mathbb{R}$ and $g_{X'} : X' \to \mathbb{R}$ are also measurable because of Definition 1.3(c) and the fact that *any* open set V in $\mathbb{R}$ is a countable union of segments of the type (α, β) so that V is also open in the extended number system $[-\infty, \infty]$.[c] In fact, we have

$$f_{X'}^{-1}(V) = f^{-1}(V) \in \mathfrak{M}_{X'} \quad \text{and} \quad g_{X'}^{-1}(V) = g^{-1}(V) \in \mathfrak{M}_{X'}.$$

We need to prove one more thing: the mapping $-g : X \to [-\infty, \infty]$ is measurable. Since g is measurable, we know from [49, Definition 11.13] that $\{x \in X \,|\, g(x) > -a\} \in \mathfrak{M}_X$ for every real a. It is obvious that $\{x \in X \,|\, -g(x) < a\} = \{x \in X \,|\, g(x) > -a\}$ for every real a, so we deduce from [49, Theorem 11.15, p. 311] that $-g$ is measurable.

Now we are ready to prove the desired results. Notice that

$$\{x \in X \,|\, f(x) = g(x)\} = \{x \in X \,|\, h(x) = 0\} = h^{-1}(0),$$

where $h = f - g$. Since f and $-g$ are measurable, the new (real) function $h = f - g$ is also measurable by Theorem 1.9(c). Since

$$h^{-1}(0) = \bigcap_{n=1}^{\infty} h^{-1}\Big(\Big(-\frac{1}{n}, \frac{1}{n}\Big)\Big)$$

and $h^{-1}((-\frac{1}{n}, \frac{1}{n})) \in \mathfrak{M}_{X'}$ for every $n \in \mathbb{N}$, we yield from this and the relation (1.12) that $h^{-1}(0) \in \mathfrak{M}_{X'} \subseteq \mathfrak{M}_X$. This shows the second assertion. For the first assertion, we note that

$$\begin{aligned}\{x \in X \,|\, f(x) < g(x)\} &= \{x \in X \,|\, h(x) = f(x) - g(x) < 0\} \\ &= \big\{h^{-1}\big((-\infty, 0)\big) \cup [S_f(-\infty) \setminus S_g(-\infty)] \cup [S_g(\infty) \setminus S_f(\infty)]\big\} \\ &\quad \setminus \big\{[S_f(\infty) \cap S_g(\infty)] \cup [S_f(-\infty) \cap S_g(-\infty)]\big\}.\end{aligned}$$

Recall that $S_f(\infty), S_f(-\infty), S_g(\infty), S_g(-\infty) \in \mathfrak{M}_X$, so we have

$$S_f(-\infty) \setminus S_g(-\infty),\ S_g(\infty) \setminus S_f(\infty),\ S_f(\infty) \cap S_g(\infty),\ S_f(-\infty) \cap S_g(-\infty) \in \mathfrak{M}_X. \tag{1.13}$$

By the measurability of h and the relation (1.12), we have

$$h^{-1}((-\infty, 0)) \in \mathfrak{M}_X. \tag{1.14}$$

Hence the facts (1.13) and (1.14) show that $\{x \,|\, f(x) < g(x)\} \in \mathfrak{M}_X$.

(b) This part is proven in [63, Problem 11.3, p. 339].

This completes the proof of the problem. ■

Problem 1.6

Rudin Chapter 1 Exercise 6.

[c] This can be seen, again, from the proof of Theorem 1.12(c) or from the comment following Proposition 1.24.

Proof. We prove the assertions one by one.

- **$\mathfrak{M}$ is a σ-algebra in X.** We check Definition 1.3(a). Since $X^c = \varnothing$, it is at most countable. Thus we have $X \in \mathfrak{M}$. Let $A \in \mathfrak{M}$. If A^c is at most countable, then $A^c \in \mathfrak{M}$. Similarly, if A is at most countable, then since $(A^c)^c = A$, we have $A^c \in \mathfrak{M}$. Suppose that $A_n \in \mathfrak{M}$ for $n = 1, 2, \ldots$. Then we have either A_n or A_n^c is at most countable for $n = 1, 2, \ldots$. If all A_n *are* at most countable, then we deduce from the corollary following [49, Theorem 2.12, p. 29] that the set
$$A = \bigcup_{n=1}^{\infty} A_n \tag{1.15}$$
is also at most countable so that $A \in \mathfrak{M}$. Otherwise, without loss of generality, we suppose that A_1 is uncountable but A_1^c is at most countable. Then we consider
$$A^c = \Big(\bigcup_{n=1}^{\infty} A_n\Big)^c = \bigcap_{n=1}^{\infty} A_n^c \subseteq A_1^c$$
which means that A^c is at most countable too. Hence $A^c \in \mathfrak{M}$ and then $\mathfrak{M}$ is a σ-algebra in X.

- **μ is a measure on $\mathfrak{M}$.** We check Definition 1.18(a). Since $\varnothing$ is at most countable, we have $\mu(\varnothing) = 0$ so that μ is *not* identically ∞. In fact, it is clear that we have $\mu : \mathfrak{M} \to \{0, 1\} \subset [0, \infty]$. Let $\{A_n\}$ be a disjoint countable collection of members of $\mathfrak{M}$.

  Case (i): All A_n are at most countable. Recall that the set A given by (1.15) is at most countable. By the definition of μ, we have $\mu(A) = \mu(A_n) = 0$ for all $n = 1, 2, \ldots$. Thus we have
$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.16}$$
  in this case.

  Case (ii): There is *at least* one A_k is uncountable. Since $A_k \in \mathfrak{M}$, A_k^c must be at most countable. Since $A_n \cap A_k = \varnothing$ for all $n \neq k$, we have $A_n \subseteq A_k^c$ for all $n \neq k$. In other words, the measurable sets A_n are at most countable for all $n \neq k$. Therefore, we have $\mu(A_k) = 1$ and $\mu(A_n) = 0$ for all $n \neq k$. Since
$$A^c = \bigcap_{n=1}^{\infty} A_n^c \subseteq A_k^c,$$
  A^c is at most countable and then $\mu(A) = 1$. Hence the equality (1.16) also holds in this case.

  This completes the proof that μ is a measure on $\mathfrak{M}$.

- **The determination of measurable functions and their integrals.** Let $f : X \to \mathbb{R}$ be a measurable function. Since X is uncountable, we have $\mu(X) = 1$ which is the only thing that we know and start with. For every $n \in \mathbb{Z}$, we know that
$$[n, n+1) = \mathbb{R} \setminus \big[(-\infty, n) \cup [n+1, \infty)\big].$$
  By the fact that[d] $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, we obtain
$$f^{-1}\big([n, n+1)\big) = f^{-1}(\mathbb{R}) \setminus \big[f^{-1}\big((-\infty, n)\big) \cup f^{-1}\big([n+1, \infty)\big)\big]. \tag{1.17}$$

[d] See [42, Exercise 2(d), p. 20].

By [49, Theorem 11.15, p. 311], each set on the right-hand side in (1.17) is measurable. This implies that

$$f^{-1}\big([n, n+1)\big) \in \mathfrak{M}.$$

Let $E_n = f^{-1}\big([n, n+1)\big)$, where $n \in \mathbb{Z}$. By the definition of $\mathfrak{M}$ and then the definition of μ, we have either $\mu(E_n) = 0$ or $\mu(E_n) = 1$. For every $x \in X$, we must have $f(x) \in [n, n+1)$ for some $n \in \mathbb{Z}$, i.e., $x \in E_n$ for some $n \in \mathbb{Z}$. Therefore, we have

$$\bigcup_{n=-\infty}^{\infty} E_n = X.$$

It is clear that $\mathbb{R} = \bigcup\limits_{n=-\infty}^{\infty} [n, n+1)$, so $\{E_n\}$ is a *disjoint* countable collection of members of $\mathfrak{M}$. By Definition 1.18(a), we see that

$$\mu(X) = \mu\Big(\bigcup_{n=-\infty}^{\infty} E_n\Big) = \sum_{n=-\infty}^{\infty} \mu(E_n). \tag{1.18}$$

The fact $\mu(X) = 1$ and the equality (1.18) force that there exists an integer n_0 such that $\mu(E_{n_0}) = 1$. Without loss of generality, we may assume that $n_0 = 0$, i.e.,

$$\mu(E_0) = \mu\big(f^{-1}([0, 1))\big) = 1. \tag{1.19}$$

Next if we write $[0, 1) = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1)$, then the above argument and the value (1.19) imply that either $\mu\big(f^{-1}([0, \frac{1}{2}))\big) = 1$ or $\mu\big(f^{-1}([\frac{1}{2}, 1))\big) = 1$. This process can be done continuously so that a sequence of intervals $\{[a_n, b_n)\}$ is constructed such that

$$\mu\big(f^{-1}([a_n, b_n))\big) = 1, \quad 0 \leq b_n - a_n \leq \frac{1}{2^n} \quad \text{and} \quad \lim_{n\to\infty} (b_n - a_n) = 0,$$

where $n = 1, 2, \ldots$. In other words, it means that

$$\mu\big(f^{-1}(a)\big) = 1 \quad \text{and} \quad \mu\big(f^{-1}(b)\big) = 0$$

for some $a \in \mathbb{R}$ and all other real numbers $b \neq a$, but this is equivalent to saying that $f^{-1}(a)$ is uncountable and $f^{-1}(b)$ is at most countable for all $b \neq a$. Now we have completely characterized every measurable function on X.

Finally, it is clear that $X \setminus f^{-1}(a)$ is a set of measure 0. Therefore, we have

$$\int_X f \,\mathrm{d}\mu = \int_{f^{-1}(a)} f \,\mathrm{d}\mu = a.$$

Hence, this completes the proof of the problem. ■

1.2 Problems related to the Lebesgue's MCT/DCT

Problem 1.7

Rudin Chapter 1 Exercise 7.

Proof. Let $E_k = \{x \in X \,|\, f_1(x) > k\} = f_1^{-1}\big((k,\infty]\big)$ and $E = \{x \in X \,|\, f_1(x) = \infty\}$, where $k = 1, 2, \ldots$. It is clear that each $(k,\infty]$ is a Borel set in $[0,\infty]$, so each E_k is measurable by Theorem 1.12(b). Since

$$E = \bigcap_{k=1}^{\infty} E_k,$$

E is also measurable by Comment 1.6(c). If $\mu(E) > 0$, then we know from the definition that

$$\int_E f_1 \,\mathrm{d}\mu = \infty. \tag{1.20}$$

By using the result (1.20), Proposition 1.24(b) and Theorem 1.33, we conclude that

$$\int_X |f_1| \,\mathrm{d}\mu \geq \left|\int_X f_1 \,\mathrm{d}\mu\right| \geq \left|\int_E f_1 \,\mathrm{d}\mu\right| = \infty$$

which contradicts the hypothesis that $f_1 \in L^1(\mu)$. In other words, we must have $\mu(E) = 0$ and thus $f_1 \in L^1(\mu)$ on $X \setminus E$.

Here we may assume that the form of Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) is also valid for measurable functions defined a.e. on X.[e] Now we see that the measurable function f_1 in the problem plays the role of g in the theorem and hence our desired result follows from Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) immediately.

For a counterexample, we consider $X = \mathbb{R}, I_n = (n,\infty), \mu = m$ the Lebesgue measure (see [49, Definition 11.5, pp. 302, 303]) and define for each $n = 1, 2, \ldots$,

$$f_n(x) = \chi_{I_n}(x) = \begin{cases} 1, & \text{if } x \in I_n; \\ 0, & \text{if } x \notin I_n. \end{cases}$$

It is clear that $f_1 \geq f_2 \geq \cdots \geq 0$ on X and

$$\int_{\mathbb{R}} |f_n| \,\mathrm{d}m = \int_{\mathbb{R}} f_n \,\mathrm{d}m = \int_{I_n} \mathrm{d}m = \infty \tag{1.21}$$

for every $n = 1, 2, \ldots$. In particular, the integrals (1.21) show that $f_1 \notin L^1(m)$. Furthermore, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \chi_{I_n}(x) = 0 \tag{1.22}$$

for every $x \in \mathbb{R}$. Thus we deduce from the expression (1.22) and Proposition 1.24(d) that

$$\int_{\mathbb{R}} f \,\mathrm{d}m = 0. \tag{1.23}$$

Hence the inconsistence of the integrals (1.21) and (1.23) show that the condition "$f_1 \in L^1(\mu)$" cannot be omitted. This completes the proof of the problem. ■

Problem 1.8

Rudin Chapter 1 Exercise 8.

[e] Actually, Rudin [51, p. 29] assumed this fact in the proof of Theorem 1.38 or the reader may refer to the comment between Theorem 11.32 and its proof in [49, p. 321].

Proof. If $x \in E$, then for all $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} f_{2k}(x) = \lim_{k\to\infty} \big(1 - \chi_E(x)\big) = 0.$$

Similarly, if $x \notin E$, then for all $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} f_{2k+1}(x) = \lim_{k\to\infty} \chi_E(x) = 0.$$

Thus we have

$$\liminf_{n\to\infty} f_n(x) = 0$$

for all $x \in X$. However, we have

$$\begin{aligned}\int_X f_n \,\mathrm{d}\mu &= \begin{cases} \displaystyle\int_X \chi_E \,\mathrm{d}\mu, & \text{if } n \text{ is odd;} \\ \displaystyle\int_X (1-\chi_E)\,\mathrm{d}\mu, & \text{if } n \text{ is even.}\end{cases} \\ &= \begin{cases} \mu(E), & \text{if } n \text{ is odd;} \\ \mu(X\setminus E), & \text{if } n \text{ is even.}\end{cases}\end{aligned} \tag{1.24}$$

Therefore we obtain from the results (1.24) that

$$\liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \min(\mu(E), \mu(X\setminus E)) \neq 0$$

if we assume that $\mu(E) > 0$ and $\mu(X\setminus E) > 0$. Hence the example here shows that the inequality in Fatou's lemma can be strict, completing the proof of the problem.[f] ■

Problem 1.9

Rudin Chapter 1 Exercise 9.

Proof. If $\mu(X) = 0$, then Proposition 1.24(e) implies that $c = \displaystyle\int_X f\,\mathrm{d}\mu = 0$, a contradiction. Let $E = \{x \in X \mid f(x) = \infty\}$. We claim that $\mu(E) = 0$. Otherwise, Proposition 1.24(b) implies that

$$c = \int_X f\,\mathrm{d}\mu \geq \int_E f\,\mathrm{d}\mu = \infty$$

which is a contradiction. Therefore, in the following discussion, we may assume that $x \in X \setminus E$ so that $0 \leq f(x) < \infty$.

For each $n = 1, 2, \ldots$, we define[g] $f_n : X\setminus E \to [0,\infty)$ by

$$f_n(x) = n\log\Big[1 + \Big(\frac{f(x)}{n}\Big)^{\alpha}\Big].$$

Since f is measurable on $X \setminus E$, $g(x) = [1 + (\frac{x}{n})^{\alpha}]$ and $h(x) = n\log(1+x)$ are continuous on $[0,\infty)$, Theorem 1.7(b) implies that

$$f_n(x) = n\log\Big[1 + \Big(\frac{f(x)}{n}\Big)^{\alpha}\Big] = h\big(g(f(x))\big)$$

[f] There is another example in [63, Problem 11.5, p. 340].

[g] $X \setminus E$ is itself a measure space by the remark following Proposition 1.24.

is also measurable on $X \setminus E$.

Now we are going to show that when $\alpha \geq 1$, there exists a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x)$$

holds for all $n = 1, 2, \ldots$ and all $x \in X \setminus E$. We first show the following lemma:

Lemma 1.2

For each $n \in \mathbb{N}$ and $\alpha \geq 1$, we have

$$n \log\left[1 + \left(\frac{x}{n}\right)^{\alpha}\right] \leq \alpha x \tag{1.25}$$

on $[0, \infty)$.

Proof of Lemma 1.2. For $x \in [0, \infty)$, we let

$$F(x) = \alpha x - n \log\left[1 + \left(\frac{x}{n}\right)^{\alpha}\right].$$

Then basic calculus gives

$$F'(x) = \alpha - \frac{\alpha\left(\frac{x}{n}\right)^{\alpha-1}}{1 + \left(\frac{x}{n}\right)^{\alpha}} = \alpha - \frac{\alpha n x^{\alpha-1}}{n^{\alpha} + x^{\alpha}} = \alpha\left(1 - \frac{n x^{\alpha-1}}{n^{\alpha} + x^{\alpha}}\right). \tag{1.26}$$

Since

$$\frac{n x^{\alpha-1}}{n^{\alpha} + x^{\alpha}} \leq \frac{x^{\alpha}}{n^{\alpha} + x^{\alpha}} < 1$$

for $n \leq x$ and since

$$\frac{n x^{\alpha-1}}{n^{\alpha} + x^{\alpha}} < \frac{n^{\alpha}}{n^{\alpha} + x^{\alpha}} < 1$$

for $n > x$ and $\alpha \geq 1$, we deduce from the derivative (1.26) that

$$F'(x) < 0$$

for all $x \in [0, \infty)$. By [49, Theorem 5.11, p. 108] and the continuity of F on $[0, \infty)$, we establish that $F(x)$ is decreasing on $[0, \infty)$ which implies the validity of the inequality (1.25). ■

Let's return to the proof of the problem. By Lemma 1.2, it is evident that for all $n \in \mathbb{N}$ and $\alpha \geq 1$, we have

$$f_n = n \log\left[1 + \left(\frac{f}{n}\right)^{\alpha}\right] \leq \alpha f \tag{1.27}$$

on $X \setminus E$. Furthermore, we apply Proposition 1.24(c) to get

$$\int_X |\alpha f| \, \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu = \alpha c < \infty.$$

This means that $\alpha f \in L^1(\mu)$ and then $g = \alpha f$ is the desired function.

Next, there are two cases for consideration:

Case (i): $\alpha = 1$. In this case, we have[h]

$$\lim_{n\to\infty} n\log\left[1+\left(\frac{f(x)}{n}\right)^\alpha\right] = \lim_{n\to\infty}\log\left(1+\frac{f(x)}{n}\right)^n = \log e^{f(x)} = f(x)$$

on $X \setminus E$.

Case (ii): $\alpha > 1$. In this case, we apply L'Hospital's rule [49, Theorem 5.13, p. 109] to conclude that

$$\lim_{n\to\infty} n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right] = \lim_{y\to 0^+}\frac{\log(1+f^\alpha y^\alpha)}{y} = \lim_{y\to 0^+}\frac{\alpha f^\alpha y^{\alpha-1}}{1+f^\alpha y^\alpha} = \frac{0}{1+0} = 0 \tag{1.28}$$

on $X \setminus E$.

Thus it follows from Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) that

$$\begin{aligned}\lim_{n\to\infty}\int_X n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right]\mathrm{d}\mu &= \begin{cases}\displaystyle\int_X f\,\mathrm{d}\mu, & \text{if } \alpha = 1;\\[2ex] \displaystyle\int_X 0\,\mathrm{d}\mu, & \text{if } \alpha > 1,\end{cases}\\ &= \begin{cases} c, & \text{if } \alpha = 1;\\ 0, & \text{if } \alpha > 1.\end{cases}\end{aligned}$$

It remains the case that $0 < \alpha < 1$. In this case, Theorem 1.28 (Fatou's Lemma) can be applied directly to get

$$\int_X \left\{\liminf_{n\to\infty}\left\{n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right]\right\}\right\}\mathrm{d}\mu \le \liminf_{n\to\infty}\int_X n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right]\mathrm{d}\mu. \tag{1.29}$$

Since $0 < \alpha < 1$, we have $y^{\alpha-1} = \frac{1}{y^{1-\alpha}}$ so that the limit (1.28) becomes

$$\lim_{n\to\infty} n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right] = \lim_{y\to 0^+}\frac{\alpha f^\alpha}{y^{1-\alpha}(1+f^\alpha y^\alpha)} = \infty. \tag{1.30}$$

Since $\lim\limits_{n\to\infty} x_n = \infty$ if and only if $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n = \infty$, the inequality (1.29) and the limit (1.30) combine to imply that

$$\liminf_{n\to\infty}\left\{n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right]\right\} = \infty$$

and then it certainly gives

$$\lim_{n\to\infty}\int_X n\log\left[1+\left(\frac{f}{n}\right)^\alpha\right]\mathrm{d}\mu = \infty.$$

This completes the proof of the problem. ■

Problem 1.10

Rudin Chapter 1 Exercise 10.

[h]We use the definition $e^x = \lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ here.

Proof. Since $f_n \to f$ uniformly on X, there exists a positive integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq 1$$

for all $x \in X$. By this, we have

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq |f(x)| + 1 \tag{1.31}$$

and

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq |f_N(x)| + 1 \tag{1.32}$$

for all $n \geq N$ and $x \in X$. Combining the inequalities (1.31) and (1.32), we see that

$$|f_n(x)| \leq |f_N(x)| + 2 \tag{1.33}$$

for all $n \geq N$ and $x \in X$.

We define $g : X \to [0, \infty)$ by

$$g(x) = \max\{|f_1(x)|, \ldots, |f_{N-1}(x)|, |f_N(x)| + 2\}. \tag{1.34}$$

Then it is easy to see from the inequality (1.33) and the definition (1.34) that

$$|f_n(x)| \leq g(x)$$

for $x \in X$ and $n = 1, 2, \ldots$.

Next we want to show that $g \in L^1(\mu)$. By Theorem 1.9(b), $|f_1(x)|, \ldots, |f_{N-1}(x)|, |f_N(x)|$ are measurable. Thus it follows from the corollaries following Theorem 1.14 that g is also measurable. Furthermore, we know from the hypothesis that $|f_n(x)| \leq M_n$ on X for some constants M_n, where $n = 1, 2, \ldots, N$. Therefore, this and the definition (1.34) certainly imply that

$$g(x) = |g(x)| \leq M$$

on X for some constant M. Since it is obvious that $g \geq 0$ on X, we get from Proposition 1.24(a) that

$$0 \leq \int_X |g| \,\mathrm{d}\mu \leq \int_X M \,\mathrm{d}\mu = M\mu(X) < \infty$$

so that $g \in L^1(\mu)$. In conclusion, our sequence of functions $\{f_n\}$ satisfies the hypotheses of Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) and hence the desired result follows immediately from this.

For a counterexample, consider $X = \mathbb{R}$ and $\mu = m$ so that $m(\mathbb{R}) = \infty$. For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{n}.$$

Then it is easy to prove that $f_n \to f \equiv 0$ uniformly on $\mathbb{R}$. Since

$$\int_{\mathbb{R}} f \,\mathrm{d}m = 0 \quad \text{and} \quad \int_{\mathbb{R}} f_n \,\mathrm{d}m = \frac{m(\mathbb{R})}{n} = \infty$$

for $n = 1, 2, \ldots$, we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \,\mathrm{d}m \neq \int_{\mathbb{R}} f \,\mathrm{d}m.$$

This finishes the proof of the problem. ∎

Problem 1.11

Rudin Chapter 1 Exercise 11.

Proof. For each $n \in \mathbb{N}$, define

$$B_n = \bigcup_{k=n}^{\infty} E_k.$$

Recall that A is the set of all $x \in X$ which lie in infinitely many E_k. On the one hand, if $x \in A$, then $x \in B_n$ for all $n \in \mathbb{N}$ so that $x \in \bigcap_{n=1}^{\infty} B_n$. In other words, we have

$$A \subseteq \bigcap_{n=1}^{\infty} B_n. \tag{1.35}$$

On the other hand, if $x \in \bigcap_{n=1}^{\infty} B_n$, then $x \in B_n$ for each positive integer n and this is equivalent to the condition that x belongs to *infinitely many* E_k, i.e.,

$$\bigcap_{n=1}^{\infty} B_n \subseteq A. \tag{1.36}$$

Hence the set relations (1.35) and (1.36) imply the desired result that

$$A = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

We have to show that $\mu(A) = 0$. By the definition of B_n, we have

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots.$$

Furthermore, our hypothesis and the *subadditive* property of a measure (see , for example, [54, Corollary 4.6, p. 26]) show that

$$\mu(B_1) = \mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Thus it follows from Theorem 1.19(e) and the subadditive property of μ again that

$$0 \leq \mu(A) = \mu\Big(\bigcap_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu\Big(\bigcup_{k=n}^{\infty} E_k\Big) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0.$$

Hence we must have $\mu(A) = 0$, completing the proof of the problem. ■

Problem 1.12

Rudin Chapter 1 Exercise 12.

Proof. Let n be a positive integer. Define $f_n : X \to [0, \infty]$ by

$$f_n(x) = \min\big(|f(x)|, n\big) \tag{1.37}$$

for every $x \in X$. This definition (1.37) clearly satisfies

$$0 \leq f_n(x) \leq n \tag{1.38}$$

for every $x \in X$. Since $f \in L^1(\mu)$, it is obviously measurable. By the corollaries following Theorem 1.14, each f_n is also measurable on X. Furthermore, if $x_0 \in X$ such that $|f(x_0)| \leq n$, then the definition (1.37) implies that

$$f_n(x_0) = |f(x_0)| \quad \text{and} \quad f_{n+1}(x_0) = |f(x_0)|; \tag{1.39}$$

if $n < |f(x_0)| \leq n+1$, then we know again from the definition (1.37) that

$$f_n(x_0) = n \quad \text{and} \quad f_{n+1}(x_0) = |f(x_0)|; \tag{1.40}$$

if $n + 1 < |f(x_0)|$, then we must have

$$f_n(x_0) = n \quad \text{and} \quad f_{n+1}(x_0) = n + 1. \tag{1.41}$$

Thus we can conclude from the computations (1.39), (1.40) and (1.41) that the inequality

$$0 \leq f_n(x) \leq f_{n+1}(x) \tag{1.42}$$

holds for all $n \in \mathbb{N}$ and $x \in X$. By Theorem 1.26 (Lebesgue's Monotone Convergence Theorem), we have

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X F \, \mathrm{d}\mu,$$

where $F = \lim\limits_{n\to\infty} f_n$. By the definition (1.37), we have $F = |f|$ and so

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X |f| \, \mathrm{d}\mu$$

or equivalently,

$$\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0.$$

Thus for every $\epsilon > 0$, there exists a positive integer N such that

$$\int_X |f_n - f| \, \mathrm{d}\mu < \frac{\epsilon}{2} \tag{1.43}$$

for all $n \geq N$.

We *fix* this N. Since $0 \leq |f| \leq |f - f_N| + |f_N|$ by the triangle inequality[i], we apply Proposition 1.24(a) and then Theorem 1.27 and the property (1.38) to the inequality (1.43) to derive

$$\int_E |f| \, \mathrm{d}\mu \leq \int_E |f - f_N| \, \mathrm{d}\mu + \int_E |f_N| \, \mathrm{d}\mu < \frac{\epsilon}{2} + \int_E N \, \mathrm{d}\mu = \frac{\epsilon}{2} + N\mu(E) \tag{1.44}$$

for every $E \in \mathfrak{M}$. Therefore, if we take $\delta = \frac{\epsilon}{2N}$ and $\mu(E) < \delta$, then our inequality (1.44) becomes

$$\int_E |f| \, \mathrm{d}\mu < \epsilon$$

as desired. Hence we complete the proof of the problem. ∎

[i] See [49, Definition 2.15, p. 30].

Problem 1.13

Rudin Chapter 1 Exercise 13.

Proof. Let $f : X \to [0,\infty] \subset [-\infty,\infty]$ be a measurable function. For each $n = 1,2,\ldots$, we define $f_n : X \to [0,\infty] \subset [-\infty,\infty]$ by

$$f_n(x) = nf(x).$$

It is clear that

$$f_n^{-1}\big((\alpha,\infty]\big) = \{x \in X \mid f_n(x) \in (\alpha,\infty]\} = \Big\{x \in X \,\Big|\, f(x) \in \Big(\frac{\alpha}{n},\infty\Big]\Big\} = f^{-1}\Big(\Big(\frac{\alpha}{n},\infty\Big]\Big)$$

for all real α. Since f is measurable, we have $f^{-1}\big((\frac{\alpha}{n},\infty]\big) \in \mathfrak{M}$ for every real α, where $\mathfrak{M}$ is a σ-algebra in X. Thus we obtain from Theorem 1.12(c) that each f_n is measurable for $n = 1,2,\ldots$. Furthermore, we have

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$$

on X and $f_n(x) \to \infty \cdot f(x)$ as $n \to \infty$. By Proposition 1.24(c), we have

$$\int_X f_n \,\mathrm{d}\mu = \int_X nf \,\mathrm{d}\mu = n\int_X f \,\mathrm{d}\mu \tag{1.45}$$

for $n = 1,2,\ldots$. Hence we conclude from the equality (1.45) and Theorem 1.26 (Lebesgue's Monotone Convergence Theorem) that

$$\int_X \infty \cdot f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \lim_{n\to\infty} n \int_X f \,\mathrm{d}\mu = \infty \cdot \int_X f \,\mathrm{d}\mu.$$

Thus Proposition 1.24(c) also holds when $c = \infty$ and so we complete the proof of the problem. ∎

CHAPTER 2

Positive Borel Measures

2.1 Properties of Semicontinuity

Problem 2.1

Rudin Chapter 2 Exercise 1.

Proof. We have $f_n : \mathbb{R} \to [0, \infty)$ for all $n \in \mathbb{N}$. In Proposition 2.5, we will prove a property which is equivalent to Definition 2.8 and our proof of **Statement (a)** below becomes simpler. However, we choose to apply Definition 2.8 to prove the statements here.

- **Statement (a):** For any real α, β_1 and β_2, let

$$E(\alpha) = \{x \in \mathbb{R} \mid f_1(x) + f_2(x) < \alpha\} \quad \text{and} \quad F_i(\beta_i) = \{x \in \mathbb{R} \mid f_i(x) < \beta_i\}, \tag{2.1}$$

where $i = 1, 2$. If $\alpha \leq 0$, then we see that $E(\alpha) = \varnothing$ which is open in $\mathbb{R}$. Similarly, $F_i(\beta_i) = \varnothing$ if $\beta_i \leq 0$. So in the following discussion, we assume that $\alpha > 0, \beta_i > 0$ and furthermore, $E(\alpha) \neq \varnothing$ and $F_i(\beta_i) \neq \varnothing$.

Now we claim that

$$E(\alpha) = \bigcup_{\substack{\beta_1+\beta_2 \leq \alpha \\ \beta_1, \beta_2 > 0}} [F_1(\beta_1) \cap F_2(\beta_2)]. \tag{2.2}$$

To prove the claim, on the one hand, if

$$x \in \bigcup_{\substack{\beta_1+\beta_2 \leq \alpha \\ \beta_1, \beta_2 > 0}} [F_1(\beta_1) \cap F_2(\beta_2)],$$

then there exist real β_1 and β_2 with $\beta_1 + \beta_2 \leq \alpha$ and $\beta_1, \beta_2 > 0$ so that $x \in F_1(\beta_1) \cap F_2(\beta_2)$. By the definition (2.1), this x satisfies $f_1(x) < \beta_1$ *and* $f_2(x) < \beta_2$ and their sum implies

$$f_1(x) + f_2(x) < \beta_1 + \beta_2 \leq \alpha.$$

Thus we have $x \in E(\alpha)$, i.e.,

$$\bigcup_{\substack{\beta_1+\beta_2 \leq \alpha \\ \beta_1, \beta_2 > 0}} [F_1(\beta_1) \cap F_2(\beta_2)] \subseteq E(\alpha).$$

On the other hand, if $x \in E(\alpha)$, then we let $\eta = f_1(x) + f_2(x)$. Define the two numbers β_1 and β_2 by

$$\beta_1 = \frac{\eta + \alpha}{2} - f_2(x) \quad \text{and} \quad \beta_2 = \frac{\eta + \alpha}{2} - f_1(x).$$

By direct computation, we know that

$$\beta_i > \frac{f_1(x) + f_2(x) + f_1(x) + f_2(x)}{2} - f_i(x) = f_1(x) + f_2(x) - f_i(x) \geq 0,$$

where $i = 1, 2$. Since $\eta = f_1(x) + f_2(x)$, it is easy to check that $f_1(x) + f_2(x) < \alpha$ if and only if $[f_1(x) + f_2(x)] + [f_1(x) + f_2(x)] < \alpha + \eta$ if and only if $f_1(x) + f_2(x) < \frac{\eta+\alpha}{2}$ if and only if

$$f_1(x) < \beta_1. \tag{2.3}$$

Similarly, the above argument can be used to show that $f_1(x) + f_2(x) < \alpha$ if and only if

$$f_2(x) < \beta_2. \tag{2.4}$$

Now we deduce from the inequalities (2.3) and (2.4) that $x \in F_1(\beta_1) \cap F_2(\beta_2)$. Furthermore, since $\beta_1 + \beta_2 = \eta + \alpha - f_1(x) - f_2(x) = \alpha$, we have

$$x \in \bigcup_{\substack{\beta_1+\beta_2 \leq \alpha \\ \beta_1, \beta_2 > 0}} [F_1(\beta_1) \cap F_2(\beta_2)],$$

i.e.,

$$E(\alpha) \subseteq \bigcup_{\substack{\beta_1+\beta_2 \leq \alpha \\ \beta_1, \beta_2 > 0}} [F_1(\beta_1) \cap F_2(\beta_2)].$$

Hence the claim (2.2) holds.

Since $F_1(\beta_1)$ and $F_2(\beta_2)$ are open in $\mathbb{R}$, $F_1(\beta_1) \cap F_2(\beta_2)$ is also open in $\mathbb{R}$. Since the union of any collection of open sets is open ([49, Theorem 2.24, p. 34]), we follow from this and the equality (2.2) that $E(\alpha)$ is open in $\mathbb{R}$. By Definition 2.8, $f_1 + f_2$ is upper semicontinuous.

- **Statement (b):** By a similar argument as in part (a), we can show that the sum of two lower semicontinuous functions is lower semicontinuous.

- **Statement (c):** This is not true in general. We use Proposition A.7 to give a counterexample. For each $n \in \mathbb{N}$, let $F_n = [\frac{1}{n+1}, \frac{1}{n}]$. Then each χ_{F_n} is upper semicontinuous because F_n is closed in $\mathbb{R}$. Consider the set

$$E = \Big\{x \in \mathbb{R} \,\Big|\, \sum_{n=1}^{\infty} \chi_{F_n}(x) < \frac{1}{2}\Big\}.$$

If $x \in F_n$ for some $n \in \mathbb{N}$, then we have

$$1 \leq \sum_{n=1}^{\infty} \chi_{F_n}(x) \leq 2. \tag{2.5}$$

In other words, we have $F_n \not\subseteq E$ for $n = 1, 2, \ldots$. Since $\bigcup_{n=1}^{\infty} F_n = (0, 1]$,[a] we see that

$$(0, 1] \not\subseteq E. \tag{2.6}$$

[a] If $x \in (0, 1]$, then there exists a positive integer k such that $\frac{1}{k+1} < x$ which implies that $x \in F_1 \cup F_2 \cup \cdots \cup F_k$.

However, if $x \leq 0$ or $x > 1$, then $x \notin F_n$ for every $n \in \mathbb{N}$ which means that $\chi_{F_n}(x) = 0$, i.e.,

$$(-\infty, 0] \cup (1, \infty) \subseteq E. \tag{2.7}$$

Hence, by combining the set relations (2.6) and (2.7), we conclude that

$$E = (-\infty, 0] \cup (1, \infty)$$

which is *not* open in $\mathbb{R}$. By Definition 2.8, $\sum_{n=1}^{\infty} \chi_{F_n}$ is not upper semicontinuous.

- **Statement (d):** The set
$$F_n(\alpha) = \{x \in \mathbb{R} \,|\, f_n(x) > \alpha\}$$
is open for every real α. By applying part (b) repeatedly, we know that $\sum_{n=1}^{N} f_n$ is lower semicontinuous for every positive integer N. For every $x \in [0, \infty)$, let
$$\begin{aligned} f(x) &= \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{\infty} f_n(x), \\ E(\alpha) &= \{x \in \mathbb{R} \,|\, f(x) > \alpha\}, \\ F_N(\alpha) &= \Big\{x \in \mathbb{R} \,\Big|\, \sum_{n=1}^{N} f_n(x) > \alpha\Big\} \end{aligned} \tag{2.8}$$
for real α and $N \in \mathbb{N}$. We claim that
$$E(\alpha) = \bigcup_{N=1}^{\infty} F_N(\alpha) \tag{2.9}$$
for every real α. Similar to the proof of part (a), we suppose that $\alpha > 0$, $E(\alpha) \neq \mathbb{R}$ and $F_N(\alpha) \neq \mathbb{R}$.

Suppose that $x \in E(\alpha)$, i.e., $f(x) > \alpha$. Then there exists a $\epsilon > 0$ such that
$$f(x) > \alpha + \epsilon.$$
Since each f_n is nonnegative, $\Big\{\sum_{n=1}^{N} f_n\Big\}$ is an increasing sequence. By this and the definition of f in (2.8), there exists a positive integer N' such that $\sum_{n=1}^{N'} f_n(x) > \alpha + \epsilon$ which implies that $x \in F_{N'}(\alpha)$, i.e.,
$$E(\alpha) \subseteq \bigcup_{N=1}^{\infty} F_N(\alpha). \tag{2.10}$$

To prove the other side, if $x \in \bigcup_{N=1}^{\infty} F_N(\alpha)$, then $x \in F_{N'}(\alpha)$ for some positive integer N', i.e., $\sum_{n=1}^{N'} f_n(x) > \alpha$. Again, the fact that $\Big\{\sum_{n=1}^{N} f_n\Big\}$ is an increasing sequence implies that
$$f(x) \geq \sum_{n=1}^{N'} f_n(x) > \alpha,$$

i.e., $x \in E(\alpha)$ and then

$$\bigcup_{N=1}^{\infty} F_N(\alpha) \subseteq E(\alpha). \tag{2.11}$$

Hence the set relations (2.10) and (2.11) definitely imply the claim (2.9) is true and since each $F_N(\alpha)$ is open in $\mathbb{R}$ by Definition 2.8, $E(\alpha)$ is also open in $\mathbb{R}$. Since α is arbitrary, f is lower semicontinuous by Definition 2.8.

In the proof of **Statements (a)** and **(b)** above, we don't use the property that f_1 and f_2 are nonnegative. Therefore, they remain valid even if the word "nonnegative" is omitted. However, **Statement (c)** cannot hold anymore if the word "nonnegative" is omitted. In fact, we consider the sequence of real functions $\{f_n\}$ defined by

$$f_1 = \chi_{[-1,1]} \quad \text{and} \quad f_n = -\chi_{[\frac{1}{n},\frac{1}{n-1}]} \quad (n = 2, 3, \ldots).$$

Since $[-1, 1$ and $[\frac{1}{n}, \frac{1}{n-1}]$ are closed in $\mathbb{R}$, we know that $f_1, f_2, \ldots$ are upper semicontinuous. It is clear that $f_1(0) = -1$, so f_1 is *not* a nonnegative function on $\mathbb{R}$. By definition, we have

$$f = \sum_{n=1}^{\infty} f_n = \chi_{[-1,1]} - \sum_{n=2}^{\infty} \chi_{[\frac{1}{n},\frac{1}{n-1}]} = \chi_{[-1,0]} + \chi_{(0,1]} - \sum_{n=1}^{\infty} \chi_{[\frac{1}{n+1},\frac{1}{n}]}. \tag{2.12}$$

By the inequalities (2.5), we see that

$$-\sum_{n=1}^{\infty} \chi_{[\frac{1}{n+1},\frac{1}{n}]}(x) = \begin{cases} -2, & \text{if } x = \frac{1}{2}, \frac{1}{3}, \ldots; \\ -1, & \text{if } x \in (0, 1] \setminus \{\frac{1}{2}, \frac{1}{3}, \ldots\}; \\ 0, & \text{if } x \leq 0 \text{ or } x > 1. \end{cases} \tag{2.13}$$

Thus we follow from the expressions (2.12) and (2.13) that

$$f(x) = \begin{cases} -1, & \text{if } x = \frac{1}{2}, \frac{1}{3}, \ldots; \\ 0, & \text{if } x \in (0, 1] \setminus \{\frac{1}{2}, \frac{1}{3}, \ldots\} \text{ or } x < -1 \text{ or } x > 1; \\ 1, & \text{if } x \in [-1, 0]. \end{cases} \tag{2.14}$$

Therefore, the expression (2.14) of f gives

$$E = \left\{x \in \mathbb{R} \,\middle|\, f(x) > \frac{1}{2}\right\} = [-1, 0]$$

which is *not* open in $\mathbb{R}$. By Definition 2.8, f is not upper semicontinuous. By Definition 2.8, a function f is lower semicontinuous if and only if $-f$ is upper semicontinuous. This observation indicates that we can deduce a counterexample to **Statement (d)** from the counterexample (2.12).

Finally, the truths of the **Statements (a)**, **(b)** and **(d)** depend *only* on the range of f and the fact that the union of any collection of open sets in a topological space X is open in X (see the set equalities (2.2) and (2.9)). This completes the proof of the problem. ■

Problem 2.2

Rudin Chapter 2 Exercise 2.

Proof. We formulate and prove the general statement: Let X be a topological space, U an open set in X containing the point x and $f : X \to \mathbb{C}$. Define

$$\varphi(x,U) = \sup\{|f(s) - f(t)| \,|\, s,t \in U\} \quad \text{and} \quad \varphi(x) = \inf\{\varphi(x,U) \,|\, U \text{ is open, } x \in U\}. \tag{2.15}$$

We claim that φ is upper semicontinuous, f is continuous at $x \in X$ if and only if $\varphi(x) = 0$ and the set of points of continuity of an arbitrary complex function is a G_δ.

- **φ is upper semicontinuous.** Let $E = \{x \in X \,|\, \varphi(x) < \alpha\}$, where $\alpha \in \mathbb{R}$. We have to show that E is open in X. If $E = \varnothing$, then there is nothing to prove. Thus we suppose that $E \neq \varnothing$. In this case, we have $p \in E$ so that

  $$\varphi(p) < \alpha.$$

  This fact shows that *there exists* an open set U containing p and[b]

  $$\varphi(p,U) < \alpha.$$

  Pick $q \in U \setminus \{p\}$. Since U is open, *there exists* an open set V containing q such that $q \in V \subseteq U$. Since $p, q \in U$, we know from the definition (2.15) that

  $$\varphi(p,U) = \varphi(q,U). \tag{2.16}$$

  Furthermore, we observe from the definition (2.15) that

  $$\varphi(x,U') \leq \varphi(x,U) \text{ for every open sets } U, U' \text{ with } U' \subseteq U. \tag{2.17}$$

  Now these facts (2.16) and (2.17) imply that $\varphi(q,V) \leq \varphi(q,U) = \varphi(p,U) < \alpha$ and then

  $$\varphi(q) \leq \varphi(q,V) < \alpha.$$

  In other words, $q \in E$. Since $q \in U \setminus \{p\}$ is arbitrary, we have shown that $p \in U \subseteq E$, i.e., E is open for every $\alpha \in \mathbb{R}$ and hence φ is upper semicontinuous by Definition 2.8.

- **f is continuous at x if and only if $\varphi(x) = 0$.** Suppose that f is continuous at x. Recall from the definition of continuity ([42, Theorem 18.1, p. 104] or [51, p. 9]) that for every $\epsilon > 0$, the neighborhood

  $$B(f(x),\epsilon) = \left\{ z \in \mathbb{C} \,\middle|\, |f(x) - z| < \frac{\epsilon}{2} \right\} \tag{2.18}$$

  has a neighborhood U_ϵ of x such that $f(U_\epsilon) \subseteq B(f(x),\epsilon)$, i.e., $f(y) \in B(f(x),\epsilon)$ for all $y \in U_\epsilon$. Now we take this U_ϵ in the definition (2.15):

  $$\varphi(x,U_\epsilon) = \sup\{|f(s) - f(t)| \,|\, s,t \in U_\epsilon\}$$

  and it follows from the definition (2.18) that[c]

  $$\varphi(x,U_\epsilon) \leq \epsilon. \tag{2.19}$$

  By the definition (2.15), we have $\varphi(x) \leq \varphi(x,U)$ for every open set U containing x. Therefore, we deduce from this and the inequality (2.19) that $\varphi(x) \leq \epsilon$. Since ϵ is arbitrary, we have the desired result that $\varphi(x) = 0$.

[b] Otherwise, we have $\varphi(p,U) \geq \alpha$ for every open set U containing p and this implies that $\alpha \leq \varphi(p)$, a contradiction.

[c] Note that $|f(s) - f(t)| \leq |f(s) - f(x)| + |f(x) - f(t)| < \epsilon$ for every $s, t \in U_\epsilon$.

Conversely, suppose that $\varphi(x) = 0$. Thus for every $\epsilon > 0$, there exists a neighborhood U_ϵ of x such that $\varphi(x, U_\epsilon) < \epsilon$. By the definition (2.15), this implies that

$$|f(s) - f(t)| < \epsilon \tag{2.20}$$

for every $s, t \in U_\epsilon$. In particular, if we take $s = x$ to be *fixed* and $t = y$ vary, then the inequality (2.20) can be rewritten as $|f(x) - f(y)| < \epsilon$ for every $y \in U_\epsilon$. By the definition ([42, Theorem 18.1, p. 104] or [51, p. 9]), f is continuous at x.

- **The set of points of continuity of an arbitrary complex function is a G_δ.** By the previous analysis, we establish that

$$G = \{x \in X \,|\, f \text{ is continuous at } x\} = \{x \in X \,|\, \varphi(x) = 0\} = \bigcap_{n=1}^{\infty} \Big\{x \in X \,\Big|\, \varphi(x) < \frac{1}{n}\Big\}.$$

Since φ is upper semicontinuous, each set $\{x \in X \,|\, \varphi(x) < \frac{1}{n}\}$ is open in X. By Definition 1.11, we see that G is actually a G_δ.

This completes the proof of the problem. ■

Problem 2.3

Rudin Chapter 2 Exercise 3.

Proof. The first part is proven in [63, Problem 4.20, pp. 73, 74]. The function in the question can be used to prove Urysohn's Lemma for any metric space X directly.

Lemma 2.1 (Urysohn's Lemma)

Suppose that X is a metric space with metric ρ, A and B are disjoint nonempty closed subsets of X. Then there exists a continuous function $f : X \to [0, 1]$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 0$ precisely on A and $f(x) = 1$ precisely on B.

Proof of Lemma 2.1. This lemma is proven in [63, Problem 4.22, pp. 75, 76]. Readers are recommended to read [42, Theorem 32.2, p. 202; Theorem 33.1, pp. 207 – 210]. ■

This ends the proof of the problem. ■

Problem 2.4

Rudin Chapter 2 Exercise 4.

Proof.

(a) Recall from [51, Eqn. (2), p. 42] that

$$\mu(E) = \inf\{\mu(V) \,|\, E \subseteq V \text{ and } V \text{ is open}\} \tag{2.21}$$

which is *defined* for every subset E of X.[d] By the result [51, Eqn. (4), p. 42], we have

$$\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2). \tag{2.22}$$

[d] Here we don't require E to be a measurable set.

We need a lemma:

Lemma 2.2

For disjoint open sets V_1 and V_2, we have

$$\mu(V_1 \cup V_2) = \mu(V_1) + \mu(V_2).$$

Proof of Lemma 2.2. The proof of the inequality $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$ can be found in [51, **STEP I**, p. 42], so we only prove the other side. Suppose that $g \in C_c(X)$ and $g \prec V_1$. Similarly, suppose that $h \in C_c(X)$ and $h \prec V_2$. By Definition 2.9, since $C_c(X)$ is a vector space, $f = g + h \in C_c(X)$. Furthermore, we know from Definition 2.9(a) that

$$\operatorname{supp}(f) = \operatorname{supp}(g+h) \subseteq \operatorname{supp}(g) + \operatorname{supp}(h) \subseteq V_1 \cup V_2. \tag{2.23}$$

By assumption, we have $V_1 \cap V_2 = \varnothing$ which means that $g(x) = 0$ on $X \setminus V_1$ and $h(x) = 0$ on $X \setminus V_2$. Thus we deduce from these facts that

$$f(x) = \begin{cases} g(x), & \text{if } x \in V_1; \\ h(x), & \text{if } x \in V_2; \\ 0, & \text{if } x \in X \setminus (V_1 \cup V_2). \end{cases}$$

In other words, we have

$$0 \leq f(x) \leq 1 \tag{2.24}$$

on X. Therefore, we can conclude from the set relation (2.23) and the inequalities (2.24) that

$$f \prec V_1 \cup V_2. \tag{2.25}$$

Now, by Definition 2.1, the relation (2.25) and then [51, Eqn. (1), p.41], we obtain that

$$\Lambda(g) + \Lambda(h) = \Lambda(g+h) = \Lambda(f) \leq \mu(V_1 \cup V_2). \tag{2.26}$$

We first *fix* the h in the inequality (2.26) and since it holds for *every* $g \prec V_1$, the definition of supremum gives

$$\mu(V_1) + \Lambda(h) \leq \mu(V_1 \cup V_2). \tag{2.27}$$

Next, the inequality (2.27) holds for *every* $h \prec V_2$, we establish from the definition of supremum that

$$\mu(V_1) + \mu(V_2) \leq \mu(V_1 \cup V_2).$$

Hence we have $\mu(V_1) + \mu(V_2) = \mu(V_1 \cup V_2)$. ■

Let's go back to the original proof. We want to compute $\mu(E_1 \cup E_2)$. By the definition (2.21), it needs to consider all open sets containing $E_1 \cup E_2$. Since $V_1 \subseteq V_2$ certainly implies $\mu(V_1) \leq \mu(V_2)$, we can restrict our attention to any open set W such that

$$E_1 \cup E_2 \subseteq W \subseteq V_1 \cup V_2.$$

Since $V_1 \cap V_2 = \varnothing$, we have $W = (W \cap V_1) \cup (W \cap V_2)$, where $(W \cap V_1) \cap (W \cap V_2) = \varnothing$.

Furthermore, since $W \cap V_1$ and $W \cap V_2$ are open sets in X, it yields from Lemma 2.2 that

$$\mu(W) = \mu(W \cap V_1) + \mu(W \cap V_2) \geq \mu(E_1) + \mu(E_2). \tag{2.28}$$

Since W is arbitrary, $\mu(E_1) + \mu(E_2)$ is a lower bound of the set in the definition (2.21). Hence we follow from this fact and the inequality (2.28) that

$$\mu(E_1 \cup E_2) \geq \mu(E_1) + \mu(E_2). \tag{2.29}$$

By the inequalities (2.22) and (2.29), we have the desired result that

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

(b) Define $E_1 = E$. By [51, **STEP V**, p. 44], there exists a compact set K_1 and an open set V_1 such that

$$K_1 \subseteq E_1 \subseteq V_1 \quad \text{and} \quad \mu(V_1 \setminus K_1) < \frac{1}{2^2}.$$

By [51, **STEP II**, p. 43], we have $K_1 \in \mathfrak{M}_F$. Then we follow from [51, **STEP VI**, p. 44] that $E_1 \setminus K_1 \in \mathfrak{M}_F$. Define $E_2 = E_1 \setminus K_1$ so that

$$E = E_1 = E_2 \cup K_1,$$

where $\mu(E_2) = \mu(E_1 \setminus K_1) \leq \mu(V_1 \setminus K_1) < \frac{1}{2^2}$. By induction, we can show that there exists a sequence of compact sets $\{K_n\}$ and a sequence of open sets $\{V_n\}$ such that

$$K_n \subseteq E_n \subseteq V_n, \tag{2.30}$$

where $E_{n+1} = E_n \setminus K_n$, $E_{n+1} \in \mathfrak{M}_F$ and $\mu(E_{n+1}) < \frac{1}{2^{n+1}}$. Therefore, the set E satisfies the following relations:

$$E = E_{n+1} \cup K_1 \cup K_2 \cup \cdots \cup K_n \quad \text{and} \quad \mu(E_{n+1}) < \frac{1}{2^{n+1}} \tag{2.31}$$

for every positive integer n. By the set relations (2.30), we deduce that

$$K_{n+1} \subseteq E_{n+1} = E_n \setminus K_n = E_{n-1} \setminus (K_{n-1} \cup K_n) = E \setminus (K_1 \cup K_2 \cup \cdots \cup K_n) \tag{2.32}$$

for all $n \in \mathbb{N}$. In other words, K_{n+1} is disjoint with all $K_1, K_2, \ldots, K_n$ and thus $\{K_1, \ldots, K_n\}$ is a disjoint collection of compact sets in X. Now, by letting $n \to \infty$ in the relations (2.31), we achieve that E is given by

$$E = N \cup K_1 \cup K_2 \cup \cdots,$$

where $\{K_n\}$ is a *disjoint* countable collection of compact sets in X and

$$N = E \setminus \bigcup_{n=1}^{\infty} K_n.$$

By the equalities in (2.32), $N \subseteq E_{n+1}$ holds for all $n \in \mathbb{N}$ and the inequality in (2.31) shows that

$$\mu(N) = \lim_{n\to\infty} \mu(E_{n+1}) = \lim_{n\to\infty} \frac{1}{2^{n+1}} = 0.$$

This completes the proof of the problem. ■

2.2 Problems on the Lebesgue Measure on $\mathbb{R}$

In Problems 2.5 to 2.8, m stands for the Lebesgue measure on $\mathbb{R}$.

Problem 2.5

Rudin Chapter 2 Exercise 5.

Proof. This problem is proven in [49, Remarks 11.11(f), p. 309]. ■

Problem 2.6

Rudin Chapter 2 Exercise 6.

Proof. Recall from [49, Theorem 2.47, p. 42] that a subset F of $\mathbb{R}$ is connected if and only if $(x, y) \subseteq F$ for every $x, y \in F$ with $x < y$. Therefore, if F is a connected subset of a totally disconnected set $K \subset \mathbb{R}$ and F consists of more than one point, then K must contain a segment, a contradiction. By this observation, we know that a totally disconnected set $K \subset \mathbb{R}$ contains no segment and this gives a hint to our construction: It is well-known that the Cantor set E is compact and contains no segment. However, Problem 2.5 shows that $m(E) = 0$, so we have to "modify" the construction of E so as to fit our problem.

Let $K_0 = [0, 1]$. The idea of the construction is to remove the "middle $\frac{1}{2^{2n}}$th" segments. In fact, we first remove the segment $(\frac{3}{8}, \frac{5}{8})$ of length $\frac{1}{2^2}$ from K_0 and let

$$K_1 = \Big[0, \frac{3}{8}\Big] \cup \Big[\frac{5}{8}, \frac{8}{8}\Big].$$

Next, we remove the middle segments $(\frac{5}{32}, \frac{7}{32})$ and $(\frac{25}{32}, \frac{27}{32})$ each of length $\frac{1}{2^4}$ from each connected component of K_1 to get

$$K_2 = \Big[0, \frac{5}{32}\Big] \cup \Big[\frac{7}{32}, \frac{12}{32}\Big] \cup \Big[\frac{20}{32}, \frac{25}{32}\Big] \cup \Big[\frac{27}{32}, \frac{32}{32}\Big].$$

Continuing in this way, we obtain a sequence of compact sets K_n such that

$$K_1 \supset K_2 \supset K_3 \supset \cdots.$$

Therefore, we have

$$m(K_n) = 1 - \sum_{k=1}^{n} \frac{1}{2^{2k}} \cdot 2^{k-1} = 1 - \sum_{k=1}^{n} \frac{1}{2^{k+1}}. \tag{2.33}$$

Define the set

$$K = \bigcap_{n=1}^{\infty} K_n.$$

We know that $K \neq \varnothing$ because of the corollary [49, Theorem 2.36, p. 38]. Applying an argument similar to that used in [49, §2.44, pp. 41, 42], we can show that K is a totally disconnected compact set in $\mathbb{R}$. Furthermore, since $m(K_1)$ is finite, Theorem 1.19(e) and the Lebesgue measure (2.33) ensure that

$$m(K) = \lim_{n\to\infty} m(K_n) = \lim_{n\to\infty} \Big(1 - \sum_{k=1}^{n} \frac{1}{2^{k+1}}\Big) = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} > 0.$$

For the second assertion, let v be lower semicontinuous and $v \leq \chi_K$. Assume that there was a $p \in \mathbb{R}$ such that $v(p) > 0$. Since v is lower semicontinuous, Definition 2.8 implies that the set $A = \{x \in \mathbb{R} \,|\, v(x) > 0\}$ is open in $\mathbb{R}$. By the assumption, we have $p \in A$, so there exists a $\delta > 0$ such that

$$(p - \delta, p + \delta) \subseteq A. \tag{2.34}$$

Now we follow from the hypothesis $v \leq \chi_K$ and the relation (2.34) that

$$\chi_K(x) > 0$$

on $(p - \delta, p + \delta)$, therefore we have $(p - \delta, p + \delta) \in K$ which contradicts to the fact that K contains no segment. Hence we have $v \leq 0$ and this completes the proof of the problem. ■

Problem 2.7

Rudin Chapter 2 Exercise 7.

Proof. By the idea used in Problem 2.6, it is not hard to see that the construction of the compact sets K_n and K still work for removing the "middle $\frac{\epsilon}{2^{2n-1}}$th" segments. In this case, instead of the Lebesgue measure (2.33), we have

$$m(K_n) = 1 - \sum_{k=1}^{n} \frac{\epsilon}{2^{2k-1}} \cdot 2^{k-1} = 1 - \epsilon \sum_{k=1}^{n} \frac{1}{2^k} \tag{2.35}$$

and so

$$m(K) = \lim_{n\to\infty} m(K_n) = 1 - \epsilon.$$

Since K is closed in $[0, 1]$, the complement

$$E = [0, 1] \setminus K \tag{2.36}$$

is definitely open in $[0, 1]$.

Now it remains to show that E is dense in $[0, 1]$. We prove an equivalent definition of a dense set: Let $A \subseteq B \subseteq \mathbb{R}$. We say A is dense in B means that for every $x, y \in B$ with $x < y$, we can find $z \in A$ such that $x < z < y$. In particular, suppose that $x, y \in [0, 1]$ with $x < y$. If $(x, y) \cap E = \varnothing$, then the definition (2.36) says that $(x, y) \subseteq K$ which is impossible. Therefore, we have

$$(x, y) \cap E \neq \varnothing$$

and this means that there exists $z \in E$ such that $x < z < y$. Hence E is dense in $[0, 1]$, completing the proof of the problem. ■

Problem 2.8

Rudin Chapter 2 Exercise 8.

Proof. Let $\{r_n\}$ be the enumeration of all rationals in $\mathbb{R}$. Let F_n be the segment given by

$$F_n = \left(r_n - \frac{1}{2^{2n}}, r_n + \frac{1}{2^{2n}}\right), \tag{2.37}$$

where $n = 1, 2, \ldots$. Define the sets

$$G_n = F_n \setminus (F_{n+1} \cup F_{n+2} \cup \cdots) = F_n \setminus \bigcup_{k=1}^{\infty} F_{n+k}. \tag{2.38}$$

We divide the proof into several steps.

- **Step 1: $m(G_n) > 0$.** By the definition (2.37), we know that $m(F_{n+1}) = \frac{m(F_n)}{2^2}$ for every $n = 1, 2, \ldots$. By the definition (2.38), for every $n \in \mathbb{N}$, we have

$$F_n = G_n \cup \bigcup_{k=1}^{\infty} F_{n+k}$$

so that the subadditive property of a measure (see the proof of Problem 1.11) implies that

$$m(F_n) \leq m(G_n) + \sum_{k=1}^{\infty} m(F_{n+k}) = m(G_n) + m(F_n) \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = m(G_n) + \frac{m(F_n)}{3}.$$

Therefore, we have

$$m(G_n) \geq \frac{2m(F_n)}{3} > 0 \tag{2.39}$$

for every positive integer n.

- **Step 2: Existence of a Borel subset $A_n \subset F_n$ with $m(A_n) = \frac{m(F_n)}{2}$.** It is easy to check from the definition (2.37) that $m(F_n) = \frac{1}{2^{2n-1}}$. By Definition 2.19, we know that[e]

$$F_n' = \frac{1}{m(F_n)} \Big[F_n - \Big(r_n - \frac{1}{2^{2n}} \Big) \Big] = (0, 1).$$

Thus, by the proof of Problem 2.7[f], there exists a (totally disconnected) compact set $A_n' \subset F_n'$ such that $m(A_n') = \frac{1}{2}$. Then Theorem 2.20(c) implies that the set A_n given by

$$A_n = m(F_n) A_n' + \Big(r_n - \frac{1}{2^{2n}} \Big)$$

is a (totally disconnected) compact subset of F_n with measure

$$m(A_n) = m\Big(m(F_n) A_n' + \Big(r_n - \frac{1}{2^{2n}} \Big) \Big) = m\big(m(F_n) A_n' \big) = m(F_n) m(A_n') = \frac{m(F_n)}{2}. \tag{2.40}$$

Since A_n is compact, it is closed by the Heine–Borel theorem [49, Theorem 2.41, p. 40]. By Definition 1.11, it is also a Borel set, completing the proof of **Step 2**.

Now we can construct a Borel set with the desired properties. The construction is as follows: For every positive integer n, we define

$$E_n = G_n \cap A_n \quad \text{and} \quad E = \bigcup_{n=1}^{\infty} E_n. \tag{2.41}$$

By the definition (2.38), since each F_n is Borel, each G_n is also Borel. Thus each E_n and their countable union E are also Borel sets. It remains to show that the E satisfies the requirements of the problem.

- **Step 3: $0 < m(E \cap F_n)$.** We first show that $m(E_n \cap F_n) > 0$ for every positive integer n. To prove this claim, we note from the definitions (2.38) and (2.41) that $E_n \subseteq G_n \subseteq F_n$, so we have

$$m(E_n \cap F_n) = m(E_n) = m(G_n \cap A_n).$$

Furthermore, since $G_n \subseteq F_n$ and $A_n \subset F_n$ (by **Step 2**), we have

$$m(G_n \cup A_n) \leq m(F_n). \tag{2.42}$$

[e] It is just a translation and an enlargement.
[f] Take $\epsilon = \frac{1}{2}$ in the measure (2.35).

Combining the inequality (2.39) and the expression (2.40), we can reduce the inequality (2.42) to

$$m(G_n \cup A_n) < m(G_n) + m(A_n). \tag{2.43}$$

Now we apply a property of additive function [49, Eqn. (7), p. 302] to the inequality (2.43) to obtain

$$m(G_n \cup A_n) < m(G_n \cup A_n) + m(G_n \cap A_n)$$

which means that $m(G_n \cap A_n) > 0$ for every positive integer n. Therefore, we follow from this and the fact that $E_n \cap F_n \subseteq E \cap F_n$

$$0 < m(G_n \cap A_n) = m(E_n \cap F_n) \leq m(E \cap F_n)$$

for every positive integer n.

- **Step 4: $m(E \cap F_n) < m(F_n)$.** Fix an integer n. For $m < n$, we apply the identity $A \setminus B = B^c \cap A$ to the definition (2.38) to get

$$G_m^c = [F_m \setminus (F_{m+1} \cup F_{m+2} \cup \cdots)]^c = (F_{m+1} \cup F_{m+2} \cup \cdots) \cup F_m^c$$

which implies that $F_n \subseteq G_m^c$ for $m = 1, 2, \ldots, n-1$. Recall from the definition (2.41) that $E_n^c = G_n^c \cup A_n^c$, so we must have

$$F_n \subseteq E_m^c$$

for $m = 1, 2, \ldots, n-1$. In other words, it says that $F_n \cap E_m = \varnothing$ for $m = 1, 2, \ldots, n-1$. Therefore, this fact implies

$$m(E \cap F_n) = m\Big(\bigcup_{m=1}^{\infty} E_m \cap F_n\Big) = m\Big(\bigcup_{m=n}^{\infty} (E_m \cap F_n)\Big) \leq m\Big(\bigcup_{m=n}^{\infty} (A_m \cap F_n)\Big). \tag{2.44}$$

By the subadditive property of a measure again, we further deduce the inequality (2.44) to

$$m(E \cap F_n) \leq \sum_{m=n}^{\infty} m(A_m \cap F_n) \leq \sum_{m=n}^{\infty} m(A_m). \tag{2.45}$$

Recall the result (2.40) and the fact $m(F_{n+1}) = \frac{m(F_n)}{2^2}$, so we derive from the inequality (2.45) that

$$m(E \cap F_n) \leq \sum_{m=0}^{\infty} \frac{m(F_{n+m})}{2} = \frac{m(F_n)}{2} \sum_{m=0}^{\infty} \frac{1}{2^{2m}} = \frac{2m(F_n)}{3} < m(F_n)$$

for every positive integer n.

- **Step 5: $0 < m(E \cap I) < m(I)$ for every nonempty segment I.** By **Step 3** and **Step 4**, the inequalities

$$0 < m(E \cap F_n) < m(F_n) \tag{2.46}$$

hold for every positive integer n, where F_n and E are given by (2.37) and (2.41) respectively.

Lemma 2.3

Let $I = (\alpha, \beta)$ with $\alpha < \beta$. Then there is a positive integer n_0 such that $F_{n_0} \subseteq I$.

Proof of Lemma 2.3. Let $\gamma = \frac{\alpha+\beta}{2}$ and $\delta = \frac{\gamma+\beta}{2}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational in (γ, δ). Let it be r_n for some positive integer n. If $F_n \subseteq I$, then we are done. Otherwise, we may find another rational in (r_n, δ), namely r_{n+1}. In fact, this process can be repeated m times, i.e., there is a rational r_{n+m} in (r_{n+m-1}, δ). It is clear that

$$r_{n+m} + \frac{1}{2^{2(n+m)}} \leq \delta + \frac{1}{2^{2(n+m)}} < \delta + \frac{1}{2^{2m}} \tag{2.47}$$

and

$$r_{n+m} - \frac{1}{2^{2(n+m)}} > \gamma - \frac{1}{2^{2(n+m)}} > \gamma - \frac{1}{2^{2m}}. \tag{2.48}$$

Now we may pick the m large enough such that

$$\frac{1}{2^{2m}} < \beta - \delta \quad \text{and} \frac{1}{2^{2m}} < \frac{\beta - \alpha}{2}$$

simultaneously. Then it follows from the inequalities (2.47) and (2.48) that

$$r_{n+m} + \frac{1}{2^{2(n+m)}} < \beta \tag{2.49}$$

and

$$r_{n+m} - \frac{1}{2^{2(n+m)}} > \gamma - \frac{1}{2^{2m}} > \frac{\alpha+\beta}{2} - \frac{\beta-\alpha}{2} = \alpha. \tag{2.50}$$

Hence the inequalities (2.49) and (2.50) together imply that $F_{n_0} \subseteq I$, where $n_0 = n+m$, completing the proof of the lemma. ■

We return to the proof of the problem. By Lemma 2.3, we have $E \cap F_{n_0} \subseteq E \cap I$ and the left-hand side of the inequalities (2.46) shows that

$$0 < m(E \cap F_{n_0}) \leq m(E \cap I). \tag{2.51}$$

Since $I = F_{n_0} \cup (I \setminus F_{n_0})$ and $F_{n_0} \cap (I \setminus F_{n_0}) = \varnothing$, we have

$$E \cap I = (E \cap F_{n_0}) \cup [E \cap (I \setminus F_{n_0})].$$

Hence, by applying the right-hand side of the inequalities (2.46) and Theorem 1.19(b) twice, we obtain

$$m(E \cap I) = m(E \cap F_{n_0}) + m\big(E \cap (I \setminus F_{n_0})\big) < m(F_{n_0}) + m(I \setminus F_{n_0}) = m(I). \tag{2.52}$$

Now our desired result follows immediately if we combine the inequalities (2.51) and (2.52).

- **Step 6: $m(E) < \infty$.** By the subadditive property of the measure m, the expression (2.40) and the definition (2.41), we get

$$m(E) \leq \sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} m(A_n) = \frac{1}{2}\sum_{n=1}^{\infty} m(F_n) = \frac{m(F_1)}{2}\sum_{n=0}^{\infty}\frac{1}{2^{2n}} = \frac{2m(F_1)}{3} < \infty.$$

This completes the proof of the problem.[g] ■

[g]Instead of Borel sets of the real number line $\mathbb{R}$, Rudin proved a similar result for measurable set A in $[0,1]$, see [50]. Furthermore, there are two interesting results ([14] and [35]) related to this problem and some of its applications can be found in [13], [18], [25], [39], [46] and [60].

2.3 Integration of Sequences of Continuous Functions

Problem 2.9

Rudin Chapter 2 Exercise 9.

Proof. For every $n \in \mathbb{N}$, we consider the function $g_n : [-1,1] \to [0,1]$ defined by

$$g_n(x) = \begin{cases} 0, & \text{if } x \notin [-\frac{1}{n}, \frac{1}{n}]; \\ \\ 1 - n|x|, & \text{if } x \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases} \tag{2.53}$$

It is clear that each g_n is continuous on $[-1,1]$ and $0 \leq g_n \leq 1$ on $[-1,1]$. The graph of g_n is shown in Figure 2.1 below.

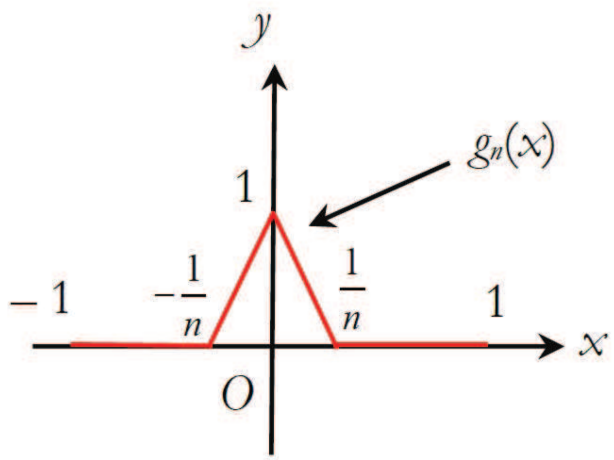

Figure 2.1: The graph of g_n on $[-1,1]$.

Next, we define $g_{n,k} : [0,1] \to [0,1]$ by

$$g_{n,k}(x) = g_n\Big(x - \frac{k}{2n}\Big), \tag{2.54}$$

where $x \in [0,1]$ and $k = 0, 1, \ldots, 2n$. It is easy to check that $x - \frac{k}{2n} \in [-1,1]$ so that each $g_{n,k}$ is well-defined by (2.54). We claim that if we define the sequence $\{f_1, f_2, \ldots\}$ to be

$$\{g_{1,0}, g_{1,1}, g_{1,2}, g_{2,0}, g_{2,1}, g_{2,2}, g_{2,3}, g_{2,4}, \ldots\},$$

then $\{f_n\}$ satisfies the hypotheses of the problem.

To this end, we first note that since each $g_{n,k}$ is continuous on $[0,1]$ and $0 \leq g_{n,k} \leq 1$ on $[0,1]$, each f_n is continuous on $[0,1]$ and $0 \leq f_n \leq 1$. To see why

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 0, \tag{2.55}$$

we have to check the behaviour of

$$\int_0^1 g_{n,k}(x)\,\mathrm{d}x$$

for every positive integer n and $k = 0, 1, \ldots, 2n$. In fact, we know from the definitions (2.53) and (2.54) that

$$g_{n,0}(x) = \begin{cases} 1 - n|x|, & \text{if } x \in [0, \frac{1}{n}); \\ 0, & \text{if } x \in [\frac{1}{n}, 1] \end{cases} \quad \text{and} \quad g_{n,1}(x) = \begin{cases} \frac{1}{2} + nx, & \text{if } x \in [0, \frac{1}{2n}); \\ \frac{3}{2} - nx, & \text{if } x \in [\frac{1}{2n}, \frac{3}{2n}]; \\ 0, & \text{if } x \in (\frac{3}{2n}, 1]. \end{cases} \tag{2.56}$$

Similarly, we have

$$g_{n,2n-1}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{2n-3}{2n}); \\ 1 - n\left|x - \frac{2n-1}{2n}\right|, & \text{if } x \in [\frac{2n-3}{2n}, 1] \end{cases} \tag{2.57}$$

and

$$g_{n,2n}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{n-1}{n}); \\ 1 + n(x-1), & \text{if } x \in [\frac{n-1}{n}, 1]. \end{cases} \tag{2.58}$$

Finally, for $k = 2, 3, \ldots, 2n-2$, we have

$$g_{n,k}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{k-2}{2n}); \\ 1 - n\left|x - \frac{k}{2n}\right|, & \text{if } x \in [\frac{k-2}{2n}, \frac{k+2}{2n}]; \\ 0, & \text{if } x \in (\frac{k+2}{2n}, 1]. \end{cases} \tag{2.59}$$

The graphs of these $g_{n,k}$ are shown as follows:

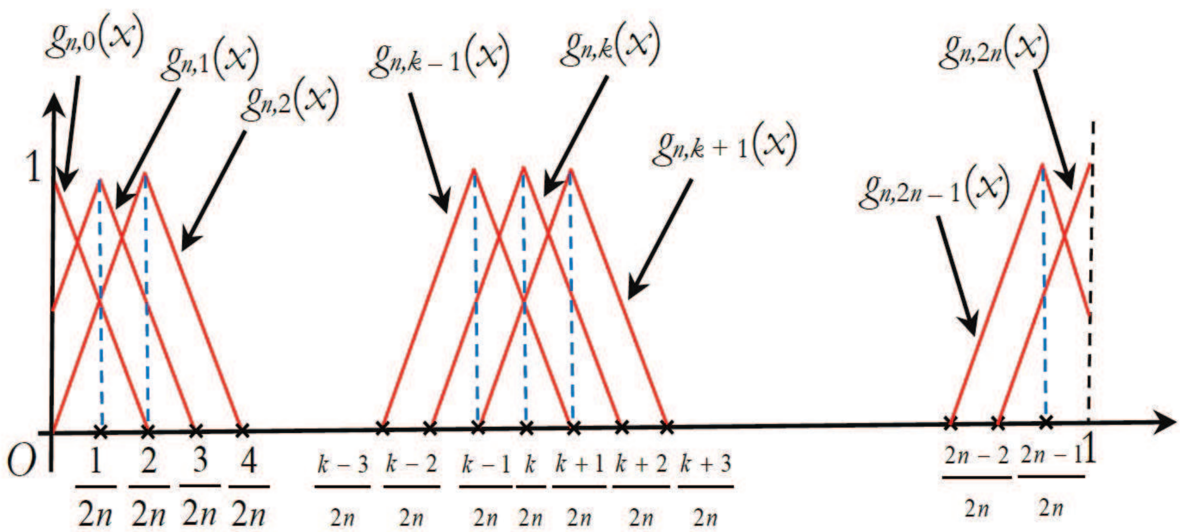

Figure 2.2: The graphs of $g_{n,k}$ on $[0, 1]$.

In other words, it is clear from Figure 2.2 that $\{g_{n,0}, g_{n,1}\ldots, g_{n,2n}\}$ is a family of tent functions centered at $\{\frac{0}{2n}, \frac{1}{2n}, \ldots, \frac{2n}{2n}\}$. Now we have the following cases.[h]

[h]We use the fact that each $g_{n,k}$ is a part or the whole of an isosceles triangle with height 1 and base less than or equal to $\frac{2}{n}$.

- When $k = 0$, we have

$$\int_0^1 g_{n,0}(x)\,\mathrm{d}x = \int_0^1 g_n(x)\,\mathrm{d}x = \int_0^{\frac{1}{n}} (1 - nx)\,\mathrm{d}x = \frac{1}{2n}. \tag{2.60}$$

- When $k = 1$, we have

$$\int_0^1 g_{n,1}(x)\,\mathrm{d}x = \int_0^{\frac{1}{2n}} \Big(\frac{1}{2} + nx\Big)\,\mathrm{d}x + \int_{\frac{1}{2n}}^{\frac{3}{2n}} \Big(\frac{3}{2} - nx\Big)\,\mathrm{d}x = \frac{7}{8n}. \tag{2.61}$$

- When $2 \leq k \leq 2n - 2$, we have

$$\int_0^1 g_{n,k}(x)\,\mathrm{d}x = \frac{1}{2} \times \frac{2}{n} \times 1 = \frac{1}{n}. \tag{2.62}$$

- When $k = 2n - 1$, we have

$$\int_0^1 g_{n,2n-1}(x)\,\mathrm{d}x = \int_{\frac{2n-3}{2n}}^1 \Big(1 - n\Big|x - \frac{2n-1}{2n}\Big|\Big)\,\mathrm{d}x = \frac{7}{8n}. \tag{2.63}$$

- When $k = 2n$, we have

$$\int_0^1 g_{n,2n}(x)\,\mathrm{d}x = \int_{1-\frac{1}{n}}^1 g_n(x-1)\,\mathrm{d}x = \int_{1-\frac{1}{n}}^1 [1 + n(x-1)]\,\mathrm{d}x = \frac{1}{2n}. \tag{2.64}$$

Combining the expressions (2.60) to (2.64), we may conclude that the limit (2.55) holds.

Now it remains to show that $\{f_n(x)\}$ converges for *no* $x \in [0,1]$. If $x = 0$, then we obtain from the definitions (2.56) and (2.57) that

$$g_{n,0}(0) = 1 \quad \text{and} \quad g_{n,2n-1}(0) = 0.$$

In other words, we can find subsequences $\{f_{n_k}(0)\}$ and $\{f_{n_l}(0)\}$ such that $f_{n_k}(0) \to 1$ and $f_{n_l}(0) \to 0$ as $k \to \infty$ and $l \to \infty$ respectively. By Definition 1.13, they imply that

$$\limsup_{n\to\infty} f_n(0) = 1 \quad \text{and} \quad \liminf_{n\to\infty} f_n(0) = 0. \tag{2.65}$$

Similarly, if $x = 1$, then the definitions (2.56) and (2.58) show that

$$g_{n,0}(1) = 0 \quad \text{and} \quad g_{n,2n}(1) = 1$$

which then imply the limits (2.65). If $x \in (0,1) \cap \mathbb{Q}$, then we have $x = \frac{p}{q} = \frac{2p}{2q}$, where $0 < p < q$. Thus it follows from the definitions (2.56) and (2.59) that

$$g_{n,0}\Big(\frac{p}{q}\Big) = 0 \quad \text{and} \quad g_{nq,2np}\Big(\frac{p}{q}\Big) = 1 \tag{2.66}$$

for $n \geq q$. Therefore, the limits (2.65) also hold for $x \in (0,1) \cap \mathbb{Q}$. For irrational x in $(0,1)$, the Archimedean Property shows that there exists $N \in \mathbb{N}$ such that $x > \frac{1}{N}$. Thus it implies that

$$g_{n,0}(x) = 0 \tag{2.67}$$

for all $n \geq N$. By the density of rationals, there exists a sequence $\{\frac{p_n}{q_n}\}$ of rationals in $(0,1)$ such that

$$\frac{p_n}{q_n} \to x$$

as $n \to \infty$, where $0 < p_n < q_n$ for all $n \in \mathbb{N}$. Since we always have

$$\frac{p_n}{q_n} \in \left[\frac{np_n - 1}{nq_n}, \frac{np_n + 1}{nq_n}\right],$$

we derive from the second expression (2.66) that

$$g_{nq_n, 2np_n}\left(\frac{p_n}{q_n}\right) = 1 \tag{2.68}$$

for all $n \in \mathbb{N}$. Combining the two results (2.67) and (2.68), we have shown that the limits (2.65) remain valid for irrationals x, completing the proof of the problem. ■

Problem 2.10

Rudin Chapter 2 Exercise 10.

Proof. Given that $\epsilon > 0$. For each positive integer n, we define

$$E_n = \{x \in [0, 1] \mid f_k(x) > \epsilon \text{ for some } k \geq n\}. \tag{2.69}$$

It is clear that each E_n is bounded and

$$E_n \supseteq E_{n+1} \supseteq \cdots .$$

If $E_{n_0} = \varnothing$ for some positive integer n_0, then we know from the definition (2.69) that

$$f_k(x) \leq \epsilon$$

for every $x \in [0, 1]$ and every positive integer $k \geq n_0$. Therefore it implies that

$$0 \leq \int_0^1 f_k(x)\,\mathrm{d}x \leq \epsilon$$

for every positive integer $k \geq n_0$. Since ϵ is arbitrary, we obtain from this that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 0.$$

Without loss of generality, we assume that $E_n \neq \varnothing$ for all $n \in \mathbb{N}$. Let's prove two properties of E_n first:

- **Property 1: E_n is open.** Let $p \in E_n$. Then we have $f_k(p) > \epsilon$ for some $k \geq n$. Define $\epsilon' = \frac{1}{2}(f_k(p) - \epsilon) > 0$. Since f_k is continuous on $[0, 1]$, it is continuous at p. Thus for this particular ϵ', there is a $\delta > 0$ such that

  $$|f_k(x) - f_k(p)| < \epsilon' \tag{2.70}$$

  for all points $x \in [0, 1]$ with $|x - p| < \delta$. Now the inequality (2.70) implies that

  $$f_k(x) > f_k(p) - \epsilon' = \frac{f_k(p) + \epsilon}{2} > \epsilon.$$

  In other words, we have $(p - \delta, p + \delta) \subseteq E_n$ as desired.

- **Property 2:** $\bigcap_{n=1}^{\infty} E_n = \varnothing$. Otherwise, there was a $p \in [0,1]$ such that $p \in E_n$ for all $n \in \mathbb{N}$. Therefore, for each positive integer n, we have
$$f_k(p) > \epsilon \tag{2.71}$$
for some $k \geq n$. If $n \to \infty$, then $k \to \infty$ and the inequality (2.71) implies that
$$0 = \lim_{n\to\infty} f_n(p) = \lim_{k\to\infty} f_k(p) \geq \epsilon > 0,$$
a contradiction.

To finish our proof, we have to study the *lengths* of certain subsets of E_n. Let F be a finite union of bounded (open or closed) intervals of $[0,1]$. Then we may write
$$F = \bigcup_{k=1}^{m} [a_k, b_k] \cup \bigcup_{r=1}^{s} (a_r, b_r) \tag{2.72}$$
where $0 \leq a_k < b_k \leq 1$ and $0 \leq a_r < b_r \leq 1$ for $k = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, s$. Define the length of F, denoted by $\ell(F)$, to be
$$\ell(F) = \sum_{k=1}^{m} \ell\big([a_k, b_k]\big) + \sum_{r=1}^{s} \ell\big((a_r, b_r)\big) = \sum_{k=1}^{m} (b_k - a_k) + \sum_{r=1}^{s} (b_r - a_r).$$
Given a nonempty finite union of bounded interval F expressed in the form (2.72) and assume that F has at least one open interval. In addition, we suppose that $\ell(F) > \epsilon$. Let $\delta = \frac{\epsilon}{4s}$. Then the set
$$G = \bigcup_{k=1}^{m} [a_k, b_k] \cup \bigcup_{r=1}^{s} [a_r + \delta, b_r - \delta]$$
is clearly a nonempty finite union of bounded and *closed* intervals of $[0,1]$. Now it is easy to see that
$$\ell(G) = \sum_{k=1}^{m} (b_k - a_k) + \sum_{r=1}^{s} (b_r - a_r - 2\delta) = \ell(F) - 2s\delta = \ell(F) - \frac{\epsilon}{2}$$
implying that
$$\ell(G) > \ell(F) - \epsilon.$$

Here we need a lemma about the sequence of bounded subsets (2.69):

Lemma 2.4

Let $S_n = \{F \,|\, F$ is a finite union of bounded intervals of $E_n\}$ for every positive integer n and
$$L_n = \sup\{\ell(F) \,|\, F \in S_n\}. \tag{2.73}$$
Then we have
$$\lim_{n\to\infty} L_n = 0.$$

Proof of Lemma 2.4. By **Property 1**, we know that each S_n is nonempty. It is obvious that $\{L_n\}$ is a bounded decreasing sequence. By the Monotone Convergence Theorem [49, Theorem 3.14, p. 55], $\{L_n\}$ converges in $\mathbb{R}$. Assume that this limit was nonzero.

- **Step 1: The construction of a compact set K_n.** There exists a $\delta > 0$ such that $L_n \geq \delta$ for every positive integer n. By the definition (2.73), $L_n - \frac{\delta}{2^n}$ is not an upper bound of $\{\ell(F) \,|\, F \in S_n\}$. In other words, *there exists* a $F_n \in S_n$ such that
$$\ell(F_n) > L_n - \delta \cdot 2^{-n} \tag{2.74}$$
for every positive integer n. By the observation preceding Lemma 2.4, we may assume further that each F_n is closed. Let $K_n = \bigcap_{k=1}^{n} F_k \subseteq F_n$. Since each F_k is closed in $\mathbb{R}$, K_n is closed in $\mathbb{R}$ too. By the Heine-Borel Theorem, the set K_n must be compact. Furthermore, we have $K_n \supseteq K_{n+1}$ for each $n = 1, 2, \ldots$.

- **Step 2: Each K_n is nonempty.** *There exists* a $F \in S_n$ such that
$$\ell(F) \geq \delta \tag{2.75}$$
for each $n \in \mathbb{N}$. Otherwise, there is a $N \in \mathbb{N}$ such that $\ell(F) < \delta$ for all $F \in S_N$ which means that δ is an upper bound of L_N, but this is a contradiction. Now we are going to show that if $K_n = \varnothing$, then it is impossible to have the inequality (2.75). This contradiction bases on the following two facts:

  - **Fact 1:** Suppose that G is a finite union of bounded intervals of $E_n \setminus F_n$ for every $n \in \mathbb{N}$, where F_n are those sets considered in **Step 1**. Then we have $G \cap F_n = \varnothing$ and $G \cup F_n \in S_n$ so that $\ell(G) + \ell(F_n) = \ell(G \cup F_n) \leq L_n$. This and the inequality (2.74) give, for every $n \in \mathbb{N}$,
$$\ell(G) < \delta \cdot 2^{-n}. \tag{2.76}$$

  - **Fact 2:** Suppose that F is a finite union of bounded intervals of $E_n \setminus K_n$ for every $n \in \mathbb{N}$. By De Morgan's law (see [42, p. 11]), we have
$$(F \setminus F_1) \cup \cdots \cup (F \setminus F_n) = F \setminus (F_1 \cap \cdots \cap F_n) = F \setminus K_n = F. \tag{2.77}$$
Note that $F \setminus F_k$ is also a finite union of bounded intervals of E_k (and hence of $E_k \setminus F_k$) for $k = 1, 2, \ldots, n$, so it follows from the inequality (2.76) that
$$\ell(F \setminus F_k) < \delta \cdot 2^{-k} \tag{2.78}$$
for $k = 1, 2, \ldots, n$. Hence it follows from the expression (2.77) and the inequality (2.78) that
$$\ell(F) < \delta. \tag{2.79}$$

  If $K_m = \varnothing$ *for some* $m \in \mathbb{N}$, then the F considered in **Fact 2** is a subset of E_m, but the inequality (2.79) definitely contradicts the inequality (2.75).

- **Step 3: A contradiction to Property 2.** By **Step 1** and **Step 2**, we deduce from [49, Theorem 2.36, p. 38] that $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$. Since $F_k \subseteq E_k$ for $k = 1, 2, \ldots, n$, we must have $K_n \subseteq E_n$. However, these two facts contradict **Property 2**.

Hence the limit must be 0 which completes the proof of the lemma. ∎

We return to the proof of the problem. By Lemma 2.4, for $\epsilon > 0$, there is a positive integer N such that for $n \geq N$, we have

$$\ell(F) \leq L_n < \epsilon, \tag{2.80}$$

where F is *any* finite union of bounded intervals of E_n. Now one may think that we can derive

$$\int_0^1 f_n(x)\,\mathrm{d}x < 2\epsilon$$

for $n \geq N$ directly from the uniform boundedness of f_n, the definition (2.69) and the inequality (2.80). However, it fails because we have no way to estimate the integral

$$\int_{E_n \setminus F} f_n(x)\,\mathrm{d}x.$$

To overcome this problem, we play the trick that since f_n is Riemann integrable on $[0,1]$, the integral must be equal to its *lower Riemann integral*, see [2, §1.17, p. 74]. Thus we have to find

$$\sup\Big\{\int_0^1 s_n(x)\,\mathrm{d}x \,\Big|\, 0 \leq s_n(x) \leq f_n(x) \text{ and } s_n \text{ is a step function}\Big\},$$

where the sup is taken over all step functions s_n below f_n on $[0,1]$.

For every $n \geq N$, we define

$$\mathcal{E}_n = \{x \in [0,1] \,|\, s_n(x) > \epsilon\} \quad \text{and} \quad \mathcal{F}_n = [0,1] \setminus \mathcal{E}_n.$$

Since s_n is a step function, it is clear that $\mathcal{E}_n$ and $\mathcal{F}_n$ are finite unions of bounded intervals. Furthermore, since $0 \leq s_n(x) \leq f_n(x)$, we have $\mathcal{E}_n \subseteq E_n$ and $\mathcal{E}_n \in S_n$. Then we deduce from the inequality (2.80) that

$$\ell(\mathcal{E}_n) < \epsilon$$

for all $n \geq N$. Hence we obtain from this and repeated uses of [49, Theorem 6.12(c), p. 128] that for all $n \geq N$, we have

$$0 \leq \int_0^1 s_n(x)\,\mathrm{d}x = \int_{\mathcal{E}_n} s_n(x)\,\mathrm{d}x + \int_{\mathcal{F}_n} s_n(x)\,\mathrm{d}x \leq \int_{\mathcal{E}_n} \mathrm{d}x + \int_{\mathcal{F}_n} \epsilon\,\mathrm{d}x \leq \ell(\mathcal{E}_n) + \epsilon < 2\epsilon.$$

Since ϵ is arbitrary, we have shown that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 0.$$

This completes the proof of the problem. ■

Remark 2.1

There are many mathematicians who have provided different proofs of Problem 2.10. See, for examples, [37], [38] and [53].

2.4 Problems on Borel Measures and Lebesgue Measures

Problem 2.11

Rudin Chapter 2 Exercise 11.

Proof. We follow the given hint. Suppose that Ω is the family of all compact subsets K_α of X with $\mu(K_\alpha) = 1$. Since X is compact and $\mu(X) = 1$, Ω is not empty. Now we define

$$K = \bigcap_{K_\alpha \in \Omega} K_\alpha. \tag{2.81}$$

Since X is Hausdorff, each K_α is closed in X by the corollaries following Theorem 2.5. Thus K is closed in X and then K is Borel (see Definition 1.11), i.e., $K \in \mathfrak{M}$. Since $K \subseteq X$, Theorem 2.4 says that K is also compact.

Suppose that $K \subseteq V$ and V is open in X. Then V^c is closed in X and so it is compact in X by Theorem 2.4. As we have mentioned in the previous paragraph that each K_α is closed in X, so each K_α^c is open in X. By the definition (2.81) and the fact that $K \subseteq V$, we have

$$V^c \subseteq K^c = \bigcup_{K_\alpha \in \Omega} K_\alpha^c.$$

In other words, $\{K_\alpha^c\}$ forms an open cover of the compact set V^c. Thus we have

$$V^c \subseteq K_{\alpha_1}^c \cup K_{\alpha_2}^c \cup \cdots \cup K_{\alpha_n}^c \tag{2.82}$$

for some positive integer n. Since $\mu(K_{\alpha_m}) = \mu(X) = 1$ for $m = 1, 2, \ldots, n$, we have $\mu(K_{\alpha_m}^c) = 0$ for $m = 1, 2, \ldots, n$. Therefore, it follows from these and the set relation (2.82) that

$$\mu(V^c) \leq \sum_{m=1}^{n} \mu(K_{\alpha_m}^c) = 0,$$

i.e., $\mu(V^c) = 0$ or equivalent $\mu(V) = 1$. Since μ is regular and $K \in \mathfrak{M}$, Definition 2.15 implies that

$$\mu(K) = \inf\{\mu(V) \mid K \subseteq V \text{ and } V \text{ is open}\} = 1$$

which proves the first assertion.

For the second assertion, let H be a proper compact subset of K, i.e., $H \subset K$. Assume that $\mu(H) = 1$. Then it means that $H \in \Omega$ and the definition (2.81) shows that $K \subseteq H$, a contradiction. Hence we must have $\mu(H) < 1$. This completes the proof of the problem. ∎

Problem 2.12

Rudin Chapter 2 Exercise 12.

Proof. Let $\mathscr{B}$ be the σ-algebra of all Borel sets in $\mathbb{R}$ and K be a nonempty compact subset of $\mathbb{R}$. We have to show that there exists a measure μ on $\mathscr{B}$ such that K is the smallest closed subset with the property $\mu(K^c) = 0$.

Since every compact set K has a countable base (see [49, Exercise 25, Chap. 2, p. 45]), there exists a countable sequence $F = \{x_1, x_2, \ldots\} \subseteq K$ such that

$$K = \overline{F}.$$

Next we define $\mu : F \to [0, \infty]$ by

$$\mu(x_n) = \frac{1}{2^n}$$

and for any $E \in \mathscr{B}$, we define $\mu : \mathscr{B} \to [0, \infty]$ by

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\{x_n\}}(E), \tag{2.83}$$

where $\chi_{\{x_n\}}$ is the characteristic function of the set $\{x_n\}$. Suppose that $\{E_i\}$ is a disjoint countable collections of members of $\mathscr{B}$ and

$$E = \bigcup_{i=1}^{\infty} E_i.$$

If $x_n \in E_i$ for some i, then $x_n \notin E_j$ for all $j \neq i$. In other words, each element of F belongs to *at most* one element of $\{E_i\}$.[i] Thus it follows from the definition (2.83) that

$$\mu(E) = \sum_{n} \frac{1}{2^n} \chi_{\{x_n\}}(E), \tag{2.84}$$

where the summation runs through all n such that $x_n \in E_i$ for some i. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges absolutely, its rearrangement converges to the same limit (see [49, Theorem 3.55, p. 78]). Therefore, we can rewrite the expression (2.84) as

$$\mu(E) = \sum_{i=1}^{\infty} \sum_{i_k} \frac{1}{2^{i_k}} \chi_{\{x_{i_k}\}}(E_i), \tag{2.85}$$

where the inner summation runs through all i_k such that $x_{i_k} \in E_i$. It may happen that the set $F \cap E_i$ is finite for some $i = 1, 2, \ldots$. Since the inner summation in the expression (2.85) is exactly $\mu(E_i)$, we deduce from the expression (2.85) that

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \mu(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

By Definitions 1.18(a) and 2.15, μ is a Borel measure on $\mathbb{R}$.

By the definition (2.83), it is easy to see that for every $E \in \mathscr{B}$, we have

$$\mu(E) > 0 \quad \text{if and only if} \quad K \cap E \neq \varnothing \tag{2.86}$$

so that $\mu(K^c) = 0$. Assume that there was a proper closed subset $H \subset K$ in X such that $\mu(H^c) = 0$. Since $H \subset K$, we have $K^c \subset H^c$. If $H^c \cap K = \varnothing$, then $H^c \subseteq K^c$, a contradiction. Thus $H^c \cap K \neq \varnothing$. Since H^c is open in X, we have $H^c \in \mathscr{B}$. Therefore, we can conclude from the condition (2.86) that

$$\mu(H^c) > 0$$

which contradicts to our hypothesis. In other words, K is the *smallest* closed subset in $\mathbb{R}$ such that $\mu(K^c) = 0$, or equivalently,

$$K = \operatorname{supp}(\mu).$$

This completes the analysis of the problem. ■

Problem 2.13

Rudin Chapter 2 Exercise 13.

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[i]However, E_i may contain *more than* one element of F.

Proof. It is clear that the point set $\{0\}$ is a compact subset of $\mathbb{R}$. Assume $f : \mathbb{R} \to \mathbb{R}$ was a continuous function such that $\{0\} = \operatorname{supp}(f)$. Let $V = (-\infty, 0) \cup (0, \infty)$. We know from Definition 2.9 that $\operatorname{supp}(f) = \overline{f^{-1}(V)}$, so

$$\{0\} = \overline{f^{-1}(V)} \tag{2.87}$$

so that $f^{-1}(V) \neq \varnothing$. Since V is open in $\mathbb{R}$ and f is continuous on $\mathbb{R}$, $f^{-1}(V)$ is open in $\mathbb{R}$ by Definition 1.2(c). Thus, if $p \in f^{-1}(V)$, then there exists a $\delta > 0$ such that

$$(p - \delta, p + \delta) \subseteq f^{-1}(V). \tag{2.88}$$

Thus we deduce from the set relations (2.87) and (2.88) that

$$[p - \delta, p + \delta] \subseteq \overline{f^{-1}(V)} = \{0\},$$

a contradiction. Hence there is no continuous function f such that $\{0\} = \operatorname{supp}(f)$.

For the second assertion, we suppose that K is a compact subset of $\mathbb{R}$. We claim that K is the support of a continuous function if and only if K is the closure of an open set V in X. If K is the support of a continuous function $f : \mathbb{R} \to \mathbb{R}$, then Definition 2.9 shows that

$$K = \overline{\{x \in \mathbb{R} \,|\, f(x) \neq 0\}} = \overline{f^{-1}\big((-\infty, 0) \cup (0, \infty)\big)} = \overline{f^{-1}\big((-\infty, 0)\big) \cup f^{-1}\big((0, \infty)\big)}.$$

Since f is continuous on $\mathbb{R}$, $f^{-1}\big((-\infty, 0)\big)$ and $f^{-1}\big((0, \infty)\big)$ are open in $\mathbb{R}$. Thus this proves one direction. Conversely, if we have

$$K = \overline{V} \tag{2.89}$$

for some open set V, then we consider $F = V^c$ which is closed in $\mathbb{R}$. Define $\rho_F : \mathbb{R} \to \mathbb{R}$ by

$$\rho_F(x) = \inf\{|x - y| \,|\, y \in F\}.$$

By Problem 2.3, we know that ρ_F is uniformly continuous on $\mathbb{R}$ and $\rho_F(x) = 0$ if and only if $x \in F$. Therefore, we must have

$$\rho_F(x) > 0 \quad \text{if and only if} \quad x \in V. \tag{2.90}$$

Applying the relation (2.90) to the expression (2.89), we have

$$K = \overline{\{x \in \mathbb{R} \,|\, \rho_F(x) \neq 0\}},$$

i.e., K is the support of the continuous function ρ_F.

We note that the third assertion is *not* valid in other topological spaces. We consider the three-element set $X = \{a, b, c\}$. By [42, Example 1, p. 76], we see that $\{\varnothing, \{b\}, X\}$ is a topology of X. Let $K = \{a, b\}$. Then it is easy to check from the definition that K is compact. Now if $K = \overline{V}$ for some open set V, then K must be closed in X (see [42, p. 95]). However, since $X \setminus K = \{c\}$ which is *not* open in X, K is *not* closed in X. This gives a counter-example to explain that the description "K is the closure of an open set V in X" *cannot* guarantee that K is the support of a continuous function in an arbitrary topological space and hence we complete the proof of the problem. ■

Problem 2.14

Rudin Chapter 2 Exercise 14.

Proof. Let $f : \mathbb{R}^k \to \mathbb{R}$. We follow the proof of Theorem 2.24 (Lusin's Theorem). Suppose that $0 \leq f < 1$. Attach a sequence $\{s_n\}$ of simple measurable functions to f, as in the proof of Theorem 1.17 (The Simple Function Approximation Theorem). Put $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for $n = 2, 3, \ldots$. Then $2^n t_n$ is the characteristic function of a measurable set $T_n \subseteq \mathbb{R}^k$ and

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} t_n(\mathbf{x}) = \sum_{n=1}^{\infty} 2^{-n}\chi_{T_n}(\mathbf{x}) \tag{2.91}$$

on $\mathbb{R}^k$. By Theorem 2.20(b), there exist $A_n, B_n \in \mathscr{B}$ in $\mathbb{R}^k$ such that

$$A_n \subseteq T_n \subseteq B_n \quad \text{and} \quad m(B_n \setminus A_n) = 0 \tag{2.92}$$

for every positive integer n.[j]

We define $g : \mathbb{R}^k \to \mathbb{R}$ and $h : \mathbb{R}^k \to \mathbb{R}$ by

$$g(\mathbf{x}) = \sum_{n=1}^{\infty} 2^{-n}\chi_{A_n}(\mathbf{x}) \quad \text{and} \quad h(\mathbf{x}) = \sum_{n=1}^{\infty} 2^{-n}\chi_{B_n}(\mathbf{x}). \tag{2.93}$$

We claim that g and h are Borel measurable functions on $\mathbb{R}^k$. It is easy to check that

$$\{\mathbf{x} \in \mathbb{R}^k \,|\, \chi_{A_n}(\mathbf{x}) < \alpha\} = \begin{cases} \varnothing, & \text{if } \alpha \leq 0; \\ A_n^c, & \text{if } 0 < \alpha \leq 1; \\ \mathbb{R}, & \text{if } \alpha > 1. \end{cases}$$

Thus each χ_{A_n} is a Borel measurable function by Definition 1.11. Similarly, each χ_{B_n} is also a Borel measurable function. Since the partial sums g_n and h_n of g and h are just simple functions and $A_n, B_n \in \mathscr{B}$, they are Borel measurable functions by the comment following Definition 1.16. Furthermore, for each $n = 1, 2, \ldots$, we have

$$|2^{-n}\chi_{A_n}(\mathbf{x})| \leq 2^{-n} \quad \text{and} \quad |2^{-n}\chi_{B_n}(\mathbf{x})| \leq 2^{-n}$$

on $\mathbb{R}^k$. Since $\displaystyle\sum_{n=1}^{\infty} 2^{-n}$ converges, it follows from Weierstrass M-test (see [49, Theorem 7.10, p. 147]) that $\sum 2^{-n}\chi_{A_n}$ and $\sum 2^{-n}\chi_{B_n}$ converge uniformly on $\mathbb{R}^k$ to g and h respectively. By the Corollary (a) following Theorem 1.14, g and h are Borel measurable. This proves our claim.

By the measure (2.92), we know that $g = h$ a.e. $[m]$ on $\mathbb{R}^k$. Obviously, if $F \subseteq E$, then we must have $\chi_F(\mathbf{x}) \leq \chi_E(\mathbf{x})$. Therefore, we observe from the set relations (2.92) that

$$\chi_{A_n}(\mathbf{x}) \leq \chi_{T_n}(\mathbf{x}) \leq \chi_{B_n}(\mathbf{x}) \tag{2.94}$$

on $\mathbb{R}^k$. Hence we apply the inequalities (2.94) to functions (2.91) and (2.93), we obtain

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) \tag{2.95}$$

on $\mathbb{R}^k$.

Next, if $0 \leq f < M$ for some positive constant M on $\mathbb{R}^k$, then the above argument can be applied to $\frac{f}{M}$ so that the inequalities (2.95) hold with g and h replaced by Mg and Mh respectively.

[j] In fact, each A_n is an F_σ and each B_n is a G_δ, see Definition 1.11.

In the general case, we consider the measurable sets $E_N = \{\mathbf{x} \in \mathbb{R}^k \,|\, -N \leq f(\mathbf{x}) \leq N\}$, where $N \in \mathbb{N}$. Now we have $\chi_{E_N} f \to f$ as $N \to \infty$ on $\mathbb{R}^k$. In fact, the above argument shows that we can find Boreal functions g_N and h_N such that $g_N(\mathbf{x}) = h_N(\mathbf{x})$ a.e. $[m]$ on $\mathbb{R}^k$ and

$$g_N(\mathbf{x}) \leq \chi_{E_N} f(\mathbf{x}) \leq h_N(\mathbf{x})$$

on $\mathbb{R}^k$. By Theorem 1.14, we see that the functions g and h defined by

$$g(\mathbf{x}) = \limsup_{N\to\infty} g_N(\mathbf{x}) \quad \text{and} \quad h(\mathbf{x}) = \limsup_{N\to\infty} h_N(\mathbf{x}) \tag{2.96}$$

are Borel measurable functions on $\mathbb{R}^k$. Since $g_N(\mathbf{x}) = h_N(\mathbf{x})$ a.e. $[m]$ on $\mathbb{R}^k$, we have $g(\mathbf{x}) = h(\mathbf{x})$ a.e. $[m]$ on $\mathbb{R}^k$. Furthermore, we note that

$$g(\mathbf{x}) = \limsup_{N\to\infty} g_N(\mathbf{x}) \leq \limsup_{N\to\infty} \chi_{E_N} f(\mathbf{x}) \leq \limsup_{N\to\infty} h_N(\mathbf{x}) = h(\mathbf{x}) \tag{2.97}$$

on $\mathbb{R}^k$. Since

$$\lim_{N\to\infty} \chi_{E_N} f(\mathbf{x}) = \limsup_{N\to\infty} \chi_{E_N} f(\mathbf{x}) = f(\mathbf{x})$$

on $\mathbb{R}^k$, the inequalities (2.95) follow immediately from the inequalities (2.96) and the fact (2.97). Hence our desired results also hold in this general case, completing the proof of the problem. ■

Problem 2.15

Rudin Chapter 2 Exercise 15.

Proof. For each positive integer n, we define $f_n : [0, \infty) \to [0, \infty]$ by

$$f_n(x) = \begin{cases} \left(1 - \dfrac{x}{n}\right)^n \mathrm{e}^{\frac{x}{2}}, & \text{if } x \in [0, n]; \\ 0, & \text{if } x > n. \end{cases}$$

It is easy to see that each f_n is continuous on $[0, \infty)$ and so Definition 1.11 shows that it is Borel measurable. Fix a $x \in \mathbb{R}$. Then the Archimedean Property ensures the existence of a positive integer N such that $N \geq x$. By this fact and the fact that $(1 + \frac{x}{n})^n \to \mathrm{e}^x$ as $n \to \infty$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n \mathrm{e}^{\frac{x}{2}} = \mathrm{e}^{-x} \cdot \mathrm{e}^{\frac{x}{2}} = \mathrm{e}^{-\frac{x}{2}}.$$

Since $\mathrm{e}^{-\frac{x}{2}} \in L^1(\mathbb{R})$ and

$$|f_n(x)| \leq \mathrm{e}^{-\frac{x}{2}} \tag{2.98}$$

for all $n = 1, 2, \ldots$ and $x \in [0, \infty)$, it follows from Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) that

$$\lim_{n\to\infty} \int_0^\infty f_n \, \mathrm{d}x = \int_0^\infty \mathrm{e}^{-\frac{x}{2}} \, \mathrm{d}x = 2. \tag{2.99}$$

By the inequality (2.98), we get

$$\left| \int_0^\infty f_n \, \mathrm{d}x - \int_0^n f_n \, \mathrm{d}x \right| = \left| \int_n^\infty f_n \, \mathrm{d}x \right| \leq \int_n^\infty |f_n| \, \mathrm{d}x \leq \int_n^\infty \mathrm{e}^{-\frac{x}{2}} \, \mathrm{d}x = 2\mathrm{e}^{-\frac{n}{2}}$$

which implies that

$$\lim_{n\to\infty} \int_0^\infty f_n \, \mathrm{d}x = \lim_{n\to\infty} \int_0^n f_n \, \mathrm{d}x. \tag{2.100}$$

Combining the results (2.99) and (2.100), we obtain

$$\lim_{n\to\infty}\int_0^n f_n\,\mathrm{d}x = 2.$$

For the second integral, we define $g_n : [0,\infty) \to [0,\infty]$ by

$$g_n(x) = \begin{cases} \left(1+\dfrac{x}{n}\right)^n \mathrm{e}^{-2x}, & \text{if } x\in[0,n];\\[2ex] \dfrac{2^x}{\mathrm{e}^{2x}}, & \text{if } x>n.\end{cases}$$

Therefore each g_n is continuous on $[0,\infty)$. Furthermore, by using a similar argument as above, we have for $x\in\mathbb{R}$,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \mathrm{e}^{-2x} = \mathrm{e}^x\cdot\mathrm{e}^{-2x} = \mathrm{e}^{-x}.$$

Since $\mathrm{e}^{-x}\in L^1(\mathbb{R})$ and $|g_n(x)|\leq \mathrm{e}^{-x}$ for all $n=1,2,\ldots$ and $x\in[0,\infty)$, it follows from Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) that

$$\lim_{n\to\infty}\int_0^\infty g_n\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1.$$

Now we apply the same trick as obtaining the result (2.100), we are able to show that

$$\lim_{n\to\infty}\int_0^n g_n\,\mathrm{d}x = 1.$$

We have completed the proof of the problem. ■

Problem 2.16

Rudin Chapter 2 Exercise 16.

Proof. Let $T:\mathbb{R}^k\to\mathbb{R}^k$ be linear, $Y=T(\mathbb{R}^k)$ with $\dim Y = r<k$. Let $\mathcal{B}=\{\mathbf{b}_1,\ldots,\mathbf{b}_r\}$ be a basis of Y. Since $\mathcal{B}$ is a finite set of linearly independent vectors of $\mathbb{R}^k$, it can be enlarged to a basis $\{\mathbf{b}_1,\ldots,\mathbf{b}_r,\mathbf{b}_{r+1},\ldots,\mathbf{b}_k\}$ for $\mathbb{R}^k$.[k] Define the linear transformation $T:\mathbb{R}^k\to\mathbb{R}^k$ by

$$T(\alpha_1\mathbf{b}_1+\cdots+\alpha_k\mathbf{b}_k) = (\alpha_1,\ldots,\alpha_k).$$

Then it is easy to see that T is one-to-one and so $\det T\neq 0$. Furthermore, we have

$$T(Y) = \{(\alpha_1,\ldots,\alpha_r,0,\ldots,0)\,|\,\alpha_1,\ldots,\alpha_r\in\mathbb{R}\}$$

so that

$$T(Y) = \bigcup_{n=1}^\infty W_n, \tag{2.101}$$

where $W_n=\{(\xi_1,\ldots,\xi_r,0,\ldots,0)\,|\,-n\leq\xi_i\leq n \text{ and } i=1,2,\ldots,r\}$. By the definition, we have

$$W_n\subseteq W_{n+1}$$

[k] See, for example, [23, Theorem 30.19, p. 279]. Sometimes, this result is called the **Basis Extension Theorem**.

for every $n \in \mathbb{N}$. It is trivial to see from Theorem 2.20(a) that

$$m(W_n) = \mathrm{vol}\,(W_n) = 0 \tag{2.102}$$

for each $n = 1, 2, \ldots$. In addition, since $\mathbb{R}^k$ is a Borel set, Y is also a Borel set. Hence we follow from this fact, the expressions (2.101), (2.102) and Theorem 1.19(d) that

$$m\big(T(Y)\big) = \lim_{n\to\infty} m(W_n) = \lim_{n\to\infty} \mathrm{vol}\,(W_n) = 0. \tag{2.103}$$

When we read the proof of Theorem 2.20(e) and §2.23 carefully, we see that the validity of the formula

$$m\big(T(E)\big) = |\det T| m(E) \tag{2.104}$$

when T is an one-to-one map of $\mathbb{R}^k$ onto $\mathbb{R}^k$ is *independent* of whether $T(\mathbb{R}^k)$ is a subspace of lower dimension or not. As a consequence, we may apply the formula (2.104) here. Hence we can conclude from $m(Y) = |\det T|^{-1} m\big(T(Y)\big)$ and the result (2.103) that $m(Y) = 0$, completing the proof of the problem. ∎

2.5 Problems on Regularity of Borel Measures

Problem 2.17

Rudin Chapter 2 Exercise 17.

Proof. Let d be the mentioned distance in the problem, $\mathbf{p} = (x_1, y_1)$ and $\mathbf{q} = (x_2, y_2)$. We are going to prove the assertions one by one.

- ***X* is a metric space with metric *d*.** We check the definition of a metric space, see [49, Definition 2.15, p. 30]:
  - If $\mathbf{p} = \mathbf{q}$, then $d(\mathbf{p}, \mathbf{q}) = |y_1 - y_2| = 0$. Otherwise, we have

$$\begin{aligned} d(\mathbf{p}, \mathbf{q}) &= \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2 \text{ and } y_1 \neq y_2; \\ 1 + |y_1 - y_2|, & \text{if } x_1 \neq x_2, \end{cases} \\ &\neq 0. \end{aligned}$$

  - It is clear that $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$ holds because $|y_1 - y_2| = |y_2 - y_1|$.
  - Let $\mathbf{r} = (x_3, y_3)$. Suppose that $\mathbf{r} \neq \mathbf{p}$ and $\mathbf{r} \neq \mathbf{q}$. Otherwise, the triangle inequality is trivial. There are two cases:
    - **Case (i): $x_1 = x_2$.** Then we have $d(\mathbf{p}, \mathbf{q}) = |y_1 - y_2|$ and

$$d(\mathbf{p}, \mathbf{r}) = \begin{cases} |y_1 - y_3|, & \text{if } x_3 = x_1 \text{ and } y_3 \neq y_1; \\ 1 + |y_1 - y_3|, & \text{if } x_3 \neq x_1. \end{cases} \tag{2.105}$$

Similarly, we have

$$d(\mathbf{q}, \mathbf{r}) = \begin{cases} |y_2 - y_3|, & \text{if } x_3 = x_2 \text{ and } y_3 \neq y_2; \\ 1 + |y_2 - y_3|, & \text{if } x_3 \neq x_2. \end{cases} \tag{2.106}$$

Since $|y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2|$, any combination of the distances (2.105) and (2.106) imply that

$$d(\mathbf{p}, \mathbf{q}) \leq d(\mathbf{p}, \mathbf{r}) + d(\mathbf{r}, \mathbf{q}). \tag{2.107}$$

* **Case (ii): $x_1 \neq x_2$.** In this case, we have $d(\mathbf{p},\mathbf{q}) = 1 + |y_1 - y_2|$. If $x_3 = x_1$, then $x_3 \neq x_2$ so that $d(\mathbf{p},\mathbf{r}) = |y_1 - y_3|$ and $d(\mathbf{q},\mathbf{r}) = 1 + |y_2 - y_3|$. The situation for $x_3 = x_1$ is similar. If $x_3 \neq x_1$ and $x_3 \neq x_2$, then we see from the distances (2.105) and (2.106) again that $d(\mathbf{p},\mathbf{r}) = 1 + |y_1 - y_3|$ and $d(\mathbf{q},\mathbf{r}) = 1 + |y_2 - y_3|$. Hence the triangle inequality (2.107) remains true.

By the above analysis, we conclude that d is a metric.

- **X is locally compact.** Let $X = (\mathbb{R}^2, d)$ and $\mathbf{p} = (x, y), \mathbf{q} = (u, v) \in \mathbb{R}^2$. Fix $\mathbf{p}$ first. By the definition, $d(\mathbf{p},\mathbf{q}) < 1$ implies that $x = u$ and $|y - v| < 1$. Thus the neighborhood $B(\mathbf{p}, 1)$ of $\mathbf{p}$ is given by

$$\begin{aligned} B(\mathbf{p}, 1) &= \{\mathbf{q} \in X \,|\, d(\mathbf{p},\mathbf{q}) < 1\} \\ &= \{\mathbf{q} \in X \,|\, x = u \text{ and } |y - v| < 1\} \\ &= \{x\} \times (y - 1, y + 1). \end{aligned} \tag{2.108}$$

Geometrically, $B(\mathbf{p}, 1)$ is the vertical line segment with half length less than 1 and (x, y) as its midpoint. In addition, this line segment is open in $\mathbb{R}$ with the usual metric. Furthermore, it is clear from the definition (2.108) that

$$\overline{B(\mathbf{p}, 1)} = \{x\} \times [y - 1, y + 1]. \tag{2.109}$$

We want to show that $\overline{B(\mathbf{p}, 1)}$ is compact with respect to the metric d. To this end, let $\{V_\alpha\}$ be an open cover of $\overline{B(\mathbf{p}, 1)}$, i.e.,

$$\overline{B(\mathbf{p}, 1)} \subseteq \bigcup_\alpha V_\alpha.$$

Let $\mathbf{a} \in \overline{B(\mathbf{p}, 1)}$, where $\mathbf{a} = (s, t)$ with $s = x$ and $t \in [y - 1, y + 1]$. Then $(x, t) \in V_\alpha$ for some α. Since V_α is open in X, there exists a $\delta_t \in (0, 1)$ such that

$$B(\mathbf{a}, \delta_t) = \{\mathbf{q} = (u, v) \,|\, u = s = x \text{ and } |v - t| < \delta_t\} = \{x\} \times (t - \delta_t, t + \delta_t) \subseteq V_\alpha \tag{2.110}$$

so that

$$\overline{B(\mathbf{p}, 1)} \subseteq \bigcup_{\substack{\mathbf{a} \in \overline{B(\mathbf{p},1)} \\ \mathbf{a}=(s,t)}} B(\mathbf{a}, \delta_t) \subseteq \bigcup_\alpha V_\alpha. \tag{2.111}$$

Applying the expressions (2.109) and (2.110) to the set relation (2.111), we see that

$$\overline{B(\mathbf{p}, 1)} = \{x\} \times [y - 1, y + 1] \subseteq \bigcup_{t \in [y-1,y+1]} \{x\} \times (t - \delta_t, t + \delta_t) \subseteq \bigcup_\alpha V_\alpha \tag{2.112}$$

which means that $\{U_t\}$, where $U_t = (t - \delta_t, t + \delta_t)$ with $t \in [y - 1, y + 1]$, is an open cover of $[y - 1, y + 1]$. Since $[y - 1, y + 1]$ is compact with respect to the *usual metric* $|\cdot|$ by the Heine-Borel Theorem, there are finitely many indices $t_1, \ldots, t_k$ such that

$$[y - 1, y + 1] \subseteq (t_1 - \delta_{t_1}, t_1 + \delta_{t_1}) \cup (t_2 - \delta_{t_2}, t_2 + \delta_{t_2}) \cup \cdots \cup (t_k - \delta_{t_k}, t_k + \delta_{t_k}).$$

Suppose that $(x, t_i) \in V_{\alpha_i}$ for $i = 1, 2, \ldots, k$. Hence we follow immediately from the set relation (2.112) that

$$\overline{B(\mathbf{p}, 1)} \subseteq V_{t_1} \cup V_{t_2} \cup \cdots \cup V_{t_k}.$$

In other words, the closure $\overline{B(\mathbf{p}, 1)}$ is compact with respect to the metric d and since $\mathbf{p}$ is an arbitrary point in X, Definition 2.3(f) shows that X is locally compact.[1]

[1] We remark that we cannot use the Heine-Borel Theorem directly to the closure (2.109) because we are working in the space X with metric d, not $\mathbb{R}^2$ with the usual metric.

- **Construction of a positive linear functional.** Let $f \in C_c(X)$, i.e., $f : (\mathbb{R}^2, d) \to \mathbb{C}$ is a continuous function and $\mathrm{supp}\,(f)$ is compact. First of all, we notice that the set

$$E = \{x \in \mathbb{R} \mid f(x,y) \neq 0 \text{ for some } y\}$$

is finite. To see this, let $K = \mathrm{supp}\,(f)$ and $0 < \delta < 1$. Since

$$K \subseteq \bigcup_{\mathbf{p} \in K} B(\mathbf{p}, \delta),$$

where $B(\mathbf{p}, \delta) = \{x\} \times (y - \delta, y + \delta)$. Now the compactness of K ensures that there exists some positive integer n such that

$$K \subseteq \bigcup_{i=1}^{n} B(\mathbf{p}_i, \delta) = \bigcup_{i=1}^{n} \{\mathbf{q} \in X \mid u = x_i \text{ and } |y_i - v| < \delta\}$$

which forces $E = \{x_1, x_2, \ldots, x_n\}$, as claimed.

Next, we claim that the mapping $\Lambda : C_c(X) \to \mathbb{R}$ defined by

$$\Lambda(f) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\,\mathrm{d}y$$

is a positive linear functional. To this end, let $f, g \in C_c(X)$. Since $C_c(X)$ is a vector space, we have $\alpha f + \beta g \in C_c(X)$ for any $\alpha, \beta \in \mathbb{C}$. Furthermore, let $f(x_i, y) \neq 0$ and $g(x_j, y) \neq 0$ for at least one y, where $1 \leq i \leq n$ and $n + 1 \leq j \leq m$, and $f(z_k, y)g(z_k, y) \neq 0$ for at least one y, where $m + 1 \leq k \leq r$. By the definition, we have

$$f(x_i, y) = g(x_j, y) = 0,$$

where $n + 1 \leq i \leq m$, $1 \leq j \leq n$ and all y, so we get

$$\begin{aligned}
\Lambda(\alpha f + \beta g) &= \sum_{s=1}^{r} \int_{-\infty}^{\infty} [\alpha f(x_s, y) + \beta g(x_s, y)]\,\mathrm{d}y \\
&= \alpha\left[\sum_{s=1}^{n} \int_{-\infty}^{\infty} f(x_s, y)\,\mathrm{d}y + \sum_{s=m+1}^{r} \int_{-\infty}^{\infty} f(z_s, y)\,\mathrm{d}y\right] \\
&\quad + \beta\left[\sum_{s=n+1}^{m} \int_{-\infty}^{\infty} g(x_s, y)\,\mathrm{d}y + \sum_{s=m+1}^{r} \int_{-\infty}^{\infty} g(z_s, y)\,\mathrm{d}y\right] \\
&= \alpha\Lambda(f) + \beta\Lambda(g).
\end{aligned}$$

By Definition 2.1, Λ is a linear functional on $C_c(X)$. If $f \geq 0$, then $\Lambda(f) \geq 0$. Thus Λ is positive. This proves the claim.

- **The measures of the x-axis and its compact subsets.** In order to apply Theorem 2.14 (The Riesz Representation Theorem), we have to show that X is Hausdorff. Given $\mathbf{p}, \mathbf{q} \in X$ and $\mathbf{p} \neq \mathbf{q}$. Let $\mathbf{p} = (x, y)$ and $\mathbf{q} = (u, v)$. There are two cases:

  – **Case (i): $x = u$ but $y \neq v$.** Then we define $\delta = |y - x|$ and it obtains from the definition (2.108) that

$$B\Big(\mathbf{p}, \frac{\delta}{2}\Big) \cap B\Big(\mathbf{q}, \frac{\delta}{2}\Big) = \Big[\{x\} \times \Big(y - \frac{\delta}{2}, y + \frac{\delta}{2}\Big)\Big] \cap \Big[\{x\} \times \Big(v - \frac{\delta}{2}, v + \frac{\delta}{2}\Big)\Big] = \varnothing.$$

– **Case (ii):** $x \neq u$ **but** $y = v$**.** Then the geometric property of the neighborhoods $B(\mathbf{p}, 1)$ and $B(\mathbf{q}, 1)$ ensures that their intersection must be empty.

In conclusion, we have shown that X is Hausdorff.

By Theorem 2.14 (The Riesz Representation Theorem), there exists a unique positive measure μ on $\mathfrak{M}$ associated with Λ. Let

$$E = \{(x, 0) \,|\, x \in \mathbb{R}\}.$$

Let $\delta \in (0, 1)$ and

$$U_\delta = \bigcup_{\mathbf{p} \in E} B(\mathbf{p}, \delta) = \bigcup_{x \in \mathbb{R}} \{x\} \times (-\delta, \delta).$$

Then U_δ is clearly an open set in X and $E \subseteq U_\delta$. Take $(x_1, 0), \ldots, (x_n, 0) \in E$ and construct

$$\begin{aligned} K &= \overline{B\big((x_1, 0), \delta\big)} \cup \overline{B\big((x_2, 0), \delta\big)} \cup \cdots \cup \overline{B\big((x_n, 0), \delta\big)} \\ &= \big(\{x_1\} \times [-\delta, \delta]\big) \cup \big(\{x_2\} \times [-\delta, \delta]\big) \cup \cdots \cup \big(\{x_n\} \times [-\delta, \delta]\big). \end{aligned}$$

Since each $\overline{B\big((x_i, 0), \delta\big)}$ is compact, K is also compact and $K \subseteq U_\delta$. By Theorem 2.12 (Urysohn's Lemma), there exists an $f \in C_c(X)$ such that

$$K \prec f \prec U_\delta.$$

Since $\mu(U_\delta) = \sup\{\Lambda(f) \,|\, f \prec U_\delta\}$ (see [51, Eqn. (1), p. 41]), we have

$$\mu(U_\delta) \geq \Lambda(f) = \sum_{j=1}^{m} \int_{-\infty}^{\infty} f(x_j', y) \, \mathrm{d}y, \tag{2.113}$$

where $x_1', \ldots, x_m'$ are those values of x for which $f(x, y) \neq 0$ for at least one y and for some positive integer m. Since $K \prec f$, we have $f(x_i, y) = 1$ for all $i = 1, 2, \ldots, n$ and $y \in (-\delta, \delta)$. Thus we have $\{x_1, x_2, \ldots, x_n\} \subseteq \{x_1', x_2', \ldots, x_m'\}$ and then the inequality (2.113) reduces to

$$\mu(U_\delta) \geq \sum_{i=1}^{n} \int_{-\delta}^{\delta} \mathrm{d}y = n\delta$$

so that $\mu(U_\delta) \to \infty$ as $n \to \infty$. If V is an open set containing E, then V must contain the open set U_δ *for some* small δ and thus $\mu(V) = \infty$ for every open set V containing E. By Theorem 2.14(c), we have

$$\mu(E) = \inf\{\mu(V) \,|\, E \subseteq V,\, V \text{ is open}\} = \infty.$$

Next, let $K \subseteq E$ be compact. Choose $\delta \in (0, 1)$. Since

$$K \subseteq W = \bigcup_{(a,0) \in K} B\big((a, 0), \delta\big), \tag{2.114}$$

the compactness of K implies that

$$K \subseteq \bigcup_{i=1}^{m} B((a_i, 0), \delta) = \bigcup_{i=1}^{m} (\{a_i\} \times (-\delta, \delta))$$

for some positive integer m. In other words, we know from the definition (2.114) that

$$K = \{(a_1, 0), (a_2, 0), \ldots, (a_m, 0)\}.$$

Besides, since W is open in X, Theorem 2.12 (Urysohn's Lemma) again ensures there is a $g \in C_c(X)$ such that

$$K \prec g \prec W.$$

Recall from [51, Eqn. (7), p. 43] that

$$\mu(K) = \inf\{\Lambda(f) \,|\, K \prec f\} \leq \Lambda(g) = \sum_{i=1}^{m} \int_{-\infty}^{\infty} g(a_i, y)\,\mathrm{d}y = 2m\delta. \tag{2.115}$$

Since m is fixed and δ is *arbitrary*, it follows from the inequality (2.115) that $\mu(K) = 0$.

This completes the proof of the problem. ■

Problem 2.18

Rudin Chapter 2 Exercise 18.

Proof. Recall that X has an order relation '$<$' and every nonempty subset A of X has a first/smallest element a, i.e., $a \leq x$ for all $x \in A$.[m] If $\omega_0 = \min X$ (ω_0 is a smallest element in X), then we have

$$X = [\omega_0, \omega_1].$$

Next, we characteristic the topology on X as follows:

The order topology of X: For $\alpha \in X$, we define

$$P_\alpha = \{x \in X \,|\, x < \alpha\} = [\omega_0, \alpha) \quad \text{and} \quad S_\alpha = \{x \in X \,|\, \alpha < x\} = (\alpha, \omega_1].$$

A subset E of X is open in X if it is in one of the following classes

$$[\omega_0, \alpha), \quad (\beta, \omega_1], \quad [\omega_0, \alpha) \cap (\beta, \omega_1] = (\beta, \alpha) \quad \text{or} \quad \text{a union of such sets.} \tag{2.116}$$

See [42, §14, pp. 84 – 86].

Now we prove the assertions one by one.

(a) **X is a compact Hausdorff space.** Suppose that $\mathcal{V}$ is an open cover of $[\omega_0, \omega_1]$. Let $\alpha_0 = \omega_1$. Then there is an open set $V_0 \in \mathcal{V}$ such that $\alpha_0 \in V_0$. If $V_0 = X$, then X is compact. Therefore, in the following discussion, we may assume that $V_0 \neq X$.

Since V_0 is open in X, it follows from the uncountability of X and the order topology (2.116) that there exists an $\alpha_1 \in X$ such that $(\alpha_1, \alpha_0] \subseteq V_0$. We note that $\alpha_1 < \alpha_0$. Now this α_1 can be chosen such that $\alpha_1 \notin V_0$. Otherwise, $\alpha_1 \in V_0$ and we may choose another $\alpha_2 \in X$ such that $(\alpha_2, \alpha_1] \subseteq V_0$. Here we have $\alpha_2 < \alpha_1$. Similarly, this α_2 can be chosen such that $\alpha_2 \notin V_0$. If not, then we continue this process in the same manner, but this must stop in a *finite* number of steps. Otherwise, we can find an infinite decreasing sequence

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots. \tag{2.117}$$

in X, which is a contradiction to the hint. Therefore, we may assume that

$$\alpha_1 \notin V_0 \quad \text{and} \quad (\alpha_1, \alpha_0] \subseteq V_0. \tag{2.118}$$

[m]See [42, p. 63].

Since $\mathcal{V}$ is a cover of $[\omega_0, \omega_1]$, there exists a $V_1 \in \mathcal{V}$ containing α_1. By the above argument, we may find an $\alpha_2 \in X$ such that

$$\alpha_2 \notin V_1 \quad \text{and} \quad (\alpha_2, \alpha_1] \subseteq V_1. \tag{2.119}$$

If $\alpha_2 \in V_0$, then since $[\alpha_2, \alpha_0] \subseteq V_0$ and $\alpha_2 < \alpha_1 < \alpha_0$, we have $\alpha_1 \in V_0$ which contradicts the result (2.118). Thus $\alpha_2 \notin V_0$ and then we deduce from this fact, the results (2.118) and (2.119) that

$$\alpha_2 \notin V_0 \cup V_1 \quad \text{and} \quad (\alpha_2, \alpha_0] \subseteq V_0 \cup V_1.$$

Inductively, we can find $\alpha_{n+1} \in X$ such that

$$\alpha_{n+1} \notin V_0 \cup V_1 \cup \cdots \cup V_n \quad \text{and} \quad (\alpha_{n+1}, \alpha_0] \subseteq V_0 \cup V_1 \cup \cdots \cup V_n.$$

If this process continues infinitely, then we can find an infinite decreasing sequence (2.117) again, a contradiction. Therefore, the sequence *must* stop eventually at the Nth step and it must stop at $\alpha_{n+1} = \omega_0$ for all $n \geq N$, i.e.,

$$\omega_0 \notin V_0 \cup V_1 \cup \cdots \cup V_N \quad \text{and} \quad (\omega_0, \alpha_0] \subseteq V_0 \cup V_1 \cup \cdots \cup V_N.$$

Since $\omega_0 \in V$ for some $V \in \mathcal{V}$, we conclude that

$$[\omega_0, \omega_1] = [\omega_0, \alpha_0] \subseteq V_0 \cup V_1 \cup \cdots \cup V_N \cup V.$$

In other words, X is compact.

Next, we prove that X is Hausdorff and we need the help of the following lemma:

Lemma 2.5

Define the mapping $\gamma : [\omega_0, \omega_1] \to [\omega_0, \omega_1]$ by

$$\gamma(\alpha) = \begin{cases} \min(S_\alpha), & \text{if } \alpha \in [\omega_0, \omega_1); \\ \omega_1, & \text{if } \alpha = \omega_1. \end{cases}$$

Then we have $X \setminus S_\alpha = P_{\gamma(\alpha)} = [\omega_0, \alpha]$.

Proof of Lemma 2.5. For every $\alpha \in [\omega_0, \omega_1)$, $S_\alpha \neq \varnothing$ so that the mapping γ is well-defined. By the definition, we have

$$S_\alpha = [\gamma(\alpha), \omega_1]$$

with $\alpha < \gamma(\alpha)$. Furthermore, there is *no* element between α and $\gamma(\alpha)$. Otherwise, $\gamma(\alpha)$ is not a smallest element of S_α anymore. Hence we deduce from this that

$$X \setminus S_\alpha = [\omega_0, \alpha] = [\omega_0, \gamma(\alpha)) = P_{\gamma(\alpha)}.$$

If $\alpha = \omega_1$, then $S_{\omega_1} = \varnothing$ and $P_{\gamma(\omega_1)} = [\omega_0, \omega_1] = X$ so that $X \setminus S_{\omega_1} = P_{\gamma(\omega_1)}$ in this case, completing the proof of the lemma. ■

Let $\alpha, \beta \in X$ and we may assume that $\alpha < \beta$. Then we see that $\alpha \in P_{\gamma(\alpha)} = [\omega_0, \alpha]$ and $\beta \in S_\alpha$. Thus $P_{\gamma(\alpha)}$ and S_α are neighborhoods of α and β respectively. By Lemma 2.5, they are disjoint and this shows that X is Hausdorff, as required.

(b) $X \setminus \{\omega_1\}$ **is open but not σ-compact.** Let $X \setminus \{\omega_1\} = [\omega_0, \omega_1)$ and $\alpha \in [\omega_0, \omega_1)$. As we have proven in part (a) that $P_{\gamma(\alpha)} = [\omega_0, \gamma(\alpha))$ is a neighborhood of α. Hence $X \setminus \{\omega_1\}$ is open in X.

Assume that $X \setminus \{\omega_1\} = [\omega_0, \omega_1)$ was σ-compact. Let K be a compact subset of $[\omega_0, \omega_1)$. We know that the family

$$\{P_x = [\omega_0, x) \,|\, x < \omega_1\}$$

is an open cover of K so that there are finitely many indices $x_1, x_2, \ldots, x_k$ such that

$$K \subseteq P_{x_1} \cup P_{x_1} \cup \cdots \cup P_{x_k}.$$

Without loss of generality, we may assume that $x_1 < x_2 < \cdots < x_k$. Thus we have

$$K \subseteq [\omega_0, x_k) \subset [\omega_0, x_k].$$

Since $x_k < \omega_1$, our hypothesis shows that K is at most countable. Since $X \setminus \{\omega_1\}$ is σ-compact, it is a countable union of compact sets so that $X \setminus \{\omega_1\}$ is countable. However, we have

$$X = \big(X \setminus \{\omega_1\}\big) \cup \{\omega_1\}$$

which means that X is countable. Evidently, this result contradicts our hypothesis. Hence $X \setminus \omega_1$ is not σ-compact.

(c) **Every $f \in C(X)$ is constant on S_α for some $\alpha \neq \omega_1$.** Suppose that $x = f(\omega_1)$ and $B(x, \frac{1}{n}) = \{z \in \mathbb{C} \,|\, |z - x| < \frac{1}{n}\}$, where n is a positive integer. Since f is continuous on X and $B(x, \frac{1}{n})$ is open in $\mathbb{C}$, $f^{-1}\big(B(x, \frac{1}{n})\big)$ is an open subset of X and $\omega_1 \in f^{-1}\big(B(x, \frac{1}{n})\big)$. By the topology (2.116), *there exists* an $\alpha_1 \in X \setminus \{\omega_1\}$ such that

$$(\alpha_1, \omega_1] \subseteq f^{-1}\Big(B\Big(x, \frac{1}{n}\Big)\Big). \tag{2.120}$$

By Lemma 2.5, we note that $(\alpha_1, \omega_1]$ is uncountable, so a sequence $\{\alpha_n\}$ exists such that

$$\alpha_1 < \alpha_2 < \cdots < \omega_1. \tag{2.121}$$

Now the results (2.120) and (2.121) together give

$$S_{\alpha_n} \subseteq S_{\alpha_{n+1}} \subseteq f^{-1}\Big(B\Big(x, \frac{1}{n}\Big)\Big)$$

for all $n \in \mathbb{N}$. Before we proceed further, we need the following result:[n]

Lemma 2.6

Every well-ordered set A has the least upper bound property.

Proof of Lemma 2.6. Let B be a nonempty subset of A having an upper bound in A. Therefore, the set U of upper bounds of B is nonempty. Since $U \subseteq A$ and A is well-ordered, U has a least element, completing the proof of Lemma 2.6. ∎

By the relation (2.121), we see that the nonempty subset $\{\alpha_n\}$ of X is bounded above by ω_1. Therefore, it follows from Lemma 2.6 that $\sup\limits_{n \in \mathbb{N}}\{\alpha_n\}$ exists in X. Call it α. It is clear that $\alpha \leq \omega_1$ and

$$S_\alpha \subseteq S_{\alpha_n} \subseteq f^{-1}\Big(B\Big(x, \frac{1}{n}\Big)\Big) \tag{2.122}$$

[n]See [42, Exercise 1, p. 66]

for all $n = 1, 2, \ldots$. In fact, it is impossible to have $\alpha = \omega_1$ because the set

$$S_{\omega_1} = \{x \in Y \,|\, \omega_1 < x\}$$

is *uncountable*, where Y is the well-ordered set as described in the "**Construction**" in the question. As $\omega_1 = \alpha$, we have $S_{\omega_1} = S_\alpha \subseteq S_{\alpha_n}$ so that S_{α_n} is uncountable too, but it contradicts to the fact that α_n is a predecessor of ω_1. Hence we obtain $\alpha < \omega_1$ and we deduce from the set relations (2.122) that $f(S_\alpha) \subseteq B(x, \frac{1}{n})$ for all $n \in \mathbb{N}$ and this means that

$$f(S_\alpha) \subseteq \bigcap_{n=1}^{\infty} B\Big(x, \frac{1}{n}\Big) = \{x\}.$$

Now this is exactly our desired result.

(d) **The intersection of $\{K_n\}$ of uncountable compact subsets of X is also uncountable compact.** We first prove a lemma which indicates a relationship between the cardinality of a compact set K and the topological property of ω_1 in K.

Lemma 2.7

Suppose that K is a nonempty compact subset of X. Then K is uncountable if and only if ω_1 is a *limit point* of K.

Proof of Lemma 2.7. Let K be uncountable. By the proof of part (b), we know that K cannot lie inside $[\omega_0, \alpha]$ for every predecessor $\alpha < \omega_1$. Otherwise, K will be at most countable, a contradiction. Hence this implies that

$$K \cap (\alpha, \omega_1] \neq \varnothing$$

for every $\alpha < \omega_1$. By the definition (see [42, p. 97]), it follows that ω_1 is a limit point of K.

Conversely, suppose that ω_1 is a limit point of K. By [42, Theorem 17.9, p. 99], K must be infinite. Assume that K was countable. Then K has the following representation

$$K = \{\alpha_n\},$$

where $\alpha_1 \leq \alpha_2 \leq \cdots$. Since K is compact and X is Hausdorff, K is closed in X by Corollary (a) following Theorem 2.5. Thus it must be $\omega_1 \in K$, i.e., every neighborhood S_α of ω_1 satisfies

$$K \cap \big(S_\alpha \setminus \{\omega_1\}\big) \neq \varnothing.$$

Let $\alpha_N \in K \cap \big(S_\alpha \setminus \{\omega_1\}\big)$ for some positive integer N, i.e., $\alpha < \alpha_N < \omega_1$. Recall that $\{\alpha_n\}$ is increasing, so we must have

$$\alpha_n \in S_\alpha$$

for all $n \geq N$. By the definition (see [42, p. 98]), $\{\alpha_n\}$ converges to ω_1. However, by an argument similar to the proof of part (c), this fact will imply the contradiction that S_{α_n} is uncountable. Hence K is uncountable and we complete the proof of the lemma. ■

Let's go back to the proof of part (d). Denote the intersection of $\{K_n\}$ by K. Since X is Hausdorff, each K_n is closed in X. Since $K \subseteq K_n$, Theorem 2.4 implies that K is compact. For each $n \in \mathbb{N}$, let $K_n' = K_1 \cap K_2 \cap \cdots \cap K_n$. It is easy to check that

$$\bigcap_{i=1}^{n} K_i' = \bigcap_{i=1}^{n}(K_1 \cap \cdots \cap K_i) = \bigcap_{i=1}^{n} K_i,$$

so we may assume that

$$K_1 \supseteq K_2 \supseteq \cdots . \tag{2.123}$$

Therefore, any finite subcollection of $\{K_n\}$ is nonempty and Theorem 2.6 shows that $K \neq \varnothing$. By Lemma 2.7, ω_1 is a limit point of every K_n. Then, using [42, Theorem 17.9, p. 99] again, it means that every neighborhood $(\alpha, \omega_1]$ of ω_1 contains infinitely many points of every K_n. Combining this fact and the sequence (2.123), every neighborhood $(\alpha, \omega_1]$ of ω_1 must contain infinitely many points of the compact set K. Hence ω_1 is a limit point of K and we finally conclude from Lemma 2.7 that K is uncountable.

(e) **$\mathfrak{M}$ is a σ-algebra containing all Borel sets in X.** Suppose that

$$\mathfrak{M} = \{E \subseteq X \mid \text{either } E \cup \{\omega_1\} \text{ or } E^c \cup \{\omega_1\} \text{ contains an uncountable compact set}\}.$$

- **$\mathfrak{M}$ is a σ-algebra.** We check Definition 1.3(a). In fact, it is obvious that $X \in \mathfrak{M}$ because X is itself compact uncountable. Let $E \in \mathfrak{M}$. Then either $E \cup \{\omega_1\}$ or $E^c \cup \{\omega_1\}$ contains an uncountable compact set. Since $(E^c)^c = E$, it must be true that $E^c \in \mathfrak{M}$.

  To verify Definition 1.3(a)(iii), let $E_n \in \mathfrak{M}$ and $E = \displaystyle\bigcup_{n=1}^{\infty} E_n$. If *there exists* an $n_0 \in \mathbb{N}$ such that $E_{n_0} \cup \{\omega_1\}$ contains an uncountable compact set K, then

  $$K \subseteq E_{n_0} \cup \{\omega_1\} \subseteq \Big(\bigcup_{n=1}^{\infty} E_n\Big) \cup \{\omega_1\} = E \cup \{\omega_1\}$$

  so that $E \in \mathfrak{M}$. Otherwise, *all* $E_n \cup \{\omega_1\}$ contain no uncountable compact sets. This forces that every $E_n^c \cup \{\omega_1\}$ contains an uncountable compact set K_n. We note that

  $$E^c \cup \{\omega_1\} = \Big(\bigcap_{n=1}^{\infty} E_n^c\Big) \cup \{\omega_1\} = \bigcap_{n=1}^{\infty} \big(E_n^c \cup \{\omega_1\}\big),$$

  so it is true that

  $$K = \bigcap_{n=1}^{\infty} K_n \subseteq \bigcap_{n=1}^{\infty} \big(E_n^c \cup \{\omega_1\}\big) = E^c \cup \{\omega_1\}.$$

  Now the result of part (d) illustrates that K is uncountable compact, thus $E^c \in \mathfrak{M}$ which implies that $E \in \mathfrak{M}$.

- **$\mathfrak{M}$ contains all Borel sets in X.** If $\alpha < \omega_1$, then $(\alpha, \omega_1) \neq \varnothing$. Let $\beta \in (\alpha, \omega_1)$. Since $X = P_\beta \cup [\beta, \omega_1]$ and P_β is countable, $[\beta, \omega_1]$ must be uncountable. Since $[\beta, \omega_1] = X \setminus P_\beta$, $[\beta, \omega_1]$ is closed in X and Theorem 2.4 verifies that $[\beta, \omega_1]$ is compact. Furthermore, we note that

  $$[\beta, \omega_1] \subseteq S_\alpha, \tag{2.124}$$

  so these facts show that $S_\alpha \in \mathfrak{M}$.

  Similarly, for every $\alpha > \omega_0$, we have $X \setminus P_\alpha = [\alpha, \omega_1]$ and then $X \setminus P_\alpha \in \mathfrak{M}$. Since $\mathfrak{M}$ is a σ-algebra, $\mathfrak{M}$ also contains P_α. Hence $\mathfrak{M}$ contains the order topology τ of X which proves that $\mathscr{B}(X) \subseteq \mathfrak{M}$ as desired.

(f) **λ is a measure on $\mathfrak{M}$ but not regular.** Suppose that $\lambda : \mathfrak{M} \to [0, \infty]$ is defined by

$$\lambda(E) = \begin{cases} 1, & \text{if } E \cup \{\omega_1\} \text{ contains an uncountable compact set;} \\ \\ 0, & \text{if } E^c \cup \{\omega_1\} \text{ contains an uncountable compact set.} \end{cases} \tag{2.125}$$

If $E \in \mathfrak{M}$, then we know from the definition of $\mathfrak{M}$ in part (e) that it is impossible for both $E \cup \{\omega_1\}$ and $E^c \cup \{\omega_1\}$ containing uncountable compact sets, so this function is well-defined.

– **λ is a measure on $\mathfrak{M}$.** Suppose that $\{E_i\}$ is a disjoint countable collection of members of $\mathfrak{M}$. We claim that *at most* one $E_i \cup \{\omega_1\}$ contains an uncountable compact set. To see this, assume that both $E_1 \cup \{\omega_1\}$ and $E_2 \cup \{\omega_1\}$ contained uncountable compact sets K_1 and K_2 respectively. Since $E_1 \cap E_2 = \varnothing$, we have

$$\begin{aligned} K_1 \cap K_2 &\subseteq \big(E_1 \cup \{\omega_1\}\big) \cap \big(E_2 \cup \{\omega_1\}\big) \\ &= \big(E_1 \cap \{\omega_1\}\big) \cup \big(E_2 \cap \{\omega_1\}\big) \cup \{\omega_1\} \\ &= \{\omega_1\}. \end{aligned} \tag{2.126}$$

However, part (d) tells us that $K_1 \cap K_2$ is uncountable so that the result (2.126) is impossible. This proves the claim.

By the previous analysis, if $E_1 \cup \{\omega_1\}$ is the *only* set containing an uncountable compact set K, then we get

$$K \subseteq \bigcup_{i=1}^{\infty} (E_i \cup \{\omega_1\}) = \Big(\bigcup_{i=1}^{\infty} E_i\Big) \cup \{\omega_1\},$$

so we follow from the definition (2.125) that

$$\lambda\Big(\bigcup_{i=1}^{\infty} E_i\Big) = 1.$$

On the other hand, we know from the facts $\lambda(E_1) = 1$ and $\lambda(E_i) = 0$ for $i = 2, 3, \ldots$ that

$$\sum_{i=1}^{\infty} \lambda(E_i) = 1.$$

Therefore, we have

$$\lambda\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \lambda(E_i) \tag{2.127}$$

in this case. Similarly, if there is *no* $E_i \cup \{\omega_1\}$ contains an uncountable compact set, then each $E_i^c \cup \{\omega_1\}$ contains an uncountable compact set K_i so that

$$\lambda(E_i) = 0$$

for every $i = 1, 2, \ldots$. Furthermore, we have

$$K = \bigcap_{i=1}^{\infty} K_i \subseteq \bigcap_{i=1}^{\infty} (E_i^c \cup \{\omega_1\}) = \Big(\bigcap_{i=1}^{\infty} E_i^c\Big) \cup \{\omega_1\} = \Big(\bigcup_{i=1}^{\infty} E_i\Big)^c \cup \{\omega_1\}$$

and part (d) ensures that K is uncountable compact. Therefore, we obtain

$$\lambda\Big(\bigcup_{i=1}^{\infty} E_i\Big) = 0$$

and then the formula (2.127) still holds in this case. Finally, since $\lambda(X) = 1 < \infty$. By Definition 1.18(a), λ is a positive measure.

- **λ is not regular.** For every $\alpha < \omega_1$, we know from the set relation (2.124) that $S_\alpha \cup \{\omega_1\}$ must contain an uncountable compact set, so the definition (2.125) gives that $\lambda(S_\alpha) = 1$. If $E = \{\omega_1\}$, then $E^c \cup \{\omega_1\} = \{\omega_1\}^c \cup \{\omega_1\} = X$ so that $\{\omega_1\} \in \mathfrak{M}$. Therefore, we derive from the definition (2.125) that

$$\lambda(\{\omega_1\}) = 0.$$

However, these facts give

$$\lambda(\{\omega_1\}) \neq \inf\{\lambda(S_\alpha) \,|\, \{\omega_1\} \subseteq S_\alpha\}.$$

By Definition 2.15, λ is *not* regular.

(g) **The validity of the integral.** Let $f \in C_c(X)$. By part (c), there exists an $\alpha_0 < \omega_1$ such that $f(x) = f(\omega_1)$ on S_{α_0}. Recall from part (e) that $S_{\alpha_0} \in \mathfrak{M}$. By the set relation (2.124), every $S_\alpha \cup \{\omega_1\}$ contains an uncountable compact set and thus, in particular, $\lambda(S_{\alpha_0}^c) = 0$. By this, we gain

$$\int_X f\,\mathrm{d}\lambda = \int_{S_{\alpha_0}^c} f\,\mathrm{d}\lambda + \int_{S_{\alpha_0}} f\,\mathrm{d}\lambda = \int_{S_{\alpha_0}} f\,\mathrm{d}\lambda = f(\omega_1)\lambda(S_{\alpha_0}) = f(\omega_1)$$

which is the required result.

(h) **The regular μ associates with the linear functional in part (f).** It is clear that $\Lambda : C_c(X) \to \mathbb{C}$ defined by

$$\Lambda(f) = f(\omega_1) \tag{2.128}$$

is a linear functional on $C_c(X)$. By Theorem 2.14 (The Riesz Representation Theorem) and Theorem 2.17(b), we have a unique regular positive Borel measure μ on X. By the result (2.128), we know that

$$\int_X f\,\mathrm{d}\mu = f(\omega_1).$$

By [51, Eqn. (1), p. 41], we have $\mu(V) = \sup\{\Lambda(f) \,|\, f \prec V\} = f(\omega_1)$ for every open set V in X. Thus this implies that

$$\mu(E) = \inf\{\mu(V) \,|\, E \subseteq V \text{ and } V \text{ is open}\} = f(\omega_1)$$

for every $E \in \mathfrak{M}$.

We have completed the proof of the problem. ■

Remark 2.2

The measure considered in Problem 2.18 is called the **Dieudonné's measure**, see [10, Exampl 7.1.3, pp. 68, 69] and [17].

Problem 2.19

Rudin Chapter 2 Exercise 19.

Proof. Suppose that X is a compact metric space with metric ρ and Λ is a positive linear functional on $C_c(X)$, the space of all continuous complex functions on X with compact support. To begin with the construction of the class μ, we use Λ to define a set function μ^* on every open set V in X by

$$\begin{aligned}\mu^*(V) &= \sup\{\Lambda(f) \,|\, f \prec V\} \\ &= \sup\{\Lambda(f) \,|\, f \in C_c(X),\, 0 \leq f \leq 1 \text{ on } X \text{ and } \operatorname{supp}(f) \subseteq V\}\end{aligned} \tag{2.129}$$

and for every *subset* $E \subseteq X$ that

$$\mu^*(E) = \inf\{\mu^*(V) \,|\, V \text{ is open in } X \text{ and } E \subseteq V\}. \tag{2.130}$$

Now we are going to present the proof by quoting several facts (some are with proofs and some are not). In fact, the idea of the following proof is stimulated by Feldman's online article [20].

- **Fact 1: The set function μ^* is an outer measure.** We check the definition [47, p. 346].
  - Since $\operatorname{supp}(f) \subseteq \varnothing$ if and only if $\operatorname{supp}(f) = \varnothing$, we get from Definition 2.9 that $f \equiv 0$ on X which implies that

    $$\mu^*(\varnothing) = \sup\{\Lambda f \,|\, f \in C_c(X),\, 0 \leq f \leq 1 \text{ on } X \text{ and } \operatorname{supp}(f) \subseteq \varnothing\} = \Lambda(0) = 0.$$

  - Suppose that $E, F \subseteq X$ and $E \subseteq F$. Since any open set V containing F must also contain E, we obtain from the definition (2.130) that

    $$\begin{aligned}\mu^*(E) &= \inf\{\mu^*(W) \,|\, W \text{ is open in } X \text{ and } E \subseteq W\} \\ &\leq \inf\{\mu^*(V) \,|\, V \text{ is open in } X \text{ and } F \subseteq V\} \\ &= \mu^*(F).\end{aligned}$$

  - The proof of the subadditivity

    $$\mu^*\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} \mu^*(E_i) \tag{2.131}$$

    follows exactly the same as the proof of **Step I**.

  Hence μ^* is an outer measure on X.

- **Fact 2: $\mu^*(K) < \infty$ for every compact set $K \subseteq X$.** Since X is open in X, we follow from the definition (2.129) that $\mu^*(X) \leq \Lambda(1)$. By **Fact 1**, we have $\mu^*(K) \leq \mu^*(X) \leq \Lambda(1)$ for every compact set $K \subseteq X$.

- **Fact 3: A topological result in metric spaces.** Now we need the following topological result about metric spaces which will be used in **Fact 4**:

  Lemma 2.8

  Let X be a metric space with metric ρ and E a nonempty proper subset of X. Then the set $V = \{x \in X \,|\, \rho_E(x) < \epsilon\}$ is open in X, where $\epsilon > 0$ and $\rho_E(x)$ is defined in Problem 2.3.

Proof of Lemma 2.8. If $V = \varnothing$, then there is nothing to prove. Thus we assume that $V \neq \varnothing$ and pick $x_0 \in V$ so that $\rho_E(x_0) < \epsilon$. Then there exists a $\delta > 0$ such that $\epsilon - \delta > 0$ and $\rho_E(x_0) < \epsilon - \delta$. We consider the ball

$$B(x_0, \delta) = \{x \in X \,|\, \rho(x, x_0) < \delta\}$$

and we claim that $B(x_0, \delta) \subseteq V$. To see this, let $x \in B(x_0, \delta)$. For every $y \in E$, we have

$$\rho(x, y) \leq \rho(x, x_0) + \rho(x_0, y) < \delta + \rho(x_0, y)$$

and therefore we obtain

$$\begin{aligned}
\rho_E(x) &= \inf\{\rho(x, y) \,|\, y \in E\} \\
&\leq \inf\{\rho(x, x_0) + \rho(x_0, y) \,|\, y \in E\} \\
&< \delta + \inf\{\rho(x_0, y) \,|\, y \in E\} \\
&= \delta + \rho_E(x_0) \\
&< \epsilon.
\end{aligned}$$

In other words, $x \in V$. Since x is arbitrary, $B(x_0, \delta) \subseteq V$ and then V is open in X, proving Lemma 2.8. ∎

- **Fact 4: If V is open in X, then V is measurable with respect to μ^*.** We recall from [47, p. 347] that $V \subseteq X$ is measurable with respect to the outer measure μ^* if for every subset E of X, we have

$$\mu^*(E) = \mu^*(E \cap V) + \mu^*(E \cap V^c). \tag{2.132}$$

It suffices to show the inequality

$$\mu^*(E) \geq \mu^*(E \cap V) + \mu^*(E \cap V^c) - \epsilon \tag{2.133}$$

holds for every $\epsilon > 0$ because the other side follows directly from the subadditivity (2.131). By the definition (2.130), there is an open set E' such that $E \subseteq E'$ and

$$\mu^*(E) \geq \mu^*(E') - \frac{\epsilon}{2}. \tag{2.134}$$

By the property in **Fact 1**, we have

$$\mu^*(E \cap V) \leq \mu^*(E' \cap V) \quad \text{and} \quad \mu^*(E \cap V^c) \leq \mu^*(E' \cap V^c). \tag{2.135}$$

Thus it is easy to see from the inequalities (2.134) and (2.135) that one can obtain the inequality (2.133) if we can show the following result holds

$$\mu^*(E') \geq \mu^*(E' \cap V) + \mu^*(E' \cap V^c) - \frac{\epsilon}{2}. \tag{2.136}$$

To this end, we first find bounds of $\mu^*(E' \cap V)$ and $\mu^*(E' \cap V^c)$. Since $E' \cap V$ is open in X, the definition (2.129) implies the existence of a continuous function $f_1 : X \to [0, 1]$ such that $\operatorname{supp}(f_1) \subseteq E' \cap V$ and

$$\mu^*(E' \cap V) \leq \Lambda(f_1) + \frac{\epsilon}{8}. \tag{2.137}$$

Let $\epsilon \in \big(0, 8\Lambda(1)\big)$ and $\delta > 0$ be a constant such that $\delta \leq \epsilon[8\Lambda(1) - \epsilon]^{-1}$. Then we have

$$\frac{\delta}{1+\delta}\Lambda(1) \leq \frac{\epsilon}{8} \quad \text{and} \quad F_1 = \frac{f_1}{1+\delta}. \tag{2.138}$$

Recall the fact that the positivity of Λ implies the monotonicity of Λ. In particular, $\Lambda(f) \leq \Lambda(1)$ for every $f \prec E' \cap V$. Thus we apply the results (2.138) and the monotonicity of Λ to the inequality (2.137) to get

$$\mu^*(E' \cap V) \leq \frac{1}{1+\delta}\Lambda(f_1) + \frac{\delta}{1+\delta}\Lambda(f_1) + \frac{\epsilon}{8} \leq \Lambda(F_1) + \frac{\delta}{1+\delta}\Lambda(1) + \frac{\epsilon}{8} = \Lambda(F_1) + \frac{\epsilon}{4}. \quad (2.139)$$

We replace E by the closed set $(E' \cap V)^c$ in Lemma 2.8 to get the open set U. Given $\delta > 0$. Since f_1 is uniformly continuous on X, there exists a $\eta > 0$ such that $|f_1(x) - f_1(y)| \leq \delta$ for all $x, y \in X$ with $\rho(x, y) < \eta$. Recall that f_1 vanishes on the closed set $(E' \cap V)^c$, so if we take $y \in (E' \cap V)^c$, then we have

$$f_1(x) \leq \delta$$

for all $x \in X$ with $\rho(x, y) < \eta$. Since $\rho_{(E' \cap V)^c}(x) < \eta$, Lemma 2.8 ensures that the set

$$U = \{x \in X \,|\, \rho_{(E' \cap V)^c}(x) < \eta\}$$

is open in X. In other words, we have established from **Fact 3** that there exists an open set U such that

$$U \supset (E' \cap V)^c \supseteq V^c \quad \text{and} \quad f_1(x) \leq \delta \text{ on } U. \quad (2.140)$$

Since $E' \cap U$ is open in X, the definition (2.129) again shows that we can find a continuous function $f_2 : X \to [0, 1]$ such that $\operatorname{supp}(f_2) \subseteq E' \cap U$ and furthermore, by using similar argument as in the proof of the inequality (2.139), we obtain that

$$\mu^*(E' \cap V^c) \leq \mu^*(E' \cap U) \leq \Lambda(f_2) + \frac{\epsilon}{4} \leq \Lambda(F_2) + \frac{\epsilon}{4}, \quad (2.141)$$

where $F_2 = \frac{f_2}{1+\delta}$.

Now we have found the bounds (2.139) and (2.141) of $\mu^*(E' \cap V)$ and $\mu^*(E' \cap V^c)$ respectively. Next, we want to show that $F_3 \in C_c(X)$, where $F_3 = F_1 + F_2$. To this end, we recall the facts that $F_1, F_2 : X \to [0, \frac{1}{1+\delta}]$ because $0 \leq f_1 \leq 1$ and $0 \leq f_2 \leq 1$. Furthermore, we note from the set relation (2.140) that

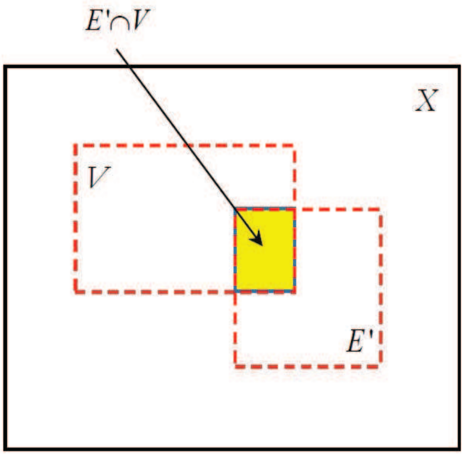
$$V \cap U \neq \varnothing.$$

Therefore, we have the following facts:

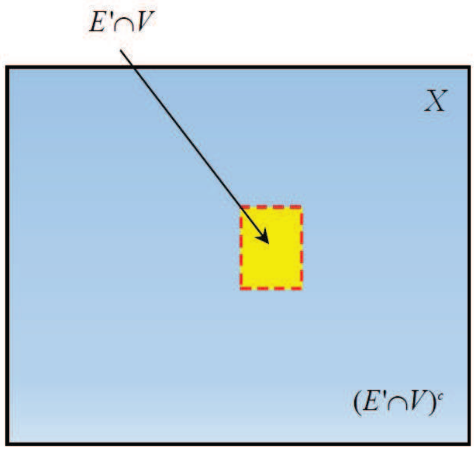
- Since $\operatorname{supp}(f_1) \subseteq E' \cap V$, $\operatorname{supp}(F_1) \subseteq E' \cap V$, i.e., $F_1(x) = 0$ on $(E' \cap V)^c$. Therefore, we conclude that
$$F_3 = F_2 \leq \frac{1}{1+\delta} \leq 1$$
on $(E' \cap V)^c$. Similarly, since $\operatorname{supp}(F_2) \subseteq E' \cap U$, we have $F_2(x) = 0$ on $(E' \cap U)^c$ and then
$$F_3 = F_1 \leq \frac{1}{1+\delta} \leq 1$$
on $(E' \cap U)^c$.
- On $(E' \cap V) \cap U$, we immediately follow from the inequality in (2.140) that $f_1(x) \leq \delta$ on $(E' \cap V) \cap U \subseteq U$. It is clear that $f_2(x) \leq 1$ on $(E' \cap V) \cap U$, so these facts imply that
$$F_3 = F_1 + F_2 = \frac{f_1}{1+\delta} + \frac{f_2}{1+\delta} \leq \frac{\delta}{1+\delta} + \frac{1}{1+\delta} = 1.$$
- On $(E' \cap V) \setminus U$, we have $F_1 \leq 1$ and $F_2 = 0$ so that $F_3 \leq 1$.

– On $U \setminus (E' \cap V)$, we have $F_1 = 0$ and $F_2 \leq 1$ so that $F_3 \leq 1$.

To have a better understanding of the above facts, we can draw some pictures. For examples, Figure 2.3(a) shows the sets V, E' and $E' \cap V$, the shaded blue part in Figure 2.3(b) indicates the closed set $(E' \cap V)^c$ and the part inside the circle in Figure 2.4 is the set $(E' \cap V)^c \setminus U$.



(a) The sets V, E' and $E' \cap V$.



(b) The set $(E' \cap V)^c$.

Figure 2.3: The pictures of $V, E', E' \cap V$ and $(E' \cap V)^c$.

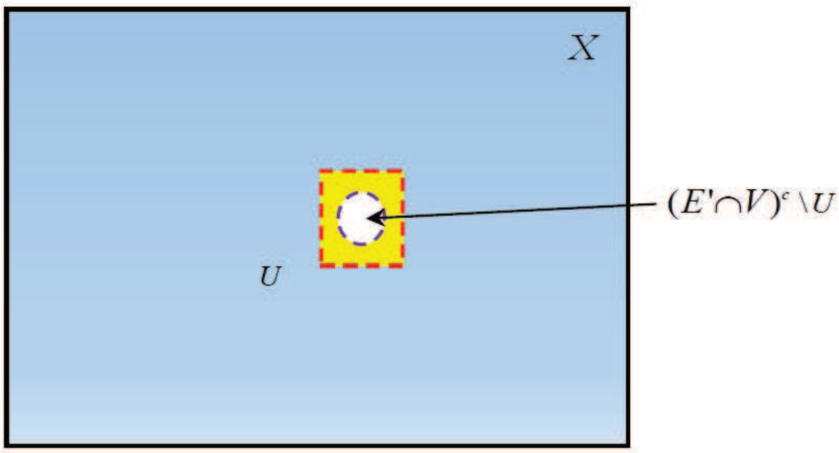


Figure 2.4: The set $(E' \cap V)^c \setminus U$.

Thus we have shown that $0 \leq F_3(x) \leq 1$ on X. Next, since F_1 and F_2 are continuous on X, F_3 is also continuous on X. By the definition, we know that $x_0 \in \operatorname{supp}(F_3)$ if and only if $x_0 \in \operatorname{supp}(F_1) \subseteq E' \cap V$ or $x_0 \in \operatorname{supp}(F_2) \subseteq E' \cap U$. In addition, we note that $(E' \cap V) \cup (E' \cap U) \subseteq E'$. On the other hand, if $y \in E'$ but $y \notin E' \cap V$, then we have $y \in (E' \cap V)^c$ and we follow from the set relation (2.140) that $y \in U$. Thus this implies that $y \in E' \cap U$ and so

$$E' = (E' \cap V) \cup (E' \cap U).$$

In other words, it gives $x_0 \in \operatorname{supp}(F_3)$ if and only if $x_0 \in \operatorname{supp}(F_1) \cup \operatorname{supp}(F_2) \subseteq E'$, i.e., $\operatorname{supp}(F_3) \subseteq E'$.

Finally, it yields from the inequalities (2.139) and (2.141) as well as the definition (2.129) that

$$\mu^*(E' \cap V) + \mu^*(E' \cap V^c) \leq \Lambda(F_1) + \Lambda(F_2) + \frac{\epsilon}{2} = \Lambda(F_3) + \frac{\epsilon}{2} \leq \mu^*(E') + \frac{\epsilon}{2}$$

which is exactly the inequality (2.136). Hence we obtain the desired result that (2.132).

- **Fact 5: The set $\mathfrak{M}^*$ of all measurable sets (with respect to μ^*) is a σ-algebra and $\mu = \mu^*|_{\mathscr{B}}(X)$ is a (complete) measure.** The first assertion follows directly from Carathéodory's Theorem [47, Theorem 8, p. 349]. By **Fact 4**, we have

$$\mathscr{B}(X) \subseteq \mathfrak{M}^*$$

which is exactly **Step VII**, where $\mathscr{B}(X)$ denotes the set of all Borel sets in X. By Carathéodory's Theorem again, it shows that μ is a (complete) measure on $\mathscr{B}(X)$ and this is the same as **Step IX**. Consequently, **Steps III**, **IV**, **V**, **VI** and **VIII** can all be skipped.[o]

- **Fact 6: Deduction of parts (b) and (d).** Let K be a compact set of X. Since X is a metric space, K is also Hausdorff. By Corollary (a) following Theorem 2.5, we know that K is closed in X and so $K \in \mathscr{B}(X)$. Therefore, we deduce from **Facts 2** and **5** that

$$\mu(K) = \mu^*(K) \leq \mu^*(X) \leq \Lambda(1)$$

which is exactly part (b) of Theorem 2.14 (The Riesz Representation Theorem).

If E is open in X, then $E \in \mathscr{B}(X)$. Thus we observe from the equation (2.132) and then the definition (2.130) that

$$\begin{aligned}
\mu(E) &= \mu^*(E) \\
&= \mu^*(X) - \mu^*(E^c) \\
&= \mu^*(X) - \inf\{\mu^*(V) \,|\, V \text{ is open in } X \text{ and } E^c \subseteq V\} \\
&= \mu^*(X) - \inf\{\mu^*(X) - \mu^*(V^c) \,|\, V^c \text{ is closed in } X \text{ and } V^c \subseteq E\} \\
&= \sup\{\mu^*(V^c) \,|\, V^c \text{ is closed in } X \text{ and } V^c \subseteq E\} \\
&= \sup\{\mu(V^c) \,|\, V^c \text{ is closed in } X \text{ and } V^c \subseteq E\}.
\end{aligned} \tag{2.142}$$

Since X is compact, V^c is compact by Theorem 2.4 so that the expression (2.142) can be expressed as

$$\mu(E) = \sup\{\mu(K) \,|\, K \text{ compact and } K \subseteq E\ \}$$

which is the same as part (d) of Theorem 2.14 (The Riesz Representation Theorem).

- **Fact 7: Deduction of part (a).** As Rudin pointed out in **Step X** that it suffices to prove

$$\Lambda(f) \leq \int_X f \,\mathrm{d}\mu \tag{2.143}$$

holds for $f \in C_c(X)$ because the opposite inequality follows by changing the sign of f. In fact, the proof of our inequality (2.143) for compact metric X is essentially the same as that of **Step X**, except that Rudin applied **Step II** to show that

$$\mu(K) \leq \sum \Lambda(h_i),$$

[o]We cannot say at this stage that **Step II** can be omitted because the proof of **Step X** still needs it, but we will see very soon in **Fact 7** that it is eventually allowable to do so.

where $h_i \prec V_i$ and $\sum h_i = 1$ on K.[p] However, this kind of argument can be replaced completely by using **Fact 6** that $\mu(K) \leq \Lambda(1)$ holds for every compact set $K \subseteq X$. Therefore, we finally arrive at the expected inequality (2.143) and furthermore, **Step II** can also be skipped.

Hence we conclude that **Steps II** to **IX** can be replaced by a simpler argument (outer measure in the sense of Carathéodory). In other words, only **Steps I** and **X** should be kept. Of course, the proof of the uniqueness of μ cannot be omitted. This completes the proof of the problem. ■

Remark 2.3

(a) We remark that a Radon measure is a measure μ defined on the σ-algebra $\mathfrak{M}$ containing all Borel sets $\mathscr{B}(X)$ of the Hausdorff space X and it satisfies $\mu(K) < \infty$ for all compact subsets K, outer regular on $\mathfrak{M}$ and inner regular on open sets in X, see [22, p. 212] or [47, p. 455]. Hence the unique positive measure μ in Theorem 2.14 (The Riesz Representation Theorem) is in fact a Radon measure when X is Hausdorff.

(b) You are also advised to read the paper [52] for a similar proof of Problem 2.19.

2.6 Miscellaneous Problems on L^1 and Other Properties

Problem 2.20

Rudin Chapter 2 Exercise 20.

Proof. Let $f = \sup\limits_n f_n : [0,1] \to [0,\infty]$. By Definition 1.30, $f \notin L^1$ on $[0,1]$ is equivalent to

$$\int_0^1 |f(x)|\,\mathrm{d}x = \infty.$$

In fact, the inspiration of the construction of $\{f_n\}$ comes from the idea of Problem 2.9. For each positive integer n, we consider the continuous functions $g_n : [-1,1] \to [0,\infty)$ given by

$$g_n(x) = \begin{cases} 0, & \text{if } x \notin [-\frac{1}{n^2}, \frac{1}{n^2}]; \\ n - n^3|x|, & \text{if } x \in [-\frac{1}{n^2}, \frac{1}{n^2}]. \end{cases}$$

Next, we define $f_n : [0,1] \to [0,\infty)$ by

$$f_n(x) = g_n\Big(x - \frac{1}{n}\Big) = \begin{cases} 0, & \text{if } x \notin [\frac{1}{n} - \frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2}]; \\ n - n^2|nx-1|, & \text{if } x \in [\frac{1}{n} - \frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2}]. \end{cases} \tag{2.144}$$

Recall that $f_n(x)$ is the tent function centered at $\frac{1}{n}$ and the graph of it is an isosceles triangle with height n and base $\frac{2}{n^2}$ so that each f_n is continuous on $[0,1]$ and its area is $\frac{1}{n}$ which implies that

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{1}{n} \to 0$$

[p]See **Step X** for the meanings of the notations used here.

as $n \to \infty$. Furthermore, if $x = 0$, then $x \notin [\frac{1}{n} - \frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2}]$ for all $n \in \mathbb{N}$. and thus

$$f_n(0) = 0$$

for all $n \in \mathbb{N}$ by the definition (2.144). If $x \in (0, 1]$, then there is a positive integer N such that $\frac{1}{n} + \frac{1}{n^2} < x$ for all $n \geq N$. In other words, we have $x \notin [\frac{1}{n} - \frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2}]$ for all $n \geq N$. Therefore, the definition (2.144) certainly shows that

$$f_n(x) \to 0$$

as $n \to \infty$ for all $x \in (0, 1]$.

Now it remains to show that $f \notin L^1$ on $[0, 1]$. To this end, we study the behaviour of f as $x \to 0$. For every $n \in \mathbb{N}$, we consider every x in $[\frac{1}{n} - \frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2}]$. If n is sufficiently large enough, we have

$$x = \frac{1}{n} + o\Big(\frac{1}{n}\Big), \tag{2.145}$$

where $o(x)$ is the little-o notation.[q] Therefore, it follows from the definition (2.144) and the relation (2.145) that

$$f_n(x) = n + o(1) = \frac{1}{x} + o(1) \tag{2.146}$$

for large enough n and small positive x. By the definition and the expression (5.102), we have

$$f(x) = \sup_n f_n(x) \geq f_n(x) = \frac{1}{x} + o(1) \tag{2.147}$$

for small positive x. Obviously, $\frac{1}{x} \notin L^1$ on $[0, 1]$, so we conclude from the estimate that (2.147) that $f \notin L^1$ on $[0, 1]$. This completes the proof of the problem. ∎

Problem 2.21

Rudin Chapter 2 Exercise 21.

Proof. Let $\alpha \in f(X) \subseteq \mathbb{R}$ and

$$E_\alpha = \{x \in X \,|\, f(x) \geq \alpha\} = f^{-1}\big([\alpha, \infty)\big).$$

By the definition, it is clear that E_α is nonempty. Furthermore, since

$$E_\alpha = X \setminus f^{-1}\big((-\infty, \alpha)\big) \subseteq X,$$

the set E_α is closed in X. Now we consider the collection $\{E_\alpha\}$ of subsets of X. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n \in f(X)$ and we define $\alpha = \max(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then we have

$$\bigcap_{k=1}^{n} E_{\alpha_k} = \bigcap_{k=1}^{n} \{x \in X \,|\, f(x) \geq \alpha_k\} = E_\alpha \neq \varnothing.$$

Therefore, $\{E_\alpha\}$ has the *finite intersection property* and then since X is compact, we conclude that[r]

$$\bigcap_\alpha E_\alpha \neq \varnothing.$$

Thus suppose that $x_0 \in \bigcap_\alpha E_\alpha$ which means that $f(x_0) \geq \alpha$ for all $\alpha \in f(X)$, i.e., $f(x_0) \geq f(x)$ for all $x \in X$. Hence this shows that f attains its maximum at some point of X, completing the proof of the problem. ∎

[q] Recall that $g(n) = o(f(n))$ means $\frac{g(n)}{f(n)} \to 0$ as $n \to \infty$.

[r] See, for instances, [42, Theorem 26.9, p. 169] or [64, Problem 4.23, p. 43].

Remark 2.4

Problem 2.21 can be treated as the **Extreme Value Theorem for upper semicontinuous functions**. Similarly, we can show that if $f : X \to \mathbb{R}$ is lower semicontinuous and X is compact, then f attains its minimum at some point of X.

Problem 2.22

Rudin Chapter 2 Exercise 22.

Proof. By the definition and the triangle inequality, for all $p \in X$, we have

$$g_n(x) \leq f(p) + nd(x,p) \leq [f(p) + nd(y,p)] + nd(x,y).$$

In other words, $g_n(x) - nd(x,y)$ is a lower bound of $\{f(p) + nd(y,p) \,|\, p \in X\}$. Thus, by the definition of infimum, we must have $g_n(x) - nd(x,y) \leq g_n(y)$ or equivalently,

$$g_n(x) - g_n(y) \leq nd(x,y). \tag{2.148}$$

Likewise, we have

$$g_n(y) - g_n(x) \leq nd(x,y). \tag{2.149}$$

Therefore, we conclude from the inequalities (2.148) and (2.149) that

$$|g_n(x) - g_n(y)| \leq nd(x,y).$$

Since $f(x) \geq 0$ on X and $d(x,y) \geq 0$ for every $x, y \in X$, we have $g_n(x) \geq 0$ for all positive integers n. Fix $x \in X$, we observe that

$$f(p) + nd(x,p) \geq f(p) + (n-1)d(x,p) \geq g_{n-1}(x)$$

for every $p \in X$ and $n = 2, 3, \ldots$. In other words, this implies that

$$g_n(x) \geq g_{n-1}(x)$$

for every $x \in X$ and $n = 2, 3, \ldots$. Besides, it is trivial that $g_n(x) \leq f(x)$ holds for all $n \in \mathbb{N}$ if $f(x) = \infty$. Without loss of generality, we may assume that $f(x) < \infty$. In this case, since $f(p) \leq f(p) + nd(x,p)$ for all $p \in X$, if we take $p = x$ in the definition of g_n, then we have

$$g_n(x) \leq f(x)$$

for every $n = 1, 2, \ldots$. This proves property (ii).

To prove property (iii), we need the following lemma:

Lemma 2.9

A function $f : X \to \mathbb{R}$ is lower semicontinuous on X if and only if given $x \in X$, for every $\{x_n\} \subseteq X \setminus \{x\}$ converging to x and for every $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies

$$f(x) < f(x_n) + \epsilon. \tag{2.150}$$

Proof of Lemma 2.9. For every $\alpha \in \mathbb{R}$, denote $E_\alpha = \{x \in X \,|\, f(x) \leq \alpha\}$. Recall that $f^{-1}\big((\alpha,\infty)\big)$ is open in X if and only if $X \setminus f^{-1}\big((\alpha,\infty)\big) = E_\alpha$ is closed in X. In other words, f is lower semicontinuous on X if and only if E_α is closed in X.

Let E_α be closed in X. Assume that the inequality (2.150) was invalid. Then it means that there exists a $x_0 \in X$ and a sequence $\{x_n\}$ in X converging to x_0 such that for some $\epsilon > 0$, the inequality

$$f(x_0) - \epsilon \geq f(x_{n_k}) \tag{2.151}$$

holds for *infinitely many* k. Take $\alpha \in (f(x_0) - \epsilon, f(x_0))$. On the one hand, since $f(x_0) > \alpha$, $x_0 \notin E_\alpha$. On the other hand, we gain from the inequality (2.151) that $f(x_{n_k}) < \alpha$ for infinitely many k and this implies that

$$\{x_{n_k}\} \subseteq E_\alpha.$$

Since the set E_α is closed and $x_{n_k} \to x_0$ as $k \to \infty$, we have $x_0 \in E_\alpha$, a contradiction. Hence the inequality (2.150) must hold for a lower semicontinuous function f.

Conversely, we prove that the inequality (2.150) implies E_α is closed in X by showing that E_α contains all its limit points. To this end, pick $\alpha \in \mathbb{R}$ such that $E_\alpha \neq \varnothing$. Let $\{x_n\} \subseteq E_\alpha \setminus \{x\}$ and $x_n \to x$ as $n \to \infty$. Given $\epsilon > 0$. By the inequality (2.150), there exists a positive integer N such that

$$f(x) - \epsilon < f(x_n) \tag{2.152}$$

for all $n \geq N$. Since $f(x_n) \leq \alpha$, we deduce immediately from the inequality (2.152) that

$$f(x) < \alpha + \epsilon.$$

Since ϵ is arbitrary, we must have $f(x) \leq \alpha$ or equivalently $x \in E_\alpha$. This ends the proof of the lemma. ∎

Remark 2.5

(a) Similarly, a function $f : X \to \mathbb{R}$ is upper semicontinuous on X if and only if given $x \in X$, for every $\{x_n\} \subseteq X \setminus \{x\}$ converging to x and for every $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies $f(x) > f(x_n) - \epsilon$.

(b) Except the equivalent definition stated in Lemma 2.9, a lower semicontinuous function f on X can also be reformulated in terms of the lower limit of f: $\liminf\limits_{y \to x} f(y) \geq f(x)$ for every $y \to x$ in $X \setminus \{x\}$, see [61, pp. 69, 70] and [65, Definition 3.62, p. 64].

Now it is time to go back to the proof of the problem. On the one hand, since f is lower semicontinuous on X, Lemma 2.9 implies that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(y) > f(x) - \epsilon \tag{2.153}$$

for all $y \in B(x, \delta)$. In this case, we always have

$$f(y) + nd(x, y) > f(x) - \epsilon \tag{2.154}$$

for every positive integer n. Since $\epsilon > 0$ is arbitrary, the inequality (2.154) becomes

$$f(y) + nd(x, y) \geq f(x). \tag{2.155}$$

On the other hand, we consider $y \notin B(x,\delta)$ so that $d(x,y) \geq \delta$. It is clear that $n\delta - \epsilon > 0$ holds for large enough n,[s] thus we get from the inequality (2.153) that

$$f(y) + nd(x,y) \geq f(y) + n\delta > f(x) - \epsilon + n\delta > f(x)$$

for large enough n. Therefore, we conclude that the inequality (2.155) always holds for all $y \in X$ and for all large enough n. By the definition of limit inferior, we have

$$\liminf_{n\to\infty} g_n(x) \geq f(x). \tag{2.156}$$

By property (ii), we see that

$$\limsup_{n\to\infty} g_n(x) \leq f(x) \tag{2.157}$$

holds for every $x \in X$. Hence it deduces from the inequalities (2.156) and (2.157) that

$$\lim_{n\to\infty} g_n(x) = f(x)$$

on X.

By property (i), each g_n is continuous on X. Next, property (ii) says that $\{g_n\}$ is increasing and bounded by f. Finally, property (iii) ensures that $\{g_n\}$ converges to f on X pointwisely. This ends the proof of the problem. ■

Remark 2.6

With the aid of Lemma 2.9, we can prove Problem 2.1(b) easily. In fact, given $x \in X$, for every $\{x_n\} \subseteq X \setminus \{x\}$ converging to x and for every $\epsilon > 0$, there exists a positive integer N_1 such that $n \geq N_1$ implies that

$$f_1(x) < f_1(x_n) + \frac{\epsilon}{2}. \tag{2.158}$$

Similarly, there exists a positive integer N_2 such that $n \geq N_2$ implies

$$f_2(x) < f_2(x_n) + \frac{\epsilon}{2}. \tag{2.159}$$

Take $N = \max(N_1, N_2)$. If $n \geq N$, then we derive from the inequalities (2.158) and (2.159) that

$$f_1(x) + f_2(x) < f_1(x_n) + f_2(x_n) + \epsilon.$$

Hence it follows from Lemma 2.9 that f is lower semicontinuous on X.

Problem 2.23

Rudin Chapter 2 Exercise 23.

Proof. Let $\mathbf{x} \in \mathbb{R}^k$ and $f : \mathbb{R}^k \to \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \mu(V + \mathbf{x}).$$

Since $V + \mathbf{x}$ is just a translation of V, it is also open in $\mathbb{R}^k$, i.e., $V + \mathbf{x} \in \mathscr{B}$.

[s]Obviously, the n depends on δ and ϵ.

We first construct a counter example which is not upper semicontinuous or continuous. By Example 1.20(b), we consider the unit mass concentrated at $\mathbf{0}$, i.e.,

$$\mu(E) = \begin{cases} 1, & \text{if } \mathbf{0} \in E; \\ 0, & \text{if } \mathbf{0} \notin E. \end{cases}$$

If $V = B(\mathbf{0}, 1)$, then we have $V + \mathbf{x} = B(\mathbf{x}, 1)$ and

$$\begin{aligned} \mu(V + \mathbf{x}) &= \begin{cases} 1, & \text{if } \mathbf{0} \in B(\mathbf{x}, 1); \\ 0, & \text{if } \mathbf{0} \notin B(\mathbf{x}, 1) \end{cases} \\ &= \begin{cases} 1, & \text{if } |\mathbf{x}| < 1; \\ 0, & \text{if } |\mathbf{x}| \geq 1. \end{cases} \end{aligned} \tag{2.160}$$

Thus it establishes from the result (2.160) that

$$f^{-1}\big((-\infty, 1)\big) = \{\mathbf{x} \in \mathbb{R}^k \,|\, f(\mathbf{x}) < 1\} = \{\mathbf{x} \in \mathbb{R}^k \,|\, \mu(V + \mathbf{x}) < 1\} = \mathbb{R}^k \setminus B(\mathbf{x}, 1)$$

which is closed in $\mathbb{R}^k$. By Definition 2.8, f is *not* upper semicontinuous or continuous.

However, we claim that f is always lower semicontinous. Let $\{\mathbf{x}_n\} \subseteq \mathbb{R}^k \setminus \{\mathbf{x}\}$ be such that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. Recall that both $V + \mathbf{x}$ and $V + \mathbf{x}_n$ are open in $\mathbb{R}^k$. We claim that $\mathbf{y} \in V + \mathbf{x}$ implies that $\mathbf{y} \in V + \mathbf{x}_n$ for all but finitely many n. Let $\mathbf{y} = \mathbf{a} + \mathbf{x}$ for some $\mathbf{a} \in V$. Since V is open in $\mathbb{R}^k$, there exists a $\epsilon > 0$ such that

$$B(\mathbf{a}, \epsilon) \subseteq V.$$

By the hypothesis, there is a positive integer N such that $n \geq N$ implies

$$\mathbf{x}_n - \mathbf{x} = \frac{\epsilon}{2} \quad \text{or} \quad \mathbf{x}_n - \mathbf{x} = -\frac{\epsilon}{2}.$$

In the first case, we have

$$\mathbf{y} = \mathbf{a} + \mathbf{x} = \mathbf{a} - \frac{\epsilon}{2} + \mathbf{x} + \frac{\epsilon}{2} = \mathbf{a} - \frac{\epsilon}{2} + \mathbf{x}_n. \tag{2.161}$$

Since $\mathbf{a} - \frac{\epsilon}{2} \in V$, the expression (2.161) guarantees that $\mathbf{y} \in V + \mathbf{x}_n$ for all $n \geq N$. The other case can be done similarly, so we omit the details here. This proves our claim.

Now if $\chi_{V+\mathbf{x}}(\mathbf{y}) = 1$, then we have $\chi_{V+\mathbf{x}_n}(\mathbf{y}) = 1$ for all but finitely many n. In other words, it means that

$$\liminf_{n\to\infty} \chi_{V+\mathbf{x}_n}(\mathbf{y}) \geq \chi_{V+\mathbf{x}}(\mathbf{y}) \tag{2.162}$$

for every $\mathbf{y} \in \mathbb{R}^k$. By Proposition 1.9(d), each $\chi_{V+\mathbf{x}_n} : \mathbb{R}^k \to [0, \infty]$ is measurable, so we may apply Theorem 1.28 (Fatou's Lemma) to conclude that

$$\liminf_{n\to\infty} \Big(\int_{\mathbb{R}^k} \chi_{V+\mathbf{x}_n} \, \mathrm{d}\mu \Big) \geq \int_{\mathbb{R}^k} \Big(\liminf_{n\to\infty} \chi_{V+\mathbf{x}_n} \Big) \, \mathrm{d}\mu. \tag{2.163}$$

By Proposition 1.24(f), we have

$$\int_{\mathbb{R}^k} \chi_{V+\mathbf{x}_n} \, \mathrm{d}\mu = \int_{V+\mathbf{x}_n} \mathrm{d}\mu = \mu(V + \mathbf{x}_n). \tag{2.164}$$

In addition, it follows from the inequality (2.162) and the application of Proposition 1.24(f) again that

$$\int_{\mathbb{R}^k} \Big(\liminf_{n\to\infty} \chi_{V+\mathbf{x}_n}\Big)\, \mathrm{d}\mu \geq \int_{\mathbb{R}^k} \chi_{V+\mathbf{x}}(\mathbf{y})\, \mathrm{d}\mu = \int_{V+\mathbf{x}} \mathrm{d}\mu = \mu(V+\mathbf{x}). \tag{2.165}$$

Thus, by substituting the results (2.164) and (2.165) into the inequality (2.163), we achieve that

$$\liminf_{n\to\infty} \mu(V+\mathbf{x}_n) \geq \mu(V+\mathbf{x})$$

or equivalently,

$$\liminf_{\mathbf{y}\to\mathbf{x}} \mu(V+\mathbf{y}) \geq \mu(V+\mathbf{x})$$

for every $\mathbf{y}\to\mathbf{x}$ in $\mathbb{R}^k\setminus\{\mathbf{x}\}$. Hence we conclude from Remark 2.5(b) that f is lower semicontinuos on $\mathbb{R}^k$, completing the proof of the problem. ■

Problem 2.24

Rudin Chapter 2 Exercise 24.

Proof. Suppose that S_1 and S_2 are the sets of simple functions and step functions on $\mathbb{R}$ respectively. We divide the proof into several steps.

- **Step 1: S_1 is dense in $L^1(\mathbb{R})$.** Let $f = f^+ - f^1$. By the Corollary (b) following Theorem 1.14, both f^+ and f^- are measurable. Since $f^+ : \mathbb{R} \to [0,\infty]$, we follow from Theorem 1.17 (The Simple Function Approximation Theorem) that there is a sequence $\{s_n\}$ of nonnegative increasing simple functions such that for every $x\in\mathbb{R}$, $s_n(x)\to f^+(x)$ as $n\to\infty$. Thus Theorem 1.26 (Lebesgue's Monotone Convergence Theorem) implies that

$$\int_{-\infty}^{\infty} s_n\, \mathrm{d}x \to \int_{-\infty}^{\infty} f^+\, \mathrm{d}x \tag{2.166}$$

as $n\to\infty$. It is clear that $|f^+(x) - s_n(x)| = f^+(x) - s_n(x)$ for every $x\in\mathbb{R}$, so we have[t]

$$\begin{aligned}\int_{-\infty}^{\infty} |f^+(x) - s_n(x)|\, \mathrm{d}x &= \int_{-\infty}^{\infty} [f^+(x) - s_n(x)]\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f^+(x)\, \mathrm{d}x - \int_{-\infty}^{\infty} s_n(x)\, \mathrm{d}x.\end{aligned} \tag{2.167}$$

If we apply the limit (2.166) to the expression (2.167), then we gain

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f^+(x) - s_n(x)|\, \mathrm{d}x = 0.$$

Likewise, the same is true for f^-, so we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f^-(x) - t_n(x)|\, \mathrm{d}x = 0$$

where $\{t_n\}$ is a sequence of nonnegative increasing simple functions such that for every $x\in\mathbb{R}$, $t_n(x)\to f^-(x)$ as $n\to\infty$. Hence $\{g_n = s_n - t_n\}$ is a sequence of simple functions such that

$$\int_{-\infty}^{\infty} |f - g_n|\, \mathrm{d}x = \int_{-\infty}^{\infty} |f^+ - f^- - s_n + t_n|\, \mathrm{d}x$$

[t]Since $|f^+(x)| \leq |f(x)|$ on $\mathbb{R}$ and $f\in L^1(\mathbb{R})$, we have $f^+\in L^1(\mathbb{R})$. Thus this makes the second equality holds because the right-hand side is *not* in the form $\infty - \infty$.

$$\leq \int_{-\infty}^{\infty} |f^+ - s_n| \,\mathrm{d}x + \int_{-\infty}^{\infty} |f^- - t_n| \,\mathrm{d}x$$
$$\to 0$$

as $n \to \infty$.

- **Step 2: S_2 is dense in S_1.** To this end, it suffices to prove that for every measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$ and every $\epsilon > 0$, there exists a step function g on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |\chi_E - g| \,\mathrm{d}x < \epsilon. \tag{2.168}$$

By Theorem 2.20(b), there exists an open set V in $\mathbb{R}$ such that $E \subseteq V$ and $m(V \setminus E) < \frac{\epsilon}{2}$. By [49, Exercise 29, Chap. 2, p. 45], we have

$$V = \bigcup_{n=1}^{\infty} (a_n, b_n), \tag{2.169}$$

where $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ if $i \neq j$. Assume that there was an unbounded segment in the expression (2.169). Then we have $m(V) = \infty$. However, since $m(V) = m(V \setminus E) + m(E)$, the facts $m(V) = \infty$ and $m(V \setminus E) < \frac{\epsilon}{2}$ will force that $m(E) = \infty$, a contradiction. Furthermore, the expression (2.169) implies that

$$m(V) = \sum_{n=1}^{\infty} m\big((a_n, b_n)\big),$$

so there exists a positive integer N such that

$$m(V) - \frac{\epsilon}{2} < \sum_{n=1}^{N} m\big((a_n, b_n)\big). \tag{2.170}$$

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{N} \chi_{(a_n, b_n)}(x)$$

and $V_N = (a_1, b_1) \cup \cdots \cup (a_N, b_N)$. Let $x \in (E \setminus V_N) \cup (V_N \setminus E)$. If $x \in E \setminus V_N$, then we have $\chi_E(x) = 1$ and $g(x) = 0$. Similarly, if $x \in V_N \setminus E$, then we have $\chi_E(x) = 0$ and $g(x) = 1$. Therefore, we have

$$|g - \chi_E| = 1 \tag{2.171}$$

on $(E \setminus V_N) \cup (V_N \setminus E)$. On the other hand, we let

$$x \in (E \setminus V_N)^c \cap (V_N \setminus E)^c = (V_N \cup E^c) \cap (E \cup V_N^c) = (V_N \cap E) \cup (V_N^c \cap E^c).$$

If $x \in V_N \cap E$, then we have $\chi_E(x) = g(x) = 1$. Similarly, $x \in V_N^c \cap E^c$ implies that $\chi_E(x) = g(x) = 0$. Both cases give

$$|g - \chi_E| = 0 \tag{2.172}$$

on $(E \setminus V_N)^c \cap (V_N \setminus E)^c$. Now the facts $E, V_N \subseteq V$ and (2.170) definitely imply that

$$m(V_N \setminus E) \leq m(V \setminus E) < \frac{\epsilon}{2} \quad \text{and} \quad m(E \setminus V_N) \leq m(V \setminus V_N) < \frac{\epsilon}{2}. \tag{2.173}$$

Hence we deduce from the results (2.171), (2.172) and the estimates (2.173) that

$$\int_{-\infty}^{\infty} |g - \chi_E| \,\mathrm{d}x = \int_{E \setminus V_N} \mathrm{d}x + \int_{V_N \setminus E} \mathrm{d}x = m(V_N \setminus E) + m(E \setminus V_N) < \epsilon.$$

This is the desired result (2.168).

- **Step 3: S_2 is dense in $L^1(\mathbb{R})$.** We need the following lemma:

Lemma 2.10

Suppose that A, B and C are subsets of a metric space X. If $A \subseteq B \subseteq C$, A is dense in B and B is dense in C, then A is dense in C.

Proof of Lemma 2.10. Recall the definition that A is dense in S if $A \subseteq S \subseteq \overline{A}$, where $\overline{A}$ denotes the closure of A in X. Then we have

$$A \subseteq B \subseteq \overline{A} \quad \text{and} \quad B \subseteq C \subseteq \overline{B}$$

which show immediately that $A \subseteq C$. Let $p \in \overline{B}$. We claim that $p \in \overline{A}$. To this end, we know from the assumption that

$$B(p,\delta) \cap (B \setminus \{p\}) \neq \varnothing$$

for every $\delta > 0$. Let $q \in B(p,\delta) \cap (B \setminus \{p\})$. Since $B \subseteq \overline{A}$, $\overline{A}$ also contains q. If $q \in A$, then we have

$$B(p,\delta) \cap (A \setminus \{p\}) \neq \varnothing \tag{2.174}$$

for every $\delta > 0$. Therefore, the definition implies that $p \in \overline{A}$. Next, suppose that $q \in A'$. Since $B(p,\delta)$ is open in X, there exists a $\epsilon > 0$ such that $B(q,\epsilon) \subseteq B(p,\delta)$. Since q is a limit point of A, the definition shows that $B(q,\epsilon) \cap A \neq \varnothing$ and thus the set relation (2.174) still holds in this case. Therefore, we have proven our claim that $p \in \overline{A}$ and this means that $\overline{B} \subseteq \overline{A}$. Hence we establish the set relations

$$A \subseteq C \subseteq \overline{A}$$

which implies that A is dense in C, completing the proof of Lemma 2.10. ■

Let's return to the original proof. By applying Lemma 2.10 to the results in **Step 1** and **Step 2**, we conclude easily that S_2 is dense in $L^1(\mathbb{R})$ which is our desired result.

Hence we have completed the proof of the problem. ■

Remark 2.7

By Definition 1.16, we remark that a simple function s is a finite linear combination of characteristic functions of an arbitrary set A_i and when all A_i are intervals, then s becomes a step function. In fact, the class of step functions is one of the building blocks for the theory of Riemann integration, see [49, Chap. 6].

Problem 2.25

Rudin Chapter 2 Exercise 25.

Proof.

(a) We note that $e^t > 0$ and $\log(1 + e^t) > 0$ for every $t > 0$. Furthermore, we know that the functions e^x and $\log x$ are increasing in their corresponding domains respectively. Thus the inequality $\log(1+\mathrm{e}^t) < c+t$ is equivalent to $1+\mathrm{e}^t < \mathrm{e}^{c+t}$ and finally, it is equivalent to

$$0 < 1+\mathrm{e}^{-t} < \mathrm{e}^c. \tag{2.175}$$

Apply log to both sides of the inequality (2.175), we have

$$\log(1+\mathrm{e}^{-t}) < c \tag{2.176}$$

for every $t > 0$. Since the left-hand side of the inequality (2.176) is a decreasing function of t, the smallest value of c for the validity of the inequality (2.176) on $(0,\infty)$ is found by

$$\lim_{\substack{t\to 0\\ t>0}} \log(1+\mathrm{e}^{-t}) = \log 2.$$

(b) Let $f \in L^1$ on $[0,1]$. Define $E = \{x \in [0,1] \,|\, f(x) > 0\} = f^{-1}\big((0,\infty)\big)$. By Theorem 1.12(b), E is a measurable set in $[0,1]$. Thus we follow from part (a) that, on E,

$$nf(x) = \log\left(\mathrm{e}^{nf(x)}\right) < \log\left(1+\mathrm{e}^{nf(x)}\right) < \log 2 + nf(x)$$

on $(0,\infty)$ and this implies that

$$\int_E f\,\mathrm{d}\mu < \frac{1}{n}\int_E \log(1+\mathrm{e}^{nf})\,\mathrm{d}\mu < \frac{\log 2}{n} + \int_E f\,\mathrm{d}\mu. \tag{2.177}$$

Next, we have $1 \leq 1+\mathrm{e}^{nf(x)} \leq 2$ on $[0,1]\setminus E$. Thus we have

$$0 \leq \log(1+\mathrm{e}^{nf}) \leq \log 2 \tag{2.178}$$

on $[0,1]\setminus E$. Since $[0,1]\setminus E$ is also a measurable set in $[0,1]$, we deduce from the inequalities (2.178) that

$$0 \leq \frac{1}{n}\int_{[0,1]\setminus E} \log(1+\mathrm{e}^{nf})\,\mathrm{d}\mu \leq \frac{\log 2}{n}.$$

Now the sum of the inequalities (2.177) and (2.178) give

$$\int_E f\,\mathrm{d}\mu < \frac{1}{n}\int_0^1 \log(1+\mathrm{e}^{nf})\,\mathrm{d}\mu < \frac{2\log 2}{n} + \int_E f\,\mathrm{d}\mu. \tag{2.179}$$

Hence, by taking $n\to\infty$ in the inequalities (2.179), we get that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^1 \log(1+\mathrm{e}^{nf})\,\mathrm{d}\mu = \int_E f\,\mathrm{d}\mu.$$

This completes the proof of the problem. ■

CHAPTER 3

L^p-Spaces

3.1 Properties of Convex Functions

Problem 3.1

Rudin Chapter 3 Exercise 1.

Proof. Let $\{\varphi_\alpha\}$ be a collection of convex functions on (a,b). Define

$$\varphi = \sup\{\varphi_\alpha\} \tag{3.1}$$

and assume that it is finite. Suppose that $x, y \in (a,b)$ and $0 \leq \lambda \leq 1$. By the definition (3.1) and the convexity of φ_α, we have

$$\varphi_\alpha\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)\varphi_\alpha(x) + \lambda\varphi_\alpha(y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

for all α. In other words, we have

$$\varphi\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y).$$

By Definition 3.1, φ is convex on (a,b).

Suppose that $\{\varphi_n\}$ is a sequence of convex functions on (a,b). For $x \in (a,b)$, we define $\varphi : (a,b) \to \mathbb{R}$ to be

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x). \tag{3.2}$$

Now for $x, y \in (a,b)$ and $0 \leq \lambda \leq 1$, we follow from the limit (3.2) that

$$\lim_{n\to\infty} [(1-\lambda)\varphi_n(x) + \lambda\varphi_n(y)] = (1-\lambda)\varphi(x) + \lambda\varphi(y) \tag{3.3}$$

and

$$\lim_{n\to\infty} \varphi_n\big((1-\lambda)x + \lambda y\big) = \varphi\big((1-\lambda)x + \lambda y\big). \tag{3.4}$$

Since $\varphi_n\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)\varphi_n(x) + \lambda\varphi_n(y)$ for every $n = 1, 2, \ldots$, we take limits to both sides and apply the results (3.3) and (3.4) to conclude that

$$\varphi\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y).$$

Hence φ is convex on (a,b) by Definition 3.1.

For each positive integer n, suppose that $f_n : (a, b) \to \mathbb{R}$ is convex. Define $f : (a, b) \to \mathbb{R}$ by

$$f(x) = \limsup_{n\to\infty} f_n(x),$$

where $x \in (a, b)$. By Definition 1.13, we have

$$f(x) = \lim_{k\to\infty} g_k(x), \tag{3.5}$$

where $g_k = \sup\{f_k, f_{k+1}, \ldots\}$ for $k = 1, 2, \ldots$. Thus if each f_n is convex on (a, b), then the first assertion shows that each g_k is convex on (a, b). By the definition (3.5), since f is the pointwise limit of the sequence of convex functions $\{g_1, g_2, \ldots\}$ defined on (a, b), we deduce from the second assertion that f is also convex on (a, b). However, the lower limit of a sequence of convex functions *may not* be convex. For example, consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = (-1)^n x.$$

It is clear that each f_n is convex on $\mathbb{R}$. However, for each $x \in \mathbb{R}$, we have

$$f(x) = \liminf_{n\to\infty} f_n(x) = -|x|.$$

This shows that f is not convex on $\mathbb{R}$ and we complete the proof of the problem. ■

Problem 3.2

Rudin Chapter 3 Exercise 2.

Proof. For every $x, y \in (a, b)$ and $0 \leq \lambda \leq 1$, since φ is convex on (a, b), we have

$$\varphi\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y). \tag{3.6}$$

Since ψ is convex and nondecreasing on $\varphi\big((a, b)\big)$, we obtain from the inequality (3.6) that

$$\psi\Big(\varphi\big((1-\lambda)x + \lambda y\big)\Big) \leq \psi\big((1-\lambda)\varphi(x) + \lambda\varphi(y)\big) \leq (1-\lambda)\psi\big(\varphi(x)\big) + \lambda\psi\big(\varphi(y)\big).$$

By Definition 3.1, $\psi \circ \varphi$ is convex on (a, b).

Let $\varphi > 0$ on (a, b). By [51, Theorem (c), p. 2], the function exp is a monotonically increasing positive function on $\mathbb{R}$. Since $(\mathrm{e}^x)' = \mathrm{e}^x$, it is convex on $\mathbb{R}$ by the paragraph following Definition 3.1. Hence, if $\log\varphi$ is convex on (a, b), then we conclude from the first assertion that $\varphi = \mathrm{e}^{\log\varphi}$ is also convex on (a, b).

However, the converse of the second assertion is false. For instance, we know that

$$\varphi(x) = x$$

is convex on $(0, \infty)$, but $\log x$ is not convex on $(0, \infty)$ because $(\log x)'' = -x^{-2} < 0$ on $(0, \infty)$, i.e., $(\log x)'$ is *not* a monotonically increasing function on $(0, \infty)$. We have completed the proof of the problem. ■

Problem 3.3

Rudin Chapter 3 Exercise 3.

Proof. This problem is proven in [63, Problem 4.24, pp. 79 – 81]. ■

3.2 Relations among L^p-Spaces and some Consequences

Problem 3.4

Rudin Chapter 3 Exercise 4.

Proof.

(a) Since $0 < r < p < s$, we can find $\lambda \in (0,1)$ such that $p = \lambda r + (1-\lambda)s$. By Theorem 3.5 (Hölder's Inequality), we have

$$\begin{aligned}\varphi(p) &= \int_X |f|^p \, d\mu \\ &= \int_X |f|^{\lambda r} \times |f|^{(1-\lambda)s} \, d\mu \\ &\leq \Big\{ \int_X (|f|^{\lambda r})^{\frac{1}{\lambda}} \, d\mu \Big\}^{\lambda} \times \Big\{ \int_X [|f|^{(1-\lambda)s}]^{\frac{1}{1-\lambda}} \, d\mu \Big\}^{1-\lambda} \\ &= [\varphi(r)]^{\lambda} \times [\varphi(s)]^{1-\lambda}. \end{aligned} \tag{3.7}$$

Since $r, s \in E$, $\varphi(r)$ and $\varphi(s)$ are finite. Hence the inequality (3.7) ensures that $\varphi(p)$ is also finite, i.e., $p \in E$.

(b) We prove the assertion one by one.

- **Case (i): $\ln \varphi$ is convex in E°.** If $E^\circ = \varnothing$, then there is nothing to prove. Assume that $E^\circ \neq \varnothing$. By [49, Theorem 2.47, p. 42] and part (a), the set E is connected. Since $E \subseteq (0,\infty)$, E is either an interval in one of the forms $[a,b]$, $[a,b)$, $(a,b]$ or (a,b) for some positive a and b with $a < b$. In each of the cases, we always have $E^\circ = (a,b)$.

  Let $x, y \in (a,b)$ and $\lambda \in [0,1]$. Since (a,b) is a convex set, $\lambda x + (1-\lambda)y \in (a,b)$. Thus it follows from the inequality (3.7) that

  $$\varphi\big(\lambda x + (1-\lambda)y\big) \leq [\varphi(x)]^{\lambda} \times [\varphi(y)]^{1-\lambda}. \tag{3.8}$$

  If $\varphi(p) = 0$ for some $p \in (0,\infty)$, then we have

  $$\int_X |f|^p \, d\mu = 0$$

  so that $|f(x)| = 0$ almost everywhere on X. By the remark following Definition 3.7, it implies that $\|f\|_\infty = 0$, a contradiction. Hence we have $\varphi(p) > 0$ and we are allowed to take the logarithm to both sides of the inequality (3.8) to get

  $$\ln \varphi\big(\lambda x + (1-\lambda)y\big) \leq \ln \big\{ [\varphi(x)]^{\lambda} \times [\varphi(y)]^{1-\lambda} \big\} \leq \lambda \ln \varphi(x) + (1-\lambda) \ln \varphi(y).$$

  By Definition 3.1, $\ln \varphi$ is convex in (a,b).

- **Case (ii): φ is continuous on E.** By Theorem 3.2, $\ln \varphi$ is continuous on (a,b). Thus φ is also continuous on (a,b), so it remains to show that φ is continuous at the endpoints. Let $a \in E$. Then E is either $[a,b)$ or $[a,b]$, but no matter which case E is, a is a limit point of E so that we can find a decreasing sequence $\{p_n\} \subseteq (a,b)$ such that $p_n \to a$ as $n \to \infty$. By this, there exists a $\epsilon > 0$ and a positive integer N such that $p_n \in (a, a+\epsilon) \subset E$ for all $n \geq N$. Fix this ϵ and then $n \geq N$ implies that

  $$|f(x)|^{p_n} \leq \begin{cases} |f(x)|^{a+\epsilon}, & \text{if } |f(x)| \geq 1; \\ |f(x)|^{a}, & \text{if } |f(x)| < 1. \end{cases}$$

Hence we obtain that
$$|f(x)|^{p_n} \le |f(x)|^{a+\epsilon} + |f(x)|^a$$
on X.

We recall that $a, a+\epsilon \in E$, so the definition implies that $\varphi(a)$ and $\varphi(a+\epsilon)$ are finite. Since f is measurable, $|f|^a$ and $|f|^{a+\epsilon}$ are measurable by Proposition 1.9(b) and Theorem 1.7(b). Thus these two facts show that $|f|^a, |f|^{a+\epsilon} \in L^1(\mu)$. By Theorem 1.32, we have $|f|^{a+\epsilon} + |f|^a \in L^1(\mu)$. For each $x \in X$, $|f(x)| \ge 0$. Since an exponential function with nonnegative base is continuous on its domain, we have
$$\lim_{n\to\infty} |f(x)|^{p_n} = |f(x)|^a$$
on X. In conclusion, we have shown that the sequence $\{|f|^{p_n}\}$ satisfies the hypotheses of Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) and then
$$\lim_{n\to\infty} \varphi(p_n) = \lim_{n\to\infty} \int_X |f|^{p_n}\,\mathrm{d}\mu = \int_X |f|^a\,\mathrm{d}\mu = \varphi(a).$$
By definition, φ is continuous at a. Similarly, φ is continuous at b if $b \in E$. Consequently, φ is continuous on E.

(c) We claim that E can be any open or closed connected subset of $(0,\infty)$. We consider $X = (0,\infty)$, $\mu = m$ and $0 < a \le b$. We need the following result (see [2, Examples 1 & 5, pp. 417, 419]):

Lemma 3.1

Let $x > 0$ and $b > 1$. Then we have
$$\int_x^1 t^{-\alpha}\,\mathrm{d}t = \begin{cases} \dfrac{1-x^{1-\alpha}}{1-\alpha}, & \text{if } \alpha \ne 1; \\ -\ln x & \text{if } \alpha = 1 \end{cases}$$
and
$$\int_1^b t^{-\beta}\,\mathrm{d}t = \begin{cases} \dfrac{b^{1-\beta}-1}{1-\beta}, & \text{if } \beta \ne 1; \\ \ln b, & \text{if } \beta = 1. \end{cases}$$
Furthermore, the improper integrals
$$\int_0^1 x^{-\alpha}\,\mathrm{d}x \quad \text{and} \quad \int_1^\infty x^{-\beta}\,\mathrm{d}x$$
are convergent if and only if $\alpha < 1$ and $\beta > 1$ respectively.

– **Case (i):** $E = (a, b)$. For $x \in (0,\infty)$, let
$$f(x) = \begin{cases} x^{-\frac{1}{b}}, & \text{if } x \in (0,1); \\ x^{-\frac{1}{a}}, & \text{if } x \in [1,\infty). \end{cases}$$
We have to find $p \in (0,\infty)$ such that
$$\varphi(p) = \int_0^\infty |f(x)|^p\,\mathrm{d}x < \infty.$$

We assume that the integral can be split as follows:

$$\varphi(p) = \int_0^1 x^{-\frac{p}{b}}\,\mathrm{d}x + \int_1^\infty x^{-\frac{p}{a}}\,\mathrm{d}x. \tag{3.9}$$

Apply Lemma 3.1 to the two improper integrals in the expression (3.9), we know that $\varphi(p) < \infty$ if and only if $\frac{p}{b} < 1$ and $\frac{p}{a} > 1$ if and only if $a < p < b$. In other words, we have $E = (a, b)$ and such split is allowable.

– **Case (ii):** $E = [a, b]$**.** In this case, we need another lemma:[a]

Lemma 3.2

The improper integral

$$\int_0^{\mathrm{e}^{-1}} \frac{\mathrm{d}x}{x^\alpha |\ln x|^\beta}$$

converges if and only if (i) $\alpha < 1$ or (ii) $\alpha = 1$ and $\beta > 1$. The improper integral

$$\int_{\mathrm{e}}^\infty \frac{\mathrm{d}x}{x^\alpha (\ln x)^\beta}$$

converges if and only if (i) $\alpha > 1$ or (ii) $\alpha = 1$ and $\beta > 1$.

We consider

$$f(x) = \begin{cases} \dfrac{1}{x^{\frac{1}{b}} |\ln x|^{1+\frac{1}{b}}}, & \text{if } x \in (0, \mathrm{e}^{-1}); \\[2ex] \mathrm{e}^{\frac{1}{b}} + \dfrac{\mathrm{e}^{-\frac{1}{a}} - \mathrm{e}^{\frac{1}{b}}}{\mathrm{e} - \mathrm{e}^{-1}}(x - \mathrm{e}^{-1}), & \text{if } x \in (\mathrm{e}^{-1}, \mathrm{e}); \\[2ex] \dfrac{1}{x^{\frac{1}{a}} (\ln x)^{1+\frac{1}{a}}}, & \text{if } x \in [\mathrm{e}, \infty). \end{cases} \tag{3.10}$$

It is clear that f is continuous on $(0, \infty)$ and so it is measurable. Similar to **Case (i)**, we assume that the integral can be split as follows:

$$\begin{aligned} \varphi(p) &= \int_0^\infty |f(x)|^p\,\mathrm{d}x \\ &= \int_0^{\mathrm{e}^{-1}} \frac{\mathrm{d}x}{x^{\frac{p}{b}} |\ln x|^{p(1+\frac{1}{b})}} + \int_{\mathrm{e}^{-1}}^{\mathrm{e}} \Big[\mathrm{e}^{\frac{1}{b}} + \frac{\mathrm{e}^{-\frac{1}{a}} - \mathrm{e}^{\frac{1}{b}}}{\mathrm{e} - \mathrm{e}^{-1}}(x - \mathrm{e}^{-1})\Big]^p\,\mathrm{d}x \\ &\quad + \int_{\mathrm{e}}^\infty \frac{\mathrm{d}x}{x^{\frac{p}{a}} (\ln x)^{p(1+\frac{1}{a})}}. \end{aligned} \tag{3.11}$$

By Lemma 3.2, the first integral in the expression (3.11) converges if and only if (i) $\frac{p}{b} < 1$ or (ii) $\frac{p}{b} = 1$ and $p(1 + \frac{1}{b}) > 1$. The first condition gives $p < b$. If $p = b$, then $p(1 + \frac{1}{b}) = 1 + b > 1$ so that the second condition is actually $p = b$. In this case, we have $p \leq b$.

Similarly, we apply Lemma 3.2 to the third integral in the expression (3.11), so it converges if and only if (i) $\frac{p}{a} > 1$ or (ii) $\frac{p}{a} = 1$ and $p(1 + \frac{1}{a}) > 1$. The first condition

[a] The integrals in Lemma 3.2 are called Bertrand's integrals, see the following webpage `https://fr.wikipedia.org/wiki/Int%C3%A9grale_impropre`.

shows $p > a$. If $p = a$, then $p(1+\frac{1}{a}) = 1+a > 1$ so that the second condition implies that $p = a$. In this case, we have $p \geq a$.

Since the second integral in the expression (3.11) is finite, we combine these observations to conclude that $\varphi(p) < \infty$ if and only if $p \in [a, b]$.

- **Case (iii):** $E = \{a\}$**.** If we take $b = a$ in the definition (3.10), then we obtain $p \geq a$ and $p \leq a$. Hence we establish $E = \{a\}$.
- **Case (iv):** $E = \varnothing$**.** Consider $f \equiv 1$ on $(0, \infty)$. Since the integral

$$\int_0^\infty |1|^p \, \mathrm{d}x$$

is obviously divergent for every $p \in (0, \infty)$, we conclude that $E = \varnothing$ in this case.

(d) Let $r < p < s$. By the proof of part (a), we see that $p = \lambda r + (1-\lambda)s$ for some $\lambda \in (0, 1)$ so that the inequality (3.7) holds. In other words, we have

$$\|f\|_p^p \leq \left(\|f\|_r^r\right)^\lambda \times \left(\|f\|_s^s\right)^{1-\lambda}. \tag{3.12}$$

It is clear that

$$\begin{aligned}\left(\|f\|_r^r\right)^\lambda \times \left(\|f\|_s^s\right)^{1-\lambda} &\leq \begin{cases} \left(\|f\|_s^r\right)^\lambda \times \left(\|f\|_s^s\right)^{1-\lambda}, & \text{if } \|f\|_r \leq \|f\|_s; \\ \left(\|f\|_r^r\right)^\lambda \times \left(\|f\|_r^s\right)^{1-\lambda}, & \text{if } \|f\|_s \leq \|f\|_r \end{cases} \\ &= \begin{cases} \|f\|_s^p, & \text{if } \|f\|_r \leq \|f\|_s; \\ \|f\|_r^p, & \text{if } \|f\|_s \leq \|f\|_r. \end{cases}\end{aligned} \tag{3.13}$$

Hence, by combining the inequalities (3.12) and (3.13), we get

$$\|f\|_p \leq \max(\|f\|_r, \|f\|_s). \tag{3.14}$$

Let $f \in L^r(\mu) \cap L^s(\mu)$. By the definition, we have

$$\|f\|_r < \infty \quad \text{and} \quad \|f\|_s < \infty.$$

Hence we follow from these and the inequality (3.14) that $\|f\|_p < \infty$, i.e., $f \in L^p(\mu)$ and then $L^r(\mu) \cap L^s(\mu) \subseteq L^p(\mu)$.

(e) Recall that $\|f\|_\infty > 0$.

- **Case (i):** $\|f\|_\infty < \infty$**.** Let $E_\alpha = \{x \in X \mid |f(x)| \geq \alpha\}$, where $\alpha \in (0, \|f\|_\infty)$. It is clear that E_α is measurable because $|f|$ is measurable. By Definition 3.7, $\|f\|_\infty$ is the smallest number such that

$$\mu(\{x \in X \mid |f(x)| > \|f\|_\infty\}) = 0.$$

Thus we have $\mu(E_\alpha) > 0$ for every $\alpha \in (0, \|f\|_\infty)$. Assume that $\mu(E_\alpha) = \infty$. Then we have

$$\|f\|_r^r = \int_X |f|^r \, \mathrm{d}\mu = \int_{E_\alpha} |f|^r \, \mathrm{d}\mu + \int_{X \setminus E_\alpha} |f|^r \, \mathrm{d}\mu \geq \alpha^r \mu(E_\alpha) = \infty$$

which contradicts to our hypothesis. Thus we have $0 < \mu(E_\alpha) < \infty$ and then Proposition 1.24(b) implies that

$$\|f\|_p^p = \int_X |f|^p \, \mathrm{d}\mu \geq \int_{E_\alpha} |f|^p \, \mathrm{d}\mu \geq \alpha^p \mu(E_\alpha)$$

which shows that

$$\|f\|_p \geq \alpha[\mu(E_\alpha)]^{\frac{1}{p}}. \tag{3.15}$$

Since $\lim_{p\to\infty} [\mu(E_\alpha)]^{\frac{1}{p}} = 1$, we obtain from the inequality (3.15) that

$$\liminf_{p\to\infty} \|f\|_p \geq \alpha.$$

In particular, since α is arbitrary in $(0, \|f\|_\infty)$, we have

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty. \tag{3.16}$$

Next, if $p > r$, then we have

$$\|f\|_p^p = \int_X |f|^p \, d\mu = \int_X |f|^{p-r} \times |f|^r \, d\mu. \tag{3.17}$$

By Definition 3.7, since $|f(x)| \leq \|f\|_\infty$ for *almost all* $x \in X$, we get from the expression (3.17) and Proposition 1.24(b) that

$$\|f\|_p^p \leq \int_{X\setminus E} \|f\|_\infty^{p-r} \times |f|^r \, d\mu = \|f\|_\infty^{p-r} \times \int_{X\setminus E} |f|^r \, d\mu = \|f\|_\infty^{p-r} \times \|f\|_r^r, \tag{3.18}$$

where $E = \{x \in X \,|\, |f(x)| > \|f\|_\infty\}$ is of measure 0. Rewrite the inequality (3.18) in the form

$$\|f\|_p \leq \|f\|_\infty^{1-\frac{r}{p}} \|f\|_r^{\frac{r}{p}}$$

which, by using the fact that $\|f\|_r < \infty$, implies

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty. \tag{3.19}$$

Hence our desired result follows immediately from the inequalities (3.16) and (3.19).

- **Case (ii):** $\|f\|_\infty = \infty$. Then we deduce immediately from the inequality (3.16) that

$$\lim_{p\to\infty} \|f\|_p = \liminf_{p\to\infty} \|f\|_p = \infty = \|f\|_\infty.$$

Hence we have completed the proof of the problem. ■

Problem 3.5

Rudin Chapter 3 Exercise 5.

Proof.

(a) Suppose that $s = \infty$. Since $|f(x)| \leq \|f\|_\infty$ holds for almost all $x \in X$, we have

$$\mu(E) = \mu(\{x \in X \,|\, |f(x)| > \|f\|_\infty\}) = 0$$

so that

$$\|f\|_r^r = \int_{X\setminus E} |f|^r \, d\mu \leq \int_{X\setminus E} \|f\|_\infty^r \, d\mu = \|f\|_\infty^r \cdot \mu(X \setminus E) = \|f\|_\infty^r. \tag{3.20}$$

Hence we have $\|f\|_r \leq \|f\|_\infty$ in this case.

Next, we suppose that $s < \infty$. Since f is measurable, $|f|^r$ is also measurable on X by Proposition 1.9(b) and Theorem 1.7(b) for every $r > 0$. We apply Theorem 3.5 (Hölder's Inequality) to the measurable functions $|f|^r$ and 1 to obtain that

$$\begin{aligned}\int_X |f|^r \, \mathrm{d}\mu &\leq \Big\{ \int_X (|f|^r)^{\frac{s}{r}} \, \mathrm{d}\mu \Big\}^{\frac{r}{s}} \times \Big\{ \int_X \mathrm{d}\mu \Big\}^{1-\frac{r}{s}} \\ &= \Big(\int_X |f|^s \, \mathrm{d}\mu \Big)^{\frac{r}{s}} \\ &= \|f\|_s^r. \end{aligned} \tag{3.21}$$

Hence we conclude that $\|f\|_r \leq \|f\|_s$.

(b) Suppose that $0 < r < s < \infty$. Since $\|f\|_r = \|f\|_s$, it means that the equality holds in the inequality (3.21). Recall that the inequality (3.21) is derived from Theorem 3.5 (Hölder's Inequality), so there are constants α and β, not both 0, such that

$$\alpha(|f|^r)^{\frac{s}{r}} = \beta \quad \text{a.e. on } X.$$

If $\alpha = 0$, then it is trivial that $\beta = 0$, a contradiction. Thus $\alpha \neq 0$ and then we have

$$|f| = \Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{s}} \quad \text{a.e.}$$

In other words, $|f|$ is a constant a.e. on X.

Next, suppose that $s = \infty$ and $\|f\|_r = \|f\|_\infty < \infty$, where $0 < r < \infty$. Then the equality holds in the inequality (3.20) so that $|f|^r = \|f\|_\infty^r < \infty$ a.e. on X, i.e.,

$$|f| = \|f\|_\infty \quad \text{a.e. on } X.$$

In other words, we achieve that $|f|$ is a constant a.e. on X in this case.

Consequently, we conclude that $0 < r < s \leq \infty$ and the conditions $\|f\|_r = \|f\|_s < \infty$ hold if and only if $|f|$ is a constant a.e. on X

(c) If $f \in L^s(\mu)$, then $\|f\|_s < \infty$ and we know from part (a) that $\|f\|_r < \infty$, i.e., $f \in L^r(\mu)$. Thus it is true that $L^s(\mu) \subseteq L^r(\mu)$. Now the other inclusion $L^r(\mu) \subseteq L^s(\mu)$ is a consequence of the following lemma[b]:

Lemma 3.3

Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space and $\mathfrak{M}_0 = \{E \in \mathfrak{M} \,|\, \mu(E) > 0\}$. Then the following conditions are equivalent:

- **Condition (1).** $\inf\{\mu(E) \,|\, E \in \mathfrak{M}_0\} > 0$,
- **Condition (2).** $L^r(\mu) \subseteq L^s(\mu)$ for all $0 < r < s \leq \infty$.

[b]This lemma is part of the statements in [62, Theorem 1] and our proof follows part of the argument there.

Proof of Lemma 3.3. Suppose that **Condition (1)** is true. Suppose that $f \in L^r(\mu)$ and $E_n = \{x \in X \,|\, |f(x)| \geq n\}$ for every $n \in \mathbb{N}$. We claim that there exists an $N \in \mathbb{N}$ such that $\mu(E_N) = 0$. Otherwise, there exists $\epsilon > 0$ such that $\mu(E_n) \geq \epsilon$ for each $n \in \mathbb{N}$. Let $E = \bigcap_{n=1}^{\infty} E_n$. By the construction of E_n, we have $E_1 \supseteq E_2 \supseteq \cdots$. Since $\mu(E_1) \leq \mu(X) < \infty$, it yields from Theorem 1.19(e) that

$$\mu(E) = \lim_{n\to\infty} \mu(E_n) \geq \epsilon > 0. \tag{3.22}$$

However, we deduce from the inequality (3.22) that

$$\|f\|_r = \Big(\int_X |f|^r \,\mathrm{d}\mu\Big)^{\frac{1}{r}} \geq \Big(\int_E |f|^r \,\mathrm{d}\mu\Big)^{\frac{1}{r}} = \infty \cdot \mu(E)^{\frac{1}{r}} = \infty,$$

a contradiction. Therefore, we have our claim that $\mu(E_N) = 0$ for some $N \in \mathbb{N}$. In other words, $|f(x)| \leq N$ holds for *almost all* $x \in X$ and we obtain from Definition 3.7 that $\|f\|_\infty \leq N$, i.e., $f \in L^\infty(\mu)$ and so $f \in L^s(\mu)$. Hence **Condition (2)** holds.

Assume that **Condition (1)** was false. Thus there exists a sequence of $\{E_n\} \subseteq \mathfrak{M}_0$ such that

$$0 < \mu(E_n) < \frac{1}{2^n}. \tag{3.23}$$

Put $F_1 = E_1$ and $F_n = E_n \setminus (E_{n-1} \cup \cdots \cup E_1)$, where $n = 2, 3, \ldots$. Then it is easy to check that $F_i \cap F_j = \varnothing$ for $i \neq j$ and

$$0 < \mu(F_n) = \mu\big(E_n \setminus (E_{n-1} \cup \cdots \cup E_1)\big) \leq \mu(E_n) < \frac{1}{2^n}$$

for every $n = 1, 2, \ldots$. Therefore, we may assume that $\{E_n\}$ is a sequence of *disjoint* measurable sets.

Next, we define $\alpha_n = \mu(E_n)$ for $n = 1, 2, \ldots$ and $E = \bigcup_{n=1}^{\infty} E_n$. Suppose first that $s < \infty$. Then we define $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \alpha_n^{-\frac{1}{s}} \chi_{E_n}(x). \tag{3.24}$$

Since $r < s$, we have $1 - \frac{r}{s} < 1$. Note that $f(x) = 0$ if $x \in X \setminus E$. Thus it reduces from the bounds (3.23) and Theorem 1.29 that

$$\|f\|_r^r = \int_{X\setminus E} |f|^r \,\mathrm{d}\mu + \int_E |f|^r \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{E_n} \alpha_n^{-\frac{r}{s}} \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \alpha_n^{1-\frac{r}{s}} < \frac{2^{1-\frac{r}{s}}}{1 - 2^{1-\frac{r}{s}}} < \infty$$

which means $f \in L^r(\mu)$. Similarly, it is easy to derive that

$$\|f\|_s^s = \int_E |f|^s \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{E_n} \alpha_n^{-1} \,\mathrm{d}\mu = \sum_{n=1}^{\infty} 1 = \infty$$

which implies that $f \notin L^s(\mu)$. Hence we have shown that

$$L^r(\mu) \nsubseteq L^s(\mu).$$

If $s = \infty$, then we replace the coefficients $\alpha_n^{-\frac{1}{s}}$ by $\alpha_n^{-\frac{1}{2r}}$ in the expression (3.24). By similar argument, since $\alpha_n^{-\frac{1}{2r}} > 2^{\frac{n}{2r}}$ for all $n \in \mathbb{N}$, we conclude $\|f\|_\infty = \infty$ and then $L^r(\mu) \nsubseteq L^\infty(\mu)$. ∎

(d) If $0 < p < q$, then part (a) says that $\|f\|_p \leq \|f\|_q < \infty$. Let $E_0 = \{x \in X \,|\, |f(x)| = 0\}$. By Problem 1.5, E_0 is measurable.

- **Case (i): $\mu(E_0) > 0$.** Now the set $E_\infty = \{x \in X \,|\, |f(x)| = \infty\}$ is measurable by Problem 1.5. Pick $p \in (0, r)$, so part (a) indicates that $\|f\|_p \leq \|f\|_r < \infty$. If $\mu(E_\infty) > 0$, then we have

$$\|f\|_p = \Big\{\int_X |f|^p \, d\mu\Big\}^{\frac{1}{p}} \geq \Big\{\int_{E_\infty} |f|^p \, d\mu\Big\}^{\frac{1}{p}} = [\infty \cdot \mu(E_\infty)]^{\frac{1}{p}} = \infty,$$

a contradiction. Thus we have $\mu(E_\infty) = 0$, i.e., $|f(x)| < \infty$ a.e. on X. Consequently, we have

$$\int_X \log|f| \, d\mu = \int_{X\setminus E_\infty} \log|f| \, d\mu = \int_{X\setminus(E_\infty \cup E_0)} \log|f| \, d\mu + \int_{E_0} \log|f| \, d\mu \tag{3.25}$$

Since $\mu(X) = 1$ and $|f(x)|$ is nonzero finite on $X \setminus (E_\infty \cup E_0)$, the first integral on the right-hand side of the equation (3.25) is finite. However, the second integral on the right-hand side of the equation (3.25) is in the form $-\infty$ so that the right-hand side of our desired result is in the form $\exp\{-\infty\}$ which is defined to be 0. On the other hand, we notice that

$$\|f\|_p = \Big\{\int_X |f|^p \, d\mu\Big\}^{\frac{1}{p}} = \Big\{\int_X |f|^p \chi_{X\setminus E_0} \, d\mu\Big\}^{\frac{1}{p}}. \tag{3.26}$$

Apply Theorem 3.5 (Hölder's Inequality) with $u > 1$ and $v > 1$ as fixed conjugate exponents to the right-hand side of the expression (3.26), we see that

$$\begin{aligned}\Big\{\int_X |f|^p \chi_{X\setminus E_0} \, d\mu\Big\}^{\frac{1}{p}} &\leq \Big\{\int_X (|f|^p)^u \, d\mu\Big\}^{\frac{1}{pu}} \Big\{\int_X (\chi_{X\setminus E_0})^v \, d\mu\Big\}^{\frac{1}{pv}} \\ &= \mu(X \setminus E_0)^{\frac{1}{pv}} \|f\|_{pu}. \end{aligned} \tag{3.27}$$

Combining the expression (3.26) and the inequality (3.27), we gain

$$\|f\|_p \leq \mu(X \setminus E_0)^{\frac{1}{pv}} \|f\|_{pu}.$$

Since $\mu(X) = 1$ and $\mu(E_0) > 0$, we have $\mu(X \setminus E_0) < 1$ and then

$$\lim_{p\to 0} \mu(X \setminus E_0)^{\frac{1}{pv}} = 0. \tag{3.28}$$

In addition, when p is chosen to be small enough such that $pu < r$, part (a) implies that $\|f\|_{pu} \leq \|f\|_r < \infty$. Hence we observe from the limit (3.28) and this fact that

$$\lim_{p\to 0} \|f\|_p \leq \lim_{p\to 0} \mu(X \setminus E_0)^{\frac{1}{pv}} \|f\|_{pu} \leq \|f\|_r \lim_{p\to 0} \mu(X \setminus E_0)^{\frac{1}{pv}} = 0.$$

In other words, the desired result holds in this case.

- **Case (ii): $\mu(E_0) = 0$.** Then $|f(x)| > 0$ a.e. on X. By the fact used in **Case (i)**, we may assume that

$$0 < |f(x)| < \infty \qquad \text{a.e. on } X. \tag{3.29}$$

This fact allows us to apply [51, Eqn. (7), p. 63] to the *almost* positive function $|f|^p$ to obtain

$$\|f\|_p = \Big\{\int_X |f|^p \, d\mu\Big\}^{\frac{1}{p}} \geq \exp\Big\{\frac{1}{p}\int_X \log|f|^p \, d\mu\Big\} = \exp\Big\{\int_X \log|f| \, d\mu\Big\}$$

for every $p \in (0, r)$. In particular,

$$\lim_{p \to 0^+} \|f\|_p \geq \exp\Big\{\int_X \log|f|\,\mathrm{d}\mu\Big\}. \tag{3.30}$$

For the reverse direction, we consider the function $\phi : (0, \infty) \to \mathbb{R}$ defined by

$$\phi(x) = x - 1 - \log x.$$

Since $\phi'(x) = 1 - \frac{1}{x} = 0$ and $\phi''(x) = \frac{1}{x^2}$, ϕ has the absolute minimum at $x = 1$, i.e., $\phi(x) \geq \phi(1)$ on $(0, \infty)$ or equivalently

$$\log x \leq x - 1 \tag{3.31}$$

on $(0, \infty)$. Recall the fact (3.29), so we replace x by $\|f\|_p^p$ in the inequality (3.31) and then using the fact that $\mu(X) = 1$ to obtain

$$\log \|f\|_p \leq \frac{\|f\|_p^p - 1}{p} = \frac{1}{p}\Big(\int_X |f|^p\,\mathrm{d}\mu - \int_X \mathrm{d}\mu\Big) = \int_X \frac{|f|^p - 1}{p}\,\mathrm{d}\mu. \tag{3.32}$$

Let $E = \{x \in X \,|\, |f(x)| = 1\}$ and $F = \{x \in X \,|\, |f(x)| \neq 1\}$. Then we get

$$\int_X \frac{|f|^p - 1}{p}\,\mathrm{d}\mu = \int_E \frac{|f|^p - 1}{p}\,\mathrm{d}\mu + \int_F \frac{|f|^p - 1}{p}\,\mathrm{d}\mu. \tag{3.33}$$

It is clear from Proposition 1.24(d) that

$$\int_E \frac{|f|^p - 1}{p}\,\mathrm{d}\mu = 0,$$

so it suffices to evaluate

$$\lim_{p \to 0+} \int_F \frac{|f|^p - 1}{p}\,\mathrm{d}\mu.$$

To this end, we need the following lemma:

Lemma 3.4

Let $|f| > 0$ and $\|f\|_r < \infty$ for some $r > 0$. Define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(p) = |f|^p$. Then we have

$$\frac{|f|^p - 1}{p} \in L^1(\mu)$$

for all $p \in (0, r)$.

Proof of Lemma 3.4. Since $\psi'' = |f|^p(\log|f|)^2 > 0$ on $\mathbb{R}$, ψ is convex on $\mathbb{R}$. Thus we see from [49, Exercise 23, p. 101] that, for $0 < p < r$,

$$\frac{\psi(p) - \psi(0)}{p - 0} \leq \frac{\psi(r) - \psi(0)}{r - 0}$$

which reduces to

$$\frac{|f|^p - 1}{p} \leq \frac{|f|^r - 1}{r}$$

and so

$$\int_X \frac{|f|^p - 1}{p}\,\mathrm{d}\mu \leq \int_X \frac{|f|^r - 1}{r}\,\mathrm{d}\mu = \frac{1}{r}\|f\|_r^r - \frac{1}{r} < \infty.$$

In other words, we have $\frac{|f|^p - 1}{p} \in L^1(\mu)$ for all $p \in (0, r)$, completing the proof of the lemma. ∎

Let's return to the proof of the problem. By Theorem 1.34 (Lebesgue's Dominated Convergence Theorem), we obtain that

$$\lim_{p\to 0+}\int_F \frac{|f|^p-1}{p}\,\mathrm{d}\mu = \int_F \lim_{p\to 0+}\Big(\frac{|f|^p-1}{p}\Big)\,\mathrm{d}\mu. \tag{3.34}$$

By L' Hôspital Rule, we derive

$$\lim_{p\to 0+}\frac{|f|^p-1}{p} = \lim_{p\to 0+}|f|^p\log|f| = \log|f| \tag{3.35}$$

on X. Therefore, when we combine the results (3.32), (3.33), (3.34) and (3.35), we establish that

$$\lim_{p\to 0+}\log\|f\|_p \leq \int_F \lim_{p\to 0+}\Big(\frac{|f|^p-1}{p}\Big)\,\mathrm{d}\mu = \int_F \log|f|\,\mathrm{d}\mu = \int_X \log|f|\,\mathrm{d}\mu. \tag{3.36}$$

Since $\log x$ is continuous for $x > 0$, we get from the inequality (3.36) that

$$\lim_{p\to 0+}\|f\|_p \leq \exp\Big\{\int_X \log|f|\,\mathrm{d}\mu\Big\}. \tag{3.37}$$

Hence the required result follows immediately from the inequalities (3.30) and (3.37).

This ends the proof of the problem. ■

Problem 3.6

Rudin Chapter 3 Exercise 6.

Proof. Let $x > 0$ and $0 \leq c \leq 1$. We claim that the function $\Phi : [0,\infty) \to \mathbb{R}$ satisfies the relation

$$c\Phi(x) + (1-c)\Phi(1) = \Phi(x^c). \tag{3.38}$$

Obviously, the relation (3.38) holds for $c = 0$ and $c = 1$. Without loss of generality, we may assume that $0 < c < 1$ in the following discussion.

Since f is bounded measurable and positive on $[0,1]$, there exists a positive constant M such that

$$\|f\|_1 = \int_0^1 |f|\,\mathrm{d}x \leq M < \infty.$$

Thus we get from Problem 3.5(d) that

$$\lim_{p\to 0}\|f\|_p = \exp\Big\{\int_0^1 \log|f(t)|\,\mathrm{d}t\Big\}. \tag{3.39}$$

By putting the expression (3.39) into the relation in question, we obtain

$$\Phi\Big(\exp\Big\{\int_0^1 \log|f(t)|\,\mathrm{d}t\Big\}\Big) = \int_0^1 \Phi\big(f(t)\big)\,\mathrm{d}t. \tag{3.40}$$

We define $f : [0,1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} x, & \text{if } x \in [0,c]; \\ 1, & \text{if } x \in (c,1]. \end{cases} \tag{3.41}$$

Since $x > 0$, the f is bounded measurable and positive on $[0, 1]$. Therefore, we substitute the function (3.41) into the relation (3.40) to yield

$$\begin{aligned}\Phi(x^c) &= \Phi\big(\exp(c\log x)\big)\\ &= \Phi\Big(\exp\Big(\int_0^c \log x\,\mathrm{d}t\Big)\Big)\\ &= \int_0^c \Phi(x)\,\mathrm{d}t + \int_c^1 \Phi(1)\,\mathrm{d}t\\ &= c\Phi(x) + (1-c)\Phi(1).\end{aligned}$$

This proves the claim (3.38).

Next, we prove a lemma which is useful of determining the explicit form of the function Φ:

Lemma 3.5

The relation (3.38) holds for all $x > 0$ and $c \geq 0$.

Proof of Lemma 3.5. The case $0 \leq c \leq 1$ is clear by the above analysis. Let $c > 1$ so that $0 < \frac{1}{c} < 1$. Then the relation (3.38) becomes

$$\Phi\big(x^{\frac{1}{c}}\big) = \frac{1}{c}\Phi(x) + \Big(1 - \frac{1}{c}\Big)\Phi(1). \tag{3.42}$$

Take $y = x^{\frac{1}{c}}$. Since $\frac{1}{c} > 0$, y will take all positive real numbers. Therefore, it follows from the expression (3.42) that

$$\Phi(y) = \frac{1}{c}\Phi(y^c) + \Big(1 - \frac{1}{c}\Big)\Phi(1)$$

and after simplification, we have

$$\Phi(y^c) = c\Phi(y) + (1-c)\Phi(1).$$

This proves the lemma. ■

We return to the proof of the problem. Define $\Psi : [0,\infty) \to \mathbb{R}$ by $\Psi(x) = \Phi(x) - \Phi(1)$, then the relation (3.38) can be rewritten as

$$\Psi(x^c) = c\Psi(x), \tag{3.43}$$

where $x > 0$ and $c \geq 0$. Take $x = \mathrm{e}$ and $c = \ln y$, where $y \geq 1$. Then the expression (3.43) becomes

$$\Psi(y) = \Psi(\mathrm{e})\ln y, \tag{3.44}$$

where $y \geq 1$. For $0 < y < 1$, we put $x = \mathrm{e}^{-1}$ and $c = \ln(y^{-1}) > 0$ into the expression (3.43) so that

$$\Psi(y) = \Psi\big(\mathrm{e}^{-\ln(y^{-1})}\big) = -\Psi(\mathrm{e}^{-1})\ln y. \tag{3.45}$$

Combining the results (3.44) and (3.45), we establish that

$$\Psi(x) = \begin{cases} \Psi(\mathrm{e})\ln x, & \text{if } x \geq 1;\\ -\Psi(\mathrm{e}^{-1})\ln x, & \text{if } 0 < x < 1\end{cases}$$

which imply that

$$\Phi(x) = \begin{cases} \Phi(1) + [\Phi(\mathrm{e}) - \Phi(1)] \ln x, & \text{if } x \geq 1; \\ \\ \Phi(1) - [\Phi(\mathrm{e}^{-1}) - \Phi(1)] \ln x, & \text{if } 0 < x < 1, \end{cases}$$

completing the proof of the problem. ■

Problem 3.7

Rudin Chapter 3 Exercise 7.

Proof. In this problem, we assume that μ is a positive measure. We give examples one by one:

Example 3.1. Let $X = \mathbb{N}$ with the counting measure μ. By Definition 3.6, we have

$$\ell^p(\mathbb{N}) = \Big\{ x = \{\xi_n\} \,\Big|\, \|x\|_p = \Big(\sum_{n=1}^{\infty} |\xi_n|^p \Big)^{\frac{1}{p}} < \infty \Big\}.$$

Let $r < s$. Thus if $\{\xi_n\} \in \ell^r(\mathbb{N})$, then $\{|\xi_n|^r\}$ is a convergent real sequence so that there exists a positive integer N such that $|\xi_n| \leq 1$ for all $n \geq N$, see [49, Theorem 3.23, p. 60]. Therefore, we must have $|\xi_n|^s \leq |\xi_n|^r$ for all $n \geq N$ and this implies that

$$\sum_{n=N}^{\infty} |\xi_n|^s \leq \sum_{n=N}^{\infty} |\xi_n|^r < \infty.$$

In other words, we prove that $\{\xi_n\} \in \ell^s(\mathbb{N})$ and then $\ell^r(\mathbb{N}) \subseteq \ell^s(\mathbb{N})$.

Example 3.2. We consider the Lebesgue measure m on $X = [0,1]$. Since $m(X) = 1$, Problem 3.5(c) shows that $L^s([0,1]) \subseteq L^r([0,1])$ if $0 < r < s$.

Example 3.3. Consider $X = (0,\infty)$ and the Lebesgue measure m. Let $1 \leq r < s \leq \infty$. Take $\frac{1}{s} < \alpha < \frac{1}{r}$ and define $f : (0,\infty) \to \mathbb{R}$ by

$$f(x) = x^{-\alpha} \chi_{(0,1)}(x).$$

Then we have

$$\Big\{ \int_0^{\infty} |f|^r \,\mathrm{d}x \Big\}^{\frac{1}{r}} = \Big\{ \int_0^{1} x^{-\alpha r} \,\mathrm{d}x \Big\}^{\frac{1}{r}} \quad \text{and} \quad \Big\{ \int_0^{\infty} |f|^s \,\mathrm{d}x \Big\}^{\frac{1}{s}} = \Big\{ \int_0^{1} x^{-\alpha s} \,\mathrm{d}x \Big\}^{\frac{1}{s}}. \tag{3.46}$$

By Lemma 3.1, the integrals on the right-hand sides of the two expressions (3.46) converge if and only if $\alpha r < 1$ and $\alpha s < 1$ respectively. Since $\frac{1}{s} < \alpha < \frac{1}{r}$, we conclude that $f \in L^r((0,\infty))$ but $f \notin L^s((0,\infty))$, i.e., $L^r((0,\infty)) \nsubseteq L^s((0,\infty))$.

Similarly, we define $g : (0,\infty) \to \mathbb{R}$ by

$$g(x) = x^{-\alpha} \chi_{(1,\infty)}(x).$$

Then we have

$$\Big\{ \int_0^{\infty} |f|^r \,\mathrm{d}x \Big\}^{\frac{1}{r}} = \Big\{ \int_1^{\infty} x^{-\alpha r} \,\mathrm{d}x \Big\}^{\frac{1}{r}} \quad \text{and} \quad \Big\{ \int_0^{\infty} |f|^s \,\mathrm{d}x \Big\}^{\frac{1}{s}} = \Big\{ \int_1^{\infty} x^{-\alpha s} \,\mathrm{d}x \Big\}^{\frac{1}{s}}.$$

Thus Lemma 3.1 again shows that $g \in L^s((0,\infty))$ but $g \notin L^r((0,\infty))$. In other words, it means that $L^s((0,\infty)) \nsubseteq L^r((0,\infty))$.

Next, we are going to find conditions on μ under which these situations will occur:

- $L^r(\mu) \subseteq L^s(\mu)$ **for** $0 < r < s \leq \infty$**.** Recall from Lemma 3.3 that it is equivalent to the condition that $\inf\{\mu(E) \,|\, E \in \mathfrak{M}_0\} > 0$, where $\mathfrak{M}_0 = \{E \in \mathfrak{M} \,|\, \mu(E) > 0\}$.
- $L^s(\mu) \subseteq L^r(\mu)$ **for** $0 < r < s < \infty$**.** Similar to the proof of Lemma 3.3, the following result is part of the statements in [62, Theorem 2]:

Lemma 3.6

Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space and $\mathfrak{M}_\infty = \{E \in \mathfrak{M} \,|\, \mu(E) \text{ is finite}\}$. Then the following conditions are equivalent:

- **Condition (1).** $\sup\{\mu(E) \,|\, E \in \mathfrak{M}_\infty\} < \infty$,
- **Condition (2).** $L^s(\mu) \subseteq L^r(\mu)$ for all $0 < r < s < \infty$.

Proof of Lemma 3.6. Suppose that **Condition (1)** holds. To begin with, let $f \in L^s(\mu)$ and $E_n = \{x \in X \,|\, \frac{1}{n+1} \leq |f(x)| < \frac{1}{n}\}$ for each $n = 1, 2, \ldots$. Then it is clear that $E_n \in \mathfrak{M}$. Now for each *fixed* n, we see that

$$\int_X |f|^s \, \mathrm{d}\mu \geq \int_{E_n} |f|^s \, \mathrm{d}\mu \geq \frac{1}{(n+1)^s} \int_{E_n} \mathrm{d}\mu = \frac{\mu(E_n)}{(n+1)^s}$$

which implies that

$$\mu(E_n) \leq (n+1)^s \int_X |f|^s \, \mathrm{d}\mu = (n+1)^s \|f\|_s^s < \infty.$$

Therefore, we conclude that $E_n \in \mathfrak{M}_\infty$. Note that $E_i \cap E_j = \varnothing$ if $i \neq j$ and μ is a positive measure, if $F_n = E_1 \cup E_2 \cup \cdots \cup E_n$, then we deduce from Theorem 1.19(b) that

$$\mu(F_n) = \mu\Big(\bigcup_{k=1}^n E_k\Big) = \sum_{k=1}^n \mu(E_k) < \infty.$$

In other words, each F_n is an element of $\mathfrak{M}_\infty$. Next, we know that $F_1 \subseteq F_2 \subseteq \cdots$ and $\bigcup_{k=1}^n F_k = \bigcup_{k=1}^n E_k$, so Theorem 1.19(d) implies that

$$\lim_{n\to\infty} \mu(F_n) = \mu\Big(\bigcup_{n=1}^\infty F_n\Big) = \mu\Big(\bigcup_{n=1}^\infty E_n\Big) = \sum_{n=1}^\infty \mu(E_n). \tag{3.47}$$

Given a positive integer n, it is clear that

$$\mu(F_n) \leq \sup\{\mu(F_k) \,|\, k \in \mathbb{N}\} \leq \sup\{\mu(E) \,|\, E \in \mathfrak{M}_\infty\}. \tag{3.48}$$

Furthermore, by combining **Condition (1)**, the expression (3.47) and the inequality (3.48), it yields that

$$\sum_{n=1}^\infty \mu(E_n) = \lim_{n\to\infty} \mu(F_n) \leq \sup\{\mu(E) \,|\, E \in \mathfrak{M}_\infty\} < \infty.$$

On $X \setminus \bigcup_{n=1}^{\infty} E_n$, we claim that $|f(x)| \geq 1$. Otherwise, there is a $x_0 \in X \setminus \bigcup_{n=1}^{\infty} E_n$ such that $|f(x_0)| < 1$. By the definition, $x_0 \in E_N$ for some $N \in \mathbb{N}$, a contradiction. This observation implies that

$$|f(x)|^r \leq |f(x)|^s$$

on $X \setminus \bigcup_{n=1}^{\infty} E_n$. Finally, we have

$$\int_X |f|^r \,\mathrm{d}\mu = \int_{X\setminus\bigcup_{n=1}^{\infty} E_n} |f|^r \,\mathrm{d}\mu + \sum_{n=1}^{\infty}\int_{E_n} |f|^r \,\mathrm{d}\mu \leq \int_X |f|^s \,\mathrm{d}\mu + \sum_{n=1}^{\infty}\frac{\mu(E_n)}{n^r} < \infty.$$

This proves that **Condition (2)** holds.

Conversely, since $L^s(\mu) \subseteq L^r(\mu)$ implies that $L^{\alpha s}(\mu) \subseteq L^{\alpha r}(\mu)$ for every $\alpha \in (0,\infty)$, we may assume that $r \geq 1$. It is known that if $r, s \in [1,\infty]$ and $L^s(\mu) \subseteq L^r(\mu)$, then the mapping

$$T : L^s(\mu) \to L^r(\mu)$$

is continuous, see [62, Lemma 1]. This result guarantees the existence of a positive constant k such that $\|f\|_r \leq k\|f\|_s$ for all $f \in L^s(\mu)$, so

$$\mu(E) \leq k^{\frac{rs}{s-r}}$$

for every $E \in \mathfrak{M}_\infty$ and this implies **Condition (1)**. ■

We have completed the proof of the problem. ■

Problem 3.8

Rudin Chapter 3 Exercise 8.

Proof. For every $n \in \mathbb{N}$, since $g(x) \to \infty$ as $x \to 0$, there exists a sequence $\{\epsilon_n\}$ of positive numbers such that

$$g(x) \geq n \tag{3.49}$$

for every $x \in (0, \epsilon_n)$. Without loss of generality, we may assume that $\{\epsilon_n\}$ is decreasing and

$$\epsilon_{n+1} < \frac{\epsilon_n}{2} \tag{3.50}$$

for every $n = 1, 2, \ldots$. Now we define $h_n : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ by

$$h_n(x) = n\Big(1 - \frac{x}{\epsilon_n}\Big)\chi_{(0,\epsilon_n)}(x) \quad \text{and} \quad h(x) = \sup_n h_n(x)$$

respectively.

Let $x \in (0,1)$ be *fixed* but arbitrary. Since $\epsilon_n \to 0$ as $n \to \infty$, there exists a positive integer N_x such that $x \leq \epsilon_n$ for all $1 \leq n \leq N_x$ and $x \geq \epsilon_n$ for all $n > N_x$. Thus we have $x \in (0, \epsilon_n)$ for all $1 \leq n \leq N_x$ and $x \notin (0, \epsilon_n)$ for all $n > N_x$. See Figure 3.1 for an illustration of the distribution of x and ϵ_n.

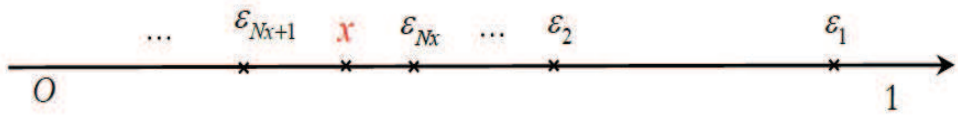


Figure 3.1: The distribution of x and ϵ_n.

These facts imply that

$$h_n(x) = \begin{cases} n\Big(1 - \dfrac{x}{\epsilon_n}\Big), & \text{if } 1 \leq n \leq N_x; \\ \\ 0, & \text{if } n > N_x \end{cases} \tag{3.51}$$

so that

$$h(x) = \sup_n h_n(x) = \sup\{h_1(x), h_2(x), \ldots\} = \max_{1\leq n\leq N_x} n\Big(1 - \frac{x}{\epsilon_n}\Big) < \infty,$$

i.e., h is finite. Since each h_n is convex on $(0,1)$, Problem 3.1 shows that the h is also convex on $(0,1)$.

Next, we want to compare the magnitudes of $h_n(x)$ and $g(x)$ at the fixed point $x \in (0,1)$. On the one hand, for $1 \leq n \leq N_x$, we have $x \in (0, \epsilon_n)$, so $n(1 - \frac{x}{\epsilon_n}) \leq n$ and we deduce from the inequality (3.49) and the definition (3.51) that

$$h_n(x) \leq g(x) \tag{3.52}$$

for $1 \leq n \leq N_x$. On the other hand, if $n > N_x$, then $x \notin (0, \epsilon_n)$ and we have $h_n(x) = 0$, but the inequality (3.49) still holds for $n = N_x$ (i.e., $x \in (0, \epsilon_{N_x})$) so that the inequality (3.52) remains valid in this case. Hence we have established that

$$h(x) \leq g(x) \tag{3.53}$$

for the fixed point $x \in (0,1)$. Since x is arbitrary, the inequality (3.53) holds on $(0,1)$.

Take $n = N_x - 1$ in the definition (3.51) to get

$$h_{N_x-1}(x) = (N_x - 1)\Big(1 - \frac{x}{\epsilon_{N_x-1}}\Big). \tag{3.54}$$

By the hypothesis (3.50), we have $2\epsilon_{N_x} < \epsilon_{N_x-1}$. Then since $x < \epsilon_{N_x}$, we have

$$\frac{x}{\epsilon_{N_x-1}} < \frac{x}{2\epsilon_{N_x}} < \frac{1}{2}. \tag{3.55}$$

By substituting the estimate (3.55) into the expression (3.54), we obtain

$$h_{N_x-1}(x) > \frac{N_x - 1}{2}.$$

Since $x \to 0$ if and only if $N_x \to \infty$, we conclude that $h_{N_x}(x) \to \infty$ as $x \to 0$ and the definition implies that $h(x) \to \infty$ as $x \to 0$ as required.

The second assertion is false. To see this, consider $g(x) = x^{\frac{1}{3}}$ which satisfies $g(x) \to \infty$ as $x \to \infty$. Assume that there was a convex function $h : (0,\infty) \to \mathbb{R}$ such that

$$h(x) \leq x^{\frac{1}{3}} \tag{3.56}$$

on $(0,\infty)$ and $h(x) \to \infty$ as $x \to \infty$. We know from [66, Exercise 6(c), p. 262][c] that if $h:(0,\infty)\to\mathbb{R}$ is convex on $(0,\infty)$, then the ratio

$$\frac{h(x)}{x}$$

tends to a finite limit or to infinity as $x \to \infty$. If this limit is finite, since $h(x)\to\infty$ as $x\to\infty$, this limit must be positive. Thus it must be true that

$$h(x) \geq kx \tag{3.57}$$

as $x \to \infty$ for some positive constant k. However, the inequality (3.57) definitely contradicts the inequality (3.56). This completes the proof of the problem. ∎

Problem 3.9

Rudin Chapter 3 Exercise 9.

Proof. Let $\Phi:(0,\infty)\to(0,\infty)$ be such that $\Phi(p)\to\infty$ as $p\to\infty$. Suppose that $E_1=(0,\frac{1}{2})$ and

$$E_n = \Big[\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}, \frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\Big) = \Big[1-\frac{1}{2^{n-1}}, 1-\frac{1}{2^n}\Big),$$

where $n\geq 2$. Obviously, we have $E_n\subset(0,1)$ for every $n\in\mathbb{N}$ and $E_i\cap E_j=\varnothing$ for $i\neq j$. Furthermore, each E_n is measurable and

$$m(E_n)=\frac{1}{2^n}>0.$$

Suppose that $x\in(0,1)$. If $x<\frac{1}{2}$, then $x\in E_1$. Otherwise, $\frac{1}{2}\leq x<1$ and there exists a $\delta>0$ such that $x<\delta<1$. Take N to be *least* positive integer such that

$$N > -\frac{\log(1-\delta)}{\log 2} > 0.$$

Then we have $x<1-\frac{1}{2^N}<1$ so that $x\in E_2\cup E_3\cup\cdots\cup E_N$. In other words, we get

$$(0,1)=\bigcup_{n=1}^{\infty} E_n.$$

Finally, we define $f_n, f:(0,1)\to(0,\infty)$ by

$$f_n(x)=\sum_{k=1}^{n} k\chi_{E_k}(x) \quad\text{and}\quad f(x)=\lim_{n\to\infty} f_n(x)=\sum_{n=1}^{\infty} n\chi_{E_n}(x) \tag{3.58}$$

respectively. Since each E_n is measurable, each χ_{E_n} is measurable by Proposition 1.9(d). Thus each function f_n is measurable and then Theorem 1.14 implies that f is also measurable. Now it remains to show that the function f satisfies the required properties of the problem.

- **Property 1: $\|f\|_p\to\infty$.** Assume that $f\in L^\infty((0,1))$. Then there exists a positive constant M such that $\|f\|_\infty<M$. By Definition 3.7, we see that

$$|f(x)|<M \quad \text{a.e. on } (0,1). \tag{3.59}$$

[c] See also [31, Theorem 126, p. 99] and [51, Eqn. (2), p. 62].

Since $E_i \cap E_j = \varnothing$ for $i \neq j$, we have

$$|f(x)| = f(x) \geq f_n(x) \geq n \tag{3.60}$$

for every $x \in E_n$. Since $n \to \infty$, there exists a positive integer N such that $N \geq M$. Recall that $m(E_N) = \frac{1}{2^N} > 0$, so we obtain from the inequality (3.60) that

$$|f(x)| \geq M$$

on E_N, but this contradicts our assumption (3.59). Therefore, we must have $f \notin L^\infty((0,1))$ and in fact $\|f\|_\infty = \infty$. Furthermore, it follows from the Ratio Test (see [49, Theorem 3.34]) that

$$\|f\|_1 = \int_0^1 f(x)\,\mathrm{d}x = \sum_{n=1}^\infty n \times m(E_n) = \sum_{n=1}^\infty \frac{n}{2^n} < \infty,$$

so we have $\|f\|_1 < \infty$. Hence we deduce from Problem 3.4(e) that

$$\|f\|_p \to \|f\|_\infty = \infty$$

as $p \to \infty$.

- **Property 2: $\|f\|_p \leq \Phi(p)$ for sufficiently large p.** Fix a positive integer n, we consider

$$\Big[\frac{n}{\Phi(p)}\Big]^p$$

for $p > 0$. Since $\Phi(p) \to \infty$ as $p \to \infty$, when p is sufficiently large, we have $\frac{n}{\Phi(p)} \leq 1$ so that

$$\Big[\frac{n}{\Phi(p)}\Big]^p \times m(E_n) \leq \frac{1}{2^n}. \tag{3.61}$$

Therefore, we see that

$$\|f\|_p^p = \int_0^1 \Big[\sum_{n=1}^\infty n\chi_{E_n}(x)\Big]^p\,\mathrm{d}x = \sum_{n=1}^\infty n^p \times m(E_n). \tag{3.62}$$

Consequently, we follow from the inequality (3.61) and the expression (3.62) that

$$\frac{\|f\|_p^p}{[\Phi(p)]^p} = \frac{1}{[\Phi(p)]^p}\sum_{n=1}^\infty n^p \times m(E_n) = \sum_{n=1}^\infty \Big[\frac{n}{\Phi(p)}\Big]^p \times m(E_n) \leq \sum_{n=1}^\infty \frac{1}{2^n} = 1$$

for sufficiently large p, or equivalently,

$$\|f\|_p \leq \Phi(p)$$

for sufficiently large p.

We end the proof of the problem. ■

3.3 Applications of Theorems 3.3, 3.5, 3.8, 3.9 and 3.12

Problem 3.10

Rudin Chapter 3 Exercise 10.

Proof. First of all, we have to show that $f \in L^p(\mu)$. Recall that $L^p(\mu)$ is a metric space with metric $\|\cdot\|$. By the triangle inequality

$$\|f_n - f_m\|_p \leq \|f_n - f\|_p + \|f - f_m\|_p$$

and the hypothesis $\|f_n - f\|_p \to 0$ as $n \to \infty$ together imply that $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$. Now the completeness of $L^p(\mu)$ ensures that $\{f_n\}$ converges to an element of $L^p(\mu)$. By the uniqueness of limits, this element *must be* f so that $f \in L^p(\mu)$. By Theorem 3.12, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. on X. Since $f_n \to g$ a.e. on X as $n \to \infty$, every subsequence of $\{f_n\}$ converges to g a.e. on X. In particular, we take this subsequence $\{f_{n_k}\}$ and the uniqueness of limits again imply that

$$f = g$$

a.e. on X. We have completed the proof of the problem. ■

Problem 3.11

Rudin Chapter 3 Exercise 11.

Proof. Since $f, g : \Omega \to [0, \infty]$, the hypothesis $fg \geq 1$ implies that $\sqrt{fg} \geq 1$. By Theorem 3.5 (Hölder's Inequality), we see that

$$\Big(\int_\Omega \sqrt{fg}\,\mathrm{d}\mu\Big)^2 \leq \int_\Omega f\,\mathrm{d}\mu \times \int_\Omega g\,\mathrm{d}\mu. \tag{3.63}$$

By Proposition 1.24(a), we have

$$\int_\Omega \sqrt{fg}\,\mathrm{d}\mu \geq \int_\Omega \mathrm{d}\mu = \mu(\Omega) = 1. \tag{3.64}$$

Now our desired result follows immediately by combining the inequalities (3.63) and (3.64). This completes the analysis of the problem. ■

Problem 3.12

Rudin Chapter 3 Exercise 12.

Proof. Suppose that $A = \infty$. Since $h \leq \sqrt{1+h^2}$ on Ω, we have

$$\int_\Omega \sqrt{1+h^2}\,\mathrm{d}\mu = \infty,$$

so the equalities definitely hold in this case. Next, suppose that $A = 0$. By the fact that $1 \leq \sqrt{1+h^2} \leq \sqrt{1+2h+h^2} = 1+h$, we achieve that

$$\int_\Omega \mathrm{d}\mu \leq \int_\Omega \sqrt{1+h^2}\,\mathrm{d}\mu \leq \int_\Omega (1+h)\,\mathrm{d}\mu = \int_\Omega \mathrm{d}\mu. \tag{3.65}$$

Since $\mu(\Omega) = 1$, we conclude from the inequalities (3.65) that

$$\int_\Omega \sqrt{1+h^2}\,\mathrm{d}\mu = 1.$$

Thus the equalities also hold in this case.

Suppose that $0 < A < \infty$ which means $h(x) \in (0, \infty)$ a.e. on Ω. We consider the function $\varphi : \mathbb{R} \to (0, \infty)$ defined by

$$\varphi(x) = \sqrt{1 + x^2}$$

which implies that

$$\varphi'(x) = \frac{x}{\sqrt{1 + x^2}} \quad \text{and} \quad \varphi''(x) = \frac{1}{(1 + x^2)^{\frac{3}{2}}}. \tag{3.66}$$

Since $\varphi''(x) \geq 0$ on $\mathbb{R}$, we know from [49, Exercise 14, p. 115] that φ is convex on $\mathbb{R}$. In particular, φ is convex on $(0, \infty)$. By Theorem 3.3 (Jensen's Inequality), we obtain that

$$\sqrt{1 + A^2} = \sqrt{1 + \int_\Omega h \, \mathrm{d}\mu} \leq \int_\Omega \sqrt{1 + h^2} \, \mathrm{d}\mu \tag{3.67}$$

which is exactly the left-hand side inequality. The right-hand side inequality is easy because we always have $\sqrt{1 + h^2} \leq 1 + h$. Hence we have verified the validity of the inequalities.

For the second assertion, we suppose that $\mu = m$ on $\Omega = [0, 1]$, $h = f'$ and h is continuous on Ω. Then we have

$$A = \int_0^1 f'(x) \, \mathrm{d}x = f(1) - f(0).$$

In addition, we notice that

$$\int_\Omega \sqrt{1 + h^2} \, \mathrm{d}\mu = \int_0^1 \sqrt{1 + [f'(x)]^2} \, \mathrm{d}x$$

which is the *arc length* of the curve from $(0, f(0))$ to $(1, f(1))$.[d] Thus the inequalities give bounds of such arc length. Geometrically, the upper bound consists of the length from $(0, f(0))$ to $(1, f(0))$ (which is 1) and the length from $(1, f(0))$ to $(1, f(1))$ (which is A). The lower bound is just the hypotenuse of the right triangle with vertices $(0, f(0))$, $(0, f(1))$ and $(1, f(1))$, see Figure 3.2 below:

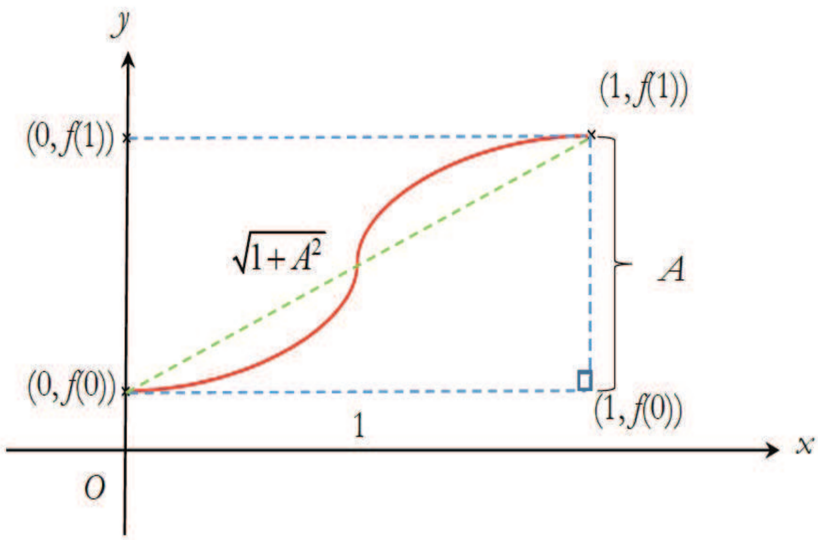


Figure 3.2: The geometric interpretation of a special case.

[d]See, for examples, [2, p. 535] and [49, Theorem 6.27, p. 137].

Our first inequality (3.67) comes from the application of Theorem 3.3 (Jensen's Inequality), so the equality of the first inequality (3.67) holds if and only if the equality in [51, Eqn. (2), p. 62] holds if and only if

$$\varphi(A) = \varphi\big(h(x)\big). \tag{3.68}$$

We see from the first derivative in (3.66) that $\varphi'(x) > 0$ in $(0, \infty)$ so that it is *injective* in $(0, \infty)$. Recall that $A \in (0, \infty)$ and $h(x) \in (0, \infty)$ a.e. on Ω, so we deduce that the equality (3.68) holds if and only if

$$h = A \quad \text{a.e. on } \Omega.$$

Next we recall the second inequality is derived from the fact that $\sqrt{1 + h^2} \leq 1 + h$ and some simple computations show that the equality holds if and only if

$$h \equiv 0 \quad \text{a.e. on } \Omega.$$

Hence we have ended the proof of the problem. ■

Problem 3.13

Rudin Chapter 3 Exercise 13.

Proof. There are two cases.

- **Case (i):** $1 < p < \infty$. Then the inequality in Theorem 3.8 is just Hölder's inequality. If $\|f\|_p = 0$ or $\|g\|_p = 0$, then we have $f = 0$ a.e. on X or $g = 0$ a.e. on X by Theorem 1.39(a). In either case, the equality holds trivially. Therefore, we may assume that both $\|f\|_p > 0$ and $\|g\|_p > 0$. Furthermore, by the remark following the proof of Theorem 3.5 (Hölder's Inequality), we may further assume that

  $$0 < \|f\|_p < \infty \quad \text{and} \quad 0 < \|g\|_p < \infty.$$

  In this case, we see that the equality in Theorem 3.8 holds if and only if there are positive constants α and β such that

  $$\alpha f^p = \beta g^q$$

  a.e. on X.

  In Theorem 3.9, the inequality is in fact Minkowski's inequality. By simple observation, it is clear that the equality holds if $f = 0$ a.e. on X or $g = 0$ a.e. on X. Therefore, we may assume that $f \neq 0$ and $g \neq 0$ on measurable sets E and F with $\mu(E) > 0$ and $\mu(F) > 0$ respectively. By examining the proof of [51, Eqn. (2), Theorem 3.5, pp. 64, 65], we know that it applies Hölder's inequality to the functions f and $(f+g)^{p-1}$ as well as the functions g and $(f+g)^{p-1}$. Thus we deduce from the previous paragraph that the equality in

  $$\int_X f \cdot (f+g)^{p-1}\,\mathrm{d}\mu \leq \Big\{\int_X f^p\,\mathrm{d}\mu\Big\}^{\frac{1}{p}} \times \Big\{\int_X (f+g)^{(p-1)q}\,\mathrm{d}\mu\Big\}^{\frac{1}{q}}$$

  holds if and only if there are positive constants α and β such that

  $$\alpha f^p = \beta(f+g)^q \quad \text{a.e. on } X. \tag{3.69}$$

  Similarly, the equality in

  $$\int_X g \cdot (f+g)^{p-1}\,\mathrm{d}\mu \leq \Big\{\int_X g^p\,\mathrm{d}\mu\Big\}^{\frac{1}{p}} \times \Big\{\int_X (f+g)^{(p-1)q}\,\mathrm{d}\mu\Big\}^{\frac{1}{q}}$$

holds if and only if there are positive constants λ and ν such that

$$\lambda g^p = \nu(f+g)^q \quad \text{a.e. on } X. \tag{3.70}$$

By combining the equations (3.69) and (3.70), we conclude that

$$f^p = \frac{\beta}{\alpha}(f+g)^q = \frac{\beta\lambda}{\alpha\nu}g^p \quad \text{a.e.}$$

which means that $f = Kg$ a.e. for some positive constant K.

- **Case (ii):** $p = \infty$**.** The inequality in Theorem 3.8 comes from the inequality[e]

$$|f(x)g(x)| \leq \|f\|_\infty \cdot |g(x)|$$

for almost all x. Therefore, it is easy to see that the equality in Theorem 3.8 holds if and only if $|f(x)g(x)| = \|f\|_\infty \cdot |g(x)|$ for almost all x if and only if $g(x) = 0$ for almost x or $f(x) = \|f\|_\infty$ for almost x.[f]

Now the triangle inequality

$$|f+g| \leq |f| + |g|$$

implies the inequality in Theorem 3.9. Thus the equality in Theorem 3.9 holds if and only if f and g are either nonnegative functions or nonpositive functions for almost all x.

This completes the proof of the problem. ■

3.4 Hardy's Inequality and Egoroff's Theorem

Problem 3.14

Rudin Chapter 3 Exercise 14.

Proof. We follow the suggestions given by Rudin.

(a) We divide the proof into two cases:

- **Special Case:** $f \geq 0$ **and** $f \in C_c\big((0,\infty)\big)$**.** By the First Fundamental Theorem of Calculus, the function $F(x)$ is differentiable on $(0,\infty)$ and

$$xF'(x) = f(x) - F(x) \tag{3.71}$$

for every $x \in (0,\infty)$. By Integration by Parts[g] and the expression (3.71), we have

$$\int_0^\infty F^p(x)\,\mathrm{d}x = [xF^p(x)]_0^\infty - \int_0^\infty pxF^{p-1}(x)F'(x)\,\mathrm{d}x. \tag{3.72}$$

Now we have to evaluate the limits:

$$\lim_{x\to 0} xF^p(x) \quad \text{and} \quad \lim_{x\to\infty} xF^p(x).$$

[e] This is [51, Eqn. (2), Theorem 3.8, p. 66].
[f] The latter case means that f is a nonnegative constant for almost all x.
[g] Here we assume that the formula for integration by parts is also valid for improper integrals, see [3, p. 278].

Since $\operatorname{supp}(f) = \overline{\{x \in (0,\infty) \,|\, f(x) \neq 0\}}$ is compact, the Heine-Borel Theorem guarantees that $\operatorname{supp}(f) = [a,b] \subset (0,\infty)$. By this and the continuity of f, we conclude that f is bounded on $(0,\infty)$, i.e., $|f(x)| \leq M$ on $(0,\infty)$ for some positive constant M. Thus we follow from this that

$$|F(x)| \leq \frac{1}{x}\int_0^x |f(t)|\,\mathrm{d}t \leq M$$

for every $x \in (0,\infty)$, so we see that

$$\lim_{x\to 0} xF^p(x) = 0. \tag{3.73}$$

Again, the boundedness of f implies that

$$|xF(x)| \leq \int_0^x |f(t)|\,\mathrm{d}t \leq \int_0^\infty |f(t)|\,\mathrm{d}t = \int_a^b |f(t)|\,\mathrm{d}t \leq M(b-a) < \infty \tag{3.74}$$

on $(0,\infty)$. Since $xF^p = \frac{(xF)^p}{x^{p-1}}$ and $p > 1$, it reduces from these and the bound (3.74) that

$$\lim_{x\to\infty} xF^p(x) = 0. \tag{3.75}$$

Therefore, we deduce from the formula (3.72) and the two limits (3.73) and (3.75) that

$$\begin{aligned}\int_0^\infty F^p(x)\,\mathrm{d}x &= -p\int_0^\infty F^{p-1}(x)[f(x)-F(x)]\,\mathrm{d}x \\ &= -p\int_0^\infty F^{p-1}(x)f(x)\,\mathrm{d}x + p\int_0^\infty F^p(x)\,\mathrm{d}x\end{aligned}$$

so that

$$(p-1)\int_0^\infty F^p(x)\,\mathrm{d}x = p\int_0^\infty F^{p-1}(x)f(x)\,\mathrm{d}x. \tag{3.76}$$

Note that $1 < p < \infty$, let q be its conjugate exponent. Obviously, our hypotheses make sure that f and F^{p-1} have range $[0,\infty]$, so we may apply Theorem 3.5 (Hölder's Inequality) to them. In fact, it yields from the fact $(p-1)q = p$ and the expression (3.76) that

$$\begin{aligned}(p-1)\|F\|_p^p &= (p-1)\int_0^\infty F^p(x)\,\mathrm{d}x \\ &\leq p\Big\{\int_0^\infty f^p(x)\,\mathrm{d}x\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty F^{(p-1)q}(x)\,\mathrm{d}x\Big\}^{\frac{1}{q}} \\ &= p\|f\|_p \cdot \|F\|_p^{\frac{p}{q}}\end{aligned}$$

which is exactly

$$\|F\|_p = \|F\|_p^{p-\frac{p}{q}} \leq \frac{p}{p-1}\|f\|_p. \tag{3.77}$$

– **General Case:** $f \in L^p\big((0,\infty)\big)$**.** By Definition 3.6, we always have

$$\|f\|_p = \big\||f|\big\|_p,$$

so we may assume that $f \geq 0$ on $(0,\infty)$. Since $1 < p < \infty$, Theorem 3.14 says that $C_c\big((0,\infty)\big)$ is dense in $L^p\big((0,\infty)\big)$. Thus there exists a sequence $\{f_n\} \subseteq C_c\big((0,\infty)\big)$ such that

$$\|f_n - f\|_p \to 0 \tag{3.78}$$

as $n \to \infty$. Notice that f_n may *not* be nonnegative, but since $\big||f_n| - |f|\big| \leq |f_n - f|$ and $f \geq 0$, we have

$$\big||f_n(x)| - f(x)\big| \leq |f_n(x) - f(x)| \tag{3.79}$$

for every $x \in (0, \infty)$. Certainly, the inequality (3.79) implies that

$$\big\||f_n| - f\big\|_p \leq \|f_n - f\|_p \tag{3.80}$$

for every $n = 1, 2, \ldots$. Now the result (3.78) and the inequality (3.80) ensure that

$$\big\||f_n| - f\big\|_p \to 0$$

as $n \to \infty$. In other words, we may also assume that $f_n \geq 0$ for each $n = 1, 2, \ldots$. Recall that $f_n \in C_c\big((0, \infty)\big)$, so we derive from the **Special Case** that

$$\|F_n\|_p \leq \frac{p}{p-1}\|f_n\|_p, \tag{3.81}$$

where

$$F_n(x) = \frac{1}{x}\int_0^x f_n(t)\,\mathrm{d}t$$

and $n = 1, 2, \ldots$. Suppose that

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t.$$

Since $f \geq 0$ and $f_n \geq 0$, it is easily seen from the definition that $F \geq 0$ and $F_n \geq 0$.

We claim that $\{F_n\}$ *converges pointwise* to F on $(0, \infty)$. To see this, fix $x \in (0, \infty)$, then their definitions give

$$|F_n(x) - F(x)| \leq \frac{1}{x}\int_0^x |f_n(t) - f(t)|\,\mathrm{d}t. \tag{3.82}$$

By the result (3.78), we know that for every $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies

$$\|f_n - f\|_p < \epsilon.$$

Apply Theorem 3.5 (Hölder's Inequality) with $\frac{1}{p} + \frac{1}{q} = 1$ to the inequality (3.82), we get immediately that

$$\begin{aligned}
|F_n(x) - F(x)| &\leq \frac{1}{x}\Big\{\int_0^x |f_n(t) - f(t)|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}} \times \Big\{\int_0^x 1^q\,\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\leq \frac{1}{x}\|f_n - f\|_p \times x^{\frac{1}{q}} \\
&= x^{-\frac{1}{p}}\|f_n - f\|_p \\
&< \epsilon x^{-\frac{1}{p}} \qquad (3.83)
\end{aligned}$$

for all $n \geq N$. Since x is *fixed* and *independent of* ϵ, the estimate (3.83) verifies the truth of the claim.

Next, recall that each f_n is continuous on $[a, x]$ for every $a > 0$, so the First Fundamental Theorem of Calculus shows that each F_n is continuous on $[a, x]$. As a continuous function, every F_n is measurable on $[a, x]$.[h] Hence, it follows from the

[h]See §1.11 or [49, Example 11.14].

pointwise convergence of $\{F_n\}$, Theorem 1.28 (Fatou's Lemma), the inequality (3.81) and then the result (3.78) that

$$\begin{aligned}\|F\|_p^p &= \int_0^\infty F^p(x)\,\mathrm{d}x\\ &= \int_0^\infty \liminf_{n\to\infty} F_n^p(x)\,\mathrm{d}x\\ &\leq \liminf_{n\to\infty}\int_0^\infty F_n^p(x)\,\mathrm{d}x\\ &\leq \lim_{n\to\infty}\int_0^\infty F_n^p(x)\,\mathrm{d}x\\ &= \lim_{n\to\infty}\|F_n\|_p^p\\ &\leq \Big(\frac{p}{p-1}\Big)^p \lim_{n\to\infty}\|f_n\|_p^p\\ &\leq \Big(\frac{p}{p-1}\Big)^p \|f\|_p^p\end{aligned}$$

which is equivalent to the desired result

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Hence the mapping $T : L^p\big((0,\infty)\big) \to L^p\big((0,\infty)\big)$ given by $T(f) = F$ is continuous.

(b) Suppose that $f \in L^p\big((0,\infty)\big)$ and

$$\|F\|_p = \frac{p}{p-1}\|f\|_p < \infty. \tag{3.84}$$

Recall that $\|f\|_p = \big\||f|\big\|_p$, so we may suppose further that $f \geq 0$ on $(0,\infty)$. By the **General Case** of the proof in part (a), there is a nonnegative sequence $\{f_n\} \subseteq C_c\big((0,\infty)\big)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Now we replace F and f by F_n and f_n in the expression (3.76) respectively, we gain

$$\|F_n\|_p^p = \int_0^\infty F_n^p(x)\,\mathrm{d}x = \frac{p}{p-1}\int_0^\infty F_n^{p-1}(x)f_n(x)\,\mathrm{d}x \tag{3.85}$$

for each $n = 1, 2, \ldots$. Since $f_n \in C_c\big((0,\infty)\big)$, $T(f_n) = F_n \in L^p\big((0,\infty)\big)$, where T is the continuous mapping considered in part (a). By Theorem 3.5 (Hölder's Inequality), it is easy to see that $f_nF_n^{p-1} \in L^p\big((0,\infty)\big)$ for each $n = 1, 2, \ldots$. Since the mapping T is continuous, it follows from the expression (3.85) that

$$\int_0^\infty F^p(x)\,\mathrm{d}x = \frac{p}{p-1}\int_0^\infty F^{p-1}(x)f(x)\,\mathrm{d}x. \tag{3.86}$$

In other words, the formula (3.76) holds for $f \in L^p\big((0,\infty)\big)$. We apply Theorem 3.5 (Hölder's Inequality) to the right-hand side of the inequality (3.86) and then using our hypothesis (3.84) to conclude that

$$\begin{aligned}\|F\|_p^p &= \frac{p}{p-1}\int_0^\infty F^{p-1}(x)f(x)\,\mathrm{d}x\\ &\leq \frac{p}{p-1}\Big\{\int_0^\infty f^p(x)\,\mathrm{d}x\Big\}^{\frac{1}{p}} \times \Big\{\int_0^\infty F^p(x)\,\mathrm{d}x\Big\}^{\frac{1}{q}} \qquad (3.87)\end{aligned}$$

$$
\begin{aligned}
&= \frac{p}{p-1}\|f\|_p \times \|F\|_p^{\frac{p}{q}} \\
&= \|F\|_p^p.
\end{aligned}
$$

Hence the equality in the inequality (3.87) must hold. Consequently, there are constants α and β, *not* both 0, such that $\alpha f^p = \beta(F^{p-1})^q = \beta F^p$ a.e. on $(0,\infty)$ and so

$$
\alpha^{\frac{1}{p}} f = \beta^{\frac{1}{p}} F \quad \text{a.e. on } (0,\infty). \tag{3.88}
$$

Here we have several cases:

- **Case (i):** $\beta = 0$**.** Then we have $\alpha \neq 0$ and $f = 0$ a.e. on $(0,\infty)$. Thus we are done.
- **Case (ii):** $\alpha = 0$**.** Then we have $\beta \neq 0$ and $F = 0$ a.e. on $(0,\infty)$. By the hypothesis (3.84), we see that $\|f\|_p = 0$ which gives $f = 0$ a.e. on $(0,\infty)$ and we are done again.
- **Case (iii):** $\alpha\beta \neq 0$**.** By Theorem 3.8, we see that $f \in L^1([a,b])$, where $[a,b]$ is any bounded interval of $(0,\infty)$. By the comment following the proof of Theorem 11.3 on [49, p. 324], we know that F is differentiable a.e. on $[a,b]$ and

$$
\big(xF(x)\big)' = f(x) \tag{3.89}
$$

a.e. on $[a,b]$. Since $[a,b]$ is arbitrary, the formula (3.89) holds a.e. on $(0,\infty)$. Combining the equations (3.88) and (3.89), we see immediately that $xF' + xF = f$ which implies

$$
xf' = (c-1)f \qquad \text{a.e. on } (0,\infty) \tag{3.90}
$$

for some nonzero constant c. By solving the differential equation (3.90), we obtain that

$$
f(x) = \gamma x^{c-1} \qquad \text{a.e. on } (0,\infty)
$$

for some constant γ. Since $\gamma x^{c-1} \in L^p((0,\infty))$, it forces that $\gamma = 0$. Hence we have shown that the equality holds only if $f = 0$ a.e. on $(0,\infty)$.

(c) Take $f(x) = x^{-\frac{1}{p}}\chi_{[1,A]}(x)$ for large A. Then we have

$$
\|f\|_p = \Big\{\int_0^\infty |x^{-\frac{1}{p}}\chi_{[1,A]}(x)|^p\,\mathrm{d}x\Big\}^{\frac{1}{p}} = \Big\{\int_1^A x^{-1}\,\mathrm{d}x\Big\}^{\frac{1}{p}} = (\log A)^{\frac{1}{p}}. \tag{3.91}
$$

Next, we know from the definition that

$$
\begin{aligned}
F(x) &= \frac{1}{x}\int_0^x t^{-\frac{1}{p}}\chi_{[1,A]}(t)\,\mathrm{d}t \\
&= \begin{cases} 0, & \text{if } x \in (0,1); \\[2ex] \dfrac{1}{x}\displaystyle\int_1^x t^{-\frac{1}{p}}\,\mathrm{d}t, & \text{if } x \in [1,A]; \\[2ex] \dfrac{1}{x}\displaystyle\int_1^A t^{-\frac{1}{p}}\,\mathrm{d}t, & \text{if } x \in (A,\infty); \end{cases} \\
&= \begin{cases} 0, & \text{if } x \in (0,1); \\[2ex] \dfrac{p}{p-1}(x^{-\frac{1}{p}} - x^{-1}), & \text{if } x \in [1,A]; \\[2ex] \dfrac{p}{p-1}\cdot\dfrac{A^{1-\frac{1}{p}}-1}{x}, & \text{if } x \in (A,\infty). \end{cases}
\end{aligned} \tag{3.92}
$$

Note that A is large and $1-\frac{1}{p}>0$, so we have

$$A^{1-\frac{1}{p}}-1>0. \tag{3.93}$$

Therefore, we deduce from the expressions (3.92) and the fact (3.93) that

$$\begin{aligned}
\|F\|_p^p &= \int_0^\infty |F(x)|^p\,\mathrm{d}x \\
&= \int_1^A |F(x)|^p\,\mathrm{d}x + \int_A^\infty |F(x)|^p\,\mathrm{d}x \\
&= \Big(\frac{p}{p-1}\Big)^p \int_1^A |x^{-\frac{1}{p}}-x^{-1}|^p\,\mathrm{d}x + \Big(\frac{p}{p-1}\Big)^p \int_A^\infty \Big|\frac{A^{1-\frac{1}{p}}-1}{x}\Big|^p\,\mathrm{d}x \\
&= \Big(\frac{p}{p-1}\Big)^p \int_1^A (x^{-\frac{1}{p}}-x^{-1})^p\,\mathrm{d}x + \frac{p^p}{(p-1)^{p+1}}\big(A^{1-\frac{1}{p}}-1\big)^p A^{1-p} \\
&> \Big(\frac{p}{p-1}\Big)^p \int_1^A (x^{-\frac{1}{p}}-x^{-1})^p\,\mathrm{d}x. \qquad (3.94)
\end{aligned}$$

Fix p first. Assume that the constant $\frac{p}{p-1}$ could be replaced by a smaller number $\frac{\delta p}{p-1}$ for some $\delta<1$, i.e.,

$$\|F\|_p \leq \frac{\delta p}{p-1}\|f\|_p.$$

Take $\epsilon>0$ such that $\delta<\epsilon<1$. Then it is clear that for large A, we have $x^{\frac{1}{p}-1}<1-\epsilon$ for $x\geq \frac{A}{2}$ or equivalently,

$$x^{-\frac{1}{p}}-x^{-1}>\epsilon x^{-\frac{1}{p}}>0 \tag{3.95}$$

for $x\geq\frac{A}{2}$. By this estimate (3.95), we obtain

$$\int_1^A (x^{-\frac{1}{p}}-x^{-1})^p\,\mathrm{d}x > \int_{\frac{A}{2}}^A \epsilon^p x^{-1}\,\mathrm{d}x = \epsilon^p\Big(\log A-\log\frac{A}{2}\Big) > \epsilon^p\log A. \tag{3.96}$$

By substituting the inequality (3.96) into the inequality (3.94) and then using the fact (3.91), we can show that

$$\|F\|_p^p > \Big(\frac{\epsilon p}{p-1}\Big)^p \log A > \Big(\frac{\delta p}{p-1}\Big)^p\|f\|_p^p \geq 0,$$

but this implies

$$\|F\|_p > \frac{\delta p}{p-1}\|f\|_p,$$

a contradiction.

(d) Since $f(x)>0$ on $(0,\infty)$, there exists a $\delta>0$ such that we can find an $\alpha\in(0,\infty)$ with

$$\int_0^\alpha f(t)\,\mathrm{d}t>\delta>0.$$

By the definition, we have $F(x)>0$ on $(0,\infty)$ so that

$$\|F\|_1=\int_0^\infty F(x)\,\mathrm{d}x \geq \int_\alpha^\infty F(x)\,\mathrm{d}x = \int_\alpha^\infty \frac{1}{x}\Big(\int_0^x f(t)\,\mathrm{d}t\Big)\,\mathrm{d}x > \int_\alpha^\infty \frac{\delta}{x}\,\mathrm{d}x=\infty.$$

Hence $F\notin L^1$. In other words, this shows that Hardy's inequality does not hold when $p=1$.

Hence we have completed the proof of the problem. ■

Problem 3.15

Rudin Chapter 3 Exercise 15.

Proof. We follow the hint. Suppose that $\{a_n\}$ is decreasing. Let $f:(0,\infty)\to(0,\infty)$ be given by

$$f(x)=\sum_{n=1}^{\infty}a_n\chi_{(n-1,n]}(x). \tag{3.97}$$

Firstly, we note that

$$\int_0^\infty f^p(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\int_{n-1}^{n}a_n^p\,\mathrm{d}x=\sum_{n=1}^{\infty}a_n^p \tag{3.98}$$

which implies that $f\in L^p\big((0,\infty)\big)$ if and only if $\sum\limits_{n=1}^{\infty}a_n^p<\infty$. This shows that the desired inequality holds if $f\notin L^p\big((0,\infty)\big)$. Without loss of generality, we assume that $\sum\limits_{n=1}^{\infty}a_n^p<\infty$ in the following discussion.

Secondly, let N be a positive integer and $x\in(N-1,N]$. Then we deduce from the definition (3.97) that

$$\begin{aligned}
F(x)&=\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\\
&=\frac{1}{x}\Big[\sum_{n=1}^{N-1}\int_{n-1}^{n}f(t)\,\mathrm{d}t+\int_{N-1}^{x}f(t)\,\mathrm{d}t\Big]\\
&=\frac{1}{x}\Big[\sum_{n=1}^{N-1}\int_{n-1}^{n}a_n\,\mathrm{d}t+\int_{N-1}^{x}a_N\,\mathrm{d}t\Big]\\
&=\frac{1}{x}[a_1+a_2+\cdots+a_{N-1}+a_N(x-N+1)].
\end{aligned}\tag{3.99}$$

It is clear from the expression (3.99) and the fact $\{a_n\}$ is decreasing that

$$\begin{aligned}
\frac{1}{x}\Big[\sum_{n=1}^{N-1}a_n+a_N(x-N+1)\Big]&=\frac{1}{x}\Big(\sum_{n=1}^{N}a_n-Na_N+xa_N\Big)\\
&=\frac{1}{x}\Big(\sum_{n=1}^{N}a_n-Na_N\Big)+a_N\\
&\geq\frac{1}{N}\Big(\sum_{n=1}^{N}a_n-Na_N\Big)+a_N\\
&=\frac{1}{N}\sum_{n=1}^{N}a_n.
\end{aligned}\tag{3.100}$$

Thus we obtain from the expression (3.99) and the inequality (3.100) that

$$\|F\|_p=\Big\{\int_0^\infty F^p(x)\,\mathrm{d}x\Big\}^{\frac{1}{p}}=\Big\{\sum_{N=1}^{\infty}\int_{N-1}^{N}F^p(x)\,\mathrm{d}x\Big\}^{\frac{1}{p}}\geq\Big\{\sum_{N=1}^{\infty}\Big(\frac{1}{N}\sum_{n=1}^{N}a_n\Big)^p\Big\}^{\frac{1}{p}}. \tag{3.101}$$

Furthermore, by Problem 3.14(a) and the expression (3.98), we derive that

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p = \frac{p}{p-1}\Big\{\int_0^\infty f^p(x)\,\mathrm{d}x\Big\}^{\frac{1}{p}} = \frac{p}{p-1}\Big\{\sum_{n=1}^\infty a_n^p\Big\}^{\frac{1}{p}}. \tag{3.102}$$

Hence our desired inequality follows immediately from the inequalities (3.101) and (3.102).

For the general case, recall the assumption that $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^\infty a_n^p < \infty$, so the sequence $\{a_n^p\}$ converges absolutely and every rearrangement converges to the same sum (see [49, Theorem 3.55, p. 78]). On the other hand, if we fix a positive integer N, then

$$\begin{aligned}\sum_{k=1}^N \Big(\frac{1}{k}\sum_{n=1}^k a_n\Big)^p &= \Big(\frac{a_1}{1}\Big)^p + \Big(\frac{a_1+a_2}{2}\Big)^p + \Big(\frac{a_1+a_2+a_3}{3}\Big)^p \\ &\quad + \cdots + \Big(\frac{a_1+a_2+\cdots+a_N}{N}\Big)^p,\end{aligned}$$

so it follows from this form that the sum attains its maximum if and only if $a_1 \geq a_2 \geq \cdots \geq a_N$. Therefore, we have

$$\sum_{N=1}^\infty \Big(\frac{1}{N}\sum_{n=1}^N a_n\Big)^p \leq \underbrace{\sum_{N=1}^\infty \Big(\frac{1}{N}\sum_{n=1}^N \alpha_n\Big)^p \leq \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^\infty \alpha_n^p}_{\text{By the special case.}} = \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^\infty a_n^p,$$

where $\{\alpha_n\}$ is the decreasing rearrangement of $\{a_n\}$. This completes the proof of the problem. ■

Problem 3.16

Rudin Chapter 3 Exercise 16.

Proof. Let's prove the assertions one by one.

- **A proof of Egoroff's Theorem.** Put

$$S(n,k) = \bigcap_{i,j>n} \Big\{x \in X \,\Big|\, |f_i(x) - f_j(x)| < \frac{1}{k}\Big\}. \tag{3.103}$$

For each $k \in \mathbb{N}$, we follow from the definition (3.103) that

$$\begin{aligned}S(n,k) = \bigcap_{i,j>n+1} \Big\{x \in X \,\Big|\, |f_i(x) - f_j(x)| < \frac{1}{k}\Big\} \cap \bigcap_{\substack{i=n+1\\ j>n}} \Big\{x \in X \,\Big|\, |f_i(x) - f_j(x)| < \frac{1}{k}\Big\} \\ \cap \bigcap_{\substack{j=n+1\\ i>n}} \Big\{x \in X \,\Big|\, |f_i(x) - f_j(x)| < \frac{1}{k}\Big\}\end{aligned}$$

which implies that $S(n,k) \subseteq S(n+1,k)$.

Next, for each *fixed* $k \in \mathbb{N}$, we claim that

$$X = \bigcup_{n=1}^\infty S(n,k). \tag{3.104}$$

To this end, we see that the set inclusion

$$\bigcup_{n=1}^{\infty} S(n,k) \subseteq X$$

is evident. To prove the reverse direction, we note that for each $k \in \mathbb{N}$ and $x \in X$, we yield from our hypothesis $f_n(x) \to f(x)$ that there exists a positive integer N such that $i, j > N$ implies that

$$|f_i(x) - f_j(x)| < \frac{1}{k}.$$

In other words, we must have $x \in S(n,k)$, i.e.,

$$X \subseteq \bigcup_{n=1}^{\infty} S(n,k).$$

Thus these prove the validity of our claim (3.104) and Theorem 1.19(d) gives

$$\lim_{n\to\infty} \mu\big(S(n,k)\big) = \mu(X)$$

for every $k = 1, 2, \ldots$.

Given that $\epsilon > 0$. For each $k \in \mathbb{N}$, this fact allows us to choose n_k such that

$$|\mu\big(S(n_k,k) - \mu(X)| < \frac{\epsilon}{2^k}.$$

Define

$$E = \bigcap_{k=1}^{\infty} S(n_k,k). \tag{3.105}$$

Then it is clear that

$$X \setminus E = \bigcup_{k=1}^{\infty} \big(X \setminus S(n_k,k)\big)$$

which implies that

$$\mu(X \setminus E) \le \sum_{k=1}^{\infty} \mu\big(X \setminus S(n_k,k)\big) \le \sum_{k=1}^{\infty} \big|\mu(X) - \mu\big(S(n_k,k)\big)\big| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

Pick a k with $\frac{1}{k} < \epsilon$. If $x \in S(n_k,k)$, then we have

$$|f_i(x) - f_j(x)| < \epsilon \tag{3.106}$$

for all $i, j > n_k$. By the definition (3.105), we know that $E \subseteq S(n_k,k)$ for every $k \in \mathbb{N}$, so the inequality (3.106) holds on E. By the definition, $f_n \to f$ uniformly on E, proving Egoroff's Theorem.

- **A counterexample on a σ-finite space.** Recall from Definition 2.16 that X is said to be a σ-finite space if X is a countable union of sets E_i with $\mu(E_i) < \infty$. It is clear that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [n-1, n) \cup \bigcup_{n=1}^{\infty} (-n, 1-n],$$

so $\mathbb{R}$ is σ-finite. Let $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$. It is clear that the sequence converges pointwise to 0 at every point of $\mathbb{R}$. Furthermore, we know that a measurable set $E \subseteq \mathbb{R}$ with $m(\mathbb{R}\setminus E) < 1$ must be unbounded. Otherwise, $E \subseteq [-M, M]$ for some $M > 0$ and this implies that

$$m(\mathbb{R} \setminus E) \ge m\big(\mathbb{R} \setminus [-M, M]\big) = \infty,$$

a contradiction. However, $\{f_n\}$ *cannot* converge uniformly to 0 on the unbounded set E. Thus this counterexample shows that Egoroff's Theorem *cannot* be extended to σ-finite spaces.

- **An extension of Egoroff's Theorem.** Suppose that $\{f_t\}$ is a family of complex measurable functions such that

  (i) $\lim\limits_{t\to\infty} f_t(x) = f(x)$ and

  (ii) Fix a $x \in X$. Then the function $F : (0,\infty) \to \mathbb{C}$ defined by
  $$F(t) = f_t(x)$$
  is continuous.

  For each $n \in \mathbb{N}$, we consider the real function $g_n : X \to \mathbb{R}$ given by
  $$g_n(x) = \sup_{t\geq n}\{|f_t(x) - f(x)|\}. \tag{3.107}$$
  Then, for every $x \in X$, $g_n(x) \to 0$ as $n \to \infty$. Now it remains to show that each g_n is measurable. To this end, let $a \in \mathbb{R}$ and
  $$\begin{aligned} E_a(n) &= \{x \in X \mid g_n(x) < a\} \\ &= \{x \in X \mid |f_t(x) - f(x)| < a \text{ for all } t \geq n\} \\ &= \bigcap_{\substack{r\geq n \\ r\in\mathbb{Q}}} \{x \in X \mid |f_r(x) - f(x)| < a\}. \end{aligned} \tag{3.108}$$
  Since f_r is measurable, f is also measurable by Corollary (a) following Theorem 1.14. Thus $|f_r - f|$ is measurable by Proposition 1.9(b) and then the set
  $$\{x \in X \mid |f_r(x) - f(x)| < a\}$$
  is measurable for every $a \in \mathbb{R}$ by Definition 1.3(c). By the expression (3.108), since there are countable such measurable sets, Comment 1.6(c) implies that $E_a(n)$ is measurable for every real a. By Definition 1.3(c) again, each g_n is measurable and thus Egoroff's Theorem can be applied to conclude that for every $\epsilon > 0$, there exists a measurable set $E \subseteq X$ such that $\mu(X \setminus E) < \epsilon$ and $\{g_n\}$ converges uniformly to 0 on E. By the definition (3.107), $\{f_t\}$ converges to f uniformly on E.

This completes the proof of the problem. ■

Problem 3.17

Rudin Chapter 3 Exercise 17.

Proof.

(a) Since the inequality is clearly holds if either $\alpha = 0$ or $\beta = 0$, we assume, without loss of generality, that α and β cannot be both zero in the following discussion.

Let $0 < p \leq 1$. We claim that the continuous function $\varphi : [0,1] \to \mathbb{R}$ defined by
$$\varphi(x) = x^p + (1-x)^p$$

satisfies $\varphi(x) \geq 1$ on $[0,1]$. To this end, we note that

$$\varphi'(x) = px^{p-1} - p(1-x)^{p-1} = 0$$

if and only if $x = \frac{1}{2}$. Since $p - 1 \leq 0$, we know that

$$\varphi'(x) \geq 0 \quad \text{and} \quad \varphi'(x) \leq 0$$

on $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively. In other words, φ is increasing and decreasing on $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively. By the continuity of φ, these are also valid on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. Thus we observe

$$\varphi(x) \geq \varphi(0) = 1 \text{ on } [0, \tfrac{1}{2}] \quad \text{and} \quad \varphi(x) \geq \varphi(1) = 1 \text{ on } [\tfrac{1}{2}, 1]$$

which mean that

$$x^p + (1-x)^p \geq 1 \tag{3.109}$$

on $[0,1]$. For arbitrary nonzero complex numbers α and β, we put $x = \frac{|\alpha|}{|\alpha|+|\beta|}$ into the inequality (3.109) to get

$$\frac{|\alpha|^p}{(|\alpha| + |\beta|)^p} + \frac{|\beta|^p}{(|\alpha| + |\beta|)^p} \geq 1$$

which implies

$$(|\alpha| + |\beta|)^p \leq |\alpha|^p + |\beta|^p. \tag{3.110}$$

By the triangle inequality, it is true that $|\alpha - \beta| \leq |\alpha| + |\beta|$, so this and the inequality (3.110) give

$$|\alpha - \beta|^p \leq |\alpha|^p + |\beta|^p. \tag{3.111}$$

Next, let $1 < p < \infty$. The function $\psi : [0,1] \to \mathbb{R}$ given by

$$\psi(x) = x^p$$

has second derivative $p(p-1)x^{p-2} > 0$ on $(0,1)$. By Definition 3.1, it is convex on $(0,1)$, i.e.,

$$[(1-\lambda)x + \lambda y]^p \leq (1-\lambda)x^p + \lambda y^p, \tag{3.112}$$

where $x, y \in (0,1)$ and $\lambda \in [0,1]$. Put $\lambda = \frac{1}{2}$, $x = \frac{|\alpha|}{|\alpha|+|\beta|}$ and $y = \frac{|\beta|}{|\alpha|+|\beta|}$ into the inequality (3.112), we obtain

$$\Big(\frac{1}{2} \cdot \frac{|\alpha| + |\beta|}{|\alpha| + |\beta|}\Big)^p \leq \frac{1}{2} \cdot \frac{|\alpha|^p}{(|\alpha| + |\beta|)^p} + \frac{1}{2} \cdot \frac{|\beta|^p}{(|\alpha| + |\beta|)^p}$$

which reduces to

$$(|\alpha| + |\beta|)^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p).$$

Again, the triangle inequality and this show that

$$|\alpha - \beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p). \tag{3.113}$$

Hence the desired inequality follows from combining the inequalities (3.111) and (3.113).

(b) (i) If $\mu(X) = 0$, then Proposition 1.24(e) implies that $\|f\|_p = 0$ for every $f \in L^p(\mu)$ and there is nothing to prove. Thus, without loss of generality, we may assume that $\mu(X) > 0$. We divide the proof into several steps:

* **Step 1: Lemma 3.7 and its application.**

Lemma 3.7

For every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $E \in \mathfrak{M}$ with $\mu(E) < \delta$, we have

$$\int_E |f|^p \, d\mu \leq \epsilon. \tag{3.114}$$

Proof of Lemma 3.7. Note that $|f|$ must be bounded a.e. on X. Otherwise, there exists an $E \in \mathfrak{M}$ such that $\mu(E) > 0$ and $|f| = \infty$ on E. Then Proposition 1.24(b) shows that

$$\int_X |f|^p \, d\mu \geq \int_E |f|^p \, d\mu = \infty,$$

a contradiction. Thus there exists a positive constant M such that $|f| \leq M$ a.e. on X. Given $\epsilon > 0$. Now for every $E \in \mathfrak{M}$ with $\mu(E) < \frac{\epsilon}{M^p}$, we have

$$\int_E |f|^p \, d\mu \leq M^p \mu(E) \leq \epsilon \quad \text{a.e. on } X$$

which is exactly what we want. ■

By Lemma 3.7, there exists a $\delta > 0$ such that the inequality (3.114) holds for every $E \in \mathfrak{M}$ with $\mu(E) < \delta$. For this $\delta > 0$, suppose that we can find a measurable set F such that $f_n \to f$ a.e. on F and $\mu(F) < \infty$. (The existence of such a F will be clear at the end of **Step 2** below.) Then Egoroff's Theorem guarantees that there exists a measurable set $B \subseteq F$ such that $\mu(F \setminus B) < \delta$ and $\{f_n\}$ converges uniformly to f on B. Since $F \setminus B \in \mathfrak{M}$ and $\mu(F \setminus B) < \delta$, we also have

$$\int_{F \setminus B} |f|^p \, d\mu \leq \frac{\epsilon}{2}. \tag{3.115}$$

Furthermore, since $f_n \to f$ uniformly on B, there is a positive integer N such that $n \geq N$ implies that

$$|f_n(x) - f(x)| \leq \left(\frac{\epsilon}{\mu(B)}\right)^{\frac{1}{p}}$$

for all $x \in B$ and this means that

$$\int_B |f_n - f|^p \, d\mu \leq \epsilon \tag{3.116}$$

for $n \geq N$.

* **Step 2: Lemma 3.8 and its application.**

Lemma 3.8

If $f \in L^1(\mu)$, then for every $\epsilon > 0$, there exists an $E \in \mathfrak{M}$ such that $F = X \setminus E$, $\mu(F) < \infty$ and

$$\int_E |f| \, d\mu \leq \epsilon.$$

Proof of Lemma 3.8. Since $|f|$ is measurable, Theorem 1.17 (The Simple Function Approximation Theorem) ensures that there exists a sequence of simple measurable functions $\{s_k\}$ on X such that

$$0 \le s_1 \le s_2 \le \cdots \le |f| \tag{3.117}$$

and s_k converges to $|f|$ pointwisely. By Theorem 1.26 (Lebesgue's Monotone Convergence Theorem), we have

$$\lim_{k\to\infty} \int_X s_k \, d\mu = \int_X |f| \, d\mu.$$

Hence, for every $\epsilon > 0$, there exists a positive integer N such that

$$\int_X (|f| - s_k) \, d\mu = \left| \int_X (|f| - s_k) \, d\mu \right| \le \epsilon. \tag{3.118}$$

By Definition 1.16, we have

$$s_k = \sum_{i=1}^{n_k} \alpha_i \chi_{F_i},$$

where each α_i is positive, F_i is measurable and $F_i \cap F_j = \varnothing$ for all $i \neq j$. Recall that $|f| \in L^1(\mu)$, so the hypothesis (3.117) implies that $s_k \in L^1(\mu)$ for every $n \in \mathbb{N}$. Since each α_i is positive, μ is positive on X (i.e., $\mu(E) \ge 0$ for every $E \in \mathfrak{M}$) and

$$\sum_{i=1}^{n_k} \alpha_i \mu(F_i) = \int_X s_k \, d\mu < \infty,$$

they force that $\mu(F_i) < \infty$ for every $i = 1, 2, \ldots, n_k$.

We suppose that

$$E = \{x \in X \,|\, s_k(x) = 0\} \quad \text{and} \quad F = \bigcup_{i=1}^{n_k} F_i.$$

Then the previous analysis shows that $\mu(F) < \infty$. Furthermore, it is clear that $s_k(x) > 0$ if and only if $x \in F_i$ for some i, so this fact implies that

$$X \setminus E = F.$$

Thus we deduce from the estimate (3.118) and the definition of E that

$$\int_E |f| \, d\mu = \int_E (|f| - s_k) \, d\mu + \int_E s_k \, d\mu \le \int_X (|f| - s_k) \, d\mu + \int_E s_k \, d\mu \le \epsilon.$$

This proves Lemma 3.8. ∎

Since $f \in L^p(\mu)$, we have $|f|^p \in L^1(\mu)$. Then Lemma 3.8 implies that there exists an $E \in \mathfrak{M}$ such that $F = X \setminus E$, $\mu(F) < \infty$ and

$$\int_E |f|^p \, d\mu \le \frac{\epsilon}{2}. \tag{3.119}$$

* **Step 3: Constructions of A and B satisfying the hypotheses.** Let E and F be defined as in Lemma 3.8 so that the estimate (3.119) holds. We remark that

$$X = E \cup F = E \cup (F \setminus B) \cup B = A \cup B,$$

where B is the measurable set guaranteed by Egoroff's Theorem in **Step 1** and $A = E \cup (F \setminus B)$. Recall that $B \subseteq F$, so it is easy to see that

$$\mu(B) \leq \mu(F) < \infty$$

and since $E \cap (F \setminus B) = \varnothing$, we obtain from Theorem 1.29 and the two estimates (3.119) and (3.115) that

$$\int_A |f|^p \,\mathrm{d}\mu = \int_E |f|^p \,\mathrm{d}\mu + \int_{F\setminus B} |f|^p \,\mathrm{d}\mu \leq \epsilon. \tag{3.120}$$

Since $f_n \to f$ a.e. on X and particularly on B, this and Theorem 1.28 (Fatou's Lemma) together show that

$$\int_B |f|^p \,\mathrm{d}\mu = \int_B \liminf_{n\to\infty} |f_n|^p \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_B |f_n|^p \,\mathrm{d}\mu \tag{3.121}$$

Next, we know from the definition that $A = (E \cup F) \setminus B = X \setminus B$ so that $A \cap B = \varnothing$. Since $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$, it deduces from this, Problem 1.4 and the inequalities (3.120) and (3.121) that

$$\begin{aligned}
\limsup_{n\to\infty} \int_A |f_n|^p \,\mathrm{d}\mu &\leq \limsup_{n\to\infty} \int_X |f_n|^p \,\mathrm{d}\mu + \limsup_{n\to\infty} \Big(- \int_B |f_n|^p \,\mathrm{d}\mu \Big) \\
&= \limsup_{n\to\infty} \int_X |f_n|^p \,\mathrm{d}\mu - \liminf_{n\to\infty} \int_B |f_n|^p \,\mathrm{d}\mu \\
&\leq \int_X |f|^p \,\mathrm{d}\mu - \int_B |f|^p \,\mathrm{d}\mu \\
&= \int_A |f|^p \,\mathrm{d}\mu \\
&\leq \epsilon.
\end{aligned} \tag{3.122}$$

* **Step 4: The establishment of $\|f_n - f\|_p \to 0$ as $n \to \infty$.** By part (a), the estimates (3.116) and (3.122), we see that

$$\begin{aligned}
\limsup_{n\to\infty} \int_X |f_n - f|^p \,\mathrm{d}\mu &\leq \limsup_{n\to\infty} \int_A |f_n - f|^p \,\mathrm{d}\mu + \limsup_{n\to\infty} \int_B |f_n - f|^p \,\mathrm{d}\mu \\
&\leq \gamma_p \limsup_{n\to\infty} \int_A (|f|^p + |f_n|^p) \,\mathrm{d}\mu + \epsilon \\
&\leq 2\gamma_p \epsilon + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we conclude that

$$\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \Big\{ \int_X |f_n - f|^p \,\mathrm{d}\mu \Big\}^{\frac{1}{p}} = 0.$$

(ii) Put $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$. By part (a), we have $h_n \geq 0$ for all $n \in \mathbb{N}$. Since f and f_n are measurable, each h_n is also measurable. By Theorem 1.28 (Fatou's Lemma), we get

$$\int_X \Big(\liminf_{n\to\infty} h_n \Big) \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X h_n \,\mathrm{d}\mu. \tag{3.123}$$

Since $f_n \to f$ a.e. on X, $|f_n - f|^p \to 0$ and $|f_n|^p \to |f|^p$ a.e. on X. Thus we must have

$$\lim_{n\to\infty} h_n = 2\gamma_p |f|^p \quad \text{a.e. on } X. \tag{3.124}$$

Putting the limit (3.124) into the inequality (3.123), we obtain from Problem 1.4 that

$$\begin{aligned}\int_X 2\gamma_p |f|^p \, d\mu &= \int_X \Big(\lim_{n\to\infty} h_n\Big)\, d\mu \\ &= \int_X \Big(\liminf_{n\to\infty} h_n\Big)\, d\mu \\ &= \liminf_{n\to\infty} \int_X h_n \, d\mu \\ &\leq \liminf_{n\to\infty} \Big[\int_X \gamma_p(|f|^p + |f_n|^p)\, d\mu - \int_X |f_n - f|^p \, d\mu\Big] \\ &\leq \gamma_p \liminf_{n\to\infty} \Big[\int_X (|f|^p + |f_n|^p)\, d\mu\Big] - \limsup_{n\to\infty} \int_X |f_n - f|^p \, d\mu \\ &= \gamma_p \liminf_{n\to\infty} \big(\|f_n\|_p^p + \|f\|_p^p\big) - \limsup_{n\to\infty} \int_X |f_n - f|^p \, d\mu. \end{aligned} \tag{3.125}$$

Since $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$, we can further reduce the inequality (3.125) to

$$0 \leq \limsup_{n\to\infty} \int_X |f_n - f|^p \, d\mu \leq \gamma_p \Big(\lim_{n\to\infty} \|f_n\|_p^p + \|f\|_p^p\Big) - 2\gamma_p \|f\|_p^p = 0.$$

Hence this leads to the following

$$\lim_{n\to\infty} \|f_n - f\|_p = \limsup_{n\to\infty} \|f_n - f\|_p = 0$$

as desired.

(c) Consider $X = (0,1)$ and $\mu = m$ the Lebesgue measure. For each $n = 1, 2, \ldots$, we let $E_n = (0, \frac{1}{n^p})$ and $f_n = n\chi_{E_n} : (0,1) \to \mathbb{R}$. If $x \in (0,1)$, then there exists a positive integer N such that $x \notin E_n$ for all $n \geq N$. In this case, we have $f_n(x) = 0$ for all $n \geq N$. In other words, we have

$$f_n(x) \to f(x) \equiv 0$$

for every $x \in (0,1)$. It is clear that $f \in L^p\big((0,1)\big)$ and $\|f\|_p = 0$. Besides, we have

$$\|f_n\|_p = \Big\{\int_0^1 |f_n(x)|^p \, dx\Big\}^{\frac{1}{p}} = \Big\{\int_{E_n} n^p \, dx\Big\}^{\frac{1}{p}} = 1$$

for every $n \in \mathbb{N}$. Thus we know that

$$\|f_n\|_p \nrightarrow \|f\|_p$$

as $n \to \infty$. Finally, since $\|f_n - f\|_p = \|f_n\|_p = 1$, we see that $\|f_n - f\|_p \nrightarrow 0$ as $n \to \infty$, i.e., the conclusion of part (b) is false.

We have completed the proof of the problem. ■

Remark 3.1

We remark that there is a short and elementary proof of Problem 3.17(b) in [45].

3.5 Convergence in Measure and the Essential Range of $f \in L^\infty(\mu)$

Problem 3.18

Rudin Chapter 3 Exercise 18.

Proof.

(a) Given small $\epsilon > 0$. For each positive integer k, we let

$$E_k = \{x \in X \,|\, |f_n(x) - f(x)| \leq \epsilon \text{ for all } n \geq k\} \quad \text{and} \quad E = \bigcup_{k=1}^{\infty} E_k.$$

The definitions of E_k and E guarantee that

$$X \setminus E = \bigcap_{k=1}^{\infty} (X \setminus E_k) = \bigcap_{k=1}^{\infty} \{x \in X \,|\, |f_n(x) - f(x)| > \epsilon \textit{ for some } n \geq k\}. \tag{3.126}$$

Since $f_n(x) \to f(x)$ a.e. on X, we deduce from the expression (3.126) that

$$\mu(X \setminus E) = 0.$$

Furthermore, we have $E_1 \subseteq E_2 \subseteq \cdots$, so Theorem 1.19 implies that $\mu(E_k) \to \mu(E)$ as $k \to \infty$. Thus it follows from this fact and the result $\mu(X \setminus E) = 0$ that

$$\lim_{k\to\infty} \mu(E_k) = \mu(E) = \mu(X) < \infty.$$

This result ensures that there exists a $N \in \mathbb{N}$ such that $\mu(X \setminus E_N) < \epsilon$.

$$\{x \in X \,|\, |f_n(x) - f(x)| > \epsilon \textit{ for all } n > N\} \subseteq X \setminus E_N.$$

Hence we must have

$$\mu(\{x \in X \,|\, |f_n(x) - f(x)| > \epsilon \text{ for all } n > N\}) < \epsilon,$$

i.e., $f_n \to f$ in measure.

(b) Suppose that $1 \leq p < \infty$. Since $\|f_n - f\|_p \to 0$ as $n \to \infty$, we must have $f_n - f \in L^p(\mu)$ for every $n \in \mathbb{N}$. Since $f_n \in L^p(\mu)$, we also have $f = (f - f_n) + f_n \in L^p(\mu)$ by Theorem 3.9. Given $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies that

$$\|f_n - f\|_p < \epsilon^{p+1}. \tag{3.127}$$

Let $F_n = \{x \in X \,|\, |f_n(x) - f(x)| > \epsilon\}$ for each $n = 1, 2, \ldots$. Now it is easy to check that[i]

$$\int_{F_n} |f_n(x) - f(x)|^p \,\mathrm{d}\mu \geq \int_{F_n} \epsilon^p \,\mathrm{d}\mu = \epsilon^p \mu(F_n). \tag{3.128}$$

[i]The inequality (3.128) is a consequence of the so-called **Chebyshev's Inequality**: If f is a nonnegative, extended real-valued measurable function on X with measure μ, $0 < p < \infty$ and $\epsilon > 0$, then

$$\mu(\{x \in X \,|\, f(x) \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \int_X f^p \,\mathrm{d}\mu.$$

See [22, Theorem 6.17, p. 193].

Hence we establish from the inequalities (3.127) and (3.128) that

$$\mu(\{x \in X \,|\, |f_n(x) - f(x)| > \epsilon\}) = \mu(F_n) < \epsilon$$

for $n \geq N$. By the definition, $f_n \to f$ in measure.

Next, we suppose that $p = \infty$. Now $\|f_n - f\|_\infty \to 0$ means the existence of a positive integer N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ a.e. on X. By the definition, we have $\mu(F_n) = 0 < \epsilon$ for all $n \geq N$, i.e., $f_n \to f$ in measure in this case.

(c) Suppose that $f_n \to f$ in measure, i.e., for every $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies that

$$\mu(\{x \in X \,|\, |f_n(x) - f(x)| > \epsilon\}) < \epsilon.$$

In particular, for each positive integer k, there exists a positive integer N_k such that

$$\mu(\{x \in X \,|\, |f_n(x) - f(x)| > 2^{-k}\}) < 2^{-k} \tag{3.129}$$

for all $n \geq N_k$. Now we may choose $n_k > N_k$ *freely* in the estimate (3.129) and consider the subsequence $\{f_{n_k}\}$. We define

$$E_k = \{x \in X \,|\, |f_{n_k}(x) - f(x)| > 2^{-k}\} \quad \text{and} \quad E = \bigcap_{m=1}^{\infty} \bigcup_{k\geq m}^{\infty} E_k. \tag{3.130}$$

Then it follows from the estimate (3.129) that

$$\mu(E) \leq \mu\Big(\bigcup_{k\geq m}^{\infty} E_k\Big) \leq \sum_{k=m}^{\infty} \mu(E_k) < \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}},$$

where $m = 1, 2, \ldots$. As a result, we must have $\mu(E) = 0$. Take any $p \in X \setminus E$, the definitions (3.130) say that

$$p \in \bigcap_{k\geq m}^{\infty} (X \setminus E_k) = \bigcap_{k\geq m}^{\infty} \{x \in X \,|\, |f_{n_k}(x) - f(x)| \leq 2^{-k}\}$$

for some $m \in \mathbb{N}$ and then

$$|f_{n_k}(p) - f(p)| \leq 2^{-k}$$

for all $k \geq m$. In other words, we have

$$\lim_{k\to\infty} f_{n_k}(p) = f(p)$$

which is our desired result.

Here we propose two examples which say that the converses of parts (a) and (b) fail:

- **Failure of the converse of part (a).** By Problem 2.9, there exists a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that $0 \leq f_n \leq 1$ and

$$\|f_n - 0\|_1 \to 0$$

as $n \to \infty$, but there is no $x \in [0,1]$ such that $\{f_n(x)\}$ converges. In other words, $f_n(x)$ *does not* converge to 0 a.e. on $[0,1]$. However, since each f_n is continuous $[0,1]$, it is $\mathscr{R}$ on $[0,1]$ so that $f_n \in L^1\big([0,1]\big)$ by [49, Theorem 11.33, p. 323]. Thus the sequence of functions $\{f_n\}$ satisfies the hypotheses of part (b) which shows that $f_n \to 0$ in measure. Hence this example implies the the converse of part (a) is false.

- **Failure of the converse of part (b).** Suppose that $X = [0,1]$, μ is the Lebesgue measure and
$$f_n = \mathrm{e}^n \chi_{[0,\frac{1}{n}]}$$
for each $n = 1, 2, \ldots$. Then $f_n \to 0$ a.e. on $[0,1]$ and part (a) implies that $f_n \to 0$ in measure. However, we note that
$$\|f_n\|_p = \left\{ \int_X |f_n(x)|^p \,\mathrm{d}\mu \right\}^{\frac{1}{p}} = \left\{ \int_0^{\frac{1}{n}} \mathrm{e}^{np} \,\mathrm{d}x \right\}^{\frac{1}{p}} = \frac{\mathrm{e}^n}{n^{\frac{1}{p}}} \to \infty$$
as $n \to \infty$, so the sequence $\{f_n\}$ fails to converge in $L^p\big([0,1]\big)$.

Finally, by examining the proofs of parts (b) and (c), we see that they remain to be true in the case $\mu(X) = \infty$. However, part (a) *does not* hold in this case. For instance, let $X = [0,\infty)$ and $E_n = [0,n]$ for each $n \in \mathbb{N}$. Consider the functions $f_n = \chi_{E_n}$. Then for every $x \in [0,\infty)$, there exists a positive integer N such that $x \in E_n$ for all $n \geq N$, so it is true that $f_n(x) = 1$ for all $n \geq N$, i.e., $f_n(x) \to f(x) \equiv 1$ on $[0,\infty)$. However, for every $\epsilon \in (0,1)$, since $f_n(x) = 0$ if and only if $x \in (n,\infty)$, we see that
$$\{x \in [0,\infty) \,|\, |f_n(x) - 1| > \epsilon\} = (n,\infty)$$
for every $n \in \mathbb{N}$. Hence it leads that
$$m(\{x \in [0,\infty) \,|\, |f_n(x) - 1| > \epsilon\}) = \infty$$
for every $n \in \mathbb{N}$, i.e., f_n does not converge in measure to f.

Hence we end the proof of the problem. ■

Problem 3.19

Rudin Chapter 3 Exercise 19.

Proof. For every $\epsilon > 0$, define $E_\epsilon(w) = \{x \in X \,|\, |f(x) - w| < \epsilon\}$ and
$$R_f = \{w \in \mathbb{C} \,|\, \mu\big(E_\epsilon(w)\big) > 0 \text{ for every } \epsilon > 0\} = \bigcap_{\epsilon>0} \{w \in \mathbb{C} \,|\, \mu\big(E_\epsilon(w)\big) > 0\}. \tag{3.131}$$
We are going to prove the assertions one by one:

- **R_f is compact.** Given $\{w_n\} \subseteq R_f$ and $w \in \mathbb{C}$ such that $w_n \to w$ as $n \to \infty$. For every $\epsilon > 0$, there exists a positive integer N such that $|w_n - w| < \frac{\epsilon}{2}$. By the triangle inequality, we have
$$|f(x) - w| \leq |f(x) - w_n| + |w_n - w| < |f(x) - w_n| + \frac{\epsilon}{2}$$
which means that $E_{\frac{\epsilon}{2}}(w_n) \subseteq E_\epsilon(w)$ or equivalently
$$\mu\big(E_\epsilon(w)\big) > \mu\big(E_{\frac{\epsilon}{2}}(w_n)\big) > 0. \tag{3.132}$$
Since ϵ is arbitrary, we know from the estimate (3.132) that $w \in R_f$. In other words, R_f is closed in $\mathbb{C}$.

  Recall that $\|f\|_\infty < \infty$, so we pick a complex number w_0 such that $|w_0| > \|f\|_\infty$ and then consider the positive number $\epsilon = |w_0| - \|f\|_\infty$. If $x \in E_\epsilon(w_0)$, then the triangle inequality shows that
$$|w_0| - |f(x)| < |f(x) - w_0| < \epsilon = |w_0| - \|f\|_\infty$$

which implies that $|f(x)| > \|f\|_\infty$. Thus we have $E_\epsilon(w_0) \subseteq \{x \in X \,|\, |f(x)| > \|f\|_\infty\}$. By Definition 3.7, $\mu(\{x \in X \,|\, |f(x)| > \|f\|_\infty\}) = 0$, so $\mu\big(E_\epsilon(w_0)\big) = 0$ too. By the definition (3.131), we have $w_0 \notin R_f$ and hence the set R_f is bounded by $\|f\|_\infty$. By the Heine–Borel Theorem, we conclude that R_f is compact.

- **A relation between R_f and $\|f\|_\infty$.** By the previous analysis, it is obvious that the relation
$$R_f \subseteq \{w \in \mathbb{C} \,|\, |w| \leq \|f\|_\infty\} \tag{3.133}$$
holds, i.e., R_f lies in $\overline{B(0, \|f\|_\infty)}$. We claim that
$$\|f\|_\infty = \max\{|z| \,|\, z \in R_f\}. \tag{3.134}$$
Given $w \in \mathbb{C}$ and $\epsilon > 0$. Let $B(w, \epsilon) = \{z \in \mathbb{C} \,|\, |z - w| < \epsilon\}$. Then we have
$$f^{-1}\big(B(w, \epsilon)\big) = \{x \in X \,|\, f(x) \in B(w, \epsilon)\} = \{x \in X \,|\, |f(x) - w| < \epsilon\} = E_\epsilon(w).$$
By the definition, we see that
$$R_f = \bigcap_{\epsilon > 0} \{w \in \mathbb{C} \,|\, \mu\big(f^{-1}\big(B(w, \epsilon)\big)\big) > 0\} = \mathbb{C} \setminus V, \tag{3.135}$$
where $V = \{w \in \mathbb{C} \,|\, \mu\big(f^{-1}\big(B(w, \epsilon)\big)\big) = 0 \textit{ for some } \epsilon > 0\}$. Now we have the following two facts:
  - **Fact 1:** It is easy to show that V is the *largest* open subset of $\mathbb{C}$ such that
  $$\mu\big(f^{-1}(V)\big) = \mu(\{x \in X \,|\, f(x) \in V\}) = 0,$$
  i.e., V is the largest open subset of $\mathbb{C}$ such that $f(x) \notin V$ for almost all $x \in X$. By this fact and the relation (3.135), we conclude that R_f is the *smallest* closed subset of $\mathbb{C}$ such that $f(x) \in R_f$ for almost all $x \in X$.
  - **Fact 2:** Recall from Definition 3.7 that the number $\|f\|_\infty$ is the minimum of the set $\{\alpha \geq 0\}$ such that $\mu(\{x \in X \,|\, |f(x)| > \alpha\}) = 0$, so we see from the relation (3.133) that it is the *minimum* radius of the closed disc centered at 0 containing the set R_f.

  Hence we follow immediately from these two observations that the equality actually holds in the set relation (3.133). Since they are equal, our claim (3.134) is established.

- **Relations between A_f and R_f.** By the definition, we have
$$A_f = \Big\{\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \,\Big|\, E \in \mathfrak{M} \text{ and } \mu(E) > 0\Big\}. \tag{3.136}$$
We claim that $R_f \subseteq \overline{A_f}$. Given $\epsilon > 0$ and $w \in R_f$. If $w \in A_f$, then there is nothing to prove. Therefore, we may assume that $w \notin A_f$. Consider $E_\epsilon = \{x \in X, | \,|f(x) - w| < \epsilon\}$. By the definition of R_f, we have $\mu(E_\epsilon) > 0$. If $\mu(E_\epsilon) = \infty$ for every ϵ, then $f(x) = w$ a.e. on X. In this case, we have
$$\frac{1}{\mu(E_\epsilon)} \int_{E_\epsilon} f \, \mathrm{d}\mu = w \in A_f,$$
a contradiction. Therefore, we may assume that $\mu(E_\epsilon) < \infty$ *for infinitely many* $\epsilon > 0$. In fact, we may take $\epsilon = \frac{1}{n}$ and let $E(n) = E_{\frac{1}{n}}$. Then, since $f \in L^\infty(\mu)$, we have $f - w \in L^1(E(n))$, where
$$L^1\big(E(n)\big) = \Big\{f : X \to \mathbb{C} \,\Big|\, \int_{E(n)} |f| \, \mathrm{d}\mu < \infty\Big\}.$$

Consequently, it follows from Theorem 1.33 and the definition of E that

$$\Big|\frac{1}{\mu(E(n))}\int_{E(n)} f\,\mathrm{d}\mu - w\Big| \leq \frac{1}{\mu(E(n))}\int_{E(n)} |f-w|\,\mathrm{d}\mu < \frac{1}{n}.$$

In other words, w is a limit point of A_f and this means that $w \in \overline{A_f}$. Hence this proves the claim that $R_f \subseteq \overline{A_f}$.

- A_f **is not always closed.** For example, we consider $X=[0,1]$, $f(x)=x$ with $\mu=m$ the Lebesque measure. By Definition 3.7, $f \in L^\infty\big([0,1]\big)$. In addition, for a measurable set E in $[0,1]$ with $m(E)>0$, we have

$$w(E) = \frac{1}{m(E)}\int_E f(x)\,\mathrm{d}x. \tag{3.137}$$

We claim that $w(E)$ can take any value in $(0,1)$. Indeed, if $a,b \in (0,1)$ and $E=(a,b)$, then we have $m(E)=b-a>0$ and it follows from the definition (3.137) that

$$w(E) = \frac{1}{m(E)}\int_E f\,\mathrm{d}x = \frac{1}{b-a}\int_a^b x\,\mathrm{d}x = \frac{1}{b-a}\times\frac{b^2-a^2}{2} = \frac{a+b}{2}$$

which implies the claim.

Next, we claim that $w(E) \neq 0$. Assume that $w(E_0)=0$ for a measurable set $E_0 \subseteq [0,1]$ with $m(E_0)>0$. Then we have

$$\int_{E_0} f(x)\,\mathrm{d}x = 0,$$

but Theorem 1.39(a) implies that $f(x)=0$ a.e. on E_0 which contradicts the fact that $f(x)=0$ only at $x=0$. Hence 0 is not a limit point of A_f and A_f is not closed in $\mathbb{R}$.

- **A measure μ such that A_f is convex for every $f \in L^\infty(\mu)$.** Consider $X=\{0\}$ and μ the counting measure (see Example 1.20(a)). Then we have $\mathfrak{M}=\{\varnothing, X\}$ and $\mu(X)=\mu(\{0\})=1>0$. By the definition, $\|f\|_\infty = |f(0)| < \infty$ for every $f \in L^\infty(\mu)$ so that $f(0)$ is either $\|f\|_\infty$ or $-\|f\|_\infty$. Now for every $f \in L^\infty(\mu)$, we have

$$w(X) = \frac{1}{\mu(X)}\int_X f\,\mathrm{d}\mu = f(0).$$

Therefore, the definition (3.136) gives either $A_f = \{\|f\|_\infty\}$ or $A_f=\{-\|f\|_\infty\}$, but both cases are also convex sets.

- **A measure μ such that A_f is not convex for some $f \in L^\infty(\mu)$.** We consider the set $X=\{0,1\}$, μ the counting measure and $f(x)=x$. Then we have $\mathfrak{M}=\{\varnothing,\{0\},\{1\},X\}$, $\mu(\{0\})=\mu(\{1\})=1>0$ and $\mu(X)=2>0$. Now we have

$$w(\{0\}) = \frac{1}{\mu(\{0\})}\int_{\{0\}} f\,\mathrm{d}\mu = f(0) = 0, \quad w(\{1\}) = \frac{1}{\mu(\{1\})}\int_{\{1\}} f\,\mathrm{d}\mu = f(1) = 1,$$
$$w(X) = \frac{1}{\mu(X)}\int_X f\,\mathrm{d}\mu = \frac{1}{2}[f(0)\mu(\{0\})+f(1)\mu(\{1\})] = \frac{1}{2}[f(0)+f(1)] = \frac{1}{2}.$$

Thus we have $A_f=\{0,1,\frac{1}{2}\}$ is *not* convex.

- **The situations when $L^\infty(\mu)$ is replaced by $L^1(\mu)$.** We consider $f:(0,\infty)\to\mathbb{C}$ by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & \text{if } x \in (0,1]; \\ \\ 0, & \text{if } x > 1 \end{cases}$$

and take $\mu = m$ the Lebesgue measure. By Lemma 3.1, we have $f \in L^1\big((0,\infty)\big)$. For any large positive integer N, if $x \in (0, \frac{1}{N^2})$, then we have $f(x) > N$ and

$$m(\{x \in (0,\infty) \,|\, f(x) > N\}) = m\Big(\Big(0, \frac{1}{N^2}\Big)\Big) = \frac{1}{N^2} > 0.$$

By Definition 3.7, we have $\|f\|_\infty = \infty$, i.e., $f \notin L^\infty\big((0,\infty)\big)$. Given that $\epsilon > 0$ and sufficiently large $N \in \mathbb{N}$.

- **R_f is unbounded and not compact.** If $x \in \big(\frac{1}{(N+\epsilon)^2}, \frac{1}{(N-\epsilon)^2}\big) \subseteq (0,1)$, then we have $|f(x) - N| < \epsilon$ so that

$$m(\{x \in (0,\infty) \,|\, |f(x) - N| < \epsilon\}) = m\Big(\Big(\frac{1}{(N+\epsilon)^2}, \frac{1}{(N-\epsilon)^2}\Big)\Big) > 0.$$

  Thus $N \in R_f$ and then $\mathbb{N} \subseteq R_f$, i.e., R_f is unbounded and not compact.
- **A_f is unbounded.** Next, if $E = (0, \frac{4}{N^2})$, then $m(E) = \frac{4}{N^2} > 0$ and

$$w(E) = \frac{1}{m(E)} \int_E f(x)\,\mathrm{d}x = \frac{N^2}{4} \int_0^{\frac{4}{N^2}} x^{-\frac{1}{2}}\,\mathrm{d}x = \frac{N^2}{4} \times 2\sqrt{x}\Big|_0^{\frac{4}{N^2}} = N.$$

  Therefore, $\mathbb{N} \subseteq A_f$, i.e., A_f is unbounded.
- **A_f is not always closed.** In fact, the previous example that $X = [0,1]$, $f(x) = x$ and $\mu = m$ satisfies both $f \in L^1\big([0,1]\big)$ and $f \in L^\infty\big([0,1]\big)$. Thus the same conclusion that 0 is *not* a limit point of A_f is achieved.
- **A measure μ such that A_f is convex for every $f \in L^1(\mu)$.** The example that $X = \{0\}$ and μ the counting measure also works for this case because if $f \in L^1(\mu)$, then $|f(0)| = \|f\|_1 < \infty$. Thus we have either $A_f = \{\|f\|_1\}$ or $A_f = \{-\|f\|_1\}$.
- **A measure μ such that A_f is not convex for some $f \in L^1(\mu)$.** The example we considered for the case $L^\infty(\mu)$ also works in this case.

We complete the proof of the problem. ■

3.6 A Converse of Jensen's Inequality

Problem 3.20

Rudin Chapter 3 Exercise 20.

Proof. We check Definition 3.1. Let $p, q \in \mathbb{R}$ and $\lambda \in [0,1]$. By the trick of Proposition 1.24(f), we see that

$$\begin{aligned}\lambda p + (1-\lambda)q &= p\int_{[0,1]} \chi_{[0,\lambda]}(x)\,\mathrm{d}x + q\int_{[0,1]} \chi_{[\lambda,1]}(x)\,\mathrm{d}x \\ &= \int_0^1 [p\chi_{[0,\lambda]}(x) + q\chi_{[\lambda,1]}(x)]\,\mathrm{d}x. \qquad (3.138)\end{aligned}$$

Suppose that

$$f(x) = p\chi_{[0,\lambda]}(x) + q\chi_{[\lambda,1]}(x) \qquad (3.139)$$

which is clearly a real and bounded function. Besides, since $[0,\lambda]$ and $[\lambda,1]$ are Borel sets in $[0,1]$, $\chi_{[0,\lambda]}$ and $\chi_{[\lambda,1]}$ are measurable by Proposition 1.9(d). By the last paragraph in [51, §1.22,

p. 19], the function f, as the sum of two measurable functions, is also measurable. Hence, by substituting the expressions (3.138) and (3.139) into the inequality in question, we obtain

$$\begin{aligned}\varphi\big(\lambda p+(1-\lambda)q\big) &\leq \int_0^1 \varphi\big(p\chi_{[0,\lambda]}(x)+q\chi_{[\lambda,1]}(x)\big)\,\mathrm{d}x\\ &= \int_0^\lambda \varphi(p)\,\mathrm{d}x+\int_\lambda^1 \varphi(q)\,\mathrm{d}x\\ &= \lambda\varphi(p)+(1-\lambda)\varphi(q).\end{aligned}$$

Hence φ is convex in $\mathbb{R}$, completing the proof of the problem. ■

Remark 3.2

We note that Problem 3.20 is a converse of Theorem 3.3 (Jensen's Inequality).

3.7 The Completeness/Completion of a Metric Space

Problem 3.21

Rudin Chapter 3 Exercise 21.

Proof. Let (X,d) and (X^*,ρ) be two metric spaces. An *isometry* of X into X^* is a mapping $\varphi : X \to Y$ such that

$$\rho\big(\varphi(p),\varphi(q)\big)=d(p,q) \tag{3.140}$$

for all $p,q\in X$. We notice immediately that an isometry φ is necessarily injective and continuous. If φ is surjective, then we call φ an *isomorphism*. Two metric spaces X and Y are called *isomorphic* if there is an isomorphism between them. Before stating the statement, we need:

Lemma 3.9

Let (X,d) be a metric space with metric d and $p\in X$. Then the function $f: X\to\mathbb{R}$ defined by $f(x)=d(x,p)$ is continuous.

Proof of Lemma 3.9. Given $\epsilon>0$. Let $x\in X$. If $y\in X$ satisfies $d(x,y)<\epsilon$, then the triangle inequality implies that

$$|f(x)-f(y)|=|d(x,p)-d(y,p)|\leq d(x,y)<\epsilon.$$

Hence f is continuous, completing the proof of Lemma 3.9. ■

Lemma 3.10

Suppose that (X,d) and (Y,ρ) are metric spaces and Y is complete. If E is dense in X and $f: E\to Y$ is an isometry, then there exists an isometry $g: X\to Y$ such that $g|_E=f$.

Proof of Lemma 3.10. If $x \in E$, we define $g(x) = f(x)$. Let $x \in X \setminus E$. Since E is dense in X, we can choose a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$. By [49, Theorem 3.11(a)], $\{x_n\}$ is Cauchy in X. Thus, given $\epsilon > 0$, there exists a positive integer N such that $n, m \geq N$ imply that

$$d(x_m, x_n) < \epsilon.$$

Since f is an isometry, it follows from the expression (3.140) that

$$\rho\big(f(x_m), f(x_n)\big) < \epsilon$$

for all $n, m \geq N$. In other words, $\{f(x_n)\}$ is Cauchy in Y. Since Y is complete, $\{f(x_n)\}$ converges to a limit y in Y and we define $g(x) = y$ in this case.

Now this y is *uniquely determined* by x. Indeed, let $\{x'_n\} \subseteq E$ be another sequence converging to x and y' be the limit of $\{f(x'_n)\}$ in Y. For each $m \in \mathbb{N}$, it follows from Lemma 3.9 and the expression (3.140) that

$$\rho\big(y, f(x'_m)\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(x'_m)\big) = \lim_{n\to\infty} d(x_n, x'_m) = d(x, x'_m).$$

Then this implies that

$$\rho(y, y') = \lim_{m\to\infty} \rho\big(y, f(x'_m)\big) = \lim_{m\to\infty} d(x, x'_m) = 0.$$

Since ρ is a metric, we must have $y = y'$ and so we obtain a mapping $g : X \to Y$ given by

$$g(x) = \begin{cases} f(x), & \text{if } x \in E; \\ \lim\limits_{n\to\infty} f(x_n), & \text{if } x \in X \setminus E,\ \{x_n\} \subseteq X \text{ and } x_n \to x. \end{cases} \tag{3.141}$$

Next, we show that g is an isometry and we consider the following situations:

- **Case (i):** $x, y \in E$**.** In this case, we know from the definition (3.141) that $g(x) = f(x)$ and $g(y) = f(y)$ and then the use of the expression (3.140) implies that
$$\rho\big(g(x), g(y)\big) = \rho\big(f(x), f(y)\big) = d(x, y).$$
- **Case (ii):** $x \in X \setminus E$ **and** $y \in E$**.** In this case, we have
$$\rho\big(g(x), g(y)\big) = \rho\big(g(x), f(y)\big).$$
Choose a sequence $\{x_n\}$ in E such that $x_n \to x$. By Lemma 3.9 and the expression (3.140), we establish that
$$\rho\big(g(x), f(y)\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(y)\big) = \lim_{n\to\infty} d(x_n, y) = d(x, y). \tag{3.142}$$
- **Case (iii):** $x, y \in X \setminus E$**.** Let $\{x_n\}, \{y_n\} \subseteq E$ be sequences converging to x and y respectively. Then we deduce from the definition (3.141) and the expression (3.142) that
$$\rho\big(g(x), f(y_n)\big) = d(x, y_n). \tag{3.143}$$
Apply Lemma 3.9 to (3.143), we see that
$$\rho\big(g(x), g(y)\big) = \lim_{n\to\infty} \rho\big(g(x), f(y_n)\big) = \lim_{n\to\infty} d(x, y_n) = d(x, y).$$

By the definition, g is an isometry and $g|_E = f$. ∎

Lemma 3.11

The mapping g in Lemma 3.9 is unique up to isometry.

Proof of Lemma 3.11. Let $g' : X \to Y$ be an isometry such that $g'|_E = f$. If $x \in E$, then

$$g(x) = f(x) = g'(x).$$

If $x \in X \setminus E$, then there is a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$ as $n \to \infty$. The triangle inequality, the expressions (3.140) and (3.142) indicate that

$$\begin{aligned}\rho\big(g(x), g'(x)\big) &\leq \rho\big(g(x), g(x_n)\big) + \rho\big(g(x_n), g'(x_n)\big) + \rho\big(g'(x_n), g'(x)\big)\\ &= \rho\big(g(x), f(x_n)\big) + \rho\big(f(x_n), f'(x_n)\big) + \rho\big(f'(x_n), g'(x)\big)\\ &= \rho\big(g(x), f(x_n)\big) + d(x_n, x_n) + \rho\big(f'(x_n), g'(x)\big)\\ &= d(x, x_n) + d(x_n, x)\end{aligned}$$

Hence when $n \to \infty$, it is clear that $\rho(g(x), g'(x)) = 0$, so $g(x) = g'(x)$ on $X \setminus E$ and our result follows from this. ■

Let's return to the proof of the problem. Let (Y_1, ρ_1) and (Y_2, ρ_2) be complete metric spaces, $\varphi_1 : X \to Y_1$ and $\varphi_2 : X \to Y_2$ be isometries such that $\varphi_1(X)$ is dense in Y_1 and $\varphi_2(X)$ is dense in Y_2. We claim that (Y_1, ρ_1) and (Y_2, ρ_2) are isomorphic. Define $f : \varphi_1(X) \to Y_2$ by

$$f = \varphi_2 \circ \varphi_1^{-1}. \tag{3.144}$$

If $p, q \in X$, $x = \varphi_1(p)$ and $y = \varphi_1(q)$, then we have

$$\begin{aligned}\rho_2\big(f(x), f(y)\big) &= \rho_2\big(\varphi_2\big(\varphi_1^{-1}(x)\big), \varphi_2\big(\varphi_1^{-1}(y)\big)\big)\\ &= \rho_2\big(\varphi_2(p), \varphi_2(q)\big)\\ &= d(p, q)\\ &= \rho_1\big(\varphi_1(p), \varphi_1(q)\big)\\ &= \rho_1(x, y).\end{aligned}$$

Consequently, f is an isometry. Since $\varphi_1(X)$ is dense in Y_1 and Y_2 is complete, Lemma 3.10 ensures that there exists an isometry $g : Y_1 \to Y_2$ such that

$$g|_{\varphi_1(X)} = f.$$

It remains to prove that g is surjective. To this end, let $y \in Y_2$. Since $\overline{\varphi_2(X)} = Y_2$, we can select a sequence $\{y_n\} \subseteq \varphi_2(X)$ such that $y_n \to y$ as $n \to \infty$. Furthermore, there exists a sequence $\{x_n\} \subseteq X$ such that

$$y_n = \varphi_2(x_n)$$

for all $n \in \mathbb{N}$. Consider the corresponding sequence

$$z_n = \varphi_1(x_n) \tag{3.145}$$

for $n \in \mathbb{N}$. Since $\{y_n\}$ is Cauchy in $\varphi_2(X)$ (and hence in Y_2) and φ_2 is an isometry, the expression (3.140) forces that $\{x_n\}$ is Cauchy in X. Then, since φ_1 is an isometry, the expression (3.140) again forces that $\{z_n\}$ is Cauchy in $\varphi_1(X)$ (and hence in Y_1). Since Y_1 is complete, we can find $z \in Y_1$ such that $z_n \to z$ as $n \to \infty$. We claim that

$$g(z) = y.$$

Thus we deduce from Lemma 3.10, the definition (3.145) and the fact $g|_{\varphi_1(X)} = f$ that

$$\rho_2\big(g(z), y\big) = \lim_{n\to\infty} \rho_2\big(g(z_n), y\big) = \lim_{n\to\infty} \rho_2\big(g\big(\varphi_1(x_n)\big), y\big) = \lim_{n\to\infty} \rho_2\big(f\big(\varphi_1(x_n)\big), y\big). \tag{3.146}$$

Finally, the definition (3.144) reduces the limit (3.146) to

$$\rho_2\big(g(z), y\big) = \lim_{n\to\infty} \rho_2\big(f\big(\varphi_1(x_n)\big), y\big) = \lim_{n\to\infty} \rho_2\big(\varphi_2(x_n), y\big) = \lim_{n\to\infty} \rho_2(y_n, y) = \rho_2(y, y) = 0.$$

Therefore, $g(z) = y$ and g is surjective. Hence g is an isomorphism and our claim follows, completing the proof of the problem. ■

Problem 3.22

Rudin Chapter 3 Exercise 22.

Proof. This problem is proven in [63, Problem 3.20, p. 50]. ■

3.8 Miscellaneous Problems

Problem 3.23

Rudin Chapter 3 Exercise 23.

Proof. If $\alpha_{n_0} = 0$ for some n_0, then we must have $f = 0$ a.e. on X by Theorem 1.39(a). However, this implies that the measure of

$$f^{-1}\big((\alpha, \infty]\big) = \{x \in X \,|\, f(x) > \alpha\}$$

is zero for every $\alpha > 0$. By Definition 3.7, it means that $\|f\|_\infty = 0$ which contradicts the hypothesis. Thus we have $\alpha_n \neq 0$ for all $n \in \mathbb{N}$ so that the limit in question is well-defined.

By Definition 3.7, we see that $|f(x)| \leq \|f\|_\infty$ holds for almost all $x \in X$. Therefore, we have

$$\alpha_{n+1} \leq \|f\|_\infty \cdot \alpha_n \quad \text{a.e. on } X \tag{3.147}$$

for all $n \in \mathbb{N}$. Since $\|f\|_\infty > 0$, we can find a $\epsilon > 0$ such that $\|f\|_\infty > \epsilon > 0$. By the definition, the measurable set

$$F = \{x \in X \,|\, |f(x)| \geq \|f\|_\infty - \epsilon > 0\}$$

satisfies $\mu(F) > 0$. Apply Theorem 3.5 (Hölder's Inequality) to the measurable functions f^n and 1 with $p = \frac{n+1}{n}$, we obtain

$$\alpha_n = \int_X |f|^n \, d\mu \leq \Big\{ \int_X (|f|^n)^{\frac{n+1}{n}} \Big\}^{\frac{n}{n+1}} \Big\{ \int_X 1^{n+1} \, d\mu \Big\}^{\frac{1}{n+1}} = (\alpha_{n+1})^{\frac{n}{n+1}} \times \mu(X)^{\frac{1}{n+1}}$$

so that

$$\begin{aligned}
\frac{\alpha_{n+1}}{\alpha_n} &\geq (\alpha_{n+1})^{\frac{1}{n+1}} \times \mu(X)^{\frac{-1}{n+1}} \\
&\geq \Big\{ \int_F |f|^{n+1} \, d\mu \Big\}^{\frac{1}{n+1}} \times \mu(X)^{\frac{-1}{n+1}} \\
&\geq \big[(\|f\|_\infty - \epsilon)^{n+1} \mu(F)\big]^{\frac{1}{n+1}} \times \mu(X)^{\frac{-1}{n+1}}
\end{aligned}$$

$$= \left(\frac{\mu(F)}{\mu(X)}\right)^{\frac{1}{n+1}} \times (\|f\|_\infty - \epsilon). \tag{3.148}$$

Thus we follow from the inequalities (3.147) and (3.148) that

$$\left(\frac{\mu(F)}{\mu(X)}\right)^{\frac{1}{n+1}} \times (\|f\|_\infty - \epsilon) \leq \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty \quad \text{a.e. on } X \tag{3.149}$$

for all $n \in \mathbb{N}$. Since $0 < \mu(F) \leq \mu(X) < \infty$, the number $(\frac{\mu(F)}{\mu(X)})^{\frac{1}{n+1}}$ tends to 1 as $n \to \infty$. By taking limit to both sides of the inequality (3.149), we see immediately that

$$\|f\|_\infty - \epsilon \leq \liminf_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \limsup_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty \quad \text{a.e. on } X.$$

Remember that ϵ is arbitrary, so this implies that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty \quad \text{a.e. on } X$$

and we complete the proof of the problem. ■

Problem 3.24

Rudin Chapter 3 Exercise 24.

Proof. Let $x \geq 0$ and $y \geq 0$. Put $\alpha - \beta = x$ and $\alpha = y$ into the equality (3.111), we gain

$$|x|^p - |y|^p \leq |x - y|^p. \tag{3.150}$$

Next, we substitute $\alpha - \beta = y$ and $\alpha = x$ into the equality (3.111) to get

$$-|x - y|^p \leq |x|^p - |y|^p. \tag{3.151}$$

Obviously, the inequalities (3.150) and (3.151) together imply that

$$|x^p - y^p| = \big||x|^p - |y|^p\big| \leq |x - y|^p \tag{3.152}$$

if $0 < p < 1$.

If $x = y$, then the inequality

$$|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1})$$

certainly holds. Without loss of generality, we may assume that $x < y$. For $p > 1$, we consider the function $\varphi(t) = t^p$ on $[x, y]$. Since φ is differentiable in (x, y), the Mean Value Theorem implies that

$$|y^p - x^p| \leq |x - y||\varphi'(\xi)| \leq p|x - y|\xi^{p-1} \tag{3.153}$$

for some $\xi \in (x, y)$. Clearly, we have $\xi^{p-1} \leq x^{p-1} + y^{p-1}$, so it reduces from the inequality (3.153) that

$$|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1}). \tag{3.154}$$

Hence we establish from the inequalities (3.152) and (3.154) that

$$|x^p - y^p| \leq \begin{cases} |x - y|^p, & \text{if } 0 < p < 1; \\ p|x - y|(x^{p-1} + y^{p-1}), & \text{if } 1 \leq p < \infty. \end{cases}$$

(a) We prove the assertions one by one:

- **The truth of the inequality.** Put $x = |f| \geq 0$ and $y = |g| \geq 0$ into the inequality (3.152) and then take integration, we get

$$\int \big||f|^p - |g|^p\big| \, \mathrm{d}\mu \leq \int \big||f| - |g|\big|^p \, \mathrm{d}\mu. \tag{3.155}$$

Since $\big||a| - |b|\big| \leq |a - b|$ holds for every $a, b \in \mathbb{R}$, we follow immediately from the inequality (3.155) that

$$\int \big||f|^p - |g|^p\big| \, \mathrm{d}\mu \leq \int \big||f| - |g|\big|^p \, \mathrm{d}\mu \leq \int |f - g|^p \, \mathrm{d}\mu, \tag{3.156}$$

where $0 < p < 1$.

- **Δ is a metric.** Define $\Delta : L^p(\mu) \times L^p(\mu) \to \mathbb{C}$ by

$$\Delta(f, g) = \int |f - g|^p \, \mathrm{d}\mu.$$

By Remark 3.10, $L^p(\mu)$ is a complex vector space, so $f - g \in L^p(\mu)$ if $f, g \in L^p(\mu)$. Thus we have

$$0 \leq \Delta(f, g) = \int |f - g|^p \, \mathrm{d}\mu < \infty$$

for all $f, g \in L^p(\mu)$. Next, Theorem 1.39(a) ensures that $\Delta(f, g) = 0$ if and only if $|f - g|^p = 0$ a.e. on X, i.e., $f = g$ a.e. on X. It is clear that

$$\Delta(f, g) = \int |f - g|^p \, \mathrm{d}\mu = \int |g - f|^p \, \mathrm{d}\mu = \Delta(g, f).$$

Finally, since $|f|^p - |g|^p \leq \big||f|^p - |g|^p\big|$, it is clear from the inequality (3.156) that

$$\int (|f|^p - |g|^p) \, \mathrm{d}\mu \leq \int \big||f|^p - |g|^p\big| \, \mathrm{d}\mu \leq \int |f - g|^p \, \mathrm{d}\mu. \tag{3.157}$$

By replacing f and g by $f - g$ and $h - g$ in the inequality (3.157), we deduce that

$$\begin{aligned} \Delta(f, g) - \Delta(h, g) &= \int (|f - g|^p - |h - g|^p) \, \mathrm{d}\mu \\ &\leq \int |(f - g) - (h - g)|^p \, \mathrm{d}\mu \\ &= \int |f - h|^p \, \mathrm{d}\mu \\ &= \Delta(f, h). \end{aligned}$$

After rearrangement, we have $\Delta(f, g) \leq \Delta(f, h) + \Delta(h, g)$. Hence, by the definition, Δ is in fact a metric.

- **$(L^p(\mu), \Delta)$ is a complete metric space.** Suppose that $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$ with respect to the metric Δ, i.e., for every $\epsilon > 0$, there exists a positive integer N such that $n, m \geq N$ implies that

$$\Delta(f_n, f_m) < \epsilon. \tag{3.158}$$

Although $\Delta(f, g) = \|f - g\|_p^p$ here, we cannot apply Theorem 3.11 directly to conclude that $L^p(\mu)$ is complete with respect to Δ because $p \geq 1$ in Theorem 3.11.

We imitate the proof of Theorem 3.11. To start with, there is a subsequence $\{f_{n_i}\}$ such that

$$\Delta(f_{n_{i+1}}, f_{n_i}) < 2^{-i} \tag{3.159}$$

for each $i = 1, 2, \ldots$. We define $g_k : X \to [0, \infty]$ and $g : X \to [0, \infty]$ by

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \quad \text{and} \quad g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

respectively. It is clear that $g_k \geq 0$ and $g \geq 0$ and furthermore, it follows from Theorem 3.5 (Minkowski's Inequality) that

$$\begin{aligned}
\Delta(g_k, 0) &= \Delta\Big(\sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, 0\Big) \\
&= \int \Big(\sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|\Big)^p \,\mathrm{d}\mu \\
&\leq \Big[\sum_{i=1}^{k} \Big\{\int |f_{n_{i+1}} - f_{n_i}|^p \,\mathrm{d}\mu\Big\}^{\frac{1}{p}}\Big]^p \\
&= \Big[\sum_{i=1}^{k} \Delta(f_{n_{i+1}}, f_{n_i})^{\frac{1}{p}}\Big]^p
\end{aligned} \tag{3.160}$$

for each $k = 1, 2, \ldots$.

To proceed further, we need the following result:

Lemma 3.12

Let $p \in (0, 1)$. For nonnegative real numbers $a_1, a_2, \ldots, a_k$, we have

$$(a_1 + a_2 + \cdots + a_k)^p \leq a_1^p + a_2^p + \cdots + a_k^p.$$

Proof of Lemma 3.12. Replace β by $-\beta$ in the inequality (3.111), we get

$$(\alpha + \beta)^p \leq |\alpha|^p + |\beta|^p$$

for $p \in (0, 1)$. In particular, if $\alpha \geq 0$ and $\beta \geq 0$, then we have

$$(\alpha + \beta)^p \leq \alpha^p + \beta^p \tag{3.161}$$

for $p \in (0, 1)$. By applying the inequality (3.161) repeatedly to $a_1, a_2, \ldots, a_k$, we derive the desired inequality. This completes the proof of Lemma 3.12. ■

Apply Lemma 3.9 to the inequality (3.160) and then using the inequality (3.159) to get

$$\Delta(g_k, 0) \leq \sum_{i=1}^{k} \Delta(f_{n_{i+1}}, f_{n_i}) < \sum_{i=1}^{k} 2^{-i} \leq 1 \tag{3.162}$$

for each $k = 1, 2, \ldots$.

Since $g_k(x) \to g(x)$ as $k \to \infty$ for every $x \in X$, we have $g_k(x)^p \to g(x)^p$ as $k \to \infty$ on X. We note that since each f_{n_i} is measurable, each g_k^p is also measurable and we may apply Theorem 1.28 (Fatou's Lemma) to $\{g_k^p\}$ and the inequality (3.162) to get

$$\int g^p \,\mathrm{d}\mu = \int \lim_{k\to\infty} g_k^p \,\mathrm{d}\mu = \int \liminf_{k\to\infty} g_k^p \,\mathrm{d}\mu \leq \liminf_{k\to\infty} \int g_k^p \,\mathrm{d}\mu = \liminf_{k\to\infty} \Delta(g_k, 0) \leq 1,$$

implying that $g \in L^p(\mu)$. In particular, $g < \infty$ a.e. on X, so the series

$$f_{n_1}(x) + \sum_{i=1}^{\infty} [f_{n_{i+1}}(x) - f_{n_i}(x)] \tag{3.163}$$

converges absolutely for almost every $x \in X$. Denote the sum (3.163) by $f(x)$ for those x at which the sum (3.163) converges and put $f(x) = 0$ on the remaining set of measure zero.

Since we have

$$f_{n_1}(x) + \sum_{i=1}^{k-1} [f_{n_{i+1}}(x) - f_{n_i}(x)] = f_{n_k}(x),$$

it implies that $f_{n_i}(x) \to f(x)$ as $i \to \infty$ a.e. on X.

Now it remains to show that $f \in L^p(\mu)$ and $\Delta(f_n, f) \to 0$ as $n \to \infty$. Given $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in $L^p(\mu)$ with respect to Δ, there exists a $N \in \mathbb{N}$ such that the estimate (3.158) holds for $n, m \geq N$. Take $m \geq N$, since $f_{n_i} \to f$ as $i \to \infty$ a.e. on X, Theorem 1.28 (Fatou's Lemma) again implies that

$$\begin{aligned}\int |f - f_m|^p \,\mathrm{d}\mu &= \int \Big(\liminf_{i\to\infty} |f_{n_i} - f_m|^p\Big) \,\mathrm{d}\mu \\ &\leq \liminf_{i\to\infty} \Big(\int |f_{n_i} - f_m|^p \,\mathrm{d}\mu\Big) \\ &= \liminf_{i\to\infty} \Delta(f_{n_i}, f_m) < \epsilon. \end{aligned} \tag{3.164}$$

so we conclude that $f - f_m \in L^p(\mu)$, hence that $f \in L^p(\mu)$.

Finally, according to the estimate (3.164), it follows that $\Delta(f, f_m) < \epsilon$ for all $m \geq N$. Since ϵ is arbitrary, we conclude that

$$\lim_{m\to\infty} \Delta(f, f_m) = 0.$$

This shows the completeness of the space $L^p(\mu)$ with respect to Δ.

(b) Similar to part (a), a direct substitution of $x = |f|$ and $y = |g|$ into the inequality (3.154) and the use of the fact $\big||x| - |y|\big| \leq |x - y|$ give

$$\begin{aligned}\int \big||f|^p - |g|^p\big| \,\mathrm{d}\mu &\leq p \int \big||f| - |g|\big| \times (|f|^{p-1} + |g|^{p-1}) \,\mathrm{d}\mu \\ &\leq p \int |f - g|(|f|^{p-1} + |g|^{p-1}) \,\mathrm{d}\mu. \end{aligned} \tag{3.165}$$

By Theorem 3.5 (Hölder's Inequality) and the facts that $\|f\|_p \leq R$ and $\|g\|_p \leq R$, we observe that

$$\begin{aligned}\int |f - g|(|f|^{p-1} + |g|^{p-1}) \,\mathrm{d}\mu &\leq \int |f - g| \cdot |f|^{p-1} \,\mathrm{d}\mu + \int |f - g| \cdot |g|^{p-1} \,\mathrm{d}\mu \\ &\leq \Big\{\int |f - g|^p \,\mathrm{d}\mu\Big\}^{\frac{1}{p}} \Big\{\int (|f|^{p-1})^{\frac{p}{p-1}} \,\mathrm{d}\mu\Big\}^{\frac{p-1}{p}} \end{aligned}$$

$$+\Big\{\int |f-g|^p\,\mathrm{d}\mu\Big\}^{\frac{1}{p}}\Big\{\int (|g|^{p-1})^{\frac{p}{p-1}}\,\mathrm{d}\mu\Big\}^{\frac{p-1}{p}}$$
$$= \|f-g\|_p\cdot(\|f\|_p^{p-1}+\|g\|_p^{p-1})$$
$$\leq 2R^{p-1}\|f-g\|_p. \tag{3.166}$$

Combining the inequalities (3.165) and (3.166), we discover that

$$\int \big||f|^p-|g|^p\big|\,\mathrm{d}\mu \leq 2pR^{p-1}\|f-g\|_p$$

which is our expected result.

We have completed the proof of the problem. ■

Problem 3.25

Rudin Chapter 3 Exercise 25.

Proof. Suppose that $E\subseteq X$ and $0<\mu(E)<\infty$. Let $\mathfrak{M}_E$ be a σ-algebra in E. Then $(E,\mathfrak{M}_E,\mu)$ is a measure space.[j] Define

$$\varphi(F)=\int_F \frac{1}{\mu(E)}\,\mathrm{d}\mu \quad (F\in\mathfrak{M}_E).$$

Then we follow from Theorem 1.29 that φ is a measure on $\mathfrak{M}_E$ and

$$\int_E f\,\mathrm{d}\varphi = \frac{1}{\mu(E)}\int_E f\,\mathrm{d}\mu \tag{3.167}$$

for every measurable f on E with range in $[0,\infty]$. It is trivial that $\varphi(E)=1$ and

$$0\leq \int_E f\,\mathrm{d}\varphi \leq \frac{1}{\mu(E)}\int_X f\,\mathrm{d}\mu = \frac{1}{\mu(E)}<\infty$$

which means $f\in L^1(\varphi)$. Since $0<f<\infty$ on E, $\log f$ is well-defined on E.

Define $\phi:(0,\infty)\to\mathbb{R}$ by $\phi(x)=-\log x$ which is convex on $(0,\infty)$. Apply Theorem 3.3 (Jensen's Inequality) with ϕ and measure φ to conclude that

$$-\log\Big(\int_E f\,\mathrm{d}\varphi\Big)\leq -\int_E \log f\,\mathrm{d}\varphi$$

or equivalently,

$$\int_E \log f\,\mathrm{d}\varphi \leq \log\Big(\int_E f\,\mathrm{d}\varphi\Big). \tag{3.168}$$

Now, by using the fact (3.167) to the inequality (3.168), we gain

$$\frac{1}{\mu(E)}\int_E \log f\,\mathrm{d}\mu \leq \log\Big(\frac{1}{\mu(E)}\int_E f\,\mathrm{d}\mu\Big)\leq \log\Big(\frac{1}{\mu(E)}\int_X f\,\mathrm{d}\mu\Big) = \log\frac{1}{\mu(E)}$$

which implies the first inequality.

For the second inequality, we define $\psi:(0,\infty)\to\mathbb{R}$ by

$$\psi(x)=-x^p,$$

[j] See the paragraphs following Proposition 1.24.

where $0 < p < 1$. Since $\psi''(x) = p(1-p)x^{p-2} > 0$ on $(0, \infty)$, we apply Theorem 3.3 (Jensen's Inequality) with ψ and measure φ to conclude that

$$-\Big\{\int_E f\,\mathrm{d}\varphi\Big\}^p \leq -\int_E f^p\,\mathrm{d}\varphi. \tag{3.169}$$

Next, we use the fact (3.167) to reduce the inequality (3.169) to

$$\frac{1}{\mu(E)}\int_E f^p\,\mathrm{d}\mu \leq \Big\{\frac{1}{\mu(E)}\int_E f\,\mathrm{d}\mu\Big\}^p \leq \Big\{\frac{1}{\mu(E)}\int_X f\,\mathrm{d}\mu\Big\}^p = \frac{1}{\mu(E)^p}$$

so that

$$\int_E f^p\,\mathrm{d}\mu \leq \mu(E)^{1-p},$$

completing the proof of the problem. ■

Problem 3.26

Rudin Chapter 3 Exercise 26.

Proof. Let $\mathfrak{M}$ be a σ-algebra in $[0,1]$. Suppose that there exists an $E \in \mathfrak{M}$ such that $f(x) = \infty$ and $m(E) > 0$, where m is the Lebesgue measure. Then

$$\int_0^1 f(x)\log f(x)\,\mathrm{d}x = \int_E f(x)\log f(x)\,\mathrm{d}x + \int_{[0,1]\setminus E} f(x)\log f(x)\,\mathrm{d}x. \tag{3.170}$$

On the one hand, since $f > 0$ on $[0,1] \setminus E$, we know from Proposition 1.24(a) that the second integral on the right-hand side of the expression (3.170) is nonnegative. Since $f(x)\log f(x) = \infty$ on E and $m(E) > 0$, we get from the expression (3.170) that

$$\int_0^1 f(x)\log f(x)\,\mathrm{d}x = \infty. \tag{3.171}$$

On the other hand, by using similar argument, we can show that

$$\int_0^1 f(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^1 \log f(t)\,\mathrm{d}t = \infty. \tag{3.172}$$

Therefore, we conclude from the results (3.171) and (3.172) that

$$\int_0^1 f(x)\log f(x)\,\mathrm{d}x = \int_0^1 f(s)\,\mathrm{d}s \int_0^1 \log f(t)\,\mathrm{d}t$$

in this case.

Next, we suppose that $0 < f(x) < \infty$ a.e. on $[0,1]$. Define $\varphi : (0,\infty) \to \mathbb{R}$ and $\psi : (0,\infty) \to \mathbb{R}$ by

$$\varphi(x) = x\log x \quad \text{and} \quad \psi(x) = -\log x$$

respectively. Since $\varphi''(x) = \frac{1}{x} > 0$ and $\psi''(x) = x^{-2} > 0$ on $(0,\infty)$, they are convex on $(0,\infty)$. Since $0 < f(x) < \infty$ a.e. on $[0,1]$ and $m([0,1]) = 1$, we apply Theorem 3.3 (Jensen's Inequality) with φ to f to obtain

$$\Big\{\int_0^1 f(x)\,\mathrm{d}x\Big\}\Big\{\log\Big(\int_0^1 f(x)\,\mathrm{d}x\Big)\Big\} \leq \int_0^1 f(x)\log f(x)\,\mathrm{d}x \tag{3.173}$$

Similarly, we apply Theorem 3.3 (Jensen's Inequality) with ψ to f to obtain

$$-\log\Big(\int_0^1 f(x)\,\mathrm{d}x\Big) \leq -\int_0^1 \log f(x)\,\mathrm{d}x. \tag{3.174}$$

Combining the inequalities (3.173) and (3.174), we see that

$$\Big\{\int_0^1 f(x)\,\mathrm{d}x\Big\}\Big\{\int_0^1 \log f(x)\,\mathrm{d}x\Big\} \leq \int_0^1 f(x)\log f(x)\,\mathrm{d}x.$$

We have ended the proof of the problem. ■

CHAPTER 4

Elementary Hilbert Space Theory

In the following problems, we use $\langle x, y\rangle$ to denote the inner product of the complex vectors x and y and $\|\cdot\|$ the norm with respect to the Hilbert space H. Besides, we use $\mathrm{span}\,(S)$ to denote the span of a set S.

4.1 Basic Properties of Hilbert Spaces

Problem 4.1

Rudin Chapter 4 Exercise 1.

Proof. Let $x \in M$. Recall that $M^{\perp} = \{y \in H \mid y \perp x \text{ for all } x \in M\}$, so $x \perp y$ for every $y \in M^{\perp}$ which means that $x \in (M^{\perp})^{\perp}$, i.e.,

$$M \subseteq (M^{\perp})^{\perp}.$$

Conversely, suppose that $x \in (M^{\perp})^{\perp}$. Since M is a closed subspace of H, it follows from Theorem 4.11(a) that $x = y + z$, where $y \in M$ and $z \in M^{\perp}$. On the one hand, since $x \in (M^{\perp})^{\perp}$ and $z \in M^{\perp}$, we have $\langle x, z\rangle = 0$. On the other hand, we deduce from Definition 4.1 that

$$\langle x, z\rangle = \langle y + z, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \langle z, z\rangle.$$

Since $\langle z, z\rangle = 0$, it must be $z = 0$ and then $x \in M$, i.e, $(M^{\perp})^{\perp} \subseteq M$. In conclusion, we obtain $M = (M^{\perp})^{\perp}$.

Suppose that M is a subspace of H which may *not* be closed. We claim that

$$\overline{M} = (M^{\perp})^{\perp}.$$

To see this, we recall from §4.8 that $\overline{M}$ is a closed subspace of H, so the first assertion implies that

$$\overline{M} = (\overline{M}^{\perp})^{\perp}. \tag{4.1}$$

Let $x \in \overline{M}^{\perp}$. By §4.9, we see that $x \perp \overline{M}$ so that $x \perp M$ particularly. In other words, we have

$$\overline{M}^{\perp} \subseteq M^{\perp}. \tag{4.2}$$

For the other direction, suppose that $x \in M^{\perp}$, we want to show that $x \in \overline{M}^{\perp}$, i.e., $x \perp \overline{M}$. To this end, given $y \in \overline{M}$, there exist a sequence $\{y_n\} \subseteq M$ such that $y_n \to y$ as $n \to \infty$. Since

$x \in M^\perp$ and $y_n \in M$, we must have $\langle y_n, x\rangle = 0$. By Theorem 4.6, the mapping $f : H \to \mathbb{C}$ defined by

$$f(y) = \langle y, x\rangle$$

is continuous on H so that

$$\langle y, x\rangle = f(y) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \langle y_n, x\rangle = 0.$$

In other words, $x \perp \overline{M}$, i.e.,

$$M^\perp \subseteq \overline{M}^\perp. \tag{4.3}$$

Now the claim is derived by combining the set relations (4.2) and (4.3) and then using the result (4.1), completing the proof of the problem. ■

Problem 4.2

Rudin Chapter 4 Exercise 2.

Proof. We note that this is actually the **Gram-Schmidt Process**. Suppose that $n = 1$. Then the set $\{u_1\}$ is clearly orthonormal and $\operatorname{span}(u_1) = \operatorname{span}(x_1)$. Thus the statement is true for $n = 1$.

Assume that the statement is true for $n = k$ for some $k \in \mathbb{N}$, i.e., $\{u_1, u_2, \ldots, u_k\}$ is an orthonormal set and

$$\operatorname{span}(u_1, u_2, \ldots, u_k) = \operatorname{span}(x_1, x_2, \ldots, x_k). \tag{4.4}$$

Let $n = k + 1$. Note that $x_{k+1} \notin \operatorname{span}(x_1, x_2, \ldots, x_k)$ because $\{x_1, x_2, \ldots\}$ is linearly independent. By the assumption (4.4), it implies that $x_{k+1} \notin \operatorname{span}(u_1, u_2, \ldots, u_k)$ so that $v_{k+1} \neq 0$. By this fact and Definition 4.1, for $j = 1, 2, \ldots, k$, we derive that

$$\begin{aligned}
\langle u_{k+1}, u_j\rangle &= \left\langle \frac{v_{k+1}}{\|v_{k+1}\|}, u_j \right\rangle \\
&= \frac{1}{\|v_{k+1}\|} \left\langle x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, u_i\rangle\, u_i, u_j \right\rangle \\
&= \frac{1}{\|v_{k+1}\|} \langle x_{k+1}, u_j\rangle - \frac{1}{\|v_{k+1}\|} \sum_{i=1}^{k} \langle x_{k+1}, u_i\rangle \langle u_i, u_j\rangle \\
&= \frac{1}{\|v_{k+1}\|} \langle x_{k+1}, u_j\rangle - \frac{1}{\|v_{k+1}\|} \langle x_{k+1}, u_j\rangle \langle u_j, u_j\rangle \\
&= 0.
\end{aligned}$$

Therefore, $\{u_1, u_2, \ldots, u_{k+1}\}$ is orthonormal. Next, by the assumption (4.4), we know that $x \in \operatorname{span}(x_1, x_2, \ldots, x_k, x_{k+1})$ if and only if

$$x \in \operatorname{span}(u_1, u_2, \ldots, u_k, x_{k+1}). \tag{4.5}$$

Since

$$v_{k+1} = x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, u_i\rangle\, u_i \quad \text{and} \quad u_{k+1} = \frac{v_{k+1}}{\|v_{k+1}\|},$$

the x_{k+1} in the result (4.5) can be replaced by v_{k+1} and ultimately by u_{k+1}. Thus we have

$$\operatorname{span}(u_1, u_2, \ldots, u_{k+1}) = \operatorname{span}(x_1, x_2, \ldots, x_{k+1}).$$

Hence the statement is true for $n = k+1$ if it is true for $n = k$. By induction, the construction yields an orthonormal set $\{u_1, u_2, \ldots\}$ with

$$\operatorname{span}(u_1, u_2, \ldots, u_n) = \operatorname{span}(x_1, x_2, \ldots, x_n)$$

for all $n \in \mathbb{N}$. This completes the proof of the problem. ■

Problem 4.3

Rudin Chapter 4 Exercise 3.

Proof. Recall the definition that a space is separable if it contains a countable dense subset.

Suppose that $1 \leq p < \infty$. Since T is compact, the second paragraph following Definition 3.16 says that $C_c(T) = C_0(T) = C(T)$. Thus we deduce from this and Theorem 3.14 that $C(T)$ is dense in $L^p(T)$ in the norm $\|\cdot\|_p$. Let $\mathcal{P}$ be the set of all trigonometric polynomials on T. By Theorem 4.25 (The Weierstrass Approximation Theorem) and Definitions 4.23 (or Remark 3.15), $\mathcal{P}$ is dense in $C(T)$ in the norm $\|\cdot\|_\infty$. By Definition 3.7, we know that $|f(x)| \leq \|f\|_\infty$ a.e. on T and then

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \, dt\right\}^{\frac{1}{p}} \leq \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f\|_\infty^p \, dt\right\}^{\frac{1}{p}} = \|f\|_\infty. \tag{4.6}$$

In other words, the inequality (4.6) implies that $\mathcal{P}$ is also dense in $C(T)$ in the norm $\|\cdot\|_p$. By Lemma 2.10, we conclude that $\mathcal{P}$ is dense in $L^p(T)$ in the norm $\|\cdot\|_p$.

Next we suppose that $\mathcal{P}_\mathbb{Q}$ is the set of all trigonometric polynomials with rational coefficients. Note that[a]

$$\begin{aligned}
\mathcal{P}_\mathbb{Q} &= \left\{P(t) = \sum_{k=-n}^{n} (a_k + ib_k)e^{ikt} \,\middle|\, a_{-n}, \ldots, a_n, b_{-n}, \ldots, b_n \in \mathbb{Q} \text{ and } n \in \mathbb{N}\right\} \\
&= \bigcup_{n=1}^{\infty} \left\{P(t) = \sum_{k=-n}^{n} (a_k + ib_k)e^{ikt} \,\middle|\, a_{-n}, \ldots, a_n, b_{-n}, \ldots, b_n \in \mathbb{Q}\right\} \\
&\sim \bigcup_{n=1}^{\infty} \mathbb{Q}^{2(2n+1)}.
\end{aligned} \tag{4.7}$$

For each $n \in \mathbb{N}$, since each $\mathbb{Q}^{2(2n+1)}$ is countable, it follows from the set relation (4.7) that $\mathcal{P}_\mathbb{Q}$ is also countable. Given $f \in \mathcal{P}$. Since the set $\{p + iq \,|\, p, q \in \mathbb{Q}\}$ is dense in $\mathbb{C}$ and f must be in the form

$$f(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

for some positive integer n, there exists a sequence $\{f_{n_k}\} \subseteq \mathcal{P}_\mathbb{Q}$ such that $\|f_{n_k} - f\|_\infty \to 0$ as $n_k \to \infty$. By the inequality (4.6), it is true that

$$\|f_{n_k} - f\|_p \to 0$$

as $n_k \to \infty$. Consequently, $\mathcal{P}_\mathbb{Q}$ is dense in $\mathcal{P}$ in the norm $\|\cdot\|_p$ and an application of Lemma 2.10 shows that $\mathcal{P}_\mathbb{Q}$ is dense in $L^p(T)$ in the norm $\|\cdot\|_p$. Since $\mathcal{P}_\mathbb{Q}$ is countable, $L^p(T)$ is separable.

By Definition 4.23, every function defined on T can be identified with a 2π-periodic function on $\mathbb{R}$, so we may identity $L^\infty(T)$ with $L^\infty\big([0, 2\pi]\big)$. Consider the set

$$E = \{f_\theta = \chi_{[0,\theta]} \,|\, \theta \in [0, 2\pi]\}.$$

[a] We write $A \sim B$ if there is a bijection between the two sets A and B, see [49, Definition 2.3, p. 25].

It is clear that $\chi_{[0,\theta]} \in L^\infty([0,2\pi])$ so that $E \subseteq L^\infty([0,2\pi])$. Since $\chi_{[0,\theta_1]} \neq \chi_{[0,\theta_2]}$ if $\theta_1 \neq \theta_2$, $E \sim [0,2\pi]$ and then it is *uncountable*. In fact, we have

$$\|\chi_{[0,\theta_1]} - \chi_{[0,\theta_2]}\|_\infty = 1 \tag{4.8}$$

if $\theta_1 < \theta_2$.

Assume that $L^\infty([0,2\pi])$ was separable. Let $F = \{f_n\}$ be a countable dense subset of $L^\infty([0,2\pi])$. Then we must have

$$E \subseteq L^\infty([0,2\pi]) \subseteq \bigcup_{f_n \in F} B\Big(f_n, \frac{1}{2}\Big), \tag{4.9}$$

where $B(f_n, \frac{1}{2}) = \{f \in L^\infty([0,2\pi]) \,|\, \|f - f_n\|_\infty < \frac{1}{2}\}$. If $\chi_{[0,\theta_1]}, \chi_{[0,\theta_2]} \in B(f_n, \frac{1}{2})$ with $\theta_1 \neq \theta_2$, then it follows from the result (4.8) that

$$1 = \|\chi_{[0,\theta_1]} - \chi_{[0,\theta_2]}\|_\infty \leq \|\chi_{[0,\theta_1]} - f_n\|_\infty + \|f_n - \chi_{[0,\theta_2]}\|_\infty < 1,$$

a contradiction. Thus each $B(f_n, \frac{1}{2})$ contains *at most* one element of E. Since F is countable but E is uncountable, we have

$$E \nsubseteq \bigcup_{f_n \in F} B\Big(f_n, \frac{1}{2}\Big)$$

which definitely contradicts the set relation (4.9). Hence $L^\infty([0,\pi])$ is not separable and we have completed the proof of the problem. ■

Problem 4.4

Rudin Chapter 4 Exercise 4.

Proof. Suppose that H is separable. By the definition, it has a countable dense subset $\{u_n\}$. By Problem 4.2, we may assume that $\{u_n\}$ is also an orthonormal set. Let $\{u_{n_k}\}$ be a *maximal* linearly independent subset of $\{u_n\}$. Note that $\{u_{n_k}\}$ is at most countable. Since we have $\operatorname{span}(\{u_{n_k}\}) = \{u_n\}$, $\operatorname{span}(\{u_{n_k}\})$ is also dense in H. By Theorem 4.18, the set $\{u_{n_k}\}$ is in fact an at most countable maximal orthonormal set in H.

Conversely, we suppose that $E = \{u_n\}$ is an at most countable maximal orthonormal set in H. By Theorem 4.18 again, $\operatorname{span}(E)$ is dense in H. Let $\operatorname{span}_{\mathbb{Q}}(E)$ be the span of the set E with rational coefficients. It is easy to see that

$$\begin{aligned}
\operatorname{span}_{\mathbb{Q}}(E) &= \Big\{\sum_{k=1}^{n}(a_k + ib_k)u_k \,\Big|\, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Q} \text{ and } n \in \mathbb{N}\Big\} \\
&= \bigcup_{n=1}^{\infty} \Big\{\sum_{k=1}^{n}(a_k + ib_k)u_k \,\Big|\, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Q}\Big\} \\
&\sim \bigcup_{n=1}^{\infty} \mathbb{Q}^{2n}.
\end{aligned}$$

As a result, $\operatorname{span}_{\mathbb{Q}}(E)$ must be countable. Next, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we conclude that $\operatorname{span}_{\mathbb{Q}}(E)$ is dense in $\operatorname{span}(E)$. By Lemma 2.10, $\operatorname{span}_{\mathbb{Q}}(E)$ is also dense in H and hence it is separable. This completes the proof of the problem. ■

Problem 4.5

Rudin Chapter 4 Exercise 5.

Proof. Suppose that $M \neq H$. If $L = 0$, then $M = H$ so that $L \neq 0$. By Theorem 4.12 (The Riesz Representation Theorem for Hilbert Spaces), there is a unique $z \in H$ such that

$$L(x) = \langle x, z\rangle$$

for all $x \in H$. This z must be nonzero, otherwise we have $L = 0$ which contradicts our assumption. It is clear that

$$M = \{x \in H \,|\, \langle x, z\rangle = 0\} = L^{-1}(0), \tag{4.10}$$

so M is closed in H. Apart from this, if $x, y \in M$, then since L is a linear functional, Definition 2.1 shows that

$$L(x + y) = L(x) + L(y) = 0 \quad \text{and} \quad L(\alpha x) = \alpha L(x) = 0$$

for a scalar α. Thus M is a closed subspace of H.

Next, we note that $\mathrm{span}\,(z) = \{\alpha z \,|\, \alpha \in \mathbb{C}\}$ and so

$$\big(\mathrm{span}\,(z)\big)^{\perp} = \{x \in H \,|\, \langle x, \alpha z\rangle = 0 \text{ for every } \alpha \in \mathbb{C}\}. \tag{4.11}$$

Since $\langle x, \alpha z\rangle = \overline{\alpha}\,\langle x, z\rangle$, the two expressions (4.10) and (4.11) give

$$M = \big(\mathrm{span}\,(z)\big)^{\perp}$$

and then

$$M^{\perp} = \Big(\big(\mathrm{span}\,(z)\big)^{\perp}\Big)^{\perp}. \tag{4.12}$$

We claim that $\mathrm{span}\,(z)$ is a closed subspace of H. In fact, it is a subspace of H follows easily from the definition of $\mathrm{span}\,(z)$. Suppose that $\{x_n\}$ is a sequence in $\mathrm{span}\,(z)$. Then we know that each x_n has the form $x_n = \alpha_n z$, where $\alpha_n \in \mathbb{C}$. Let $\alpha = \lim\limits_{n\to\infty} \alpha_n$ and $x = \alpha z$. We want to show that $x_n \to x$ in H with respect to the norm $\|\cdot\|$. Indeed, we have

$$\|x_n - x\|^2 = \langle(\alpha_n - \alpha)z, (\alpha_n - \alpha)z\rangle = |\alpha_n - \alpha^2| \times \|z\|$$

so that $\|x_n - x\| \to 0$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} x_n = x$$

as desired. In conclusion, $\mathrm{span}\,(z)$ contains all its limit points and therefore it is closed in H. By Problem 4.1, we have $\Big(\big(\mathrm{span}\,(z)\big)^{\perp}\Big)^{\perp} = \mathrm{span}\,(z)$. Combining this fact and the expression (4.12), we have

$$M^{\perp} = \mathrm{span}\,(z)$$

and since $\{z\}$ is a basis of $\mathrm{span}\,(z)$, we have $\dim\big(\mathrm{span}\,(z)\big) = 1$ and our required result follows. We end the proof of the problem. ∎

Problem 4.6

Rudin Chapter 4 Exercise 6.

Proof. We prove the assertions one by one:

- $E = \{u_n\}$ **is closed and bounded, but not compact.** Since E is orthonormal, $\|u_n\| \leq 1$ for every $n \in \mathbb{N}$. Thus E is bounded by 1. If $n \neq m$, then it follows from Definition 4.1 that
$$\|u_n - u_m\|^2 = \langle u_n - u_m, u_n - u_m\rangle = \langle u_n, u_n\rangle + \langle u_m, u_m\rangle = 2. \tag{4.13}$$
Assume that u was a limit point of E. Then there exists a positive integer N such that $n \geq N$ implies that $\|u_n - u\| < \frac{\sqrt{2}}{2}$. If $n, m \geq N$, then we have
$$\|u_n - u_m\| \leq \|u_n - u\| + \|u - x_m\| < \sqrt{2}$$
which contradicts the result (4.13). In other words, E has *no* limit point and then it is closed in H.

  Suppose that $B(u_n, 1) = \{x \in H \mid \|u_n - x\| < 1\}$ which is clearly open in H. Furthermore, it is obvious that $\{B(u_n, 1)\}$ is an open cover of E. By the fact (4.13), we conclude that each $B(u_n, 1)$ contains *exactly* one element of E, so it is impossible to have
$$E \subseteq \bigcup_{i=1}^{k} B(u_{n_i}, 1)$$
for any finite k, i.e., E is *not* compact.

- S **is compact if and only if** $\sum_{n=1}^{\infty} \delta_n^2 < \infty$. Suppose that $\sum_{n=1}^{\infty} \delta_n^2 < \infty$. It is well-known that a metric space X is compact if and only if X is *sequentially compact*.[b] As a subset of a metric space, S is itself a metric space. Let $\{x^k\} \subseteq S$ be a sequence of the form
$$x^k = \sum_{n=1}^{\infty} c_n^k u_n, \tag{4.14}$$
where $k = 1, 2, \ldots$. Note that we have $|c_n^k| \leq \delta_n$ for all $k \in \mathbb{N}$. Since $\{c_1^k\} \subseteq \overline{B(0, \delta_1)} \subseteq \mathbb{C}$ for all $k \in \mathbb{N}$ and $\overline{B(0, \delta_1)}$ is compact, $\overline{B(0, \delta_1)}$ is sequentially compact and then there exists a subsequence $k_1(m)$ such that
$$c_1^{k_1(m)} \to c_1 \in \overline{B(0, \delta_1)}$$
as $m \to \infty$. Next, we consider the sequence $\{c_2^{k_1(m)}\} \subseteq \overline{B(0, \delta_2)}$. Then we can find a subsequence $\{k_2(m)\} \subseteq \{k_1(m)\}$ such that
$$c_2^{k_2(m)} \to c_2 \in \overline{B(0, \delta_2)}$$
as $m \to \infty$. Furthermore, we also have
$$c_1^{k_2(m)} \to c_1 \in \overline{B(0, \delta_1)}$$
as $m \to \infty$. By this observation, *for every* positive integer N, we can select a sequence $\{k_{N+1}(m)\} \subseteq \{k_N(m)\}$ such that for all $n \leq N$,
$$c_n^{k_N(m)} \to c_n \in \overline{B(0, \delta_n)} \tag{4.15}$$

[b]We say that the space X is sequentially compact if every sequence of points of X has a convergent subsequence, see [42, Theorem 28.2, p. 179].

as $m \to \infty$. Now the limit (4.15) makes sure that

$$x = \sum_{n=1}^{\infty} c_n u_n \in S.$$

We claim that the sequence $\{x^{k_m(m)}\}$, which is clearly a subsequence of $\{x^k\}$ converges to the point x. To this end, given $\epsilon > 0$, there is a $N' \in \mathbb{N}$ such that

$$\sum_{n>N'} \delta_n^2 < \frac{\epsilon}{8}. \tag{4.16}$$

Then we obtain from the definition (4.14), the fact $|c_n^k| \leq \delta_n$ for all k and the estimate (4.16) that

$$\begin{aligned}
\|x^{k_m(m)} - x\|^2 &= \Big\| \sum_{n=1}^{\infty} [c_n^{k_m(m)} - c_n] u_n \Big\|^2 \\
&= \sum_{n=1}^{\infty} |c_n^{k_m(m)} - c_n|^2 \\
&= \sum_{n=1}^{N'} |c_n^{k_m(m)} - c_n|^2 + \sum_{n=N'+1}^{\infty} |c_n^{k_m(m)} - c_n|^2 \\
&\leq \sum_{n=1}^{N'} |c_n^{k_m(m)} - c_n|^2 + \sum_{n=N'+1}^{\infty} (|c_n^{k_m(m)}|^2 + 2|c_n^{k_m(m)}| \cdot |c_n| + |c_n|^2) \\
&\leq \sum_{n=1}^{N'} |c_n^{k_m(m)} - c_n|^2 + 4 \times \sum_{n=N'+1}^{\infty} \delta_n^2 \\
&< \sum_{n=1}^{N'} |c_n^{k_m(m)} - c_n|^2 + \frac{\epsilon}{2}.
\end{aligned} \tag{4.17}$$

To finish our proof, we have to find an upper bound of the summation in the inequality (4.17). Recall from the limit (4.15) that for each $n \leq N'$

$$c_n^{k_{N'}(m)} \to c_n \in \overline{B(0, \delta_n)}$$

as $m \to \infty$. In other words, for each $n \leq N'$, there is a $M_n \in \mathbb{N}$ such that $m \geq M_n$ implies

$$|c_n^{k_{N'}(m)} - c_n|^2 < \frac{\epsilon}{2N'}. \tag{4.18}$$

Since $\{c_n^{k_m(m)}\} \subseteq \{c_n^{k_{N'}(m)}\}$ for $m \geq N'$, this means that the inequality (4.18) also holds when $c_n^{k_{N'}(m)}$ is replaced by $c_n^{k_m(m)}$. Let $M = \max(M_1, M_n, \ldots, M_{N'})$. If $m \geq M$, then we conclude from the inequalities (4.17) and (4.18) that

$$\|x^{k_m(m)} - x\|^2 < \sum_{n=1}^{N'} \frac{\epsilon}{2N'} + \frac{\epsilon}{2} = \epsilon,$$

proving our claim. Hence S is sequentially compact and then compact.

Conversely, we suppose that S is compact. Consider the point

$$x_N = \sum_{n=1}^{N} \delta_n u_n,$$

where $N \in \mathbb{N}$. Then we have $\|x_N\|^2 = \sum_{n=1}^{N} \delta_n^2$. Since S is a compact metric space, it is bounded. Thus there exists a positive constant M such that

$$\sum_{n=1}^{N} \delta_n^2 \leq M$$

for all $N \in \mathbb{N}$ and hence $\sum_{n=1}^{\infty} \delta_n^2 < \infty$.

- **Q is compact.** Obviously, we have[c]

  $$\sum_{n=1}^{\infty} |c_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

  Therefore, the previous assertion implies that Q is compact.

- **H is not locally compact.** Recall that H is itself a vector space, so it contains the zero element 0. Assume that H was locally compact. Then the closure

  $$\overline{B(0,r)} = \{x \in H \mid \|x\| \leq r\}$$

  must be compact *for every* $r > 0$. Note that $\{ru_n\} \subseteq \overline{B(0,r)}$. Since $E_r = \{ru_n\}$ is closed in H, Theorem 2.4 implies that E_r is compact. Similar to part (a), $\{B(ru_n, r)\}$ is an open cover of E_r. A direct computation shows that

  $$\|ru_n - ru_m\| = \sqrt{2}r. \tag{4.19}$$

  If $ru_m \in B(ru_n, r)$ for $n \neq m$, then we have

  $$\|ru_n - ru_m\| < r$$

  which contradicts the fact (4.19). Therefore, each $B(ru_n, r)$ contains *exactly* one element of E_r so that $\{B(ru_n, r)\}$ *does not* have a finite subcover for E_r. In other words, E_r is not compact, a contradiction. Hence we establish the fact that H is not locally compact.

We have completed the proof of the problem. ■

Problem 4.7

Rudin Chapter 4 Exercise 7.

Proof. We follow the suggestion given by Rudin. Assume that $\sum a_n^2 = \infty$. Therefore, for every $k \in \mathbb{N}$, there are disjoint sets $E_1, E_2, \ldots$ such that

$$\sum_{n \in E_k} a_n^2 > k > 0. \tag{4.20}$$

Now for $n \in E_k$, we define

$$b_n = \frac{a_n}{\left\{\sum_{n \in E_k} a_n^2\right\}^{\frac{1}{2}}} \times \frac{1}{k}.$$

[c]See [49, Theorem 3.28, p. 62].

On the one hand, we have

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{k=1}^{\infty}\sum_{n\in E_k} b_n^2 = \sum_{k=1}^{\infty}\left(\frac{1}{\sum\limits_{n\in E_k} a_n^2}\times \sum_{n\in E_k} a_n^2 \times \frac{1}{k^2}\right) = \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

On the other hand, we obtain from the inequality (4.20) that

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{\infty}\sum_{n\in E_k} a_n b_n = \sum_{k=1}^{\infty}\left(\sum_{n\in E_k}\frac{a_n^2}{\left\{\sum\limits_{n\in E_k} a_n^2\right\}^{\frac{1}{2}}}\times\frac{1}{k}\right) = \sum_{k=1}^{\infty}\left\{\sum_{n\in E_k} a_n^2\right\}^{\frac{1}{2}}\frac{1}{k} > \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$$

which is divergent, a contradiction. Hence we must have $\sum a_n^2 < \infty$ and it completes the proof of the problem.

■

Problem 4.8

Rudin Chapter 4 Exercise 8.

Proof. By Problem 4.2, H contains an orthonormal set A and then Theorem 4.22 (The Hausdorff Maximality Theorem) implies that A is contained in a maximal orthonormal set in H. Suppose that $\{u_\alpha \,|\, \alpha \in A\}$ and $\{v_\beta \,|\, \beta \in B\}$ are maximal orthonormal sets in H_1 and H_2 respectively. Without loss of generality, we may assume that $|A| \leq |B|$. Then we take a subset B' of B which is of the same cardinality as A. Now we consider

$$H = \overline{\operatorname{span}(\{v_\beta \,|\, \beta \in B'\})} \subseteq H_2.$$

By Definition 4.13, $\operatorname{span}(\{v_\beta \,|\, \beta \in B'\})$ is a subspace of H_2. Thus we follow from the last paragraph in §4.7 that H is a closed subspace of H_2. By the given hint, H is also Hilbert. Finally, we know from §4.19 that H_1 and H are *Hilbert space isomorphic* to $\ell^2(A)$ and $\ell^2(B')$ respectively. Since A and B have the same cardinality, $\ell^2(A)$ is Hilbert space isomorphic to $\ell^2(B')$. In other words, H_1 is Hilbert space isomorphic to $H \subseteq H_2$, completing the proof of the problem. ■

Problem 4.9

Rudin Chapter 4 Exercise 9.

Proof. Since $A \subseteq [0, 2\pi]$ and A is measurable, we have $\chi_A \in L^2(T)$. By Theorem 3.11, $L^2(T)$ is a complete metric space, so it is a Hilbert space by Definition 4.4. Since $\{u_n \,|\, n \in \mathbb{Z}\}$ is a maximal orthonormal set in the Hilbert space $L^2(T)$, Theorem 4.18(iii) implies that

$$\sum_{n\in\mathbb{Z}} \big|\langle u_n, \chi_A\rangle\big|^2 = \|\chi_A\|^2 = \int_0^{2\pi} |\chi_A(x)|^2\,dx = m(A) < \infty.$$

By [49, Theorem 3.23, p. 60], we know that

$$\lim_{n\to\infty}\big|\langle u_n, \chi_A\rangle\big| = \lim_{n\to-\infty}\big|\langle u_n, \chi_A\rangle\big| = 0. \tag{4.21}$$

By Definition 4.1, we have

$$\begin{aligned}
\langle \cos nx, \chi_A\rangle &= \left\langle \frac{u_n + u_{-n}}{2}, \chi_A\right\rangle = \frac{1}{2}(\langle u_n, \chi_A\rangle + \langle u_{-n}, \chi_A\rangle),\\
\langle \sin nx, \chi_A\rangle &= \left\langle \frac{u_n - u_{-n}}{2i}, \chi_A\right\rangle = \frac{1}{2i}(\langle u_n, \chi_A\rangle - \langle u_{-n}, \chi_A\rangle).
\end{aligned} \tag{4.22}$$

By applying the limits (4.21) to the inner products (4.22), we obtain that

$$\lim_{n\to\infty}\int_A \cos nt\,\mathrm{d}t = \lim_{n\to\infty}\int_{-\pi}^{\pi}(\cos nt)\cdot\chi_A(t)\,\mathrm{d}t = 2\pi \lim_{n\to\infty}\langle \cos nx, \chi_A\rangle = 0$$

and

$$\lim_{n\to\infty}\int_A \sin nt\,\mathrm{d}t = \lim_{n\to\infty}\int_{-\pi}^{\pi}(\sin nt)\cdot\chi_A(t)\,\mathrm{d}t = 2\pi \lim_{n\to\infty}\langle \sin nx, \chi_A\rangle = 0.$$

Thus we have completed the proof of the problem. ■

Problem 4.10

Rudin Chapter 4 Exercise 10.

Proof. This problem is proven in [63, Problem 11.16, p. 352]. ■

Problem 4.11

Rudin Chapter 4 Exercise 11.

Proof. For each $n = 1, 2, \ldots$, we define $x_n = \frac{n+1}{n}u_n$ and $E = \{x_n\}$ which is obviously a subset of $L^2(T)$ because, by Definition 4.23,

$$\|x_n\| = \|x_n\|_2 = \frac{n+1}{n} < \infty$$

for every $n \in \mathbb{N}$. It is clear that $E \neq \varnothing$. Now we are going to apply a similar argument as in the proof of the first assertion in Problem 4.6. In fact, we note that $\langle x_n, x_m\rangle = 0$ for $n \neq m$, so we follow from Definition 4.1 that

$$\begin{aligned}
\|x_m - x_n\|^2 &= \langle x_m - x_n, x_m - x_n\rangle\\
&= \langle x_m, x_m - x_n\rangle - \langle x_n, x_m - x_n\rangle\\
&= \langle x_m, x_m\rangle - \langle x_m, x_n\rangle - \langle x_n, x_m\rangle + \langle x_n, x_n\rangle\\
&= \left(\frac{m+1}{m}\right)^2\langle u_m, u_m\rangle + \left(\frac{n+1}{n}\right)^2\langle u_n, u_n\rangle\\
&= \left(\frac{m+1}{m}\right)^2 + \left(\frac{n+1}{n}\right)^2\\
&\geq 2
\end{aligned}$$

which implies that

$$\|x_m - x_n\| \geq \sqrt{2}. \tag{4.23}$$

Assume that x was a limit point of E. Then there exists a positive integer N such that $n \geq N$ implies $\|x_n - x\| < \frac{\sqrt{2}}{2}$. If $n, m \geq N$, then we have

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \sqrt{2}$$

which contradicts the result (4.23). Thus E has *no* limit point and hence it is closed in $L^2(T)$.

Furthermore, assume that x_N was an element of E such that $\|x_N\| \leq \|x_n\|$ for all $x_n \in E$. However, since we have

$$\|x_N\| = \frac{N+1}{N} > \frac{N+2}{N+1} = \|x_{N+1}\|,$$

this contradiction shows that E has no element of smallest norm. Hence we end the proof of the problem. ■

Remark 4.1

Problem 4.11 indicates that the condition "convexity" in Theorem 4.10 *cannot* be omitted.

Problem 4.12

Rudin Chapter 4 Exercise 12.

Proof. By the half-angle formula for cosine function, we have

$$1 = \frac{c_k}{\pi}\int_0^{\pi}\Big(\frac{1+\cos t}{2}\Big)^k\,\mathrm{d}t = \frac{2c_k}{\pi}\int_0^{\frac{\pi}{2}}\cos^{2k}x\,\mathrm{d}x. \tag{4.24}$$

A further substitution with $z = \sin x$ reduces the expression (4.24) to

$$1 = \frac{2c_k}{\pi}\int_0^1 (1-z^2)^{k-\frac{1}{2}}\,\mathrm{d}z. \tag{4.25}$$

If we substitute $y = z^2$ and use the integral form of the beta function (see [49, Theorem 8.20, p. 193]), then the expression (4.25) can be written as

$$1 = \frac{c_k}{\pi}\int_0^1 (1-y)^{k-\frac{1}{2}}y^{-\frac{1}{2}}\,\mathrm{d}y = \frac{c_k}{\pi}\cdot\frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k+1)} = \frac{c_k}{\sqrt{\pi}}\cdot\frac{\Gamma(k+\frac{1}{2})}{k\Gamma(k)}. \tag{4.26}$$

Finally, we recall from [49, Exercise 30, p. 203] that

$$\lim_{k\to\infty}\frac{\Gamma(k+c)}{k^c\Gamma(k)} = 1 \tag{4.27}$$

for every real constant c. Take $c = \frac{1}{2}$ in the limit (4.27) and then substitute the corresponding result into the expression (4.26), we get

$$\lim_{k\to\infty} k^{-\frac{1}{2}}c_k = \lim_{k\to\infty}\sqrt{\pi}\cdot\frac{k^{\frac{1}{2}}\Gamma(k)}{\Gamma(k+\frac{1}{2})} = \sqrt{\pi}.$$

This completes the proof of the problem. ■

Problem 4.13

Rudin Chapter 4 Exercise 13.

Proof. We follow the hint. Consider the function $f(t) = \mathrm{e}^{2\pi ikt}$, where $k = 0, \pm 1, \pm 2, \ldots$. When $k = 0$, we have $f \equiv 1$ and then the formula holds trivially. Suppose that $k \neq 0$. Since α is irrational, $k\alpha \notin \mathbb{Z}$ and then $\mathrm{e}^{2\pi ik\alpha} \neq 1$. By this, we have

$$\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2\pi ikn\alpha} = \frac{1}{N}\sum_{n=1}^{N} (\mathrm{e}^{2\pi ik\alpha})^n = \frac{\mathrm{e}^{2\pi ik\alpha}}{N} \cdot \frac{\mathrm{e}^{2\pi ikN\alpha} - 1}{\mathrm{e}^{2\pi ik\alpha} - 1}$$

so that

$$\lim_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2\pi ikn\alpha}\right| \leq \lim_{N\to\infty} \frac{2}{N|\mathrm{e}^{2\pi ik\alpha} - 1|} = 0. \tag{4.28}$$

On the other hand, we have

$$\int_0^1 \mathrm{e}^{2\pi ikt}\,\mathrm{d}t = \frac{\mathrm{e}^{2\pi ikt}}{2\pi ik}\Big|_0^1 = 0. \tag{4.29}$$

Hence we deduce from the inequality (4.28) and the integral (4.29) that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2\pi ikn\alpha} = \int_0^1 \mathrm{e}^{2\pi ikt}\,\mathrm{d}t \tag{4.30}$$

for every $k = 0, \pm 1, \pm 2, \ldots$. Therefore the formula (4.30) holds for every trigonometric polynomials in the form

$$P(t) = \sum_{k=-N}^{N} c_k \mathrm{e}^{2\pi ikt}, \tag{4.31}$$

where $c_{-N}, \ldots, c_N \in \mathbb{C}$.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x + 1) = f(x)$ for every $x \in \mathbb{R}$. By Theorem 4.25 (The Weierstrass Approximation Theorem), for every $\epsilon > 0$, there exists a trigonometric polynomial P in the form (4.31) such that

$$|f(t) - P(t)| < \frac{\epsilon}{3} \tag{4.32}$$

for every $t \in \mathbb{R}$. Next, the analysis in the previous paragraph shows that there is a positive integer N such that $n \geq N$ implies

$$\left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) - \int_0^1 P(t)\,\mathrm{d}t\right| < \frac{\epsilon}{3}. \tag{4.33}$$

Hence we follow from the estimates (4.32) and (4.33) that $n \geq N$ implies

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \int_0^1 f(t)\,\mathrm{d}t\right| &= \left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \frac{1}{N}\sum_{n=1}^{N} P(n\alpha) + \frac{1}{N}\sum_{n=1}^{N} P(n\alpha)\right. \\
&\quad \left. - \int_0^1 P(t)\,\mathrm{d}t + \int_0^1 P(t)\,\mathrm{d}t - \int_0^1 f(t)\,\mathrm{d}t\right| \\
&\leq \frac{1}{N}\sum_{n=1}^{N} \left|f(n\alpha) - P(n\alpha)\right| + \left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) - \int_0^1 P(t)\,\mathrm{d}t\right| \\
&\quad + \int_0^1 |f(t) - P(t)|\,\mathrm{d}t \\
&< \epsilon.
\end{aligned}$$

Hence the formula (4.30) holds for every continuous function on $\mathbb{R}$ with period 1, completing the proof of the problem. ∎

Remark 4.2

A question similar to Problem 4.13 is also proven in [63, Problem 8.19, pp. 194 – 196]. In fact, Problem 4.13 is a very important tool in the study of the Weyl's Equidistribution Theorem, see [57, §4.2, pp. 105 – 113].

4.2 Application of Theorem 4.14

Problem 4.14

Rudin Chapter 4 Exercise 14.

Proof. Let H be the Hilbert space $L^2\big([-1,1]\big)$. Define the inner product of $f, g \in H$ by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,\mathrm{d}x. \tag{4.34}$$

It is clear that $x^n \in H$ for every $n = 0, 1, 2, \ldots$ and the set $\{1, x, x^2, \ldots\}$ is a linearly independent set of vectors in H. By Problem 4.2, we can obtain an orthonormal set $\{u_1(x), u_2(x), \ldots\}$ such that $\{1, x, \ldots, x^{N-1}\}$ and $\{u_1(x), u_2(x), \ldots, u_N(x)\}$ have the same span for all positive integers N. Next, we let $F = \{0, 1, 2\}$ and we consider the closed subspace

$$M_F = \overline{\operatorname{span}\left(\{1, x, x^2\}\right)} \tag{4.35}$$

of H. Then the orthonormal set $\{u_1(x), u_2(x), u_3(x)\}$ obtained by $\{1, x, x^2\}$ is given by

$$u_1(x) = \frac{1}{\sqrt{2}}, \quad u_2(x) = \sqrt{\frac{3}{2}}x \quad \text{and} \quad u_3(x) = \frac{3}{2}\sqrt{\frac{5}{2}}\Big(x^2 - \frac{1}{3}\Big).$$

Let $x^3 \in H$. Define

$$s_F(x^3) = \big\langle x^3, u_1(x)\big\rangle \cdot u_1(x) + \big\langle x^3, u_2(x)\big\rangle \cdot u_2(x) + \big\langle x^3, u_3(x)\big\rangle \cdot u_3(x).$$

By Theorem 4.14(b) and the inner product (4.34), we derive

$$\|x^3 - s_F\| = \min\{\|x^3 - s\| \mid s \in M_F\} = \left[\min_{a,b,c}\int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,\mathrm{d}x\right]^{\frac{1}{2}}. \tag{4.36}$$

Since $\big\langle x^3, u_1(x)\big\rangle = 0$, $\big\langle x^3, u_2(x)\big\rangle = \frac{2}{5}\sqrt{\frac{3}{2}}$ and $\big\langle x^3, u_3(x)\big\rangle = 0$, we have

$$s_F(x^3) = \frac{3}{5}x$$

and then

$$\Big\|x^3 - \frac{3}{5}x\Big\| = \Big[\int_{-1}^{1}\Big(x^3 - \frac{3}{5}x\Big)^2\,\mathrm{d}x\Big]^{\frac{1}{2}} = \sqrt{\frac{8}{175}}. \tag{4.37}$$

Combining the results (4.36) and (4.37), it yields that

$$\min_{a,b,c}\int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,\mathrm{d}x = \frac{8}{175}. \tag{4.38}$$

For the second assertion, we first notice that the definite integral must be a real number because we are considering its maximum. Secondly, the conditions on g actually mean that

$$\langle 1, \overline{g}\rangle = \langle x, \overline{g}\rangle = \langle x^2, \overline{g}\rangle = 0 \quad \text{and} \quad \|\overline{g}\| = 1.$$

Thus, by the discussion in §4.9, we have

$$\overline{g} \in M_F^{\perp} \quad \text{and} \quad \|\overline{g}\| = 1,$$

where M_F is defined by (4.35). Now we follow from these facts that

$$\max_{\substack{\overline{g}\in M_F^{\perp}\\ \|\overline{g}\|=1}} \int_{-1}^{1} x^3 g(x)\,\mathrm{d}x \leq \max_{\substack{\overline{g}\in M_F^{\perp}\\ \|\overline{g}\|=1}} \left|\int_{-1}^{1} x^3 g(x)\,\mathrm{d}x\right| = \max\{|\langle x^3, \overline{g}\rangle| \mid \overline{g} \in M_F^{\perp} \text{ and } \|\overline{g}\| = 1\}. \tag{4.39}$$

By Problem 4.16[d], we see that

$$\max\{|\langle x^3, \overline{g}\rangle| \mid \overline{g} \in M_F^{\perp} \text{ and } \|\overline{g}\| = 1\} = \min\{\|x^3 - s\| \mid s \in M_F\}. \tag{4.40}$$

Therefore, we combine the expressions (4.39) and (4.40) and then using the value (4.38) to get

$$\max_{\substack{\overline{g}\in M_F^{\perp}\\ \|\overline{g}\|=1}} \int_{-1}^{1} x^3 g(x)\,\mathrm{d}x \leq \sqrt{\frac{8}{175}}. \tag{4.41}$$

If $g(x) = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$, then we have

$$\int_{-1}^{1} x^3 g(x)\,\mathrm{d}x = \sqrt{\frac{8}{175}}$$

so that the equality in the inequality (4.41) is attainable. Hence we have completed the proof of the problem. ∎

Problem 4.15

Rudin Chapter 4 Exercise 15.

Proof. Now we work on the space $H = L^2([0,\infty])$ and it is easy to check that the formula given by

$$\langle f, g\rangle = \int_0^{\infty} f\overline{g}e^{-x}\,\mathrm{d}x \tag{4.42}$$

satisfies Definition 4.1, so it is indeed an inner product for the space H.

Now we follow the idea of proof in Problem 4.14. Evidently, we have $x^n \in H$ for every $n = 0, 1, 2, \ldots$ and the set $\{1, x, x^2, \ldots\}$ is a linearly independent set of vectors in H. By Problem 4.2, we can obtain an orthonormal set $\{u_1(x), u_2(x), \ldots\}$ such that $\{1, x, \ldots, x^{N-1}\}$ and $\{u_1(x), u_2(x), \ldots, u_N(x)\}$ have the same span for all positive integers N. Next, we let $F = \{0, 1, 2\}$ and we consider the closed subspace

$$M_F = \overline{\operatorname{span}(\{1, x, x^2\})}$$

of H. Then the orthonormal set $\{u_1(x), u_2(x), u_3(x)\}$ obtained by $\{1, x, x^2\}$ is given by

$$u_1(x) = 1, \quad u_2(x) = x - 1 \quad \text{and} \quad u_3(x) = \frac{1}{2}(x^2 - 4x + 2).$$

[d] We assume its truth here.

Let $x^3 \in H$. Define

$$s_F(x^3) = \langle x^3, u_1(x)\rangle \cdot u_1(x) + \langle x^3, u_2(x)\rangle \cdot u_2(x) + \langle x^3, u_3(x)\rangle \cdot u_3(x).$$

By Theorem 4.14(b) and the inner product (4.42), we derive

$$\|x^3 - s_F\| = \min\{\|x^3 - s\| \mid s \in M_F\} = \left[\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \mathrm{e}^{-x}\,\mathrm{d}x\right]^{\frac{1}{2}}. \tag{4.43}$$

Since $\langle x^3, u_1(x)\rangle = 6$ and $\langle x^3, u_2(x)\rangle = \langle x^3, u_3(x)\rangle = 18$, we have

$$s_F(x^3) = 6 + 18(x-1) + 9(x^2 - 4x + 2) = 9x^2 - 18x + 6$$

so that

$$\|x^3 - 9x^2 + 18x - 6\| = \left[\int_{-1}^{1} (x^3 - 9x^2 + 18x - 6)^2 \mathrm{e}^{-x}\,\mathrm{d}x\right]^{\frac{1}{2}} = 6. \tag{4.44}$$

Combining the results (4.43) and (4.44), it yields that

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \mathrm{e}^{-x}\,\mathrm{d}x = 36. \tag{4.45}$$

For the second assertion, we observe again that the definite integral is a real number and the conditions on g imply that

$$g \in M_F^{\perp} \quad \text{and} \quad \|g\| = 1.$$

Therefore, it yields that

$$\begin{aligned}\max_{\substack{\overline{g} \in M_F^{\perp} \\ \|\overline{g}\|=1}} \int_{-1}^{1} x^3 g(x)\mathrm{e}^{-x}\,\mathrm{d}x &\leq \max_{\substack{\overline{g} \in M_F^{\perp} \\ \|\overline{g}\|=1}} \left|\int_{-1}^{1} x^3 g(x)\mathrm{e}^{-x}\,\mathrm{d}x\right| \\ &= \max\{|\langle x^3, \overline{g}\rangle| \mid \overline{g} \in M_F^{\perp} \text{ and } \|\overline{g}\| = 1\}.\end{aligned} \tag{4.46}$$

By Problem 4.16 again, we see that

$$\max\{|\langle x^3, \overline{g}\rangle| \mid \overline{g} \in M_F^{\perp} \text{ and } \|\overline{g}\| = 1\} = \min\{\|x^3 - s\| \mid s \in M_F\}. \tag{4.47}$$

By combining the expressions (4.46) and (4.47) and then using the value (4.45), we gain

$$\max_{\substack{\overline{g} \in M_F^{\perp} \\ \|\overline{g}\|=1}} \int_{-1}^{1} x^3 g(x)\mathrm{e}^{-x}\,\mathrm{d}x \leq 6. \tag{4.48}$$

If $g(x) = \frac{1}{6}(x^3 - 9x^2 + 18x - 6)$, then we have

$$\int_{-1}^{1} x^3 g(x)\mathrm{e}^{-x}\,\mathrm{d}x = 6$$

so that the equality in the inequality (4.48) is attainable and we end the proof of the problem. ■

4.3 Miscellaneous Problems

Problem 4.16

Rudin Chapter 4 Exercise 16.

Proof. Since M is a closed linear subspace of H, we follow from Theorem 4.11(a) that x_0 has the unique decomposition
$$x_0 = Px_0 + Qx_0,$$
where $Px_0 \in M$ and $Qx_0 \in M^\perp$. In addition, Theorem 4.11(b) ensures that
$$\|Px_0 - x_0\| = \min\{\|x - x_0\| \,|\, x \in M\}. \tag{4.49}$$
On the other hand, if $y \in M^\perp$ and $\|y\| = 1$, then we have
$$\langle x_0, y\rangle = \langle Px_0 + Qx_0, y\rangle = \langle Qx_0, y\rangle$$
and then Theorem 4.2 (The Schwarz Inequality) implies that
$$|\,\langle x_0, y\rangle\,| = |\,\langle Qx_0, y\rangle\,| \leq \|Qx_0\| \times \|y\| = \|Qx_0\|. \tag{4.50}$$
In other words, the inequality (4.50) shows that
$$\max\{|\,\langle x_0, y\rangle\,| \,|\, y \in M^\perp \text{ and } \|y\| = 1\} \leq \|Qx_0\|. \tag{4.51}$$
If $Qx_0 = 0$, then it means that $x_0 = Px_0 \in M$. In this case, we deduce from the expression (4.49) and the inequality (4.51) that
$$\min\{\|x - x_0\| \,|\, x \in M\} = \|Px_0 - x_0\| = \max\{|\,\langle x_0, y\rangle\,| \,|\, y \in M^\perp \text{ and } \|y\| = 1\} = 0.$$
If $Qx_0 \neq 0$, since $Qx_0 \in M^\perp$, we may consider $y_0 = \frac{Qx_0}{\|Qx_0\|}$ which satisfies the conditions $y_0 \in M^\perp$ and $\|y_0\| = 1$. A direct computation shows that
$$|\,\langle x_0, y_0\rangle\,| = \frac{1}{\|Qx_0\|} \times |\,\langle x_0, Qx_0\rangle\,| = \frac{1}{\|Qx_0\|} \times |\,\langle Qx_0, Qx_0\rangle\,| = \|Qx_0\|.$$
Thus the inequality (4.51) actually becomes
$$\max\{|\,\langle x_0, y\rangle\,| \,|\, y \in M^\perp \text{ and } \|y\| = 1\} = \|Qx_0\|. \tag{4.52}$$
Hence our desired result follows from the comparison between the expressions (4.49) and (4.52) and the fact that $\|Px_0 - x_0\| = \|Qx_0\|$. We have ended the proof of the problem. ■

Problem 4.17

Rudin Chapter 4 Exercise 17.

Proof. Take $H = L^2\big([0,1]\big)$ which is a Hilbert space with the inner product given by
$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t, \tag{4.53}$$
where $f, g \in H$, see Example 4.5(b). For $t \in [0,1]$, we note that $\chi_{[0,t]} \in H$ because
$$\|\chi_{[0,t]}\| = \Big\{\int_0^1 |\chi_{[0,t]}(x)|^2\,\mathrm{d}x\Big\}^{\frac{1}{2}} = \Big\{\int_0^t \mathrm{d}x\Big\}^{\frac{1}{2}} = \sqrt{t} \leq 1.$$
Thus we can consider the mapping $\gamma : [0,1] \to H$ defined by
$$\gamma(t) = \chi_{[0,t]}$$

which is well-defined. It is trivial that γ is injective. Furthermore, for every $p \in [0,1]$, if $t > p$, then we have

$$\|\gamma(t)-\gamma(p)\| = \left\{\int_0^1 |\chi_{(p,t]}(x)|^2\,\mathrm{d}x\right\}^{\frac{1}{2}} = \left\{\int_p^t \mathrm{d}x\right\}^{\frac{1}{2}} = (t-p)^{\frac{1}{2}}. \tag{4.54}$$

Similarly, if $t < p$, then we have

$$\|\gamma(t)-\gamma(p)\| = \left\{\int_t^p \mathrm{d}x\right\}^{\frac{1}{2}} = (p-t)^{\frac{1}{2}}. \tag{4.55}$$

By combining the two expressions (4.54) and (4.55), we conclude that

$$\|\gamma(t)-\gamma(p)\| = |t-p|^{\frac{1}{2}}$$

and thus γ is continuous on $[0,1]$.

Let $0 \le a \le b \le c \le d \le 1$. If $a = b$, then $\gamma(a) - \gamma(b) = 0$ and

$$\langle \gamma(b)-\gamma(a), \gamma(d)-\gamma(c)\rangle = \langle 0, \gamma(d)-\gamma(c)\rangle = \int_0^1 0 \times \overline{\chi_{(c,d]}(x)}\,\mathrm{d}x = 0.$$

Similarly, if $c = d$, then $\langle \gamma(b)-\gamma(a), \gamma(d)-\gamma(c)\rangle = 0$. Without loss of generality, we may assume that $a < b$ and $c < d$. Then $(a,b] \cap (c,d] = \varnothing$ and so we have

$$\langle \gamma(b)-\gamma(a), \gamma(d)-\gamma(c)\rangle = \left\langle \chi_{(a,b]}, \chi_{(c,d]}\right\rangle = \int_0^1 \chi_{(a,b]}(x)\overline{\chi_{(c,d]}(x)}\,\mathrm{d}x = 0.$$

Hence the desired result holds in the special case that $H = L^2\big([0,1]\big)$.

To treat the general case, we first consider the function $u_n(t) = \mathrm{e}^{2\pi i n t}$ for every $n \in \mathbb{Z}$. It is clear that $u_n \in H$ and

$$\langle u_n, u_m\rangle = \int_0^1 \mathrm{e}^{2\pi i(n-m)t}\,\mathrm{d}t = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases}$$

Thus $\{u_n \,|\, n \in \mathbb{Z}\}$ is an orthonormal set in H. By Theorem 3.14, $C\big([0,1]\big)$ is dense in H with respect to the norm induced by the inner product (4.53). By Theorem 4.25 (The Weierstrass Approximation Theorem) and Definition 4.23, we know that the set $\mathcal{P}$ of all trigonometric polynomials in the form

$$P(t) = \sum_{n=-N}^{N} c_n u_n(t) = \sum_{n=-N}^{N} c_n \mathrm{e}^{2\pi i n t}$$

is dense in $C\big([0,1]\big)$ with respect to the norm $\|\cdot\|_\infty$. Note that $\mathcal{P} = \operatorname{span}\big(\{u_n \,|\, n \in \mathbb{Z}\}\big)$. By Definition 3.7, it can be shown easily that $\mathcal{P}$ is also dense in $C\big([0,1]\big)$ with respect to the norm induced by the inner product (4.53). Therefore, we derive from Lemma 2.10 that $\mathcal{P}$ is also dense in H in the norm induced by the inner product (4.53). Consequently, we conclude from Theorem 4.18(i) that $\{u_n \,|\, n \in \mathbb{Z}\}$ is in fact maximal and then[e]

$$H \cong \ell^2(\mathbb{N}). \tag{4.56}$$

Suppose that H' is a Hilbert space with an *infinite* maximal orthonormal set $\{v_\alpha \,|\, \alpha \in A\}$, where A is countable or uncountable infinite. Then we have

$$H' \cong \ell^2(A). \tag{4.57}$$

[e]We say $A \cong B$ if A is Hilbert space isomorphic to B.

Applying Problem 4.8 to the Hilbert space isomorphisms (4.56) and (4.57), we conclude that H must be Hilbert space isomorphic to a subspace of H' and then the special case is applicable to this general case. Indeed, let $\Lambda : H \to K$ be a Hilbert space isomorphism, where K is a Hilbert subspace of H'. If we define $\lambda : [0,1] \to H'$ by

$$\lambda = \Lambda \circ \gamma,$$

then λ is injective and continuous. Furthermore, since $\langle \Lambda(f), \Lambda(g)\rangle = \langle f, g\rangle$ for all $f, g \in H$, we deduce from the special case above that

$$\begin{aligned}\langle \lambda(b) - \lambda(a), \lambda(d) - \lambda(c)\rangle &= \big\langle \Lambda\big(\gamma(b)\big) - \Lambda\big(\gamma(a)\big), \Lambda\big(\gamma(d)\big) - \Lambda\big(\gamma(c)\big)\big\rangle \\ &= \big\langle \Lambda\big(\gamma(b)\big), \Lambda\big(\gamma(d)\big)\big\rangle - \big\langle \Lambda\big(\gamma(b)\big), \Lambda\big(\gamma(c)\big)\big\rangle \\ &\quad - \big\langle \Lambda\big(\gamma(a)\big), \Lambda\big(\gamma(d)\big)\big\rangle + \big\langle \Lambda\big(\gamma(a)\big), \Lambda\big(\gamma(c)\big)\big\rangle \\ &= \langle \gamma(b), \gamma(d)\rangle - \langle \gamma(b), \gamma(c)\rangle - \langle \gamma(a), \gamma(d)\rangle + \langle \gamma(a), \gamma(c)\rangle \\ &= \langle \gamma(b) - \gamma(a), \gamma(c) - \gamma(d)\rangle \\ &= 0.\end{aligned}$$

This completes the proof of the problem. ■

Problem 4.18

Rudin Chapter 4 Exercise 18.

Proof. If $r, s \in \mathbb{R}$ and $r \neq s$, then we have

$$\langle u_r, u_s\rangle = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \mathrm{e}^{i(r-s)t}\,\mathrm{d}t = \lim_{A\to\infty} \frac{2i\sin[(r-s)A]}{2A(r-s)} = 0. \tag{4.58}$$

If $r = s$, then

$$\langle u_s, u_s\rangle = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \mathrm{e}^{ist}\mathrm{e}^{-ist}\,\mathrm{d}t = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \mathrm{d}t = 1. \tag{4.59}$$

Thus $\{u_s \,|\, s \in \mathbb{R}\}$ is an orthonormal set of H by Definition 4.13. Since f and g are finite combinations of u_s, we may suppose that

$$f(t) = \sum_{p=1}^{n} c_p \mathrm{e}^{is_p t} \quad \text{and} \quad g(t) = \sum_{q=1}^{m} d_q \mathrm{e}^{ir_q t}. \tag{4.60}$$

Therefore, we obtain

$$\lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \Big(\sum_{p=1}^{n} c_p \mathrm{e}^{is_p t}\Big)\Big(\sum_{q=1}^{m} \overline{d_q}\mathrm{e}^{-ir_q t}\Big)\,\mathrm{d}t = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \sum_{p=1}^{n}\sum_{q=1}^{m} c_p\overline{d_q}\mathrm{e}^{i(s_p - r_q)t}\,\mathrm{d}t$$

which is undoubtedly finite by the values (4.58) and (4.59). As a consequence, we have shown that $\langle f, g\rangle$ is well-defined for all $f, g \in X$.

Next we are going to show that $\langle\,\cdot\,\rangle$ satisfies Definition 4.1. Firstly, it is clear that

$$\langle f, g\rangle = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} f(t)\overline{g(t)}\,\mathrm{d}t = \overline{\lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} g(t)\overline{f(t)}\,\mathrm{d}t} = \overline{\langle g, f\rangle}.$$

Secondly, for $f, g, h \in X$, we observe that

$$\langle f + g, h\rangle = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} [f(t) + g(t)]\overline{h(t)}\,\mathrm{d}t$$

$$= \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{h(t)}\,\mathrm{d}t + \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} g(t)\overline{h(t)}\,\mathrm{d}t$$
$$= \langle f, h\rangle + \langle g, h\rangle .$$

Thirdly, for any scalar α, we have

$$\langle \alpha f, g\rangle = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \alpha f(t)\overline{h(t)}\,\mathrm{d}t = \alpha \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{h(t)}\,\mathrm{d}t = \alpha \langle f, g\rangle .$$

Fourthly, if f takes the representation (4.60), then we establish from the results (4.58) and (4.59) that

$$\langle f, f\rangle = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{f(t)}\,\mathrm{d}t = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} |f(t)|^2\,\mathrm{d}t = \sum_{p=1}^{n} |c_p|^2 \geq 0. \tag{4.61}$$

Finally, if $\langle f, f\rangle = 0$, then the result (4.61) definitely shows that $c_1 = c_2 = \cdots = c_p = 0$. In other words, we have $f \equiv 0$ in this case. Hence this inner product certainly makes X into a unitary space.

Let H be the completion of X, i.e., X is dense in H and $H = \overline{X}$. Since span $(\{u_s \,|\, s \in \mathbb{R}\}) = X$, Theorem 4.18(i) says that the set $\{u_s \,|\, s \in \mathbb{R}\}$ is in fact a maximal orthonormal in H. By the discussion in §4.19, we have

$$H \cong \ell^2(\mathbb{R}). \tag{4.62}$$

Assume that H was separable. By Problem 4.4, H has an at most countable maximal orthonormal system $\{v_n \,|\, n \in \mathbb{N}\}$. By the discussion in §4.19 again, we know that

$$H \cong \ell^2(\mathbb{N}). \tag{4.63}$$

Now the two isomorphic relations (4.62) and (4.63) imply that $\ell^2(\mathbb{R}) \cong \ell^2(\mathbb{N})$, but then $\mathbb{R}$ and $\mathbb{N}$ have the same cardinal number, a contradiction. Hence we complete the proof of the problem. ■

Problem 4.19

Rudin Chapter 4 Exercise 19.

Proof. If $N = 1$, then $\omega = \mathrm{e}^{2\pi i} = 1$ and k can only take the value 0. Thus the orthogonality relation holds in this special case. Suppose that $N > 1$. If $k = 0$, then $\omega^0 = 1$ so that

$$\frac{1}{N} \sum_{n=1}^{N} \omega^0 = 1.$$

Let $k = 1, 2, \ldots, N-1$. We remark that

$$z^N - 1 = (z-1)(z^{N-1} + \cdots + z + 1). \tag{4.64}$$

Putting $z = \omega^k$ into the expression (4.64), we obtain

$$(\omega^k)^N - 1 = (\omega^k - 1)\big[(\omega^k)^{N-1} + \cdots + \omega^k + 1\big] = (\omega^k - 1)\big(\omega^{k(N-1)} + \cdots + \omega^k + 1\big). \tag{4.65}$$

It is clear that $(\omega^k)^N = (\omega^N)^k = 1$. Furthermore, since $1 \leq k \leq N-1$, k is evidently not divisible by N which implies that $\omega^k \neq 1$. Therefore, it deduces from the expression (4.65) that

$$\omega^{k(N-1)} + \cdots + \omega^k + 1 = 0$$

or equivalently

$$\omega^{kN} + \omega^{k(N-1)} + \cdots + \omega^k = 0.$$

In conclusion, we have

$$\frac{1}{N}\sum_{n=1}^{N}\omega^{nk} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } 1 \leq k \leq N-1. \end{cases}$$

Let $N \geq 3$. It is clear from the orthogonality relations and the properties in Definition 4.1 that

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N}\|x+\omega^n y\|^2\omega^n &= \frac{1}{N}\sum_{n=1}^{N}\langle x+\omega^n y, x+\omega^n y\rangle\,\omega^n \\
&= \frac{1}{N}\Big[\sum_{n=1}^{N}\langle x,x\rangle\,\omega^n + \sum_{n=1}^{N}\langle x,\omega^n y\rangle\,\omega^n + \sum_{n=1}^{N}\langle \omega^n y, x\rangle\,\omega^n \\
&\quad + \sum_{n=1}^{N}\langle \omega^n y,\omega^n y\rangle\,\omega^n\Big] \\
&= \frac{1}{N}\Big[\langle x,x\rangle\sum_{n=1}^{N}\omega^n + \sum_{n=1}^{N}\langle x,y\rangle\,\omega^{-n}\times\omega^n + \sum_{n=1}^{N}\langle y,x\rangle\,\omega^n\times\omega^n \\
&\quad + \langle y,y\rangle\sum_{n=1}^{N}\omega^n\Big] \\
&= \frac{1}{N}\Big[\langle x,x\rangle\sum_{n=1}^{N}\omega^n + N\langle x,y\rangle + \langle y,x\rangle\sum_{n=1}^{N}\omega^{2n} + \langle y,y\rangle\sum_{n=1}^{N}\omega^n\Big] \\
&= \langle x,y\rangle
\end{aligned} \tag{4.66}$$

as desired.

Finally, Definition 4.1 gives

$$\|x+e^{i\theta}y\|^2 = \Big\langle x+e^{i\theta}y, x+e^{i\theta}y\Big\rangle = \langle x,x\rangle + e^{-i\theta}\langle x,y\rangle + e^{i\theta}\langle y,x\rangle + \langle y,y\rangle$$

so that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\|x+e^{i\theta}y\|^2 e^{i\theta}\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big[\|x\|^2+\|y\|^2+e^{-i\theta}\langle x,y\rangle + e^{i\theta}\langle y,x\rangle\big]e^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\langle x,y\rangle\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{2i\theta}\langle y,x\rangle\,d\theta \\
&= \langle x,y\rangle\,.
\end{aligned} \tag{4.67}$$

This completes the proof of the problem. ■

Remark 4.3

We note that the polarization identity (see [51, p. 86]) can be written in the form

$$\langle x,y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^{-n}\|x+i^n y\|^2.$$

Thus the two identities (4.66) and (4.67) in Problem 4.19 are actually generalizations of this.

CHAPTER 5

Examples of Banach Space Techniques

5.1 The Unit Ball in a Normed Linear Space

Problem 5.1

Rudin Chapter 5 Exercise 1.

Proof. For $0 < p \leq \infty$, we suppose that

$$B = \{f \in L^p(\mu) \,|\, \|f\|_p \leq 1\}.$$

There are two cases:

- **Case (i): $0 < p < \infty$.** In this case, we have

$$\begin{aligned} \|f\|_p &= \Big\{ \int_X |f|^p \,\mathrm{d}\mu \Big\}^{\frac{1}{p}} \\ &= \big(|f(a)|^p \times \mu(\{a\}) + |f(b)|^p \times \mu(\{b\})\big)^{\frac{1}{p}} \\ &= \frac{1}{2^{\frac{1}{p}}} \{|f(a)|^p + |f(b)|^p\}^{\frac{1}{p}}. \end{aligned}$$

  Consequently, $\|f\|_p \leq 1$ if and only if

$$|f(a)|^p + |f(b)|^p \leq 2. \tag{5.1}$$

  Therefore, it follows from the inequality (5.1) that the unit ball B is a circle if and only if $p = 2$. Besides, the unit ball B becomes a square if and only if $p = 1$.

- **Case (ii): $p = \infty$.** By Definition 3.7, we know that

$$\|f\|_\infty = \max(|f(a)|, |f(b)|) \tag{5.2}$$

  so that $\|f\|_\infty \leq 1$ if and only if

$$\max(|f(a)|, |f(b)|) \leq 1 \tag{5.3}$$

  which is a square.

See Figure 5.1 for an illustration.

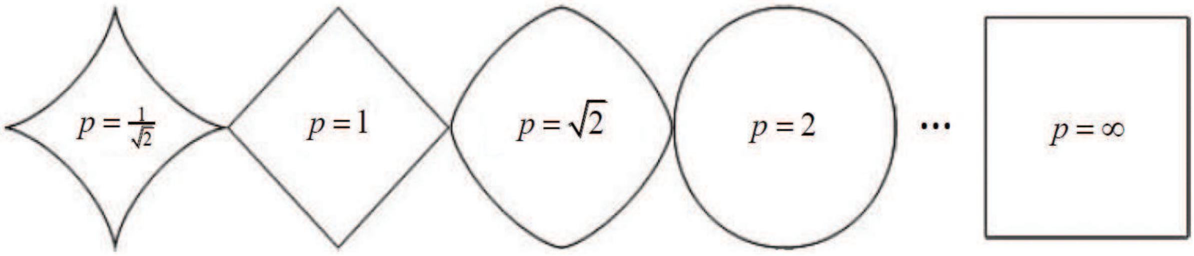


Figure 5.1: The unit circle in different p-norm.

If $\mu(\{a\}) \neq \mu(\{b\})$, then the expression (5.1) is replaced by

$$\mu(\{a\})|f(a)|^p + \mu(\{b\})|f(b)|^p \leq 1$$

which cannot be a circle or a square for any $p \in (0, \infty)$. However, we note that the expression (5.2) is still valid even in the case $\mu(\{a\}) \neq \mu(\{b\})$, so we have the inequality (5.3) and then B is a square when $p = \infty$. This ends the analysis of the problem. ■

Problem 5.2

Rudin Chapter 5 Exercise 2.

Proof. Let X be a normed linear space with norm $\|\cdot\|$ and $B(0,1) = \{x \in X \mid \|x\| < 1\}$ be the open unit ball. For every $x, y \in B(0,1)$ and $t \in [0,1]$, we follow from Definition 5.2 that

$$\|(1-t)x + ty\| \leq (1-t)\|x\| + t\|y\| < 1$$

so that $(1-t)x + ty \in B(0,1)$. By §4.8, $B(0,1)$ is convex. Similarly, the convexity of the closed unit ball $\overline{B(0,1)} = \{x \in X \mid \|x\| \leq 1\}$ can be proven similarly. Hence we complete the proof of the problem. ■

Problem 5.3

Rudin Chapter 5 Exercise 3.

Proof. Suppose that $\|f\|_p = \|g\|_p = 1$ and $f \neq g$. By Theorem 3.9, we always have

$$\|h\|_p = \left\|\frac{1}{2}(f+g)\right\|_p \leq \frac{1}{2}\|f\|_p + \frac{1}{2}\|g\|_p = 1.$$

Assume that $\|h\|_p = 1$. Then it follows from Problem 3.13 that $f = \lambda g$ a.e. on X for some $\lambda > 0$. Since $\lambda\|g\|_p = \|f\|_p = \|g\|_p$, we have $\lambda = 1$ implying $f = g$ a.e. on X, a contradiction.

However, strictly convexity *does not* hold for $L^1(\mu)$, $L^\infty(\mu)$ and $C(X)$. Ignore trivialities, we suppose that X contains more than one point. In fact, we can pick $E, F \in \mathfrak{M} \setminus \{\varnothing\}$ such that $E \cap F = \varnothing$ and $\mu(E), \mu(F) \in (0, \infty)$. Define $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ by

$$f(x) = \frac{1}{\mu(E)}\chi_E(x) \quad \text{and} \quad g(x) = \frac{1}{\mu(F)}\chi_F(x) \tag{5.4}$$

respectively. It is obvious that $f \neq g$. Furthermore, we have

$$\|f\|_1 = \int_X |f(x)|\,dx = \frac{1}{\mu(E)}\int_E \chi_E(x)\,dx = 1,$$

$$\|g\|_1 = \int_X |g(x)|\,\mathrm{d}x = \frac{1}{\mu(F)}\int_F \chi_F(x)\,\mathrm{d}x = 1,$$
$$\Big\|\frac{f+g}{2}\Big\|_1 = \int_X \Big|\frac{f(x)+g(x)}{2}\Big|\,\mathrm{d}x = \frac{1}{2}\int_E f(x)\,\mathrm{d}x + \frac{1}{2}\int_F g(x)\,\mathrm{d}x = 1.$$

These show that the invalidity of strictly convexity in $L^1(\mu)$. For $L^\infty(\mu)$, we replace the functions f and g in the definition (5.4) by

$$f(x) = \chi_E(x) \quad \text{and} \quad g(x) = \chi_{E\cup F}(x)$$

respectively. Then it is also true that $f \neq g$. By Definition 3.7, we have

$$\|f\|_\infty = \|g\|_\infty = \Big\|\frac{f+g}{2}\Big\|_\infty = 1.$$

Therefore, the strictly convexity does not hold in $L^\infty(\mu)$.

Finally, we consider the space $C(X)$, where X is Hausdorff. By Definition 3.16, we see that X is assumed to be compact and so the norm $\|\cdot\|_\infty$ of $C(X)$ is given by

$$\|f\|_\infty = \sup_{x\in X} |f(x)|. \tag{5.5}$$

Take $f \in C(X)$ and suppose that f is nonconstant. Thus one can find a $x_0 \in X$ such that

$$|f(x_0)| < \|f\|_\infty. \tag{5.6}$$

Otherwise, we have $|f(x)| \geq \|f\|_\infty$ on X, but this and the definition (5.5) will imply that $|f(x)| = \|f\|_\infty$ on X, i.e., f is a constant which is a contradiction. Define

$$a = \frac{3}{4}\|f\|_\infty + \frac{1}{4}|f(x_0)| \quad \text{and} \quad b = \frac{1}{4}\|f\|_\infty + \frac{3}{4}|f(x_0)|.$$

Consider the sets

$$K = \Big\{x \in X \,\Big|\, |f(x)| \geq a\Big\} = |f|^{-1}\big([a,\infty)\big) \quad \text{and} \quad E = \Big\{x \in X \,\Big|\, |f(x)| \leq b\Big\} = |f|^{-1}\big((-\infty, b]\big).$$

It is clear that $x_0 \in E$, i.e., $E \neq \varnothing$. Since $|f| : X \to \mathbb{R}$ is continuous, the Extreme Value Theorem ([42, Theorem 27.4, p. 174]) implies that $\|f\|_\infty = \max_{x\in X} |f(x)| = |f(p)|$ for some $p \in X$. Consequently, we have

$$p \in K. \tag{5.7}$$

Furthermore, both K and E are closed in X by [42, Theorem 18.1, p. 104]. If $K \cap E \neq \varnothing$, then we have

$$\frac{3}{4}\|f\|_\infty + \frac{1}{4}|f(x_0)| \leq \frac{1}{4}\|f\|_\infty + \frac{3}{4}|f(x_0)|$$

but these imply that $\|f\|_\infty \leq |f(x_0)|$ which contradicts the hypothesis (5.6). Hence we have $K \cap E = \varnothing$.

What we have done in the previous paragraph is that $V = E^c$ is open in X and $K \subseteq V$. Since X is compact, Theorem 2.4 ensures that K is also compact. By Theorem 2.12 (Urysohn's Lemma), there exists a $h \in C_c(X)$ such that $K \prec h \prec V$, i.e., $0 \leq h(x) \leq 1$ on X, $h(x) = 1$ on K and $\operatorname{supp}(h) \subseteq V$ (or equivalently $h(x) = 0$ on E). If we define $g = h \times f : X \to \mathbb{C}$, then we must have $g \not\equiv f$ and furthermore, the fact (5.7) shows that

$$\|g\|_\infty = \sup_{x\in X} |h(x)f(x)| = |h(p)| \cdot |f(p)| = |f(p)| = \|f\|_\infty$$

and

$$\Big\|\frac{f+g}{2}\Big\|_\infty = \frac{1}{2}\|f + hf\|_\infty = \frac{1}{2}\sup_{x\in X}\big|[1+h(x)]f(x)\big| = \frac{1}{2}|1+h(p)| \cdot |f(p)| = |f(p)| = \|f\|_\infty.$$

Hence strictly convexity does not hold for $C(X)$, completing the proof of the problem. ■

5.2 Failure of Theorem 4.10 and Norm-preserving Extensions

Problem 5.4

Rudin Chapter 5 Exercise 4.

Proof. Define $f : [0,1] \to \mathbb{C}$ by

$$f(x) = \begin{cases} -8x+4, & \text{if } x \in [0, \frac{1}{2}]; \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then it is easy to see that $f \in M$, i.e., M is nonempty. Suppose that $f, g \in M$, $t \in [0,1]$ and $h = (1-t)f + tg$. Then we have

$$\begin{aligned} \int_0^{\frac{1}{2}} h(x)\,\mathrm{d}x - \int_{\frac{1}{2}}^1 h(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} [(1-t)f(x) + tg(x)]\,\mathrm{d}x - \int_{\frac{1}{2}}^1 [(1-t)f(x) + tg(x)]\,\mathrm{d}x \\ &= (1-t)\Big[\int_0^1 f(x)\,\mathrm{d}x - \int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x\Big] \\ &\quad + t\Big[\int_0^1 g(x)\,\mathrm{d}x - \int_{\frac{1}{2}}^1 g(x)\,\mathrm{d}x\Big] \\ &= 1 \end{aligned}$$

so that $h \in M$, i.e., M is convex.

Next, we note that C and M are metric spaces. By [49, Theorem 7.15, p. 151], we see that C is a complete metric space. Since $M \subseteq C$, M is also a complete metric space. Suppose that $\{f_n\} \subseteq M$ and

$$\|f_n - f\| = \sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0$$

as $n \to \infty$. Then it follows from[a] [49, Theorem 7.9, p. 148] that $f_n \to f$ uniformly on $[0,1]$. Since each f_n is continuous on $[0,1]$, f is also continuous on $[0,1]$. Furthermore, we have

$$\begin{aligned} \lim_{n\to\infty} \Big[\int_0^{\frac{1}{2}} f_n(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 f_n(t)\,\mathrm{d}t\Big] &= 1 \\ \int_0^{\frac{1}{2}} \Big(\lim_{n\to\infty} f_n(t)\Big)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 \Big(\lim_{n\to\infty} f_n(t)\Big)\,\mathrm{d}t &= 1 \\ \int_0^{\frac{1}{2}} f(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 f(t)\,\mathrm{d}t &= 1. \end{aligned}$$

In other words, $f \in M$ and M is closed in the space C.

Finally, we want to show that M contains *no* element of minimal norm. To this end, we notice that

$$1 = \Big|\int_0^{\frac{1}{2}} f(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 f(t)\,\mathrm{d}t\Big| \leq \int_0^{\frac{1}{2}} |f(t)|\,\mathrm{d}t + \int_{\frac{1}{2}}^1 |f(t)|\,\mathrm{d}t = \|f\|_1 \leq \|f\|_\infty \tag{5.8}$$

[a]Or we can see the rephrased Theorem 7.9 on p. 151 in [49].

for every $f \in M$. Assume that there was a $f \in M$ such that $\|f\|_\infty = 1$. Then we deduce from the inequality (5.8) that

$$\int_0^1 |f(t)|\,\mathrm{d}t = 1 \quad \text{or} \quad \int_0^1 (1 - |f(t)|)\,\mathrm{d}t = 0. \tag{5.9}$$

Since $\|f\|_\infty = 1$, we have $|f(x)| \in [0,1]$ for all $x \in [0,1]$ so that the function $g(x) = 1 - |f(x)|$ is continuous and nonnegative on $[0,1]$. Applying Theorem 1.39(a) to the second integral in (5.9), we obtain

$$g(x) = 0 \quad \text{a.e. on } [0,1].$$

Now the continuity of g forces that $g(x) = 0$ on $[0,1]$, i.e., $|f(x)| = 1$ on $[0,1]$. In particular, we have

$$-1 \leq \mathrm{Re}\,\big(f(x)\big) \leq 1 \tag{5.10}$$

on $[0,1]$.

On the other hand, since $f \in M$, the definition gives

$$1 = \mathrm{Re}\,\Big(\int_0^{\frac{1}{2}} f(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 f(t)\,\mathrm{d}t\Big) = \int_0^{\frac{1}{2}} \mathrm{Re}\,\big(f(t)\big)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 \mathrm{Re}\,\big(f(t)\big)\,\mathrm{d}t$$

or equivalently,

$$\int_0^{\frac{1}{2}} \big[\mathrm{Re}\,\big(f(t)\big) - 1\big]\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \big[-1 - \mathrm{Re}\,\big(f(t)\big)\big]\,\mathrm{d}t = 0. \tag{5.11}$$

Observe from the inequalities (5.10) that $\mathrm{Re}\,\big(f(x)\big) - 1 \leq 0$ and $-1 - \mathrm{Re}\,\big(f(x)\big) \leq 0$ on $[0,1]$. Therefore, the equation (5.11) and the continuities of $\mathrm{Re}\,\big(f(x)\big) - 1$ and $-1 - \mathrm{Re}\,\big(f(x)\big)$ imply that

$$\mathrm{Re}\,\big(f(x)\big) = \begin{cases} 1, & \text{if } [0, \frac{1}{2}]; \\ -1, & \text{if } [\frac{1}{2}, 1]. \end{cases}$$

However, this says that $\mathrm{Re}\,\big(f(x)\big)$ is discontinuous at $x = \frac{1}{2}$, a contradiction. Hence *no* such f exists and this completes the proof of the problem. ■

Problem 5.5

Rudin Chapter 5 Exercise 5.

Proof. It is clear that $1 \in M$, so $M \neq \varnothing$. Suppose that $f, g \in M \subseteq L^1\big([0,1]\big)$, $t \in [0,1]$ and $h = (1-t)f + tg$. Then we have

$$\int_0^1 |h(x)|\,\mathrm{d}x \leq (1-t)\int_0^1 |f(x)|\,\mathrm{d}x + t\int_0^1 |g(x)|\,\mathrm{d}x < \infty$$

which means that $h \in L^1\big([0,1]\big)$. Besides, we also have

$$\int_0^1 h(x)\,\mathrm{d}x = \int_0^1 [(1-t)f(x) + tg(x)]\,\mathrm{d}x = (1-t)\int_0^1 f(x)\,\mathrm{d}x + t\int_0^1 g(x)\,\mathrm{d}x = 1$$

so that $h \in M$, i.e., M is convex. Next, suppose that $\{f_n\} \subseteq M$ and there exists a function f on $[0,1]$ such that $\|f_n - f\|_1 \to 0$ as $n \to \infty$. Thus $f_n - f \in L^1\big([0,1]\big)$ and it yields from Theorem 1.33 that

$$\Big|\int_0^1 f_n(t)\,\mathrm{d}t - \int_0^1 f(t)\,\mathrm{d}t\Big| \leq \int_0^1 |f_n(t) - f(t)|\,\mathrm{d}t = \|f_n - f\|_1$$

which implies that

$$\lim_{n\to\infty}\int_0^1 f_n(t)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t.$$

In other words, we have $f \in M$ and M is closed in $L^1([0,1])$.

For every $f \in M \subseteq L^1([0,1])$, we see from Theorem 1.33 that $\|f\|_1 \geq 1$. For each $n = 1, 2, \ldots$, we define $f_n : [0,1] \to \mathbb{C}$ by

$$f_n(x) = nx^{n-1}.$$

By direct checking, it is clear that

$$\|f_n\|_1 = \int_0^1 nt^{n-1}\,\mathrm{d}t = 1.$$

In addition, $f_n \neq f_m$ if $n \neq m$. Hence M contains infinitely many elements of minimal norm, completing the proof of the problem. ■

Problem 5.6

Rudin Chapter 5 Exercise 6.

Proof. We note that M is *not* necessarily closed in H. Let $f : M \subseteq H \to \mathbb{C}$ be a bounded linear functional. If we check the proof of Theorem 5.4 carefully, we see that the bounded linear transformation $\Lambda : X \to Y$ is in fact uniformly continuous on X. Thus $f : M \to \mathbb{C}$ is *uniformly continuous* on M. Since H is a metric (in fact Hilbert) space, $\overline{M}$ is also a metric space. Since $\mathbb{C}$ is a complete metric space and M is dense in $\overline{M}$, we follow from [49, Exercises 4 & 13, pp. 98, 99] that f can be *uniquely* extended to a continuous function $\widetilde{f} : \overline{M} \to \mathbb{C}$.[b] Without loss of generality, we may assume that M is closed in H.

As a closed subspace of a Hilbert space H, M is itself a Hilbert space (see the note in Problem 4.8). By Theorem 4.12 (The Riesz Representation Theorem for Hilbert Spaces), there corresponds a *unique* $x_0 \in M$ such that

$$f(x) = \langle x, x_0\rangle \tag{5.12}$$

for all $x \in M$. Recall from Definition 5.3 that $\|f\|$ is the *smallest* number such that

$$|f(x)| \leq \|f\| \cdot \|x\|$$

for every $x \in M$. In particular, we must have $\|f(x_0)\| \leq \|f\| \cdot \|x_0\|$. On the other hand, the representation (5.12) gives

$$|f(x_0)| = f(x_0) = \langle x_0, x_0\rangle = \|x_0\|^2 = \|x_0\| \cdot \|x_0\|,$$

so we must have

$$\|x_0\| = \|f\|. \tag{5.13}$$

By Theorem 4.11, every $x \in H$ can be expressed uniquely as $x = Px + Qx$, where $Px \in M$ and $Qx \in M^\perp$. This means that $H = M \oplus M^\perp$.[c] Now if we define $F : H \to \mathbb{C}$ by

$$F(x) = \langle x, x_0\rangle$$

[b] In fact, such $\widetilde{f}$ is uniformly continuous on $\overline{M}$. See also [42, Exercise 2, p. 270].

[c] Let U and W be two vector subspaces of the vector space V. Then V is said to be the *direct sum* of U and W, denoted by $V = U \oplus W$, if $V = U + W$ and $U \cap W$. See [4, pp. 95, 96].

for every $x \in H$, then we obtain

$$F(x) = \langle x, x_0 \rangle = \begin{cases} f(x), & \text{if } x \in M; \\ 0, & \text{if } x \in M^{\perp}. \end{cases}$$

Furthermore, by the same analysis in obtaining the representation (5.13), we get $\|F\| = \|x_0\|$ and then using the representation (5.13) again, we conclude that

$$\|F\| = \|f\| = \|x_0\|, \tag{5.14}$$

i.e., F is a norm-preserving extension on H of f which vanishes on $M^{\perp}$.

Suppose that $F' : H \to \mathbb{C}$ is *another* norm-preserving extension of f that vanishes on $M^{\perp}$. Then F' is bounded and thus continuous by Theorem 5.4. By Theorem 4.12 (The Riesz Representation Theorem for Hilbert Spaces), there corresponds a unique $x_1 \in H$ such that

$$F'(x) = \langle x, x_1 \rangle$$

for all $x \in H$ and also

$$\|F'\| = \|f\| = \|x_1\|. \tag{5.15}$$

Since $F'(x) = f(x)$ on M, we have

$$\langle x, x_1 \rangle = \langle x, x_0 \rangle$$

on M. By Definition 4.1, $\langle x, x_1 - x_0 \rangle = 0$ and then $x_1 - x_0 \in M^{\perp}$. Since $x_1 = x_0 + (x_1 - x_0)$, where $x_0 \in M$ and $x_1 - x_0 \in M^{\perp}$, Theorem 4.11 shows that x_1 and $x_1 - x_0$ are unique and

$$\|x_0\|^2 = \|x_1\|^2 + \|x_0 - x_1\|^2. \tag{5.16}$$

By putting the numbers (5.14) and (5.15) into the formula (5.16), we see that

$$\|f\|^2 = \|f\|^2 + \|x_0 - x_1\|^2$$

and this reduces to $\|x_0 - x_1\|^2 = 0$. By Definition 4.1, we must have $x_0 = x_1$, i.e., $F' = F$. This proves the uniqueness part and so we have completed the proof of the problem. ■

Problem 5.7

Rudin Chapter 5 Exercise 7.

Proof. Consider $X = [-1, 1]$ with $\mu = m$, the Lebesgue measure. Then the vector space $L^1\big([-1, 1]\big)$ consists of all complex measurable function on $[-1, 1]$ such that

$$\|f\|_1 = \int_{-1}^{1} |f(x)|\,\mathrm{d}x < \infty.$$

We define

$$M = \{f \in L^1\big([-1, 1]\big) \mid f \text{ is real-valued and } f(x) = 0 \text{ on } [-1, 0]\}.$$

Now the function $\chi_{[0,1]}$ is obviously an element of M, so $M \neq \varnothing$ and also $\|\chi_{[0,1]}\|_1 = 1$. It is also clear that M is a subspace of $L^1\big([-1, 1]\big)$. Next, we define the functional $\Lambda : M \to \mathbb{C}$ by

$$\Lambda(f) = \int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{0}^{1} f(x)\,\mathrm{d}x \tag{5.17}$$

which is linear. By Definition 5.3, the definition (5.17) and the fact that $\chi_{[0,1]} \in M$, we have

$$\begin{aligned}\|\Lambda\| &= \sup\{|\Lambda(f)| \mid f \in M \text{ and } \|f\|_1 = 1\} \\ &= \sup\Big\{\int_0^1 f(x)\,\mathrm{d}x \,\Big|\, f \in M \text{ and } \int_{-1}^1 |f(x)|\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x = 1\Big\} \\ &= 1.\end{aligned}$$

Thus Λ is bounded.

Let $\delta \in (0,1)$. If we define $\Lambda_\delta : L^1\big([-1,1]\big) \to \mathbb{C}$ by

$$\Lambda_\delta(f) = \int_{-1}^1 \mathrm{Re}\,[f(x)]\chi_{(-\delta,1)}(x)\,\mathrm{d}x = \int_{-\delta}^1 \mathrm{Re}\,[f(x)]\,\mathrm{d}x,$$

then it satisfies Definition 2.1 so that Λ_δ is linear on $L^1\big([-1,1]\big)$. It is also clear that $\Lambda_\delta|_M = \Lambda$. A direct computation also shows that

$$|\Lambda_\delta(f)| = \Big|\int_{-\delta}^1 \mathrm{Re}\,[f(x)]\,\mathrm{d}x\Big| \leq \int_{-\delta}^1 |\mathrm{Re}\,[f(x)]|\,\mathrm{d}x \leq \int_{-1}^1 |f(x)|\,\mathrm{d}x$$

which implies that

$$\|\Lambda_\delta\| = \sup\{|\Lambda_\delta(f)| \mid f \in L^1\big([-1,1]\big) \text{ and } \|f\|_1 = 1\} = 1 = \|\Lambda\|.$$

Thus Λ_δ is a norm-preserving extension on $L^1\big([-1,1]\big)$ of Λ. Hence if $\alpha \neq \beta$, then we have $\Lambda_\alpha \neq \Lambda_\beta$. This completes the proof of the problem. ■

5.3 The Dual Space of X

Problem 5.8

Rudin Chapter 5 Exercise 8.

Proof.

(a) We first show that X^* is a normed linear space by checking Definition 5.2. For all $f, g \in X^*$, since $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all $x \in X$, it must be true that

$$\|f + g\| \leq \|f\| + \|g\|.$$

Next, for a scalar α and $f \in X^*$, we have $|\alpha f(x)| = |\alpha| \cdot |f(x)|$ for all $x \in X$ which implies that

$$\|\alpha f\| = |\alpha| \cdot \|f\|.$$

Finally, suppose that $f \in X^*$ is such that $\|f\| = 0$. By [51, Eqn. (3), p. 96], we have

$$|f(x)| \leq \|f\| \cdot \|x\| = 0$$

holds for every $x \in X$. Thus $f \equiv 0$ on X. By Definition 5.2, X^* is a normed linear space.

It remains to show that X^* is complete. To this end, let $\{\Lambda_n\} \subseteq X^*$ be Cauchy. Then given $\epsilon > 0$, there corresponds a positive integer N such that $n, m \geq N$ imply that

$$\|\Lambda_n - \Lambda_m\| < \epsilon$$

and for *arbitrary but fixed* $x \in X$, this and [51, Eqn. (3), p. 96] together imply that

$$|\Lambda_n(x) - \Lambda_m(x)| = |(\Lambda_n - \Lambda_m)(x)| \leq \|\Lambda_n - \Lambda_m\| \cdot \|x\| < \epsilon\|x\|. \tag{5.18}$$

Thus $\{\Lambda_n(x)\} \subseteq \mathbb{C}$ is Cauchy. Since $\mathbb{C}$ is complete, $\{\Lambda_n(x)\}$ converges to a point in $\mathbb{C}$ and it is reasonable to define the pointwise limit $\Lambda : X \to \mathbb{C}$ by

$$\Lambda(x) = \lim_{n\to\infty} \Lambda_n(x).$$

Now our target is to show that $\Lambda \in X^*$ and $\{\Lambda_n\}$ converges to Λ in X^*, i.e., $\|\Lambda_n - \Lambda\| \to 0$ as $n \to \infty$. Let $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$. Since X is a normed linear space, $\alpha x + \beta y \in X$ and thus

$$\begin{aligned}\Lambda(\alpha x + \beta y) &= \lim_{n\to\infty} \Lambda_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} [\alpha\Lambda_n(x) + \beta\Lambda_n(y)] \\ &= \alpha \lim_{n\to\infty} \Lambda_n(x) + \beta \lim_{n\to\infty} \Lambda_n(y) \\ &= \alpha\Lambda(x) + \beta\Lambda(y).\end{aligned}$$

By Definition 2.1, Λ is linear. By the hypothesis, $\{\Lambda_n\}$ is a Cauchy sequence in X^* so that it is bounded, i.e., there exists a positive constant M such that

$$\|\Lambda_n\| \leq M \tag{5.19}$$

for every $n = 1, 2, \ldots$. Therefore, it follows from the triangle inequality, then [51, Eqn. (3), p. 96] and finally the inequalities (5.19) that

$$\begin{aligned}|\Lambda(x)| &\leq |\Lambda_n(x)| + |\Lambda(x) - \Lambda_n(x)| \\ &\leq \|\Lambda_n\| \cdot \|x\| + |\Lambda(x) - \Lambda_n(x)| \\ &\leq M\|x\| + |\Lambda(x) - \Lambda_n(x)| \end{aligned} \tag{5.20}$$

for all $n > N$ and for all $x \in X$. Recall that if we take $m \to \infty$ in the inequality (5.18), then we have

$$|\Lambda_n(x) - \Lambda(x)| \leq \epsilon\|x\| \tag{5.21}$$

for all $n > N$. Thus, by putting the inequality (5.19) into the inequality (5.20), we get

$$|\Lambda(x)| \leq (M + \epsilon)\|x\|$$

for every $x \in X$. Hence we obtain $\|\Lambda\| < \infty$, i.e., $\Lambda \in X^*$. Since the inequality (5.21) is true for all $x \in X$, it must be true for those x with $\|x\| \leq 1$ and so $n > N$ implies that

$$\|\Lambda_n - \Lambda\| = \sup\{|\Lambda_n(x) - \Lambda(x)| \,|\, x \in X \text{ and } \|x\| \leq 1\} \leq \epsilon.$$

Since ϵ is arbitrary, we have $\{\Lambda_n\}$ converges to Λ in X^*. In other words, X^* is complete and we conclude that X^* is a Banach space.

(b) Fix the $x \in X$, define $\Lambda_x^* : X^* \to \mathbb{C}$ by

$$\Lambda_x^*(f) = f(x).$$

If $x = 0$, then it is clear that $\Lambda_0^*(f) = f(0) = 0$ on X^*. Therefore, Λ_0^* must be linear and of norm 0. Without loss of generality, we may assume that $x \neq 0$ in the following discussion. For any $f, g \in X^*$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\Lambda_x^*(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha\Lambda_x^*(f) + \beta\Lambda_x^*(g).$$

By Definition 2.1, Λ_x^* is a linear functional on X^*. On the one hand, by [51, Eqn. (3), p. 96], we know that

$$|\Lambda_x^*(f)| = |f(x)| \leq \|f\| \cdot \|x\| \leq \|x\| \tag{5.22}$$

for all $f \in X^*$ with $\|f\| = 1$. Thus it means that $\|\Lambda_x^*\| \leq \|x\|$. On the other hand, since X is a normed linear space, and $x \neq 0$, Theorem 5.20 ensures the existence of a $g \in X^*$ such that $\|g\| = 1$ and $g(x) = \|x\|$, so we have

$$\|\Lambda_x^*\| \geq |\Lambda_x^*(g)| = |g(x)| \geq \|x\|, \tag{5.23}$$

where $\|g\| = 1$. Combining the inequalities (5.22) and (5.23), we get the desired result that

$$\|\Lambda_x^*\| = \|x\|.$$

(c) Suppose that $\{x_n\} \subseteq X$ and $\{f(x_n)\}$ is bounded for every $f \in X^*$. We consider the mapping $\Lambda_{x_n}^* : X^* \to \mathbb{C}$ given by

$$\Lambda_{x_n}^*(f) = f(x_n).$$

By part (a), X^* is a Banach space. By part (b), $\|\Lambda_{x_n}^*\| = \|x_n\|$ for every $n \in \mathbb{N}$. By the hypothesis, *for each* $f \in X^*$, the set $\{\Lambda_{x_n}^*(f)\} = \{f(x_n)\}$ is bounded by a positive constant M_n, so you can't find a $f \in X^*$ such that

$$\sup_{n\in\mathbb{N}} |\Lambda_{x_n}^*(f)| = \infty.$$

Therefore, we deduce from Theorem 5.8 (The Banach-Steinhaus Theorem) that

$$\|\Lambda_{x_n}^*\| \leq M$$

for all $n \in \mathbb{N}$, where M is a positive constant. Hence the sequence $\{\|x_n\|\}$ is bounded.

This completes the proof of the problem. ■

Problem 5.9

Rudin Chapter 5 Exercise 9.

Proof.

(a) We are going to prove the assertions one by one.

- $\|\Lambda\| = \|y\|_1$. Fix $y = \{\eta_i\} \in \ell^1$. Define $\Lambda : c_0 \to \mathbb{C}$ by

$$\Lambda(x) = \sum_{i=1}^{\infty} \xi_i \eta_i, \tag{5.24}$$

for every $x = \{\xi_i\} \in c_0 \subseteq \ell^\infty$. Since $\xi_i \to 0$ as $i \to \infty$, the sequence $\{\xi_i\}$ is bounded by a positive constant M (see [49, Theorem 3.2(c), p. 48]). Since $y \in \ell^1$, we have

$$\Big|\sum_{i=1}^{\infty} \xi_i \eta_i\Big| \leq \sum_{i=1}^{\infty} |\xi_i \eta_i| \leq M \sum_{i=1}^{\infty} |\eta_i| < \infty$$

so that the series (5.24) converges (absolutely) and thus the map Λ is well-defined.

Let $z = \{\omega_i\} \in c_0$ and $\alpha, \beta \in \mathbb{C}$. Since c_0 is a vector space, we have $\alpha x + \beta z \in c_0$. By the definition, we have

$$\Lambda(x+z) = \sum_{i=1}^{\infty} (\alpha\xi_i + \beta\omega_i)\eta_i \tag{5.25}$$

By the analysis in the previous paragraph, the series (5.25) is allowed to split into two series so that

$$\Lambda(x+z) = \alpha \sum_{i=1}^{\infty} \xi_i\eta_i + \beta \sum_{i=1}^{\infty} \omega_i\eta_i = \alpha\Lambda(x) + \beta\Lambda(z),$$

i.e., Λ is linear. By the definition (5.24), it is easy to see that

$$|\Lambda(x)| \leq \|x\|_\infty \cdot \|y\|_1$$

for every $x \in c_0$. Thus it is true that $\|\Lambda\| \leq \|y\|_1$.

For the reverse direction, since $y \in \ell^1$, we know that $|\eta_i| \to 0$ as $i \to \infty$. Next, for each $n \in \mathbb{N}$, we consider the sequence $x_n = \{\xi_{n,i}\}$ defined by

$$\xi_{n,i} = \begin{cases} \dfrac{\overline{\eta_i}}{|\eta_i|}, & \text{if } i = 1, 2, \ldots, n \text{ and } \eta_i \neq 0; \\[2ex] 0, & \text{if } i > n \text{ or } \eta_i = 0. \end{cases} \tag{5.26}$$

Since $|\frac{\overline{\eta_i}}{|\eta_i|}| \leq 1$ for all $i \in \mathbb{N}$, we have $x_n \in \ell^\infty$ for each $n \in \mathbb{N}$. In fact, we have $\|x_n\|_\infty \leq 1$. Furthermore, *for every fixed* $n \in \mathbb{N}$, one can find $i \in \mathbb{N}$ such that $i > n$, thus we deduce easily from the definition (5.26) that

$$\xi_{n,i} \to 0$$

as $i \to \infty$, i.e., $x_n \in c_0$. Now we derive from Definition 5.2 that

$$\|\Lambda\| = \sup\{|\Lambda(x)| \mid x \in c_0 \text{ and } \|x\|_\infty \leq 1\} \geq |\Lambda(x_n)| = \sum_{i=1}^{n} \xi_{n,i}\eta_i = \sum_{i=1}^{n} |\eta_i|$$

for every $n \in \mathbb{N}$, i.e., $\|\Lambda\| \geq \|y\|_1$. In conclusion, we have $\|\Lambda\| = \|y\|_1 < \infty$ and thus

$$\Lambda \in (c_0)^*.$$

– **$(c_0)^*$ is isometric isomorphic to ℓ^1.** In the first assertion, we have shown that given a $y \in \ell^1$, if Λ is defined in the form (5.24), then $\Lambda \in (c_0)^*$. More precisely, the mapping $\Phi : \ell^1 \to (c_0)^*$ given by

$$\Phi(y) = \Lambda_y$$

is well-defined, where Λ_y is in the form (5.24).

* **Step 1: Φ is linear.** For every $y = \{\eta_i\} \in \ell^1, z = \{\theta_i\} \in \ell^1$ and $\alpha, \beta \in \mathbb{C}$, we have $\alpha y + \beta z = \{\alpha\eta_i + \beta\theta_i\} \in \ell^1$. If $x = \{\xi_i\} \in c_0$, then we deduce from the definition (5.24) that

$$\Lambda_{\alpha y + \beta z}(x) = \sum_{i=1}^{\infty} \xi_i(\alpha\eta_i + \beta\theta_i) = \alpha \sum_{i=1}^{\infty} \xi_i\eta_i + \beta \sum_{i=1}^{\infty} \xi_i\theta_i = \alpha\Lambda_y(x) + \beta\Lambda_z(x).$$

Consequently, $\Phi(\alpha y + \beta z) = \alpha\Phi(y) + \beta\Phi(z)$.

* **Step 2: Φ is surjective.** Given $\Lambda \in (c_0)^*$. We want to show that *there exists* a $y \in \ell^1$ such that $\Phi(y) = \Lambda$.

  It is well-known that the sequence $\{e_i\}_{i=1}^{\infty}$ of standard unit vectors of c_0 is a basis for c_0 and that $\{\xi_i\} = \sum \xi_i e_i$ whenever $\{\xi_i\}_{i=1}^{\infty} \in c_0$.[d] In fact, we have $e_i = \{\delta_{i,k}\}_{k=1}^{\infty}$, where $\delta_{i,k}$ is the Kronecker delta function. Then it is routine to check that $e_i \in c_0$ and $\|e_i\|_\infty = 1$.

  Next, we define the sequence $y = \{\eta_i\}_{i=1}^{\infty}$ by

$$\eta_i = \Lambda(e_i) \tag{5.27}$$

  for every $i = 1, 2, \ldots$. Our aim is to show that $y \in \ell^1$. If the sequence $x_n = \{\xi_{n,i}\}_{i=1}^{n}$ is given by the form (5.26), then we immediately know that $x_n \in c_0$ and $\|x_n\|_\infty = 1$. Since

$$x_n = \sum_{i=1}^{\infty} \xi_{n,i} e_i = \sum_{i=1}^{n} \xi_{n,i} e_i,$$

  the linearity of Λ shows that

$$\Lambda(x_n) = \sum_{i=1}^{n} \xi_{n,i} \Lambda(e_i) = \sum_{i=1}^{n} \xi_{n,i} \eta_i = \sum_{i=1}^{n} |\eta_i|. \tag{5.28}$$

  Therefore, we apply [51, Eqn. (3), p. 96] to the expression (5.28) to obtain

$$\sum_{i=1}^{n} |\eta_i| = |\Lambda(x_n)| \leq \|\Lambda\| \cdot \|x_n\|_\infty = \|\Lambda\|$$

  holds *for every* $n = 1, 2, \ldots$. In particular, we have

$$\|y\|_1 = \sum_{i=1}^{\infty} |\eta_i| \leq \|\Lambda\| < \infty,$$

  i.e, $y \in \ell^1$.

  Finally, for every $x = \{\xi_i\} \in c_0$, we have

$$x = \sum_{i=1}^{\infty} \xi_i e_i$$

  and the application of the expressions (5.27) and the linearity of Λ indicate that

$$\Lambda(x) = \Lambda\Big(\sum_{i=1}^{\infty} \xi_i e_i\Big) = \sum_{i=1}^{\infty} \xi_i \Lambda(e_i) = \sum_{i=1}^{\infty} \xi_i \eta_i$$

  which is exactly in the form (5.24). Hence the map Φ is surjective.

* **Step 3: Φ is injective.** Let $y = \{\eta_i\} \in \ell^1, z = \{\theta_i\} \in \ell^1$. Suppose further that $\Phi(y) = \Phi(z)$. Since Φ is linear, we have $\Lambda_{y-z} = \Phi(y - z) = 0$. By this, for every $j = 1, 2, \ldots$, we get

$$\Lambda_{y-z}(e_j) = \sum_{i=1}^{\infty} \delta_{j,i}(\eta_i - \theta_i) = \eta_j - \theta_j = 0$$

  so that $\eta_j = \theta_j$. In other words, we conclude that $y = z$, as desired.

[d] The sequence $\{e_i\}$ is a *Schauder basis* for c_0 (and also for ℓ^1). In fact, a sequence $\{x_i\}_{i=1}^{\infty}$ in a Banach space X is called a Schauder basis if for each $x \in X$, there is a unique sequence $\{\alpha_i\}$ of scalars such that $x = \sum_{i=1}^{\infty} \alpha_i x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i x_i$. See, for instance, [40, Example 4.1.3, p. 351].

* **Step 4: Φ is an isometry.** This follows from the first assertion directly because $\|\Phi(y)\| = \|\Lambda_y\| = \|y\|_1$.

Hence we have $(c_0)^* = \ell^1$.

(b) For each $y = \{\eta_i\} \in \ell^\infty$, we define $\Theta : \ell^1 \to \mathbb{C}$ by

$$\Theta(x) = \sum_{i=1}^{\infty} \xi_i \eta_i \tag{5.29}$$

for every $x = \{\xi_i\} \in \ell^1$. Similar to the proof of the first assertion in part (a), we can show that the series (5.29) converges (absolutely) and Θ is linear. It is clear from the definition that $|\Theta(x)| \leq \|y\|_\infty \cdot \|x\|_1$, so we have

$$\|\Theta\| \leq \|y\|_\infty < \infty. \tag{5.30}$$

This means that $\Theta \in (\ell^1)^*$.

On the other hand, suppose that $\Theta \in (\ell_1)^*$, i.e., $\|\Theta\| < \infty$. If we consider the Schader basis $\{e_i\}$ for ℓ^1 and the relationships (5.27) with Λ replaced by Θ, then we have

$$|\eta_i| = |\Theta(e_i)| \leq \|\Theta\| \cdot \|e_i\|_1 = \|\Theta\| < \infty \tag{5.31}$$

for every $i = 1, 2, \ldots$. Put $z = \{\eta_i\}$. Then the inequality (5.31) immediately implies that $z \in \ell^\infty$. In fact, it shows further that $\|z\|_\infty \leq \|\Theta\|$. Now for every $x = \{\xi_i\} \in \ell^1$, we have

$$x = \sum_{i=1}^{\infty} \xi_i e_i$$

and then

$$\Theta(x) = \sum_{i=1}^{\infty} \xi_i \eta_i$$

which is exactly in the form (5.29), so the inequality (5.30) holds for z. Therefore, we have $\|\Theta\| = \|z\|_\infty$, i.e., $\Theta \in \ell^\infty$.

If we define the mapping $\Psi : \ell^\infty \to (\ell^1)^*$ by

$$\Psi(y) = \Theta_y,$$

where Θ_y is given by the form (5.29), then the previous paragraph actually shows that Ψ is surjective. By using similar argument as in the proofs of **Step 1**, **Step 3** and **Step 4** in part (a), it is easy to conclude that Ψ is an isometric vector space isomorphism.

(c) Let $y = \{\eta_i\} \in \ell^1$. Define the functional $\Lambda : \ell^\infty \to \mathbb{C}$ by

$$\Lambda(x) = \sum_{i=1}^{\infty} \xi_i \eta_i,$$

where $x = \{\xi_i\} \in \ell^\infty$. Since this series converges (absolutely), it is a linear functional. Since $|\Lambda(x)| \leq \|x\|_\infty \cdot \|y\|_1$, we know from [51, Eqn. (2), p. 96] that $\|\Lambda\| \leq \|y\|_1$, i.e., $\Lambda \in (\ell^\infty)^*$.

Assume that ℓ^1 was isometric isomorphic to $(\ell^\infty)^*$. It is clear that $x = \{1, 1, \ldots\} \in \ell^\infty \setminus c_0$. By [40, Example 1.2.13, p. 14], we know that c_0 is a *proper* closed subspace of

ℓ^∞ so that $\ell^\infty \setminus c_0 \neq \varnothing$. Take $x_0 \in \ell^\infty \setminus c_0$. Then Theorem 5.19 implies that there is a $f \in (\ell^\infty)^*$ such that $\|f\| = 1$, $f(x) = 0$ on c_0 and

$$f(x_0) \neq 0. \tag{5.32}$$

If $f \in \ell^1$, then we have $f \in (c_0)^*$ by part (a). Since $f(x) = 0$ on c_0, f is in fact the trivial bounded functional on c_0. By the assumption and part (a), $(c_0)^*$ is isometric isomorphic to $(\ell^\infty)^*$ so that the trivial bounded functional on c_0 corresponds to the trivial bounded functional on ℓ^∞. In other words, $f(x) = 0$ on ℓ^∞ which contradicts the result (5.32). Hence we conclude that ℓ^1 is *not* isometric isomorphic to $(\ell^\infty)^*$.

(d) We prove the assertions one by one.

- c_0 **is separable.** For each $k \in \mathbb{N}$, we define

$$\mathcal{S}_k = \{\{\xi_1, \ldots, \xi_k, 0, 0, \ldots\} \,|\, \xi_1, \ldots, \xi_k \in \mathbb{Q}\} \quad \text{and} \quad \mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k.$$

Since each $\mathcal{S}_k$ is countable, $\mathcal{S}$ is also countable. We claim that $\mathcal{S}$ is dense in c_0. Pick $y = \{\eta_i\} \in c_0$. Given $\epsilon > 0$. Since $\eta_i \to 0$ as $i \to \infty$, there exists a positive integer N such that $i \geq N$ implies

$$|\eta_i| < \epsilon. \tag{5.33}$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can take $x = \{\xi_1, \ldots, \xi_N, 0, 0, \ldots\} \in \mathcal{S}_N$ such that

$$|\xi_i - \eta_i| < \epsilon \tag{5.34}$$

for all $i = 1, 2, \ldots, N$. Thus we obtain from the definition of $\|\cdot\|_\infty$, the inequalities (5.33) and (5.34) that

$$\|x - y\|_\infty = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i| \leq \epsilon.$$

Hence this proves the claim and then c_0 is separable.

- ℓ^1 **is separable.** Take $y = \{\eta_i\} \in \ell^1$. Given that $\epsilon > 0$. Since $\sum_{i=1}^{\infty} |\eta_i| < \infty$, the Cauchy Criterion (see [49, Theorem 3.22, p. 59]) shows that one can find a positive integer N' such that $i \geq N'$ implies

$$\sum_{i=N'}^{\infty} |\eta_i| < \frac{\epsilon}{2}. \tag{5.35}$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$ again, we can take $x = \{\xi_1, \ldots, \xi_{N'}, 0, 0, \ldots\} \in \mathcal{S}_{N'}$ such that

$$\sum_{i=1}^{N'} |\xi_i - \eta_i| < \frac{\epsilon}{2}. \tag{5.36}$$

Combining the inequalities (5.35) and (5.36), we get

$$\|x - y\|_1 = \sum_{i=1}^{\infty} |\xi_i - \eta_i| \leq \sum_{i=1}^{N'} |\xi_i - \eta_i| + \sum_{i=N'}^{\infty} |0 - \eta_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\mathcal{S}$ is also dense in ℓ^1 and thus ℓ^1 is separable.

– ℓ^∞ **is not separable.** Let $\mathcal{A}$ be the set of all sequences $x = \{\xi_i\}$ such that $\xi_i \in \{0,1\}$. Clearly, $\mathcal{A}$ is uncountable (see [49, Theorem 2.14]) and $\mathcal{A} \subseteq \ell^\infty$. Let $x, y \in \mathcal{A}$ be such that $x = \{\xi_i\}$ and $y = \{\eta_i\}$. If $x \neq y$, then there is *at least one* i such that $\xi_i \neq \eta_i$ which implies that
$$\|x - y\|_\infty = 1.$$
By this and the fact that $B(x, \frac{1}{2}) \cap B(y, \frac{1}{2}) = \varnothing$ if $x \neq y$, we conclude that the collection of open balls $\{B(x, \frac{1}{2}) \,|\, x \in \mathcal{A}\}$ is uncountable. If S was a dense subset of ℓ^∞, then we have
$$S \cap B\Big(x, \frac{1}{2}\Big) \neq \varnothing$$
for every $x \in \mathcal{A}$ so that S must be uncountable. Hence ℓ^∞ is *not* separable.

■

Remark 5.1

We say two normed linear spaces X and Y *isometric isomorphism* if there is a bijective linear map $T : X \to Y$ such that
$$\|T(x)\| = \|x\|$$
for every $x \in X$.

5.4 Applications of Baire's and other Theorems

Problem 5.10

Rudin Chapter 5 Exercise 10.

Proof. For each positive integer n, we define $\Lambda_n : c_0 \to \mathbb{C}$ by
$$\Lambda_n(x) = \sum_{i=1}^{n} \alpha_i \xi_i, \tag{5.37}$$
where $x = \{\xi_i\} \in c_0$. By the hypotheses of Problem 5.9, we know that c_0 is a Banach space. On the one hand, we note from the definition (5.37) that Λ_n is linear and the analysis in Problem 5.9(a) first assertion shows that
$$|\Lambda_n(x)| \leq \|x\|_\infty \cdot \sum_{i=1}^{n} |\alpha_i|$$
for every $x \in c_0 \subseteq \ell^\infty$ and therefore,
$$\|\Lambda_n\| \leq \sum_{i=1}^{n} |\alpha_i|. \tag{5.38}$$
On the other hand, if we define the sequence $x_n = \{\xi_{n,i}\}$ by (5.26) with η_i replaced by α_i, then, using the argument as in the proof of Problem 5.9(a), we conclude that $x_n \in c_0$. Thus we obtain from Definition 5.2 that
$$\|\Lambda_n\| = \sup\{|\Lambda_n(x)| \,|\, x \in c_0 \text{ and } \|x\|_\infty \leq 1\} \geq |\Lambda_n(x_n)| = \sum_{i=1}^{n} |\alpha_i| \tag{5.39}$$

for every $n \in \mathbb{N}$. Combining the inequalities (5.38) and (5.39), we conclude that

$$\|\Lambda_n\| = \sum_{i=1}^{n} |\alpha_i| \tag{5.40}$$

which is bounded for each $n \in \mathbb{N}$. By the hypothesis and the definition (5.37), there is *no* $x \in c_0$ such that $\sup\limits_{n\in\mathbb{N}} |\Lambda_n(x)| = \infty$. Thus Theorem 5.8 (The Banach-Steinhaus Theorem) implies that there exists a positive constant M such that

$$\|\Lambda_n\| \leq M \tag{5.41}$$

for every $n \in \mathbb{N}$. Consequently, we deduce from the two results (5.40) and (5.41) that

$$\sum_{i=1}^{\infty} |\alpha_i| < \infty,$$

as required. We have completed the proof of the problem. ■

Problem 5.11

Rudin Chapter 5 Exercise 11.

Proof. The proof will be divided into several steps:

- **Step 1: $\mathrm{Lip}\,\alpha$ is a vector space.** For every $f, g \in \mathrm{Lip}\,\alpha$ and $\mu, \nu \in \mathbb{C}$, we have

$$\begin{aligned} M_{\mu f+\nu g} &= \sup \frac{|\mu f(s) + \nu g(s) - \mu f(t) - \nu g(t)|}{|s-t|^{\alpha}} \\ &\leq |\mu| \cdot \sup \frac{|f(s) - f(t)|}{|s-t|^{\alpha}} + |\nu| \cdot \sup \frac{|g(s) - g(t)|}{|s-t|^{\alpha}} \\ &= |\mu| \cdot M_f + |\nu| \cdot M_g \qquad (5.42) \\ &< \infty \end{aligned}$$

  which implies that $\mu f + \nu g \in \mathrm{Lip}\,\alpha$, i.e., $\mathrm{Lip}\,\alpha$ is a vector space.

- **Step 2: $\mathrm{Lip}\,\alpha$ is a normed linear space with the mentioned norms.** If we define $\|f\|_1 = |f(a)| + M_f$, then for $f, g \in \mathrm{Lip}\,\alpha$, it is easy to see from the inequality (5.42) that

$$\|f+g\|_1 = |f(a) + g(a)| + M_{f+g} \leq |f(a)| + |g(a)| + M_f + M_g = \|f\|_1 + \|g\|_1.$$

  Next, if $\mu \in \mathbb{C}$, then we have

$$\|\mu f\|_1 = |\mu f(a)| + M_{\mu f} = |\mu| \cdot |f(a)| + |\mu| \cdot M_f = |\mu| \cdot \|f\|_1.$$

  Finally, if $\|f\|_1 = 0$, then $|f(a)| = M_f = 0$ so that $|f(s) - f(a)| = 0$ for all $s \in (a, b]$. Thus we establish that $f(x) = 0$ on $[a, b]$ and we can say that $\mathrm{Lip}\,\alpha$ is a normed linear space with respect to the first norm.

  For the second norm, if $f, g \in \mathrm{Lip}\,\alpha$, then we see that

$$\|f+g\|_2 = M_{f+g} + \sup_{x\in[a,b]} |f(x)+g(x)| \leq M_f + M_g + \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| = \|f\|_2 + \|g\|_2.$$

Next, if $\mu \in \mathbb{C}$, then we have

$$\|\mu f\|_2 = M_{\mu f} + \sup_{x\in[a,b]} |\mu f(x)| = |\mu| \cdot M_f + |\mu| \cdot \sup_{x\in[a,b]} |f(x)| = |\mu| \cdot \|f\|_2.$$

Finally, if $\|f\|_2 = 0$, then we have $M_f = \sup_{x\in[a,b]} |f(x)| = 0$ implying that $f(x) = 0$ on $[a,b]$. Hence $\mathrm{Lip}\,\alpha$ is also a normed linear space with respect to the second norm.

- **Step 3:** $\mathrm{Lip}\,\alpha$ **is complete.** Suppose that $\{f_n\} \subseteq \mathrm{Lip}\,\alpha$ is Cauchy with respect to the norm $\|\cdot\|_2$. Given $\epsilon > 0$. There exists a positive integer N such that $n, m \geq N$ imply that

$$\sup_{x\in[a,b]} |f_n(x) - f_m(x)| + M_{f_n - f_m} = \|f_n - f_m\|_2 < \frac{\epsilon}{2}. \tag{5.43}$$

Particularly, we have

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \tag{5.44}$$

for all $x \in [a,b]$. Thus $\{f_n(x)\}$ is a Cacuhy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, there exists a function $f : [a,b] \to \mathbb{C}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{5.45}$$

for each $x \in [a,b]$.

We claim that $\|f_n - f\|_2 \to 0$ as $n \to \infty$ and $f \in \mathrm{Lip}\,\alpha$. On the one hand, we take $m \to \infty$ in the inequality (5.44) to get

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2} \tag{5.46}$$

for all $x \in [a,b]$. Thus we have

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \frac{\epsilon}{2}. \tag{5.47}$$

On the other hand, we know from the inequality (5.43) that $M_{f_n - f_m} < \frac{\epsilon}{2}$, or explicitly

$$\frac{|f_n(s) - f_m(s) - f_n(t) + f_m(t)|}{|s-t|^\alpha} < \frac{\epsilon}{2} \tag{5.48}$$

for all $s, t \in [a,b]$ with $s \neq t$. Taking $m \to \infty$ in the inequality (5.48), we observe that

$$\frac{|f_n(s) - f(s) - f_n(t) + f(t)|}{|s-t|^\alpha} \leq \frac{\epsilon}{2}$$

for all $s, t \in [a,b]$ with $s \neq t$. In other words, we have

$$M_{f_n - f} \leq \frac{\epsilon}{2}. \tag{5.49}$$

Hence we combine the inequalities (5.47) and (5.49) to establish that

$$\|f_n - f\|_2 = \sup_{x\in[a,b]} |f_n(x) - f(x)| + M_{f_n - f} \leq \epsilon,$$

i.e., $\|f_n - f\|_2 \to 0$ as $n \to \infty$. Furthermore, for $s, t \in [a,b]$ with $s \neq t$, since we have

$$\frac{|f(s) - f(t)|}{|s-t|^\alpha} = \frac{|f(s) - f_N(s) + f_N(t) - f(t) + f_N(s) - f_N(t)|}{|s-t|^\alpha}$$

$$\leq \frac{|f(s) - f_N(s) + f_N(t) - f(t)|}{|s-t|^\alpha} + \frac{|f_N(s) - f_N(t)|}{|s-t|^\alpha}$$
$$\leq M_{f_N - f} + M_{f_N},$$

where N is the positive integer which makes the inequality (5.43) holds. Therefore, we deduce from the definition and the inequality (5.49) that

$$M_f \leq M_{f_N - f} + M_{f_N} \leq \frac{\epsilon}{2} + M_{f_N} < \infty,$$

i.e., $f \in \operatorname{Lip}\alpha$. In other words, $\operatorname{Lip}\alpha$ is complete with respect to the norm $\|\cdot\|_2$.

For the first norm $\|\cdot\|_1$, we note that

$$\|f\|_1 = |f(a)| + M_f \leq \sup_{x\in[a,b]} |f(x)| + M_f = \|f\|_2. \tag{5.50}$$

If $\{f_n\} \subseteq \operatorname{Lip}\alpha$ is Cauchy with respect to the norm $\|\cdot\|_1$, then we deduce from the inequality (5.50) that

$$\|f_n - f\|_1 \leq \|f_n - f\|_2 < \epsilon$$

for large enough n, where the f is defined by the limit (5.45). Since $f \in \operatorname{Lip}\alpha$, we have shown that $\operatorname{Lip}\alpha$ is complete with respect to the norm $\|\cdot\|_1$.

Hence $\operatorname{Lip}\alpha$ is also a Banach space with respect to the two norms and this completes the proof of the problem. ∎

Problem 5.12

Rudin Chapter 5 Exercise 12.

Proof. We have $A = \{f : K \to \mathbb{R} \,|\, f(x,y) = \alpha x + \beta y + \gamma, \text{ where } \alpha, \beta, \gamma \in \mathbb{R}\}$.

- **Case (i): K is a triangle.** Suppose that $P = (x_1, y_1), Q = (x_2, y_2), R = (x_3, y_3)$ and $H = \{P, Q, R\}$. Then it is well-known that

$$\begin{aligned} K &= \operatorname{Conv}(P, Q, R) \\ &= \{\lambda P + \tau Q + \nu R \,|\, \lambda, \tau, \nu \geq 0 \text{ and } \lambda + \tau + \nu = 1\} \\ &= \{(\lambda x_1 + \tau x_2 + \nu x_3, \lambda y_1 + \tau y_2 + \nu y_3) \,|\, \lambda, \tau, \nu \geq 0 \text{ and } \lambda + \tau + \nu = 1\}. \end{aligned}$$

Therefore, for each $(x_0, y_0) \in K$, we have the representation

$$(x_0, y_0) = \lambda P + \tau Q + \nu R = (\lambda x_1 + \tau x_2 + \nu x_3, \lambda y_1 + \tau y_2 + \nu y_3) \tag{5.51}$$

and thus

$$\begin{aligned} f(x_0, y_0) &= f(\lambda x_1 + \tau x_2 + \nu x_3, \lambda y_1 + \tau y_2 + \nu y_3) \\ &= \alpha(\lambda x_1 + \tau x_2 + \nu x_3) + \beta(\lambda y_1 + \tau y_2 + \nu y_3) + (\lambda + \tau + \nu)\gamma \\ &= \lambda(\alpha x_1 + \beta y_1 + \gamma) + \tau(\alpha x_2 + \beta y_2 + \gamma) + \nu(\alpha x_3 + \beta y_3 + \gamma) \\ &= \lambda f(x_1, y_1) + \tau f(x_2, y_2) + \nu f(x_3, y_3). \end{aligned} \tag{5.52}$$

If we define $\mu(P) = \mu(x_1, y_1) = \lambda$, $\mu(Q) = \mu(x_2, y_2) = \tau$ and $\mu(R) = \mu(x_3, y_3) = \nu$, then the expression (5.52) becomes

$$f(x_0, y_0) = \int_H f \, \mathrm{d}\mu. \tag{5.53}$$

To prove the uniqueness of μ, we suppose that the integral representation (5.53) holds for the measure μ'. Then the integral (5.53) implies that

$$\begin{aligned} f(x_0, y_0) &= \mu'(P) f(x_1, y_1) + \mu'(Q) f(x_2, y_2) + \mu'(R) f(x_3, y_3) \\ &= f\big(\mu'(P)x_1 + \mu'(Q)x_2 + \mu'(R)x_3, \mu'(P)y_1 + \mu'(Q)y_2 + \mu'(R)y_3\big). \end{aligned} \tag{5.54}$$

By the definition of f, the expression (5.54) means that

$$(x_0, y_0) = \mu'(P) \cdot P + \mu'(Q) \cdot Q + \mu'(R) \cdot R. \tag{5.55}$$

Comparing the expressions (5.51) and (5.55), we derive that

$$[\mu'(P) - \lambda]P + [\mu'(Q) - \tau]Q + [\mu'(R) - \nu]R = 0. \tag{5.56}$$

Since the set $\{P, Q, R\}$ is linearly independent, it follows from the equation (5.56) that

$$\mu'(P) = \lambda, \quad \mu'(Q) = \tau \quad \text{and} \quad \mu'(R) = \nu.$$

In other words, we establish the fact that $\mu' = \mu$.

- **Case (ii): K is a square.** Without loss of generality, we may assume that the vertices of K are $P(0,0), Q(1,0), R(1,1)$ and $S(0,1)$. Let $A(a,b)$ be a point of K, see Figure 5.2 below.

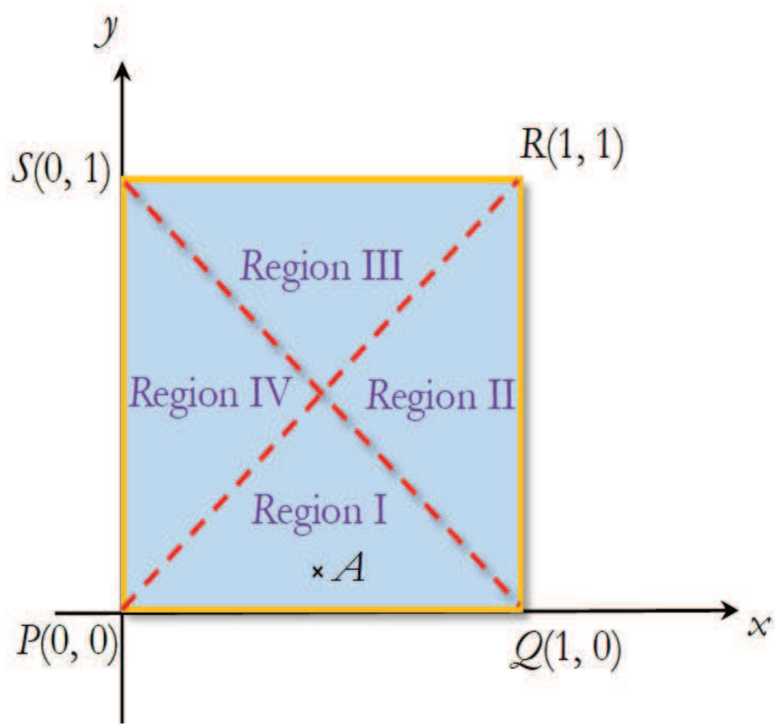


Figure 5.2: The square K.

Suppose that $(a,b) = \lambda P + \tau Q + \nu R + \theta S$, where $\lambda, \tau, \nu, \theta \in \mathbb{R}$. Then we have

$$(a,b) = \tau(1,0) + \nu(1,1) + \theta(0,1) = (\tau + \nu, \nu + \theta).$$

Thus we have $\nu + \theta = b$ and $\tau + \nu = a$. Solving the equations, we have

$$\nu = b - \theta \quad \text{and} \quad \tau = a - b + \theta. \tag{5.57}$$

Now we have five subcases below:

- **Subcase (i):** A **belongs to Region I.** Here we have $0 < b \le a < 1$ and $a+b-1 \le 0$. If $\theta = 0$, then we obtain from the equations (5.57) that $\nu = b$ and $\tau = a - b \ge 0$. If $\lambda = 1 - a \ge 0$, then it is easy to see that

$$(a, b) = (1-a)P + (a-b)Q + bR + 0 \cdot S.$$

Similarly, $\theta = b$, $\nu = 0$, $\tau = a$ and $\lambda = 1 - a - b \ge 0$ imply that

$$(a, b) = (1-a-b)P + aQ + 0 \cdot R + bS.$$

Hence, every point in Region I has two different convex combinations of P, Q, R and S.

- **Subcase (ii):** A **belongs to Region II.** In this case, we have $0 < b \le a < 1$ and $a + b - 1 \ge 0$. Since we have

$$\begin{aligned}(a, b) &= (1-a)P + (a-b)Q + bR + 0 \cdot S \\ &= 0 \cdot P + (1-b)Q + (a+b-1)R + (1-a)S,\end{aligned}$$

every point in Region II has two distinct convex combinations of P, Q, R and S.

- **Subcase (iii):** A **belongs to Region III.** In this case, we have $0 < a \le b < 1$ and $a + b - 1 \ge 0$. It is easy to see that

$$\begin{aligned}(a, b) &= 0 \cdot P + (1-b)Q + (a+b-1)R + (1-a)S \\ &= (1-b)P + 0 \cdot Q + aR + (b-a)S,\end{aligned}$$

so every point in Region III also has two different convex combinations of P, Q, R and S.

- **Subcase (iv):** A **belongs to Region IV.** Now we have $0 < a \le b < 1$ and $a + b - 1 \le 0$. Then we certainly have

$$\begin{aligned}(a, b) &= (1-b)P + 0 \cdot Q + aR + (b-a)S \\ &= (1-a-b)P + aQ + 0 \cdot R + bS\end{aligned}$$

which means that it has two different convex combinations of P, Q, R and S in Region IV.

- **Subcase (v):** A **belongs to the boundary of** K**.** Suppose that $A = (a, b)$. We note that

$$(a, 0) = (1-a) \cdot P + aQ + 0 \cdot R + 0 \cdot S = 1 \cdot P + aQ + 0 \cdot R + 0 \cdot S,$$

thus every point on PQ can be expressed as two distinct linear (but *not* convex) combinations of P, Q, R and S. Similarly, all points lie on QR, RS and SP also have two different linear combinations of P, Q, R and S.

In conclusion, every point (x_0, y_0) in the square K has *at least two* distinct linear combinations of P, Q, R and S. Now, instead of the expression (5.52), we have

$$f(x_0, y_0) = \lambda f(P) + \tau f(Q) + \nu f(R) + \theta f(S) = \lambda' f(P) + \tau' f(Q) + \nu' f(R) + \theta' f(S),$$

where $\{\lambda, \tau, \nu, \theta\}$ and $\{\lambda', \tau', \nu', \theta'\}$ are two sets of *distinct* numbers corresponding to two *distinct* linear combinations of P, Q, R and S. Thus $\{\lambda, \tau, \nu, \theta\}$ and $\{\lambda', \tau', \nu', \theta'\}$ give *distinct* measures μ and μ' such that the representation (5.53) holds.

- **Case (iii): The general situation.** Suppose that $H = \{v_0, v_1, \ldots, v_n\}$ and
$$K = \{\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n \subseteq \mathbb{R}^{n+1} \,|\, \lambda_0, \lambda_1, \ldots, \lambda_n \geq 0 \text{ and } \lambda_0 + \cdots + \lambda_n = 1\}.$$
Let $f : K \to \mathbb{R}$ be defined by
$$f(v_0, v_1, \ldots, v_n) = \alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_n v_n + \gamma,$$
where $\alpha_0, \alpha_1, \ldots, \alpha_n, \gamma$ are real. For each $(v_0, v_1, \ldots, v_n) \in K$, there corresponds a measure μ on H such that
$$f(v_0, v_1, \ldots, v_n) = \int_H f \, \mathrm{d}\mu.$$
Furthermore, the measure μ is unique if K is a n-simplex, i.e., $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$ is linearly independent, see [32, pp. 102, 103].

We have completed the proof of the problem. ■

Problem 5.13

Rudin Chapter 5 Exercise 13.

Proof.

(a) For each $N = 1, 2, \ldots,$ we define
$$E_N = \{x \in X \,|\, |f_n(x)| \leq N \text{ for all } n \in \mathbb{N}\}.$$
Since each f_n is continuous on X, each E_N is closed in X. On the one hand, we have
$$E_N \subseteq X. \tag{5.58}$$
On the other hand, for each *fixed* $x \in X$, since $f(x)$ is well-defined as a complex number, it must be true that
$$|f(x)| \leq m$$
for some $m > 0$. By the convergence of $\{f_n(x)\}$, there is a positive integer N_0 such that $n \geq N_0$ implies that
$$|f(x) - f_n(x)| < 1$$
Thus, for $n \geq N_0$, the triangle inequality shows that
$$|f_n(x)| \leq |f(x)| + |f(x) - f_n(x)| < m + 1. \tag{5.59}$$
Take $m' = \max\big(f_1(x), f_2(x), \ldots, f_{N_0 - 1}(x)\big)$. Combining this and the inequality (5.59), we obtain that
$$|f_n(x)| < \max(m', m + 1)$$
for all $n \in \mathbb{N}$. Hence we have $x \in E_{N_1}$ for some $N_1 \geq \max(m', m + 1)$ and then we follow from the set relation (5.58) that
$$X = \bigcup_{N=1}^{\infty} E_N.$$
By Theorem 5.6 (Baire's Theorem), X is *not* of the first category, i.e., X is not a countable union of nowhere dense sets. Therefore, *some* $E_{N'}$ contains a nonempty open subset V of X, i.e., for all $x \in V$ and $n = 1, 2, \ldots,$ we have
$$|f_n(x)| \leq N',$$
where $N' < \infty$.

(b) We follow the hint. For each $N = 1, 2, 3, \ldots$, we put

$$A_N = \{x \in X \,|\, |f_m(x) - f_n(x)| \leq \epsilon \text{ for all } n, m \geq N\}.$$

Since f_m and f_n are continuous on X, $f_m - f_n$ is also continuous on X. Thus for each pair $n, m \in \mathbb{N}$, the set

$$S(m, n) = (f_m - f_n)^{-1}\big([-\epsilon, \epsilon]\big) = \{x \in X \,|\, |f_m(x) - f_n(x)| \leq \epsilon\}$$

is closed in X. Since

$$A_N = \bigcap_{n,m \geq N} S(m, n),$$

A_N is definitely closed in X.

It is clear that

$$A_N \subseteq X \tag{5.60}$$

for each $N = 1, 2, \ldots$. Now for every $x \in X$, by the convergence of $\{f_n(x)\}$, it is Cauchy in $\mathbb{C}$, i.e., there exists a $N_0 \in \mathbb{N}$ such that $n, m \geq N_0$ imply that

$$|f_m(x) - f_n(x)| \leq \epsilon$$

or equivalently, $x \in A_{N_0}$. In other words, we have

$$X \subseteq \bigcup_{N=1}^{\infty} A_N. \tag{5.61}$$

Combining the set relations (5.60) and (5.61), we conclude that

$$X = \bigcup_{N=1}^{\infty} A_N.$$

By Theorem 5.6 (Baire's Theorem), X is *not* of the first category, i.e., X is not a countable union of nowhere dense sets. Therefore, *some* $A_{N'}$ contains a nonempty open subset V of X. Particularly, if $x \in V$ and $n, m \geq N'$, then we have

$$|f_m(x) - f_n(x)| \leq \epsilon. \tag{5.62}$$

Taking $m \to \infty$ in the inequality (5.62), we have established that

$$|f(x) - f_n(x)| \leq \epsilon$$

for all $x \in V$ and $n \geq N'$.

This has completed the proof of the problem. ■

Problem 5.14

Rudin Chapter 5 Exercise 14.

Proof. Let $C = \{f : [0, 1] \to \mathbb{R} \,|\, f \text{ is continuous on } [0, 1]\}$ and $\|f\| = \sup\limits_{x \in [0,1]} |f(x)|$. Furthermore, for a positive integer n, let

$$X_n = \{f \in C \,|\, \text{there exists a } t \in I \text{ such that } |f(s) - f(t)| \leq n|s - t| \text{ for all } s \in I\}.$$

To prove the first assertion, we fix $n \in \mathbb{N}$. We follow the hint given by Rudin and the construction of g will be shown in several steps below.

Let V be an open set in C and $f \in V$. Then there exists a $\epsilon > 0$ such that $B(f, 2\epsilon) \subseteq V$. Since f is continuous on the compact $[0,1]$, it is uniformly continuous on $[0,1]$, i.e., there exists a $k \in \mathbb{N}$ such that

$$|f(x) - f(y)| < \epsilon \tag{5.63}$$

for all $x, y \in [0,1]$ with $|x - y| \leq \frac{1}{k}$. In particular, we consider the partition $P = \{0 = x_0, x_1, \ldots, x_k = 1\}$ of $[0,1]$, where $x_i = \frac{i}{k}$ for $i = 0, 1, \ldots, k$. Then we know from the inequality (5.63) that

$$|f(x) - f(x_i)| < \epsilon \tag{5.64}$$

for all $x \in [x_{i-1}, x_{i+1}]$. To prove the first assertion, we are going to divide the proof into several steps:

- **Step 1: Construction of a $g \in C$ with $\|g - f\| < \epsilon$.** Suppose that

$$g_i(x) = g_i\big(\lambda_x x_i + (1-\lambda_x)x_{i+1}\big) = \lambda_x f(x_i) + (1-\lambda_x)f(x_{i+1}) \tag{5.65}$$

on $[x_i, x_{i+1}]$, where $\lambda_x \in [0,1]$. Since $x = \lambda_x x_i + (1-\lambda_x)x_{i+1}$, we have

$$\lambda_x = \frac{x - x_{i+1}}{x_i - x_{i+1}} = -k(x - x_{i+1}) = -kx + (i+1). \tag{5.66}$$

Thus λ_x is a continuous function of x. Therefore, each zigzag function g_i is continuous on $[x_i, x_{i+1}]$ and furthermore, we have

$$g_i(x_{i+1}) = f(x_{i+1}) = g_{i+1}(x_{i+1}).$$

Thus if we "glue" all the zigzag functions $g_0, g_1, \ldots, g_k$ together and let it be g, then we have $g \in C$. In addition, if $x \in [x_i, x_{i+1}]$, then we see from the definition (5.65) and the inequality (5.64) that

$$|g(x) - f(x)| \leq \lambda_x \cdot |f(x_i) - f(x)| + (1-\lambda_x) \cdot |f(x) - f(x_{i+1})| < \epsilon.$$

In other words, $\|g - f\| < \epsilon$ as required.

- **Step 2: Construction of a continuous function with large one-sided derivatives.** To complete the construction, we have to consider the function $\varphi : \mathbb{R} \to [0,1]$ defined by $\varphi(x+1) = \varphi(x)$, where

$$\varphi(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}]; \\ 2 - 2x, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, φ is a zigzag continuous function on $\mathbb{R}$.[e] Now our desired function is constructed in the following lemma:

Lemma 5.1

For each $n \in \mathbb{N}$, if we define

$$\varphi_n(x) = 2^{-n}\varphi(4^n x),$$

then φ_n is a zigzag continuous function on $\mathbb{R}$ such that $|\varphi_n'(x+)| \geq 2^n$ and $|\varphi_n'(x-)| \geq 2^n$ for all $x \in \mathbb{R}$.

[e]Such zigzag function φ is something looks like the graph in Figure 2.1.

Proof of Lemma 5.1. Obviously, φ_n is a zigzag continuous function on $\mathbb{R}$. Suppose that $x \in [N, N+1]$ for some $N \in \mathbb{N}$. Then we have $4^n x \in [4^n N, 4^n(N+1)]$. Since φ is a function of period 1, we have

$$\varphi(4^n x) = \varphi(4^n x - 4^n N), \tag{5.67}$$

where $4^n x - 4^n N \in [0, 4^n]$. It is clear that there exists $N' \in \mathbb{N} \cup \{0\}$ such that $4^n x - 4^n N - N' \in [0, 1]$, then the expression (5.67) and the periodicity of φ again shows that

$$\varphi(4^n x) = \varphi(4^n x - 4^n N) = \varphi(4^n x - 4^n N - N').$$

In other words, we may assume that $4^n x \in [0, 1)$. Let $h > 0$ be so small such that $4^n(x + h) \in [0, 1]$. Then we obtain

$$\begin{aligned}
&\frac{\varphi_n(x+h) - \varphi_n(x)}{h} \\
&= \frac{1}{2^n} \cdot \frac{\varphi\big(4^n(x+h)\big) - \varphi(4^n x)}{h} \\
&= \begin{cases} \dfrac{1}{2^n} \cdot \dfrac{2 \cdot 4^n(x+h) - 2 \cdot 4^n x}{h}, & \text{if } 4^n x, 4^n(x+h) \in [0, \frac{1}{2}]; \\[2ex] \dfrac{1}{2^n} \cdot \dfrac{1 - 2 \cdot 4^n h - 1}{h}, & \text{if } 4^n x = \frac{1}{2} \text{ and } 4^n(x+h) = \frac{1}{2} + 4^n h; \\[2ex] \dfrac{1}{2^n} \cdot \dfrac{-2 \cdot 4^n(x+h) + 2 \cdot 4^n x}{h}, & 4^n x, 4^n(x+h) \in [\frac{1}{2}, 1], \end{cases} \\
&= \begin{cases} 2 \cdot 2^n, & \text{if } 4^n x, 4^n(x+h) \in [0, \frac{1}{2}]; \\[1ex] -2 \cdot 2^n, & \text{if } 4^n x = \frac{1}{2} \text{ and } 4^n(x+h) = \frac{1}{2} + 4^n h; \\[1ex] -2 \cdot 2^n, & 4^n x, 4^n(x+h) \in [\frac{1}{2}, 1] \end{cases}
\end{aligned}$$

which implies that $|\varphi_n'(x+)| \geq 2^n$. Similarly, $|\varphi_n'(x-)| \geq 2^n$, completing the proof of Lemma 5.1. ∎

- **Step 3: Construction of a $\widetilde{g} \in C$ with large slopes and $\|\widetilde{g} - f\| < 2\epsilon$.** We define

$$\widetilde{g} = g + \varphi_k,$$

where the k will be determined later. We claim that $\widehat{g}$ satisfies the required properties. To this end, let

$$M_k = \max_{0 \leq i \leq k-1} k|f(x_{i+1}) - f(x_i)|.$$

Now for every $x \in [0, 1]$, if we take $h > 0$ small enough such that $x + h \in [x_i, x_{i+1}]$, then we see from the definition (5.65) and the expression (5.66) that

$$\left|\frac{g(x+h) - g(x)}{h}\right| = \frac{|\lambda_{x+h} - \lambda_x| \cdot |f(x_{i+1}) - f(x_i)|}{h} = k|f(x_{i+1}) - f(x_i)| \leq M_k \tag{5.68}$$

which implies that $|g'(x+)| \leq M_k$ on $[0, 1]$. Similarly, we have $|g'(x-)| \leq M_k$ for every $x \in [0, 1]$.[f]

[f]If $x = 1$, then $g'(1+)$ does not exist. Similarly, if $x = 0$, then $g'(0-)$ does not exist.

Suppose that $N' \in \mathbb{N}$. Since f is bounded on $[0,1]$, we must have $M_k \leq kM$ for some constant $M > 0$ and then k can be chosen so large that

$$2^k \geq kM \geq M_k + N' \quad \text{and} \quad \frac{1}{2^k} < \frac{\epsilon}{2}. \tag{5.69}$$

Since g and φ_k are continuous on $[0,1]$, $\widetilde{g}$ is also continuous on $[0,1]$. Next, for $x \in [0,1)$ and $h > 0$ so small such that $x + h \in [0,1]$, we derive from Lemma 5.1 and the inequality (5.68) that

$$\begin{aligned}\Big|\frac{\widetilde{g}(x+h) - \widetilde{g}(x)}{h}\Big| &= \Big|\frac{g(x+h) - g(x)}{h} + \frac{\varphi(x+h) - \varphi(x)}{h}\Big| \\ &\geq \Big|\frac{\varphi(x+h) - \varphi(x)}{h}\Big| - \Big|\frac{g(x+h) - g(x)}{h}\Big| \\ &\geq 2^k - M_k \\ &> N' \end{aligned} \tag{5.70}$$

which yields $|\widetilde{g}'(x+)| > N'$ for all $x \in [0,1)$. Similarly, we have $|\widetilde{g}'(x-)| > N'$ for all $x(0,1]$. Since N' can be chosen arbitrary large, our $\widetilde{g}$ is continuous function on $[0,1]$ with large slopes.

Finally, it remains to show that $\|\widetilde{g} - f\| < \epsilon$. On $[0,1]$, we know from **Steps 1** and **2** that

$$|\widetilde{g}(x) - f(x)| \leq |g(x) - f(x)| + |\varphi_k(x)| < \epsilon + 2^{-k}|\varphi(4^k x)| < \epsilon + \frac{\epsilon}{2} < 2\epsilon.$$

Thus we conclude that $\|\widetilde{g} - f\| < 2\epsilon$. By the definition of X_n and the lower bound (5.70), we note that $\widetilde{g} \notin X_n$ if we pick $N' \geq n$.

- **Step 4: X_n is closed in C.** Let $\{f_k\} \subseteq X_n$ and $f_k \to f$ in C. Then one can find a sequence $\{t_k\} \subseteq [0,1]$ such that

$$|f_k(s) - f_k(t_k)| \leq n|s - t_k| \tag{5.71}$$

for every $s \in [0,1]$ and $k \in \mathbb{N}$. Since $[0,1]$ is compact, we may assume that $t_k \to t \in [0,1]$. By Theorem 3.17, C is a metric space and then we recall from the rephrased Theorem 7.9 [49, p. 151] that $f_k \to f$ uniformly on $[0,1]$. Hence it follows from [49, Exercise 9, p. 166] that

$$\lim_{k\to\infty} f_k(t_k) = f(t)$$

and we deduce from the inequality (5.71) that

$$|f(s) - f(t)| \leq n|s - t|,$$

for every $s \in [0,1]$. In other words, we conclude that $f \in X_n$ which means X_n is closed in C.

- **Step 5: Construction of $B(\widetilde{g}, \epsilon') \subseteq C$ such that $B(\widetilde{g}, \epsilon') \cap X_n = \varnothing$.** By **Step 3** and our choice of $\epsilon > 0$, we know that $\widetilde{g} \in B(f, 2\epsilon) \subseteq V$ and $\widetilde{g} \notin X_n$. Since $\widetilde{g} \in X_n^c$ and X_n^c is open in C by **Step 4**. There exists a $\delta > 0$ such that $B(\widetilde{g}, \delta) \subseteq X_n^c$, i.e., $B(\widetilde{g}, \delta) \cap X_n = \varnothing$. If we take $\epsilon' = \frac{1}{2}\min(\|\widetilde{g} - f\|, 2\epsilon - \|\widetilde{g} - f\|)$ and $h \in B(\widetilde{g}, \epsilon')$, then we have

$$\|h - f\| \leq \|h - \widetilde{g}\| + \|\widetilde{g} - f\| < \epsilon' + \|\widetilde{g} - f\| \leq \frac{1}{2}(2\epsilon - \|\widetilde{g} - f\|) + \|\widetilde{g} - f\| < 2\epsilon$$

which means $B(\widetilde{g}, \epsilon') \subseteq B(f, 2\epsilon) \subseteq V$.

Hence we have shown the first assertion.

To prove the second assertion, we notice that $X_n^\circ = \varnothing$; otherwise, since X_n° is open in C, the first assertion shows the existence of a non-empty open set $G \subseteq X_n^\circ \subseteq X_n$ such that $G \cap X_n = \varnothing$, a contradiction. Hence each X_n^c is open and dense in C. Since C is a complete metric space, we establish from Theorem 5.6 (Baire's Theorem) that the set

$$X = \bigcap_{n=1}^{\infty} X_n^c$$

is a dense G_δ in C. Suppose that $f \in C$ and f is differentiable at $p \in [0,1)$. Then there exists a $\delta > 0$ such that for all $0 < |h| < \delta$, we have[g]

$$\Big|\frac{f(p+h) - f(p)}{h} - f'(p)\Big| \leq 1$$

which implies that

$$\Big|\frac{f(p+h) - f(p)}{h}\Big| \leq 1 + |f'(p)|. \tag{5.72}$$

Obviously, if $|h| \geq \delta$, then

$$\Big|\frac{f(p+h) - f(p)}{h}\Big| \leq \frac{|f(p+h)| + |f(p)}{h} \leq \frac{2\|f\|}{\delta}. \tag{5.73}$$

Combining the inequalities (5.72) and (5.73), we obtain that

$$|f(x) - f(p)| \leq N|x - p|$$

if $N = \max(1 + |f'(p)|, 2\delta^{-1}\|f\|)$. In the case that $p = 1$, we will replace $f(p+h)$ by $f(p-h)$ in the inequalities (5.72) and (5.73).

Consequently, $f \in X_n$ for every $n \geq N$. By this, we must have $f \notin X$. Otherwise, $f \in X_N^c$ or equivalently, $f \notin X_N$ which is a contradiction. Hence we have completed the proof of the problem. ■

Problem 5.15

Rudin Chapter 5 Exercise 15.

Proof. Recall from Problem 5.9 that c_0 is the subspace of ℓ^∞ consisting of all $x = \{\xi_i\} \in \ell^\infty$ for which $\xi_i \to 0$ as $i \to \infty$. By the definition, we have

$$\{\sigma_i\} = \mathbf{A}(x)$$

whenever the series converges (but ξ_i *not* necessarily converging to 0). We are going to prove the results one by one:

- **Necessity part:** Suppose that $\mathbf{A}$ transforms every convergent sequence $\{s_j\}$ to a sequence $\{\sigma_i\}$ which converges to the same limit.

  To prove **Condition (a)**, it suffices to prove the case when both $\{s_j\}$ and $\{\sigma_i\}$ converge to 0 because we may replace s_j and σ_i by $s_j - L$ and $\sigma_i - L$ respectively, where L is the

[g] Of course, only one-sided derivative $f'(0+)$ will be considered if $p = 0$.

common limit of $\{s_j\}$ and $\{\sigma_i\}$. In other words, we work on the space c_0. In fact, for each *fixed* $i \in \mathbb{N}$, we define $\Lambda_i : c_0 \to \mathbb{C}$ by

$$\Lambda_i(x) = \sigma_i = \sum_{j=0}^{\infty} a_{ij} s_j$$

for all $x \in c_0$ so that $\mathbf{A}(x) = \{\Lambda_i(x)\}$. Since $\{\sigma_i\}$ is convergent, it is bounded so that

$$|\Lambda_i(x)| \leq M \tag{5.74}$$

for all $x \in c_0$ and $i = 1, 2, \ldots$ for some positive constant M. For every $k = 0, 1, 2, \ldots$, we take $s_j = \delta_{jk}$ (the Kronecker delta function). Then we have $s_j \to 0$ as $j \to \infty$ and we obtain

$$0 = \lim_{i \to \infty} \sigma_i = \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ij} s_j = \lim_{i \to \infty} a_{ik}$$

which is **Condition (a)**.

If $s_j = 1$ for all $j \in \mathbb{N}$, then $s_j \to 1$ as $j \to \infty$ and thus

$$1 = \lim_{i \to \infty} \sigma_i = \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ij}$$

which is **Condition (c)**.

To prove **Condition (b)**, we first show that $\sum\limits_{j=0}^{\infty} |a_{ij}| < \infty$ for every $i = 1, 2, \ldots$. Assume that it was not the case. Then one is able to find a strictly increasing sequence of integers $\{n_k\}$ such that

$$\sum_{j=n_k+1}^{n_{k+1}} |a_{ij}| > k.$$

Let $k \in \mathbb{N}$ and define

$$s_j = \begin{cases} 0, & \text{if } j = 1, 2, \ldots, n_k; \\ \\ \dfrac{\operatorname{sgn}(a_{ij})}{k}, & \text{if } j = n_k + 1, n_k + 2, \ldots, n_{k+1}. \end{cases} \tag{5.75}$$

Obviously, $j \to \infty$ if and only if $k \to \infty$. Then we have $s_j \to 0$ as $j \to \infty$. Thus $x = \{s_j\} \in c_0$ and so $\lim\limits_{i \to \infty} \sigma_i = 0$. However, we notice from the construction (5.75) that

$$|\Lambda_i(x)| = |\sigma_i| = \Big| \sum_{j=0}^{\infty} a_{ij} s_j \Big| = \sum_{k=1}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} \frac{|a_{ij}|}{k} = \infty$$

which contradicts the inequality (5.74). Thus we must have

$$\sum_{j=0}^{\infty} |a_{ij}| < \infty.$$

By the inequality (5.74) again, we have $\Lambda_i \in c_0^*$ and since c_0 is Banach, we may apply Theorem 5.8 (The Banach-Steinhaus Theorem) to conclude that

$$\|\Lambda_i\| = \sup\{|\Lambda_i(x)| \,|\, x \in c_0 \subseteq \ell^\infty \text{ and } \|x\|_\infty = 1\} \leq M \tag{5.76}$$

for all $i \in \mathbb{N}$. Recall that $\sum_{j=0}^{\infty} |a_{ij}| < \infty$ for every $i = 1, 2, \ldots$, so $y_i = \{a_{ij}\} \in \ell^1$ for every $i = 1, 2, \ldots$ and Problem 5.9(a) implies that

$$\|\Lambda_i\| = \|y_i\|_1 = \sum_{j=0}^{\infty} |a_{ij}| \tag{5.77}$$

for every $i = 1, 2, \ldots$. Hence **Condition (b)** follows from the inequality (5.76) and the equality (5.77).

- **Sufficiency part:** Let $x = \{s_j\}$ and $s_j \to L$ as $j \to \infty$. By **Condition (b)**, the series $\sigma_i = \sum_{j=0}^{\infty} a_{ij} s_j$ converges absolutely for every $i \in \mathbb{N}$. We write

$$\sigma_i = \sum_{j=0}^{\infty} a_{ij}(s_j - L) + L \cdot \sum_{j=0}^{\infty} a_{ij}. \tag{5.78}$$

By **Condition (c)**, we have

$$\lim_{i\to\infty} L \cdot \sum_{j=0}^{\infty} a_{ij} = L. \tag{5.79}$$

Let the supremum in **Condition (b)** be M. Given $\epsilon > 0$, we choose a $N \in \mathbb{N}$ such that $|s_j - L| < \frac{\epsilon}{M}$ for all $n \geq N$. Thus we see that

$$\Big|\sum_{j=0}^{\infty} a_{ij}(s_j - L)\Big| \leq \sum_{j=0}^{N} |a_{ij}| \cdot |s_j - L| + \sum_{n=N+1}^{\infty} |a_{ij}| \cdot |s_j - L| \leq \sum_{j=0}^{N} |a_{ij}| \cdot |s_j - L| + \epsilon.$$

By **Condition (a)**, we know that

$$\lim_{i\to\infty} \sum_{j=0}^{N} |a_{ij}| \cdot |s_j - L| = 0$$

so that

$$\lim_{i\to\infty} \Big|\sum_{j=0}^{\infty} a_{ij}(s_j - L)\Big| \leq \epsilon.$$

Since ϵ is arbitrary, we conclude from the representation (5.78) and the limit (5.79) that

$$\lim_{i\to\infty} \sigma_i = L.$$

- **Two examples.** It is easy to check that the examples satisfy the conditions of the problem.

  To show the last assertion, we consider the sequence $\{s_j\}$ defined by

$$s_j = \begin{cases} \sqrt{k}, & \text{if } j = 2k; \\ -\sqrt{k}, & \text{if } j = 2k-1. \end{cases}$$

Then it is clear that $\{s_j\}$ is unbounded and direct computation shows that, for each $i = 1, 2, \ldots$,

$$\sigma_i = \sum_{j=0}^{\infty} a_{ij} s_j$$

$$= \sum_{j=0}^{i} \frac{1}{i+1} s_j$$
$$= \begin{cases} \dfrac{1}{2k+1} \displaystyle\sum_{j=0}^{2k} s_j, & \text{if } i = 2k; \\ \dfrac{1}{2k} \displaystyle\sum_{j=0}^{2k-1} s_j, & \text{if } i = 2k-1; \end{cases}$$
$$= \begin{cases} 0, & \text{if } i = 2k; \\ -\dfrac{1}{2\sqrt{k}}, & \text{if } i = 2k-1. \end{cases}$$

Since $i \to \infty$ if and only if $k \to \infty$, we have $\sigma_i \to 0$ as $i \to \infty$, i.e., $\{\sigma_i\}$ is convergent.

For the other example, pick δ to be a number such that $r_i < \delta < 1$ for every $i \in \mathbb{N}$. If $s_j = (-\delta)^{-j}$, then $\{s_j\}$ is divergent and we have

$$\sigma_i = \sum_{j=0}^{\infty} a_{ij} s_j = (1 - r_i) \sum_{j=0}^{\infty} \Big(-\frac{r_i}{\delta} \Big)^j = \frac{1 - r_i}{1 + \delta^{-1} r_i} \to 0$$

as $i \to \infty$.

We complete the proof of the problem. ■

Remark 5.2

Classically, Problem 5.15 is called the Silverman-Toepliotz Theorem. For more information or other proofs about this theorem, please refer to [30, §3.2], [44, Chap. 4] and [68, §1.2].

5.5 Miscellaneous Problems

Problem 5.16

Rudin Chapter 5 Exercise 16.

Proof. We follow Rudin's hint. Let $X \oplus Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$ with addition and scalar multiplication defined componentwise. We define the norm $\|\cdot\|$ on $X \oplus Y$ by[h]

$$\|(x, y)\| = \|x\|_X + \|y\|_Y. \tag{5.80}$$

We check Definition 5.2. For $(x_1, y_1), (x_2, y_2) \in X \oplus Y$, we have

$$\begin{aligned} \|(x_1, y_1) + (x_2, y_2)\| &= \|(x_1 + x_2, y_1 + y_2)\| \\ &= \|x_1 + x_2\|_X + \|y_1 + y_2\|_Y \\ &\leq \|x_1\|_X + \|x_2\|_X + \|y_1\|_Y + \|y_2\|_Y \\ &= \|(x_1, y_1)\| + \|(x_2, y_2)\|. \end{aligned}$$

[h] Of course, $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norms in X and Y respectively.

Next, if $(x, y) \in X \oplus Y$ and α is a scalar, then we have

$$\|\alpha(x,y)\| = \|(\alpha x, \alpha y)\| = \|\alpha x\|_X + \|\alpha y\|_Y = |\alpha| \cdot \|x\|_X + |\alpha| \cdot \|y\|_Y = |\alpha| \cdot \|(x,y)\|.$$

Finally, if $\|(x,y)\| = 0$, then $\|x\|_X + \|y\|_Y = 0$. Since $\|x\|_X$ and $\|y\|_Y$ are nonnegative for every $x \in X$ and $y \in Y$, $\|x\|_X + \|y\|_Y = 0$ implies that $\|x\|_X = \|y\|_Y = 0$ and thus $(x, y) = (0, 0)$. Hence $X \oplus Y$ is a normed linear space.

Suppose that $\{(x_n, y_n)\} \subseteq X \oplus Y$ is Cauchy. Given $\epsilon > 0$, Then there exists a positive integer N such that $n, m \geq N$ imply that

$$\|(x_n, y_n) - (x_m, y_m)\| < \epsilon.$$

By the definition (5.80), we must have $\|x_n - x_m\|_X < \epsilon$ and $\|y_n - y_m\|_Y < \epsilon$ for all $n, m \geq N$. In other words, $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ are also Cauchy. Since X and Y are Banach, there exist $x \in X$ and $y \in Y$ such that

$$\|x_n - x\|_X \to 0 \quad \text{and} \quad \|y_n - y\|_Y \to 0$$

as $n \to \infty$. Now these limits and the definition (5.80) show that

$$\|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\|_X + \|y_n - y\|_Y \to 0$$

as $n \to \infty$. By Definition 5.2, $X \oplus Y$ ia Banach.

Suppose that $G = \{\big(x, \Lambda(x)\big) \,|\, x \in X\} \subseteq X \oplus Y$. Furthermore, we let $\alpha, \beta \in \mathbb{C}$ and $\big(x, \Lambda(x)\big), \big(y, \Lambda(y)\big) \in G$. Then the linearity of Λ says that

$$\alpha\big(x, \Lambda(x)\big) + \beta\big(y, \Lambda(y)\big) = \big(\alpha x + \beta y, \alpha\Lambda(x) + \beta\Lambda(y)\big) = \big(\alpha x + \beta y, \Lambda\big(\alpha x + \beta y\big)\big) \in G.$$

Thus G is a linear subspace of $X \oplus Y$ and it is also a metric space. We claim that G is closed in $X \oplus Y$. To see this, let $\{\big(x_n, \Lambda(x_n)\big)\} \subseteq G$ and $\|\big(x_n, \Lambda(x_n)\big) - (x, y)\| \to 0$ as $n \to \infty$. By the definition (5.80), we have $\|x_n - x\|_X \to 0$ and $\|\Lambda(x_n) - y\|_Y \to 0$ as $n \to \infty$, i.e.,

$$x = \lim_{n \to \infty} x_n \quad \text{and} \quad y = \lim_{n \to \infty} \Lambda(x_n).$$

By the hypothesis, we gain that $y = \Lambda(x)$ and so $(x, y) = \big(x, \Lambda(x)\big) \in G$ by the definition. Therefore, we have shown the claim that G is closed in $X \oplus Y$. Since $X \oplus Y$ is complete, we deduce from [49, Theorem 3.11, p. 53] that G is also complete and thus Banach.

Consider the mapping $\Phi : G \to X$ defined by

$$\Phi\big(x, \Lambda(x)\big) = x.$$

Then it is clear that Φ is linear and bijective. Furthermore, for every $x \in X$, we see that

$$\|\Phi\big(x, \Lambda(x)\big)\|_X = \|x\|_X \leq \|x\|_X + \|\Lambda(x)\|_Y = \|\big(x, \Lambda(x)\big)\|$$

which implies that

$$\|\Phi\| = \sup\{\|\Phi\big(x, \Lambda(x)\big)\|_X \,|\, \big(x, \Lambda(x)\big) \in G \text{ and } \|\big(x, \Lambda(x)\big)\| = 1\} \leq 1,$$

i.e., Φ is bounded. By Theorem 5.10, Φ^{-1} is also a bounded linear transformation of X onto G and it follows from Theorem 5.10's proof that there exists a positive constant δ such that

$$\|\Phi^{-1}\| \leq \frac{1}{\delta}.$$

Besides, Theorem 5.4 shows that Φ^{-1} is continuous. Combining these two facts and [51, Eqn. (3), p. 96], we obtain

$$\|x\|_X + \|\Lambda(x)\|_Y = \|\big(x, \Lambda(x)\big)\| = \|\Phi^{-1}(x)\| \leq \|\Phi^{-1}\| \cdot \|x\|_X \leq \frac{1}{\delta}\|x\|_X < \infty \tag{5.81}$$

for every $x \in X$. Since $\|x\|_X$ and $\|\Lambda(x)\|_Y$ are nonnegative, we get from the inequality (5.81) that $\delta < 1$ and

$$0 \leq \|\Lambda(x)\|_Y \leq \Big(\frac{1}{\delta} - 1\Big) \cdot \|x\|_X \tag{5.82}$$

on X. Hence we conclude from the inequality (5.82) that

$$\|\Lambda\| = \sup\{\|\Lambda(x)\|_Y \,|\, x \in X \text{ and } \|x\|_X = 1\} \leq \frac{1}{\delta} < \infty - 1.$$

Consequently, Λ is bounded and it is continuous on X by Theorem 5.4, completing the proof of the problem. ■

Problem 5.17

Rudin Chapter 5 Exercise 17.

Proof. We prove the assertions one by one.

- **$\|M_f\| \leq \|f\|_\infty$.** Given $f \in L^\infty(\mu)$. Define $M_f : L^2(\mu) \to L^2(\mu)$ by

$$M_f(g) = fg.$$

  By Definition 3.7, we have $|fg| \leq \|f\|_\infty \cdot |g|$ for all $f \in L^2(\mu)$ and then Remark 3.10 shows that

$$\|M_f(g)\|_2 = \|fg\|_2 = \Big\{\int_X |fg|^2\,d\mu\Big\}^{\frac{1}{2}} \leq \Big\{\int_X \|f\|_\infty^2 \cdot |g|^2\,d\mu\Big\}^{\frac{1}{2}} = \|f\|_\infty \cdot \|g\|_2. \tag{5.83}$$

  By the inequality (5.83) and Definition 5.3, we obtain immediately that

$$\|M_f\| = \sup\{\|M_f(g)\|_2 \,|\, g \in L^2(\mu) \text{ and } \|g\|_2 = 1\} \leq \|f\|_\infty. \tag{5.84}$$

- **Measures μ with $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$.** We call the measure μ **semifinite** if for each $E \in \mathfrak{M}$ with $\mu(E) = \infty$ one can find a $F \in \mathfrak{M}$ with $F \subset E$ such that $0 < \mu(F) < \infty$, see [22, p. 25].

  Now we claim that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$ if and only if the measure μ is semifinite.

  - Suppose that μ is semifinite. Since $f \in L^\infty(\mu)$, we have $\|f\|_\infty < \infty$. Let $\alpha = \|f\|_\infty$. If $\alpha = 0$, then the inequality (5.84) forces that

$$\|M_f\| = \|f\|_\infty = 0$$

    and we are done. If $\alpha > 0$, then we see from Problem 3.19 that $\alpha = \max\{|z| \,|\, z \in R_f\}$, where R_f denotes the essential range of f. Thus there exists a $z_0 \in \mathbb{C}$ such that $\alpha = |z_0|$. Without loss of generality, we may assume that $z_0 = \alpha$ and so

$$\mu\{x \in X \,|\, |f(x) - \alpha| < \epsilon\} > 0 \tag{5.85}$$

    for every $\epsilon > 0$. Combining Definition 3.7 and the result (5.85), we establish that the measure of the set

$$E = \{x \in \mathbb{R} \,|\, |f(x)| > \alpha - \epsilon\}$$

    is nonzero.

* **Case (i): $\mu(E) < \infty$.** Now it is trivial that the function

$$g = \frac{1}{\sqrt{\mu(E)}}\chi_E$$

satisfies the conditions that $g \in L^2(\mu)$ and $\|g\|_2 = 1$. Furthermore, we derive from the definition that

$$\|M_f(g)\|_2 = \left\{\frac{1}{\mu(E)}\int_E |f\chi_E|^2 \,\mathrm{d}\mu\right\}^{\frac{1}{2}} > \left\{\frac{(\alpha-\epsilon)^2}{\mu(E)}\int_E \mathrm{d}\mu\right\}^{\frac{1}{2}} = \alpha - \epsilon. \tag{5.86}$$

Since ϵ is arbitrary, the estimate (5.86) implies that

$$\|M_f\| = \sup\{\|M_f(g)\|_2 \,|\, g \in L^2(\mu) \text{ and } \|g\|_2 = 1\} \geq \alpha$$

and hence $\|M_f\| = \|f\|_\infty$ by the inequality (5.84).

* **Case (ii): $\mu(E) = \infty$.** Then the hypothesis ensures that there exists a $F \in \mathfrak{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$. Now the function

$$g = \frac{1}{\sqrt{F}}\chi_F$$

also satisfies the estimate (5.86) and thus $\|M_f\| = \|f\|_\infty$ holds in this case.

– Suppose that μ is *not* semifinite. Then there is a $E \in \mathfrak{M}$ such that $\mu(E) = \infty$ and every $F \subset E$ satisfies either $\mu(F) = 0$ or $\mu(F) = \infty$. Take $g \in L^2(\mu)$ so that

$$\int_E |g(x)|^2 \,\mathrm{d}\mu \leq \int_X |g(x)|^2 \,\mathrm{d}\mu < \infty.$$

If there is a $F \subset E$ such that $|g(x)| > 0$ on F and $g(x) = 0$ on $E \setminus F$, then $\mu(F) \neq \infty$ by Proposition 1.24(a). Recall that μ is not semifinite, we have $\mu(F) = 0$. In other words, it must be true that $g = 0$ a.e. on E. Let $f = \chi_E$. Then we have

$$fg = 0 \text{ a.e. on } E$$

for every $g \in L^2(\mu)$. By the definition, we obtain

$$\|M_f\| = 0.$$

On the other hand, $\|f\|_\infty = 1$ so that $\|M_f\| < \|f\|_\infty$.

Hence we have proven the claim.

- **Functions $f \in L^\infty(\mu)$ such that M_f is onto.** Suppose that μ satisfies the condition that every measurable set E of positive measure contains a measurable subset F with $0 < \mu(F) < \infty$.[i] We claim that the map M_f is onto if and only if $\frac{1}{f} \in L^\infty(\mu)$.

– If M_f is onto, then we let

$$E = \{x \in X \,|\, f(x) = 0\}. \tag{5.87}$$

We claim that $\mu(E) = 0$. Assume that $\mu(E) > 0$. By the hypothesis, there exists a $F \subseteq E$ with $0 < \mu(F) < \infty$. Thus $\chi_F \in L^2(\mu)$. Since $fg = 0$ on F for every $g \in L^2(\mu)$, it implies that

$$\chi_F \notin M_f\big(L^2(\mu)\big)$$

[i] Some books take this as the definition of a semifinite measure μ. See, for example, [7, Exercise 25.9]

which contradicts the surjective property of M_f. Thus we conclude that $\mu(E) = 0$.

If $g \in L^2(\mu)$ is such that $M_f(g) = fg = 0$, then it follows from the definition (5.87) that $g = 0$ a.e. on E^c. Since $\mu(E) = 0$, we obtain fact that

$$g = 0 \text{ a.e. on } X$$

and this means that M_f is one-to-one. Since M_f is assumed to be onto, it is bijective.

On the one hand, recall that $L^2(\mu)$ is Banach and $\|M_f\| \leq \|f\|_\infty < \infty$ by the first assertion, Theorem 5.10 ensures that there corresponds a $\delta > 0$ such that

$$\|M_f(g)\|_2 \geq \delta \|g\|_2 \tag{5.88}$$

for every $g \in L^2(\mu)$. On the other hand, we consider $F = \{x \in X \,|\, |f(x)| < \frac{\delta}{2}\}$. If $\mu(F) \neq 0$, then our hypothesis tells us that there is a $G \subset F$ such that $0 < \mu(G) < \infty$ and thus

$$\|\chi_G\|_2 = [\mu(G)]^{\frac{1}{2}}$$

which verifies that

$$\|M_f(\chi_G)\|_2 = \|f\chi_G\|_2 = \Big\{\int_G |f|^2 \,\mathrm{d}\mu\Big\}^{\frac{1}{2}} < \Big\{\int_G \frac{\delta^2}{4}\,\mathrm{d}\mu\Big\}^{\frac{1}{2}} = \frac{\delta}{2}[\mu(G)]^{\frac{1}{2}} = \frac{\delta}{2}\|\chi_G\|_2,$$

but it contradicts the inequality (5.88). Hence we have $\mu(F) = 0$ if and only if $|f(x)| \geq \frac{\delta}{2} > 0$ a.e. on X if and only if $\frac{1}{|f(x)|} \leq \frac{2}{\delta}$ a.e. on X if and only if $\frac{1}{f} \in L^\infty(\mu)$.

- Suppose that $\frac{1}{f} \in L^\infty(\mu)$. Then it is clear that $M_{\frac{1}{f}}$ is the inverse operator of M_f so that M_f is bijective. In particular, M_f is surjective.

Hence we have completed the proof of the problem. ■

Problem 5.18

Rudin Chapter 5 Exercise 18.

Proof. Let $x \in X$ and $\epsilon > 0$. Since E is dense in the normed linear space X, we can find $y \in E$ such that

$$\|x - y\| < \frac{\epsilon}{2M}. \tag{5.89}$$

Since $\{\Lambda_n(y)\}$ converges in the Banach space Y, there exists a positive integer N such that $n, m \geq N$ imply that

$$\|\Lambda_n(y) - \Lambda_m(y)\| < \frac{\epsilon}{2}. \tag{5.90}$$

Therefore, if $n, m \geq N$, then we deduce immediately from the inequalities (5.89) and (5.90) that

$$\begin{aligned}
\|\Lambda_n(x) - \Lambda_m(x)\| &\leq \|\Lambda_n(x) - \Lambda_n(y)\| + \|\Lambda_n(y) - \Lambda_m(y)\| + \|\Lambda_m(y) - \Lambda_m(x)\| \\
&< \|\Lambda_n(x - y)\| + \frac{\epsilon}{2} + \|\Lambda_m(y - x)\| \\
&\leq \|\Lambda_n\| \cdot \|x - y\| + \|\Lambda_m\| \cdot \|x - y\| + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This means that $\{\Lambda_n(x)\}$ is Cauchy in Y. Since Y is Banach, $\{\Lambda_n(x)\}$ converges in Y and we complete the analysis of the problem. ■

Problem 5.19

Rudin Chapter 5 Exercise 19.

Proof. Given $f \in C(T)$, i.e., f is a continuous complex function on T. Recall that

$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,\mathrm{d}t \quad \text{and} \quad D_n(t) = \sum_{k=-n}^{n} \mathrm{e}^{ikt} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}. \tag{5.91}$$

By §5.11, $C(T)$ is Banach relative to the supremum norm $\|f\|_\infty$.[j] Next, since $f \in C(T)$, we follow from the equations (5.91) that $s_n(f;x) \in C(T)$ for each $n \in \mathbb{N}$.

Let $X = Y = C(T)$ and for each $n = 2, 3, \ldots$, we define $\Lambda_n : C(T) \to C(T)$ by

$$\Lambda_n(f) = \frac{s_n(f;x)}{\log n}.$$

Since $f \in C(T)$, there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in T$ so that

$$|s_n(f;x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)| \cdot |D_n(x-t)|\,\mathrm{d}t \leq \frac{M}{2\pi}\int_{-\pi}^{\pi} |D_n(x-t)|\,\mathrm{d}t. \tag{5.92}$$

Since $D_n(x-t) = \mathrm{e}^{ikx}D_n(t)$ and $D_n(t)$ is an even function, the estimate (5.92) can be replaced by

$$|s_n(f;x)| \leq \frac{M}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,\mathrm{d}t = \frac{M}{\pi}\int_{0}^{\pi} |D_n(t)|\,\mathrm{d}t = \frac{M}{\pi}\int_{0}^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right|\mathrm{d}t. \tag{5.93}$$

Next, we consider $f(x) = \sin\frac{x}{2} - \frac{x}{4}$ on $[0,\pi]$. Using differentiation, we can show that $\sin\frac{x}{2} \geq \frac{x}{4}$ on $[0,\pi]$. Thus the estimate (5.93) can be further written as

$$\begin{aligned} |s_n(f;x)| &\leq \frac{4M}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{t}\,\mathrm{d}t \\ &= \frac{4M}{\pi}\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin t|}{t}\,\mathrm{d}t \\ &\leq \frac{4M}{\pi}\Big(\int_0^{\pi}\frac{\sin t}{t}\,\mathrm{d}t + \sum_{k=1}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,\mathrm{d}t\Big). \end{aligned} \tag{5.94}$$

On the interval $[k\pi, (k+1)\pi]$, it is easy to see that $\frac{1}{t} \leq \frac{1}{k\pi}$, where $k = 1, 2, \ldots, n$, so we have

$$\sum_{k=1}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,\mathrm{d}t \leq \sum_{k=1}^{n}\frac{1}{k\pi}\Big(\int_{k\pi}^{(k+1)\pi}|\sin t|\,\mathrm{d}t\Big) = \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k}. \tag{5.95}$$

Besides, we get from [49, Problem 8.6, p. 197] that $\frac{2}{\pi} < \frac{\sin t}{t} < 1$ for $0 < t < \frac{\pi}{2}$, so the integral

$$\int_0^{\frac{\pi}{2}}\frac{\sin t}{t}\,\mathrm{d}t$$

is bounded by $\frac{\pi}{2}$ and thus

$$\int_0^{\pi}\frac{\sin t}{t}\,\mathrm{d}t = \int_0^{\frac{\pi}{2}}\frac{\sin t}{t}\,\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi}\frac{\sin t}{t}\,\mathrm{d}t \leq \frac{\pi}{2} + \int_{\frac{\pi}{2}}^{\pi}\frac{\mathrm{d}t}{t} = \frac{\pi}{2} + \ln\pi - \ln\frac{\pi}{2} = \frac{\pi}{2} + \ln 2. \tag{5.96}$$

[j] To see this, since T is compact, we follow from Definition 3.16 that $C(T) = C_0(T)$ and Theorem 3.17 implies that $C(T)$ is complete. By Definition 5.2, we see that $C(T)$ is Banach.

Now, by putting the estimates (5.95) and (5.96) into the estimate (5.94), we derive that

$$|\Lambda_n(f)| = \Big|\frac{s_n(f;x)}{\log n}\Big| \leq \frac{4M}{\pi \log n} \times (\frac{\pi}{2} + \ln 2) + \frac{8M}{\pi^2 \log n} \cdot \sum_{k=1}^{n} \frac{1}{k}. \tag{5.97}$$

When $\|f\|_\infty = 1$, we may take $M = 1$ in the estimate (5.97). It is well-known (see [63, Eqn. (6.27), p. 124]) that $\log n$ and $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ are of the same growth as $n \to \infty$, so we follow from the estimate (5.97) that there exists a $M' > 0$ such that

$$\|\Lambda_n\| \leq M'$$

for all $n = 2, 3, \ldots$. That is, $\{\Lambda_n\}$ satisfies the first hypothesis of Problem 5.18.

By Theorem 4.25 (The Weierstrass Approximation Theorem), the set of all trigonometric polynomials, namely $\mathcal{P}$, is dense in $C(T)$. Let $P_m(t) = \mathrm{e}^{imt}$ for some $m \in \mathbb{Z}$. If $n \geq m$, then we follow from the result [51, Eqn. (8), p. 89] that

$$s_n(P_m; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-imt} \sum_{k=-n}^{n} \mathrm{e}^{ikt}\, \mathrm{d}t = \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{i(k-m)t}\, \mathrm{d}t = 1. \tag{5.98}$$

Therefore, if $P(t) = \displaystyle\sum_{m=-N}^{N} c_m P_m(t)$, then we get immediately from the result (5.98) that

$$s_n(P; x) = \sum_{m=-N}^{N} c_m s_n(P_m; x) = \sum_{m=-N}^{N} c_m$$

for every $n \geq N$ and this implies that

$$\Lambda_n(P) = \frac{s_n(P; x)}{\log n} = \frac{1}{\log n} \cdot \sum_{m=-N}^{N} c_m \to 0$$

as $n \to \infty$. Thus $\{\Lambda_n\}$ satisfies the second hypothesis of Problem 5.18. Hence we establish from Problem 5.18 that $\{\Lambda_n(f)\}$ converges to 0 for each $f \in C(T)$ and as a consequence, we have

$$\lim_{n\to\infty} \frac{\|s_n\|_\infty}{\log n} = \lim_{n\to\infty} \|\Lambda_n\| = 0.$$

For the second assertion, we note that if we define

$$\Lambda_n f = \frac{s_n(f; 0)}{\lambda_n},$$

then we may apply an argument similar to that used in §5.11 to show that

$$\|\Lambda_n\| = \frac{\|D_n\|_1}{|\lambda_n|}$$

holds. Furthermore, we follow from the hypothesis $\frac{\lambda_n}{\log n} \to 0$ as $n \to \infty$ that[k]

$$\frac{\|D_n\|_1}{|\lambda_n|} \geq \frac{4}{\pi^2 |\lambda_n|} \sum_{k=1}^{n} \frac{1}{k} \geq \frac{4}{\pi^2 |\lambda_n|} (\log n + \gamma) = \frac{4}{\pi^2} \Big(\frac{\log n}{|\lambda_n|} + \frac{\gamma}{|\lambda_n|} \Big) \to \infty \tag{5.99}$$

[k]We have applied the estimate of $\|D_n\|_1$ used in [51, p. 102] in the first inequality in (5.99).

as $n \to \infty$, where γ is the famous Euler constant. Hence Theorem 5.8 (the Banach-Steinhaus Theorem) ensures that the sequence

$$\left\{\frac{s_n(f;0)}{\lambda_n}\right\}$$

is unbounded for every f in some dense G_δ set in $C(T)$, completing the proof of the problem. ■

Problem 5.20

Rudin Chapter 5 Exercise 20.

Proof.

(a) Assume that such a sequence of continuous positive functions $\{f_n\}$ existed. Let $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$. Furthermore, let

$$U = \{x \in \mathbb{R} \,|\, \{f_n(x)\} \text{ is unbounded}\} \quad \text{and} \quad U_k = \{x \in \mathbb{R} \,|\, f_n(x) > k \text{ for some } n \in \mathbb{N}\}.$$

We claim that

$$U = \bigcap_{k=1}^{\infty} U_k. \tag{5.100}$$

On the one hand, if $x \in U_k$ for all $k \in \mathbb{N}$, then for every $k \in \mathbb{N}$, there exists a $n \in \mathbb{N}$ such that $f_n(x) > k$. In other words, we have $x \in U$. On the other hand, if $x \in U$, then since $\{f_n(x)\}$ is unbounded, for every $k \in \mathbb{N}$, there exists a $n \in \mathbb{N}$ such that $f_n(x) > k$ which is equivalent to saying that $x \in U_k$ for all $k \in \mathbb{N}$. Thus this proves the claim. Next, it is easy to see that

$$U_k = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \,|\, f_n(x) > k\}.$$

Since each f_n is continuous, the set $\{x \in \mathbb{R} \,|\, f_n(x) > k\}$ is open in $\mathbb{R}$ and therefore, each U_k is open in $\mathbb{R}$. By the definition (5.100), U is a G_δ set. Thus we follow from the assumption that

$$\mathbb{Q} = U$$

is also a G_δ set. We deduce from the definition (5.100) that

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c = \mathbb{Q} \cup \bigcup_{k=1}^{\infty} U_k^c = \bigcup_{k=1}^{\infty} \{q_k\} \cup \bigcup_{k=1}^{\infty} U_k^c \tag{5.101}$$

Clearly, each q_k is a nowhere dense subset of $\mathbb{R}$. Furthermore, it is trivial that each U_k^c is closed in $\mathbb{R}$. Assume that $U_{k_0}^c$ was not a nowhere dense subset of $\mathbb{R}$ for some k_0. Let V_{k_0} be a nonempty open subset of $U_{k_0}^c$. Then there exists a $p \in V_{k_0}$ and a $\delta > 0$ such that

$$(p-\delta, p+\delta) \subseteq V_{k_0} \subseteq U_{k_0}^c \subseteq \mathbb{Q}^c,$$

but this means that $\mathbb{Q} \cap \mathbb{Q}^c \neq \varnothing$, a contradiction. Thus every U_k^c is also a nowhere dense subset of $\mathbb{R}$ and the representation (5.101) shows that $\mathbb{R}$ is a set of the first category which contradicts Theorem 5.6 (Baire's Theorem) that *no complete metric space is of the first category.*

(b) Suppose that $\mathbb{Q} = \{q_1, q_2, \ldots\}$ and for each $n \in \mathbb{N}$, we define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \min(n|x-q_1| + 1, n|x-q_2| + 2, \ldots, n|x-q_n| + n). \tag{5.102}$$

It is clear that f_n is continuous on $\mathbb{R}$ and $f_n(x) \geq 1$ for all $x \in \mathbb{R}$, i.e., $\{f_n\}$ is a sequence of continuous positive functions on $\mathbb{R}$.

Suppose that θ is irrational and $N \in \mathbb{N}$. Consider the number

$$\alpha = \min(|\theta - q_1|, |\theta - q_2|, \ldots, |\theta - q_N|) > 0.$$

Then the Archimedean Property ([49, Theorem 1.20(a)]) implies that there is a positive integer n such that $n\alpha > N$ and thus

$$f_n(\theta) = \min(n|\theta - q_1| + 1, n|\theta - q_2| + 2, \ldots, n|\theta - q_n| + n) > N + 1.$$

Since N is arbitrary, we have $f_n(\theta) \to \infty$ as $n \to \infty$ so that $\{f_n(\theta)\}$ is unbounded.

To prove the other direction (i.e., $\{f_n(x)\}$ is unbounded implies x is irrational), we prove its contrapositive. Let q_k be a rational number. If $n \geq k$, then we have

$$f_n(q_k) = \min(n|q_k - q_1| + 1, \ldots, n|q_k - q_k| + k, \ldots, n|q_k - q_n| + n) \leq k$$

so that $f_n(q_k) \leq \min\big(f_1(q_k), \ldots, f_{k-1}(q_k), k\big)$. In other words, the set $\{f_n(q_k)\}$ is bounded.

(c) The sequence of the functions given by (5.102) shows that the assertion is true for irrationals. For the rational numbers, we first prove the case on $[0, 1]$.[l] To begin with, suppose that $Q_n = \{q_1, q_2, \ldots, q_n\} \subseteq [0, 1] \cap \mathbb{Q}$, where $q_1 = 0$. Furthermore, we let

$$\delta_n = \frac{1}{2^{n+1}} \min\{|q_i - q_j| \,|\, 1 \leq i < j \leq n\} > 0.$$

Clearly, we have

$$\lim_{n\to\infty} \delta_n = 0 \quad \text{and} \quad (q_i - \delta_n, q_i + \delta_n) \cap (q_j - \delta_n, q_j + \delta_n) = \varnothing \tag{5.103}$$

for all $1 \leq i < j \leq n$. (If $q_i = 0$ or $q_i = 1$, then $(q_i - \delta_n, q_i + \delta_n)$ are replaced by $[0, \delta_n)$ or $(1 - \delta_n, 1]$ respectively.)

Suppose that $E_n = \bigcup_{i=1}^{n} (q_i - \delta_n, q_i + \delta_n)$ and $f_n : [0, 1] \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} \dfrac{n}{\delta_n}(x - q_i + \delta_n), & \text{if } x \in (q_i - \delta_n, q_i] \text{ and } 1 \leq i \leq n; \\[2ex] -\dfrac{n}{\delta_n}(x - q_i - \delta_n), & \text{if } x \in [q_i, q_i + \delta_n) \text{ and } 1 \leq i \leq n; \\[2ex] 0, & \text{if } x \in [0, 1] \setminus E_n. \end{cases}$$

Thus f_n is a continuous function on $[0, 1]$, zig-zag on $(q_i - \delta_n, q_i + \delta_n)$ and $f_n(q_i) = n$ for each $i = 1, 2, \ldots, n$.[m] Therefore, if $x \in [0, 1] \cap \mathbb{Q}$, then $x = q_k$ for some $k \in \mathbb{N}$ and thus we obtain

$$\lim_{n\to\infty} f_n(x) = \infty.$$

Next, suppose that x is irrational. Assume that there was an $N \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq N$. By the definition, it means that

$$x \in (q_k - \delta_N, q_k + \delta_N) \tag{5.104}$$

[l]The following argument is stimulated by the papers of Fabrykowski [19] and Myerson [43].
[m]The graph of the f_n looks like the graph shown in Figure 2.1.

for some $k \in \{1, 2, \ldots, N\}$. Since $x \in E_n$ for all $n \geq N$, the set relation (5.104) shows that

$$x \in (q_k - \delta_n, q_k + \delta_n)$$

for all $n \geq N$. By the limit (5.103), we know that $x = q_k \in \mathbb{Q}$, a contradiction. Hence we must have $x \notin E_n$ for infinitely many n, i.e., $f_n(x) = 0$ for *infinitely many* n so that $\lim\limits_{n\to\infty} f_n(x) = \infty$ is impossible.

In conclusion, we have constructed the sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ which satisfies $f_n(x) \to \infty$ as $n \to \infty$ if and only if $x \in \mathbb{Q}$. If we suppose that f_n is a function of period 1, then the domain of f_n can be extended to $\mathbb{R}$ and we obtain the desired result.

Hence we have completed the proof of the problem. ■

Problem 5.21

Rudin Chapter 5 Exercise 21.

Proof. Since $\mathbb{Q}$ is a countable union of closed sets of $\mathbb{R}$, we have $\mathbb{Q} \in \mathscr{B}$. It is well-known that $m(\mathbb{Q}) = 0$. Obviously, the translate $\mathbb{Q} + \sqrt{2}$ does not intersect $\mathbb{Q}$. This gives an affirmative answer to the first assertion.

Assume that there was a homeomorphism[n] $h : \mathbb{R} \to \mathbb{R}$ such that

$$\widetilde{E} = h(E) \cap E \neq \varnothing$$

for every measurable $E \subseteq \mathbb{R}$ with $m(E) = 0$. Given $\mathbb{Q} = \{q_k\}$ and $\epsilon > 0$. We first construct a particular measurable set E with $m(E) = 0$. To this end, for all $q_k \in \mathbb{Q}$, we consider the neighborhoods $(q_k - 2^{-k}\epsilon, q_k + 2^{-k}\epsilon)$ and their union

$$E(\epsilon) = \bigcup_{k=1}^{\infty} (q_k - 2^{-k}\epsilon, q_k + 2^{-k}\epsilon).$$

It is easy to see that every $E(\epsilon)$ is nonempty open in $\mathbb{R}$. In addition, since $\mathbb{Q} \subsetneq E(\epsilon)$, each $E(\epsilon)$ is dense in $\mathbb{R}$ with $m\big(E(\epsilon)\big) \leq 2\epsilon$. Hence it follows from Theorem 5.6 (Baire's Theorem) that the set

$$E = \bigcap_{n=1}^{\infty} E\Big(\frac{1}{n}\Big) \tag{5.105}$$

is a dense G_δ set in $\mathbb{R}$ and $m(E) = 0$.

Suppose that E is the set (5.105). Since h is a homeomorphism, it is an open map which implies that

$$h(E) = \bigcap_{n=1}^{\infty} h\Big(E\Big(\frac{1}{n}\Big)\Big)$$

is also a dense G_δ set in $\mathbb{R}$ with $m\big(h(E)\big) = 0$. Thus the intersection $\widetilde{E}$ is also a dense G_δ set of measure zero. Since $E\big(\frac{1}{n}\big)$ is countable for all $n \in \mathbb{N}$, E is also countable and thus $\widetilde{E} \subseteq E$ is a countable dense G_δ in $\mathbb{R}$ of measure zero. However, $\mathbb{R}$ is a complete metric space which has *no* isolated points, the existence of $\widetilde{E}$ certainly contradicts Theorem 5.13. Hence no such homeomorphism h exists and this completes the proof of the problem. ■

[n] By the definition, h is a continuous bijection and h^{-1} is continuous.

Problem 5.22

Rudin Chapter 5 Exercise 22.

Proof. Suppose that $f \in C(T)$, $f \in \mathrm{Lip}\,\alpha$ and $f(0) = 0$. We need to show that

$$\lim_{n\to\infty} s_n(f;x) = f(x). \tag{5.106}$$

To achieve the goal, we first quote the following stronger form of the Riemann-Lebesgue Lemma whose proof can be found in [3, Theorem 11.16, p. 313].

Lemma 5.2 (Riemann-Lebesgue Lemma)

For every $f \in L^1(T)$, we define

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-i(n+\beta)t}\,\mathrm{d}t,$$

where $\beta \in \mathbb{R}$. Then we have $\widehat{f}(n) \to 0$ as $|n| \to \infty$.

To begin with, since $f \in C(T)$, we have $f \in L^1(T)$. Furthermore, the hypothesis $f \in \mathrm{Lip}\,\alpha$ implies that $|f(s) - f(t)| \leq M_f|s-t|^\alpha$ for all $s,t \in [-\pi,\pi]$, where M_f is finite and $\alpha \in (0,1]$. Particularly, take $s = 0$ so that $|f(t)| \leq M_f|t|^\alpha$ for all $t \in [-\pi,\pi]$ which implies

$$\int_{-\pi}^{\pi}\Big|\frac{f(t)}{t}\Big|\,\mathrm{d}t \leq M_f\int_{-\pi}^{\pi}|t|^{\alpha-1}\,\mathrm{d}t = M_f\Big[\int_{-\pi}^{0}(-1)^{\alpha-1}t^{\alpha-1}\,\mathrm{d}t + \int_0^{\pi}t^{\alpha-1}\,\mathrm{d}t\Big] = \frac{2M_f\pi^\alpha}{\alpha} < \infty.$$

In other words, we have $\frac{f(t)}{t} \in L^1(T)$. Since we have

$$\sin\Big(n+\frac{1}{2}\Big)t = \frac{1}{2i}\Big[\mathrm{e}^{i(n+\frac{1}{2})t} - \mathrm{e}^{-i(n+\frac{1}{2})t}\Big],$$

it yields

$$\begin{aligned}\Big|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\frac{\sin(n+\frac{1}{2})t}{t}\,\mathrm{d}t\Big| &= \Big|\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(t)}{t}\mathrm{e}^{i(n+\frac{1}{2})t}\,\mathrm{d}t - \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(t)}{t}\mathrm{e}^{-i(n+\frac{1}{2})t}\,\mathrm{d}t\Big| \\ &\leq \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{t}\mathrm{e}^{i(n+\frac{1}{2})t}\,\mathrm{d}t\Big| + \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{t}\mathrm{e}^{-i(n+\frac{1}{2})t}\,\mathrm{d}t\Big|. \end{aligned} \tag{5.107}$$

By Lemma 5.2, each of the integrals on the right-hand side of the inequality (5.107) tends to 0 as $|n| \to \infty$. Thus it is true that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\frac{\sin(n+\frac{1}{2})t}{t}\,\mathrm{d}t \to 0 = f(0)$$

as $|n| \to \infty$.[o]

Next, we claim that

$$s_n(f;0) \to f(0) \tag{5.108}$$

[o]Integrals of the form $\displaystyle\int_0^b g(t)\frac{\sin\alpha t}{t}\,dt$ are called **Dirichlet integrals**, where $\alpha > 0$ and g is defined on $[0,b]$.

as $n \to \infty$. To this end, we consider

$$\begin{aligned}\left|s_n(f;0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\frac{\sin(n+\frac{1}{2})t}{t}\,\mathrm{d}t\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(t)\,\mathrm{d}t - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})t}{t}\,\mathrm{d}t\right| \\ &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{\sin\frac{t}{2}} - \frac{1}{\frac{t}{2}}\Big) f(t)\sin\Big(n+\frac{1}{2}\Big)t\,\mathrm{d}t\right| \\ &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)f(t)\sin\Big(n+\frac{1}{2}\Big)t\,\mathrm{d}t\right|, \end{aligned} \tag{5.109}$$

where $F : [0,\pi] \to \mathbb{R}$ is defined by

$$F(t) = \begin{cases} \frac{1}{\sin\frac{t}{2}} - \frac{1}{\frac{t}{2}}, & \text{if } t \in [-\pi,\pi]\setminus\{0\}; \\ 0, & \text{if } t = 0. \end{cases}$$

Clearly, F is continuous on $[-\pi,\pi]$ and so $Ff \in L^1(T)$. Using Lemma 5.2 to the right-hand side of the equation (5.109), we see that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)f(t)\sin\Big(n+\frac{1}{2}\Big)t\,\mathrm{d}t = 0$$

and this guarantees the validity of the claim (5.108).

For the general case, we consider the function $g : [-\pi,\pi] \to \mathbb{R}$ defined by

$$g(t) = f(x+t) - f(x) \tag{5.110}$$

for every $x \in \mathbb{R}$. Clearly, the real function g satisfies the conditions $g \in C(T)$, $g \in \operatorname{Lip}\alpha$ and $g(0) = 0$. By the above argument, we obtain

$$\lim_{n\to\infty} s_n(g;0) = 0. \tag{5.111}$$

By the definition (5.110), we gain

$$\begin{aligned} s_n(g;0) &= s_n\big(f(x+t) - f(x);0\big) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)D_n(-t)\,\mathrm{d}t - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_n(-t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)D_n(-t)\,\mathrm{d}t - f(x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,\mathrm{d}t - f(x) \\ &= s_n(f;x) - f(x). \end{aligned} \tag{5.112}$$

By applying the limit (5.111) to the equation (5.112), we obtain our desired result (5.106), completing the proof of the problem. ■

CHAPTER 6

Complex Measures

6.1 Properties of Complex Measures

Problem 6.1

Rudin Chapter 6 Exercise 1.

Proof. By the definition, we have

$$\lambda(E) = \sup\Big\{\sum_{k=1}^{n} |\mu(E_i)| \,\Big|\, E_1, E_2, \ldots, E_k \text{ are mutually disjoint and } E = \bigcup_{j=1}^{k} E_j\Big\}.$$

By Definition 6.1, we have

$$|\mu|(E) = \sup\Big\{\sum_{k=1}^{\infty} |\mu(E_i)| \,\Big|\, E_1, E_2, \ldots \text{ are mutually disjoint and } E = \bigcup_{j=1}^{\infty} E_j\Big\}.$$

Obviously, we have $\lambda(E) \leq |\mu|(E)$ for every $E \in \mathfrak{M}$.

For the other direction, we suppose that $\{E_i\}$ is a partition of $E \in \mathfrak{M}$. Given $\epsilon > 0$. Since μ is a complex measure, we get from Definition 6.1 that the series

$$\sum_{i=1}^{\infty} \mu(E_i)$$

converges absolutely. Thus there exists a $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |\mu(E_n)| < \epsilon. \tag{6.1}$$

Define $E_1' = E_1$, $E_2' = E_2, \ldots, E_{N-1}' = E_{N-1}$ and $E_N' = E_N \cup E_{N+1} \cup \cdots$. By Definition 1.3(a), we have $E_N' \in \mathfrak{M}$. Then we have

$$\bigcup_{i=1}^{\infty} E_i = E_1' \cup E_2' \cup \cdots \cup E_{N-1}' \cup E_N'$$

which implies, with the aid of the estimate (6.1), that

$$\sum_{i=1}^{\infty} |\mu(E_i)| = \sum_{i=1}^{N-1} |\mu(E_i')| + \sum_{i=N}^{\infty} |\mu(E_i)| < \sum_{i=1}^{N-1} |\mu(E_i')| + |\mu(E_N')| + \epsilon \leq \lambda(E) + \epsilon. \tag{6.2}$$

Since the inequality (6.2) holds for every partition of E, we have $|\mu|(E) \leq \lambda(E) + \epsilon$. Since ϵ is arbitrary, we obtain $|\mu|(E) \leq \lambda(E)$. Hence we conclude that $\lambda = |\mu|$ which completes the proof of the problem. ■

Problem 6.2

Rudin Chapter 6 Exercise 2.

Proof. Let μ be Lebesgue measure on $(0,1)$ and λ be the counting measure on the σ-algebra $\mathfrak{M}$ of all Lebesgue measurable sets in $(0,1)$. Assume that λ was σ-finite. Then we have

$$(0,1) = \bigcup_{n=1}^{\infty} E_n, \tag{6.3}$$

where $\lambda(E_n) < \infty$ for each $n \in \mathbb{N}$. Clearly, we have $\lambda(E) < \infty$ if and only if E is finite. Therefore, the representation (6.3) implies that $(0,1)$ is countable, a contradiction. Hence λ is not σ-finite.

Next, we check $\mu \ll \lambda$. Suppose that $\lambda(E) = 0$, where $E \in \mathfrak{M}$. By the definition of λ, we know that $E = \varnothing$ which implies definitely that $\mu(E) = \mu(\varnothing) = 0$. In addition, the definition of μ ensures that $\mu\big((0,1)\big) = 1$, i.e., μ is a bounded measure.

Assume that there was a $h \in L^1(\lambda)$ such that $\mathrm{d}\mu = h\,\mathrm{d}\lambda$, i.e.,

$$\mu(E) = \int_E h\,\mathrm{d}\lambda \tag{6.4}$$

for every $E \in \mathfrak{M}$. Note that $\mu(E) \geq 0$, so $h(x) \geq 0$ a.e. $[\lambda]$ on E.[a] Particulary, if $E = (0,1)$, then $h(x) \geq 0$ a.e. $[\lambda]$ on $(0,1)$. Suppose that $E_n = \{x \in (0,1) \,|\, h(x) \geq \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Since $h \in L^1(\lambda)$, Proposition 1.24(a) implies that

$$\infty > \int_{E_n} h\,\mathrm{d}\lambda \geq \frac{1}{n}\lambda(E_n) \geq 0.$$

Thus we must have $\lambda(E_n) \in [0,\infty)$ or equivalently, E_n is finite or $E_n = \varnothing$. Let

$$E = \bigcup_{n=1}^{\infty} E_n = \{x \in (0,1) \,|\, h(x) > 0\}. \tag{6.5}$$

Therefore, the set E is countable. There are two cases for consideration:

- **Case (i): $E \neq \varnothing$.** Let $E = \{x_1, x_2, \ldots\}$. On the one hand, we know from [49, Remark 11.11(f), p. 309] that $\mu(F) = 0$ *for every* countable subset $F \subset (0,1)$, so we deduce from the representation (6.4) and the definition (6.5) that

$$0 = \mu(E) = \int_E h\,\mathrm{d}\lambda = \sum_{n=1}^{\infty} h(x_n) \neq 0,$$

  a contradiction.

- **Case (ii): $E = \varnothing$.** In this case, we have $h(x) = 0$ a.e. $[\lambda]$ on $(0,1)$ and the representation (6.4) again shows that

$$1 = \mu\big((0,1)\big) = \int_{(0,1)} h\,\mathrm{d}\lambda = 0,$$

  a contradiction.

[a] See Definition 1.35 for the meaning of the notation a.e. $[\lambda]$.

Hence *no* such h exists and this ends the proof of the problem. ■

Problem 6.3

Rudin Chapter 6 Exercise 3.

Proof. We first show that if μ and λ are complex regular Borel measures, then both $\mu+\lambda$ and $\alpha\mu$ are complex regular Borel measures too. By §6.18, it is equivalent to show that both $|\mu+\nu|$ and $|\alpha\mu|$ are regular Borel measures. To this end, we follow from §6.18 again that both $|\mu|$ and $|\lambda|$ are regular Borel measures. Let $E \in \mathscr{B}$ and $\epsilon > 0$. By Definition 2.15, there exist open sets $V_1, V_2 \supseteq E$ such that

$$|\mu|(V_1) < |\mu|(E) + \frac{\epsilon}{2} \quad \text{and} \quad |\lambda|(V_2) < |\lambda|(E) + \frac{\epsilon}{2}. \tag{6.6}$$

Similarly, there are compact sets $K_1, K_2 \subseteq E$ such that

$$|\mu|(E) < |\mu|(K_1) + \frac{\epsilon}{2} \quad \text{and} \quad |\lambda|(E) < |\lambda|(K_2) + \frac{\epsilon}{2}. \tag{6.7}$$

The triangle inequality certainly implies $|\mu(E)+\lambda(E)| \leq |\mu(E)|+|\lambda(E)|$ and then Definition 6.1 gives

$$|\mu+\lambda|(E) = \sup \sum_{i=1}^{\infty} |\mu(E)+\lambda(E)| \leq \sup \sum_{i=1}^{\infty} \big[|\mu(E)| + |\lambda(E)|\big], \tag{6.8}$$

where the supremum being taken over all partitions $\{E_i\}$ of E. Since the series $\sum_{i=1}^{\infty} |\mu(E)|$ and $\sum_{i=1}^{\infty} |\lambda(E)|$ converge, we deduce from the expression (6.8) that

$$|\mu+\lambda|(E) \leq \sup \sum_{i=1}^{\infty} |\mu(E)| + \sup \sum_{i=1}^{\infty} |\lambda(E)| = |\mu|(E) + |\lambda|(E).$$

Thus we have

$$|\mu+\lambda| \leq |\mu| + |\lambda|. \tag{6.9}$$

Let $V = V_1 \cap V_2$ and $K = K_1 \cup K_2$. Then K is compact and V is open in X. Now we follow from the estimates (6.6) and (6.7) and the inequality (6.9) that

$$\begin{aligned}
|\mu+\lambda|(V) &= |\mu+\lambda|(E) + |\mu+\lambda|(V \setminus E) \\
&\leq |\mu+\lambda|(E) + |\mu|(V \setminus E) + |\lambda|(V \setminus E) \\
&\leq |\mu+\lambda|(E) + |\mu|(V_1 \setminus E) + |\lambda|(V_2 \setminus E) \\
&< |\mu+\lambda|(E) + \epsilon.
\end{aligned}$$

Since ϵ and E are arbitrary, the measure $|\mu+\lambda|(E)$ is in fact outer regular. Similarly, we have

$$\begin{aligned}
|\mu+\lambda|(K) &= |\mu+\lambda|(E) - |\mu+\lambda|(E \setminus K) \\
&\geq |\mu+\lambda|(E) - |\mu+\lambda|(E \setminus K_1) - |\mu+\lambda|(E \setminus K_2) \\
&> |\mu+\lambda|(E) - \epsilon.
\end{aligned}$$

In other words, $|\mu+\lambda|$ is also inner regular. By Definition 2.15, $\mu+\lambda$ is a regular complex Borel measure, i.e., $\mu+\lambda \in M(X)$. Now the regularity of $|\alpha\mu|$ is easy to prove, so we omit the details here.

Now it is time to prove the assertion in the question. Since X is a locally compact Hausdorff space, Theorem 6.19 (The Riesz Representation Theorem) ensures every $\Phi \in C_0(X)^*$ is represented by a unique $\mu_\Phi \in M(X)$ in the sense that

$$\Phi(f) = \int_X f \,\mathrm{d}\mu_\Phi \tag{6.10}$$

for every $f \in C_0(X)$. By [51, Eqn. (3), p. 130], we see that

$$\int_X f \,\mathrm{d}\mu_{\Phi+\Psi} = (\Phi+\Psi)(f) = \Phi(f) + \Psi(f) = \int_X f \,\mathrm{d}\mu_\Phi + \int_X f \,\mathrm{d}\mu_\Psi = \int_X f \,\mathrm{d}(\mu_\Phi + \mu_\Psi)$$

and

$$\int_X f \,\mathrm{d}\mu_{\alpha\Phi} = \alpha\Phi(f) = \int_X f \,\mathrm{d}(\alpha\mu_\Phi).$$

Therefore, they imply that

$$\mu_{\Phi+\Psi} = \mu_\Phi + \mu_\Psi \quad \text{and} \quad \mu_{\alpha\Phi} = \alpha\mu_\Phi. \tag{6.11}$$

By the previous analysis, we see that $\mu_{\Phi+\Psi}, \mu_{\alpha\mu} \in M(X)$, so we may define the mapping $F : C_0(X)^* \to M(X)$ by

$$F(\Phi) = \mu_\Phi$$

and it is easy to see from the results (6.11) that

$$F(\Phi+\Psi) = F(\Phi) + F(\Psi) \quad \text{and} \quad F(\alpha\Phi) = \alpha F(\Phi).$$

Furthermore, Theorem 6.19 (The Riesz Representation Theorem) also implies that F is a bijection and $\|\Phi\| = |\mu_\Phi|(X) = \|\mu_\Phi\| = \|F(\Phi)\|$. Consequently, F is actually an isometric vector space isomorphism, i.e.,

$$C_0(X)^* \cong M(X).$$

Since $C_0(X)$ is a Banach space with the supremum norm, Problem 5.8 guarantees that $C_0(X)^*$ is Banach. Hence $M(X)$ is Banach and we have completed the proof of the problem. ■

6.2 Dual Spaces of $L^p(\mu)$

Problem 6.4

Rudin Chapter 6 Exercise 4.

Proof. Since μ is positive and σ-finite, we can write

$$X = \bigcup_{n=1}^{\infty} X_n, \tag{6.12}$$

where $\{X_n\}$ is an increasing sequence of measurable sets and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$, see [54, Definition 4.22, p. 22]. Let $A = \{x \in X \,|\, |g(x)| = \infty\}$ and $A_n = \{x \in X_n \,|\, |g(x)| = \infty\}$, where $n \in \mathbb{N}$. Since g is measurable, every A_n and A are measurable by Problem 1.5. Besides, it is evident that $A_1 \subseteq A_2 \subseteq \cdots$ and if $x \in A$, then $x \in X_{N_0}$ for some $N_0 \in \mathbb{N}$ so that $x \in A_{N_0}$. As a result, we get

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Now we are going to divide the proof into several steps:

- **Step 1: $\mu(A) = 0$.** Otherwise, it follows from the construction and Theorem 1.19(c) that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) > 0.$$

This means that one can find a $N_1 \in \mathbb{N}$ such that $0 < \mu(A_{N_1}) \leq \mu(X_{N_1}) < \infty$. We know that $\chi_{A_{N_1}} \in L^p(\mu)$, so our hypothesis implies that $\chi_{A_{N_1}} g \in L^1(\mu)$, but

$$\|\chi_{A_{N_1}} g\|_1 = \int_X |\chi_{A_{N_1}} g| \, \mathrm{d}\mu = \int_{A_{N_1}} |g| \, \mathrm{d}\mu = \mu(A_{N_1}) \cdot \infty = \infty$$

which is a contradiction. Hence $\mu(A) = 0$, i.e., g is finite a.e. on X.

- **Step 2: $g \in L^q(\mu)$ when $1 < p < \infty$.** In this case, we have $q \in (1, \infty)$. Here we define

$$E_n = \{x \in X_n \,|\, |g(x)| \leq n\}.$$

Obviously, $\{E_n\}$ is also an increasing sequence of measurable sets. Furthermore, if $x_0 \in X \setminus A$, then $x_0 \in X_{N_2}$ for some $N_2 \in \mathbb{N}$ by the definition (6.12) so that $x_0 \in X_n$ for all $n \geq N_2$. Since g is finite a.e. on X, there exists a positive integer N_3 such that $|g(x)| \leq N_3$ for almost all $x \in X$. Take $N_4 = \max(N_2, N_3)$, then $x_0 \in X_{N_4}$ and $|g(x_0)| \leq N_4$ which imply that $x_0 \in E_{N_4}$. Consequently, we have shown that

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Let $g_n = \chi_{E_n} g : X \to \mathbb{C}$, where $n = 1, 2, \ldots$. Then each g_n is measurable on X and

$$\|g_n\|_q^q = \int_X |g_n|^q \, \mathrm{d}\mu = \int_{E_n} |g|^q \, \mathrm{d}\mu \leq n^q \mu(E_n) \leq n^q \mu(X_n) < \infty \tag{6.13}$$

for each $n = 1, 2, \ldots$. Thus we have $g_n \in L^q(\mu)$ for all $n \in \mathbb{N}$. Next, we define $\Lambda_n : L^p(\mu) \to \mathbb{C}$ by

$$\Lambda_n(f) = \int_X f g_n \, \mathrm{d}\mu$$

which is linear for $n \in \mathbb{N}$. Since $fg_n = (f\chi_{E_n})g$ and $f\chi_{E_n} \in L^p(\mu)$, we have $fg_n \in L^1(\mu)$. By Theorem 1.33 and Theorem 3.8, we obtain

$$\left| \int_X f g_n \, \mathrm{d}\mu \right| \leq \int_X |f g_n| \, \mathrm{d}\mu = \|f g_n\|_1 \leq \|f\|_p \times \|g_n\|_q, \tag{6.14}$$

where $n = 1, 2, \ldots$. By Definition 5.3 and the inequality (6.14), we know that

$$\|\Lambda_n\| = \sup\{|\Lambda_n(f)| \,|\, f \in L^p(\mu) \text{ and } \|f\|_p = 1\} \leq \|g_n\|_q, \tag{6.15}$$

where $n = 1, 2, \ldots$. Recall that $1 < q < \infty$, so we may consider

$$f_0 = \|g_n\|_q^{-\frac{q}{p}} |g_n|^{q-2} \overline{g_n}.$$

Then we have $|f_0|^p = \|g_n\|_q^{-q} |g_n|^{(q-1)p}$ so that

$$\|f_0\|_p = \int_X |f_0|^p \, \mathrm{d}\mu = \|g_n\|_q^{-q} \int_X |g_n|^{(q-1)p} \, \mathrm{d}\mu = \|g_n\|_q^{-q} \int_X |g_n|^q \, \mathrm{d}\mu = 1$$

and

$$\Lambda_n(f_0) = \int_X \|g_n\|_q^{-\frac{q}{p}} |g_n|^{q-2} \overline{g_n} \cdot g_n \, \mathrm{d}\mu$$

$$
\begin{aligned}
&= \|g_n\|_q^{-\frac{q}{p}} \int_X |g_n|^q \, \mathrm{d}\mu \\
&= \|g_n\|_q^{-\frac{q}{p}} \times \|g_n\|_q^q \\
&= \|g_n\|_q \qquad (6.16)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Combining the results (6.13), (6.15) and (6.16), we have established the fact that

$$\|\Lambda_n\| = \|g_n\|_q < \infty, \tag{6.17}$$

where $n \in \mathbb{N}$. Hence $\{\Lambda_n\}$ is a family of bounded linear transformations of $L^p(\mu)$ into $\mathbb{C}$.

Since $L^p(\mu)$ is Banach (see Definition 5.2) and

$$|\Lambda_n(f)| = \Big| \int_X fg_n \, \mathrm{d}\mu \Big| \leq \int_X |fg_n| \, \mathrm{d}\mu = \int_{E_n} |fg| \, \mathrm{d}\mu \leq \int_X |fg| \, \mathrm{d}\mu = \|fg\|_1 < \infty$$

for every $f \in L^p(\mu)$ and $n \in \mathbb{N}$, Theorem 5.8 (The Banach-Steinhaus Theorem) implies that there is a $M > 0$ such that

$$\|\Lambda_n\| \leq M \tag{6.18}$$

for all $n \in \mathbb{N}$. Using the results (6.17) and (6.18), we conclude that

$$\|g_n\|_q \leq M \tag{6.19}$$

for all $n \in \mathbb{N}$.

By **Step 1** and the definition of g_n, we have $|g_n(x)| \leq |g(x)| < \infty$ a.e. on X for all $n \in \mathbb{N}$ and

$$|g_1(x)| \leq |g_2(x)| \leq \cdots$$

for every $x \in X$. Now the Monotone Convergence Theorem [49, Theorem 3.14, p. 55] implies that $|g_n(x)| \to |g(x)| < \infty$ a.e. on X as $n \to \infty$. Hence we gain from Theorem 1.26 (Lebesgue's Monotone Convergence Theorem) that

$$\lim_{n\to\infty} \|g_n\|_q^q = \lim_{n\to\infty} \int_X |g_n|^q \, \mathrm{d}\mu = \int_X |g|^q \, \mathrm{d}\mu = \|g\|_q^q. \tag{6.20}$$

Hence it follows from the inequality (6.19) and the limit (6.20) that

$$\|g\|_q \leq M < \infty,$$

i.e. $g \in L^q(\mu)$.

- **Step 3: $g \in L^\infty(\mu)$ when $p = 1$.** In this case, $q = \infty$. We recall from **Step 1** that g is finite a.e. on X, so there exists a $M > 0$ such that $|g(x)| \leq M$ a.e. on X. By Definition 3.7, we assert that $\|g\|_\infty \leq M$, i.e., $g \in L^\infty(\mu)$.

- **Step 4: $g \in L^1(\mu)$ when $p = \infty$.** Define $f_1 : X \to \mathbb{C}$ by

$$f_1(x) = \begin{cases} \dfrac{\overline{g(x)}}{|g(x)|}, & \text{if } g(x) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $|f_1(x)| \leq 1$ for all $x \in X$, we get from Definition 3.7 that $\|f_1\|_\infty \leq 1$, i.e., $f_1 \in L^\infty(\mu)$. By the hypothesis, we have $f_1 g \in L^1(\mu)$. Since $f_1 g = |g|$, we conclude that $g \in L^1(\mu)$.

Now we have completed the proof of the problem. ■

Problem 6.5

Rudin Chapter 6 Exercise 5.

Proof. The answer is negative. On the one hand, if $f_1, f_2 : X \to \mathbb{C}$ are defined by

$$f_1(a) = 0, \quad f_1(b) = 1 \quad \text{and} \quad f_2(a) = 1, \quad f_2(b) = 0, \tag{6.21}$$

then we have $f_1, f_2 \in L^\infty(\mu)$ and $f_1 \not\equiv f_2$. For every $f \in L^\infty(\mu)$, we have $f(a) = A + Bi$ and $f(b) = C + Di$ for some $A, B, C, D \in \mathbb{R}$. Obviously, we get from the definition (6.21) the representation

$$f = (C + Di)f_1 + (A + Bi)f_2.$$

As a result, $L^\infty(\mu)$ is a two-dimensional space spanned by f_1 and f_2, i.e., $\dim L^\infty(\mu) = 2$.

On the other hand, if $f \in L^1(\mu)$, then $\|f\|_1 < \infty$ and we follow from Definition 3.6 that $f(b) = 0$ and $\|f\|_1 = |f(a)| = |f(a)|f_2(a)$. Therefore, $L^1(\mu)$ is an one-dimensional space spanned by f_2. As a vector space (see Remark 3.10), we have

$$\dim L^1(\mu)^* = \dim L^1(\mu) = 1.$$

Hence, $L^\infty(\mu) \neq L^1(\mu)^*$, completing the proof of the problem. ■

Problem 6.6

Rudin Chapter 6 Exercise 6.

Proof. We want to show that $L^p(\mu)^* \cong L^q(\mu)$ for $1 < p < \infty$. Equivalently, we have to show that for each $\Lambda \in L^p(\mu)^*$, there exists a unique $g \in L^q(\mu)$ such that

$$\Lambda f = \int_X fg \, d\mu \tag{6.22}$$

for all $f \in L^p(\mu)$. The following proof follows mainly Follan's argument ([22, p. 190]):

- **Step 1: μ is finite.** Let s be a simple function on X. Since μ is finite, we have $\mu(\{x \in X \mid s(x) \neq 0\}) \leq \mu(X) < \infty$ and then it follows from Theorem 3.13 that $s \in L^p(\mu)$. Let $\Lambda \in L^p(\mu)^*$, E be measurable and $\lambda(E) = \Lambda(\chi_E)$. For every partition $\{E_i\}$ of E, if we let $F_n = E_1 \cup E_2 \cup \cdots \cup E_n$, then we obtain

$$\int_X |\chi_E - \chi_{F_n}|^p \, d\mu = \int_X |\chi_{E \setminus F_n}|^p \, d\mu = \int_X \chi_{E \setminus F_n} \, d\mu = \mu(E \setminus F_n) \tag{6.23}$$

for all $n \in \mathbb{N}$. In fact, the expression (6.23) can be rewritten as

$$\|\chi_E - \chi_{F_n}\|_p = \mu(E \setminus F_n)^{\frac{1}{p}}. \tag{6.24}$$

Since $E \setminus F_1 \supseteq E \setminus F_2 \supseteq \cdots$ and $\mu(E \setminus F_1)$ is finite, it establishes from Theorem 1.19(e) that

$$\lim_{n\to\infty} \mu(E \setminus F_n) = \mu\Big(\bigcap_{n=1}^{\infty} (E \setminus F_n)\Big) = \mu(\varnothing) = 0. \tag{6.25}$$

Combining the results (6.24) and (6.25) and using the fact $p \in (1, \infty)$, we derive that

$$\lim_{n\to\infty} \|\chi_{F_n} - \chi_E\|_p = 0.$$

Since $\chi_E \in L^p(\mu)$ and each $\chi_{E_i} g$ is measurable, we deduce from the representation (6.22) and then Theorem 1.27 that

$$\lambda(E) = \Lambda(\chi_E) = \int_X \chi_E g \,\mathrm{d}\mu = \int_X \sum_{i=1}^{\infty} (\chi_{E_i} g) \,\mathrm{d}\mu = \sum_{i=1}^{\infty} \int_X \chi_{E_i} g \,\mathrm{d}\mu = \sum_{i=1}^{\infty} \lambda(\chi_{E_i}).$$

By Definition 6.1, λ is a complex measure.

If $E \in \mathfrak{M}$ satisfies $\mu(E) = 0$, then $\chi_E = 0$ in $L^p(\mu)$ and so $\lambda(E) = \Lambda(0) = 0$. By Definition 6.7, we have $\lambda \ll \mu$. Thus we know from Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem) that there exists a unique $h \in L^1(\mu)$ such that

$$\Lambda(\chi_E) = \lambda(E) = \int_E g \,\mathrm{d}\mu = \int_X \chi_E g \,\mathrm{d}\mu$$

for every $E \in \mathfrak{M}$. Recall that $\Lambda \in L^p(\mu)^*$, Λ is linear and bounded so that

$$\Lambda(s) = \int_X sg \,\mathrm{d}\mu$$

for every simple function $s \in L^p(\mu)$. By [51, Eqn. (3), p. 96], we know that

$$\Big| \int_X sg \,\mathrm{d}\mu \Big| = \|\Lambda(s)\| \leq \|\Lambda\| \cdot \|s\|_p < \infty.$$

By [22, Theorem 6.14, p. 189], we conclude that $g \in L^q(\mu)$. Given that $f \in L^p(\mu)$. By Theorem 3.8, we have $fg \in L^1(\mu)$. By Theorem 3.13, there exists a sequence of simple functions $\{s_n\}$ such that $|s_n| \leq |f|$ for every $n \in \mathbb{N}$ and $s_n \to f$ in $L^p(\mu)$ as $n \to \infty$. Since $|s_n g| \leq |fg|$ and $fg \in L^1(\mu)$, we obtain from Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) and Theorem 5.4 that

$$\Lambda(f) = \lim_{n\to\infty} \Lambda(s_n) = \lim_{n\to\infty} \int_X s_n g \,\mathrm{d}\mu = \int_X fg \,\mathrm{d}\mu.$$

- **Step 2: μ is σ-finite.** This is exactly the hypothesis of Theorem 6.16, so we have $L^p(\mu)^* \cong L^q(\mu)$ in this case.

- **Step 3: μ is any measure and $p > 1$.** In this case, $q \in (1, \infty)$. As in **Step 2**, for each σ-finite subset E of X, there corresponds a *unique* $g_E \in L^q(E)$ such that

$$\Lambda(f) = \int_X f g_E \,\mathrm{d}\mu$$

for all $f \in L^p(E)$, where $L^p(E) = \{f \in L^p(\mu) \,|\, f(x) = 0 \text{ for all } x \notin E\}$ and $L^q(E)$ is defined similarly. By Theorem 6.16, we know that

$$\|g_E\|_q = \big\|\Lambda|_{L^p(E)}\big\| \leq \|\Lambda\|. \tag{6.26}$$

If F is a σ-finite subset of X containing E, then the uniqueness of g_E implies that $g_F = g_E$ a.e. on E so that

$$\|g_E\|_q \leq \|g_F\|_q. \tag{6.27}$$

Now we define

$$M = \sup\{\|g_E\|_q \,|\, E \text{ a } \sigma\text{-finite subset of } X\}.$$

Then it follows from the inequality (6.26) that

$$M \leq \|\Lambda\|$$

holds. By the definition, we may select a sequence $\{E_n\}$ of σ-finite subsets of X such that $\|g_{E_n}\|_q \to M$ as $n \to \infty$. Define $F = \bigcup_{n=1}^{\infty} E_n$ which is obviously σ-finite subset of X so that $\|g_F\|_q \leq M$. Furthermore, we deduce from the inequality (6.27) that

$$\|g_{E_n}\|_q \leq \|g_F\|_q$$

for all $n \in \mathbb{N}$. Therefore, the definition of M implies that $M \leq \|g_F\|_q$ and then

$$M = \|g_F\|_q.$$

Finally, if A is a σ-finite subset of X containing F, we get

$$\int_X |g_F|^q \,\mathrm{d}\mu + \int_X |g_{A\setminus F}|^q \,\mathrm{d}\mu = \int_X |g_A|^q \,\mathrm{d}\mu = \|g_A\|_q^q \leq M^q = \int_X |g_F|^q \,\mathrm{d}\mu$$

which means that $g_{A\setminus F} = 0$ a.e. on X or equivalently,

$$g_A = g_F \text{ a.e. on } X. \tag{6.28}$$

If $f \in L^p(\mu)$, then the set $A = F \cup \{x \in X \mid f(x) \neq 0\}$ is clearly σ-finite. Therefore, we observe from the result (6.28) that

$$\Lambda(f) = \int_X f g_A \,\mathrm{d}\mu = \int_X f g_F \,\mathrm{d}\mu.$$

Hence we may pick $g = g_F$ and the above argument makes sure this g is *unique*.

We complete the proof of the problem. ■

6.3 Fourier Coefficients of Complex Borel Measures

Problem 6.7

Rudin Chapter 6 Exercise 7.

Proof. If μ is a real measure, then we have

$$\widehat{\mu}(-n) = \int e^{int} \,\mathrm{d}\mu(t) = \overline{\int e^{-int} \,\mathrm{d}\mu(t)} = \overline{\widehat{\mu}(n)}$$

for every $n \in \mathbb{Z}$. Therefore, our result follows in this case.

Since μ is a complex Borel measure, Theorem 6.12 implies the existence of a Borel measurable function h such that $|h(t)| = 1$ on $[0, 2\pi)$ and $\mathrm{d}\mu = h \,\mathrm{d}|\mu|$. Thus we have

$$\mathrm{d}|\mu| = 1 \cdot \mathrm{d}|\mu| = \overline{h} \cdot h \,\mathrm{d}|\mu| = \overline{h} \,\mathrm{d}\mu. \tag{6.29}$$

Since $|\mu|$ is real, the previous paragraph and the expression (6.29) imply that

$$\overline{\widehat{\mu}(-n)} = \overline{\int e^{int} h \,\mathrm{d}|\mu|} = \int e^{-int} \overline{h(t)} \,\mathrm{d}|\mu|(t) = \int e^{-int} [\overline{h(t)}]^2 \,\mathrm{d}\mu(t)$$

for every $n \in \mathbb{Z}$. By Theorem 3.14, $C(T)$ is dense in $L^1(|\mu|)$. Using Theorem 4.25 (The Weierstrass Approximation Theorem) and then Lemma 2.10, we deduce that the set of all trigonometric polynomials $\mathcal{P}$ is dense in $L^1(|\mu|)$.

We want to apply Problem 5.18. For any $f \in L^1(|\mu|)$, we define $\Lambda_n : L^1(|\mu|) \to \mathbb{C}$ by

$$\Lambda_n(f) = \int f(t)\mathrm{e}^{-int}\,\mathrm{d}\mu(t)$$

for all $n \in \mathbb{N}$. Then we have

$$|\Lambda_n(f)| \leq \int |f| \cdot |\,\mathrm{d}\mu(t)| = \int |f| \cdot |h|\,\mathrm{d}|\mu(t)| = \int |f|\,\mathrm{d}|\mu(t)| = \|f\|_1. \tag{6.30}$$

By Definition 5.3, we see from the inequality (6.30) that

$$\|\Lambda_n\| = \sup\{|\Lambda_n(f)| \mid f \in L^1(|\mu|) \text{ and } \|f\|_1 = 1\} \leq 1$$

for every $n \in \mathbb{N}$. Next, if $g(t) = \mathrm{e}^{ikt}$, then we have

$$\Lambda_n(g) = \int g(t)\mathrm{e}^{-int}\,\mathrm{d}\mu(t) = \int \mathrm{e}^{-i(n-k)t}\,\mathrm{d}\mu(t).$$

Now our hypothesis ensures that $\Lambda_n(g) \to 0$ as $n \to \infty$ so that

$$\Lambda_n(f) \to 0 \tag{6.31}$$

as $n \to \infty$ for each $f \in \mathcal{P}$. Therefore, Problem 5.18 says that the limit (6.31) also holds for every $f \in L^1(|\mu|)$. Obviously, we have $\overline{h}^2 \in L^1(|\mu|)$ and so

$$\overline{\widehat{\mu}(-n)} = \int \mathrm{e}^{-int}[\overline{h}(t)]^2\,\mathrm{d}\mu(t) = \Lambda_n(\overline{h}^2).$$

Hence we conclude from the result (6.31) that

$$\lim_{n\to\infty} \widehat{\mu}(-n) = \lim_{n\to\infty} \Lambda_n(\overline{h}^2) = 0.$$

This completes the proof of the problem. ■

Problem 6.8

Rudin Chapter 6 Exercise 8.

Proof. Suppose that $X = [0, 2\pi)$, $\mathscr{B}$ is the collection of all Borel sets in X and $\widehat{\mu}$ is periodic with period $k \in \mathbb{Z}$. Denote $\mathrm{d}\lambda(t) = (\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu(t)$. Clearly, we have

$$\widehat{\mu}(n+k) - \widehat{\mu}(n) = \int \mathrm{e}^{-int}(\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu(t) = \int \mathrm{e}^{-int}\,\mathrm{d}\lambda(t) = \widehat{\lambda}(n) \tag{6.32}$$

for all $n \in \mathbb{Z}$. Particularly, take $n = 0$ in the result (6.32) and use the periodicity of $\widehat{\mu}$ and Theorem 6.12 to get

$$\int h\,\mathrm{d}|\lambda| = \int \mathrm{d}\lambda = 0, \tag{6.33}$$

where h is a measurable function such that $|h(x)| = 1$ on X. If $|\lambda|(X) \neq 0$, then Theorem 1.39(a) and the form (6.33) together imply that $h = 0$ a.e. on X, a contradiction. This means that $|\lambda|(X) = 0$ and we assert from the fact $|\lambda(E)| \leq |\lambda|(X) = 0$ that

$$\lambda(E) = 0 \tag{6.34}$$

for every $E \in \mathscr{B}$.

Using Theorem 6.12 to write $\mathrm{d}\mu = H\,\mathrm{d}|\mu|$, where H is a measurable function with $|H(x)| = 1$ on X. Next, we recall the meaning of the notation $\mathrm{d}\lambda(t) = (\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu(t)$ and we observe from the result (6.34) that

$$0 = \lambda(E) = \int_E (\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu = \int_E (\mathrm{e}^{-ikt} - 1)H\,\mathrm{d}|\mu| \tag{6.35}$$

for every $E \in \mathscr{B}$. Let $A_1 = \{t \in X \mid \cos kt - 1 = 0\}$, $A_2 = \{t \in X \mid \sin kt = 0\}$ and

$$A = \{t \in X \,|\, \mathrm{e}^{-ikt} - 1 = 0\} = A_1 \cap A_2.$$

Since $f_1(t) = \cos kt - 1$, $f_2(t) = \sin kt$ and $g(t) = 0$ are continuous on X, they are Borel measurable.[b] By Problem 1.5(a), we have $A_1, A_2 \in \mathscr{B}$ and also $A \in \mathscr{B}$.

We claim that μ is concentrated on A. Assume that it was not the case, i.e., there is $E_0 \in \mathscr{B}$ such that a $E_0 \cap A = \varnothing$ but $\mu(E_0) \neq 0$. We recall from the result (6.35) that $(\mathrm{e}^{-ikt} - 1)H = 0$ a.e. on *any* E with $|\mu|(E) > 0$. Since $|H| = 1$ on X, we have $\mathrm{e}^{-ikt} = 1$ a.e. on any E with $|\mu|(E) > 0$. Particularly, since $\mu(E_0) \neq 0$, we have $|\mu|(E_0) > 0$ and then

$$\mathrm{e}^{-ikt} = 1 \tag{6.36}$$

a.e. on E_0. However, $E_0 \cap A = 0 = \varnothing$ means that $\mathrm{e}^{-ikt} \neq 1$ on E_0 which contradicts the expression (6.36). Hence we have proven our claim that μ is concentrated on A.

On the other hand, if μ is concentrated on A, then for every $E \in \mathscr{B}$, we write $E = (E \setminus A) \cup A$ and then

$$\int_E (\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu = \int_{E\setminus A} (\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu + \int_A (\mathrm{e}^{-ikt} - 1)\,\mathrm{d}\mu. \tag{6.37}$$

Notice that $\mathrm{e}^{-ikt} - 1 = 0$ on A and $(E \setminus A) \cap A = \varnothing$ so that $\mu(E \setminus A) = 0$. Thus we deduce from Proposition 1.24(d) and (e) that all the integrals in the expression (6.37) are zero. This certainly implies that $\lambda(E) = 0$ for every $E \in \mathscr{B}$ and then

$$\widehat{\lambda}(n) = \int \mathrm{e}^{-int}\,\mathrm{d}\lambda = 0 \tag{6.38}$$

for every $n \in \mathbb{Z}$. Combining the expression (6.32) and the result (6.38), we may conclude that

$$\widehat{\mu}(n + k) = \widehat{\mu}(n)$$

for all $n \in \mathbb{Z}$.

Hence we have shown that μ is a complex Borel measure with periodic Fourier coefficient $\widehat{\mu}$ with period k if and only if μ is concentrated on

$$A = \{t \in X \,|\, \mathrm{e}^{-ikt} - 1 = 0\} = \Big\{t = \frac{2n\pi}{k} \,\Big|\, n \in \mathbb{N} \cup \{0\}\Big\},$$

completing the proof of the problem. ∎

Problem 6.9

Rudin Chapter 6 Exercise 9.

[b] See §1.11.

Proof. The assertion is false. Take $\mu = m$ the Lebesgue measure on I. Since $m \ll m$, if $m \perp m$, then Proposition 6.8(g) implies that $m = 0$, a contradiction. Now it remains to construct a sequence of $\{g_n\}$ satisfying the required properties.

Suppose that $n, k \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Let $\delta_n = \frac{2}{n(2n+3)}$. Define $g_{n,k} : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ by

$$g_{n,k}(x) = \begin{cases} n+1, & \text{if } x \in \Big[\dfrac{k}{n+1} - \dfrac{\delta_n}{2}, \dfrac{k}{n+1} + \dfrac{\delta_n}{2}\Big]; \\ \text{linear}, & \text{if } x \in \Big(\dfrac{k}{n+1} - \dfrac{\delta_n}{2} - \dfrac{\delta_n}{2n+2}, \dfrac{k}{n+1} - \dfrac{\delta_n}{2}\Big); \\ \text{linear}, & \text{if } x \in \Big(\dfrac{k}{n+1} + \dfrac{\delta_n}{2}, \dfrac{k}{n+1} + \dfrac{\delta_n}{2} + \dfrac{\delta_n}{2n+2}\Big); \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_n(x) = \sum_{k=1}^{n} g_{n,k}(x)$$

respectively. By direct computation, it is easy to check that

$$0 < \frac{1}{n+1} - \frac{\delta_n}{2} - \frac{\delta_n}{2n+2}, \quad \frac{n}{n+1} + \frac{\delta_n}{2} + \frac{\delta_n}{2n+2} < 1$$

and

$$\frac{k}{n+1} + \frac{\delta_n}{2} + \frac{\delta_n}{2n+2} < \frac{k+1}{n+1} - \frac{\delta_n}{2} - \frac{\delta_n}{2n+2}$$

for $k = 1, 2, \ldots, n-1$. See Figure 6.1 for $g_{n,1}(x)$ and $g_{n,2}(x)$ below:

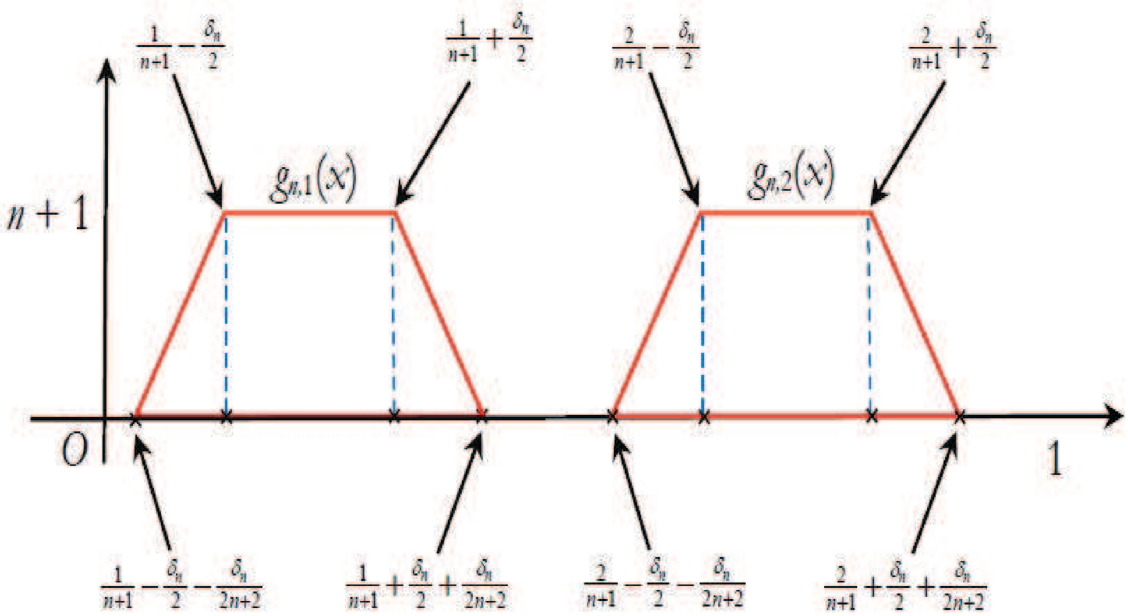


Figure 6.1: The graphs of $g_{n,1}(x)$ and $g_{n,2}(x)$.

Now the Lebesgue measure of the support of each $g_{n,k}$ is given by

$$\frac{k}{n+1} + \frac{\delta_n}{2} + \frac{\delta_n}{2n+2} - \frac{k}{n+1} + \frac{\delta_n}{2} + \frac{\delta_n}{2n+2} = \Big(1 + \frac{1}{n+1}\Big)\delta_n = \frac{2(n+2)}{n(n+1)(2n+3)}$$

so that the Lebesgue measure of the support of g_n is given by

$$\sum_{k=1}^{n}\frac{2(n+2)}{n(n+1)(2n+3)}=\frac{2(n+2)}{(n+1)(2n+3)}\to 0$$

as $n\to\infty$. This shows that condition (i) holds for this g_n. Since each $g_{n,k}$ is in fact a trapezium (see Figure 6.1 again) with area

$$\int_0^1 g_{n,k}(x)\,\mathrm{d}x=\frac{(n+1)}{2}\Big(\frac{n+2}{n+1}\delta_n+\delta_n\Big)=\frac{2n+3}{2}\delta_n=\frac{1}{n}$$

which implies that

$$\int_0^1 g_n(x)\,\mathrm{d}x=\sum_{k=1}^{n}\int_0^1 g_{n,k}(x)\,\mathrm{d}x=1.$$

Thus this g_n satisfies condition (ii). Finally, note that for large n, we have

$$\frac{1}{(n+1)^2\delta_n}=\frac{n(2n+3)}{2(n+1)^2}\approx 1.$$

Thus for every $f\in C(I)$ and sufficiently large n, we have

$$\begin{aligned}\Big|\int_0^1 fg_n\,\mathrm{d}x-\frac{1}{n+1}\sum_{k=1}^{n+1}f\Big(\frac{k}{n+1}\Big)\Big|&=\Big|\sum_{k=1}^{n}\int_0^1 fg_{n,k}\,\mathrm{d}x-\frac{1}{n+1}\sum_{k=1}^{n}f\Big(\frac{k}{n+1}\Big)+\frac{f(1)}{n+1}\Big|\\&\leq\Big|\sum_{k=1}^{n}\Big[\int_0^1 fg_{n,k}\,\mathrm{d}x-\frac{1}{n+1}f\Big(\frac{k}{n+1}\Big)\Big]\Big|+\frac{|f(1)|}{n+1}.\end{aligned}\tag{6.39}$$

Since $f\in C(I)$, there is a $M>0$ such that $|f(x)|\leq M$ on I. On the linear part, we have $g_{n,k}=a_kx+b_k$ for some $a_k,b_k\in\mathbb{R}$. Then it asserts that, for example,

$$\Big|\int_{\frac{k}{n+1}-\frac{\delta_n}{2}-\frac{\delta_n}{2n+2}}^{\frac{k}{n+1}-\frac{\delta_n}{2}}(a_kx+b_k)f(x)\,\mathrm{d}x\Big|\leq M|a_k|\cdot\Big|\int_{\frac{k}{n+1}-\frac{\delta_n}{2}-\frac{\delta_n}{2n+2}}^{\frac{k}{n+1}-\frac{\delta_n}{2}}x\,\mathrm{d}x\Big|+M|b_k|\cdot\frac{\delta_n}{2n+2}\tag{6.40}$$

for every $k=1,2,\ldots,n$. By routine computation, it can be shown easily that the integral on the right-hand side of the inequality (6.40) is of the growth $\frac{M'}{n^4}$, so the integral on the left-hand side of the inequality (6.40) is bounded by $\frac{M'}{n^3}$ for some positive constant M'. Using this fact, the inequality (6.39) can be further reduced to

$$\begin{aligned}&\Big|\int_0^1 fg_n\,\mathrm{d}x-\frac{1}{n+1}\sum_{k=1}^{n+1}f\Big(\frac{k}{n+1}\Big)\Big|\\&\leq\Big|\sum_{k=1}^{n}\Big[\int_{\frac{k}{n+1}-\frac{\delta_n}{2}}^{\frac{k}{n+1}+\frac{\delta_n}{2}}(n+1)f(x)\,\mathrm{d}x-\frac{1}{n+1}f\Big(\frac{k}{n+1}\Big)\Big]\Big|+\frac{\widetilde{M}}{n^2}+\frac{\|f\|_\infty}{n+1}\\&=\Big|\sum_{k=1}^{n}\Big[\int_{\frac{k}{n+1}-\frac{\delta_n}{2}}^{\frac{k}{n+1}+\frac{\delta_n}{2}}(n+1)f(x)\,\mathrm{d}x-\frac{1}{(n+1)\delta_n}\int_{\frac{k}{n+1}-\frac{\delta_n}{2}}^{\frac{k}{n+1}+\frac{\delta_n}{2}}f\Big(\frac{k}{n+1}\Big)\,\mathrm{d}x\Big]\Big|+\frac{\widetilde{M}}{n^2}+\frac{\|f\|_\infty}{n+1}\\&\leq\sum_{k=1}^{n}\Big[\int_{\frac{k}{n+1}-\frac{\delta_n}{2}}^{\frac{k}{n+1}+\frac{\delta_n}{2}}(n+1)\Big|f(x)-\frac{1}{(n+1)^2\delta_n}f\Big(\frac{k}{n+1}\Big)\Big|\,\mathrm{d}x\Big]+\frac{\widetilde{M}}{n^2}+\frac{\|f\|_\infty}{n+1}\\&\leq\sum_{k=1}^{n}\Big[(n+1)\int_{\frac{k}{n+1}-\frac{\delta_n}{2}}^{\frac{k}{n+1}+\frac{\delta_n}{2}}\Big|f(x)-f\Big(\frac{k}{n+1}\Big)\Big|\,\mathrm{d}x\Big]+\frac{\widetilde{M}}{n^2}+\frac{\|f\|_\infty}{n+1},\end{aligned}\tag{6.41}$$

where $\widetilde{M}$ is a positive constant.

Applying the Mean-Value Theorem for Integrals (see [3, Theorem 7.30, pp. 160, 161]) to the integral in the inequality (6.39), we see that

$$\int_{\frac{k}{n+1}-\frac{\delta_n}{2}}^{\frac{k}{n+1}+\frac{\delta_n}{2}} \left|f(x) - f\left(\frac{k}{n+1}\right)\right| \mathrm{d}x = \left|f(\xi) - f\left(\frac{k}{n+1}\right)\right|\delta_n, \tag{6.42}$$

where $\xi_k \in [\frac{k}{n+1} - \frac{\delta_n}{2}, \frac{k}{n+1} + \frac{\delta_n}{2}]$. Substituting the expression (6.42) into the inequality (6.41), we get

$$\begin{aligned}\left|\int_0^1 fg_n \,\mathrm{d}x - \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right)\right| &\leq \sum_{k=1}^{n}(n+1)\left|f(\xi_k) - f\left(\frac{k}{n+1}\right)\right|\delta_n + \frac{\|f\|_\infty}{n+1} \\ &\leq n(n+1)\delta_n M_n + \frac{\widetilde{M}}{n^2} + \frac{\|f\|_\infty}{n+1},\end{aligned} \tag{6.43}$$

where

$$M_n = \max\left(\left|f(\xi_1) - f\left(\frac{1}{n+1}\right)\right|, \left|f(\xi_2) - f\left(\frac{2}{n+1}\right)\right|, \ldots, \left|f(\xi_n) - f\left(\frac{n}{n+1}\right)\right|\right).$$

When $n \to \infty$, $\xi_k \to \frac{k}{n+1}$ for each $k = 1, 2, \ldots, n$ so that $M_n \to 0$ as $n \to \infty$. By this observation and the fact $\lim\limits_{n\to\infty} n(n+1)\delta_n = 1$, we derive from the inequality (6.43) that

$$\lim_{n\to\infty} \int_0^1 fg_n \,\mathrm{d}x = \int_0^1 f \,\mathrm{d}x$$

which is exactly condition (iii).

We have completed the proof of the problem. ■

6.4 Problems on Uniformly Integrable Sets

Problem 6.10

Rudin Chapter 6 Exercise 10.

Proof.

(a) Let $\Phi = \{f_1, f_2, \ldots, f_N\} \subseteq L^1(\mu)$ for some $N \in \mathbb{N}$ and $\epsilon > 0$. By Problem 1.12, there exists a $\delta_k > 0$ such that

$$\int_{E_k} |f_k| \,\mathrm{d}\mu < \epsilon \tag{6.44}$$

whenever $\mu(E_k) < \delta_k$, where $k = 1, 2, \ldots, N$. Suppose that $\delta = \min(\delta_1, \delta_2, \ldots, \delta_k)$. Then it yields from Theorem 1.33 and the estimates (6.44) that if $\mu(E) < \delta \leq \delta_k$, then

$$\left|\int_E f_k \,\mathrm{d}\mu\right| \leq \int_E |f_k| \,\mathrm{d}\mu < \epsilon$$

for each $k = 1, 2, \ldots, N$. By the definition, Φ is uniformly integrable.

(b) We first show that

Lemma 6.1

Suppose that $\Phi \subseteq L^1(\mu)$ is uniformly integrable. Given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_E |f|\,\mathrm{d}\mu < \epsilon$$

whenever $\mu(E) < \delta$ and $f \in \Phi$.

Proof of Lemma 6.1. Suppose that Φ is a collection of *real* measurable functions in $L^1(\mu)$ and $E \in \mathfrak{M}$. Given $\epsilon > 0$. Since Φ is uniformly integrable, there corresponds a $\delta > 0$ such that

$$\Big|\int_E f\,\mathrm{d}\mu\Big| < \frac{\epsilon}{4} \tag{6.45}$$

whenever $f \in \Phi$ and $\mu(E) < \delta$.

Define $E_+ = \{x \in E \,|\, f(x) \geq 0\}$ and $E_- = \{x \in E \,|\, f(x) < 0\}$. It is clear that $E = E_+ \cup E_-$, $E_+ \cap E_- = \varnothing$ and $\mu(E_\pm) \leq \mu(E) < \delta$. Thus we gain from Theorem 1.29 that

$$\int_E |f|\,\mathrm{d}\mu = \int_{E_+} |f|\,\mathrm{d}\mu + \int_{E_-} |f|\,\mathrm{d}\mu = \int_{E_+} f\,\mathrm{d}\mu - \int_{E_-} f\,\mathrm{d}\mu$$

and then the estimate (6.45) yields that

$$\int_E |f|\,\mathrm{d}\mu = \Big|\int_E |f|\,\mathrm{d}\mu\Big| \leq \Big|\int_{E_+} f\,\mathrm{d}\mu\Big| + \Big|\int_{E_-} f\,\mathrm{d}\mu\Big| < \frac{\epsilon}{2} \tag{6.46}$$

for all $f \in \Phi$ and $\mu(E) < \delta$. In other words, Lemma 6.1 holds for subsets of *real* measurable functions in $L^1(\mu)$.

Next, suppose that Φ is a collection of *complex* measurable functions in $L^1(\mu)$. If $f \in \Phi$, then $f = u + iv$, where u and v are real measurable functions in $L^1(\mu)$. Let

$$\Psi = \{\mathrm{Re}\,(f) \,|\, f \in \Phi\} \quad \text{and} \quad \Omega = \{\mathrm{Im}\,(f) \,|\, f \in \Phi\}.$$

Since $0 < |u| \leq |f|$ and $0 < |v| \leq |f|$, we see that $u, v \in L^1(\mu)$ and then $\Psi, \Omega \subseteq L^1(\mu)$. Besides, by the estimate (6.46), we have

$$\int_E |u|\,\mathrm{d}\mu < \frac{\epsilon}{2} \quad \text{and} \quad \int_E |v|\,\mathrm{d}\mu < \frac{\epsilon}{2} \tag{6.47}$$

when $\mu(E) < \delta$ and for all $u \in \Psi$ and $v \in \Omega$. By the definition, the sets Ψ and Ω are uniformly integrable. Since $|f| \leq |u| + |v|$, we deduce from the estimates (6.47) that

$$\int_E |f|\,\mathrm{d}\mu \leq \int_E |u|\,\mathrm{d}\mu + \int_E |v|\,\mathrm{d}\mu < \epsilon$$

whenever $\mu(E) < \delta$ and for all $f \in \Phi$. This completes the proof of Lemma 6.1. ■

Now it is time to return to the proof of the problem.

– **Proof of $\|f\|_1 < \infty$.** Since $\{f_n\}$ is uniformly integrable, it follows from Lemma 6.1 that given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\int_E |f_n|\,\mathrm{d}\mu < \epsilon \tag{6.48}$$

whenever $\mu(E) < \delta$ and for all $n \in \mathbb{N}$. Employing Theorem 1.28 (Fatou's Lemma) to the estimate (6.48), we know that

$$\int_E |f|\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_E |f_n|\,\mathrm{d}\mu \leq \epsilon \tag{6.49}$$

whenever $\mu(E) < \delta$.

By hypothesis (iii), there exists a $E_1 \in \mathfrak{M}$ such that $\mu(X \setminus E_1) = 0$ and $f_n(x) \to f(x)$ pointwisely a.e. on E_1. Since $\mu(X) < \infty$ and hypothesis (iii) again, Egoroff's Theorem asserts the existence of a measurable set $E_2 \subseteq X$ with $\mu(X \setminus E_2) < \delta$ such that

$$f_n(x) \to f(x) \tag{6.50}$$

uniformly a.e. on E_2 as $n \to \infty$. Since uniform convergence implies pointwise convergence, we may assume that $E_2 \subseteq E_1$. Otherwise, we can replace E_2 by the set $E_2 \cap E_1$. In this case, De Morgan's Laws give

$$\mu\big(X \setminus (E_1 \cap E_2)\big) = \mu\big((X \setminus E_1) \cup (X \setminus E_2)\big) \leq \mu(X \setminus E_1) + \mu(X \setminus E_2) < \delta$$

so that the limit (6.50) also holds on $E_2 \cap E_1$.
By hypothesis (iv), there exists a $E_3 \in \mathfrak{M}$ such that $\mu(X \setminus E_3) = 0$ and $|f(x)| < \infty$ on E_3. Take $E = E_2 \cap E_3 \subseteq E_2$ so that the uniform convergence (6.50) also holds on E. This means that there exists a positive integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \epsilon$$

a.e. on E and then hypothesis (i) implies

$$\int_E |f_n - f|\,\mathrm{d}\mu < \epsilon\mu(E) \leq \epsilon\mu(X) < \infty \tag{6.51}$$

for all $n \geq N$. By De Morgan's Laws again, we obtain

$$\mu(X \setminus E) \leq \mu(X \setminus E_2) + \mu(X \setminus E_3) < \delta,$$

so we deduce from the estimates (6.49) that

$$\int_{X\setminus E} |f|\,\mathrm{d}\mu \leq \epsilon. \tag{6.52}$$

Now using the estimates (6.51), (6.52) and the fact $f_N \in L^1(\mu)$ that

$$\int_X |f|\,\mathrm{d}\mu = \int_E |f|\,\mathrm{d}\mu + \int_{X\setminus E} |f|\,\mathrm{d}\mu \leq \int_E |f_N - f|\,\mathrm{d}\mu + \int_E |f_N|\,\mathrm{d}\mu + \int_{X\setminus E} |f|\,\mathrm{d}\mu < \infty.$$

In other words, we have $f \in L^1(\mu)$.

– **Proof of $\|f_n - f\|_1 \to 0$ as $n \to \infty$.** Again the estimates (6.48), (6.51) and (6.52) together show that

$$\begin{aligned}\int_X |f_n - f|\,\mathrm{d}\mu &= \int_E |f_n - f|\,\mathrm{d}\mu + \int_{X\setminus E} |f_n - f|\,\mathrm{d}\mu \\ &< \epsilon\mu(X) + \int_{X\setminus E} |f_n|\,\mathrm{d}\mu + \int_{X\setminus E} |f|\,\mathrm{d}\mu \\ &< [2 + \mu(X)] \cdot \epsilon\end{aligned}$$

for all $n \geq N$. Hence we conclude that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

(c) For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \dfrac{1}{n}, & \text{if } x \in [0, n]; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that

$$\|f_n\|_1 = \int_{\mathbb{R}} |f_n| \,\mathrm{d}m = \int_0^n \frac{1}{n} \,\mathrm{d}m = 1, \tag{6.53}$$

i.e., $\{\|f_n\|_1\}$ is bounded. Next, for every $x \in \mathbb{R}$, we have $f_n(x) \to 0$ as $n \to \infty$. So we let $f(x) = 0$. Since $|f_n(x)| \leq 1$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, for each $\epsilon > 0$, if $\delta = \epsilon$, then for every $E \in \mathscr{B}$ with $m(E) < \delta$, we obtain

$$\Big| \int_E f_n \,\mathrm{d}m \Big| \leq \int_E |f_n| \,\mathrm{d}m \leq m(E) < \epsilon.$$

Thus $\{f_n\}$ is uniformly integrable. However, the expression (6.53) indicates that

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f_n - 0| \,\mathrm{d}m \neq 0.$$

Hence hypothesis (i) cannot be omitted in part (b).

(d) We construct two examples showing that hypothesis (iv) is redundant in some cases, but not in other cases.

- **Example 1.** Let $\mathscr{B}$ be the collection of Borel sets of $[0, 1]$. We employ the concept of **atomicless measures** . A set $E \in \mathfrak{M}$ is called an **atom** of the measure μ if $\mu(E) > 0$ and every measurable subset $F \subset E$ has measure either 0 or $\mu(E)$. If μ has no atoms, then it is called atomless. Lebesgue measure m on $[0, 1]$ is atomless. In other words, for every $\epsilon \in (0, 1)$, if $E \in \mathscr{B}$ satisfies $m(E) = \epsilon > 0$, then there exists a $F \in \mathscr{B}$ and $F \subset E$ such that $0 < m(F) < \epsilon$. See [9, Definition 1.12.7, Example 1.12.8, p. 55].

  We claim that Vitali's Convergence Theorem holds for m without the hypothesis (iv). To this end, it suffices to show that hypotheses (i) to (iii) imply hypothesis (iv). Suppose that

  $$E = \{x \in [0, 1] \,|\, |f(x)| = \infty\}.$$

  If $m(E) = 0$, then there is nothing to prove. So we assume that $m(E) > 0$. Since $\{f_n\}$ is uniformly integrable, Lemma 6.1 implies the existence of a $\delta > 0$ such that

  $$\int_E |f_n| \,\mathrm{d}x < 1$$

  whenever $m(E) < \delta$ and all $n \in \mathbb{N}$. Since m is atomless and $m(E) > 0$, there exists a $F \in \mathscr{B}$ such that $0 < m(F) < \delta$ and thus

  $$\int_F |f_n| \,\mathrm{d}x < 1 \tag{6.54}$$

  for all $n \in \mathbb{N}$. By Theorem 1.28 (Fatou's Lemma) and the fact that $f_n(x) \to f(x)$ a.e. on $[0, 1]$ as $n \to \infty$, we deduce from the inequality (6.54) that

  $$\int_F |f| \,\mathrm{d}x \leq \liminf_{n \to \infty} \int_F |f_n| \,\mathrm{d}x \leq 1. \tag{6.55}$$

  However, since $F \subset E$ and $m(F) > 0$, the left-most integral in the inequality (6.55) is actually ∞, a contradiction. Hence $m(E) = 0$ or equivalently, $|f(x)| < \infty$ a.e. on $[0, 1]$.

– **Example 2.** Let $\mathfrak{M}$ be the collection of all sets $E \subseteq \mathbb{R}$ such that either E or E^c is at most countable and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. By Problem 1.6, we see that $\mathfrak{M}$ is a σ-algebra in $\mathbb{R}$ and μ is a measure on $\mathfrak{M}$. Clearly, μ is a finite measure. We define $f(x) \equiv \infty$ on $\mathbb{R}$ and $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = n$ for every $n \in \mathbb{N}$ so that $f_n(x) \to f(x)$ as $n \to \infty$ on $\mathbb{R}$, but hypothesis (iv) *does not* hold.

To each $\epsilon > 0$, we pick $\delta = 1$, so if $E \in \mathfrak{M}$ satisfies $\mu(E) < 1$, then it must be $\mu(E) = 0$ and thus Proposition 1.24(e) implies that

$$\Big|\int_E f_n \,\mathrm{d}\mu\Big| = 0 < \epsilon$$

for all $n \in \mathbb{N}$. By the definition, $\{f_n\}$ is uniformly integrable. However, it is clear that $f \notin L^1(\mu)$ because $\mathbb{R}^c = \varnothing$ is at most countable. Consequently, conclusion of part (b) is false.

(e) Suppose that the hypotheses of Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) hold. Then hypotheses (i) and (iii) are true definitely. On the one hand, since $g \in L^1(\mu)$, Problem 1.12 says that each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\int_E |g| \,\mathrm{d}\mu < \epsilon \tag{6.56}$$

whenever $\mu(E) < \delta$. On the other hand, $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ will imply that

$$\int_X |f_n| \,\mathrm{d}\mu \leq \int_X |g| \,\mathrm{d}\mu < \infty. \tag{6.57}$$

Therefore, $\{f_n\} \subseteq L^1(\mu)$ and then we gain from this, Theorem 1.33 and the estimate (6.56) that

$$\Big|\int_E f_n \,\mathrm{d}\mu\Big| \leq \int_E |f_n| \,\mathrm{d}\mu < \epsilon$$

whenever $\mu(E) < \delta$ and all $n \in \mathbb{N}$. Hence $\{f_n\}$ is uniformly integrable, i.e., hypothesis (ii) is true. Since $|f_n|$ are measurable by Proposition 1.9(b), it follows from Theorem 1.28 (Fatou's Lemma) and the estimate (6.57) that

$$\int_X |f| \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X |f_n| \,\mathrm{d}\mu < \infty.$$

Thus $|f(x)| < \infty$ a.e. on X, i.e., hypothesis (iv) is valid. Hence part (b) verifies that Vitali's Convergence Theorem implies Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) if $\mu(X) < \infty$.

Now we are going to construct an example in which Vitali's Theorem applies, but *not* Theorem 1.34 (Lebesgue's Dominated Convergence Theorem). For every $n \in \mathbb{N}$, we consider $f_n : (0, 1) \to \mathbb{R}$ defined by

$$f_n(x) = \frac{1}{x}\chi_{(\frac{1}{n+1},\frac{1}{n})} \geq 0.$$

Evidently, we have $m\big((0, 1)\big) = 1$ and $|f(x)| \leq$ on $(0, 1)$ which are hypotheses (i) and (iv) respectively. For each $x \in (0, 1)$, we have $x \notin \chi_{(\frac{1}{n+1},\frac{1}{n})}$ for all sufficiently large n so that $f_n(x) \to 0 = f(x)$ as $n \to \infty$. Thus we have hypothesis (iii). It is trivial to check that

$$\int_0^1 |f_n| \,\mathrm{d}x = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x} \,\mathrm{d}x = \ln\Big(1 + \frac{1}{n}\Big) < \infty \tag{6.58}$$

so that $\{f_n\} \in L^1\big((0,1)\big)$. Given $\epsilon > 0$. One can find a positive integer N such that $\ln(1+\frac{1}{n}) < \epsilon$ for all $n \geq N$. By this result, we observe from the representation (6.58) that

$$\Big|\int_E f_n\,\mathrm{d}x\Big| \leq \Big|\int_0^1 f_n\,\mathrm{d}x\Big| \leq \int_0^1 |f_n|\,\mathrm{d}x < \epsilon$$

for every $n \geq N$ and every measurable subset E of $(0,1)$. By part (a), the finite set $\{f_1, f_2, \ldots, f_{N-1}\}$ is uniformly integrable. Therefore, these two facts confirm that $\{f_n\}$ is uniformly integrable, i.e., hypothesis (ii). Hence the set $\{f_n\}$ satisfies all the hypotheses of Vitali's Convergence Theorem.

However, for every $x \in (0,1)$, we note from Definition 1.13 that

$$\sup_{n\in\mathbb{N}}(f_n(x)) = \frac{1}{x},$$

so if $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$, then $\frac{1}{x} \leq g(x)$. Since $\frac{1}{x} \notin L^1$, $g \notin L^1$ too and this means that $\{f_n\}$ *does not* satisfy the hypotheses of Theorem 1.34 (Lebesgue's Dominated Convergence Theorem).

(f) For each $n \in \mathbb{N}$, we consider $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = n\chi_{(0,\frac{1}{n})}(x) - n\chi_{(1-\frac{1}{n},1)}(x).$$

Notice that each f_n is measurable and satisfies $f_n(0) = f_n(1) = 0$. If $x \in (0,1)$, then there exists a positive integer N such that $x \notin (0, \frac{1}{n})$ and $x \notin (1-\frac{1}{n}, 1)$ for all $n \geq N$. Thus we obtain $f_n(x) = 0$ for all $n \geq N$. Next,

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_0^{\frac{1}{n}} n\,\mathrm{d}x - \int_{1-\frac{1}{n}}^1 n\,\mathrm{d}x = n \times \frac{1}{n} - n \times \frac{1}{n} = 0$$

for each $n \in \mathbb{N}$.

However, $\{f_n\}$ is *not* uniformly integrable. To see this, assume that there corresponds a $\delta > 0$ such that

$$\Big|\int_E f_n\,\mathrm{d}x\Big| < 1 \tag{6.59}$$

whenever $m(E) < \delta$ and for all $n \in \mathbb{N}$. In particular, fix $n = N > \frac{1}{\delta}$ and take $E_N = (0, \frac{1}{N})$ so that $m(E_n) = \frac{1}{N} < \delta$ and

$$\Big|\int_{E_N} f_N\,\mathrm{d}x\Big| = \Big|\int_0^{\frac{1}{N}} f_N\,\mathrm{d}x\Big| = \Big|\int_0^{\frac{1}{N}} N\,\mathrm{d}x\Big| = 1$$

which contradicts to the inequality (6.59).

(g) If $\mu(X) = 0$, then there is nothing to prove. Without loss of generality, we may assume that $\mu(X) > 0$. We follow the hint and prove it into several steps.

– **Step 1: ρ is a metric (modulo sets of measure 0)[c].** Define

$$\rho(A,B) = \int_X |\chi_A - \chi_B|\,\mathrm{d}\mu \geq 0. \tag{6.60}$$

[c] Remember that sets A and B with $A = B$ a.e. on X are treated to be identical.

Since $\mu(X) > 0$, $\rho(A, B) = 0$ if and only if $\chi_A = \chi_B$ a.e. on X if and only if $A = B$ a.e. on X. It is trivial that $\rho(A, B) = \rho(B, A)$. Finally, since

$$|\chi_A - \chi_B| \leq |\chi_A - \chi_C| + |\chi_C - \chi_B|$$

for every $A, B, C \in \mathfrak{M}$, we must have

$$\rho(A, B) \leq \rho(A, C) + \rho(C, B).$$

By the definition, ρ is a metric (modulo sets of measure 0).

– **Step 2: $(\mathfrak{M}, \rho)$ is a complete metric space (modulo sets of measure 0).** Let $\{E_n\} \subseteq \mathfrak{M}$ be a Cauchy sequence, i.e., given $\epsilon > 0$, there exists a positive integer N such that $n, m \geq N$ imply that

$$\rho(E_n, E_m) < \epsilon.$$

We note from the definition (6.60) that $\rho(A, B) = \|\chi_A - \chi_B\|_1$, so we obtain

$$\|\chi_{E_n} - \chi_{E_m}\|_1 < \epsilon$$

for all $n, m \geq N$, i.e., $\{\chi_{E_n}\}$ is Cauchy in $L^1(\mu)$. By Theorem 3.11, there exists $f \in L^1(\mu)$ such that $\{\chi_{E_n}\}$ converges to f in $L^1(\mu)$. By Theorem 3.12, $\{\chi_{E_n}\}$ has a subsequence $\{\chi_{E_{n_k}}\}$ such that

$$\chi_{E_{n_k}}(x) \to f(x)$$

pointwise a.e. on X as $n \to \infty$. By the definition of $\chi_{E_{n_k}}$, we get immediately that either $f(x) = 0$ or 1 for almost every $x \in X$. Now if we define $E = \{x \in X \mid f(x) = 1\}$, then $E \in \mathfrak{M}$, $f = \chi_E$ a.e. on X and therefore

$$\rho(E_n, E) = \|\chi_{E_n} - \chi_E\|_1 \to 0$$

as $n \to \infty$. Hence $(\mathfrak{M}, \rho)$ is a complete metric space (modulo sets of measure 0).

– **Step 3: The map $E \to \int_E f_n \,\mathrm{d}\mu$ is continuous for each n.** Denote this map by φ_n. Recall that $f_n \in L^1(\mu)$, so Problem 1.12 asserts that for every $\epsilon > 0$, there corresponds a $\delta > 0$ such that

$$\int_E |f_n| \,\mathrm{d}\mu < \epsilon \tag{6.61}$$

whenever $\mu(E) < \delta$. Suppose that $E, F \in \mathfrak{M}$ such that $\rho(E, F) < \delta$. We deduce from the fact $E \cup F = (E \setminus F) \cup (E \cap F) \cup (F \setminus E)$ that

$$\begin{aligned}
\rho(E, F) &= \int_X |\chi_E - \chi_F| \,\mathrm{d}\mu \\
&= \int_{E \cup F} |\chi_E - \chi_F| \,\mathrm{d}\mu \\
&= \int_{E \setminus F} |\chi_E - \chi_F| \,\mathrm{d}\mu + \int_{F \setminus E} |\chi_E - \chi_F| \,\mathrm{d}\mu \\
&= \mu(E \setminus F) + \mu(F \setminus E) \qquad (6.62)
\end{aligned}$$

so that $\mu(E \setminus F) < \delta$ and $\mu(F \setminus E) < \delta$. Since $(\chi_E - \chi_F)f_n \in L^1(\mu)$, Theorem 1.33 and the estimate (6.61) imply that

$$|\varphi_n(E) - \varphi_n(F)| = \Big| \int_E f_n \,\mathrm{d}\mu - \int_F f_n \,\mathrm{d}\mu \Big|$$

$$
\begin{aligned}
&= \Big| \int_X \chi_E f_n \, \mathrm{d}\mu - \int_X \chi_F f_n \, \mathrm{d}\mu \Big| \\
&= \Big| \int_X (\chi_E - \chi_F) f_n \, \mathrm{d}\mu \Big| \\
&\leq \int_X |\chi_E - \chi_F| \cdot |f_n| \, \mathrm{d}\mu \\
&= \int_{E \setminus F} |f_n| \, \mathrm{d}\mu + \int_{F \setminus E} |f_n| \, \mathrm{d}\mu \\
&< 2\epsilon.
\end{aligned}
$$

Hence φ_n is continuous on $\mathfrak{M}$ for every $n \in \mathbb{N}$.

– **Step 4: Completion of the proof.** Now $\{\varphi_n\}$ is a sequence of continuous complex functions on the complete metric space $(\mathfrak{M}, \rho)$. Denote

$$\varphi(E) = \lim_{n\to\infty} \varphi_n(E)$$

as a complex number for every $E \in \mathfrak{M}$. If $\epsilon > 0$, it follows from Problem 5.13 (particularly the inequality (5.62)) that there exist $E_0 \in \mathfrak{M}$, $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\Big| \int_E (f_n - f_N) \, \mathrm{d}\mu \Big| = |\varphi_n(E) - \varphi_N(E)| < \epsilon \tag{6.63}$$

for all $E \in \mathfrak{M}$ with $\rho(E, E_0) < \delta$ and $n \geq N$.

If $\mu(A) < \delta$ and $B = E_0 \setminus A$, then the expression (6.62) indicates that

$$\rho(B, E_0) = \mu\big((E_0 \setminus A) \setminus E_0\big) + \mu(E_0 \setminus (E_0 \setminus A)) = \mu(E_0 \cap A) \leq \mu(A) < \delta.$$

Similarly, if $C = E_0 \cup A$, then we know that

$$\rho(C, E_0) = \mu\big((E_0 \cup A) \setminus E_0\big) + \mu(E_0 \setminus (E_0 \cup A)) = \mu(A) < \delta.$$

Thus the estimate (6.63) holds with B and C in place of E.

Since $E_0 \cup A = (E_0 \setminus A) \cup A$ and $(E_0 \setminus A) \cap A = \varnothing$, we get

$$\int_{E_0 \cup A} (f_n - f_N) \, \mathrm{d}\mu = \int_A (f_n - f_N) \, \mathrm{d}\mu + \int_{E_0 \setminus A} (f_n - f_N) \, \mathrm{d}\mu$$

which implies that

$$
\begin{aligned}
\Big| \int_A (f_n - f_N) \, \mathrm{d}\mu \Big| &= \Big| \int_{E_0 \cup A} (f_n - f_N) \, \mathrm{d}\mu - \int_{E_0 \setminus A} (f_n - f_N) \, \mathrm{d}\mu \Big| \\
&\leq \Big| \int_{E_0 \cup A} (f_n - f_N) \, \mathrm{d}\mu \Big| + \Big| \int_{E_0 \setminus A} (f_n - f_N) \, \mathrm{d}\mu \Big| \\
&< 2\epsilon
\end{aligned} \tag{6.64}
$$

for all $n > N$. By part (a), the set $\{f_1, f_2, \ldots, f_N\}$ is uniformly integrable, i.e., there corresponds a $\delta' > 0$ such that

$$\Big| \int_A f_n \, \mathrm{d}\mu \Big| < \epsilon \tag{6.65}$$

whenever $\mu(A) < \delta'$ and $n = 1, 2, \ldots, N$. Define $\delta'' = \min(\delta, \delta')$. If $\mu(A) < \delta''$, then since $\mu(A) < \delta'$, the inequality (6.65) still holds for $n = 1, 2, \ldots, N$. Next, for

$n = N+1, N+2, \ldots$, since $\mu(A) < \delta$, it yields from the estimates (6.64) and (6.65) that

$$\left|\int_A f_n\,\mathrm{d}\mu\right| \leq \left|\int_A (f_n - f_N)\,\mathrm{d}\mu\right| + \left|\int_A f_N\,\mathrm{d}\mu\right| < 3\epsilon. \tag{6.66}$$

By combining the results (6.65) and (6.66), we conclude that

$$\left|\int_A f_n\,\mathrm{d}\mu\right| < 3\epsilon$$

if $\mu(A) < \delta''$ and for all $n \in \mathbb{N}$, i.e., $\{f_n\}$ is uniformly integrable.

We have completed the proof of the problem. ■

Problem 6.11

Rudin Chapter 6 Exercise 11.

Proof. We note that if $\mu(X) = 0$, then Proposition 1.24(e) shows that the result holds trivially. Therefore, without loss of generality, we may assume that $\mu(X) \neq 0$. Now we want to apply Problem 6.10(b). By our hypotheses, it suffices to show that $\Phi = \{f_n\}$ is uniformly integrable and $|f(x)| < \infty$ a.e. on X.

Since $f_n \in L^1(\mu)$ for every $n \in \mathbb{N}$, we apply Theorem 1.33 and then Theorem 3.5 (Hölder's Inequality) to obtain

$$\left|\int_E f_n\,\mathrm{d}\mu\right| \leq \int_E |f_n|\,\mathrm{d}\mu \leq \left\{\int_E |f_n|^p\,\mathrm{d}\mu\right\}^{\frac{1}{p}}\left\{\int_E \mathrm{d}\mu\right\}^{\frac{1}{q}} < C^{\frac{1}{p}} \times \mu(E)^{\frac{1}{q}} \tag{6.67}$$

for every $E \in \mathfrak{M}$. Since $p > 1$, $q < \infty$. Take $\delta = \epsilon^q C^{-\frac{q}{p}}$. If $\mu(E) < \delta$, then it follows from the inequality (6.67) that

$$\left|\int_E f_n\,\mathrm{d}\mu\right| < \epsilon.$$

Thus the set $\Phi = \{f_n\}$ is uniformly integrable.

It remains to show that $|f(x)| < \infty$ a.e. on X. Since $f_n(x) \to f(x)$ a.e. on X, $|f_n(x)| \to |f(x)|$ a.e. on X. By Theorem 1.28 (Fatou's Lemma) and the inequality (6.67), we know that

$$\int_X |f|\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X |f_n|\,\mathrm{d}\mu < C^{\frac{1}{p}} \times \mu(X)^{\frac{1}{q}} < \infty. \tag{6.68}$$

Since $\mu(X) \neq 0$, the inequality (6.68) ensures that $|f(x)| < \infty$ a.e. on X. Hence we deduce from Problem 6.10(b) that

$$\lim_{n\to\infty} \int_X |f - f_n|\,\mathrm{d}\mu = 0$$

holds, completing the proof of the problem. ■

6.5 Dual Spaces of $L^p(\mu)$ Revisit

Problem 6.12

Rudin Chapter 6 Exercise 12.

Proof. We note that Problem 1.6 establishes that $\mathfrak{M}$ is a σ-algebra. Let $E = (\frac{1}{4}, \frac{1}{2})$ which is open in $[0,1]$. It is clear that $g^{-1}(E) = E$. Since E and E^c are uncountable, we have $g^{-1}(E) \notin \mathfrak{M}$ and thus g is *not* $\mathfrak{M}$-measurable by Definition 1.3(c). We have completed the proof of the problem. ■

Problem 6.13

Rudin Chapter 6 Exercise 13.

Proof. Let $C(I)$ be the space of all continuous functions on I. It is clear that $\chi_{[0,\delta]} \in L^\infty$ but $\chi_{[0,\delta]} \notin C(I)$ for some $\delta > 0$. We need the following result:

Lemma 6.2

The space $C(I)$ is closed in the metric space L^∞.

Proof of Lemma 6.2. Let $\{f_n\} \subseteq C(I)$ and $\{f_n\}$ converges to f in L^∞. Given $\epsilon > 0$. It means that there exists a positive integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \sup_{x\in I} |f_n(x) - f(x)| = \|f_n - f\|_\infty < \epsilon$$

for all $x \in I$. In other words, $\{f_n\}$ converges uniformly to f on I and thus f is continuous on I (see [49, Theorem 7.12, p. 150]), i.e., $f \in C(I)$. Hence $C(I)$ is closed in L^∞, completing the proof of the lemma. ■

We return to the proof of the problem. Since L^∞ is a Banach space, we get from Lemma 6.2 and the equivalent form of Theorem 5.19 that *there exists* a bounded linear functional Λ on L^∞ such that

$$\Lambda(f) = 0 \tag{6.69}$$

for every $f \in C(I)$, but $\Lambda(\chi_{[0,\delta]}) \neq 0$.

Assume that there was a $g \in L^1(\mu)$ such that

$$\Lambda(f) = \int_I fg \, \mathrm{d}m \tag{6.70}$$

for every $f \in L^\infty$. In particular, the result (6.69) and the representation (6.70) imply that

$$\int_I fg \, \mathrm{d}m = 0 \tag{6.71}$$

on $C(I)$. Since $f \in C(I)$ and $g \in L^1(\mu)$, we have $fg \in L^1(\mu)$ and then we apply Theorem 1.39(b) to the integral (6.71) to conclude that $fg = 0$ a.e. on I. If we take $f(x) = e^x$ on I which belongs to $C(I)$, then we obtain the fact that

$$g = 0$$

a.e. on I. Hence we follow from the representation (6.70) that the result (6.69) actually holds on the whole L^∞, a contradiction to the fact that $\Lambda(\chi_{[0,\delta]}) \neq 0$. This completes the proof of the problem. ■

CHAPTER 7

Differentiation

7.1 Lebesgue Points and Metric Densities

Problem 7.1

Rudin Chapter 7 Exercise 1.

Proof. Let x be a Lebesgue point of f. Since $f \in L^1(\mathbb{R}^k)$, we deduce from the triangle inequality and Theorem 1.33 that

$$\begin{aligned}
|f(x)| &\leq \left|f(x) - \frac{1}{m(B_r)}\int_{B_r} f(y)\,\mathrm{d}y\right| + \left|\frac{1}{m(B_r)}\int_{B_r} f(y)\,\mathrm{d}y\right| \\
&\leq \left|\frac{1}{m(B_r)}\int_{B_r} [f(x) - f(y)]\,\mathrm{d}y\right| + \frac{1}{m(B_r)}\int_{B_r} |f(y)|\,\mathrm{d}y \\
&\leq \frac{1}{m(B_r)}\int_{B_r} |f(y) - f(x)|\,\mathrm{d}y + (Mf)(x). \qquad (7.1)
\end{aligned}$$

Since x is a Lebesgue point of f, if we let $r \to 0+$ in the inequality (7.1), then we obtain

$$|f(x)| \leq (Mf)(x)$$

as required, completing the proof of the problem. ■

Problem 7.2

Rudin Chapter 7 Exercise 2.

Proof. We have $I = (-\delta, \delta)$. Now we are going to prove the existence of a measurable set $E \subseteq \mathbb{R}$ such that

$$\liminf_{\delta\to 0} \frac{m\big(E \cap I(\delta)\big)}{2\delta} = \alpha \quad \text{and} \quad \limsup_{\delta\to 0} \frac{m\big(E \cap I(\delta)\big)}{2\delta} = \beta. \qquad (7.2)$$

In fact, it suffices to prove that

$$\liminf_{\delta\to 0} \frac{m\big(E \cap [0,\delta)\big)}{\delta} = \alpha \quad \text{and} \quad \limsup_{\delta\to 0} \frac{m\big(E \cap [0,\delta)\big)}{\delta} = \beta, \qquad (7.3)$$

where $E \subseteq [0, \infty)$ because we can get the desired result (7.2) by reflecting about the origin. To start our proof, we consider five cases:

- **Case (i):** $0 < \alpha < \beta < 1$. In this case, we consider the numbers

$$\theta = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} < \frac{\alpha}{\beta} < 1 \quad \text{and} \quad d_n = \frac{\beta - \alpha}{1-\alpha} \cdot \theta^n > 0$$

for every $n \in \mathbb{N}$. Then it is easy to see that

$$\theta^n - \theta^{n+1} = \theta^n \Big[1 - \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big)\Big] = \frac{\beta - \alpha}{\beta(1-\alpha)}\theta^n = \frac{d_n}{\beta} > d_n. \tag{7.4}$$

Next, for each $n = 1, 2, \ldots$, we define the closed interval

$$E_n = [\theta^n - d_n, \theta^n].$$

We claim that the measurable set

$$E = \bigcup_{n=1}^{\infty} E_n$$

satisfies the requirement (7.3). To see this, we first note from the inequality (7.4) that $d_n \to 0$ as $n \to \infty$ and $E_n \cap E_{n+1} = \varnothing$ for every $n \in \mathbb{N}$, see Figure 7.1 below:

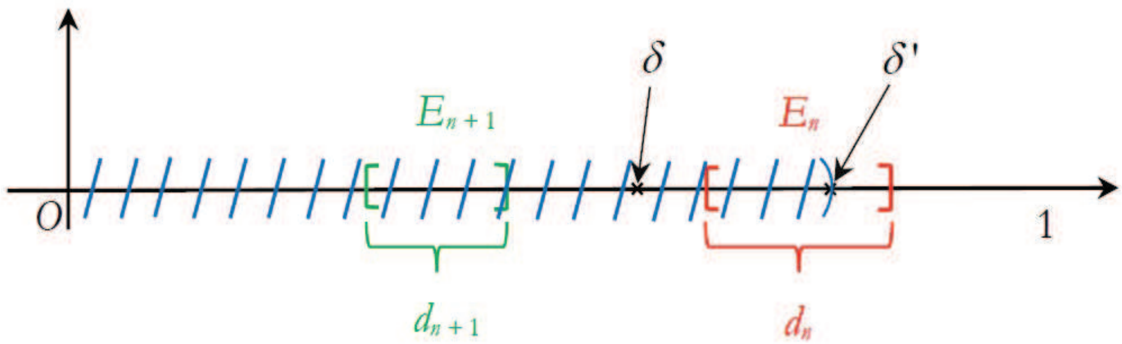


Figure 7.1: The closed intervals E_n and E_{n+1}.

Next, choose $\delta' > \delta > 0$ as shown in Figure 7.1, then it is easy to know that

$$[0, \theta^n - d_n) \cap E = \bigcup_{k=n+1}^{\infty} E_k, \quad [0, \theta^n) \cap E = \bigcup_{k=n}^{\infty} E_k, \quad [0, \delta) \cap E = \bigcup_{k=n+1}^{\infty} E_k$$

and

$$[0, \delta') \cap E = [\theta^n - d_n, \delta') \cup \bigcup_{k=n+1}^{\infty} E_k$$

so that

$$\begin{aligned}
\frac{m\big([0, \theta^n - d_n) \cap E\big)}{\theta^n - d_n} &= \frac{1-\alpha}{(1-\beta)\theta^n} \sum_{k=n+1}^{\infty} d_k = \frac{\beta - \alpha}{1-\beta} \cdot \frac{\theta}{1-\theta} = \alpha, \\
\frac{m\big([0, \theta^n) \cap E\big)}{\theta^n} &= \frac{1}{\theta^n} \sum_{k=n}^{\infty} d_k = \frac{\beta - \alpha}{1-\alpha} \cdot \frac{1}{1-\theta} = \beta, \\
\frac{m\big([0, \delta) \cap E\big)}{\delta} &= \frac{m\big([0, \theta^{n+1}) \cap E\big)}{\delta} = \frac{1}{\delta} \sum_{k=n+1}^{\infty} d_k = \frac{\beta\theta^{n+1}}{\delta}
\end{aligned}$$

and

$$\frac{m\big([0,\delta')\cap E\big)}{\delta'}=\frac{m\big([0,\theta^n)\cap E\big)-m\big([\delta',\theta^n]\big)}{\delta'}=\frac{\beta\theta^n+\delta'-\theta^n}{\delta'}=1-(1-\beta)\frac{\theta^n}{\delta'}.$$

Therefore, the above analysis shows trivially that the function $f:(0,\infty)\to\mathbb{R}$ defined by

$$f(\delta)=\frac{m([0,\delta)\cap E)}{\delta} \tag{7.5}$$

attains the values α and β at the end-points of each E_n, decreases from β to α in the interval $[\theta^{n+1},\theta^n-d_n]$ and increases from α to β in E_n. Hence, we have

$$\alpha=\min_{x\in(0,\delta)}f(x)\quad\text{and}\quad\beta=\max_{x\in(0,\delta)}f(x)$$

which mean that the limits (7.3) hold.

- **Case (ii): $\alpha=0$ and $\beta=1$.** In this case, we defined the measurable set

$$E=\bigcup_{n=1}^{\infty}\Big[\frac{1}{(2n)!},\frac{1}{(2n-1)!}\Big].$$

By similar analysis as in **Case (i)**, we see that

$$0\leq\frac{m\big(E\cap[0,\frac{1}{(2n)!})\big)}{\frac{1}{(2n)!}}\leq(2n)!\times m\Big(\Big[0,\frac{1}{(2n+1)!}\Big]\Big)=(2n)!\times\frac{1}{(2n+1)!}=\frac{1}{2n+1}$$

and

$$\frac{2n-1}{2n}=(2n-1)!\times m\Big(\Big[\frac{1}{(2n)!},\frac{1}{(2n-1)!}\Big]\Big)\leq\frac{m\big(E\cap[0,\frac{1}{(2n-1)!})\big)}{\frac{1}{(2n-1)!}}\leq 1.$$

Hence we have proven the limits (7.3) in this case.

- **Case (iii): $0\leq\alpha=\beta\leq1$.** For each $n\in\mathbb{N}$, we take

$$E_n=\Big[\frac{\alpha}{n+1}+\frac{1-\alpha}{n},\frac{1}{n}\Big].$$

Thus it is easy to see that $E_n\cap E_{n+1}=\varnothing$ for all $n\in\mathbb{Z}$. We define $E=\displaystyle\bigcup_{n=1}^{\infty}E_n$. If we take $\delta=\frac{1}{n}$, then we follow similar argument as in **Case (i)** that

$$m\Big(E\cap\Big[0,\frac{1}{n}\Big)\Big)=\sum_{k=n}^{\infty}m(E_k)=\alpha\sum_{k=n}^{\infty}\frac{1}{k(k+1)}. \tag{7.6}$$

Since $\displaystyle\sum_{k=n}^{N}\frac{1}{k(k+1)}=\frac{1}{n}-\frac{1}{N}$, we have

$$\sum_{k=n}^{\infty}\frac{1}{k(k+1)}=\frac{1}{n}. \tag{7.7}$$

Now we put the infinite series (7.7) into the expression (7.6) to get

$$\frac{m\big(E\cap[0,\frac{1}{n})\big)}{\frac{1}{n}}=\alpha$$

which induces exactly the limits (7.3).

- **Case (iv): $0 = \alpha < \beta < 1$.** Similar to the construction in **Case (i)**, we suppose that
$$E_n = \Big[\frac{1-\beta}{(2n)!} + \frac{\beta}{(2n+2)!}, \frac{1}{(2n)!}\Big] \tag{7.8}$$
where $n \in \mathbb{N}$. Similarly, we have $E_n \cap E_{n+1} = \varnothing$ for all $n \in \mathbb{N}$. Let $E = \bigcup_{n=1}^{\infty} E_n$. On the one hand, since
$$m\Big(E \cap \Big[0, \frac{1-\beta}{(2n)!} + \frac{\beta}{(2n+2)!}\Big)\Big) = \beta \sum_{k=n+1}^{\infty} \Big[\frac{1}{(2k)!} - \frac{1}{(2k+2)!}\Big] = \frac{\beta}{(2n+2)!},$$
we have
$$\frac{m(E \cap [0, \frac{1-\beta}{(2n)!} + \frac{\beta}{(2n+2)!}))}{\frac{1-\beta}{(2n)!} + \frac{\beta}{(2n+2)!}} = \frac{\frac{\beta}{(2n+2)!}}{\frac{1-\beta}{(2n)!} + \frac{\beta}{(2n+2)!}} = \frac{\beta}{(1-\beta)(2n+2)(2n+1) + \beta} \to 0$$
as $n \to \infty$. On the other hand, we have
$$m\Big(E \cap \Big[0, \frac{1}{(2n)!}\Big)\Big) = \beta \sum_{k=n}^{\infty} \Big[\frac{1}{(2k)!} - \frac{1}{(2k+2)!}\Big] = \frac{\beta}{(2n)!},$$
we have
$$\frac{m(E \cap [0, \frac{1}{(2n)!}))}{\frac{1}{(2n)!}} = \frac{\frac{\beta}{(2n)!}}{\frac{1}{(2n)!}} = \beta.$$
Besides, the function (7.5) decreases from β to 0 in the interval $[\frac{1}{(2n+2)!}, \frac{1-\beta}{(2n)!} + \frac{\beta}{(2n+2)!}]$ and increases from 0 to β in E_n. Hence these imply the results (7.3).

- **Case (v): $0 < \alpha < \beta \leq 1$.** Instead of the intervals (7.8), we consider the intervals
$$E_n = \Big[\frac{1}{(2n)!}, \frac{\alpha}{(2n)!} + \frac{1-\alpha}{(2n+2)!}\Big],$$
where $n \in \mathbb{N}$. Thus it can be shown similar to **Case (iv)** that the limits (7.3) hold for the measurable set $E = \bigcup_{n=1}^{\infty} E_n$ and we omit the details here.

This completes the proof of the problem. ∎

Remark 7.1

The limits defined in (7.2) are called the upper and lower densities of E at 0 respectively.

7.2 Periods of Functions and Lebesgue Measurable Groups

Problem 7.3

Rudin Chapter 7 Exercise 3.

Proof. We follow Rudin's hint. Pick $\alpha \in \mathbb{R}$ and put

$$F(x) = m(E \cap [\alpha, x])$$

for $x > \alpha$. If $y > x > \alpha + p_i$, then we deduce from Theorem 2.20(c) and the fact

$$[\alpha - p_i, \alpha + p_i] = [\alpha - p_i, x] \setminus (\alpha + p_i, x]$$

that

$$\begin{aligned} F(x+p_i) - F(x-p_i) &= m(E \cap [\alpha, x+p_i]) - m(E \cap [\alpha, x-p_i]) \\ &= m(E \cap [\alpha, x+p_i] - p_i) - m(E \cap [\alpha, x-p_i] + p_i) \\ &= m(E \cap [\alpha - p_i, x]) - m(E \cap (\alpha + p_i, x]) \\ &= m(E \cap [\alpha - p_i, \alpha + p_i]) \\ &= F(y+p_i) - F(y-p_i). \end{aligned} \tag{7.9}$$

If F is differentiable at x, then we have

$$\lim_{i\to\infty} \frac{F(x+p_i) - F(x-p_i)}{2p_i} = \lim_{i\to\infty} \frac{1}{2}\Big[\frac{F(x+p_i) - F(x)}{p_i} + \frac{F(x-p_i) - F(x)}{-p_i}\Big] = F'(x). \tag{7.10}$$

Combining the results (7.9) and (7.10), we see that $F'(x) = F'(y)$ for all $x, y > \alpha$ and thus

$$F'(x) = c_\alpha \tag{7.11}$$

for all $x > \alpha$, where c_α is a constant. Next, we recall from the definition of F that

$$F(x) = m(E \cap [\alpha, x]) = \int_\alpha^x \chi_E(t)\,\mathrm{d}t.$$

Since $\chi_E \in L^1(\mathbb{R})$, it establishes from Theorem 7.11 that F is differentiable and $F'(x) = \chi_E(x)$ a.e. for $x > \alpha$. Using this and the fact (7.11), we get

$$\chi_E(x) = c_\alpha \tag{7.12}$$

a.e. for $x > \alpha$. Since χ_E takes only 0 or 1, we see that $c_\alpha \in \{0, 1\}$. If $x > \beta > \alpha$, then

$$c_\beta = F'(x) = \chi_E(x) = F'(x) = c_\alpha,$$

so c_α *does not* depend on α and we can simply replace c_α by c in the results (7.11) and (7.12). Finally, if $c = 0$, then we conclude from the result (7.12) that

$$m(E) = \int_{\mathbb{R}} \chi_E\,\mathrm{d}m = 0.$$

Otherwise, $c = 1$ and the result (7.12) implies that $\chi_{E^c}(x) = 0$ and hence $m(E^c) = 0$. We have completed the proof of the problem. ∎

Problem 7.4

Rudin Chapter 7 Exercise 4.

Proof. Our proof does not follow Rudin's hint. In fact, we mainly follow the argument of Cignoli and Hounie [15] and now we first quote one of their results that will be used very soon:

Lemma 7.1

Let D be a dense subset of $\mathbb{R}$ and μ a Borel measure on $\mathbb{R}$ such that

$$\mu\big(d + [a,b)\big) = \mu\big([a,b)\big)$$

for every $d \in D$ and $a, b \in \mathbb{R}$. Then we have $\mu = km$, where $k = \mu\big((0,1]\big)$.

Next, it is clear that the set $G(f) = \{ns + mt \,|\, n, m \in \mathbb{Z}\}$ is an additive group of $\mathbb{R}$. Without loss of generality, we assume that $t > 0$. Since $\alpha = s/t$ is irrational, it follows from the Kronecker's Approximation Theorem[a] that for every $\theta \in \mathbb{R}$ and every $\epsilon t^{-1} > 0$, there exists $n, m \in \mathbb{Z}$ such that

$$\Big| n\alpha + m - \frac{\theta}{t} \Big| < \frac{\epsilon}{t}$$

which is equivalent to saying that

$$|ns + mt - \theta| < \epsilon.$$

Hence $G(f)$ is dense in $\mathbb{R}$.

We first consider the special case that f is nonnegative and bounded a.e. on $\mathbb{R}$. Note that if $F(x) = f(xt)$, then $F(x+1) = f(xt + t) = f(xt) = F(x)$, so F is a function of period 1 and we may assume that f has period 1. Consequently, it suffices to prove the problem when f is a nonnegative, bounded a.e. function on $[0,1]$ having period 1. Suppose that $\mu : \mathscr{B} \to [0, \infty)$ is defined by

$$\mu(E) = \int_E f(x)\,\mathrm{d}x \tag{7.13}$$

for all $E \in \mathscr{B}$. By Theorem 1.29, μ is a Borel measure. Let p be a period of f. Then we deduce from the representation (7.13) that

$$\mu(E + p) = \int_{E+p} f(x)\,\mathrm{d}x = \int_E f(x+p)\,\mathrm{d}x = \int_E f(x)\,\mathrm{d}x = \mu(E).$$

In other words, μ has the same periods as f. Furthermore, if $p \in G(f)$, then we obtain from Theorem 7.26 (The Change-of-variables Theorem)[b] that

$$\mu\big(p + [a,b)\big) = \mu\big([a+p, b+p)\big) = \int_{a+p}^{b+p} f(x)\,\mathrm{d}x = \int_a^b f(x+p)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x = \mu\big([a,b)\big).$$

By Lemma 7.1, we have

$$\mu = km, \tag{7.14}$$

where $0 < k = \mu\big((0,1]\big) < \infty$. By combining the results (7.13) and (7.14), we establish that

$$\int_E [f(x) - k]\,\mathrm{d}x = 0$$

for every measurable subset E of $[0,1]$. Since f is bounded a.e. on $[0,1]$, it is clear that $f - k \in L^1\big([0,1]\big)$. Now Theorem 1.39(b) implies that $f = k$ a.e. on $[0,1]$ and the periodicity of f shows that

$$f = k$$

[a] See [63, Lemma 4.6, p. 82]

[b] It is applied to the third equality. Recall that we are discussing Lebesgue integral, *not* Riemann integral.

a.e. on $\mathbb{R}$ in this special case.

For the general case, the composite function $g(x) = \frac{\pi}{2} + \tan^{-1}\big(f(x)\big)$ is clearly a measurable function on $\mathbb{R}$. Furthermore, g is also nonnegative and bounded a.e. on $\mathbb{R}$. If $f(x+1) = f(x)$, then $g(x+1) = \frac{\pi}{2} + \tan^{-1}\big(f(x+1)\big) = \frac{\pi}{2} + \tan^{-1}\big(f(x)\big) = g(x)$ so that g is also a function of period 1. Applying the previous part to g to conclude that $g = k$ a.e. on $\mathbb{R}$ for a constant k and this implies that

$$f(x) = \tan\Big(k - \frac{\pi}{2}\Big)$$

a.e. on $\mathbb{R}$.

Finally, if we define $G = \{ns + mt \,|\, m, n \in \mathbb{Z}\}$ and $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in G; \\ 1, & \text{otherwise,} \end{cases}$$

then it is trivial that f has periods s and t as well as $f = 1$ a.e. on $\mathbb{R}$.[c] This completes the proof of the problem. ∎

Problem 7.5

Rudin Chapter 7 Exercise 5.

Proof. We prove the assertions one by one.

- $A + B$ **contains a segment.** Since $m(A) > 0$ and $m(B) > 0$, A and B are Lebesgue measurable. By the comment in §7.12, the metric densities of A and B are 1 at *almost every* point of A and B. Thus we pick $a_0 \in A$ and $b_0 \in B$ such that

$$\lim_{r\to 0+} \frac{m\big(A \cap B(a_0, r)\big)}{m\big(B(a_0, r)\big)} = \lim_{r\to 0+} \frac{m\big(B \cap B(b_0, r)\big)}{m\big(B(b_0, r)\big)} = 1. \tag{7.15}$$

  Since $\chi_A, \chi_B \in L^1(\mathbb{R})$, we may assume with the aid of Theorem 7.7 that a_0 and b_0 are Lebesgue points of χ_A and χ_B respectively.

  Set $c_0 = a_0 + b_0$. For each $\epsilon > 0$, define

$$B_\epsilon = \{c_0 + \epsilon - b \,|\, b \in B \text{ and } |b - b_0| < \delta\}, \tag{7.16}$$

  where $\delta > 0$ is small. For every $c_0 + \epsilon - b$, we have

$$|c_0 + \epsilon - b - (a_0 + \epsilon)| = |c_0 + \epsilon - b - (c_0 - b_0 + \epsilon)| = |b - b_0| < \delta$$

  which implies that $B_\epsilon \subseteq (a_0 + \epsilon - \delta, a_0 + \epsilon + \delta)$. If we take $\delta = 2\epsilon$ and ϵ is sufficiently small, then we have

$$B_\epsilon \subseteq (a_0 + \epsilon - \delta, a_0 + \epsilon + \delta) = (a_0 - \epsilon, a_0 + 3\epsilon) \subseteq (a_0 - 4\epsilon, a_0 + 4\epsilon) = B(a_0, 4\epsilon). \tag{7.17}$$

  By the definition (7.16), we note that

$$B_\epsilon - (c_0 + \epsilon) = \{-b \,|\, b \in B \text{ and } |b - b_0| < 2\epsilon\} = B \cap B(b_0, 2\epsilon).$$

  Since $B_\epsilon - (c_0 + \epsilon)$ is just a translate of B_ϵ, Theorem 2.20(c) says that

$$m(B_\epsilon) = m\big(B_\epsilon - (c_0 + \epsilon)\big) = m\big(B \cap B(b_0, 2\epsilon)\big). \tag{7.18}$$

[c] By Problem 7.6, we have $m(G) = 0$.

By the metric density of B at b_0 in (7.15), we know that

$$m\big(B\cap B(b_0,2\epsilon)\big) > \frac{m\big(B(b_0,2\epsilon)\big)}{2} = 2\epsilon. \tag{7.19}$$

Combining the expression (7.18) and the inequality (7.19), we arrive at

$$m(B_\epsilon) > 2\epsilon > 0. \tag{7.20}$$

We claim that $A\cap B_\epsilon \neq \varnothing$ *for all* small enough $\epsilon > 0$. Otherwise, no matter how small $\epsilon > 0$ is, we can find a $\eta \in (0,\epsilon)$ such that $A\cap B_\eta = \varnothing$. In fact, we may pick η small enough so that the metric density of A at a_0 in (7.15) implies

$$m\big(A\cap B(a_0,4\eta)\big) > \frac{7}{8}m\big(B(a_0,4\eta)\big) = 7\eta. \tag{7.21}$$

Since $A\cap B_\eta = \varnothing$, we have $A\cap[B(a_0,4\eta)\setminus B_\eta] = [A\cap B(a_0,4\eta)]\setminus[A\cap B_\eta] = A\cap B(a_0,4\eta)$ so that

$$m\big(A\cap B(a_0,4\eta)\big) = m\big(A\cap[B(a_0,4\eta)\setminus B_\eta]\big) \leq m\big(B(a_0,4\eta)\setminus B_\eta\big). \tag{7.22}$$

Recall from the set relation (7.17), we have the fact $B(a_0,4\eta) = [B(a_0,4\eta)\setminus B_\eta]\cup B_\eta$. By this fact and the inequality (7.20), we may further reduce the inequality (7.22) to

$$m\big(A\cap B(a_0,4\eta)\big) \leq m\big(B(a_0,4\eta)\big) - m(B_\eta) < 8\eta - 2\eta = 6\eta. \tag{7.23}$$

However, the inequalities (7.21) and (7.23) are contradictory and this proves our claim that $A\cap B_\epsilon \neq \varnothing$ for all small $\epsilon > 0$.

Pick $x\in(c_0-\epsilon',c_0+\epsilon')$ for sufficiently small ϵ' and without loss of generality, suppose that $\epsilon = x - c_0 > 0$. Then we have $0<\epsilon<\epsilon'$ and the above claim implies that

$$A\cap B_\epsilon \neq \varnothing.$$

Choose $a\in A\cap B_\epsilon$. By the definition (7.16) and then using the fact $\epsilon = x - c_0$, we have

$$a = c_0 + \epsilon - b = x - b$$

for some $b\in B$ with $|b-b_0|<2\epsilon$. In other words, we have the representation

$$x = a + b \in A + B,$$

i.e., $(c_0-\epsilon',c_0+\epsilon')\subseteq A+B$.

- $\frac{C}{2}+\frac{C}{2}=[0,1]$. Recall from [49, Exercise 19, p. 81] that if $c\in C$, then

$$c = \sum_{n=1}^{\infty}\frac{\alpha_n}{3^n},$$

where $\alpha_n = 0$ or 2. On the one hand, let $x\in[0,1]$ and we have

$$x = \sum_{n=1}^{\infty}\frac{a_n}{3^n}, \tag{7.24}$$

where $a_n\in\{0,1,2\}$ for all $n=1,2,\ldots$. Define

$$\beta_n = \begin{cases} 0, & \text{if } a_n\in\{0,1\};\\ 1, & \text{otherwise,}\end{cases} \quad\text{and}\quad \gamma_n = \begin{cases} 0, & \text{if } a_n\in\{0\};\\ 1, & \text{otherwise.}\end{cases}$$

Then it is always true that $a_n = \beta_n + \gamma_n$ for all $n = 1, 2, \ldots$. Thus the x given in (7.24) can be represented as

$$x = y + z.$$

Here y and z are given by

$$y = \sum_{n=1}^{\infty} \frac{\beta_n}{3^n} \quad \text{and} \quad z = \sum_{n=1}^{\infty} \frac{\gamma_n}{3^n}, \tag{7.25}$$

where $\beta_n, \gamma_n \in \{0, 1\}$ for all $n = 1, 2, \ldots$.[d] In other words, the analysis implies that $[0, 1] \subseteq \frac{1}{2}C + \frac{1}{2}C$.

On the other hand, let $y, z \in \frac{1}{2}C$ be expressed by the series (7.25), where $\beta_n, \gamma_n \in \{0, 1\}$. Then we have

$$y + z = \sum_{n=1}^{\infty} \frac{\beta_n + \gamma_n}{3^n},$$

where $\beta_n + \gamma_n \in \{0, 1, 2\}$ for all $n = 1, 2, \ldots$. If $\beta_n + \gamma_n = 0$ for all $n = 1, 2, \ldots$, then $y + z = 0$. If $\beta_n + \gamma_n = 2$ for all $n = 1, 2, \ldots$, then

$$y + z = \sum_{n=1}^{\infty} \frac{2}{3^n} = 1.$$

Otherwise, $y + z \in (0, 1)$. Therefore, we have $\frac{1}{2}C + \frac{1}{2}C \subseteq [0, 1]$. Hence we have obtained the required result that $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$.

We have completed the proof of the problem. ■

Remark 7.2

The result in Problem 7.5 is called the **Steinhaus Theorem**. It has a generalization in $\mathbb{R}^k$ which says that if $A \in \mathscr{B}^k$ and $m_k(A) > 0$, then $\mathbf{0}$ is an interior point of $A - A$, see [5, Theorem 26.6, p. 163] for details.

Problem 7.6

Rudin Chapter 7 Exercise 6.

Proof. Assume that $m(G) > 0$. By Problem 7.5, $G + G$ contains a segment (a, b). Since G is a group with addition, we have $G + G \subseteq G$ so that $(a, b) \subseteq G$. Let $m = \frac{a+b}{2} \in (a, b)$ so that $m \in G$. Since G is a group, $-m \in G$ so that $(a - m, b - m) \subseteq G$. Since m is the mid-point of a and b, the segment $(a - m, b - m)$ is in the form $(-c, c)$ for some *fixed* $c \in \mathbb{R} \setminus \{0\}$. Recall that $G + G \subseteq G$, so we have $(-2c, 2c) \subseteq G$. Repeat this kind of argument, we may conclude that

$$(-nc, nc) \subseteq G$$

for all $n \in \mathbb{N}$, but this implies that $G = \mathbb{R}$, a contradiction. Hence $m(G) = 0$ and this completes the proof of the problem. ■

[d] Since the series (7.25) converge, we can add them together.

7.3 The Cantor Function and the Non-measurability of $f \circ T$

Problem 7.7

Rudin Chapter 7 Exercise 7.

Proof. The construction has been done in [26, Example 15, pp. 96 – 98]. We won't repeat it here. See also [26, Example 30, pp. 105, 195]. ■

Problem 7.8

Rudin Chapter 7 Exercise 8.

Proof. We have $T : V = (a, b) \to \mathbb{R}$. Since V is a bounded segment, we define $\varphi(a) = \varphi(b) = 0$ so that the domain of T can be extended to $[a, b]$ easily.

(a) Obviously, V is open in $\mathbb{R}$ and Lebesgue measurable. Since $0 \leq \varphi_n(x) < 2^{-n}$ in V and $\sum_{n=1}^{\infty} 2^{-n} < \infty$, the Weierstrass M-test [49, Theorem 7.10, p. 148] indicates that

$$\varphi = \sum_{n=1}^{\infty} \varphi_n \tag{7.26}$$

converges uniformly on V. Since each φ_n is continuous on V, φ is continuous on V. Furthermore, $0 \leq \varphi(x) \leq 1$ on V so that φ is also bounded on $[a, b]$. By [49, Theorem 11.33, p. 323], we see that $\varphi \in \mathscr{R}$ on $[a, x]$ and

$$T(x) = \int_a^x \varphi(t)\,\mathrm{d}t = \mathscr{R}\int_a^x \varphi(t)\,\mathrm{d}t$$

for every $x \in [a, b]$. Therefore, it follows from the First Fundamental Theorem of Calculus that T is differentiable at every point of the continuity of φ in $[a, x]$. Consequently, T is differentiable (and hence continuous) at every point of V.

Next, if $T(x) = T(y)$ for $x < y$, then we have

$$T(y) = \int_a^y \varphi(t)\,\mathrm{d}t = T(x) + \int_x^y \varphi(t)\,\mathrm{d}t$$

which implies that

$$\int_x^y \varphi(t)\,\mathrm{d}t = 0. \tag{7.27}$$

Since W is dense in V, $(x, y) \cap W \neq \varnothing$ so that $(x, y) \cap W_N \neq \varnothing$ *for some* positive integer N. Let $p \in (x, y) \cap W_N$. By the definition, we have $\varphi(p) > 0$ and the sign-preserving property of continuous functions (see [64, Problem 7.15, p. 112]) assures that there exists a $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq (x, y) \cap W_n$ so that $\varphi(q) > 0$ for all $q \in (p - \delta, p + \delta)$, but this implies that

$$\int_x^y \varphi(t)\,\mathrm{d}t \geq \int_{p-\delta}^{p+\delta} \varphi(t)\,\mathrm{d}t > 0,$$

a contradiction to the result (7.27). Thus T is one-to-one and condition (ii) is satisfied. Since T is differentiable at every point of V, it is continuous on V and condition (i) is also satisfied. Condition (iii) is obvious because $X \setminus X = \varnothing$.

(b) By the First Fundamental Theorem of Calculus, we know that $T' = \varphi$ which is continuous on V. If $p \in K$, then $p \notin W_n$ for all $n \in \mathbb{N}$ which implies that $\varphi_n(p) = 0$ for all $n \in \mathbb{N}$. Therefore, we have

$$T'(p) = \varphi(p) = \sum_{n=1}^{\infty} \varphi_n(p) = 0. \tag{7.28}$$

For the third assertion, we first show that $T(K)$ is a Borel set. To this end, since T is injective, we note from [42, Exercise 2(h), p. 21] that

$$T(K) = T(V \setminus W) = T(V) \setminus T(W) = T(V) \setminus \bigcup_{n=1}^{\infty} T(W_n).$$

Since T is continuous and injective on V, the property of **Invariance of Domain** [32, Theorem 2B.3, p. 172] shows that T is an open map. Thus $T(V)$ and each $T(W_n)$ is open in $\mathbb{R}$ so that $T(K) \in \mathscr{B}$, as desired.

By the previous result, the function $f = \chi_{T(K)}$ is Lebesgue measurable. By part (a), T satisfies the hypotheses of Theorem 7.26 (The Change-of-variables Theorem), so we have

$$m\big(T(K)\big) = \int_{T(V)} \chi_{T(K)}\,\mathrm{d}m = \int_{T(V)} f\,\mathrm{d}m = \int_V (f \circ T)|J_T|\,\mathrm{d}m, \tag{7.29}$$

where $|J_T(x)|$ denotes the Jacobian of T at x. Recall that T is injective, so $T(x) \in T(K)$ if and only if $x \in K$ and then we reduce from the expression (7.29) that

$$m\big(T(K)\big) = \int_V \chi_{T(K)}\big(T(x)\big)|J_T|\,\mathrm{d}m = \int_K \chi_K(x)|J_T|\,\mathrm{d}m. \tag{7.30}$$

Notice from the result (7.28) that $|J_T(x)| = |\det T'(x)| = 0$ on K, so we obtain from the integral (7.30) that $m\big(T(K)\big) = 0$.

(c) Since $m(K) > 0$, it follows from Corollary to Theorem 2.22 that K contains a nonmeasurable subset E. Let $A = T(E)$. Since $E \subset K$, we have $A \subset T(K)$. Since $m\big(T(K)\big) = 0$, we recall from Theorem 1.36 that all subsets of sets of measure 0 are measurable, thus A is a measurable set and χ_A is a measurable function. By the definition, we know that

$$\chi_A\big(T(x)\big) = \chi_{T(E)}\big(T(x)\big) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, we have $\chi_A \circ T = \chi_E$. Since E is nonmeasurable, χ_E is not measurable.

(d) By part (b), we know that $T'(x) = \varphi(x)$ on V. Thus if we can construct a φ which is *infinitely differentiable*, then we are done. To see this, we are going to construct an infinitely differentiable function φ_n on W_n for each positive integer n. Let's recall the **bump function** $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} \exp\Big(-\dfrac{1}{1-x^2}\Big), & \text{if } x \in (-1,1); \\ 0, & \text{otherwise.} \end{cases}$$

The ψ has derivatives of all orders in $\mathbb{R}$, i.e., ψ is an element of $\mathscr{C}^\infty(\mathbb{R})$ the space of all infinitely differentiable functions on $\mathbb{R}$. Since each W_n is a segment, we let $W_n = (a_n, b_n)$ for some $a_n, b_n \in (a,b)$ with $a_n < b_n$. Therefore the bump function $\varphi_n : V \to \mathbb{R}$ given by

$$\varphi_n(x) = 2^{-n}\psi\Big(\frac{2}{b_n - a_n}(x - b_n) + 1\Big) = 2^{-n}\psi\Big(\frac{x - \frac{a_n+b_n}{2}}{\frac{b_n-a_n}{2}}\Big)$$

is what we want. Put $U_1 = W_1$ and $U_n = W_n \setminus (W_1 \cup W_2 \cup \cdots \cup W_{n-1})$ for $n = 2, 3, \ldots$. Then we have $U_i \cap U_j = \varnothing$ for all $i \neq j$ and their union is also W. Since each W_n is a segment, each U_n is a union of at most countable *disjoint* segments. In other words, we may assume that $W_i \cap W_j = \varnothing$ for all $i \neq j$. Therefore, the function $\varphi : V \to \mathbb{R}$ defined by (7.26) satisfies

$$\varphi(x) = \begin{cases} \varphi_n(x), & \text{if } x \in W_n \subseteq W; \\ 0, & \text{if } x \in V \setminus W. \end{cases}$$

Hence, as we have mentioned above, φ is infinitely differentiable in V and we complete the construction of the required integral T.

This completes the proof of the problem. ■

7.4 Problems related to the AC of a Function

Problem 7.9

Rudin Chapter 7 Exercise 9.

Proof. The function f discussed in §7.16 is called a singular function (see [8, p. 161] or [34, Definition 7.1.44, p. 170]). Let $0 < \alpha < 1$. Pick t so that $t^\alpha = 2$. Let $\delta_n = (\frac{2}{t})^n = 2^n t^{-n}$ for all $n \in \mathbb{N}$.

We remark that comprehensive materials about the relations between singular functions and Cantor sets can be found in [33, Chap. 4] and [34, Chap. 8]. In fact, our proof here relies heavily on the known results there.

We fix some notations first. Let $n \geq 0$. At the nth step, we have

$$E_n = \bigcup_{k=1}^{2^n} I_n(k), \tag{7.31}$$

where $I_n(1), I_n(2), \ldots, I_n(2^n)$ are the 2^n disjoint closed intervals, each of length $2^{-n}\delta_n = t^{-n}$. Next the deleted open segment between the intervals $I_n(k)$ and $I_n(k+1)$ is denoted by $J(k\cdot 2^{-n})$, where $k = 1, 2, \ldots, 2^n - 1$. Suppose that

$$D = \{m \cdot 2^{-n} \,|\, 0 \leq m \leq 2^n \text{ and } n \geq 0\}$$

which is the set of dyadic rationals in $[0, 1]$.

We claim that the function f discussed in §7.16 satisfies

$$f(x) = k \cdot 2^{-n} \tag{7.32}$$

for all $x \in J(k \cdot 2^{-n})$ and $k \cdot 2^{-n} \in D$. To this end, the hypothesis $x \in J(k \cdot 2^{-n})$ implies that there are k closed disjoint intervals $I_n(1), I_n(2), \ldots, I_n(k)$ on its left-hand side, so we derive from this and the fact [51, Eqn. (5), p. 145] that

$$f_n(x) = \int_0^x g_n(t)\,\mathrm{d}t = \sum_{m=1}^{k} \int_{I_n(m)} g_n(t)\,\mathrm{d}t = k \cdot 2^{-n}. \tag{7.33}$$

Since $x \notin E_n$ by the definition (7.31), we see from [51, Eqn. (6), p. 145] and the values (7.33) that

$$f(x) = \lim_{n\to\infty} f_n(x) = k \cdot 2^{-n}.$$

This proves our claim. By [51, Eqn. (3), p. 145], we have $E^c = \bigcup_{n=1}^{\infty} E_n^c$. Since $x \notin E_n$, we have $x \in E_n^c \subseteq E^c$. Therefore, the f is well-defined on E^c by the values (7.32).

By [33, p. 84] or [34, p. 202], E is *nowhere dense* in $[0,1]$ so that E^c is a dense subset of $[0,1]$. Next, since f is uniformly continuous on $[0,1]$ and particularly on E^c, it follows from [49, Exercise 13, pp. 99, 100] that f has a unique continuous extension on $[0,1]$ which agrees with f on E^c. Hence the uniqueness shows that our f considered in §7.16 is the so-called **Lebesgue function associated with the Cantor-like set** E. Let $h : [0,1] \to \mathbb{R}$ be defined by

$$h(x) = x^{\alpha}.$$

Since $\alpha \in (0,1)$, it is immediate from basic calculus that h is strictly concave on $[0,1]$. Clearly, we know that

$$h(2^{-n}\delta_n) = (2^{-n}\delta_n)^{\alpha} = \frac{1}{t^{n\alpha}} = 2^{-n}, \tag{7.34}$$

where $n = 0, 1, 2, \ldots$. Thus it gives

$$h\big(2^{-(n-1)}\delta_{n-1}\big) = 2^{-(n-1)} = 2 \cdot 2^{-n} = 2h(2^{-n}\delta_n) > h(2^{-n+1}\delta_n)$$

for all $n = 1, 2, \ldots$. Since h is strictly increasing on $[0,1]$, we obtain $2^{-(n-1)}\delta_{n-1} > 2^{-n+1}\delta_n$ which is equivalent to $\delta_{n-1} > \delta_n$ for each $n = 1, 2, \ldots$. By the formula (7.34), we know that $h(\delta_0) = 1$ which implies $\delta_0 = 1$. In other words, the Cantor-like set E in §7.16 is constructed by means of our fixed concave function h. By [33, p. 97] or [34, Theorem 8.5.12, p. 205], it concludes that

$$|f(x) - f(y)| \leq h\big(|x-y|\big) = |x-y|^{\alpha}$$

for all $x, y \in [0,1]$. Hence f belongs to $\mathrm{Lip}\,\alpha$ on $[0,1]$, completing the proof of the problem. ■

Problem 7.10

Rudin Chapter 7 Exercise 10.

Proof. By Problem 5.11, there exists a constant $M > 0$ such that

$$|f(s) - f(t)| \leq M|s-t| \tag{7.35}$$

for all $s, t \in [a,b]$. For every $\epsilon > 0$, if we take $\delta = \frac{\epsilon}{M}$, then for any n and any disjoint collection of segments $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ in $[a,b]$ with

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \frac{\epsilon}{M},$$

the inequality (7.35) implies that

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| \leq M \sum_{i=1}^{n} |\beta_i - \alpha_i| = M \sum_{i=1}^{n} (\beta_i - \alpha_i) < \epsilon.$$

By Definition 7.17, f is AC on $[a,b]$.

By Theorem 7.20, f' exists a.e. on $[a,b]$. Let f be differentiable at $x \in [a,b]$. Then we follow from the inequality (7.35) that

$$|f'(x)| = \lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq \lim_{h \to 0} M \left| \frac{(x+h) - x}{h} \right| = M,$$

i.e., $|f'| \leq M$ a.e. on $[a,b]$. Hence $f' \in L^{\infty}$ and we have completed the proof of the problem. ■

Problem 7.11

Rudin Chapter 7 Exercise 11.

Proof. Since f is AC on $[a,b]$, we get from Theorem 7.20 that

$$f(x)-f(a)=\int_a^x f'(t)\,\mathrm{d}t \quad \text{and} \quad f(y)-f(a)=\int_a^y f'(t)\,\mathrm{d}t$$

for any $a\leq y\leq x\leq b$. Now their difference gives

$$f(x)-f(y)=\int_y^x f'(t)\,\mathrm{d}t.$$

By Theorem 7.20 again, we have $f'\in L^1(m)$ so that Theorem 1.33 implies

$$|f(x)-f(y)|=\Big|\int_y^x f'(t)\,\mathrm{d}t\Big|\leq \int_y^x |f'(t)|\,\mathrm{d}t. \tag{7.36}$$

Since $f'\in L^p$, we apply Theorem 3.5 (Hölder's Inequality) to the right-hand side of the inequality (7.36) to obtain

$$|f(x)-f(y)|\leq\Big\{\int_y^x |f'(t)|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_y^x \mathrm{d}t\Big\}^{\frac{1}{q}}\leq \|f'\|_p\cdot|x-y|^\alpha$$

and thus

$$\sup_{\substack{x,y\in[a,b]\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\leq\|f'\|_p<\infty.$$

By the definition (see Problem 5.11), we conclude that $f\in \mathrm{Lip}\,\alpha$, completing the proof of the problem. ■

Problem 7.12

Rudin Chapter 7 Exercise 12.

Proof.

(a) Since φ is nondecreasing, we follow from [49, Theorems 4.29 & 4.30, pp. 95, 96] that $\varphi(x+)$ and $\varphi(x-)$ exist for every point x of $[a,b]$ and *the set of discontinuities of* φ *is at most countable*. Let this set be $A=\{x_1,x_2,\ldots\}$ and $x_1<x_2<\cdots$. We define $f:[a,b]\to\mathbb{R}$ by $f(x)=\sup\limits_{t<x}\varphi(t)$. In fact, we have

$$f(x)=\begin{cases}\varphi(x), & \text{if } x\in[a,b]\setminus A;\\ \lim\limits_{\substack{t\to0\\ t>0}}\varphi(x_k-t)=\varphi(x_k-), & \text{if } x=x_k \text{ for some } k\in\mathbb{N}.\end{cases} \tag{7.37}$$

Let $x<y$ and $x,y\in[a,b]$. Then the nondecreasing property of φ ensures that

$$f(x)=\sup_{t<x}\varphi(t)\leq\sup_{t<y}\varphi(t)=f(y)$$

which implies that f is nondecreasing on $[a,b]$.

Clearly, $\{x \in [a,b] \,|\, f(x) \neq \varphi(x)\} \subseteq A$ is at most countable. Given $x \in (a,b]$ and $\epsilon > 0$. By the definition (7.37), if $x = x_k$, then there is a $\delta > 0$ such that

$$|f(x_k) - \varphi(x_k - t)| < \epsilon \tag{7.38}$$

for all $t \in (0,\delta)$. By the construction of A, the number δ can be chosen so small such that $x_k - t \notin A$ for all $t \in (0,\delta)$, i.e., $x_k - t \in [a,b] \setminus A$ for all $t \in (0,\delta)$. Then $\varphi(x)$ is continuous at every $x_k - t$ and $\varphi(x_k - t) = f(x_k - t)$. Thus the inequality (7.38) reduces to

$$|f(x_k) - f(x_k - t)| < \epsilon$$

for all $t \in (0,\delta)$. In other words, f is a left-continuous function.

(b) Consider the function $g : [a,b] \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} \varphi(t+) - \varphi(t-), & \text{if } t \in (a,b); \\ 0, & \text{if } t = a; \\ 0, & \text{if } t = b. \end{cases} \tag{7.39}$$

Since φ is nondecreasing, we have $g(t) \geq 0$ on $[a,b]$.

Firstly, we claim that

$$\sum_{t\in[a,b]} g(t) \leq \varphi(b) - \varphi(a). \tag{7.40}$$

Since $g(t) = 0$ on $[a,b] \setminus A$, we have

$$\sum_{t\in[a,b]} g(t) = \sum_{t\in A} g(t).$$

Let $A_N = \{x_1, x_2, \ldots, x_N\} \subseteq A$ for a positive integer N. Insert $\{y_0, y_1, y_2, \ldots, y_N\}$ into A_N in such a way that

$$a \leq y_0 \leq x_1 < y_1 < x_2 < \cdots < y_{N-1} < x_N \leq y_N \leq b. \tag{7.41}$$

Since φ is nondecreasing, we have

$$\varphi(x_k+) \leq \varphi(y_k) \leq \varphi(x_{k+1}-)$$

for $k = 1, 2, \ldots, N-1$. For the end points x_1 and x_N, if $a \leq y_0 < x_1$, then $\varphi(x_1-) \geq \varphi(y_0) \geq \varphi(a)$; if $a = y_0 = x_1$, then we have obviously $\varphi(x_1) = \varphi(y_0) = \varphi(a)$. Similarly, if $x_N < y_N \leq b$, then $\varphi(x_N+) \leq \varphi(y_N) \leq \varphi(b)$; if $x_N = y_N = b$, then we certainly have $\varphi(x_N) = \varphi(y_N) = \varphi(b)$. Hence this analysis implies that

$$\varphi(b) - \varphi(a) \geq \varphi(y_N) - \varphi(y_0) \geq \sum_{k=1}^{N} [\varphi(x_k+) - \varphi(x_k-)] = \sum_{k=1}^{N} g(x_k). \tag{7.42}$$

Now our claim (7.40) follows directly by taking limit $N \to \infty$ in the inequality (7.42).

Secondly, we define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = \sum_{t\in A} g(t)\chi_{(t,b]}(x) = \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x). \tag{7.43}$$

By the claim (7.40) (or the second inequality in (7.42)) and the Weierstrass M-test, we know that the sum (7.43) actually converges uniformly on $[a, b]$. If $a \leq x < y \leq b$, then we have $\chi_{(t,b]}(x) \leq \chi_{(t,b]}(y)$ and this shows that

$$h(x) = \sum_{t \in A} g(t)\chi_{(t,b]}(x) \leq \sum_{t \in A} g(t)\chi_{(t,b]}(y) = h(y),$$

i.e., h is nondecreasing on $[a, b]$.

Thirdly, we need two lemmas about the properties of the function $F : [a, b] \to \mathbb{R}$ defined by $F(x) = f(x) - h(x)$.

Lemma 7.2

The function F is continuous on $[a, b]$.

Proof of Lemma 7.2. If $x \notin A$, then $f = \varphi$ is continuous at x. Furthermore, if $x_m < x < x_{m+1}$ for some $m \in \mathbb{N}$, then for all small enough δ, we have $(x - \delta, x + \delta) \subset (x_m, x_{m+1})$. In this case, we obtain

$$\lim_{\delta \to 0} h(x \pm \delta) = \lim_{\delta \to 0} \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x \pm \delta) = \lim_{\delta \to 0} \sum_{k=1}^{m} g(x_k)\chi_{(x_k,b]}(x \pm \delta) = h(x),$$

i.e., h is continuous at x. By the definition, F is also continuous at x.

If $x \in A$, then $x = x_m$ for some $m \in \mathbb{N}$. On the one hand, the definition (7.37) gives

$$f(x_m+) = \lim_{\substack{\delta \to 0 \\ \delta > 0}} f(x_m + \delta) = \varphi(x_m+) \quad \text{and} \quad f(x_m-) = \varphi(x_m-).$$

On the other hand, the definitions (7.39) and (7.43) show

$$h(x_m+) = \lim_{\substack{\delta \to 0 \\ \delta > 0}} h(x_m + \delta) = \lim_{\substack{\delta \to 0 \\ \delta > 0}} \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x_m + \delta) = \sum_{k=1}^{m} [\varphi(x_k+) - \varphi(x_k-)]$$

and

$$h(x_m-) = \lim_{\substack{\delta \to 0 \\ \delta < 0}} h(x_m + \delta) = \lim_{\substack{\delta \to 0 \\ \delta < 0}} \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x_m + \delta) = \sum_{k=1}^{m-1} [\varphi(x_k+) - \varphi(x_k-)].$$

Consequently, they imply that

$$\begin{aligned} F(x_m+) &= f(x_m+) - h(x_m+) \\ &= \varphi(x_m+) - \sum_{k=1}^{m} [\varphi(x_k+) - \varphi(x_k-)] \\ &= \varphi(x_m-) - \sum_{k=1}^{m-1} [\varphi(x_k+) - \varphi(x_k-)] \\ &= f(x_m-) - h(x_m-) \\ &= F(x_m-), \end{aligned}$$

i.e., F is continuous at x_m and this ends the proof of the lemma. ■

Lemma 7.3

The function F is nondecreasing on $[a,b]$.

Proof of Lemma 7.3. Let $x, y \in [a,b]$ be $x < y$. Suppose that $x_m < x \leq x_{m+1}$ for some $m \in \mathbb{N}$. If $x_m < x < y \leq x_{m+1}$, then the definition (7.43) gives

$$\begin{aligned}
h(y) - h(x) &= \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(y) - \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x) \\
&= \sum_{k=1}^{m} g(x_k)\chi_{(x_k,b]}(y) - \sum_{k=1}^{m} g(x_k)\chi_{(x_k,b]}(x) \\
&= 0 \\
&\leq f(y) - f(x).
\end{aligned}$$

Suppose that

$$x_m < x \leq x_{m+1} < \cdots < x_{m+p} \leq y \tag{7.44}$$

for some $p \in \mathbb{N}$, then we get from the definition (7.39) that

$$\begin{aligned}
h(y) - h(x) &= \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(y) - \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x) \\
&= \sum_{k=1}^{m} g(x_k)\chi_{(x_k,b]}(y) + \sum_{k=m+1}^{m+p} g(x_k)\chi_{(x_k,b]}(y) - \sum_{k=1}^{m} g(x_k)\chi_{(x_k,b]}(x) \\
&= \sum_{k=m+1}^{m+p} g(x_k)\chi_{(x_k,b]}(y) \\
&\leq \sum_{k=m+1}^{m+p} [\varphi(x_k+) - \varphi(x_k-)]. \qquad (7.45)
\end{aligned}$$

If we apply the same trick (the sequence (7.41) is replaced by the sequence (7.44) with a and b are replaced by x and y respectively) as in proving the inequality (7.42), we can easily deduce from the inequality (7.45) that $h(y) - h(x) \leq f(y) - f(x)$.

In conclusion, we have shown that

$$h(y) - h(x) \leq f(y) - f(x)$$

for all $x, y \in [a,b]$ with $x < y$. Equivalently, it means that

$$F(x) = f(x) - h(x) \leq f(y) - h(y) \leq F(y)$$

and so F is nondecreasing, as desired. ∎

Fourthly, we express F in terms of a finite and positive Borel measure. By Lemmas 7.2 and 7.3, we know that the function

$$g(x) = x + F(x)$$

is continuous and strictly increasing on $[a,b]$. Then g is one-to-one on $[a,b]$ so that its inverse function $g^{-1} : [g(a), g(b)] \to [a,b]$ is a continuous and bijective function on $[g(a), g(b)]$

by [49, Theorem 4.17, p. 90]. For every $E \subseteq [a,b]$ and $E \in \mathscr{B}$, since g^{-1} is continuous on $[g(a), g(b)]$, Definition 1.2(c) tells us that $g(E) = (g^{-1})^{-1}(E) \in \mathscr{B}$ so that we are able to define

$$\nu(E) = m\big(g(E)\big). \tag{7.46}$$

We have to show that ν is a positive Borel measure, but it follows exactly the same reasons as in [51, p. 147]. Furthermore, since m is a finite measure, the construction (7.46) implies that ν is also a finite measure. By Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem), there exists a unique pair of positive and finite Borel measures ν_{ac} and ν_{sc} such that

$$\nu_{\text{ac}} \ll m, \quad \nu_{\text{sc}} \perp m \quad \text{and} \quad \nu = \nu_{\text{ac}} + \nu_{\text{sc}}.$$

Now the function F and the measures ν and m are linked up in the following lemma:

Lemma 7.4

Let $\mu' = \nu - m$. Then μ' is finite and positive Borel measure. Furthermore, we have

$$\mu'\big([a,x)\big) = F(x) - F(a). \tag{7.47}$$

Proof of Lemma 7.4. Clearly, the measure μ' is a complex Borel measure. The finiteness of μ' is also clear. If $x \in [a,b]$, then we deduce from the definition (7.46) that

$$\mu'\big([a,x)\big) = m\big(g[a,x)\big) - m([a,x)) = g(x) - g(a) - x + a = F(x) - F(a).$$

It remains to verify that μ' is positive. To this end, it is obvious from the definition and Lemma 7.3 that for $x, y \in [a,b]$ with $x < y$, we have

$$\nu\big((x,y)\big) = m\big(g\big((x,y)\big)\big) = g(y) - g(x) = y + F(y) - x - F(x) \geq y - x = m\big((x,y)\big).$$

Suppose that V is an open set in $[a,b]$. By [49, Exercise 29, p. 45], V can be expressed as a union of an at most countable collection of *disjoint* segments $\{V_i\}$. Since μ and m are measures, we have

$$\nu(V) = \nu\Big(\bigcup_{i=1}^{\infty} V_i\Big) = \bigcup_{i=1}^{\infty} \nu(V_i) \geq \bigcup_{i=1}^{\infty} m(V_i) = m\Big(\bigcup_{i=1}^{\infty} V_i\Big) = m(V). \tag{7.48}$$

Next, it is well-known that m is regular. Since $[a,b]$ is compact, it is σ-compact. Recall that ν is a finite positive Borel measure, so $\nu(K) < \infty$ for every compact set $K \subseteq [a,b]$. Therefore, ν is regular by Theorem 2.18. Thus if $E \in \mathscr{B}$ and $E \subseteq [a,b]$, then Theorem 2.14(c) implies that given $\epsilon > 0$, there exists an open set V containing E such that

$$\nu(V) < \nu(E) + \epsilon. \tag{7.49}$$

Combining the inequalities (7.48) and (7.49), we obtain

$$\nu(E) + \epsilon > \nu(V) \geq m(V) \geq m(E).$$

Since ϵ is arbitrary, we establish the fact that $\nu(E) \geq m(E)$ and then $\mu'(E) \geq 0$ for all $E \in \mathscr{B}$ and $E \subseteq [a,b]$. This proves the positivity of the measure μ' and ends the proof of Lemma 7.4. ∎

We come to the final step of the proof of part (b). Recall that a complex Borel measure μ'' is called a **discrete measure** if there is a countable set $\{x_j\} \subset \mathbb{R}$ and complex numbers c_j such that $\sum |c_j| < \infty$ and $\mu'' = \sum c_j \delta_{x_j}$, where δ_x is the point mass at x, see [22, p. 106] or [51, Example 1.20(b), p. 17]. Now if we let

$$\mu''(E) = \sum_{k=1}^{\infty} g(x_k)\delta_{x_k}(E) \geq 0$$

for any $E \subseteq [a,b]$ and $E \in \mathscr{B}$, then it follows from the fact (7.40) that μ'' is finite. Besides, we deduce from this and the definition (7.43) easily that

$$\mu''\big([a,x)\big) = \sum_{k=1}^{\infty} g(x_k)\delta_{x_k}\big([a,x)\big) = \sum_{k=1}^{\infty} g(x_k)\chi_{(x_k,b]}(x) = h(x) - h(a). \tag{7.50}$$

Therefore, if we further define $\mu = \mu' + \mu''$, then μ must be a finite positive Borel measure on $[a,b]$ and it yields from the representations (7.47) and (7.50) that

$$\mu([a,x)) = \mu'([a,x)) + \mu''([a,x)) = F(x) - F(a) + h(x) - h(a) = f(x) - f(a). \tag{7.51}$$

This proves that μ is the desired measure.

(c) By Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem), we have

$$\mu = \mu_{\rm ac} + \mu_{\rm sc},$$

where $\mu_{\rm ac} \ll m$ and $\mu_{\rm sc} \perp m$. Suppose that ω is the Radon-Nikodym derivative of $\mu_{\rm ac}$ with respect to m. Since $\omega \in L^1(m)$ on $[a,b]$, almost every $x \in [a,b]$ is a Lebesgue point of ω by Theorem 7.7.

Let $x \in (a,b)$ be a Lebesgue point of ω. Define $r = \min(x-a, b-x)$ and we can choose $\{r_i\} \subseteq (0,r)$ such that $r_i \to 0$ as $i \to \infty$. Then the two sequences of sets $\{E_i\}$ and $\{E_i'\}$ given by

$$E_i = [x, x+r_i) \quad \text{and} \quad E_i' = [x-r_i, x)$$

satisfy the inequalities

$$m(E_i) = r_i \geq \frac{1}{2}\cdot 2r_i = \frac{1}{2}m\big((x-r_i, x+r_i)\big) \quad \text{and} \quad m(E_i') \geq \frac{1}{2}m\big((x-r_i, x+r_i)\big)$$

for all $i = 1, 2, \ldots$. In other words, they mean that $\{E_i\}$ and $\{E_i'\}$ shrink to x nicely. Since μ is a complex Borel measure on $[a,b]$ and $\mathrm{d}\mu = \omega\,\mathrm{d}m + \mathrm{d}\mu_s$, Theorem 7.14 and the formula (7.51) imply that

$$f_+'(x) = \lim_{i\to\infty} \frac{f(x+r_i) - f(x)}{r_i} = \lim_{i\to\infty} \frac{\mu\big([x, x+r_i)\big)}{m\big([x, x+r_i)\big)} = \omega(x) \tag{7.52}$$

a.e. $[m]$ on $[a,b]$. Similarly, we have

$$f_-'(x) = \lim_{i\to\infty} \frac{f(x-r_i) - f(x)}{-r_i} = \lim_{i\to\infty} \frac{\mu\big([x-r_i, x)\big)}{m\big([x-r_i, x)\big)} = \omega(x) \tag{7.53}$$

a.e. $[m]$ on $[a,b]$. Now the results (7.52) and (7.53) definitely give $f'(x)$ exists a.e. $[m]$ on $[a,b]$ and $f' = \omega \in L^1(m)$ on $[a,b]$.

Finally, since $\mathrm{d}\mu_{\rm ac} = \omega\,\mathrm{d}m$, we have

$$\mu_{\rm ac}\big([a,x)\big) = \int_a^x \omega\,\mathrm{d}m = \int_a^x f'\,\mathrm{d}m = \int_a^x f'(t)\,\mathrm{d}t$$

so that

$$f(x) - f(a) = \mu\big([a, x)\big) = \mu_{\rm ac}\big([a, x)\big) + \mu_{\rm sc}\big([a, x)\big) = \int_a^x f'(t)\,\mathrm{d}t + s(x),$$

where $s(x) = \mu_{\rm sc}\big([a, x)\big)$ for all $x \in [a, b]$. Now $\mu_{\rm sc}\big([a, x)\big)$ is positive finite because μ is positive finite, so s is nondecreasing on $[a, b]$. Since $f' \in L^1(m)$ on $[a, b]$, Theorem 7.11 shows that

$$s'(x) = [f(x) - f(a)]' - \Big(\int_a^x f'(t)\,\mathrm{d}t\Big)' = f'(x) - f'(x) = 0$$

a.e. $[m]$ on (a, b).

(d) By Theorem 6.8(e), $\mu \perp m$ if and only if $\mu_{\rm ac} = 0$ if and only if

$$\int_E f'\,\mathrm{d}m = 0 \tag{7.54}$$

for every $E \in \mathscr{B}$ and $E \subseteq [a, b]$. Recall that $f' \in L^1(m)$. By Theorem 1.39(b), the result (7.54) is equivalent to $f'(x) = 0$ a.e. $[m]$ on $[a, b]$.

Similarly, $\mu \ll m$ if and only if $\mu_{\rm sc} = 0$ if and only if $s = 0$ on $[a, b]$ if and only if

$$f(x) - f(a) = \int_a^x f'\,\mathrm{d}m \tag{7.55}$$

for all $x \in [a, b]$. As Rudin pointed out in the discussion preceding Definition 7.17 (on p. 145), one may apply Theorem 6.11 to show that f is AC on $[a, b]$ if the formula (7.55) holds. Conversely, if f is AC on $[a, b]$, then f is automatically continuous on $[a, b]$ and since f is nondecreasing, Theorem 7.18 implies that the formula (7.55) holds.

(e) Suppose that $f'(p)$ exists at $p \in (a, b) \setminus A$. Then f is continuous at p and the definition (7.37) implies the existence of a $\delta > 0$ such that

$$f(x) = \varphi(x)$$

for all $x \in (p - \delta, p + \delta)$. If $h \in (0, \delta)$, then $p + h \in (p - \delta, p + \delta)$ and so

$$\Big|\frac{\varphi(p + h) - \varphi(p)}{h} - f'(p)\Big| = \Big|\frac{f(p + h) - f(p)}{h} - f'(p)\Big|. \tag{7.56}$$

By taking $h \to 0+$ in the expression (7.56), we get

$$|\varphi'_+(p) - f'(p)| = |f'_+(p) - f'(p)| = 0,$$

i.e., $\varphi'_+(p) = f'(p)$. The other side $\varphi'_-(p) = f'(p)$ can be done similarly. Thus we conclude that $\varphi'(p) = f'(p)$. Since $f'(x)$ exists a.e. $[m]$ on $[a, b]$, we have $\varphi'(x) = f'(x)$ a.e. $[m]$ on $[a, b]$.

Hence we have completed the proof of the problem. ∎

Remark 7.3

By the result of part (b) in Problem 7.12, we have

$$\mu = \mu' + \mu'' = v - m + \mu'' = \underbrace{(\nu_a - m)}_{\text{absolutely continuous}} + \overbrace{\nu_s}^{\text{singular continuous}} + \underbrace{\mu''}_{\text{discrete}}.$$

See also the decomposition in [22, Theorem 3.35, p. 106].

Problem 7.13

Rudin Chapter 7 Exercise 13.

Proof. By the definition,

$$BV = \left\{ f : [a,b] \to \mathbb{C} \,\middle|\, F(b) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| < \infty \right\},$$

where the supremum is taken over all N and over all choices of $\{t_i\}$ such that

$$a = t_0 < t_1 < \cdots < t_N = b. \tag{7.57}$$

(a) Let $f : [a,b] \to \mathbb{C}$ be a bounded nondecreasing function. Then we have

$$\sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{N} [f(t_i) - f(t_{i-1})] = f(b) - f(a) < \infty,$$

where $\{t_i\}$ satisfies the requirement (7.57). Hence we have $f \in BV$. If f is a bounded nonincreasing function, then we consider $-f$ which is a bounded nondecreasing function.

(b) Let $f \in BV$ be real. Define

$$f_1(x) = \frac{F(x) + f(x)}{2} \quad \text{and} \quad f_2(x) = \frac{F(x) - f(x)}{2}, \tag{7.58}$$

where F is given by [51, Eqn. (1), p. 147]. It is easy to see that $f = f_1 - f_2$, so it remains to prove that they are bounded and monotonic.

Suppose first that f is nondecreasing. By Theorem 7.19, $F+f$ and $F-f$ are nondecreasing,[e] so are f_1 and f_2. By the definition of the space BV, we have $0 \leq F(x) \leq F(b) < \infty$ on $[a,b]$. If $x \in [a,b]$, then the triangle inequality gives

$$|f(x)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + F(b).$$

Thus f is bounded on $[a,b]$. By the definition (7.59), f_1 and f_2 are also bounded on $[a,b]$. If f is nonincreasing, then the function $-f$ is nondecreasing but the functions $f_1 = \frac{F+f}{2} = \frac{F-(-f)}{2}$ and $f_2 = \frac{F-f}{2} = \frac{F+(-f)}{2}$ remain nondecreasing.[f]

(c) Suppose that $f \in BV$ and it is left-continuous. It suffices to prove that F is left-continuous at p, i.e., if $a < p \leq b$ and $\epsilon > 0$, then there is a $\delta > 0$ so that

$$|F(p) - F(p-h)| < \epsilon \tag{7.59}$$

whenever $0 < h < \delta$. To start with, we define

$$F(x-h;x) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|,$$

where $x - h = t_0 < t_1 < \cdots < t_N = x$. We claim that

$$F(x) - F(x-h) = F(x-h;x)$$

[e] The conclusion of this assertion *does not* depend on the assumption that f is AC on $[a,b]$.
[f] If we write $f = (-f_2) - (-f_1)$, then $-f_1$ and $-f_2$ are nonincreasing.

for every small $h > 0$ such that $x - h \in [a, b]$. By the definition, if $\{t_i\}$ and $\{s_j\}$ are partitions of $[a, x-h]$ and $[x-h, x]$ respectively, then $\{t_i\} \cup \{s_j\}$ is certainly a partition of $[a, b]$ and so

$$F(x) \geq F(x-h) + F(x-h; x). \tag{7.60}$$

On the other hand, let $\{t_0, t_1, \ldots, t_N\}$ be a partition of $[a, x]$. If $t_M < x - h < t_{M+1}$, then we may insert a number t' such that

$$a = t_0 < t_1 < \cdots \leq t_M < t' = x - h < t_{M+1} < \cdots < t_N = x.$$

Clearly, the triangle inequality implies

$$\begin{aligned}\sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| &\leq \Big[\sum_{i=1}^{M} |f(t_i) - f(t_{i-1})| + |f(t') - f(t_M)|\Big] \\ &\quad + \Big[|f(t_{M+1}) - f(t')| + \sum_{i=M+2}^{N} |f(t_i) - f(t_{i-1})|\Big] \\ &\leq F(x-h) + F(x-h; x). \end{aligned} \tag{7.61}$$

Since the partition $\{t_0, t_1, \ldots, t_N\}$ is *arbitrary*, the inequality (7.61) asserts that

$$F(x) \leq F(x-h) + F(x-h; x). \tag{7.62}$$

Hence our desired claim follows immediately from the inequalities (7.60) and (7.62).

Since f is left-continuous at $p \in (a, b]$, given $\epsilon > 0$, there exists a $\delta > 0$ such that $h \in (0, \delta)$ implies

$$|f(p) - f(p-h)| < \frac{\epsilon}{2}.$$

For this same ϵ, we can find a partition $\{t_i\}$ of $[a, p]$, say

$$a = t_0 < t_1 < \cdots < t_N = p,$$

such that

$$F(a; p) - \frac{\epsilon}{2} < \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| \tag{7.63}$$

Since adding more points to the partition $\{t_i\}$ results in an increase of the summation (7.63), we may assume that $t_N - t_{N-1} = h \in (0, \delta)$ and this implies that

$$|f(t_N) - f(t_{N-1})| = |f(t_N) - f(t_N - h)| = |f(p) - f(p-h)| < \frac{\epsilon}{2}$$

and the inequality (7.63) becomes

$$F(a; p) - \frac{\epsilon}{2} < \frac{\epsilon}{2} + \sum_{i=1}^{N-1} |f(t_i) - f(t_{i-1})| \leq \frac{\epsilon}{2} + F(a; p-h)$$

or equivalently,

$$F(a; p) - F(a; p-h) < \epsilon.$$

However, we note that

$$0 \leq F(p) - F(p-h) = F(p-h; p) = F(a; p) - F(a; p-h) < \epsilon$$

which is exactly the expected result (7.59). Hence F and then f_1 as well as f_2 are left-continuous on $[a, b]$.

(d) By parts (b) and (c), $f = f_1 - f_2$, where f_1 and f_2 are left-continuous and nondecreasing on $[a, b]$. Now Problem 7.12(b) shows that there exist positive and finite Borel measures λ and ν on $[a, b]$ such that

$$f_1(x) - f_1(a) = \lambda\big([a, x)\big) \quad \text{and} \quad f_2(x) - f_2(a) = \nu\big([a, x)\big),$$

where $x \in [a, b]$. Then the difference $\mu = \lambda - \nu$ is easily seen to be a Borel measure on $[a, b]$ satisfying the equation in the question. In fact, the finiteness of λ and ν imply the finiteness of μ. (However, μ may *not* be positive.)

As Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem) tells us that there exists a unique pair of positive and finite measures $\mu_a \ll m$ and $\mu_s \perp m$ such that $\mu = \mu_a + \mu_s$. Furthermore, there is a unique $h \in L^1(m)$ on $[a, b]$ such that

$$\mu_a(E) = \int_E h \, dm \tag{7.64}$$

for every $E \in \mathscr{B}$.

If $\mu \ll m$, then Proposition 6.8(e) implies that $\mu_s = 0$ and thus the μ_a in the representation (7.64) can be replaced by μ. Now we put $E = [a, x)$ into the representation and use the result $f(x) - f(a) = \mu([a, x))$ to obtain

$$f(x) - f(a) = \mu([a, x)) = \int_{[a,x)} h \, dm = \int_a^x h \, dm.$$

Since $h \in L^1(m)$ on $[a, b]$, we follow from [9, Theorem 5.3.6, p. 339] or [22, Theorem 3.35, p. 106] that f is AC on $[a, b]$.[g] Conversely, suppose that f is AC on $[a, b]$. By Theorem 7.19, F, f_1 and f_2 are AC and nondecreasing on $[a, b]$. By Problem 7.12(d), we conclude that

$$\lambda \ll m \quad \text{and} \quad \nu \ll m.$$

Since $\mu = \lambda - \nu$, Proposition 6.8(c) ensures that $\mu \ll m$.

(e) By part (b), we have $f = f_1 - f_2$, where f_1 and f_2 are nondecreasing. By Problem 7.12(e), f_1' and f_2' exist a.e. $[m]$ on $[a, b]$. By Problem 7.12(c), we further know that $f_1', f_2' \in L^1(m)$ on $[a, b]$. Hence we conclude that f' exists a.e. $[m]$ on $[a, b]$ and $f' \in L^1(m)$.

We complete the proof of the problem. ■

Problem 7.14

Rudin Chapter 7 Exercise 14.

Proof. Suppose that f and g are AC on $[a, b]$. By Definition 7.17, they are continuous on $[a, b]$, so the Extreme Value Theorem ([49, Theorem 4.16, p. 89]) ensures that we may assume that there exists a $M > 0$ such that

$$|f(x)| \leq M \quad \text{and} \quad |g(x)| \leq M \tag{7.65}$$

on $[a, b]$. Given $\epsilon > 0$, there corresponds a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(\beta_k) - f(\alpha_k)| < \frac{\epsilon}{2M} \quad \text{and} \quad \sum_{k=1}^{n} |g(\beta_k) - g(\alpha_k)| < \frac{\epsilon}{2M} \tag{7.66}$$

[g]These theorems *do not* need the assumptions of Theorem 7.18: f is continuous and nondecreasing on $[a, b]$.

for every n and any disjoint collection of segments $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ in $[a, b]$ with

$$\sum_{k=1}^{n}(\beta_k - \alpha_k) < \delta.$$

It is clear that

$$\begin{aligned}\sum_{k=1}^{n}|f(\beta_k)g(\beta_k) - f(\alpha_k)g(\alpha_k)| &= \sum_{k=1}^{n}|f(\beta_k)g(\beta_k) - f(\beta_k)g(\alpha_k) + f(\beta_k)g(\alpha_k) - f(\alpha_k)g(\alpha_k)| \\ &\leq \sum_{k=1}^{n}|f(\beta_k)g(\beta_k) - f(\beta_k)g(\alpha_k)| + \sum_{k=1}^{n}|f(\beta_k)g(\alpha_k) - f(\alpha_k)g(\alpha_k)| \\ &\leq M\sum_{k=1}^{n}|g(\beta_k) - g(\alpha_k)| + M\sum_{k=1}^{n}|f(\beta_k) - f(\alpha_k)|. \end{aligned} \tag{7.67}$$

By substituting the inequalities (7.66) into the inequality (7.67), we gain

$$\sum_{k=1}^{n}|f(\beta_k)g(\beta_k) - f(\alpha_k)g(\alpha_k)| < \epsilon$$

which implies that fg is AC on $[a, b]$.

Now Theorem 7.20 shows that f, g and fg are differentiable a.e. on $[a, b]$ and $f', g', (fg)' \in L^1$ on $[a, b]$. Apply Theorem 7.20 to fg, we have

$$f(b)g(b) - f(a)g(a) = \int_a^b \big(f(x)g(x)\big)' \,\mathrm{d}x. \tag{7.68}$$

By the bounds (7.65), we see that $f'g, fg' \in L^1$ on $[a, b]$. Furthermore, since $(fg)' = f'g + fg'$ a.e. on $[a, b]$, we deduce from the formula (7.68) that

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,\mathrm{d}x,$$

completing the proof of the problem. ■

Problem 7.15

Rudin Chapter 7 Exercise 15.

Proof. It is well-known [49, Example 5.6(b), p. 106] that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

has derivative given by

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus f is differentiable at all points but f' is *not* a continuous function. However, such function oscillates around 0 so that it is not monotonic and we have to "modify" it. To do this, we notice that since

$$\lim_{x\to\pm\infty} x\sin\frac{1}{x} = \lim_{x\to\pm\infty}\frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1,$$

$f'(x)$ is bounded on $\mathbb{R}$. Let $M > 0$ be a bound of f' on $\mathbb{R}$, i.e.,

$$-M \leq f'(x) \leq M \tag{7.69}$$

for all $x \in \mathbb{R}$. We consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = f(x) + Mx.$$

Now $F'(x) = f'(x) + M$ which exists *finitely* on $\mathbb{R}$. In addition, the inequalities (7.69) implies that

$$F'(x) \geq -M + M \geq 0$$

for all $x \in \mathbb{R}$. Hence [49, Theorem 5.11, p. 108] ensures that F is monotonically increasing on $\mathbb{R}$. Since f' is not continuous at 0, F' is also discontinuous at 0 and thus F is our desired function. This completes the proof of the problem. ∎

Problem 7.16

Rudin Chapter 7 Exercise 16.

Proof. We follow the hint given by Rudin. Since m is outer regular (see Theorem 2.14(c) and Theorem 2.20(b)) and $m(E) = 0$, we can find a sequence $\{V_n\}$ of open subsets of $\mathbb{R}$ such that $E \subseteq \cdots \subseteq V_2 \subseteq V_1$ and $m(V_n) < 2^{-n}$. This construction gives

$$E \subseteq \bigcap_{n=1}^{\infty} V_n. \tag{7.70}$$

Consider $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \sum_{n=1}^{\infty} \chi_{V_n}(x) \geq 0.$$

Since each χ_{V_n} is measurable, it follows from Theorem 1.27 that

$$0 \leq \int_a^b F(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_a^b \chi_{V_n}(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_{V_n}\mathrm{d}x = \sum_{n=1}^{\infty} m(V_n) < \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

In other words, $F \in L^1\big([a, b]\big)$. Next we define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \int_a^x F(t)\,\mathrm{d}t \geq 0 \tag{7.71}$$

which is clearly nondecreasing. By Theorem 7.11, f is differentiable a.e. on $[a, b]$ and then we follow from Theorem 7.18 that f is AC on $[a, b]$.

Now it remains to show that $f'(x) = \infty$ on E. To see this, if $x \in E$, then the set relation (7.70) tells us that $x \in V_n$ for all $n \in \mathbb{N}$. Since each V_n is open, there exists a $\delta_n > 0$ such that

$$(x - \delta_n, x + \delta_n) \subseteq V_n$$

for all $n \in \mathbb{N}$. In fact, we may assume that $\{\delta_n\}$ is a decreasing sequence of positive numbers. Pick $y \in (x - \delta_n, x + \delta_n)$ so that

$$|f(y) - f(x)| = \Big|\int_x^y F(t)\,\mathrm{d}t\Big| \geq \Big|\int_x^y \sum_{k=1}^{n}[\chi_{V_1}(t) + \cdots + \chi_{V_n}(t)]\,\mathrm{d}t\Big| \tag{7.72}$$

Since $V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_2 \subseteq V_1$ and $t \in [x,y] \subseteq (x - \delta_n, x + \delta_n) \subseteq V_n$,[h] the inequality (7.72) becomes

$$|f(y) - f(x)| \geq n|y - x|.$$

Notice that $y \to x$ if and only if $n \to \infty$, so we have

$$|f'(x)| = \lim_{y \to x}\Big|\frac{f(y) - f(x)}{y - x}\Big| = \infty.$$

By the definition (7.71), we conclude that $f'(x) = \infty$ on E. We have completed the proof of the problem. ■

7.5 Miscellaneous Problems on Differentiation

Problem 7.17

Rudin Chapter 7 Exercise 17.

Proof. We prove the assertions one by one:

- **μ is a Borel measure.** Suppose that $\{E_i\}$ is a collection of mutually disjoint Borel subsets of $\mathbb{R}^k$. Let $E = \bigcup_{i=1}^{\infty} E_i$. Since $\mu_n(E_i) \geq 0$ for all $i \in \mathbb{N}$, we know from [49, Exericse 3, p. 196] that

$$\mu(E) = \sum_{n=1}^{\infty}\mu_n(E) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\mu_n(E_i) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n(E_i) = \sum_{i=1}^{\infty}\mu(E_i).$$

  Clearly, we have $\mu(E) \geq 0$ for all $E \in \mathscr{B}$. By Definition 1.18(a), μ is a positive Borel measure on $\mathbb{R}^k$.

- **The relation between the Lebesgue decompositions of the μ_n and that of μ.** Since m is a positive σ-finite measure on $\mathscr{B}$, it follows from Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem) that

$$\mu = \mu_a + \mu_s \quad \text{and} \quad \mu_n = \mu_{n,a} + \mu_{n,s},$$

  where

$$\mu_a, \mu_{n,a} \ll m \quad \text{and} \quad \mu_s, \mu_{n,s} \perp m \tag{7.73}$$

  for all $n \in \mathbb{N}$. Let

$$\lambda_a = \sum_{n=1}^{\infty}\mu_{n,a} \quad \text{and} \quad \lambda_s = \sum_{n=1}^{\infty}\mu_{n,s}.$$

[h] Or $t \in [y, x]$.

Since μ and μ_n are positive, so are $\mu_a, \mu_{n,a}, \mu_s$ and $\mu_{n,s}$ and it deduces from the first assertion that both λ_a and λ_s are positive Borel measures of $\mathbb{R}^k$. Furthermore, we have

$$\mu = \sum_{n=1}^{\infty} \mu_n = \sum_{n=1}^{\infty} (\mu_{n,a} + \mu_{n,s}) = \sum_{n=1}^{\infty} \mu_{n,a} + \sum_{n=1}^{\infty} \mu_{n,s} = \lambda_a + \lambda_s.$$

By Theorem 6.8 and the relations (7.73), we assert that $\lambda_a \ll m$ and $\lambda_s \perp m$. Recall that the Lebesgue decomposition of μ relative to m is *unique*, it must be true that

$$\mu_a = \lambda_a = \sum_{n=1}^{\infty} \mu_{n,a} \quad \text{and} \quad \mu_s = \lambda_s = \sum_{n=1}^{\infty} \mu_{n,s}.$$

- **The truth of the formula.** By Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem) again, there exists a unique $h \in L^1(m)$ such that

$$\mu_a(E) = \int_E h \, \mathrm{d}m \tag{7.74}$$

for all $E \in \mathscr{B}$. Similarly, for each $n \in \mathbb{N}$, there exists a unique $h_n \in L^1(m)$ such that

$$\mu_{n,a}(E) = \int_E h_n \, \mathrm{d}m \tag{7.75}$$

for all $E \in \mathscr{B}$. Since $\mu_a, \mu_{n,a} \ll m$ for all $n \in \mathbb{N}$, we observe from Theorem 7.8 that

$$D\mu_a = h \quad \text{and} \quad D\mu_{n,a} = h_n \tag{7.76}$$

a.e. $[m]$ for all $n \in \mathbb{N}$. Since $\mu_s \perp m$ and $\mu_{n,s} \perp m$ for all $n \in \mathbb{N}$, Theorem 7.14 ensures that

$$D\mu_s = 0 \quad \text{and} \quad D\mu_{n,s} = 0 \tag{7.77}$$

a.e. $[m]$ for all $n \in \mathbb{N}$. Hence the two results (7.76) and (7.77) say that

$$D\mu = h \quad \text{and} \quad D\mu_n = h_n$$

a.e. $[m]$ for all $n \in \mathbb{N}$.

Recall from the second assertion and the application of the integrals (7.74) and (7.75) that

$$\int_E h \, \mathrm{d}m = \mu_a(E) = \sum_{n=1}^{\infty} \mu_{n,a}(E) = \sum_{n=1}^{\infty} \int_E h_n \, \mathrm{d}m \tag{7.78}$$

for all $E \in \mathscr{B}$. Since $h_n \in L^1(m)$, each h_n is measurable and Theorem 1.27 implies that the expression (7.78) can be further reduced to

$$\int_E h \, \mathrm{d}m = \int_E \sum_{n=1}^{\infty} h_n \, \mathrm{d}m$$

for all $E \in \mathscr{B}$, or equivalently,

$$\int_E \Big(h - \sum_{n=1}^{\infty} h_n \Big) \, \mathrm{d}m = 0 \tag{7.79}$$

for all $E \in \mathscr{B}$.

Since $\mu_a(E) \geq \sum_{n=1}^{N} \mu_{n,a}(E)$ for every $N \in \mathbb{N}$, the integrals (7.74) and (7.75) show that

$$\int_E \Big(h - \sum_{n=1}^{N} h_n\Big)\,\mathrm{d}m \geq 0 \tag{7.80}$$

for every $E \in \mathscr{B}$. Assume that $h < \sum_{n=1}^{N} h_n$ on E' with $m(E') > 0$. For this Borel set E', we derive from Proposition 1.24(a) that

$$\int_{E'} \Big(h - \sum_{n=1}^{N} h_n\Big)\,\mathrm{d}m < 0$$

which contradicts the result (7.80). Consequently, we always have $h \geq \sum_{n=1}^{N} h_n$ a.e. $[m]$ on $\mathbb{R}^k$ and since N is arbitrary, this implies that

$$h - \sum_{n=1}^{\infty} h_n \geq 0$$

a.e. $[m]$ on $\mathbb{R}^k$. Now we apply Theorem 1.39(a) to the integral (7.79) to conclude that

$$h(\mathbf{x}) = \sum_{n=1}^{\infty} h_n(\mathbf{x})$$

a.e. $[m]$ on $\mathbb{R}^k$, then this shows the formula

$$D\mu(\mathbf{x}) = \sum_{n=1}^{\infty} (D\mu_n)(\mathbf{x})$$

a.e. $[m]$ on $\mathbb{R}^k$.

- **A corresponding theorem for a sequence $\{f_n\}$ of positive nondecreasing functions on $\mathbb{R}$.** Suppose that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges on $\mathbb{R}$. We claim that $f'(x)$ exists and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \tag{7.81}$$

a.e. $[m]$ on $\mathbb{R}$. By the definition, it is trivial that f is *nondecreasing* and $0 < f(x) < \infty$ on $\mathbb{R}$. Let $[a, b]$ be a compact interval of $\mathbb{R}$. Suppose that

$$E = \{x \in (a, b) \,|\, f'(x) \text{ and } f_n'(x) \text{ exist}\}.$$

By Problem 7.12(e), we see that f and each f_n are differentiable a.e. $[m]$ on $[a, b]$ so that $[a, b] \setminus E$ is of measure zero.

Let $x \in E$ and $h > 0$ be sufficiently small so that $x + h \in [a, b]$. Then for any $N \in \mathbb{N}$, we have

$$\frac{f(x+h) - f(x)}{h} \geq \sum_{n=1}^{N} \frac{f_n(x+h) - f_n(x)}{h}$$

so if we let $h \to 0$, then we have

$$f'(x) \geq \sum_{n=1}^{N} f_n'(x). \tag{7.82}$$

Now we take $N \to \infty$ in the inequality (7.82) to conclude that

$$f'(x) \geq \sum_{n=1}^{\infty} f_n'(x) \tag{7.83}$$

a.e. $[m]$ on $[a, b]$. Recall that $x \in E$ so that the inequality (7.83) ensures that the series of its right-hand side is convergent a.e. $[m]$ on $[a, b]$. Consequently, it implies that

$$\lim_{n\to\infty} f_n'(x) = 0 \tag{7.84}$$

a.e. $[m]$ on $[a, b]$.

Next we define

$$s_N(x) = \sum_{n=1}^{N} f_n(x).$$

Since $s_N(b) \to f(b)$ as $N \to \infty$, there is a sequence $\{N_k\} \subseteq \mathbb{N}$ such that

$$0 \leq f(b) - s_{N_k}(b) < \frac{1}{2^k}. \tag{7.85}$$

Obviously, we have

$$g_k(x) = f(x) - s_{N_k}(x) = \sum_{n=N_k+1}^{\infty} f_n(x),$$

so each g_k is a *positive nondecreasing* function on $[a, b]$. To apply the first part of the proof, we have to show that the series $\sum_{k=1}^{\infty} g_k(x)$ converges on $[a, b]$. Combining this fact and the inequality (7.85), we obtain

$$0 \leq g_k(x) < \frac{1}{2^k}$$

on $[a, b]$. As a result, the series

$$\sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} (f - s_{N_k})$$

converges (uniformly) to $g(x)$ on $[a, b]$ by the Weierstrass M-test. Now we are able to apply the result (7.84) to the sequence $\{g_k\}$ of positive nondecreasing functions and conclude that

$$\lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} [f'(x) - s_{N_k}'(x)] = 0$$

a.e. $[m]$ on $[a, b]$ or equivalently,

$$\lim_{k\to\infty} \sum_{n=1}^{N_k} f_n'(x) = f'(x) \tag{7.86}$$

a.e. $[m]$ on $[a,b]$. Since each f_n is nondecreasing on $[a,b]$, $f_n'(x) \geq 0$ for all $x \in E$. Hence the limit (7.86) implies that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

a.e. $[m]$ on $[a,b]$. Since any real number is contained in some compact interval $[a,b]$, our desired result (7.81) follows from this.

It completes the proof of the problem. ■

Remark 7.4

The last assertion in Problem 7.17 is called **Fubini's Theorem on Differentiation**. See [33, Sec. C, pp. 527 – 529].

Problem 7.18

Rudin Chapter 7 Exercise 18.

Proof. First of all, we show that $\{\varphi_n\}$ is orthonormal on $[0,1]$. Suppose that $m,n \in \mathbb{N}$ and $m \geq n$. If $m = n$, then $\varphi_n(t)\varphi_n(t) = \varphi_n^2(t) = 1$ and so

$$\int_0^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = 1.$$

If $m > n$, then we deduce from Theorem 7.26 (The Change-of-variables Theorem) that

$$\begin{aligned}\int_0^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t &= \int_0^1 \varphi_0(2^n t)\varphi_0(2^m t)\,\mathrm{d}t \\ &= \frac{1}{2^n}\int_0^{2^n} \varphi_0(y)\varphi_0(2^{m-n}y)\,\mathrm{d}y \\ &= \frac{1}{2^n}\sum_{i=1}^{2^{n-1}}\int_{2(i-1)}^{2i} \varphi_0(y)\varphi_0(2^{m-n}y)\,\mathrm{d}y. \end{aligned} \tag{7.87}$$

Since $\varphi_0(y)$ is periodic with period 2, we have $\varphi_0(y) = 1$ on $[2i-2, 2i-1)$ and $\varphi_0(y) = -1$ on $[2i-1, 2i)$. Therefore, we can reduce the expression (7.87) to

$$\int_0^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = \frac{1}{2^n}\sum_{i=1}^{2^{n-1}}\Big[\int_{2(i-1)}^{2i-1} \varphi_0(2^{m-n}y)\,\mathrm{d}y - \int_{2i-1}^{2i} \varphi_0(2^{m-n}y)\,\mathrm{d}y\Big]. \tag{7.88}$$

Note that the interval $[2^{m-n+1}i - 2^{m-n}, 2^{m-n+1}i)$ is of length 2^{m-n}, so the integrals on the right-hand side of the equation (7.88) are 0 and this gives

$$\int_0^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = 0$$

if $m \neq n$. By Definition 4.16, $\{\varphi_n\}$ is orthonormal on $[0,1]$.

Since $\sum |c_n|^2 < \infty$, it yields from the Riesz-Fischer Theorem [49, Theorem 11.43, p. 330] that

$$f \sim \sum_{n=1}^{\infty} c_n\varphi_n \tag{7.89}$$

for some $f \in L^2([0,1])$, i.e., the series in (7.89) is the Fourier series of f.

Consider $a = j \cdot 2^{-N}$, $b = (j+1) \cdot 2^{-N}$, $a < t < b$ and $s_N = c_1\varphi_1 + \cdots + c_N\varphi_N$, where $j = 0, 1, 2, \ldots$. On the one hand, if $1 \leq k \leq N$, then $a \leq t \leq b$ implies that

$$j \cdot 2^{k-N} \leq 2^k t \leq (j+1)2^{k-N} \quad \text{and} \quad m\big([2^k a, 2^k b]\big) = 2^{k-N} \leq 1$$

so that each $\varphi_k(t) = \varphi_0(2^k t)$ (of course, it is 1 or -1) is constant in $[a,b]$. This implies that

$$\frac{1}{b-a}\int_a^b s_N(t)\,\mathrm{d}t = s_N(t) \cdot \frac{1}{b-a}\int_a^b \mathrm{d}t = s_N(t).$$

On the other hand, if $k > N$, then $2^{k-N} > 1$ and we apply Theorem 7.26 (The Change-of-variables Theorem) to get

$$\int_a^b \varphi_k(t)\,\mathrm{d}t = \int_a^b \varphi_0(2^k t)\,\mathrm{d}t = \frac{1}{2^k}\int_{2^k a}^{2^k b} \varphi_0(y)\,\mathrm{d}y = \frac{1}{2^k}\int_{j \cdot 2^{k-N}}^{(j+1)\cdot 2^{k-N}} \varphi_0(y)\,\mathrm{d}y.$$

Using the same argument as in evaluating the integrals in the equation (7.88), we can show that

$$\frac{1}{2^k}\int_{j \cdot 2^{k-N}}^{(j+1)\cdot 2^{k-N}} \varphi_0(y)\,\mathrm{d}y = 0.$$

In conclusion, we arrive at

$$s_N(t) = \frac{1}{b-a}\int_a^b s_N\,\mathrm{d}m = \frac{1}{b-a}\int_a^b s_n\,\mathrm{d}m \tag{7.90}$$

for every $n = 1, 2, \ldots$. By the relation (7.89), $s_n \to f$ in L^2 on $[0,1]$. By Theorem 3.5 (Hölder's Inequality), we obtain

$$\int_a^b |s_n - f|\,\mathrm{d}m \leq \Big\{\int_a^b |s_n - f|^2\,\mathrm{d}m\Big\}^{\frac{1}{2}}\Big\{\int_a^b \mathrm{d}m\Big\}^{\frac{1}{2}} = (b-a)^{\frac{1}{2}}\|s_n - f\|_2 \to 0$$

as $n \to \infty$. Thus this implies that $s_n - f \in L^1$ on $[0,1]$ for large enough n and so we derive from Theorem 1.33 that

$$\lim_{n\to\infty}\int_a^b s_n\,\mathrm{d}m = \int_a^b f\,\mathrm{d}m. \tag{7.91}$$

By combining the representation (7.90) and the limit (7.91), we establish

$$s_N(t) = \frac{1}{b-a}\int_a^b f\,\mathrm{d}m. \tag{7.92}$$

Since $f \in L^2$ on $[0,1]$, we have $f \in L^1$ on $[0,1]$ by Theorem 3.8. By Theorem 7.7, almost every $x \in [0,1]$ is a Lebesgue point of f. Let $p \in [0,1]$ be such a point and for each $N \in \mathbb{N}$, we define $E_N(p) = [2^{-N}, 2^{-N+1}]$. Clearly, we have

$$m\big(E_N(p)\big) = 2^{-N} \quad \text{and} \quad \sum_{N=1}^{\infty} 2^{-N} = 1.$$

Therefore, one can find a $N \in \mathbb{N}$ such that $p \in [2^{-N}, 2^{-N+1}]$. Furthermore, it is easy to check that the sequence $\{E_N(p)\}$ shrinks to p nicely because

$$E_N(p) \subseteq B(p, 2^{-N+1}) \quad \text{and} \quad m\big(E_N(p)\big) = \frac{1}{4}m\big(B(p, 2^{-N+1})\big)$$

for every $N \in \mathbb{N}$. Using Theorem 7.10 (Lebesgue Differentiation Theorem) and then the representation (7.92), we find that

$$f(p) = \lim_{N\to\infty} \frac{1}{m\big([2^{-N}, 2^{-N+1}]\big)} \int_{[2^{-N},2^{-N+1}]} f\,\mathrm{d}m = \lim_{N\to\infty} s_N(p) = \sum_{N=1}^{\infty} c_N\varphi_N(p).$$

In other words, the series $\sum_{n=1}^{\infty} c_n\varphi_n$ converges to f a.e. in $[0,1]$.

Finally, we know that

$$\varphi_n(t+1) = \varphi_0\big(2^n(t+1)\big) = \varphi_0(2^n t + 2^n) = \varphi_0(2^n t) = \varphi_n(t)$$

for every $n \in \mathbb{N}$, the series and f are functions with period 1. Now for every $t \in \mathbb{R}$, we can write $t = t' + m$ with $t' \in [0,1]$ and $m \in \mathbb{Z}$. Furthermore, if t is a Lebesgue point of f, then it can be shown from the definition and the periodicity of f that t' is also a Lebesgue point of f. By this fact (for the third equality below), we obtain

$$\sum_{n=1}^{\infty} c_n\varphi_n(t) = \sum_{n=1}^{\infty} c_n\varphi_n(t'+m) = \sum_{n=1}^{\infty} c_n\varphi_n(t') = f(t') = f(t'+m) = f(t),$$

completing the proof of the problem. ■

Problem 7.19

Rudin Chapter 7 Exercise 19.

Proof. Suppose that $c \in \mathbb{R}$. Since f is continuous on $\mathbb{R}$, it is Boreal measurable. Thus the function $f_n(x) = n^c f(nx)$ is also Borel measurable for every $n \in \mathbb{N}$. By Theorem 1.14, $h_c(x)$ is Borel measurable.

(a) By the hypotheses, it is true that

$$\max_{x\in\mathbb{R}} f(x) = \max_{x\in[0,1]} f(x) = \max_{x\in(0,1)} f(x) \tag{7.93}$$

exists and it is positive. Let this number be M. If $x \leq 0$, then $nx \leq 0$ for all $n \in \mathbb{N}$. Similarly, if $x \geq 1$, then $nx \geq 1$ for all $n \in \mathbb{N}$. In other words, we have

$$h_c(x) = 0$$

for all $x \in \mathbb{R} \setminus (0,1)$.

Fix $x \in (0,1)$, then $0 < nx < 1$ if and only if

$$0 < n < \frac{1}{x}. \tag{7.94}$$

In this case, there are only *finitely many* n satisfying the inequalities (7.94), so we have

$$h_c(x) = \sup\{n^c f(nx) \,|\, n = 1,2,3,\ldots\} = \max\Big\{n^c f(np) \,\Big|\, n \leq \Big[\frac{1}{x}\Big]\Big\} \leq \frac{M}{x^c}. \tag{7.95}$$

Since $c \in (0,1)$, direct computation shows that

$$\int_0^1 \frac{M}{x^c}\,\mathrm{d}x = Mx^{1-c}|_0^1 = M$$

and then

$$0 \leq \int_{\mathbb{R}} h_c(x)\,\mathrm{d}x = \int_0^1 h_c(x)\,\mathrm{d}x \leq M.$$

In conclusion, we have $h_c \in L^1(\mathbb{R})$ for $c \in (0,1)$.

(b) By the inequality (7.95), we see that

$$h_1(x) \leq \frac{M}{x} \tag{7.96}$$

for $x \in (0,1)$ and $h_1(x) = 0$ otherwise. Take $\lambda > 0$. If the real number x satisfies $|h_1(x)| = h_1(x) > \lambda$, then we must have $x \in (0,1)$ and furthermore $0 < x < \frac{M}{\lambda}$ by the inequality (7.96). In other words, we have

$$\{x \in \mathbb{R} \,|\, h_1(x) > \lambda\} \subseteq \Big\{x \in \mathbb{R} \,\Big|\, 0 < x < \frac{M}{\lambda}\Big\}$$

and so

$$\lambda \cdot m\{|h_1| > \lambda\} \leq \lambda \cdot m\Big\{x \in \mathbb{R} \,\Big|\, 0 < x < \frac{M}{\lambda}\Big\} = \lambda \cdot \frac{M}{\lambda} = M.$$

By the definition in §7.5, we see that h_1 is in weak L^1.

Recall the fact (7.93), so the Extreme Value Theorem [49, Theorem 4.16, p. 89] ensures that $M = f(p)$ for some $p \in (0,1)$. Given $\epsilon > 0$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f(x) - M + \epsilon$$

which is continuous on $\mathbb{R}$. Since $g(p) = f(p) - M + \epsilon = \epsilon > 0$, the sign-preserving property for continuous functions [64, Problem 7.15, p. 112] says that there exists a $\delta > 0$ small enough such that $g(x) > 0$ on $(p-\delta, p+\delta) \subset (0,1)$, i.e., $f(x) > M - \epsilon$ on $(p-\delta, p+\delta) \subset (0,1)$. Particularly, we may take $\epsilon = \frac{M}{2}$ so that

$$f(x) > \frac{M}{2} \tag{7.97}$$

on $(p-\delta, p+\delta)$. Fix this δ. If $0 < y < 2\delta$, then $\frac{2\delta}{y} > 1$. Suppose that $N_y = [\frac{p-\delta}{y}] + 1$. Then it is easy to check that

$$\frac{p-\delta}{y} < N_y \leq \frac{p-\delta}{y} + 1 = \frac{p-\delta+y}{y} < \frac{p+\delta}{y} \tag{7.98}$$

which is equivalent to saying that $N_y y \in (p-\delta, p+\delta)$. Hence it follows from the inequalities (7.97) and (7.98) that

$$h_1(y) \geq N_y f(N_y y) > \frac{N_y M}{2} > \frac{(p-\delta)M}{2y}.$$

However, we know that $\displaystyle\int_0^{2\delta} \frac{\mathrm{d}t}{t} = \infty$, so we conclude that $h_1 \notin L^1(\mathbb{R})$.

(c) Let $c > 1$ and $\delta > 0$ be the number such that the inequality (7.97) holds on the segment $(p-\delta, p+\delta) \subset (0,1)$. Now for each $k \in \mathbb{N}$, if $x \in (\frac{p-\delta}{k}, \frac{p+\delta}{k})$, then $kx \in (p-\delta, p+\delta)$ and thus

$$h_c(x) \geq k^c f(kx) > \frac{Mk^c}{2}$$

which means $(\frac{p-\delta}{k}, \frac{p+\delta}{k}) \subset \{x \in \mathbb{R} \,|\, h_c(x) > \frac{Mk^c}{2}\}$. Consequently, if we take $\lambda_k = \frac{Mk^c}{2}$, then we obtain

$$\lambda_k \cdot m\{h_c > \lambda_k\} > \lambda_k \cdot m\Big(\Big(\frac{p-\delta}{k}, \frac{p+\delta}{k}\Big)\Big) = \frac{Mk^c}{2} \cdot \frac{2\delta}{k} = M\delta k^{c-1}. \tag{7.99}$$

As M and δ are fixed as well as $c > 1$, the inequality (7.99) implies that

$$\lambda_k \cdot m\{h_c > \lambda_k\} \to \infty$$

as $k \to \infty$. Then we have proved that h_c is *not* in weak L^1 if $c > 1$.

We have ended the proof of the problem. ■

Problem 7.20

Rudin Chapter 7 Exercise 20.

Proof. By the definition, we have $\partial E = \overline{E} \setminus E^\circ$.

(a) If $m(\partial E) = 0$, then we have $m(\overline{E} \setminus E^\circ) = 0$. Since E° is the union of all open sets contained in E and $\overline{E}$ is the intersection of all closed sets containing E, we have $E^\circ \subseteq E \subseteq \overline{E}$ and $E^\circ, \overline{E} \in \mathscr{B}$. In fact, we have

$$m(E) = m(E^\circ)$$

by Theorem 1.36.[i] Furthermore, since m is a complete measure, E is also Lebesgue measurable.[j]

(b) To prove this part, we need stronger versions of Theorems 7.7 and 7.10 (The Lebesgue Differentiation Theorem). Roughly speaking, the hypothesis $f \in L^1(\mathbb{R}^k)$ can be replaced by $f \in L^1_{\text{loc}}(\mathbb{R}^k)$, where $L^1_{\text{loc}}(\mathbb{R}^k)$ is the space of all **locally integrable functions**:

$$\int_K |f| \, dm < \infty$$

for every bounded measurable set $K \subseteq \mathbb{R}^k$, see [22, p. 95]. For convenience, we state the stronger versions of these theorems in a single lemma:

Lemma 7.5

(a) If $f \in L^1_{\text{loc}}(\mathbb{R}^k)$, then almost every $\mathbf{x} \in \mathbb{R}^k$ is a Lebesgue point of f.

(b) Associate to each $\mathbf{x} \in \mathbb{R}^k$ a sequence $\{E_i(\mathbf{x})\}$ that shrinks to $\mathbf{x}$ nicely, and let $f \in L^1_{\text{loc}}(\mathbb{R}^k)$. Then

$$f(\mathbf{x}) = \lim_{i\to\infty} \frac{1}{m\big(E_i(\mathbf{x})\big)} \int_{E_i(\mathbf{x})} f \, dm$$

at every Lebesgue point of f, hence a.e. $[m]$ on $\mathbb{R}^k$.

[i] See also [49, Theorem 2.27, p. 35; Exercise 9, p. 43].

[j] Notice the difference between a Borel measurable set and a Lebesgue measurable set: *Not* every subset of a Borel set with measure 0 is also Borel measurable, but Lebesgue measure is obtained by enlarging $\mathscr{B}$ to include all subsets of sets of Borel measure 0.

Let A be a (possibly uncountable) set. Suppose that

$$E = \bigcup_{\alpha \in A} \overline{B(\mathbf{x}_\alpha, r_\alpha)}, \tag{7.100}$$

where $\overline{B(x_\alpha, r_\alpha)}$ is a closed disc in $\mathbb{R}^2$ and $r_\alpha \geq 1$.

Assume that $m(\partial E) > 0$. Clearly, E° is an open subset of $\mathbb{R}^2$, so it is a Borel subset in $\mathbb{R}^2$. As in §7.12, we consider the characteristic function $f = \chi_{E^\circ}$. Denote $D_A(\mathbf{x})$ to be the density of the Lebesgue measurable set A at $\mathbf{x}$. Since $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, we apply Lemma 7.5 to conclude that

$$D_{E^\circ}(\mathbf{x}) = \lim_{r \to 0} \frac{m\big(E^\circ \cap B(\mathbf{x}, r)\big)}{m\big(B(\mathbf{x}, r)\big)} = \lim_{r \to 0} \frac{1}{m\big(B(\mathbf{x}, r)\big)} \int_{B(\mathbf{x},r)} \chi_{E^\circ} \, \mathrm{d}m = \chi_{E^\circ}(\mathbf{x}) = 0$$

for almost every point $\mathbf{x} \in \partial E \subseteq (E^\circ)^c$. In particular, we choose a $\mathbf{p} \in \partial E$ such that

$$D_{E^\circ}(\mathbf{p}) = 0. \tag{7.101}$$

To continue our proof, we need the following result:

Lemma 7.6

There is a sequence $\{E_i(\mathbf{p})\}$ that shrinks to $\mathbf{p}$ nicely.

Proof of Lemma 7.6. For every $i \in \mathbb{N}$, we have $E \cap B(\mathbf{p}, \frac{1}{i}) \neq \varnothing$. (In fact, this is an equivalent definition of a boundary point of E, see [3, Definition 3.40, p. 64].) Pick $\{\mathbf{q}_i\}$ to be an *arbitrary* sequence temporarily such that $\mathbf{q}_i \in E \cap B(\mathbf{p}, \frac{1}{i})$. By the definition (7.100), we know that $\mathbf{q}_i \in \overline{B(\mathbf{x}_{\alpha_i}, r_{\alpha_i})}$ for some $\alpha_i \in A$. If $|\mathbf{p} - \mathbf{q}_i| \leq \frac{1}{2i}$, then we have

$$B\Big(\mathbf{q}_i, \frac{1}{2i}\Big) \subset B\Big(\mathbf{p}, \frac{1}{i}\Big) \tag{7.102}$$

and we define $E_i(\mathbf{p}) = B(\mathbf{q}_i, \frac{1}{2i})$. Otherwise, we consider the point

$$\mathbf{s}_{2i} \in \overline{B(\mathbf{x}_{\beta_{2i}}, r_{\beta_{2i}})} \cap B\Big(\mathbf{p}, \frac{1}{2i}\Big)$$

which satisfies $|\mathbf{p} - \mathbf{s}_{2i}| < \frac{1}{2i}$ so that if we let $\mathbf{q}_i = \mathbf{s}_{2i}$ and $\alpha_i = \beta_{2i}$, then we gain the set inclusion (7.102) in this case. (See Figure 7.2 below.) Now this process can be done continually and then we obtain a sequence $\{\mathbf{q}_i\}$ satisfying

$$\mathbf{q}_i \in E \cap B\Big(\mathbf{p}, \frac{1}{i}\Big) \quad \text{and} \quad E_i(\mathbf{p}) = B\Big(\mathbf{q}_i, \frac{1}{2i}\Big) \subset B\Big(\mathbf{p}, \frac{1}{i}\Big).$$

Clearly, we have

$$m(E_i(\mathbf{p})) = \pi \times \Big(\frac{1}{2i}\Big)^2 = \frac{1}{4} m\Big(B\Big(\mathbf{p}, \frac{1}{i}\Big)\Big) \tag{7.103}$$

for every $i \in \mathbb{N}$, therefore $\{E_i(\mathbf{p})\}$ shrinks to $\mathbf{p}$ nicely by the definition in §7.9, completing the proof of the lemma. ∎

We go back to the proof of part (b). Recall that $\mathbf{q}_i \in \overline{B(\mathbf{x}_{\alpha_i}, r_{\alpha_i})}$. There are two cases:

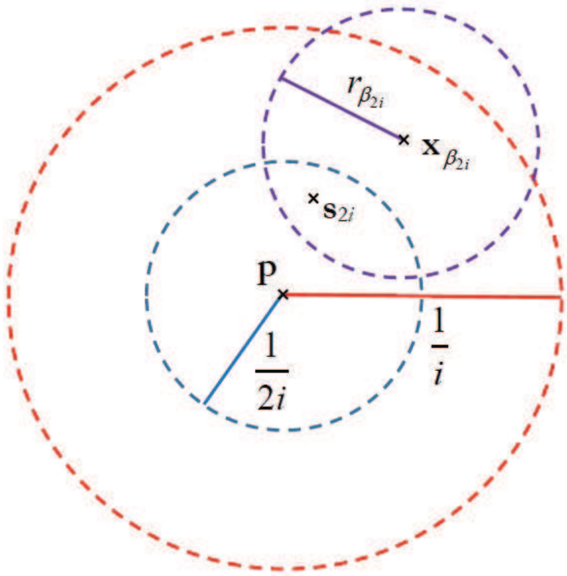


Figure 7.2: Construction of the sequence $E_i(\mathbf{p})$.

– **Case (i): $\mathbf{q}_i \in \partial B(\mathbf{x}_{\alpha_i}, r_{\alpha_i})$.** Geometrically, the area of $B(\mathbf{q}_i, r) \cap B(\mathbf{x}_{\alpha_i}, r_{\alpha_i})$ is always greater than that of $\frac{1}{4}B(\mathbf{q}_i, r)$ whenever $0 < r < r_{\alpha_i}$ so that

$$m\big(B(\mathbf{q}_i, r) \cap B(\mathbf{x}_{\alpha_i}, r_{\alpha_i})\big) \geq \frac{1}{4} m\big(B(\mathbf{q}_i, r)\big). \tag{7.104}$$

Since $B(\alpha, r_\alpha) \subseteq E$ for all $\alpha \in A$, we have $B(\alpha, r_\alpha) \subseteq E^\circ$ for all $\alpha \in A$. Furthermore, since $r_\alpha \geq 1$ for all $\alpha \in A$, we certainly have $\frac{1}{2i} < r_{\alpha_i}$ for all i. Hence these observations allow us to apply the equality (7.103) and the inequality (7.104) to get

$$\begin{aligned}
D_{E^\circ}(\mathbf{p}) &= \lim_{i\to\infty} \frac{m\big(E^\circ \cap B(\mathbf{p}, \frac{1}{i})\big)}{m\big(B(\mathbf{p}, \frac{1}{i})\big)} \\
&= \lim_{i\to\infty} \frac{m\big(E^\circ \cap B(\mathbf{p}, \frac{1}{i})\big)}{4m\big(E_i(\mathbf{p})\big)} \\
&\geq \frac{1}{4} \cdot \lim_{i\to\infty} \frac{m\big(B(\mathbf{x}_{\alpha_i}, r_{\alpha_i}) \cap B(\mathbf{q}_i, \frac{1}{2i})\big)}{m\big(B(\mathbf{q}_i, \frac{1}{2i})\big)} \\
&\geq \frac{1}{16}
\end{aligned}$$

which contradicts the result (7.101).

– **Case (ii): $\mathbf{q}_i \in B(\mathbf{x}_{\alpha_i}, r_{\alpha_i})$.** Since $|\mathbf{p} - \mathbf{q}_i| < \frac{1}{2i}$ and $\mathbf{q}_i \in B(\mathbf{x}_{\alpha_i}, r_{\alpha_i}) \cap B(\mathbf{p}, \frac{1}{i})$, the geometry tells us that $r_{\alpha_i} > \frac{1}{2i}$ so that the inequality (7.104) holds in this case. Finally, we have $D_{E^\circ}(\mathbf{p}) \geq \frac{1}{16}$, a contradiction again.

In conclusion, we have proved that $m(\partial E) = 0$.

(c) The crux of the analysis in part (b) is the inequalities $r_{\alpha_i} > \frac{1}{2i}$. For unrestricted radii r_α, we can pick a sequence $\{r_{\alpha_i}\}$ satisfying this condition for *sufficiently large* i and thus $D_{E^\circ}(\mathbf{p}) \geq \frac{1}{16}$

(d) By the comments in §2.21, there exists a non-Borel subset of $\mathbb{R}$. Let this set be A. Suppose that

$$E = \bigcup_{x\in A} \overline{B\big((x,0),1\big)}.$$

Assume that E was Borel in $\mathbb{R}^2$. If we can show that $\mathbb{R}\times\{1\}$ is Borel in $\mathbb{R}^2$, then since

$$E\cap(\mathbb{R}\times\{1\})=A\times\{1\},$$

A is also Borel in $\mathbb{R}$, a contradiction. Hence E must be non-Borel in $\mathbb{R}^2$. Now it remains to show that $\mathbb{R}\times\{1\}$ is Borel in $\mathbb{R}^2$ and this is the content of the following lemma:

Lemma 7.7

Let A and B be Boreal subsets of $\mathbb{R}$. Then $A\times B$ is also Borel in $\mathbb{R}^2$.

Proof of Lemma 7.7. Suppose that $\mathfrak{M}_1=\{A\subseteq\mathbb{R}\,|\,A\times\mathbb{R}\text{ is Borel in }\mathbb{R}^2\}$. It is obvious that $\mathbb{R}\in\mathfrak{M}_1$. Let $A\in\mathfrak{M}_1$, i.e., $A\times\mathbb{R}$ is Borel in $\mathbb{R}^2$. Since $A^c\times\mathbb{R}=\mathbb{R}^2\setminus(A\times\mathbb{R})$, $A^c\times\mathbb{R}$ is Borel in $\mathbb{R}^2$ by Comment 1.6(d). Suppose that $A_n\in\mathfrak{M}_1$ for all $n=1,2,\ldots$, i.e., each $A_n\times\mathbb{R}$ is Borel in $\mathbb{R}^2$ so that $\displaystyle\bigcup_{n=1}^{\infty}(A_n\times\mathbb{R})$ is Borel in $\mathbb{R}^2$. Since

$$\begin{aligned}\Big(\bigcup_{n=1}^{\infty}A_n\Big)\times\mathbb{R}&=\Big\{(a,y)\,\Big|\,a\in\bigcup_{n=1}^{\infty}A_n,y\in\mathbb{R}\Big\}\\&=\{(a_1,y)\,|\,a_1\in A_1,y\in\mathbb{R}\}\cup\{(a_2,y)\,|\,a_2\in A_1,y\in\mathbb{R}\}\cup\cdots\\&=(A_1\times\mathbb{R})\cup(A_2\times\mathbb{R})\cup\cdots\\&=\bigcup_{n=1}^{\infty}(A_n\times\mathbb{R}),\end{aligned}$$

we establish that

$$\bigcup_{n=1}^{\infty}A_n\in\mathfrak{M}_1.$$

By Definition 1.3(a), $\mathfrak{M}_1$ is a σ-algebra in $\mathbb{R}$. Next, let V be open in $\mathbb{R}$. Since $V\times\mathbb{R}$ is open in $\mathbb{R}^2$ (see [42, p. 86]), it is Borel in $\mathbb{R}^2$ and thus $V\in\mathfrak{M}_1$. Hence $\mathscr{B}\subseteq\mathfrak{M}_1$.

Similarly, if we define $\mathfrak{M}_2=\{B\subseteq\mathbb{R}\,|\,\mathbb{R}\times B\text{ is Borel in }\mathbb{R}^2\}$, then in the same way as above we can show that $\mathfrak{M}_2$ is also a σ-algebra in $\mathbb{R}$ containing $\mathscr{B}$. Finally, if $A,B\in\mathscr{B}$, then $A\times\mathbb{R}$ and $\mathbb{R}\times B$ are Borel in $\mathbb{R}^2$. Since $(A\times\mathbb{R})\cap(\mathbb{R}\times B)=A\times B$, our desired result follows. ■

(e) As Balcerzak and Kharazishvili [6, p. 205] pointed out that the union of an arbitrary family of convex bodies (compact convex sets with non-empty interiors) in $\mathbb{R}^2$ is Lebesgue measurable. Therefore, the closed discs can be replaced by arbitrary closed polygons provided that they are convex.[k]

Hence we have completed the proof of the problem. ■

Problem 7.21

Rudin Chapter 7 Exercise 21.

[k]More generally, it is known that if $E=\bigcup X_\alpha$ and for each X_α, there exists an open ball B_α such that $B_\alpha\subseteq X_\alpha\subseteq\overline{B_\alpha}$, then E is Lebesgue measurable. See [33, Problem 22, p. 482].

Proof. Notice that the triangle inequality implies that $f, g \in BV$ on $[a,b]$ if and only if $f+g \in BV$ on $[a,b]$. Clearly, the function $g(t) = t$ belongs to BV on $[0,1]$, so $\gamma \in BV$ on $[0,1]$ if and only if $f \in BV$ on $[0,1]$.

By the definition, the length of the graph of f is the total variation of γ on $[0,1]$, say $V_\gamma(1)$. We want to show that

$$V_\gamma(1) = \int_0^1 \sqrt{1 + [f'(t)]^2}\,\mathrm{d}t. \tag{7.105}$$

Suppose that f is AC on $[0,1]$. Then γ is also AC on $[0,1]$. Without loss of generality, we may assume that $f(0) = 0$; otherwise, we consider the function $\widetilde{f} : [0,1] \to \mathbb{R}$ defined by $\widetilde{f} = f - f(0)$. This assumption implies that $\gamma(0) = 0$. By Theorem 7.18,[1] we know that $\gamma' \in L^1$ a.e. $[m]$ on $[0,1]$ and

$$\gamma(x) = \int_0^x \gamma'(t)\,\mathrm{d}t.$$

Therefore, if $x, y \in [0,1]$ and $x < y$, then the above fact and Theorem 1.33 establish

$$|\gamma(y) - \gamma(x)| = \Big|\int_x^y \gamma'(t)\,\mathrm{d}t\Big| \leq \int_x^y |\gamma'(t)|\,\mathrm{d}t = \int_x^y \sqrt{1 + f'(t)}\,\mathrm{d}t. \tag{7.106}$$

Finally, it follows from the definition and the inequality (7.106) that if $0 = t_0 < t_1 < \cdots < t_N = 1$, then we have

$$V_\gamma(1) = \sup \sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1}| \leq \sup \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \sqrt{1 + f'(t)}\,\mathrm{d}t = \int_0^1 \sqrt{1 + f'(t)}\,\mathrm{d}t.$$

This proves one direction.

For the other direction, since γ is AC on $[0,1]$, V_γ is also AC on $[0,1]$ by Theorem 7.19. Thus Theorem 7.18 and the fact $V_\gamma(0) = 0$ imply that

$$\int_0^1 V_\gamma'(t)\,\mathrm{d}t = V_\gamma(1). \tag{7.107}$$

Employing the notation in the proof of Problem 7.13(c), if $0 \leq x < y \leq 1$, then we have

$$V_\gamma(y) - V_\gamma(x) = V_\gamma(x; y) = \sup \sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1})|, \tag{7.108}$$

where the supremum is taken over all positive integer N and over all partitions $\{t_i\}$ such that $x = t_0 < t_1 < \cdots < t_N = y$. By repeated use of the triangle inequality, we certainly have

$$\sqrt{(y-x)^2 + [f(y) - f(x)]^2} = |\gamma(y) - \gamma(x)| \leq \sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1})| \leq V_\gamma(x; y). \tag{7.109}$$

Therefore, it follows from the expressions (7.108) and (7.109) that

$$\sqrt{(y-x)^2 + [f(y) - f(x)]^2} \leq V_\gamma(y) - V_\gamma(x)$$

and then

$$\sqrt{1 + \Big[\frac{f(y) - f(x)}{y - x}\Big]^2} \leq \frac{V_\gamma(y) - V_\gamma(x)}{y - x}. \tag{7.110}$$

[1] Monotonicity of γ is *not* necessary for the implication (a) $\to$ (c).

Recall that f and V_γ are AC on $[0,1]$, f' and V'_γ exist a.e. $[m]$ on $[0,1]$ by Theorem 7.18, so the inequality (7.110) ensures that

$$\sqrt{1+[f'(x)]^2} \leq V'_\gamma(x)$$

a.e. $[m]$ on $[0,1]$. Substituting this into the expression (7.107), we get

$$\int_0^1 \sqrt{1+[f'(t)]^2}\,\mathrm{d}t \leq V_\gamma(1)$$

which proves the other direction. Hence we have completed the proof of the problem. ∎

Problem 7.22

Rudin Chapter 7 Exercise 22.

Proof.

(a) Assume that $f \neq 0$ on a measurable set $E \subseteq \mathbb{R}^k$ of positive measure. We claim that there corresponds a constant $c = c(f) > 0$ such that

$$(Mf)(\mathbf{x}) \geq c|\mathbf{x}|^{-k}$$

for sufficiently large $|\mathbf{x}|$. If it was not the case, then for each $n \in \mathbb{N}$, there exists a $\mathbf{x}_n \in \mathbb{R}^k$ with $|\mathbf{x}_n| > n$ such that

$$(Mf)(\mathbf{x}_n) < \frac{|\mathbf{x}_n|^{-k}}{n}.$$

Recall the basic fact that the Lebesgue measure of the ball $B(\mathbf{x}, r)$ in $\mathbb{R}^k$ is given by

$$\frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}r^k.$$

Thus it follows from the definition [51, Eqn. (4), p. 138] and this fact that

$$\int_{B(\mathbf{x}_n,r)} |f|\,\mathrm{d}m \leq m(B_r) \times (Mf)(\mathbf{x}_n) < \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}r^k \times \frac{|\mathbf{x}_n|^{-k}}{n} \tag{7.111}$$

for all $r > 0$. Particularly, we pick the sequence $\{r_n\}$, where $r_n = \sqrt[2k]{n} \cdot |\mathbf{x}_n|$, and the inequality (7.111) becomes

$$\int_{\mathbb{R}^k} \chi_{B(\mathbf{x}_n,r_n)}|f|\,\mathrm{d}m = \int_{B(\mathbf{x}_n,r_n)} |f|\,\mathrm{d}m < \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} \times \frac{1}{\sqrt{n}}. \tag{7.112}$$

By the Archimedean Property [49, Theorem 1.20(a), p. 9], for every $\mathbf{x} \in \mathbb{R}^k$, there is a positive integer n such that $n \geq |\mathbf{x}|$, so we have $|\mathbf{x}| < n < |\mathbf{x}_n|$ and then

$$|\mathbf{x} - \mathbf{x}_n| \leq |\mathbf{x}| + |\mathbf{x}_n| < 2|\mathbf{x}_n| < \sqrt[2k]{n} \cdot |\mathbf{x}_n|$$

for large enough n (remember that k is *fixed*). In other words, this means that $\mathbf{x} \in B(\mathbf{x}_n, \sqrt[2k]{n} \cdot |\mathbf{x}_n|)$ and thus

$$\lim_{n\to\infty} \chi_{B(\mathbf{x}_n,\, \sqrt[2k]{n}|\mathbf{x}_n|)}(\mathbf{x}) = 1. \tag{7.113}$$

Let $|f_n| = \chi_{B(\mathbf{x}_n,r_n)}|f|$ for all $n \in \mathbb{N}$. Then each $|f_n|$ is measurable, $|f_n| \leq |f|$ and the limit (7.113) implies

$$|f|(\mathbf{x}) = \lim_{n\to\infty} |f_n|(\mathbf{x})$$

for every $\mathbf{x} \in \mathbb{R}^k$. Since $f \in L^1(\mathbb{R}^k)$, the sequence $\{|f_n|\}$ satisfies the hypotheses of Theorem 1.34 (Lebesgue's Dominated Convergence Theorem). Thus we have

$$\lim_{n\to\infty} \int_{B(\mathbf{x}_n, r_n)} |f| \,\mathrm{d}m = \lim_{n\to\infty} \int_{\mathbb{R}^k} |f_n| \,\mathrm{d}m = \int_{\mathbb{R}^k} |f| \,\mathrm{d}m. \tag{7.114}$$

Now we combine the inequality (7.112) and the result (7.114) to get

$$\int_{\mathbb{R}^k} |f| \,\mathrm{d}m \leq \lim_{n\to\infty} \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} \times \frac{1}{\sqrt{n}} = 0.$$

By Theorem 1.39(a), $f = 0$ a.e. on $\mathbb{R}^k$, a contradiction. Hence there exists a constant $c = c(f) > 0$ and $r > 0$ such that for all $|\mathbf{x}| \geq r$, we have

$$(Mf)(\mathbf{x}) \geq c|\mathbf{x}|^{-k}$$

which implies that $\|Mf\|_1 = \infty$, a contradiction. Hence we must have $f = 0$ a.e. on $\mathbb{R}^k$.

(b) For every $x \in (0, \frac{1}{4})$, if $r \in (0, x]$, then we have $x - r \geq 0$ and $x + r < \frac{1}{2}$. Particularly, we take $r = x$ so that

$$\frac{1}{m(B_r)} \int_{B(x,r)} |f| \,\mathrm{d}m = \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \,\mathrm{d}t = \frac{1}{2x} \int_0^{2x} \frac{1}{t(\log t)^2} \,\mathrm{d}t. \tag{7.115}$$

Since $t(\log t)^2 > 0$ on $(0, \frac{1}{2})$ and $t^{-1}(\log t)^{-2} \in \mathscr{R}$ on $[\epsilon, 2x]$, we apply [49, Theorem 11.33, p. 323] to obtain

$$\int_0^{2x} \frac{1}{t(\log t)^2} \,\mathrm{d}t \geq \int_\epsilon^{2x} \frac{1}{t(\log t)^2} \,\mathrm{d}t = \mathscr{R} \int_\epsilon^{2x} \frac{1}{t(\log t)^2} \,\mathrm{d}t = -\frac{1}{\log t}\Big|_\epsilon^{2x} = -\frac{1}{\log(2x)} + \frac{1}{\log \epsilon}$$

for every $\epsilon > 0$. As $\epsilon \to 0+$, we know that $\frac{1}{\log \epsilon} \to 0-$ and so

$$\int_0^{2x} \frac{1}{t(\log t)^2} \,\mathrm{d}t \geq -\frac{1}{\log(2x)}. \tag{7.116}$$

Since $2x \in (0, \frac{1}{2})$, we have $-\frac{1}{\log(2x)} > 0$ and then we substitute the inequality (7.116) into the expression (7.115) to obtain

$$(Mf)(x) \geq \sup_{0<r<\infty} \frac{1}{m(B_r)} \int_{B(x,r)} |f| \,\mathrm{d}m \geq -\frac{1}{2x\log(2x)} = \Big|\frac{1}{2x\log(2x)}\Big|,$$

as required.

Furthermore, direct computation gives

$$\int_0^{\frac{1}{4}} \frac{\mathrm{d}x}{|2x\log(2x)|} = \infty.$$

Recall the basic fact that $(Mf)(x) \geq 0$, so it implies that

$$\int_0^1 (Mf)(x) \,\mathrm{d}x \geq \int_0^{\frac{1}{4}} \frac{\mathrm{d}x}{|2x\log(2x)|} = \infty,$$

as desired.

Hence we have completed the proof of the problem. ■

Problem 7.23

Rudin Chapter 7 Exercise 23.

Proof. We first show that $(SF)(\mathbf{x})$ is well-defined. To this end, let α_1 and α_2 be two complex numbers such that

$$\lim_{r\to 0+} \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |f_1 - \alpha_1|\,\mathrm{d}m = \lim_{r\to 0+} \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |f_2 - \alpha_2|\,\mathrm{d}m = 0 \tag{7.117}$$

for some $f_1, f_2 \in F$. Then we see that

$$\begin{aligned} |\alpha_1 - \alpha_2| &= \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |\alpha_1 - \alpha_2|\,\mathrm{d}m \\ &\leq \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} (|\alpha_1 - f| + |f - g| + |g - \alpha_2|)\,\mathrm{d}m \\ &= \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |\alpha_1 - f|\,\mathrm{d}m + \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |g - \alpha_2|\,\mathrm{d}m \\ &\quad + \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |f - g|\,\mathrm{d}m. \end{aligned} \tag{7.118}$$

Since f and g belong to the *same* equivalence class F, $f = g$ a.e. on $\mathbb{R}^k$ and the last integral in the inequality (7.118) is actually 0. By taking $r \to 0+$ to the integrals in the inequality (7.118) and using the hypotheses (7.117), we obtain $\alpha_1 = \alpha_2$, i.e., $(SF)(\mathbf{x})$ is unique if $\mathbf{x}$ is a Lebesgue point of F.

Next, if $\mathbf{x}$ is a Lebesgue point of f and $f \in F$, then Definition 7.6 gives

$$\lim_{r\to 0+} \frac{1}{m(B_r)} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\,\mathrm{d}m(\mathbf{y}) = 0$$

so that $\mathbf{x}$ is also a Lebesgue point of F and the uniqueness of Lebesgue points of F in the previous paragraph shows immediately that $(SF)(\mathbf{x}) = f(\mathbf{x})$. This completes the proof of the problem. ∎

CHAPTER 8

Integration on Product Spaces

8.1 Monotone Classes and Ordinate Sets of Functions

Problem 8.1

Rudin Chapter 8 Exercise 1.

Proof. Suppose that

$$\mathfrak{M} = \{\varnothing, \mathbb{R}, (-\infty, a), (-\infty, a], (b, \infty), [b, \infty) \,|\, a, b \in \mathbb{R}\}.$$

We claim that $\mathfrak{M}$ is a monotone class. To see this, let $A_i \in \mathfrak{M}$, $A_i \subset A_{i+1}$ and $A = \bigcup_{i=1}^{\infty} A_i$. We have several cases to consider.

- **Case (i):** $A_i = (-\infty, a_i)$**.** Then the condition $A_i \subset A_{i+1}$ forces that A_{i+1} is in one of the following forms:
$$(-\infty, a_i], \quad (-\infty, a_{i+1}) \quad \text{and} \quad (-\infty, a_{i+1}] \tag{8.1}$$
for some $a_{i+1} > a_i$. Thus A is in one of the following forms:
$$(-\infty, a), \quad (-\infty, a] \quad \text{and} \quad \mathbb{R} \tag{8.2}$$
for some $a \in \mathbb{R}$. Both cases say that $A \in \mathfrak{M}$.

- **Case (ii):** $A_i = (-\infty, a_i]$**.** Now A_{i+1} is still in one of the forms (8.1), so similar analysis shows that A is in one of the forms (8.2). Therefore, $A \in \mathfrak{M}$.

- **Case (iii):** $A_i = (b_i, \infty)$**.** In this case, A_{i+1} is in one of the forms:
$$[b_i, \infty), \quad (b_{i+1}, \infty) \quad \text{and} \quad [b_{i+1}, \infty) \tag{8.3}$$
for some $b_{i+1} < b_i$. Then it can be shown easily that A is in one of the following forms:
$$(b, \infty), \quad [b, \infty) \quad \text{and} \quad \mathbb{R} \tag{8.4}$$
for some $b \in \mathbb{R}$. Consequently, we have $A \in \mathfrak{M}$.

- **Case (iv):** $A_i = [b, \infty)$**.** The A_{i+1} is expressed in one of the forms (8.3) so that A is in one of the forms (8.4). In conclusion, we have $A \in \mathfrak{M}$.

Next, let $B_i \in \mathfrak{M}$, $B_i \supset B_{i+1}$ and $B = \bigcup_{i=1}^{\infty} B_i$. Similar to the sequence of sets $\{A_i\}$, there are also four cases for $\{B_i\}$. Since the analysis of these cases is very similar to those above, so we omit the details here. By Definition 8.1, we see that $\mathfrak{M}$ is a monotone class.

Clearly, $\mathbb{R} \in \mathfrak{M}$. Since $\mathbb{R} \setminus (-\infty, a) = [a, \infty)$, $\mathbb{R} \setminus (-\infty, a] = (a, \infty)$, $\mathbb{R} \setminus (b, \infty) = (-\infty, b]$ and $\mathbb{R} \setminus [b, \infty) = (-\infty, b)$, we immediately have $\mathbb{R} \setminus A \in \mathfrak{M}$ for every $A \in \mathfrak{M}$. However, $\mathfrak{M}$ is *not* a σ-algebra because $(-\infty, 0), (1, \infty) \in \mathfrak{M}$ but $(-\infty, 0) \cup (1, \infty) \notin \mathfrak{M}$ which violates Definition 1.3(a). This completes the proof of the problem. ■

Problem 8.2

Rudin Chapter 8 Exercise 2.

Proof. We have $f : \mathbb{R} \to [0, \infty)$ and

$$A(f) = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < y < f(x)\}. \tag{8.5}$$

Now all the answers are affirmative.

(a) Since f is Lebesgue measurable, Theorem 1.17 (The Simple Function Approximation Theorem) ensures the existence of simple measurable functions s_n on $\mathbb{R}$ such that $0 \leq s_1 \leq s_2 \leq \cdots \leq f$ and $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in \mathbb{R}$.

Suppose that s is a nonnegative simple function on $\mathbb{R}$. Then, by Definition 1.6, we have

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where $A_1, A_2, \ldots, A_n$ can be assumed to be *mutually disjoint* Lebesgue measurable sets of $\mathbb{R}$. Therefore, we obtain

$$\begin{aligned} A(s) &= \{(x, y) \in \mathbb{R}^2 \,|\, 0 < y < s(x)\} \\ &= \bigcup_{i=1}^{n} \{(x, y) \in \mathbb{R}^2 \,|\, x \in A_i \text{ and } 0 < y < \alpha_i\} \\ &= \bigcup_{i=1}^{n} [A_i \times (0, \alpha_i)]. \end{aligned}$$

By Definition 8.1, $A(s)$ must be a Lebesgue measurable subset of $\mathbb{R}^2$. It is easy to check from the definition of the ordinate set of a function that the following lemma holds:

Lemma 8.1

If f and g are functions on $\mathbb{R}$ such that $f(x) \leq g(x)$ for every $x \in \mathbb{R}$, then $A(f) \subseteq A(g)$.

Now we claim that

$$A(f) = \bigcup_{n=1}^{\infty} A(s_n). \tag{8.6}$$

The direction

$$\bigcup_{n=1}^{\infty} A(s_n) \subseteq A(f)$$

holds trivially because of the fact that $s_n \leq f$ for every $n = 1, 2, \ldots$ and Lemma 8.1. For the other direction, let $(x, y) \in A(f)$. This implies that $x \in \mathbb{R}$ and $0 < y < f(x)$. Since $s_n(x) \to f(x)$ as $n \to \infty$, there exists a $N \in \mathbb{N}$ such that $0 < y < s_N(x) \leq f(x)$ which means $(x, y) \in A(s_N)$ so that

$$A(f) \subseteq \bigcup_{n=1}^{\infty} A(s_n)$$

also holds. Hence we have established the claim (8.6). Finally, since each $A(s_n)$ is a Lebesgue measurable subset of $\mathbb{R}^2$, the relation (8.6) implies that $A(f)$ is also a Lebesgue measurable subset of $\mathbb{R}^2$.

(b) By the definition (8.5), the x-section of $A(f)$ is given by

$$[A(f)]_x = \{y \in \mathbb{R} \,|\, (x, y) \in A(f)\} = \{y \in \mathbb{R} \,|\, 0 < y < f(x)\} = \big(0, f(x)\big).$$

Since $\mathbb{R}$ is obviously σ-finite, it follows from Theorem 8.6 (see also Definition 8.7) that

$$m_2\big(A(f)\big) = (m \times m)\big(A(f)\big) = \int_{\mathbb{R}} m\big([A(f)]_x\big)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

(c) Recall that the graph of f, namely G, is given by

$$G = \{(x, y) \in \mathbb{R}^2 \,|\, y = f(x)\}.$$

For each $i \in \mathbb{N}$, the set $A(f + \frac{1}{i})$ is Lebesgue measurable in $\mathbb{R}^2$ by part (a). Then the set

$$A = \bigcap_{i=1}^{\infty} A\Big(f + \frac{1}{i}\Big) = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < y \leq f(x)\}$$

is also Lebesgue measurable by Comment 1.6(c). Therefore, the set

$$A \setminus A(f) = \{(x, y) \in \mathbb{R}^2 \,|\, y = f(x) \text{ but } y > 0\}$$

is Lebesgue measurable in $\mathbb{R}^2$. Obviously, we have

$$\{(x, 0) \in \mathbb{R}^2 \,|\, 0 = f(x)\} = \{x \in \mathbb{R} \,|\, f(x) = 0\} \times \{0\} = f^{-1}(0) \times \{0\}.$$

Since f is Lebesgue measurable, $f^{-1}(0)$ is a Lebesgue measurable set in $\mathbb{R}$ by Definition 1.3(c).[a] Since $\{0\}$ is a Lebesgue measurable set in $\mathbb{R}$, we use Definition 8.1 to conclude that $f^{-1}(0) \times \{0\}$ is Lebesgue measurable in $\mathbb{R}^2$. Since

$$G = \big(A \setminus A(f)\big) \cup [f^{-1}(0) \times \{0\}],$$

the preceding analysis shows that G is a Lebesgue measurable subset of $\mathbb{R}^2$.

(d) For each *fixed* $x \in \mathbb{R}$, we consider the x-section of G:

$$G_x = \{y \in \mathbb{R} \,|\, (x, y) \in G\} = \{y \in \mathbb{R} \,|\, y = f(x)\} = \{f(x)\}.$$

By a similar argument as in part (b), we conclude easily from this that

$$m_2(G) = (m \times m)(G) = \int_{\mathbb{R}} m(G_x)\,\mathrm{d}x = 0$$

as required.

This completes the proof of the problem. ■

[a]Of course, we replace "for every open set V in Y" by "for every closed set V in Y" here.

8.2 Applications of the Fubini Theorem

Problem 8.3

Rudin Chapter 8 Exercise 3.

Proof. Note that

$$\varphi(x) = \int_0^1 f(x,y)\,\mathrm{d}y$$

for each $x \in (0,1)$. Suppose that

$$f(x,y) = y^{(x-\frac{1}{2})^2-1}$$

in $(0,1)\times(0,1)$. It is clear that f is a positive continuous function in $(0,1)\times(0,1)$. If $x=\frac{1}{2}$, then $f(\frac{1}{2},y)=y^{-1}$ so that

$$\varphi\Big(\frac{1}{2}\Big) = \int_0^1 y^{-1}\,\mathrm{d}y = \ln y|_0^1 = \infty.$$

If $x \neq \frac{1}{2}$, then we have

$$\varphi(x) = \int_0^1 y^{(x-\frac{1}{2})^2-1}\,\mathrm{d}y = \frac{1}{(x-\frac{1}{2})^2}$$

and this implies that

$$\int_0^1 \varphi(x)\,\mathrm{d}x = \int_0^1 \frac{1}{(x-\frac{1}{2})^2}\,\mathrm{d}x = -\Big(x-\frac{1}{2}\Big)^{-1}\Big|_0^1 = -4.$$

This is a required example, completing the proof of the problem. ■

Problem 8.4

Rudin Chapter 8 Exercise 4.

Proof.

(a) As Rudin pointed out, we may assume that f and g are Borel functions on $\mathbb{R}$. If $p=1$, then it is in fact Theorem 8.14. If $p=\infty$, then $\|g\|_\infty < \infty$ and so

$$\begin{aligned}
|(f*g)(x)| &= \Big|\int_{-\infty}^{\infty} f(x-t)g(t)\,\mathrm{d}t\Big| \\
&\leq \int_{-\infty}^{\infty} |f(x-t)|\cdot|g(t)|\,\mathrm{d}t \\
&\leq \|g\|_\infty \int_{-\infty}^{\infty} |f(x-t)|\,\mathrm{d}t \\
&= \|g\|_\infty \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t \\
&= \|g\|_\infty \cdot \|f\|_1.
\end{aligned}$$

Thus we have $\|f*g\|_\infty \leq \|f\|_1\cdot\|g\|_\infty$.

Suppose that $1 < p < \infty$. Since f and g are measurable on $\mathbb{R}$, the functions $|f(x-t)|^{\frac{1}{q}}$ and $|f(x-t)|^{\frac{1}{p}} \cdot |g(y)|$ are also measurable on $\mathbb{R}$ by Theorem 1.7, where q is the conjugate exponent of p. By Theorem 3.5 (Hölder's Inequality), we derive that

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x-y)g(y)|\,\mathrm{d}y &= \int_{-\infty}^{\infty} |f(x-y)|^{\frac{1}{q}} \cdot \big(|f(x-y)|^{\frac{1}{p}}|g(y)|\big)\,\mathrm{d}y \\ &\leq \Big\{\int_{-\infty}^{\infty} |f(x-y)|\,\mathrm{d}y\Big\}^{\frac{1}{q}} \times \Big\{\int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}y\Big\}^{\frac{1}{p}} \\ &= \|f\|_1^{\frac{1}{q}} \cdot \Big\{\int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}y\Big\}^{\frac{1}{p}}\end{aligned}$$

which gives

$$\Big\{\int_{-\infty}^{\infty} |f(x-y)g(y)|\,\mathrm{d}y\Big\}^p \leq \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}y. \tag{8.7}$$

Define $F : \mathbb{R}^2 \to \mathbb{C}$ by

$$F(x,y) = f(x-y)g(y).$$

Using similar argument as in the proof of Theorem 8.14, we are able to show that the function $F \in L^1(\mathbb{R}^2)$. Since $\mathbb{R}$ is a σ-finite measure space, we get from Theorem 8.8 (The Fubini Theorem) that

$$\varphi(x) = \int_{-\infty}^{\infty} F_x(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y = (f*g)(x)$$

a.e. is in $L^1(\mathbb{R})$ so that the integral defining $(f*g)(x)$ exists for almost all x. This proves the first assertion.

To verify the second and the third assertions, we recall that $|f|, |g|^p \in L^1(\mathbb{R})$, so Theorem 8.14 ensures that the function $h : \mathbb{R} \to [0,\infty]$ given by

$$h(x) = \int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}y$$

belongs to $L^1(\mathbb{R})$ which means that h is a measurable function on $\mathbb{R}$. By Theorem 1.33, we observe from the inequality (8.7) that

$$\begin{aligned}|(f*g)(x)|^p &= \Big|\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y\Big|^p \\ &\leq \Big\{\int_{-\infty}^{\infty} |f(x-y)g(y)|\,\mathrm{d}y\Big\}^p \\ &\leq \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}y.\end{aligned} \tag{8.8}$$

Now we apply Theorem 8.8 (The Fubini Theorem) to the right-hand side of the inequality (8.8) to get

$$\begin{aligned}\int_{-\infty}^{\infty} |(f*g)(x)|^p\,\mathrm{d}x &\leq \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}y\,\mathrm{d}x \\ &= \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p\,\mathrm{d}x\,\mathrm{d}y \\ &= \|f\|_1^{\frac{p}{q}} \cdot \|f\|_1 \int_{-\infty}^{\infty} |g(y)|^p\,\mathrm{d}y \\ &= \|f\|_1^p \cdot \|g\|_p^p\end{aligned}$$

which implies that $\|f*g\|_p \leq \|f\|_1 \cdot \|g\|_p < \infty$.

(b) Let $p = 1$. If $f \geq 0$ and $g \geq 0$, then the definition gives

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y \geq 0. \tag{8.9}$$

Therefore we deduce from Theorem 8.8 (The Fubini Theorem) that

$$\begin{aligned}\|f * g\|_1 &= \|h\|_1 \\ &= \int_{-\infty}^{\infty} h(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} g(y)\,\mathrm{d}y \int_{-\infty}^{\infty} f(x-y)\,\mathrm{d}x \\ &= \|g\|_1 \cdot \|f\|_1\end{aligned}$$

as required.

Let $p = \infty$. If $f \geq 0$ and $g = c \geq 0$, then the inequality (8.9) holds in this case. Recall that $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, so we have

$$|h(x)| = \Big|\int_{-\infty}^{\infty} f(x-y)c\,\mathrm{d}y\Big| = c\|f\|_1 < \infty$$

for almost all $x \in \mathbb{R}$. By Definition 3.7, we obtain

$$\|f * g\|_\infty = \|h\|_\infty = \|f\|_1 \times c = \|f\|_1 \cdot \|g\|_\infty.$$

(c) Suppose that $1 < p < \infty$ and $\|f * g\|_p = \|f\|_1 \cdot \|g\|_p$. By the definition of $f * g$ and the application of part (a), we observe that

$$\|f * g\|_p \leq \big\||f| * |g|\big\|_p \leq \big\||f|\big\|_1 \cdot \big\||g|\big\|_p = \|f\|_1 \cdot \|g\|_p = \|f * g\|_p$$

which forces $\big\||f| * |g|\big\|_p = \big\||f|\big\|_1 \cdot \big\||g|\big\|_p$. Thus we may assume that $f, g \geq 0$.

Recall that the inequality $\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p$ derives from (see the deduction of the inequality (8.8))

$$\begin{aligned}(f * g)(x)^p &= \Big\{\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y\Big\}^p \\ &= \Big\{\int_{-\infty}^{\infty} f(x-y)^{\frac{1}{q}} \cdot [f(x-y)^{\frac{1}{p}} g(y)]\,\mathrm{d}y\Big\}^p \\ &\leq \Big\{\int_{-\infty}^{\infty} f(x-y)\,\mathrm{d}y\Big\}^{\frac{p}{q}} \times \Big\{\int_{-\infty}^{\infty} f(x-y)g(y)^p\,\mathrm{d}y\Big\} \qquad (8.10) \\ &= \|f\|_1^{\frac{p}{q}} \times \Big\{\int_{-\infty}^{\infty} f(x-y)g(y)^p\,\mathrm{d}y\Big\}\end{aligned}$$

and so the equality $\|f * g\|_p = \|f\|_1 \cdot \|g\|_p$ forces that the equality (8.10) holds, but this means that the equality of Theorem 3.5 (Hölder's Inequality) holds there i.e., there are constants (with respect to y) $\alpha(x)$ and $\beta(x)$, not both zero, such that

$$\alpha(x)f(x-y) = \alpha(x)f(x-y)^{\frac{q}{q}} = \beta(x)[f(x-y)^{\frac{1}{p}}g(y)]^p = \beta(x)f(x-y)g(y)^p \tag{8.11}$$

for almost all $y \in \mathbb{R}$. Let $E = \{z \in \mathbb{R} \,|\, f(-z) > 0\}$. If $\|f\|_1 = 0$, then since $m(\mathbb{R}) = \infty$, $f = 0$ a.e. on $\mathbb{R}$ and we are done. Therefore, without loss of generality, we may assume that $\|f\|_1 = 1$ which gives $m(E) > 0$.

On the one hand, if $\beta(x) = 0$ *for some* $x \in \mathbb{R}$, then $\alpha(x) \neq 0$ and so the equation (8.11) implies that $f(x-y) = 0$ for almost all $y \in \mathbb{R}$ which is equivalent to $f = 0$ a.e. on $\mathbb{R}$. On the other hand, if $\alpha(x) = 0$ *for some* $x \in \mathbb{R}$, then $\beta(x) \neq 0$ and so

$$f(x-y)g(y)^p = 0 \tag{8.12}$$

for almost all $y \in \mathbb{R}$. Now if $y \in E_x = x + E = \{x + z \,|\, z \in E\}$, then $y = x + z$ or $-z = x - y$ for some $z \in E$ so that

$$f(x-y) > 0$$

for all $y \in E_x$. Therefore, it follows from the equation (8.12) that $g(y) = 0$ for almost all $y \in E_x$. Assume that $m(E_x) < \infty$. Then $m(E_x^c) = \infty$. Furthermore, we have $g(y) > 0$ for all $y \in E_x^c$ and this implies that

$$\|g\|_p^p = \int_{\mathbb{R}} g(y)^p \, \mathrm{d}y = \int_{E_x^c} g(y)^p \, \mathrm{d}y = \infty$$

which contradicts $g \in L^p(\mathbb{R})$.

(d) Suppose that $p = \infty$. We simply take $f(x) = \chi_{[0,1]}$ and $g(x) = 1$ on $\mathbb{R}$. Then we have $\|f\|_1 = \|g\|_\infty = 1$. Since $f \geq 0$ and $g \geq 0$ on $\mathbb{R}$, we deduce from Theorem 7.26 (The Change-of-variables Theorem) that

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y) \, \mathrm{d}y = \int_0^1 \mathrm{d}y = 1.$$

Hence we always have

$$\|f * g\|_\infty = 1 > (1-\epsilon)\|f\|_1 \cdot \|g\|_\infty.$$

Next, we suppose that $p < \infty$. Let $0 < \alpha < 1 - (1-\epsilon)^p < 1$ and define

$$f(x) = \frac{1}{2\alpha}\chi_{[-\alpha,\alpha]}(x) \quad \text{and} \quad g(x) = \frac{1}{2^p}\chi_{[-1,1]}(x).$$

It is easy to see that

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x = \int_{-\alpha}^{\alpha} \frac{1}{2\alpha}\chi_{[-\alpha,\alpha]}(x) \, \mathrm{d}x = 1$$

and

$$\|g\|_p = \int_{-\infty}^{\infty} |g(x)|^p \, \mathrm{d}x = \int_{-1}^{1} \frac{1}{2}\chi_{[-1,1]}(x) \, \mathrm{d}x = 1.$$

Since $f \geq 0$ and $g \geq 0$ on $\mathbb{R}$, we deduce from Theorem 7.26 (The Change-of-variables Theorem) that

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y) \, \mathrm{d}y = \frac{1}{2\alpha \cdot 2^{\frac{1}{p}}} \int_{-\alpha}^{\alpha} \chi_{[-1,1]}(x-y) \, \mathrm{d}y. \tag{8.13}$$

Since $-1 \leq x - y \leq 1$ if and only if $x - 1 \leq y \leq x + 1$, we follow from the expression (8.13) that

$$(f * g)(x) = \frac{1}{2\alpha \cdot 2^{\frac{1}{p}}} \int_{-\alpha}^{\alpha} \chi_{[x-1,x+1]}(y) \, \mathrm{d}y$$

$$= \frac{1}{2\alpha \cdot 2^{\frac{1}{p}}} m\big([-\alpha, \alpha] \cap [x-1, x+1]\big)$$

$$= \begin{cases} 0, & \text{if } x \in (-\infty, -1-\alpha]; \\[1em] \dfrac{x+1+\alpha}{2\alpha \cdot 2^{\frac{1}{p}}} & \text{if } x \in (-1-\alpha, -1+\alpha]; \\[1em] \dfrac{2\alpha}{2\alpha \cdot 2^{\frac{1}{p}}} & \text{if } x \in (-1+\alpha, 1-\alpha]; \\[1em] \dfrac{1+\alpha-x}{2\alpha \cdot 2^{\frac{1}{p}}} & \text{if } x \in (1-\alpha, 1+\alpha]; \\[1em] 0, & \text{if } x \in (1+\alpha, \infty). \end{cases}$$

Hence direct computation shows that

$$\begin{aligned} \|f * g\|_p^p &= \int_{-\infty}^{\infty} |(f*g)(x)|^p \, dx \\ &= \int_{-1-\alpha}^{1+\alpha} |(f*g)(x)|^p \, dx \\ &= \int_{-1-\alpha}^{-1+\alpha} \frac{1}{2(2\alpha)^p} (x+1+\alpha)^p \, dx + \int_{-1+\alpha}^{1-\alpha} \frac{(2\alpha)^p}{2(2\alpha)^p} \, dx \\ &\quad + \int_{1-\alpha}^{1+\alpha} \frac{1}{2(2\alpha)^p} (1+\alpha-x)^p \, dx \\ &= 2\int_{-1-\alpha}^{-1+\alpha} \frac{1}{2(2\alpha)^p} (x+1+\alpha)^p \, dx + (1-\alpha). \end{aligned} \tag{8.14}$$

Since $(x+1+\alpha)^p \geq 0$ on $[-1-\alpha, -1+\alpha]$, the integral in the expression (8.14) is nonnegative so that

$$\|f * g\|_p^p \geq 1 - \alpha > (1-\epsilon)^p$$

and this is equivalent to saying that $\|f * g\|_p > (1-\epsilon)\|f\|_1 \cdot \|g\|_p$.

This completes the proof of the problem. ■

Remark 8.1

The result in Problem 8.4(a) is sometimes called Minkowski's Inequality which is a special case of the Young's Convolution Inequality.

Problem 8.5

Rudin Chapter 8 Exercise 5.

Proof.

(a) Clearly, the function $\alpha : \mathbb{R}^2 \to \mathbb{R}$ defined by $\alpha(x, y) = x + y$ is continuous. We want to show the following result first:

Lemma 8.2

It is true that $\alpha^{-1}(E) \in \mathscr{B}_2$ if $E \in \mathscr{B}_1$. (See Problem 8.11 for the meaning of the notation $\mathscr{B}_k$ for any $k \in \mathbb{N}$.)

Proof of Lemma 8.2. Let $\mathfrak{M} = \{E \subseteq \mathbb{R} \,|\, \alpha^{-1}(E) \in \mathscr{B}_2\}$. Since α is continuous, $\alpha^{-1}(V) \in \mathscr{B}_2$ for every open set V in $\mathbb{R}$. In other words, $\mathfrak{M}$ contains the *standard topology* of $\mathbb{R}$. It is easy to check that

- $\mathbb{R} \in \mathfrak{M}$.
- If $E \in \mathfrak{M}$, then $E^c \in \mathfrak{M}$ because $\alpha^{-1}(E^c) = \alpha^{-1}(\mathbb{R}) \setminus \alpha^{-1}(E) = \mathbb{R}^2 \setminus \alpha^{-1}(E) \in \mathscr{B}_2$.
- If $E_n \in \mathfrak{M}$ so that $\alpha^{-1}(E_n) \in \mathscr{B}_2$ for every $n \in \mathbb{N}$, then we have

$$\alpha^{-1}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \bigcup_{n=1}^{\infty} \alpha^{-1}(E_n) \in \mathscr{B}_2.$$

By Definition 1.3(a), $\mathfrak{M}$ is a σ-algebra and so $\mathfrak{M} = \mathscr{B}_1$ which completes the proof of Lemma 8.2 ■

Let's return to the proof of the problem. We want to check Definition 6.1:

$$(\mu * \lambda)(E) = \sum_{i=1}^{\infty} (\mu * \lambda)(E^i) \tag{8.15}$$

for every partition $\{E^i\} \subseteq \mathscr{B}_1$ of $E \in \mathscr{B}_1$. For each $i = 1, 2, \ldots$, we denote

$$E_2^i = \{(x, y) \in \mathbb{R}^2 \,|\, x + y \in E^i\} \subseteq \mathbb{R}^2 \quad \text{and} \quad E_2 = \{(x, y) \in \mathbb{R}^2 \,|\, x + y \in E\} \subseteq \mathbb{R}^2.$$

Then we have

$$E_2 = \bigcup_{i=1}^{\infty} E_2^i.$$

Next, if $(x_0, y_0) \in E_2^i \cap E_2^j$ for $i \neq j$, then $x_0 + y_0 \in E^i \cap E^j$ but $E^i \cap E^j = \varnothing$, a contradiction. Thus the set $\{E_2^1, E_2^2, \ldots\}$ must be a partition of E_2 and since $\mu \times \lambda$ is a measure by Definition 8.7, we get

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2) = \sum_{i=1}^{\infty} (\mu \times \lambda)(E_2^i) = \sum_{i=1}^{\infty} (\mu * \lambda)(E^i), \tag{8.16}$$

i.e., $\mu * \nu \in M$ and this proves the first assertion.

For the second assertion, since μ and λ are complex measures on $\mathbb{R}$, Theorem 6.12 implies that

$$\mathrm{d}\mu = h_1\,\mathrm{d}|\mu| \quad \text{and} \quad \mathrm{d}\lambda = h_2\,\mathrm{d}|\lambda|,$$

where $|h_1| = |h_2| = 1$ on $\mathbb{R}$. For any $A \in \mathscr{B}(\mathbb{R}^2)$, we observe from Definition 8.7 that

$$\begin{aligned}(\mu \times \lambda)(A) &= \int_X \lambda(A_x)\,\mathrm{d}\mu(x) \\ &= \int_X \lambda(A_x) h_1(x)\,\mathrm{d}|\mu|(x)\end{aligned}$$

$$= \int_X \Big\{ \int_Y \chi_{A_x}(y)\,\mathrm{d}\lambda(y) \Big\} h_1(x)\,\mathrm{d}|\mu|(x)$$
$$= \int_X \Big\{ \int_Y \chi_A(x,y) h_2(y)\,\mathrm{d}|\lambda|(y) \Big\} h_1(x)\,\mathrm{d}|\mu|(x). \tag{8.17}$$

Since $|\mu|$ and $|\lambda|$ are finite by Theorem 6.4, we obtain

$$\varphi^*(x) = \int_Y |g|_x\,\mathrm{d}|\lambda| = |\lambda|(Y) \quad \text{and} \quad \int_X \varphi^*\,\mathrm{d}|\mu| = |\lambda|(Y)|\mu|(X) < \infty.$$

Thus, by Theorem 8.8 (The Fubini Theorem), the integral (8.17) can be reduced to

$$\begin{aligned}(\mu \times \lambda)(A) &= \int_X \int_Y \chi_A(x,y) h_2(y) h_1(x)\,\mathrm{d}|\lambda|(y)\,\mathrm{d}|\mu|(x) \\ &= \int_{X\times Y} \chi_A(x,y) h_2(y) h_1(x)\,\mathrm{d}(|\mu| \times |\nu|) \\ &= \int_A f(x,y)\,\mathrm{d}(|\mu| \times |\nu|),\end{aligned} \tag{8.18}$$

where $f(x,y) = h_2(y)h_1(x)$. Obviously, Theorem 8.8 (The Fubini Theorem) also says that $f(x,y) \in L^1(|\mu| \times |\lambda|)$ so that Theorem 6.13 may be applied to the result (8.18) to get

$$|\mu \times \lambda|(A) = \int_A |f(x,y)|\,\mathrm{d}(|\mu| \times |\lambda|) = \int_A \mathrm{d}(|\mu| \times |\lambda|) = (|\mu| \times |\lambda|)(A), \tag{8.19}$$

i.e., the total variation of $\mu \times \lambda$ is $|\mu| \times |\lambda|$.

Since $\mu * \lambda \in M$, the series (8.16) converges absolutely and then the fact (8.16) implies

$$|(\mu * \lambda)|(\mathbb{R}) \leq \sum_{i=1}^{\infty} |\mu * \lambda|(E^i) \leq \sum_{i=1}^{\infty} |(\mu \times \lambda)|(E_2^i) \tag{8.20}$$

where $\{E^i\}$ is a partition of $\mathbb{R}$. By Theorem 6.2, $|\mu \times \lambda|$ is a positive measure. Recall that $\{E_2^i\}$ is a partition of E_2, so if we apply these facts and the result (8.19) to the inequality (8.20), then it becomes

$$|(\mu * \lambda)|(\mathbb{R}) \leq |(\mu \times \lambda)|(E_2) = (|\mu| \times |\lambda|)(E_2) \leq (|\mu| \times |\lambda|)(\mathbb{R}^2).$$

Hence we follow from the definition of the norm in M and Definition 8.7 ($\mathbb{R}$ is σ-finite) that

$$\|\mu * \lambda\| = |\mu * \lambda|(\mathbb{R}) \leq (|\mu| \times |\lambda|)(\mathbb{R} \times \mathbb{R}) = |\mu|(\mathbb{R}) \times |\lambda|(\mathbb{R}) = \|\mu\| \cdot \|\lambda\|. \tag{8.21}$$

(b) Firstly, for $E \in \mathscr{B}_1$, we know from the definition of $\mu * \lambda$, Definition 8.7 and then [51, Eqn. (3), p. 163] that

$$\begin{aligned}\int_{\mathbb{R}} \chi_E(x)\,\mathrm{d}(\mu * \lambda) &= (\mu * \lambda)(E) \\ &= (\mu \times \lambda)(E_2) \\ &= \int_{\mathbb{R}} \lambda(E_2^y)\,\mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y).\end{aligned}$$

Thus the formula holds for χ_E and then it also holds for any simple function $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where each A_i is a Borel set in $\mathbb{R}$, i.e.,

$$\int_{\mathbb{R}} s(x)\,\mathrm{d}(\mu * \lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} s(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y). \tag{8.22}$$

Next, we let $f \in C_0(\mathbb{R})$ so that $f : \mathbb{R} \to \mathbb{C}$ is continuous. By Definition 3.16, f is bounded by a positive constant M on $\mathbb{R}$ and Borel measurable. By [22, Theorem 2.10(b), p. 47], there exist simple Borel measurable functions $s_n : \mathbb{R} \to \mathbb{C}$ such that

$$0 \leq |s_1| \leq |s_2| \leq \cdots \leq |f| \tag{8.23}$$

and $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in \mathbb{R}$. Now we replace s by s_n in the formula (8.22) to get

$$\int_{\mathbb{R}} s_n(x)\,\mathrm{d}(\mu * \lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} s_n(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y). \tag{8.24}$$

By Theorem 6.12, there is a Borel measurable function h such that $|h(x)| = 1$ on $\mathbb{R}$ and $\mathrm{d}(\mu * \lambda) = h\,\mathrm{d}|\mu * \lambda|$. If we consider the sequence $\{s_n h\}$ of Borel measurable functions, then the inequalities (8.23) are still valid. Furthermore, we have $s_n(x)h(x) \to f(x)h(x)$ as $n \to \infty$ for every $x \in \mathbb{R}$. By the inequality (8.21), we see that

$$\|fh\|_1 = \int_{\mathbb{R}} |f(x)h(x)|\,\mathrm{d}|\mu * \lambda| \leq M|(\mu * \lambda)|(\mathbb{R}) \leq M|\mu|(\mathbb{R})| \cdot \lambda|(\mathbb{R}) < \infty.$$

Therefore, it means $fh \in L^1(\mu * \lambda)$ and we apply Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) to the left-hand side of the formula (8.24) to obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} s_n(x)\,\mathrm{d}(\mu * \lambda) = \lim_{n\to\infty} \int_{\mathbb{R}} s_n(x)h(x)\,\mathrm{d}|\mu * \lambda| = \int_{\mathbb{R}} f(x)h(x)\,\mathrm{d}|\mu * \lambda| = \int_{\mathbb{R}} f\,\mathrm{d}(\mu * \lambda).$$

Now we may apply similar analysis to show that the limit of the right-hand side of the formula (8.24) is exactly

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y).$$

Hence we conclude that

$$\int_{\mathbb{R}} f\,\mathrm{d}(\mu * \lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y). \tag{8.25}$$

Since $\mu * \lambda \in M$, §6.18 on p. 130 indicates that the mapping

$$f \mapsto \int_{\mathbb{R}} f\,\mathrm{d}(\mu * \lambda)$$

is a *bounded linear functional* on $C_0(\mathbb{R})$. Consequently, Theorem 6.19 (The Riesz Representation Theorem) ensures that $\mu * \nu$ is the *unique* complex Borel measure satisfying the formula (8.25).

(c) Since $\mathbb{R}$ is commutative, $x + y \in E$ if and only if $y + x \in E$. Thus it follows from the formula (8.25) and Theorem 8.8 (The Fubini Theorem) that

$$\int f\,\mathrm{d}(\mu * \lambda) = \int\int f(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y) = \int\int f(y+x)\,\mathrm{d}\lambda(y)\,\mathrm{d}\mu(x) = \int f\,\mathrm{d}(\lambda * \mu).$$

Therefore, the uniqueness property in part (b) implies that $\mu * \lambda = \lambda * \mu$.

On the one hand, it is clear from the formula (8.25) and Theorem 8.8 (The Fubini Theorem) that

$$\int_{\mathbb{R}} f \, \mathrm{d}[(\mu * \lambda) * \nu] = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y+z) \, \mathrm{d}\mu(x) \, \mathrm{d}\lambda(y) \, \mathrm{d}\nu(z). \tag{8.26}$$

On the other hand, we also have

$$\begin{aligned}\int_{\mathbb{R}} f \, \mathrm{d}[\mu * (\lambda * \nu)] &= \int_{\mathbb{R}} \underbrace{\Big\{ \int_{\mathbb{R}} f(x+Y) \, \mathrm{d}\mu(x) \Big\}}_{F} \, \mathrm{d}(\lambda * \nu)(Y) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y+z) \, \mathrm{d}\mu(z) \, \mathrm{d}\lambda(y) \, \mathrm{d}\nu(z). \end{aligned} \tag{8.27}$$

Thus the associativity of the convolution in M follows immediately from the formulas (8.26) and (8.27), i.e., $(\mu * \lambda) * \nu = \mu * (\lambda * \nu)$.

By the formula (8.25) again, we see that

$$\begin{aligned}\int_{\mathbb{R}} f \, \mathrm{d}\big(\mu * (\lambda + \nu)\big) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) \, \mathrm{d}\mu(x) \, \mathrm{d}[\lambda(y) + \nu(y)] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) \, \mathrm{d}\mu(x) \, \mathrm{d}\lambda(y) + \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &= \int_{\mathbb{R}} f \, \mathrm{d}(\mu * \lambda) + \int_{\mathbb{R}} f \, \mathrm{d}(\mu * \nu) \\ &= \int_{\mathbb{R}} f \, \mathrm{d}(\mu * \nu + \mu * \nu)\end{aligned}$$

which shows immediately that the convolution in M is distributive with respect to addition, i.e., $\mu * (\lambda + \nu) = \mu * \lambda + \mu * \nu$.

(d) By Definition 8.7, we have

$$(\mu \times \lambda)(Q) = \int_{\mathbb{R}} \mu(Q^y) \, \mathrm{d}\lambda(y) = \int \Big\{ \int_{Q^y} \mathrm{d}\mu(x) \Big\} \, \mathrm{d}\lambda(y), \tag{8.28}$$

where $Q^y = \{x \in \mathbb{R} \,|\, (x, y) \in Q\}$. Let $E \in \mathscr{B}_1$ and $y \in \mathbb{R}$. Then we have

$$E_2^y = \{x \in \mathbb{R} \,|\, (x, y) \in E_2\} = \{x \in \mathbb{R} \,|\, x + y \in E\} = \{x - y \in \mathbb{R} \,|\, x \in E\} = E - y.$$

By the formula (8.28), we have

$$(\mu \times \lambda)(E_2) = \int \Big\{ \int_{E_2^y} \mathrm{d}\mu(x) \Big\} \, \mathrm{d}\lambda(y) = \int \Big\{ \int_{E-y} \mathrm{d}\mu(x) \Big\} \, \mathrm{d}\lambda(y) = \int \mu(E - y) \, \mathrm{d}\lambda(y)$$

as required.

(e) Suppose that μ and λ are concentrated on the countable sets A and B respectively, i.e.,

$$\mu(E) = \lambda(F) = 0$$

whenever $E \cap A = \varnothing$ and $F \cap B = \varnothing$, where $A, B \in \mathscr{B}_1$. Let $Q \in \mathscr{B}_2$ and $Q \cap (A \times B) = \varnothing$. Then Theorem 8.2 implies that $Q_x \in \mathscr{B}_1$ for every $x \in \mathbb{R}$. Assume that $y_0 \in Q_x \cap B$. Then $(x, y_0) \in Q$ for every $x \in \mathbb{R}$. However, this means that $(x_0, y_0) \in Q \cap (A \times B)$ for some

$x_0 \in \mathbb{R}$, a contradiction. As a result, it must be true that $Q_x \cap B = \varnothing$. Similarly, we can show that $Q^y \cap A = \varnothing$. Therefore, we have

$$\lambda(Q_x) = 0 \quad \text{and} \quad \mu(Q^y) = 0$$

which definitely give

$$(\mu \times \lambda)(Q) = \int_{\mathbb{R}} \lambda(Q_x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}} \mu(Q^y)\,\mathrm{d}\lambda(y) = 0 \tag{8.29}$$

by Definition 8.7. In other words, $\mu \times \lambda$ is concentrated on $A \times B$.[b]

To continue the proof, we need the following lemma:

Lemma 8.3

$E_2 \cap (A \times B) = \varnothing$ if and only if $E \cap (A + B) = \varnothing$.

Proof of Lemma 8.3. It is easy to see that $x = a + b \in E \cap (A + B)$ if and only if $(a, b) \in E_2 \cap (A \times B)$, so our expected result follows. ■

By the definition, if $E \cap (A + B) = \varnothing$, then Lemma 8.3 implies that $E_2 \cap (A \times B) = \varnothing$ and the analysis in the preceding paragraph shows that

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2) = 0.$$

Since $A + B$ is countable, we have established the result that $\mu * \lambda$ is discrete.

Next, suppose that μ is continuous and $\lambda \in M$. By part (d), if $x \in \mathbb{R}$, then we have

$$(\mu * \lambda)(\{x\}) = \int_{\mathbb{R}} \mu(\{x\} - t)\,\mathrm{d}\lambda(t) = \int_{\mathbb{R}} \mu(\{x - t\})\,\mathrm{d}\lambda(t) = \int_{\mathbb{R}} 0\,\mathrm{d}\lambda(t) = 0.$$

Thus $\mu * \lambda$ is continuous.

Finally, let $\mu \ll m$ and $E \in \mathscr{B}_1$ satisfy $m(E) = 0$. By Theorem 2.20(c), we have $m(E - t) = m(E) = 0$ for every $t \in \mathbb{R}$ and then $\mu(E - t) = 0$ for every $t \in \mathbb{R}$. By this fact and part (d), we achieve

$$(\mu * \lambda)(E) = \int_{\mathbb{R}} \mu(E - t)\,\mathrm{d}\lambda(t) = \int_{\mathbb{R}} 0\,\mathrm{d}\lambda(t) = 0.$$

By Definition 6.7, we conclude that $\mu * \lambda \ll m$.

(f) By part (b) and the assumptions, we note that

$$\int_{\mathbb{R}} \mathrm{d}(\mu * \lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{d}\mu(x)\,\mathrm{d}\lambda(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(y)\,\mathrm{d}m\,\mathrm{d}m. \tag{8.30}$$

By Theorem 8.14, since

$$f * g = \int_{\mathbb{R}} f(x - y)g(y)\,\mathrm{d}m,$$

we may further reduce the expression (8.30) to

$$\int_{\mathbb{R}} \mathrm{d}(\mu * \lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x - y)g(y)\,\mathrm{d}m\,\mathrm{d}m. = \int_{\mathbb{R}} (f * g)\,\mathrm{d}m$$

and this is equivalent to $\mathrm{d}(\mu * \lambda) = (f * g)\,\mathrm{d}m$.

[b] It seems that it is enough to assume either μ or λ is discrete for the validity of the result (8.29). However, as Definition 8.7 indicates, $(\mu \times \lambda)(Q)$ can be computed by using two different integrals, so both μ and λ must be assumed to be discrete in order that the two integrals there are equal and thus $(\mu \times \lambda)(Q)$ makes sense.

(g) To being with, suppose that $X_1 = \{\mu \in M \,|\, \mu \text{ is discrete}\}$, $X_2 = \{\mu \in M \,|\, \mu \text{ is continuous}\}$ and $X_3 = \{\mu \in M \,|\, \mu \ll m\}$.

Recall from [40, Definition 3.3.1, p. 305] that a set X with binary operations $+$ and $*$ from $X \times X$ to X, and that $\cdot$ is a binary operation from $\mathbb{F} \times X$ to X is called an **algebra** if

- $(X, +, \cdot)$ is a vector space;
- for all $x, y, z \in X$ and every scalar $\alpha \in \mathbb{F}$, we have

  Condition (1). $x * (y * z) = (x * y) * z$;
  Condition (2). $x * (y + z) = (x * y) + (x * z)$ and $(x + y) * z = (x * z) + (y * z)$;
  Condition (3). $\alpha \cdot (x * y) = (\alpha \cdot x) * y = x * (\alpha \cdot y)$.

  A subset of X is called a **subalgebra** if it is an algebra.

Now let's prove the assertions one by one:

- **Case(1): $(X_1, +, *, \cdot)$ is an algebra.** If μ and λ are concentrated on countable (Borel) sets A and B respectively, then $\mu(E) = 0$ and $\lambda(F) = 0$ whenever $E \cap A = \varnothing$ and $F \cap B = \varnothing$. Let $C = A \cup B$. If $E \cap (A \cup B) = \varnothing$, then we must have $E \cap A = \varnothing$ and $E \cap B = \varnothing$ which imply immediately that

$$(\alpha\mu + \beta\lambda)(E) = \alpha\mu(E) + \beta\lambda(E) = 0,$$

  where α and β are real. Thus $(X_1, +, \cdot)$ is a vector space. Next, by part (e), we know that the map $X_1 \times X_1 \to X_1$ given by

$$(\mu, \lambda) \mapsto \mu * \lambda$$

  is well-defined. Thus it can be seen that **Conditions (1)** to **(3)** come directly from the results of part (c). By the definition, $(X_1, +, *, \cdot)$ is an algebra.
- **Case (2): X_2 is an ideal in M.** Recall that Y is called an **ideal** of an algebra X if Y is a vector subspace of X and $xY, Yx \subseteq Y$ for every $x \in X$. Here

$$xY = \{x * y \,|\, y \in Y\} \quad \text{and} \quad Yx = \{y * x \,|\, y \in Y\}.$$

  Let $\mu, \lambda \in X_2$ so that $\mu(\{x\}) = \lambda(\{x\}) = 0$ for every $x \in \mathbb{R}$, but these imply trivially that

$$(\alpha\mu + \beta\lambda)(\{x\}) = \alpha\mu(\{x\}) + \beta\lambda(\{x\}) = 0$$

  for every real α and β. In other words, $(X_2, +, \cdot)$ is a vector space. Since M is commutative by part (c), it suffices to prove $X_2\lambda \subseteq X_2$ for every $\lambda \in M$, but this fact follows immediately from part (e). Hence we may conclude that X_2 is an ideal in M.
- **Case(3): X_3 is an ideal in M.** Let $\mu, \lambda \in X_3$. Then $\mu(E) = \lambda(E) = 0$ whenever $m(E) = 0$. For every $\alpha, \beta \in \mathbb{R}$, if $m(E) = 0$, then

$$(\alpha\mu + \beta\lambda)(E) = \alpha\mu(E) + \beta\lambda(E) = 0.$$

  Therefore, it means that $(\alpha\mu + \beta\lambda) \ll m$ and so $(X_3, +, \cdot)$ is a vector space. Next, for any $\mu \in X_3$ and $\lambda \in M$, $\mu * \lambda \ll m$ by part (e), so we have $\mu\lambda \in X_3$ and X_3 is an ideal in M by the definition.

  Recall that m is a positive σ-finite measure, thus for every $\mu \in X_3$, we have$\mu \ll m$ so that Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem) ensures that there is a unique $f \in L^1(\mathbb{R})$ such that $\mathrm{d}\mu = f\,\mathrm{d}m$. Therefore, it is reasonable to define $\Phi : (X_3, *) \to \big(L^1(\mathbb{R}), *\big)$ by

$$\Phi(\mu) = f.$$

We claim that Φ is in fact an isomorphism. Let λ be another element in X_3. Then there is a unique $g \in L^1(\mathbb{R})$ such that $\mathrm{d}\lambda = g\,\mathrm{d}m$. By part (f), we have

$$\mathrm{d}(\mu * \lambda) = (f * g)\,\mathrm{d}m. \tag{8.31}$$

By Theorem 6.10 (The Lebesgue-Radon-Nikodym Theorem) again, the $h \in L^1(\mathbb{R})$ satisfying the formula (8.31) must be unique. Hence we have

$$\Phi(\mu * \nu) = f * g,$$

i.e., Φ is a homomorphism. Next, if $\Phi(\mu) = \Phi(\lambda)$, then $\mathrm{d}\mu = \mathrm{d}\lambda$ which implies exactly $\mu = \lambda$. Thus Φ is injective. Next, suppose that $f \in L^1(\mathbb{R})$. By Definition 1.31, if $f = u + iv$, where u and v are real measurable functions, then we have

$$\mu(E) = \int_E f\,\mathrm{d}m = \int_E u^+\,\mathrm{d}\mu - \int_E u^-\,\mathrm{d}m + i\int_E v^+\,\mathrm{d}m - i\int_E v^-\,\mathrm{d}m. \tag{8.32}$$

where $E \in \mathscr{B}_1$. Since $u^\pm$ and $v^\pm$ are measurable and nonnegative, Theorem 1.29 guarantees that every integral in the expression (8.32) is a measure on $\mathscr{B}_1$. Thus, if $m(E) = 0$, then Proposition 1.24(e) tells us that $\mu(E) = 0$. By Definition 6.7, $\mu \ll m$ or equivalently $\mu \in X_3$ and Φ is surjective.

In conclusion, Φ is an isomorphism, as required.

(h) Consider the Dirac measure $\delta_0 : \mathscr{B}_1 \to [0, \infty]$, that is

$$\delta_0(E) = \begin{cases} 1, & \text{if } 0 \in E; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\delta_0 \in M$ and part (d) shows that

$$(\delta_0 * \mu)(E) = \int_{\mathbb{R}} \mu(E - t)\,\mathrm{d}\delta_0(t) = \mu(E)$$

for all $\mu \in M$. Hence δ_0 is a unit of M.

(i) Suppose that M is defined to be the Banach space of all complex Borel measures on $\mathbb{R}^k$ with norm $\|\mu\| = |\mu|(\mathbb{R}^k)$ and associate to each Borel set $E \subseteq \mathbb{R}^k$ the set

$$E_2 = \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x} + \mathbf{y} \in E\} \subseteq \mathbb{R}^k \times \mathbb{R}^k.$$

If $\mu, \lambda \in M$, we define

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$$

for every Borel set $E \subseteq \mathbb{R}^k$. Then all the assertions from parts (a) to (h) also hold when we replace $\mathbb{R}$ and $\mathscr{B}_1$ by $\mathbb{R}^k$ and $\mathscr{B}_k$ respectively because $\mathbb{R}^k$ is always a commutative group and m_k is a translation invariant Borel measure on $\mathbb{R}^k$. The proofs go exactly the same, so we omit the details here.

For the k-dimensional torus T^k, we need to seek a measure similar to the Lebesgue measure m_k on $\mathbb{R}^k$. Note that T^k is a locally compact group with the σ-algebra $\mathscr{B}(T^k)$, so the **Haar measure** denoted by σ_k is what we want. (For the existence, uniqueness and basic properties of such a measure, the reader is suggested to read, for examples, [16, Chap. 9], [22, §11.1, pp. 339 – 348] and [48, pp. 128 – 132, 211].) Once we have such a measure, we can follow the same line as above to obtain the corresponding results. Again, we omit the details here.

This completes the analysis of the problem. ■

Problem 8.6

Rudin Chapter 8 Exercise 6.

Proof. Let $\mathbf{x} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ and $r = |\mathbf{x}|$. Then we have $r > 0$ so that it is meaningful to define $\mathbf{u} = \frac{\mathbf{x}}{r}$ which certainly gives the representation

$$\mathbf{x} = r\mathbf{u} \quad \text{and} \quad |\mathbf{u}| = 1. \tag{8.33}$$

Suppose that $\mathbf{x} = r'\mathbf{u}'$, where $r' > 0$ and $|\mathbf{u}'| = 1$. Then the equality $r\mathbf{u} = r'\mathbf{u}'$ implies that

$$r = |r\mathbf{u}| = |r'\mathbf{u}'| = r'$$

and so $\mathbf{u} = \mathbf{u}'$. In other words, the representation (8.33) is unique. Therefore, we may "identify" $\mathbb{R}^k \setminus \{\mathbf{0}\}$ as $(0, \infty) \times S_{k-1}$, i.e., the map

$$\varphi : \mathbb{R}^k \setminus \{\mathbf{0}\} \to (0, \infty) \times S_{k-1}$$

is a homeomorphism with the continuous inverse

$$\varphi^{-1} : (0, \infty) \times S_{k-1} \to \mathbb{R}^k \setminus \{\mathbf{0}\}$$

and they are given by

$$\varphi(\mathbf{x}) = \Big(|\mathbf{x}|, \frac{\mathbf{x}}{|\mathbf{x}|}\Big) \quad \text{and} \quad \varphi^{-1}(r, \mathbf{u}) = r\mathbf{u} \tag{8.34}$$

respectively. To proceed further, we need the following result:[c]

Lemma 8.4

Let $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ be σ-algebras in X and Y respectively. Suppose that $\varphi : X \to Y$ is measurable and $\mu : \mathfrak{M}_X \to [0, \infty]$ is a measure. Define $\nu = \varphi_* \mu$ by

$$\nu(E) = \mu\big(\varphi^{-1}(E)\big)$$

for $E \in \mathfrak{M}_Y$. Then the set function ν is a measure on $\mathfrak{M}_Y$ and for every measurable function $g : Y \to [0, \infty]$, we have

$$\int_Y g \, \mathrm{d}\nu = \int_X (g \circ \varphi) \, \mathrm{d}\mu.$$

Let ρ be the measure on $(0, \infty)$ given by $\mathrm{d}\rho = r^{k-1} \, \mathrm{d}r$. Now it suffices to prove that

$$\nu(E) = m_k\big(\varphi^{-1}(E)\big) = (\rho \times \sigma_{k-1})(E) \tag{8.35}$$

for every $E \in \mathscr{B}\big((0, \infty) \times S_{k-1}\big)$, where σ_{k-1} is the measure defined on S_{k-1} in the question. This is because if the formula (8.35) holds, then for every nonnegative Borel function f on $\mathbb{R}^k$,

[c] The measure ν in Lemma 8.4 is called the **push-forward measure** of μ. See [9, §3.6].

we deduce immediately from Lemma 8.4 (with $X = \mathbb{R}^k \setminus \{\mathbf{0}\}$, $Y = (0, \infty) \times S_{k-1}$, φ is given by the formula (8.34), $\mu = m_k$ and $g = f \circ \varphi^{-1}$) and Theorem 8.8 (The Fubini Theorem) that

$$\begin{aligned}\int_{\mathbb{R}^k \setminus \{\mathbf{0}\}} f \, \mathrm{d}m_k &= \int_{(0,\infty)\times S_{k-1}} (f \circ \varphi^{-1}) \, \mathrm{d}(\rho \times \sigma_{k-1}) \\ &= \int_{(0,\infty)\times S_{k-1}} f\big(\varphi^{-1}(r, \mathbf{u})\big) \, \mathrm{d}\sigma_{k-1} \, \mathrm{d}\rho \\ &= \int_0^\infty \int_{S_{k-1}} r^{k-1} f(r\mathbf{u}) \, \mathrm{d}\sigma_{k-1} \, \mathrm{d}r. \end{aligned} \tag{8.36}$$

Since $\{\mathbf{0}\}$ is a Borel set in $\mathbb{R}^k$ and $m_k(\mathbf{0}) = 0$, the integral on the left-hand side of the equation (8.36) can be replaced by

$$\int_{\mathbb{R}^k} f \, \mathrm{d}m_k$$

and we have what we want. To prove the formula (8.35), we have several steps:

- **Step 1: The formula (8.35) holds for $(r_1, r_2) \times A$, where $A \in \mathscr{B}(S_{k-1})$.** On the one hand, it is clear from Definition 8.7 that

$$\begin{aligned}(\rho \times \sigma_{k-1})\big((r_1, r_2) \times A\big) &= \rho\big((r_1, r_2)\big)\sigma_{k-1}(A) \\ &= \sigma_{k-1}(A) \int_{r_1}^{r_2} r^{k-1} \, \mathrm{d}r \\ &= \frac{r_2^k - r_1^k}{k} \sigma_{k-1}(A). \end{aligned} \tag{8.37}$$

  On the other hand, we see from the definition (8.34) that

$$\varphi\big((r_1, r_2) \times A\big) = \{r\mathbf{u} \,|\, 0 < r_1 < r < r_2 \text{ and } \mathbf{u} \in A\} = (r_2\widetilde{A}) \setminus [(r_1\widetilde{A}) \cup \{r_1\mathbf{u} \,|\, \mathbf{u} \in A\}]$$

  so that

$$\begin{aligned}\nu\big((r_1, r_2) \times A\big) &= m_k\big(\varphi\big((r_1, r_2) \times A\big)\big) \\ &= m_k\big((r_2\widetilde{A}) \setminus [(r_1\widetilde{A}) \cup \{r_1\mathbf{u} \,|\, \mathbf{u} \in A\}]\big) \\ &= m_k(r_2\widetilde{A}) - m_k(r_1\widetilde{A}) - 0 \\ &= (r_2^k - r_1^k) m_k(\widetilde{A}) \\ &= \frac{r_2^k - r_1^k}{k} \sigma_{k-1}(A). \end{aligned} \tag{8.38}$$

  Hence our claim follows by combining the two expressions (8.37) and (8.38).

- **Step 2: The formula (8.35) holds for $B \times A$, where $A \in \mathscr{B}(S_{k-1})$ and $B \in \mathscr{B}\big((0, \infty)\big)$.** For any open V in $(0, \infty)$, we know from [49, Exercise 29, Chapter 2] that it can be represented as an at most countable union of disjoint segments in the form (a, b). Since both ν (by Lemma 8.4) and $\rho \times \sigma_{k-1}$ are measures on $\mathscr{B}\big((0, \infty) \times S_{k-1}\big)$, **Step 1** ensures that the formula (8.35) also holds for $V \times A$ and hence for all $K \times A$, where K is closed in $(0, \infty)$.

  Next, let U be a bounded Borel set in $(0, \infty)$. The proof of Theorem 2.17(c) says that you can find an increasing sequence of closed sets K_n and a decreasing sequence of bounded open sets V_n such that

$$K_1 \subseteq K_2 \subseteq \cdots \subset U \subset \cdots \subset V_2 \subset V_1 \tag{8.39}$$

and $m(V_n \setminus K_n) \to 0$ as $n \to \infty$. Clearly, it is true that $V_1 \setminus K_1 \supseteq V_2 \setminus K_2 \supseteq \cdots$ and $V_1 \setminus K_1$ is a bounded set. Let $r^{k-1} \leq M$ on $V_1 \setminus K_1$ for some positive constant M. Then we have

$$\rho(V_n \setminus K_n) = \int_{V_n \setminus K_n} r^{k-1}\,\mathrm{d}m \leq M \int_{V_n \setminus K_n} \mathrm{d}m = M \cdot m(V_n \setminus K_n) \to 0 \tag{8.40}$$

as $n \to \infty$. Therefore, we apply the set relations (8.39), then the fact that φ^{-1} sends Borel sets in $(0,\infty) \times S_{k-1}$ to Borel sets in $\mathbb{R}^k \setminus \{\mathbf{0}\}$ and finally Definition 8.7 to get

$$\begin{aligned}\nu\big((V_n \setminus U) \times A\big) &\leq \nu\big((V_n \setminus K_n) \times A\big)\\ &= (\rho \times \sigma_{k-1})\big((V_n \setminus K_n) \times A\big)\\ &= \rho(V_n \setminus K_n)\sigma_{k-1}(A).\end{aligned} \tag{8.41}$$

By using the inequality (8.40), the inequality (8.41) shows that $\nu(V_n \setminus U) \times A) \to 0$ as $n \to \infty$. In other words, it means that

$$\nu(V_n \times A) \to \nu(U \times A) \tag{8.42}$$

as $n \to \infty$. Furthermore, if we suppose

$$K = \bigcup_{n=1}^{\infty} K_n \quad \text{and} \quad V = \bigcap_{n=1}^{\infty} V_n$$

which are an F_σ set and a G_δ set respectively, then Theorem 1.19(d) and (e) imply that

$$\rho(V) - \rho(K) = \lim_{n\to\infty} [\rho(V_n) - \rho(K_n)] = \lim_{n\to\infty} \rho(V_n \setminus K_n) = 0$$

so that $\rho(V) = \rho(K)$. Since $K, V \in \mathscr{B}\big((0,\infty)\big)$ and $K \subset U \subset V$, Theorem 1.36 asserts that ρ is complete and then

$$\rho(U) = \rho(K) = \rho(U) = \lim_{n\to\infty} \rho(V_n). \tag{8.43}$$

Thus we establish from Definition 8.7 and the limits (8.42) and (8.43) that

$$\begin{aligned}(\rho \times \sigma_{k-1})(U \times A) &= \rho(U)\sigma_{k-1}(A)\\ &= \lim_{n\to\infty} [\rho(V_n)\sigma_{k-1}(A)]\\ &= \lim_{n\to\infty} (\rho \times \sigma_{k-1})(V_n \times A)\\ &= \lim_{n\to\infty} \nu(V_n \times A)\\ &= \nu(U \times A).\end{aligned}$$

Finally, if U is any bounded or unbounded Borel set in $(0,\infty)$, then we consider the sets

$$U_n = U \cap (n-1, n]$$

for $n \in \mathbb{N}$. It is trivial that each U_n is a bounded Borel set, so the preceding paragraph gives

$$(\rho \times \sigma_{k-1})(U_n \times A) = \nu(U_n \times A) \tag{8.44}$$

for each $n = 1, 2, \ldots$. Since $U_i \cap U_j = \varnothing$ if $i \neq j$, $(U_i \times A) \cap (U_j \times A) = \varnothing$ for all $i \neq j$ and then the result (8.44) shows that

$$(\rho \times \sigma_{k-1})(U \times A) = (\rho \times \sigma_{k-1})\Big(\Big(\bigcup_{n=1}^{\infty} U_n\Big) \times A\Big)$$

$$\begin{aligned}
&= (\rho\times\sigma_{k-1})\Big(\bigcup_{n=1}^{\infty}(U_n\times A)\Big)\\
&= \sum_{n=1}^{\infty}(\rho\times\sigma_{k-1})(U_n\times A)\\
&= \sum_{n=1}^{\infty}\nu(U_n\times A)\\
&= \nu\Big(\bigcup_{n=1}^{\infty}U_n\times A\Big)\\
&= \nu(U\times A),
\end{aligned}$$

i.e., the formula (8.35) holds for every $U\times A$, where $U\in\mathscr{B}\big((0,\infty)\big)$ and $A\in\mathscr{B}(S_{k-1})$.

- **Step 3: The formula (8.35) holds for $E\in\mathscr{B}\big((0,\infty)\times S_{k-1}\big)$.** Suppose that

$$\mathfrak{M}=\{E\in\mathscr{B}\big((0,\infty)\times S_{k-1}\big)\,|\,\text{the formula (8.35) holds for } E\}\subseteq\mathscr{B}\big((0,\infty)\times S_{k-1}\big). \tag{8.45}$$

By using similar argument as in the proof of Problem 8.11, we can prove that

$$\mathscr{B}\big((0,\infty)\big)\times\mathscr{B}(S_{k-1})=\mathscr{B}\big((0,\infty)\times S_{k-1}\big).$$

By **Step 2**, we see that $\mathscr{B}\big((0,\infty)\big)\times\mathscr{B}(S_{k-1})\subseteq\mathfrak{M}$. Thus it concludes from the set inclusion (8.45) that

$$\mathfrak{M}=\mathscr{B}\big((0,\infty)\times S_{k-1}\big).$$

When $k=2$, for a nonnegative Borel function f on $\mathbb{R}^2$, the formula becomes

$$\int_{\mathbb{R}^2}f(\mathbf{x})\,\mathrm{d}m_2=\int_0^{\infty}r\Big\{\int_0^{2\pi}f(r\cos\theta,r\sin\theta)\,\mathrm{d}\theta\Big\}\,\mathrm{d}r.$$

We have to show that

$$\int_{S_1}f(r\mathbf{u})\,\mathrm{d}\sigma_1(\mathbf{u})=\int_0^{2\pi}f(r\cos\theta,r\sin\theta)\,\mathrm{d}\theta \tag{8.46}$$

for every $r\in(0,\infty)$. To this end, let $\mathbf{u}\in S_1$. Then $\mathbf{u}$ is a point on the unit circle so that $\mathbf{u}=(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}})$. We define $F:B(\mathbf{0},1)\setminus\{\mathbf{0}\}\to[0,\infty]$ by

$$F(\mathbf{u})=f(r\mathbf{u})=f\Big(\frac{rx}{\sqrt{x^2+y^2}},\frac{ry}{\sqrt{x^2+y^2}}\Big)=f(r\cos\theta,r\sin\theta),$$

where $\theta\in[0,2\pi)$. Recall the definition of σ_1 so that

$$\begin{aligned}
\int_{S_1}f(r\mathbf{u})\,\mathrm{d}\sigma_1&=2\int_{B(\mathbf{0},1)\setminus\{\mathbf{0}\}}F(\mathbf{u})\,\mathrm{d}m_2\\
&=2\int_0^1 a\int_0^{2\pi}f(r\cos\theta,r\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}a\\
&=\int_0^{2\pi}f(r\cos\theta,r\sin\theta)\,\mathrm{d}\theta
\end{aligned}$$

which is exactly the formula (8.46). When $k=3$, the formula becomes

$$\int_{\mathbb{R}^3}f(\mathbf{x})\,\mathrm{d}m_3=\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi}r^2f(r\cos\varphi\sin\theta,r\sin\varphi\sin\theta,r\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r,$$

where $\theta\in[0,\pi)$ and $\varphi\in[0,2\pi)$. This completes the proof of the problem. ■

Remark 8.2

We note that it can be further shown that the Borel measure σ_{k-1} on S_{k-1} is unique, see [22, Theorem 2.49, pp. 78, 79] for a proof of this. For similar or other proofs of Problem 8.6, you are suggested to read [36, pp. 172, 173, 322], [54, Theorem 15.13, pp. 154, 155] or [58, §3.2, pp. 279 – 281].

8.3 The Product Measure Theorem and Sections of a Function

Problem 8.7

Rudin Chapter 8 Exercise 7.

Proof. Since X and Y are σ-finite, X is the union of countably many disjoint sets X_n with $\mu(X_n) < \infty$, and that Y is the union of countably many disjoint sets Y_m with $\lambda(Y_m) < \infty$. For each $X_n \times Y_m$, we have

$$\psi(X_m \times Y_m) = \mu(X_n)\lambda(Y_m) < \infty. \tag{8.47}$$

Suppose that Ω is the class of all $E \in \mathscr{S} \times \mathscr{T}$ for which the conclusion of the problem holds, i.e.,

$$\Omega = \{E \in \mathscr{S} \times \mathscr{T} \,|\, \psi(E) = (\mu \times \lambda)(E)\}.$$

If $E = A \times B$ for some $A \in \mathscr{S}$ and $B \in \mathscr{T}$, then $E_x = B$ if $x \in A$ and $E_x = \varnothing$ if $x \in B$. Thus we deduce from Definition 8.7 that

$$(\mu \times \lambda)(E) = \int_X \lambda(E_x)\,\mathrm{d}\mu(x) = \int_A \lambda(B)\,\mathrm{d}\mu(x) = \mu(A)\lambda(B) = \psi(A \times B) = \psi(E). \tag{8.48}$$

In other words, $A \times B \in \Omega$. Besides, if $A_i \in \Omega$, $A_i \cap A_j = \varnothing$ for $i \neq j$ and $A = \bigcup_{i=1}^{\infty} A_i$, then since ψ and $\mu \times \mu$ are measures,

$$\psi(A) = \psi\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \psi(A_i) = \sum_{i=1}^{\infty}(\mu \times \lambda)(A_i) = (\mu \times \lambda)\Big(\bigcup_{i=1}^{\infty} A_i\Big) = (\mu \times \lambda)(A). \tag{8.49}$$

Suppose that

$$\mathfrak{M} = \{E \in \mathscr{S} \times \mathscr{T} \,|\, E \cap (X_n \times Y_m) \in \Omega \text{ for all } n, m \in \mathbb{N}\} \subseteq \mathscr{S} \times \mathscr{T}. \tag{8.50}$$

Our goal is to show that $\mathfrak{M}$ is a monotone class containing $\mathscr{E}$ so that Theorem 8.3 (The Monotone Class Theorem) can be applied to obtain the reverse direction of the set inclusion (8.50).

We claim that $\mathfrak{M}$ is a monotone class. To see this, let $A_i \in \mathfrak{M}$, $A_i \subseteq A_{i+1}$ and $A = \bigcup_{i=1}^{\infty} A_i$. By the definition (8.50), we know that

$$\psi\big(A_i \cap (X_n \times Y_m)\big) = (\mu \times \lambda)\big(A_i \cap (X_n \times Y_m)\big) \tag{8.51}$$

for all $n, m \in \mathbb{N}$. Since $A_i \cap (X_n \times Y_m) \in \Omega$ and $\mathscr{S} \times \mathscr{T}$ is a σ-algebra in $X \times Y$, we have

$$A \cap (X_n \times Y_m) = \bigcup_{i=1}^{\infty} [A_i \cap (X_n \times Y_m)] \in \mathscr{S} \times \mathscr{T}.$$

Furthermore, since $A_i \cap (X_n \times Y_m) \subseteq A_{i+1} \cap (X_n \times Y_m)$ for all $n, m \in \mathbb{N}$, Theorem 1.19(d) and the equality (8.51) imply that

$$\begin{aligned}\psi\big(A \cap (X_n \times Y_m)\big) &= \psi\Big(\bigcup_{i=1}^{\infty}[A_i \cap (X_n \times Y_m)]\Big) \\ &= \lim_{i\to\infty} \psi\big(A_i \cap (X_n \times Y_m)]\big) \\ &= \lim_{i\to\infty} (\mu \times \lambda)\big(A_i \cap (X_n \times Y_m)\big) \\ &= (\mu \times \lambda)\Big(\bigcup_{i=1}^{\infty}[A_i \cap (X_n \times Y_m)]\Big) \\ &= (\mu \times \lambda)\big(A \cap (X_n \times Y_m)\big).\end{aligned}$$

In other words, $A \cap (X_n \times Y_m) \in \Omega$ so that $A \in \mathfrak{M}$. Next, if $B_i \in \mathfrak{M}$, $B_i \supseteq B_{i+1}$ and $B = \bigcap_{i=1}^{\infty} B_i$. By the definition (8.50) again, we have

$$\psi\big(B_i \cap (X_n \times Y_m)\big) = (\mu \times \lambda)\big(B_i \cap (X_n \times Y_m)\big) \tag{8.52}$$

for all $n, m \in \mathbb{N}$. Since $B_i \cap (X_n \times Y_m) \in \Omega$ and $\mathscr{S} \times \mathscr{T}$ is a σ-algebra in $X \times Y$, we have

$$B \cap (X_n \times Y_m) = \bigcap_{i=1}^{\infty}[B_i \cap (X_n \times Y_m)] \in \mathscr{S} \times \mathscr{T}$$

by Comment 1.6(c). In addition, we follow from the inequality (8.47) that

$$\psi\big(B_1 \cap (X_n \times Y_m)\big) \leq \psi(X_n \times Y_m) < \infty$$

and

$$(\mu \times \lambda)\big(B_1 \cap (X_n \times Y_m)\big) \leq (\mu \times \lambda)(X_n \times Y_m) < \infty.$$

Since $B_i \cap (X_n \times Y_m) \supseteq B_{i+1} \cap (X_n \times Y_m)$ for all $n, m \in \mathbb{N}$, Theorem 1.19(e) and the equality (8.52) imply that

$$\begin{aligned}\psi\big(B \cap (X_n \times Y_m)\big) &= \psi\Big(\bigcap_{i=1}^{\infty}[B_i \cap (X_n \times Y_m)]\Big) \\ &= \lim_{i\to\infty} \psi\big(B_i \cap (X_n \times Y_m)]\big) \\ &= \lim_{i\to\infty} (\mu \times \lambda)\big(B_i \cap (X_n \times Y_m)\big) \\ &= (\mu \times \lambda)\Big(\bigcap_{i=1}^{\infty}[B_i \cap (X_n \times Y_m)]\Big) \\ &= (\mu \times \lambda)\big(B \cap (X_n \times Y_m)\big).\end{aligned}$$

In other words, $B \cap (X_n \times Y_m) \in \Omega$ so that $B \in \mathfrak{M}$. By Definition 8.1, $\mathfrak{M}$ is a monotone class as claimed.

We claim that $\mathscr{E} \subseteq \mathfrak{M}$. Denote $Q = R_1 \cup R_2 \cup \cdots \cup R_n$, where each R_i is a measurable rectangle and $R_i \cap R_j = \varnothing$ for $i \neq j$. Then each R_i is in the form $A_i \times B_i$ for some $A_i \in \mathscr{S}$ and $B_i \in \mathscr{T}$, so it is clear from the equality (8.48) that $R_i \in \Omega$. Since

$$R_i \cap (X_n \times Y_m) = (A_i \cap X_n) \times (B_i \cap Y_m),$$

we have $R_i \cap (X_n \times Y_m) \in \Omega$ and then $R_i \in \mathfrak{M}$. Since $R_i \neq R_j$ for $i \neq j$, we observe from the fact (8.49) that $Q \in \mathfrak{M}$. Hence we have our claim that $\mathscr{E} \subseteq \mathfrak{M}$.

Now we have proven our goal that $\mathfrak{M}$ is a monotone class containing $\mathscr{E}$. These fact imply that

$$\mathscr{S} \times \mathscr{T} \subseteq \mathfrak{M}. \tag{8.53}$$

Consequently, our proof is completed if we compare the set inclusions (8.50) and (8.53). ■

Remark 8.3

Problem 8.7 is sometimes called the **Product Measure Theorem**.

Problem 8.8

Rudin Chapter 8 Exercise 8.

Proof. We follow the hint given by Rudin.

(a) Given $x \in \mathbb{R}$. We take

$$\alpha_n(x) = \frac{[nx]}{n},$$

where $[x]$ denotes the greatest integer function. Since $x \in [\frac{i}{n}, \frac{i+1}{n})$ if and only if $nx \in [i, i+1)$ so that

$$\alpha_n(x_0) = \frac{i}{n}$$

if $x_0 \in [\frac{i}{n}, \frac{i+1}{n})$. Thus this implies that

$$0 \leq x - \alpha_n(x) < \frac{1}{n}$$

for every $n \in \mathbb{N}$ and $\alpha_n(x) \to x$ as $n \to \infty$. Therefore, it is illegal to say that there are $n \in \mathbb{N}$ and $i(n) \in \mathbb{Z}$ such that

$$\frac{i(n)-1}{n} = a_{i(n)-1} \leq x \leq a_{i(n)} = \frac{i(n)}{n}$$

and $a_{i(n)} \to x$ as $n \to \infty$. Now we simply write $a_i = a_{i(n)}$ and define $f_n : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} f_n(x,y) &= \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y) \\ &= \frac{a_i - x}{a_i - a_{i-1}} f_{a_{i-1}}(y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f_{a_i}(y). \end{aligned} \tag{8.54}$$

By the hypotheses, $f_{a_{i-1}}(y)$ and $f_{a_i}(y)$ are Borel measurable. Obviously, both

$$\frac{a_i - x}{a_i - a_{i-1}} \quad \text{and} \quad \frac{x - a_{i-1}}{a_i - a_{i-1}}$$

are continuous in x so that they are Borel measurable. By Proposition 1.19(c), each $f_n(x,y)$ is Borel measurable. Furthermore, it satisfies

$$|f_n(x,y) - f(x,y)| = \Big| \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y) - \frac{a_i - x}{a_i - a_{i-1}} f(x,y)$$

$$\begin{aligned}&-\frac{x-a_{i-1}}{a_i-a_{i-1}}f(x,y)\Big|\\&\leq |f(a_{i-1},y)-f(x,y)|+|f(a_i,y)-f(x,y)|\\&=|f^y(a_{i-1})-f^y(x)|+|f^y(a_i)-f^y(x)|,\end{aligned}\tag{8.55}$$

where $y\in\mathbb{R}$. Since a_i and a_{i-1} tend to x as $n\to\infty$ and f^y is continuous for all $y\in\mathbb{R}$, we deduce from the inequality (8.55) that

$$f(x,y)=\lim_{n\to\infty}f_n(x,y).$$

By Theorem 1.14, f is also Borel measurable in $\mathbb{R}^2$.

(b) We prove it by induction on k. The case for $k=1$ is obviously true. Assume that the statement is also true for $k=m$, i.e., if $g(x_1,x_2,\ldots,x_m)$ is continuous in each of the m variables separately, then g is a Borel function in $\mathbb{R}^m$. For $k=m+1$, $g(x_1,x_2,\ldots,x_{m+1})$ is continuous in each of the $m+1$ variables separately. Now for each $x_{m+1}\in\mathbb{R}$, the function

$$g_{x_{m+1}}(x_1,x_2,\ldots,x_m)$$

is continuous separately in $x_1,\ldots,x_m$. Thus we get from our assumption that every $g_{x_{m+1}}$ is a Borel function in $\mathbb{R}^m$. By our hypothesis of the question, *for every* choice $y=(x_2,x_3,\ldots,x_{m+1})$, the function

$$g^y(x_1)=g(x_1,x_2,\ldots,x_{m+1})$$

is continuous in x_1.

Now if we define $\frac{i-1}{n}=a_{i-1}\leq x_1\leq a_i=\frac{i}{n}$ and $g_n:\mathbb{R}^{m+1}\to\mathbb{R}$ by

$$g_n(x,y)=\frac{a_i-x_1}{a_i-a_{i-1}}g(a_{i-1},y)+\frac{x_1-a_{i-1}}{a_i-a_{i-1}}g(a_i,y),$$

then we use a similar argument as part (a) to conclude that each g_n is Borel measurable in $\mathbb{R}^{m+1}$ and the limit

$$g(x,y)=\lim_{n\to\infty}g_n(x,y)$$

implies that g is a Borel function in $\mathbb{R}^{m+1}$. Hence our desired result follows from induction.

This completes the proof of the problem. ■

Problem 8.9

Rudin Chapter 8 Exercise 9.

Proof. For each $x\in\mathbb{R}$, let $\{p_n\}\subseteq E$ such that $p_n\to x$. By the hypotheses, let N be the Lebesgue measurable set such that $m(N)=0$ and f^y is continuous on $\mathbb{R}$ for all $y\in\mathbb{R}\setminus N$. Now for every $y\in\mathbb{R}\setminus N$, these facts imply that

$$\lim_{n\to\infty}f_{p_n}(y)=\lim_{n\to\infty}f(p_n,y)=\lim_{n\to\infty}f^y(p_n)=f^y(x)=f(x,y)=f_x(y).$$

Since each f_{p_n} is Lebesgue measurable, Theorem 1.14 ensures that f_x is also Lebesgue measurable. In other words, what we have shown is that for each $x\in\mathbb{R}$, the function f_x is Lebesgue measurable on $\mathbb{R}\setminus N$.

Using the same idea of the proof of Problem 8.8, we define $f_n : \mathbb{R} \times (\mathbb{R} \setminus N) \to \mathbb{R}$ by

$$f_n(x,y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y),$$

where

$$\frac{i(n) - 1}{n} = a_{i(n)-1} \leq x \leq a_{i(n)} = \frac{i(n)}{n}.$$

Imitate Problem 8.8's proof, we conclude that each $f_n(x,y)$ is Lebesgue measurable and the inequality (8.55) holds for every $y \in \mathbb{R} \setminus N$ which further imply that

$$f(x,y) = \lim_{n\to\infty} f_n(x,y)$$

for every $y \in \mathbb{R} \setminus N$. By Theorem 1.14, the function $f : \mathbb{R} \times \mathbb{R} \setminus N \to \mathbb{R}$ is clearly Lebesgue measurable. Eventually, the completeness of m_2 guarantees that $f : \mathbb{R}^2 \to \mathbb{R}$ is Lebesgue measurable which ends the analysis of the proof. ■

Problem 8.10

Rudin Chapter 8 Exercise 10.

Proof. Define each $f_n : \mathbb{R}^2 \to \mathbb{R}$ by the representation (8.54), put[d]

$$h_n(y) = f_n\big(g(y), y\big) = \frac{a_i - g(y)}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{g(y) - a_{i-1}}{a_i - a_{i-1}} f(a_i, y)$$

if $g(y) \in [a_{i-1}, a_i]$. Similar to the proof of Problem 8.8,

$$f_{a_{i-1}}(y) = f(a_{i-1}, y) \quad \text{and} \quad f_{a_i}(y) = f(a_i, y)$$

are Lebesgue measurable on $\mathbb{R}$ by the hypotheses. Furthermore, since g is Lebesgue measurable on $\mathbb{R}$, both

$$\frac{a_i - g(y)}{a_i - a_{i-1}} \quad \text{and} \quad \frac{g(y) - a_{i-1}}{a_i - a_{i-1}}$$

are also Lebesgue measurable on $\mathbb{R}$. By Proposition 1.19(c), each $h_n(y)$ is Lebesgue measurable on $\mathbb{R}$.

Now our h_n and h satisfy

$$\begin{aligned}
|h_n(y) - h(y)| &= \Big|\frac{a_i - g(y)}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{g(y) - a_{i-1}}{a_i - a_{i-1}} f(a_i, y) - \frac{a_i - g(y)}{a_i - a_{i-1}} f\big(g(y), y\big) \\
&\quad - \frac{g(y) - a_{i-1}}{a_i - a_{i-1}} f\big(g(y), y\big)\Big| \\
&\leq |f(a_{i-1}, y) - f\big(g(y), y\big)| + |f(a_i, y) - f\big(g(y), y\big)| \\
&= |f^y(a_{i-1}) - f^y\big(g(y)\big)| + |f^y(a_i) - f^y\big(g(y)\big)|.
\end{aligned}$$

By using similar argument as in the proof of Problem 8.8, we can show that

$$h(y) = \lim_{n\to\infty} h_n(y)$$

which implies that h is Lebesgue measurable on $\mathbb{R}$ by Theorem 1.14, completing the proof of the problem. ■

[d] Remember that i actually depends on n.

8.4 Miscellaneous Problems

Problem 8.11

Rudin Chapter 8 Exercise 11.

Proof. Recall from the definition on [42, p. 86] that the product topology on $\mathscr{T}_{m+n} = \mathbb{R}^{m+n}$ is $\mathbb{R}^m \times \mathbb{R}^n$ which is the topology having as basis the collection

$$\mathcal{B}_{m+n} = \{U \times V \,|\, U \in \mathscr{T}_m \text{ and } V \in \mathscr{T}_n\}.$$

Furthermore, the result [42, Theorem 15.1, p. 86] shows that the basis $\mathcal{B}_{m+n}$ can be replaced by the basis

$$\mathcal{B}'_{m+n} = \{U \times V \,|\, U \in \mathcal{B}_m \text{ and } V \in \mathcal{B}_n\}.$$

For every $k \geq 1$, since $\mathbb{R}^k$ is second-countable, it has a countable basis.[e] Thus we may assume that $\mathcal{B}_m$ and $\mathcal{B}_n$ are countable and consequently $\mathcal{B}'_{m+n}$ is also countable. Obviously, we have $U \in \mathcal{B}_m \subseteq \mathscr{T}_m \subseteq \mathscr{B}_m$ and $V \in \mathcal{B}_n \subseteq \mathscr{T}_n \subseteq \mathscr{B}_n$ so that

$$U \times V \in \mathcal{B}_m \times \mathcal{B}_n \subseteq \mathscr{T}_m \times \mathscr{T}_n \subseteq \mathscr{B}_m \times \mathscr{B}_n. \tag{8.56}$$

If $W \in \mathscr{T}^{m+n}$, then it can be expressed as a union of basis elements of $\mathcal{B}'_{m+n}$. Since $\mathcal{B}'_{m+n}$ is countable, this union must be countable too. Therefore, it follows from this fact, the fact $\mathscr{B}_m \times \mathscr{B}_n$ is an σ-algebra and the set relation (8.56) that $\mathscr{T}_{m+n} \subseteq \mathscr{B}_m \times \mathscr{B}_n$ and this means that

$$\mathscr{B}_{m+n} \subseteq \mathscr{B}_m \times \mathscr{B}_n. \tag{8.57}$$

For the other direction, the projections $\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\pi_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings because $\pi_1^{-1}(U) = U \times \mathbb{R}^n$ and $\pi_2^{-1}(V) = \mathbb{R}^m \times V$ which are open in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ if U and V are open in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. If $A \in \mathscr{B}_m$, then $A \times \mathbb{R}^n$ is a Borel set in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Similarly, if $B \in \mathscr{B}_n$, then $\mathbb{R}^m \times B$ ia a Borel set in $\mathbb{R}^{m+n}$. Therefore, we get $A \times \mathbb{R}^n, \mathbb{R}^m \times B \in \mathscr{B}_{m+n}$ and these imply that

$$A \times B = (A \times \mathbb{R}^n) \cap (\mathbb{R}^m \times B) \in \mathscr{B}_{m+n}. \tag{8.58}$$

We recall from Definition 8.1 that $\mathscr{B}_m \times \mathscr{B}_n$ is the *smallest* σ-algebra in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ containing every measurable rectangle $A \times B$, where $A \in \mathscr{B}_m$ and $B \in \mathscr{B}_n$. Then we deduce from the set relation (8.58) that

$$\mathscr{B}_m \times \mathscr{B}_n \subseteq \mathscr{B}_{m+n}. \tag{8.59}$$

Hence our desired result follows if we combine the set relations (8.57) and (8.59), completing the proof of the problem. ∎

Problem 8.12

Rudin Chapter 8 Exercise 12.

Proof. Let $f(x, y) = (\sin x)\mathrm{e}^{-xt}$. Since f is continuous in $\mathbb{R}^2$, f is a m_2-measurable function on $\mathbb{R}^2$. It is trivial that $(0, A)$ and $(0, \infty)$ are σ-finite, where $A > 0$. Furthermore, we have

$$\int_0^A \mathrm{d}x \int_0^\infty |f(x,t)|\,\mathrm{d}t = \int_0^A |\sin x|\,\mathrm{d}x \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}t = \int_0^A \frac{|\sin x|}{x}\,\mathrm{d}x. \tag{8.60}$$

[e] Read [42, §30] for details.

By [49, Exercise 7, p. 197], we have $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ on $(0, \frac{\pi}{2})$. Therefore, the integral (8.60) becomes

$$\begin{aligned}\Big|\int_0^A \mathrm{d}x \int_0^\infty |f(x,t)|\,\mathrm{d}t\Big| &= \Big|\int_0^A \frac{|\sin x|}{x}\,\mathrm{d}x\Big| \\ &\leq \Big|\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,\mathrm{d}x\Big| + \Big|\int_{\frac{\pi}{2}}^A \frac{|\sin x|}{x}\,\mathrm{d}x\Big| \\ &< \frac{\pi}{2} + \int_{\frac{\pi}{2}}^A \frac{\mathrm{d}x}{x} \\ &= \frac{\pi}{2} + \ln A - \ln\frac{\pi}{2} \\ &< \infty\end{aligned}$$

for large $A > 0$.

By Theorem 8.8 (The Fubini Theorem)[f] and the given relation, we see that

$$\begin{aligned}\int_0^A \frac{\sin x}{x}\,\mathrm{d}x &= \int_0^A \sin x\,\mathrm{d}x\Big(\int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}t\Big) \\ &= \int_0^A \mathrm{d}x \int_0^\infty (\sin x)\mathrm{e}^{-xt}\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{d}t \int_0^A (\sin x)\mathrm{e}^{-xt}\,\mathrm{d}x \\ &= \int_0^\infty \Big[\frac{1}{1+t^2} - \frac{\mathrm{e}^{-At}}{1+t^2}(\cos A + t\sin A)\Big]\,\mathrm{d}t \\ &= \frac{\pi}{2} - \int_0^\infty \frac{\mathrm{e}^{-At}}{1+t^2}(\cos A + t\sin A)\,\mathrm{d}t.\end{aligned} \tag{8.61}$$

For $A \geq 1$, we notice that

$$\Big|\frac{\mathrm{e}^{-At}}{1+t^2}(\cos A + t\sin A)\Big| \leq \Big|\frac{1}{1+t^2}\Big|\mathrm{e}^{-At} + \Big|\frac{t}{1+t^2}\Big|\mathrm{e}^{-At} \leq 2\mathrm{e}^{-At} \leq 2\mathrm{e}^{-t}. \tag{8.62}$$

Then the given relation implies that $\mathrm{e}^{-t} \in L^1(m)$ on $(0,\infty)$. Obviously, for $t \geq 0$, we note from the second inequality in (8.62) that

$$\lim_{A\to\infty} \frac{\mathrm{e}^{-At}}{1+t^2}(\cos A + t\sin A) = 0,$$

so we deduce from this and Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) that

$$\lim_{A\to\infty} \int_0^\infty \frac{\mathrm{e}^{-At}}{1+t^2}(\cos A + t\sin A)\,\mathrm{d}t = 0. \tag{8.63}$$

Hence our desired result follows from the representation (8.61) and the limit (8.63), completing the proof of the problem. ■

Remark 8.4

The improper integral in Problem 8.12 is known as **Dirichlet integral**.

[f] See the *Notes* on [51, p. 165]

Problem 8.13

Rudin Chapter 8 Exercise 13.

Proof. We follow Rudin's hint. Since μ is a complex measure on a σ-algebra $\mathfrak{M}$ in X, Theorem 6.12 implies that *there is* a measurable function h such that $|h(x)| = 1$ on X and $\mathrm{d}\mu = h\,\mathrm{d}|\mu|$. This asserts that there exists a real measurable function θ such that $h = \mathrm{e}^{i\theta}$ and so

$$\mathrm{d}\mu = \mathrm{e}^{i\theta}\,\mathrm{d}|\mu|.$$

Given $E \in \mathfrak{M}$ and we define A_α to be the subset of E where $\cos(\theta - \alpha) > 0$. Then we have

$$\mu(A_\alpha) = \int_{A_\alpha} \mathrm{d}\mu = \int_{A_\alpha} \mathrm{e}^{i\theta}\,\mathrm{d}|\mu| = \int_E \chi_{A_\alpha} \mathrm{e}^{i\theta}\,\mathrm{d}|\mu|$$

and this gives

$$|\mu(A_\alpha)| = |\mathrm{e}^{-i\alpha}\mu(A_\alpha)| \geq \mathrm{Re}\left[\mathrm{e}^{-i\alpha}\mu(A_\alpha)\right] = \mathrm{Re}\left(\int_E \chi_{A_\alpha} \mathrm{e}^{i(\theta-\alpha)}\,\mathrm{d}|\mu|\right). \tag{8.64}$$

Since $\chi_{A_\alpha}\cos(\theta - \alpha) = \cos^+(\theta - \alpha)$ by the Corollary following Theorem 1.14, the inequality (8.64) becomes

$$|\mu(A_\alpha)| \geq \int_E \chi_{A_\alpha}\cos(\theta-\alpha)\,\mathrm{d}|\mu| = \int_E \cos^+(\theta-\alpha)\,\mathrm{d}|\mu|. \tag{8.65}$$

As in the proof of Lemma 6.3 from Rudin, we may choose α_0 so as to maximize the integral in the inequality (8.65). Since this maximum is at least as large as the average of the sum over $[-\pi, \pi]$, we obtain

$$|\mu(A_{\alpha_0})| \geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_E \cos^+(\theta-\alpha)\,\mathrm{d}|\mu|\,\mathrm{d}\theta.$$

Since $|\mu|(E) < \infty$ by Theorem 6.4, E is σ-finite. It is clear that $[-\pi, \pi]$ is also σ-finite. Thus we deduce from Theorem 8.8 (The Fubini Theorem) and the integral in Lemma 6.3 that

$$|\mu(A_{\alpha_0})| \geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_E \cos^+(\theta-\alpha)\,\mathrm{d}|\mu|\,\mathrm{d}\theta = \int_E \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^+(\theta-\alpha)\,\mathrm{d}\theta\right]\mathrm{d}|\mu| = \frac{1}{\pi}|\mu|(E).$$

By [16, Example 9.2.1(c), p. 285], if T is the circle group and $f : [0, 2\pi] \to T$ is given by $f(t) = (\cos t, \sin t)$, then the Haar measure μ on T is given by

$$\mu(S) = \frac{1}{2\pi}m\big(f^{-1}(S)\big). \tag{8.66}$$

In particular, we have $\mu(T) = 1$. Since f is continuous and $2 < 2\pi$, if we consider $f([0, 2]) = A$, then we see from the definition (8.66) that

$$\mu(A) = \frac{1}{2\pi}m\big(f^{-1}(A)\big) = \frac{1}{2\pi}m([0,2]) = \frac{1}{\pi} = \frac{1}{\pi}\mu(S).$$

Since μ is a positive measure, we have $|\mu| = \mu$ so that the Haar measure considered here gives an example showing that $\frac{1}{\pi}$ is the beast constant. We have completed the proof of the problem. ∎

Problem 8.14

Rudin Chapter 8 Exercise 14.

Proof. Since $f(t)t^\alpha$ and $t^{-\alpha}$ satisfy the hypotheses of Theorem 3.5 (Hölder's Inequality), we have

$$xF(x)=\int_0^x f(t)t^\alpha t^{-\alpha}\,\mathrm{d}t\leq\Big\{\int_0^x f^pt^{\alpha p}\,\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_0^x t^{\alpha q}\,\mathrm{d}t\Big\}^{\frac{1}{q}}=\Big\{\int_0^x f^pt^{\alpha p}\,\mathrm{d}t\Big\}^{\frac{1}{p}}\times\Big(\frac{x^{-\alpha q+1}}{-\alpha q+1}\Big)^{\frac{1}{q}}$$

which gives

$$F^p(x)\leq(1-\alpha q)^{1-p}x^{-1-\alpha p}\int_0^x f^pt^{\alpha p}\,\mathrm{d}t$$

and then

$$\int_0^\infty F^p(x)\,\mathrm{d}x\leq(1-\alpha q)^{1-p}\int_0^\infty x^{-1-\alpha p}\int_0^x f^pt^{\alpha p}\,\mathrm{d}t\,\mathrm{d}x \tag{8.67}$$

$$=(1-\alpha q)^{1-p}\int_0^\infty x^{-1-\alpha p}\int_0^\infty \chi_{(0,x)}(t)f^pt^{\alpha p}\,\mathrm{d}t\,\mathrm{d}x. \tag{8.68}$$

As the space $(0,\infty)$ is σ-finite, $\chi_{(0,x)}x^{-1-\alpha p}f^pt^{\alpha p}\geq 0$ and measurable on $(0,\infty)\times(0,\infty)$, Theorem 8.8 (The Fubini Theorem) allows the inequality (8.67) can be rewritten as

$$\begin{aligned}\int_0^\infty F^p(x)\,\mathrm{d}x&\leq(1-\alpha q)^{1-p}\int_0^\infty f^pt^{\alpha p}\int_0^\infty \chi_{(0,x)}(t)x^{-1-\alpha p}\,\mathrm{d}x\,\mathrm{d}t\\&=(1-\alpha q)^{1-p}\int_0^\infty f^pt^{\alpha p}\int_t^\infty x^{-1-\alpha p}\,\mathrm{d}x\,\mathrm{d}t\\&=(1-\alpha q)^{1-p}\int_0^\infty f^pt^{\alpha p}\times\frac{t^{-\alpha p}}{\alpha p}\,\mathrm{d}t\\&=(1-\alpha q)^{1-p}(\alpha p)^{-1}\int_0^\infty f^p\,\mathrm{d}t.\end{aligned}$$

If we define $g(\alpha)=(1-\alpha q)^{1-p}(\alpha p)^{-1}$ on $(0,\frac{1}{q})$, then we can apply basic calculus to verify that g will attain the absolute minimum at $\alpha=\frac{1}{pq}$ and its minimum value is

$$\Big(\frac{p}{p-1}\Big)^p.$$

Hence we have again proven the result of Problem 3.14 and this completes the proof of the problem. ■

Problem 8.15

Rudin Chapter 8 Exercise 15.

Proof. Notice that

$$\varphi(x)=\begin{cases}1-\cos x, & \text{if } x\in[0,2\pi];\\ 0, & \text{otherwise.}\end{cases} \tag{8.69}$$

For all $x\in\mathbb{R}$, we have

$$(f*g)(x)=\int_{-\infty}^\infty f(x-y)g(y)\,\mathrm{d}y=\int_{-\infty}^\infty \sin y\,\mathrm{d}y=\int_0^{2\pi}\sin y\,\mathrm{d}y=0.$$

This proves **Statement (i)**.

Clearly, if $x \in \mathbb{R}$, then it is easy to see that

$$(g * h)(x) = \int_{-\infty}^{\infty} g(x-y)h(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} g(y)h(x-y)\,\mathrm{d}y = \int_{0}^{2\pi} \varphi'(y)h(x-y)\,\mathrm{d}y. \tag{8.70}$$

Applying the Mean Value Theorem to $\varphi(t) = 1 - \cos t$, we get

$$|\varphi(x) - \varphi(y)| \leq |x-y|$$

for all $x, y \in [0, 2\pi]$. Next, for $x \in [0, 2\pi]$, we follow from the definition (8.69) that

$$h(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t = \int_{0}^{x} (1-\cos t)\,\mathrm{d}t = x - \sin x.$$

Using the Mean Value Theorem to h, we obtain

$$|h(x) - h(y)| \leq 2|x-y|$$

for all $x, y \in [0, 2\pi]$. Hence we have

$$\varphi, h \in \mathrm{Lip}\,1$$

on $[0, 2\pi]$. By Problem 7.10, φ and h are AC on $[0, 2\pi]$ and then we may apply Problem 7.14 to the integral (8.70) to establish

$$\begin{aligned}
(g * h)(x) &= \int_{0}^{2\pi} \varphi'(y)h(x-y)\,\mathrm{d}y \\
&= \varphi(2\pi)h(x-2\pi) - \varphi(0)h(x) + \int_{0}^{2\pi} \varphi(y)h'(x-y)\,\mathrm{d}y \\
&= \int_{0}^{2\pi} \varphi(y)\varphi(x-y)\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \varphi(y)\varphi(x-y)\,\mathrm{d}y \\
&= (\varphi * \varphi)(x)
\end{aligned}$$

for all $x \in \mathbb{R}$. Suppose that $x \in (0, 2\pi]$. Then we have

$$(\varphi * \varphi)(x) = \int_{0}^{x} \varphi(y)\varphi(x-y)\,\mathrm{d}y.$$

It is obvious that $\varphi(y) = 1 - \cos y > 0$ on $(0, x)$. Similarly, if $y \in (0, x)$, then $x - y \in (0, x)$ and so $\varphi(x-y) = 1 - \cos(x-y) > 0$ for all $y \in (0, x)$. Consequently, we deduce that

$$(\varphi * \varphi)(x) > 0 \tag{8.71}$$

on $(0, 2\pi]$. Suppose that $x \in (2\pi, 4\pi)$. Then the definition (8.69) implies that

$$(\varphi * \varphi)(x) = \int_{0}^{x} \varphi(y)\varphi(x-y)\,\mathrm{d}y = \int_{0}^{x-2\pi} \varphi(y)\varphi(x-y)\,\mathrm{d}y.$$

Obviously, $\varphi(y) = 1 - \cos y > 0$ on $(0, x - 2\pi)$. If $0 < y < x - 2\pi$, then $2\pi < x - y < x$ and so $\varphi(x-y) = 1 - \cos(x-y) > 0$ for all $y \in (0, x - 2\pi)$. Therefore, the inequality (8.71) is till valid whenever $x \in (2\pi, 4\pi)$. In conclusion, we establish that

$$(g * h)(x) = (\varphi * \varphi)(x) > 0$$

on $(0, 4\pi)$. This is **Statement (ii)**.

On the one hand, **Statement (i)** says that $f * g = 0$ which implies that $(f * g) * h = 0$. On the other hand, if $x \in \mathbb{R}$, since $f(x) = 1$ for all $x \in \mathbb{R}$, we have

$$[f * (g * h)](x) = \int_{-\infty}^{\infty} f(y)(g * h)(x - y)\,\mathrm{d}y = \int_{-\infty}^{\infty} (g * h)(x - y)\,\mathrm{d}y. \tag{8.72}$$

By Theorem 7.26 (The Change-of-variables Theorem) (or its special case on p. 156), we see that the last integral in the expression (8.72) becomes

$$\int_{-\infty}^{\infty} (g * h)(y)\,\mathrm{d}y. \tag{8.73}$$

Recall from the definition (8.69) that $\varphi(y) = 0$ outside $[0, 2\pi]$ and $\varphi(x-y) = 0$ outside $[x-2\pi, x]$. If $x \leq 0$ or $x \geq 4\pi$, then $[0, 2\pi] \cap [x - 2\pi, x]$ is of measure 0 so that $(\varphi * \varphi)(x) = 0$ outside $(0, 4\pi)$. Hence it follows from this fact and the expression (8.73) that

$$[f * (g * h)](x) = \int_{-\infty}^{\infty} (g * h)(y)\,\mathrm{d}y = \int_{0}^{4\pi} (g * h)(y)\,\mathrm{d}y = \int_{0}^{4\pi} (\varphi * \varphi)(y)\,\mathrm{d}y > 0$$

by **Statement (ii)**.

Now the inconsistence in **Statement (iii)** is due to the fact that $f \notin L^1(\mathbb{R})$. This completes the proof of the problem. ■

Problem 8.16

Rudin Chapter 8 Exercise 16.

Proof. Suppose that $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{T}, \nu)$ are σ-finite measure spaces. Let f be an $(\mathscr{L} \times \mathscr{T})$-measurable function on $X \times Y$. If $0 \leq f \leq \infty$ and $1 \leq p < \infty$, then we have the analogy of Minkowski's inequality in the question.

If $p = 1$, then it is exactly Theorem 8.8 (The Fubini Theorem). Suppose that $1 < p < \infty$ and q is the conjugate exponent of p. Since the inequality holds trivially when

$$\int \Big[\int f^p(x, y)\,\mathrm{d}\mu(x)\Big]^{\frac{1}{p}}\,\mathrm{d}\lambda(y) = \infty,$$

so without loss of generality, we assume further that this integral is finite. Let $g \in L^q(\mu)$. By Theorem 8.8 (The Fubini Theorem) and Theorem 3.5 (Hölder's Inequality), we have

$$\begin{aligned}\int \Big[\int f(x, y)\,\mathrm{d}\lambda(y)\Big] \cdot |g(x)|\,\mathrm{d}\mu(x) &= \int \int f(x, y) \cdot |g(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y) \\ &\leq \int \Big\{\int f^p(x, y)\,\mathrm{d}\mu(x)\Big\}^{\frac{1}{p}} \Big\{\int |g|^q\,\mathrm{d}\mu(x)\Big\}^{\frac{1}{q}}\,\mathrm{d}\lambda(y) \\ &= \|g\|_q \int \Big[\int f^p(x, y)\,\mathrm{d}\mu(x)\Big]^{\frac{1}{p}}\,\mathrm{d}\lambda(y).\end{aligned} \tag{8.74}$$

Thus if we define $\Phi : L^q(\mu) \to [0, \infty]$ by

$$\Phi(g) = \int \Big[\int f(x, y)\,\mathrm{d}\lambda(y)\Big] \cdot |g(x)|\,\mathrm{d}\mu(x), \tag{8.75}$$

then the inequality (8.74) and the assumption imply that Φ is a bounded linear functional on $L^q(\mu)$. Applying Theorem 6.16 to Φ, there exists a unique $h \in L^p(\mu)$ such that

$$\Phi(g) = \int hg\,\mathrm{d}\mu(x). \tag{8.76}$$

Since $\Phi(|g|) = \Phi(g)$, the uniqueness of h and the comparison of the representations (8.75) and (8.76) show that

$$h = \int f(x,y)\,\mathrm{d}\lambda(y).$$

Furthermore, Theorem 6.16 gives that $\|\Phi\| = \|h\|_p$ and we deduce from this and the inequality (8.74) that

$$\begin{aligned}
\left\{\int \left[\int f(x,y)\,\mathrm{d}\lambda(y)\right]^p \mathrm{d}\mu(x)\right\}^{\frac{1}{p}} &= \|h\|_p \\
&= \|\Phi\| \\
&= \sup\{|\Phi(g)| \mid g \in L^q(\mu) \text{ and } \|g\|_q = 1\} \\
&\leq \int \left[\int f^p(x,y)\,\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \mathrm{d}\lambda(y)
\end{aligned}$$

which completes the proof of the problem. ■

CHAPTER 9

Fourier Transforms

9.1 Properties of The Fourier Transforms

Problem 9.1

Rudin Chapter 9 Exercise 1.

Proof. For every $y \in \mathbb{R}$, $|f(x)e^{-ixy}| = |f(x)| = f(x)$ so that $f(x)e^{-ixy} \in L^1$. By Definition 9.1 and Theorem 1.33, we have

$$|\widehat{f}(y)| = \frac{1}{\sqrt{2\pi}}\Big|\int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx\Big| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,dx = \widehat{f}(0)$$

for every $y \in \mathbb{R}$. Assume that $|\widehat{f}(y)| = \widehat{f}(0)$ for some $y \neq 0$, i.e.,

$$\Big|\int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx\Big| = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} |f(x)e^{-ixy}|\,dx.$$

By Theorem 1.39(c), there is a constant α such that

$$\alpha f(x)e^{-ixy} = |f(x)e^{-ixy}| = |f(x)| = f(x) \tag{9.1}$$

a.e. on $\mathbb{R}$. Since $f(x) > 0$ on $\mathbb{R}$, the expression (9.1) implies that $\alpha e^{-ixy} = 1$ a.e. on $\mathbb{R}$. Thus it gives $y = 0$, a contradiction. This completes the proof of the problem. ■

Problem 9.2

Rudin Chapter 9 Exercise 2.

Proof. We prove the assertions one by one.

- **The Fourier transform of $\chi_{[a,b]}$.** By Definition 9.1, we have

$$\begin{aligned}\widehat{\chi_{[a,b]}}(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi_{[a,b]}(x)e^{-ixt}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{a}^{b} e^{-ixt}\,dx\end{aligned}$$

$$= \begin{cases} -\dfrac{(\mathrm{e}^{-ibt} - \mathrm{e}^{-iat})}{\sqrt{2\pi} it}, & \text{if } t \neq 0; \\ \\ \dfrac{b-a}{\sqrt{2\pi}}, & \text{otherwise.} \end{cases} \tag{9.2}$$

By similar argument, we see that $\widehat{\chi_{(a,b)}}$, $\widehat{\chi_{(a,b]}}$ and $\widehat{\chi_{[a,b)}}$ also give the same answer (9.2).

- **The expression of $\chi_{[-n,n]} * \chi_{[-1,1]}$.** We have $g_n = \chi_{[-n,n]}$ and $h = \chi_{[-1,1]}$. By Definition 9.1, we have

$$\begin{aligned} (g_n * h)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[-n,n]}(x-y)\chi_{[-1,1]}(y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \chi_{[-n,n]}(x-y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} m\big([x-n, x+n] \cap [-1,1]\big). \end{aligned} \tag{9.3}$$

If $x \leq -n-1$, then $x+n \leq -1$ so that

$$m\big([x-n, x+n] \cap [-1,1]\big) = 0.$$

If $-n-1 \leq x \leq -n+1$, then $-1 \leq x+n \leq 1$ and $x-n \leq -2n+1 \leq -1$ so that

$$m\big([x-n, x+n] \cap [-1,1]\big) = x+n-(-1) = x+n+1.$$

If $-n+1 \leq x \leq n-1$, then $x+n \geq 1$ and $x-n \leq -1$ so that $[-1,1] \subseteq [x-n, x+n]$ and

$$m\big([x-n, x+n] \cap [-1,1]\big) = 2.$$

If $n-1 \leq x \leq n+1$, then $-1 \leq x-n \leq 1$ and $x+n \geq 2n-1 \geq 1$ so that

$$m\big([x-n, x+n] \cap [-1,1]\big) = 1-(x-n) = 1-x+n.$$

If $x \geq n+1$, then $x-n \geq 1$ so that

$$m\big([x-n, x+n] \cap [-1,1]\big) = 0.$$

- **Proof of $\dfrac{2}{\pi}\widehat{f_n} = g_n * h$.** We first claim that $f_n \in L^1$ for each $n = 1, 2, \ldots$. Recall the basic fact that $\dfrac{\sin x}{x} \to 1$ as $x \to 0$. Thus if we define $f_n(0) = n$, then f_n is continuous on $\mathbb{R}$. Particularly, f_n is bounded by a positive constant M_n on $[-1,1]$. Since $|f_n(x)| \leq \frac{1}{x^2}$ if $|x| \geq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}} |f_n(x)|\,\mathrm{d}x &= \int_{|x|\geq 1} |f_n(x)|\,\mathrm{d}x + \int_{|x|\leq 1} |f_n(x)|\,\mathrm{d}x \\ &\leq \int_1^{\infty} \frac{\mathrm{d}x}{x^2} + \int_{-\infty}^{-1} \frac{\mathrm{d}x}{x^2} + M_n \\ &= 2 + M_n. \end{aligned}$$

Therefore, our claim is true.

Next, since $g_n, h \in L^1$, Theorem 9.2(c) implies that $\widehat{g_n * h} = \widehat{g_n} \cdot \widehat{h}$. By the formulas (9.2), we see that

$$\widehat{g_n}(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{\sin nx}{x}, & \text{if } x \neq 0; \\ \sqrt{\dfrac{2}{\pi}}, & \text{otherwise} \end{cases} \quad \text{and} \quad \widehat{h}(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{\sin x}{x}, & \text{if } x \neq 0; \\ \sqrt{\dfrac{2}{\pi}}, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$(\widehat{g_n * h})(x) = \begin{cases} \dfrac{2}{\pi} \cdot \dfrac{\sin x \sin nx}{x^2}, & \text{if } x \neq 0; \\ \dfrac{2}{\pi}, & \text{otherwise} \end{cases}$$

and then

$$(\widehat{g_n * h})(x) = \frac{2}{\pi} \cdot f_n(x) \tag{9.4}$$

for all $x \neq 0$. Now the equation (9.4) clearly gives

$$\underbrace{\frac{2}{\pi}\widehat{f_n}(-x)}_{\substack{\text{It is } g \text{ in the} \\ \text{theorem}}} = \int_{-\infty}^{\infty} \frac{2}{\pi} f(t) e^{ixt}\, dm(x) = \int_{-\infty}^{\infty} \underbrace{(\widehat{g_n * h})(t)}_{\substack{\text{It is } f \text{ in the} \\ \text{theorem}}} e^{ixt}\, dm(x)$$

Furthermore, the relation (9.4) also gives $(\widehat{g_n * h}) \in L^1$. Recall the expression (9.3) ensures that $g_n * h \in L^1$. In other words, the function $g_n * h$ satisfies the hypotheses of Theorem 9.11 (The Inversion Theorem), so we conclude that

$$\frac{2}{\pi}\widehat{f_n}(-x) = (g_n * h)(x) \tag{9.5}$$

a.e. on $\mathbb{R}$. Since f_n is an odd function, the equation (9.5) reduces to

$$\frac{2}{\pi}\widehat{f_n}(x) = (g_n * h)(x)$$

a.e. on $\mathbb{R}$, as required.

- $\|f_n\|_1 \to \infty$ **as** $n \to \infty$**.** By [49, Exercise 7, p. 197], we have

$$\frac{\sin x}{x} > \frac{2}{\pi} \tag{9.6}$$

if $0 < x < \frac{\pi}{2}$. Again the fact $\frac{\sin x}{x} \to 1$ as $x \to 0$ implies that the inequality (9.6) also holds at $x = 0$. By the definition, we have

$$|f_n(x)| > \frac{2}{\pi} \cdot \frac{|\sin nx|}{x}$$

if $0 \leq x < \frac{\pi}{2}$ and so

$$\|f_n\|_1 = \int_{-\infty}^{\infty} |f_n(x)|\, dm(x) \geq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{|\sin nx|}{x}\, dm(x) = \frac{2}{\pi} \int_0^{\frac{n\pi}{2}} \frac{|\sin x|}{x}\, dm(x). \tag{9.7}$$

It is well-known that

$$\int_0^{\infty} \frac{|\sin x|}{x}\, dx \geq \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\, dx \geq \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi}\, dx = \frac{2}{\pi} \sum_{k=0}^{n} \frac{1}{k+1} \to \infty$$

as $n \to \infty$, so the right-most integral in the inequality (9.7) tends to ∞ as $n \to \infty$.

- **The mapping $f \mapsto \widehat{f}$ maps L^1 into a proper subset of C_0.** Suppose that $\mathcal{F} : L^1 \to C_0$ is given by
$$\mathcal{F}(f) = \widehat{f}.$$
By Definition 9.1, $\mathcal{F}$ is linear. If $f \in L^1$ and $\|f\|_1 = 1$, then Theorem 9.6 gives $\|\widehat{f}\|_\infty \leq 1$ so that
$$\|\mathcal{F}\| = \sup\{\|\widehat{f}\|_\infty \,|\, f \in L^1 \text{ and } \|f\|_1 = 1\} \leq 1,$$
i.e., $\mathcal{F}$ is bounded. If $\mathcal{F}(f) = \mathcal{F}(g)$, then the linearity of $\mathcal{F}$ gives
$$\widehat{f - g} = \mathcal{F}(f - g) = \mathcal{F}(f) - \mathcal{F}(g) = 0.$$
By Theorem 9.12 (The Uniqueness Theorem), we obtain $f(x) = g(x)$ for almost all $x \in \mathbb{R}$ which means that $\mathcal{F}$ is injective.

  Assume that $\mathcal{F}$ was surjective. Since both L^1 and C_0 are Banach, we know from Theorem 5.10 that there is a $\delta > 0$ such that
$$\|\widehat{f}\|_\infty \geq \delta \|f\|_1 \tag{9.8}$$
for all $f \in L^1$. Recall the facts from the above assertions that
$$f_n \in L^1, \quad g_n * h \in L^1, \quad \frac{2}{\pi}\widehat{f_n} = g_n * h \quad \text{and} \quad \|\widehat{f_n}\|_\infty = \frac{\pi}{2}\|g_n * h\|_\infty = \frac{\pi}{2} \cdot \sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{\pi}},$$
so we get from the inequality (9.8) that
$$\|f_n\|_1 \leq \frac{1}{\delta}\sqrt{\frac{2}{\pi}} < \infty$$
but this contradicts the fact $\|f_n\|_1 \to \infty$. Hence $\mathcal{F}$ is *not* surjective.

- **The set $\mathcal{B} = \{\widehat{f} \,|\, f \in L^1\}$ is dense in C_0.** To solve this part, we need the locally compact Hausdorff version of the **Stone-Weierstrass Theorem**[a]. In fact, it is

Lemma 9.1 (The Stone-Weierstrass Theorem)

If $\mathcal{A}$ is a subalgebra of C_0 that separates points on $\mathbb{R}$ and vanishes at no point of $\mathbb{R}$, then $\mathcal{A}$ is dense in C_0.

  It is clear that $\mathcal{B}$ is a vector subspace of C_0. For every pair of $f, g \in L^1$, Theorem 8.14 says that L^1 is closed under convolution, i.e., $f * g \in L^1$. By Theorem 9.2(c), we get
$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} \in \mathcal{B}.$$
By the definition, $\mathcal{B}$ is an algebra. Suppose that $p, q \in \mathbb{R}$ and we have two cases for consideration.

  - **Case (i): $p < q$ and $p \neq -q$.** Consider the function
$$f(x) = \mathrm{e}^{-|x|}. \tag{9.9}$$
By [51, Eqn. (3), p. 183], since f is even in $\mathbb{R}$, we have
$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ixt}\,\mathrm{d}m(x) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{ixt}\,\mathrm{d}m(x) = h_1(t) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+t^2}. \tag{9.10}$$
Since $\widehat{f}(x) = \widehat{f}(y)$ if and only if $x = \pm y$, we obtain $\widehat{f}(p) \neq \widehat{f}(q)$ in our case.

[a] See [49, pp. 159 – 165] for the compact cases and [22, §4.7] for the locally compact Hausdorff cases

– **Case (ii): $p < q$ and $p = -q$.** We consider $g(x) = f(x)e^{ix}$, where f is the function (9.9). By Theorem 9.2(a), we have

$$\widehat{g}(t) = \widehat{f}(t-1) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+(t-1)^2}.$$

If $\widehat{g}(p) = \widehat{g}(-q) = \widehat{g}(q)$, then we have $(q-1)^2 = (q+1)^2$ which implies that $q = 0$, a contradiction. Hence we have $\widehat{g}(-q) \neq \widehat{g}(q)$.

In other words, $\mathcal{B}$ separates points on $\mathbb{R}$. Furthermore, we see from the expression (9.10) that $\widehat{f}(t) \neq 0$ for each $t \in \mathbb{R}$. Thus $\mathcal{B}$ vanishes at no point of $\mathbb{R}$. Hence our desired result follows immediately from Lemma 9.1.

This completes the proof of the problem. ■

Remark 9.1

The result in Problem 9.2 that $\mathcal{F}(L^1) \subset C_0$ is called the **Riemann-Lebesgue Lemma** ([58, Theorem 1.4, p. 80]). See also §5.14 from Rudin.

Problem 9.3

Rudin Chapter 9 Exercise 3.

Proof. By similar analysis as in Problem 9.2, if $g_\lambda = \chi_{[-\lambda,\lambda]}$, then for $t \neq 0$, we have

$$\widehat{g_\lambda} = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin \lambda t}{t}.$$

Note that $\widehat{g}_\lambda \notin L^1$, so we can't apply any inversion theorems (Theorems 9.11 or 9.14) directly. To solve this problem, we apply Theorem 9.13.

It is clear that $g_\lambda, \widehat{g_\lambda} \in L^2$. Using the same notation as in Theorem 9.13(d), we have

$$\psi_A(x) = \int_{-A}^{A} \widehat{g_\lambda}(t)e^{ixt}\,dm(t) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{ixt}\,dt = \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{ixt}\,dt$$

By Theorem 9.13(d) again,

$$\|\psi_A - g_\lambda\|_2 \to 0 \tag{9.11}$$

as $A \to \infty$. If we restrict A to be a positive integer, then the result (9.11) means that $\{\psi_A\}$ is a Cauchy sequence in L^2 with limit g_λ.[b] Thus it follows from Theorem 3.12 that $\{\psi_A\}$ has a subsequence which converges pointwise almost everywhere to g_λ, i.e.,

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{ixt}\,dt = \lim_{k\to\infty} \int_{-A_k}^{A_k} \frac{\sin \lambda t}{t} e^{ixt}\,dt = \pi\chi_{[-\lambda,\lambda]}(x) = \begin{cases} \pi, & \text{if } x \in [-\lambda,\lambda]; \\ 0, & \text{otherwise} \end{cases} \tag{9.12}$$

for almost all real x, where $\lambda > 0$.

Since the result (9.12) only holds for almost all real x, if we want to evaluate the limit *for all* $x \in \mathbb{R}$, we need an inversion theorem which is applicable to g_λ. Fortunately, there is such a

[b]Here we use the basic fact [49, Theorem 3.11(a)].

theorem for **piecewise smooth functions**[c]. Indeed, we have (see [21, Theorem 7.6, p. 220] or [55, Theorem 6.6.2, p. 330])

Lemma 9.2

If f is a piecewise smooth function on $\mathbb{R}$ such that $f \in L^1$, then we have

$$\lim_{A\to\infty} \int_{-A}^{A} \widehat{f}(t)e^{ixt}\,dm(t) = \frac{\pi}{2}[f(x-) + f(x+)]$$

for every $x \in \mathbb{R}$.

Now the g_λ has discontinuities only at $x = \pm\lambda$, so it is piecewise smooth on $\mathbb{R}$ and $g_\lambda \in L^1$. Apply Lemma 9.2 to g_λ, we establish at once that

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{ixt}\,dt = \frac{\pi}{2}[g_\lambda(x-) + g_\lambda(x+)] = \begin{cases} \pi, & \text{if } x \in (-\lambda, \lambda); \\ \frac{\pi}{2} & \text{if } x = \pm\lambda; \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have ended the analysis of the proof. ■

Problem 9.4

Rudin Chapter 9 Exercise 4.

Proof. Suppose that there exists a $g \in L^1 \cap L^2$ such that $\widehat{g} \notin L^1$ and

$$f(x) = \widehat{g}(-x) \tag{9.13}$$

on $\mathbb{R}$. Then the hypotheses immediately give $f \notin L^1$. Since $g \in L^2$, Theorem 9.13 (The Plancherel Theorem) implies that

$$\|\widehat{g}\|_2 = \|g\|_2 < \infty.$$

Thus we have $\widehat{g} \in L^2$ and we deduce immediate from the hypothesis (9.13) that $f \in L^2$. Consequently, we must have $f \in L^2 \setminus L^1$. Now it remains to show that $\widehat{f} \in L^1$. To this end, we need the following property of Fourier transform ([55, Theorem 6.5.1, p. 324]):

Lemma 9.3

If $f \in L^2$, then we have $\widehat{\widehat{f}} = f^-$, where $\widehat{\widehat{f}}$ denotes the Fourier transform of $\widehat{f}$ and $f^-(x) = f(-x)$.

Now the expression (9.13) can be written as $f^- = \widehat{g}$. Since $g \in L^2$, this and Lemma 9.3 imply that

$$\widehat{f^-} = \widehat{\widehat{g}} = g^-.$$

Since $g \in L^1$, we have $g^- \in L^1$ and then $\widehat{f^-} \in L^1$. Recall from Definition 9.1 that

$$\widehat{f^-}(t) = \int_{-\infty}^{\infty} f^-(x)e^{-ixt}\,dm(x) = \int_{-\infty}^{\infty} f(-x)e^{-ixt}\,dm(x) = \int_{-\infty}^{\infty} f(x)e^{ixt}\,dm(x) = \overline{\widehat{f}(t)}.$$

[c]For the definition, see [55, Definition 1.4.2, p. 29].

Hence, we may conclude that $\widehat{f} \in L^1$.

Finally, we give examples of the problem. Consider $g_n = \chi_{[-n,n]}$ which is clearly an element of $L^1 \cap L^2$ for every $n \in \mathbb{N}$. Since

$$\widehat{g_n}(t) = \int_{-\infty}^{\infty} g_n(x)e^{-ixt}\,dm(x) = \frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{-ixt}\,dx = \sqrt{\frac{2}{\pi}} \cdot \frac{2\sin nt}{t},$$

we have $\widehat{g_n} \notin L^1$. Therefore, the expression (9.13) ensures that each function

$$f_n(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{2\sin nx}{x}$$

satisfies the hypotheses, completing the proof of the problem.

■

Problem 9.5

Rudin Chapter 9 Exercise 5.

Proof. By Theorem 9.6, we have $\widehat{f}$ is continuous on $\mathbb{R}$ which vanishes at infinity. Thus $\widehat{f}$ is bounded by a positive constant, say M, on $[-1,1]$. In addition, it is obvious that

$$|\widehat{f}(t)| \leq |t\widehat{f}(t)|$$

if $|t| > 1$ and so our hypothesis implies that

$$\int_{\mathbb{R}} |\widehat{f}(t)|\,dt = \int_{|t|\leq 1} |\widehat{f}(t)|\,dt + \int_{|t|>1} |\widehat{f}(t)|\,dt \leq M + \int_{|t|>1} |t\widehat{f}(t)|\,dt < \infty.$$

Therefore, $\widehat{f} \in L^1$ and Theorem 9.11 (The Inversion Theorem) ensures that

$$f(x) = g(x) = \int_{-\infty}^{\infty} \widehat{f}(t)e^{ixt}\,dm(t) \tag{9.14}$$

a.e. on $\mathbb{R}$.

It remains to show that g is differentiable with the mentioned derivative in the question. To this end, we define $F(t) = \widehat{f}(t)$ and $G(t) = -it\widehat{f}(t)$ so that

$$G(x) = -ixF(x)$$

for all $x \in \mathbb{R}$. Since $|G(t)| = |t\widehat{f}(t)|$, the hypotheses give $G \in L^1$. Thus we deduce from Theorem 9.2(f) that $\widehat{F}$ is differentiable and

$$\widehat{F}'(t) = \widehat{G}(t) \tag{9.15}$$

for all $t \in \mathbb{R}$. By the integral (9.14) and Definition 9.1, we see that

$$g(x) = \int_{-\infty}^{\infty} F(t)e^{ixt}\,dm(t) = \widehat{F}(-x).$$

Using the equation (9.15) and the definition of G, we conclude that

$$g'(x) = -\widehat{F}'(-x) = -\widehat{G}(-x) = -\int_{-\infty}^{\infty} G(t)e^{ixt}\,dm(t) = i\int_{-\infty}^{\infty} t\widehat{f}(t)e^{ixt}\,dm(t).$$

Hence we have completed the proof of the problem. ■

Problem 9.6

Rudin Chapter 9 Exercise 6.

Proof. Let $f = \chi_{[-1,1]}$. Clearly, $f \in L^1$ and $f' = 0$ a.e. on $\mathbb{R}$. Thus we have $f' \in L^1$ too. However, we see from Definition 9.1 that

$$\widehat{f'}(t) = \int_{-\infty}^{\infty} f'(x)e^{-ixt}\,dm(x) = 0$$

a.e. on $\mathbb{R}$ and

$$ti\widehat{f}(t) = ti\int_{-\infty}^{\infty} f(x)e^{-ixt}\,dm(x) = \frac{ti}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ixt}\,dx = \frac{ti}{\sqrt{2\pi}}\cdot\frac{2\sin t}{t} = i\sqrt{\frac{2}{\pi}}\sin t.$$

Hence we conclude that $\widehat{(f')}(t) \neq ti\widehat{f}(t)$ which completes the proof of the problem. ∎

Problem 9.7

Rudin Chapter 9 Exercise 7.

Proof. Let $f \in S$. Denote $\mathcal{F}[f(x)](t) = \widehat{f}(t)$, or simply $\mathcal{F}[f(x)] = \widehat{f}$, to be the Fourier transform of f. We want to show that, for every m and $n = 0, 1, 2, \ldots$, there are positive numbers $A_{mn}\big(\mathcal{F}[f(x)]\big) < \infty$ such that

$$|x^n D^m \mathcal{F}[f(x)]| \leq A_{mn}(\mathcal{F}[f(x)]).$$

Since $|x^2 f(x)| \leq A_{02}(f)$ for every $x \in \mathbb{R}$, it means that

$$|f(x)| \leq \frac{A_{02}}{x^2}$$

for every $x \neq 0$. By this, we have $f \in L^1$ and thus Theorem 9.6 gives

$$\big|\mathcal{F}[f(x)](t)\big| \leq \big\|\mathcal{F}[f(x)]\big\|_\infty \leq \|f\|_1 < \infty.$$

It is easy to prove that if $f \in S$, then $f'(x) \in S$ and $xf(x) \in S$. These facts show that $D^n\big((-ix)^m f(x)\big)$ belongs to S for every $m, n = 0, 1, 2, \ldots$ and so

$$\Big|\mathcal{F}\big[D^n\big((-ix)^m f(x)\big)\big]\Big| < \infty \tag{9.16}$$

for every $m, n = 0, 1, 2, \ldots$.

By Theorem 9.2(f), we observe that

$$\frac{d}{dt}\{\mathcal{F}[f(x)](t)\} = \mathcal{F}[-ixf(x)](t) \quad \text{and} \quad \frac{d^2}{dt^2}\{\mathcal{F}[f(x)](t)\} = \mathcal{F}[(-ix)^2 f(x)](t).$$

Therefore, we see that

$$D^m\mathcal{F}[f(x)](t) = \frac{d^m}{dt^m}\{\mathcal{F}[f(x)](t)\} = \mathcal{F}[(-ix)^m f(x)](t) \tag{9.17}$$

for every $m = 0, 1, 2, \ldots$. To continue the proof, we need the following lemma:

Lemma 9.4

Let $f \in S$. Then we have $\mathcal{F}[f'(x)](t) = it\mathcal{F}[f(x)](t)$.

Proof of Lemma 9.4. By Integration by Parts, we obtain

$$\begin{aligned}\int_{-N}^{N} f'(x)\mathrm{e}^{-ixt}\,\mathrm{d}x &= f(x)\mathrm{e}^{-ixt}\Big|_{-N}^{N} - \int_{-N}^{N} f(x)\,\mathrm{d}(\mathrm{e}^{-ixt})\\ &= \big[f(N)\mathrm{e}^{-iNt} - f(-N)\mathrm{e}^{iNt}\big] + it\int_{-N}^{N} f(x)\mathrm{e}^{-ixt}\,\mathrm{d}x. \qquad (9.18)\end{aligned}$$

Since $f \in S$, $f(x) \to 0$ as $x \to \pm\infty$. Thus we follow from the expression (9.18) by letting $N \to \infty$ that

$$\mathcal{F}[f'(x)](t) = it\mathcal{F}[f(x)](t),$$

completing the proof of the lemma. ■

Using Lemma 9.4 repeatedly, we establish the relation

$$\mathcal{F}[D^n f(x)](t) = (it)^n \mathcal{F}[f(x)](t) \tag{9.19}$$

for every $n = 0, 1, 2, \ldots$. Particularly, by replacing $f(x)$ by $(-ix)^m f(x)$ in the formula (9.19), we gain

$$\mathcal{F}[D^n\big((-ix)^m f(x)\big)](t) = (it)^n \mathcal{F}[(-ix)^m f(x)](t)$$

or equivalently,

$$\mathcal{F}[(-ix)^m f(x)](t) = (it)^{-n}\mathcal{F}[D^n\big((-ix)^m f(x)\big)](t). \tag{9.20}$$

Now we combine formulas (9.17) and (9.20) to conclude

$$(it)^n D^m \mathcal{F}[f(x)](t) = \mathcal{F}[D^n\big((-ix)^m f(x)\big)](t)$$

and we use the result (9.16) to get

$$\big|t^n D^m \mathcal{F}[f(x)](t)\big| = \big|\mathcal{F}[D^n\big((-ix)^m f(x)\big)]\big| < \infty$$

for every $m, n = 0, 1, 2, \ldots$. Hence we have proven that $\mathcal{F}[f(x)] = \widehat{f} \in S$.

The function $f(x) = \mathrm{e}^{-x^2}$ is called a **Gaussian function** which is an element of S. Similarly, the function $f(x) = P(x)\mathrm{e}^{-x^2}$ belongs to S for any polynomial $P(x)$. This completes the proof of the problem. ■

Remark 9.2

The class S in Problem 9.7 is called the **Schwartz space** or **space of rapidly decreasing functions on** $\mathbb{R}$.

Problem 9.8

Rudin Chapter 9 Exercise 8.

Proof. We are going to show the assertions one by one.

- $h = f * g$ **is uniformly continuous on** $\mathbb{R}$. We claim that $(f*g)_z = f_z * g$. To see this, we note from the definition that

$$(f*g)_z(x) = (f*g)(x-z) = \int_{-\infty}^{\infty} f(x-z-y)g(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f_z(x-y)g(y)\,\mathrm{d}y = (f_z*g)(x).$$

It is clear that either p or q must be finite. Suppose that $1 \leq p < \infty$. We need the following result:

Lemma 9.5

Let $p, q \in [1, \infty]$ be conjugate exponents. If $f \in L^p$ and $g \in L^q$, then $f * g \in L^\infty$ and

$$\|f * g\|_\infty \leq \|f\|_p \cdot \|g\|_q. \tag{9.21}$$

Proof of Lemma 9.5. Suppose that $\sigma(f)(x) = f(-x)$. By Theorem 3.8 and the fact that $\sigma(f_{-x})(y) = f_{-x}(-y) = f(-y + x)$ for every $y \in \mathbb{R}$, if $x \in \mathbb{R}$, then

$$|(f * g)(x)| = \Big|\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}m(y)\Big| = \|\sigma(f_{-x}) \times g\|_1 \leq \|\sigma(f_{-x})\|_p \cdot \|g\|_q. \tag{9.22}$$

In addition, we have

$$\|\sigma(f_x)\|_p = \Big\{\int_{-\infty}^{\infty} |f(-y-x)|^p\,\mathrm{d}y\Big\}^{\frac{1}{p}} = \Big\{\int_{-\infty}^{\infty} |f(y)|^p\,\mathrm{d}y\Big\}^{\frac{1}{p}} = \|f\|_p. \tag{9.23}$$

Hence we derive (9.23) by simply putting the result (9.23) back into the inequality (9.22). This proves Lemma 9.5. ■

Now it is easily seen that the convolution of functions f and g is distributive, so for every $x \in \mathbb{R}$, it follows from Lemma 9.5 that

$$\begin{aligned}
|(f * g)(x-\delta) - (f * g)(x)| &= |(f * g)_\delta(x) - (f * g)(x)| \\
&\leq \|(f * g)_\delta - f * g\|_\infty \\
&= \|(f_\delta - f) * g\|_\infty \\
&\leq \|f_\delta - f\|_p \cdot \|g\|_q
\end{aligned}$$

which implies that $|(f * g)(x-\delta) - (f * g)(x)| \to 0$ as $\delta \to 0$. Hence $f * g$ is uniformly continuous on $\mathbb{R}$.

- **$h \in C_0$ if $1 < p < \infty$.** Recall from Theorem 3.14 that $C_c(\mathbb{R})$ is dense in L^p and L^q. Choose $\{f_n\} \subseteq C_c(\mathbb{R}) \subset L^p$ and $\{g_n\} \subseteq C_c(\mathbb{R}) \subset L^q$ such that $\|f_n - f\|_p \to 0$ and $\|g_n - g\|_q \to 0$ as $n \to \infty$.

  Let M be a positive constant such that $\operatorname{supp}(f_n), \operatorname{supp}(g_n) \subseteq [-M, M]$. Thus for every $|x| > 2M$ and every $y \in \mathbb{R}$, we have either $y \in \mathbb{R} \setminus [-M, M]$ or $x - y \in \mathbb{R} \setminus [-M, M]$. Otherwise, it will give the contradiction that $|x| = |x - y + y| \leq 2M$. Thus this fact gives $f_n(x-y)g_n(y) = 0$ and then
  $$(f_n * g_n)(x) = 0$$
  for every x such that $|x| > 2M$, i.e., $\operatorname{supp}(f_n * g_n) \subseteq [-2M, 2M]$ which means that each $f_n * g_n$ is an element of $C_c(\mathbb{R})$.

  Next, by Lemma 9.5 and then Theorem 3.9, we obtain

$$\begin{aligned}
|(f_n * g_n)(x) - (f_m * g_m)(x)| &\leq |(f_n * g_n)(x) - (f_n * g_m)(x)| \\
&\quad + |(f_n * g_m)(x) - (f_m * g_m)(x)| \\
&= \big|[f_n * (g_n - g_m)](x)\big| + \big|[(f_n - f_m) * g_m](x)\big| \\
&\leq \|f_n\|_p \cdot \|g_n - g_m\|_q + \|f_n - f_m\|_p \cdot \|g_m\|_q
\end{aligned}$$

$$\leq (\|f_n - f\|_p + \|f\|_p) \times \|g_n - g_m\|_q + \|f_n - f_m\|_p \times (\|g_m - g\|_q + \|g\|_q). \tag{9.24}$$

Given $\epsilon > 0$. Now there exists a $N \in \mathbb{N}$ such that $n, m \geq N$ imply that

$$\|f_n - f\|_p \leq 1, \quad \|g_m - g\|_p \leq 1, \quad \|g_n - g_m\|_q < \frac{\epsilon}{2(1 + \|f\|_p)} \quad \text{and} \quad \|f_n - f_m\|_p < \frac{\epsilon}{2(1 + \|g\|_q)}.$$

Hence if $n, m \geq N$, then we deduce from the inequality (9.24) that

$$|(f_n * g_n)(x) - (f_m * g_m)(x)| < \epsilon$$

for every $x \in \mathbb{R}$. In other words, $\{f_n * g_n\}$ is Cauchy in $C_c(\mathbb{R})$ relative to the metric induced by the supremum norm.

Finally, using similar argument as in proving the inequality (9.24), we can show that one can find a positive integer N' such that $n \geq N'$ implies

$$|(f_n * g_n)(x) - (f * g)(x)| < \epsilon.$$

Consequently, it means that

$$\|f_n * g_n - f * g\|_\infty \to 0 \tag{9.25}$$

as $n \to \infty$. By Theorem 3.17, we know that $C_0(\mathbb{R}) = \overline{C_c(\mathbb{R})}$ so the limit (9.25) establishes the fact that $f * g \in C_0(\mathbb{R})$.

- **A counter-example.** Consider $f = \chi_{[-1,1]}$ and $g = 1$ so that $f \in L^1$ and $g \in L^\infty$. However, for any $x \in \mathbb{R}$, we note that

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} \chi_{[x-1,x+1]}(y)\,\mathrm{d}y = m\big([x-1, x+1]\big) = 2.$$

  Thus h *does not* vanish at ∞, i.e., $h \notin C_0(\mathbb{R})$.

This completes the proof of the problem. ■

Problem 9.9

Rudin Chapter 9 Exercise 9.

Proof. We have

$$g(x) = \int_{\mathbb{R}} \chi_{[x,x+1]}(t) f(t)\,\mathrm{d}t = \int_{\mathbb{R}} \chi_{[0,1]}(t-x) f(t)\,\mathrm{d}t. \tag{9.26}$$

By the change of variable formula [51, p. 156], the integral (9.26) reduces to

$$g(x) = \int_{\mathbb{R}} \chi_{[0,1]}(t) f(t+x)\,\mathrm{d}t = \int_{\mathbb{R}} \chi_{[0,1]}(t) f_{-x}(t)\,\mathrm{d}t.$$

For every $x \in \mathbb{R}$, by Theorem 9.5, the mapping $x \mapsto f_{-x}$ is continuous at x. In other words, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f_{-x} - f_{-y}\|_p < \epsilon$ for all $y \in \mathbb{R}$ with $|x - y| < \delta$. Since $f_{-x} - f_{-y} \in L^p$ and $\chi_{[0,1]} \in L^q$, where q be the conjugate exponent of p, we deduce from Theorem 3.8 and then Theorem 1.33 that

$$\begin{aligned} |g(x) - g(y)| &= \left| \int_{\mathbb{R}} \chi_{[0,1]}(t)[f_{-x}(t) - f_{-y}(t)]\,\mathrm{d}t \right| \\ &= \big\| \chi_{[0,1]} \cdot (f_{-x} - f_{-y}) \big\|_1 \end{aligned}$$

$$\leq \|\chi_{[0,1]}\|_q \cdot \|f_{-x} - f_{-y}\|_p$$
$$< \epsilon$$

for all $y \in \mathbb{R}$ with $|x-y| < \delta$. Thus g is continuous at x.

Next, suppose that a and b are real numbers with $a < b$. Since $f \in L^1$, we have $f \in L^p([a,b])$ and then Theorem 3.5 (Hölder's Inequality) implies that

$$\int_a^b |f(t)|\,\mathrm{d}t \leq \Big\{\int_a^b \mathrm{d}t\Big\}^{\frac{1}{q}}\Big\{\int_a^b |f(t)|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}} = (b-a)^{\frac{1}{q}} \cdot \Big\{\int_a^b |f(t)|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}}. \tag{9.27}$$

In particular, if $a = n$ and $b = n+1$, then we know from the inequality (9.27) that

$$\int_n^{n+1} |f(t)|\,\mathrm{d}t \leq \Big\{\int_n^{n+1} |f(t)|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}}. \tag{9.28}$$

Given $\epsilon > 0$. Again $f \in L^p$ gives

$$\int_{-\infty}^{\infty} |f(t)|^p\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} \int_{-n}^{n} |f(t)|^p\,\mathrm{d}t < \infty.$$

Thus there is a positive integer N such that for all $n \in \mathbb{Z}$, $|n| \geq N$ implies that

$$\int_n^{n+1} |f(t)|^p\,\mathrm{d}t < \epsilon^p. \tag{9.29}$$

Combining the inequalities (9.28) and (9.29), we get

$$\int_n^{n+1} |f(t)|^p\,\mathrm{d}t < \epsilon$$

if $|n| \geq N$. In other words, we have

$$\lim_{|n|\to\infty} \int_n^{n+1} |f(t)|^p\,\mathrm{d}t = 0$$

and equivalently,

$$\lim_{|x|\to\infty} \int_x^{x+1} |f(t)|^p\,\mathrm{d}t = 0. \tag{9.30}$$

Finally, we use Theorem 1.33 and then apply the limit (9.30) to the inequality (9.27) with $a = x$ and $b = x+1$ to establish

$$\lim_{|x|\to\infty} |g(x)| \leq \lim_{|x|\to\infty} \int_x^{x+1} |f(t)|\,\mathrm{d}t \leq \lim_{|x|\to\infty} \Big\{\int_x^{x+1} |f(t)|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}} = 0.$$

Hence we conclude that g vanishes at infinity and then $g \in C_0(\mathbb{R})$.

Suppose that $f \in L^\infty$. Therefore, we can say that $|f(x)| \leq \|f\|_\infty < \infty$ for almost all $x \in \mathbb{R}$. Thus if $x < y$, then we have

$$\begin{aligned}
|g(x) - g(y)| &= \Big|\int_x^{x+1} f(t)\,\mathrm{d}t - \int_y^{y+1} f(t)\,\mathrm{d}t\Big| \\
&= \Big|\int_x^0 f(t)\,\mathrm{d}t + \int_0^{x+1} f(t)\,\mathrm{d}t - \int_y^0 f(t)\,\mathrm{d}t - \int_0^{y+1} f(t)\,\mathrm{d}t\Big| \\
&= \Big|\int_x^y f(t)\,\mathrm{d}t - \int_{x+1}^{y+1} f(t)\,\mathrm{d}t\Big|
\end{aligned}$$

$$\leq \int_x^y |f(t)|\,\mathrm{d}t + \int_{x+1}^{y+1} |f(t)|\,\mathrm{d}t$$
$$\leq 2(y-x)\|f\|_\infty.$$

Hence this means that g is uniformly continuous on $\mathbb{R}$ and furthermore,

$$|g(x)| = \Big|\int_x^{x+1} f(t)\,\mathrm{d}t\Big| \leq \int_x^{x+1} |f(t)|\,\mathrm{d}t \leq \int_x^{x+1} \|f\|_\infty\,\mathrm{d}t = \|f\|_\infty$$

for almost every $x \in \mathbb{R}$ so that $\|g\|_\infty \leq \|f\|_\infty$. This completes the proof of the problem. ■

Problem 9.10

Rudin Chapter 9 Exercise 10.

Proof. Let's prove the results one by one.

- **C_c^∞ does not consist of 0 alone.** In Problem 7.8(d), we define the bump function $\psi : \mathbb{R} \to [0,\infty)$ by

$$\psi(x) = \begin{cases} \exp\Big(-\dfrac{1}{1-x^2}\Big), & \text{if } x \in (-1,1); \\ 0, & \text{otherwise.} \end{cases} \tag{9.31}$$

  This is an element of C^∞. Since $\mathrm{supp}\,(\psi) = [-1,1]$, it must be true that $\psi \in C_c^\infty$.

- **If $f \in L^1_{\mathrm{loc}}$ and $g \in C_c^\infty$, then $f * g \in C^\infty$.** To begin with, we have to show that $f * g$ is well-defined first. Since $\mathrm{supp}\,(g)$ is compact in $\mathbb{R}$, let M be a positive number such that $\mathrm{supp}\,(g) \subseteq [-M, M]$. Thus we have

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y = \int_{-M}^{M} f(x-y)g(y)\,\mathrm{d}y. \tag{9.32}$$

  As g is continuous on $[-M, M]$, the Heine-Borel Theorem says that it is bounded. Let a bound of g be K. Then $|g(y)| \leq K$ on $[-M, M]$. Therefore, the boundedness of g and the fact $f \in L^1_{\mathrm{loc}}$ indicate that the integral in the equation (9.32) is finite. Consequently, $f * g$ is well-defined.

  Next, we claim that $(f * g)' = f * g'$. In fact, if $x \in \mathbb{R}$ and $0 < \delta < 1$, then we write

$$\begin{aligned} &\frac{(f * g)(x+\delta) - (f * g)(x)}{\delta} \\ &= \frac{1}{\delta}\Big[\int_{-\infty}^{\infty} f(x+\delta-y)g(y)\,\mathrm{d}y - \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y\Big] \\ &= \frac{1}{\delta}\Big[\int_{-M}^{M} f(x+\delta-y)g(y)\,\mathrm{d}y - \int_{-M}^{M} f(x-y)g(y)\,\mathrm{d}y\Big] \\ &= \frac{1}{\delta}\Big[\int_{x-(M-\delta)}^{x+(M+\delta)} f(y)g(x+\delta-y)\,\mathrm{d}y - \int_{x-M}^{x+M} f(y)g(x-y)\,\mathrm{d}y\Big]. \end{aligned} \tag{9.33}$$

  Since $x+\delta-[x+(M+\delta)] = -M$, $x+\delta-[x-(M+\delta)] = M+2\delta$ and $\mathrm{supp}\,(g) \subseteq [-M, M]$, the first integral in (9.33) can be replaced by

$$\int_{x-(M+1)}^{x+(M+1)} f(y)g(x+\delta-y)\,\mathrm{d}y.$$

Similarly, we may extend the lower limit and the upper limit of the second integral in (9.33) to $x-(M+1)$ and $x+(M+1)$ respectively. Therefore, the expression (9.33) can be reduced to

$$\frac{(f*g)(x+\delta)-(f*g)(x)}{\delta} = \int_{x-(M+1)}^{x+(M+1)} \frac{g(x-y+\delta)-g(x-y)}{\delta}\cdot f(y)\,\mathrm{d}y. \tag{9.34}$$

Now we *fix* $x\in\mathbb{R}$. Let $g=g_1+ig_2$. Since $g\in C_c^\infty$, g_1 and g_2 are real differentiable functions. It follows from the Mean Value Theorem [49, Theorem 5.10, p. 108] that there exist $\xi,\eta\in(x-y,x-y+\delta)$ such that

$$g_1(x-y+\delta)-g_1(x-y)=\delta g_1'(\xi) \quad \text{and} \quad g_2(x-y+\delta)-g_2(x-y)=\delta g_2'(\eta).$$

Then they imply that $g(x-y+\delta)-g(x-y)=\delta[g_1'(\xi)+ig_2'(\eta)]$ which gives

$$\Big|\frac{g(x-y+\delta)-g(x-y)}{\delta}\Big| \le \sqrt{[g_1'(\xi)]^2+[g_2'(\eta)]^2} \le \sqrt{|g'(\xi)|^2+|g'(\eta)|^2}. \tag{9.35}$$

Since $g\in C_c^\infty$, g' is continuous and thus bounded on $[x-y,x-y+\delta]$. Recall that $y\in[x-(M+1),x+(M+1)]$ and $0<\delta<1$. Thus, there is an $A>0$ such that $|g'(t)|\le A$ *for every* $t\in[-M-2,M+2]$[d] and then we observe from the inequality (9.35) that

$$\Big|\frac{g(x-y+\delta)-g(x-y)}{\delta}\cdot f(y)\Big| \le \sqrt{2}A|f(y)|.$$

Since f is integrable on $[x-(M+1),x+(M+1)]$, Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) ensures that the expression (9.34) gives

$$\begin{aligned}
(f*g)'(x) &= \lim_{\delta\to 0}\frac{(f*g)(x+\delta)-(f*g)(x)}{\delta}\\
&= \lim_{\delta\to 0}\int_{x-(M+1)}^{x+(M+1)} \frac{g(x-y+\delta)-g(x-y)}{\delta}\cdot f(y)\,\mathrm{d}y\\
&= \int_{x-(M+1)}^{x+(M+1)} \lim_{\delta\to 0}\frac{g(x-y+\delta)-g(x-y)}{\delta}\cdot f(y)\,\mathrm{d}y\\
&= \int_{x-(M+1)}^{x+(M+1)} g'(x-y)\cdot f(y)\,\mathrm{d}y\\
&= (g'*f)(x).
\end{aligned}$$

Since x is arbitrary, we have $f*g\in C^1$ and $(f*g)'=g'*f$. Since g has compact support, $g^{(n)}$ also has compact support for every $n\in\mathbb{N}$. By induction, we have obtained

$$(f*g)^{(n)}=g^{(n)}*f.$$

- **Existence of a sequence $\{g_n\}\subseteq C_c^\infty$ such that $\|f*g_n-f\|_1\to 0$ as $n\to\infty$ for every $f\in L^1$.** If $f\in L^1$, then $f\in L^1_{\mathrm{loc}}$ and so $f*g\in C^\infty$ for any $g\in C_c^\infty$. Let

$$c=\int_{-1}^{1}\psi(x)\,\mathrm{d}x,$$

where ψ is the bump function (9.31). It is clear that c is a finite positive number because $\psi>0$. If we define $\varphi:\mathbb{R}\to[0,\infty)$ by

$$\varphi(x)=c^{-1}\psi(x),$$

[d] Notice that A *does not* depend on x, y or δ.

then φ satisfies the conditions that $\varphi \in C_c^\infty$, $\operatorname{supp}(\varphi) = [-1, 1]$, $\varphi \in L^\infty$ and $\|\varphi\|_1 = 1$. Next, for $\epsilon > 0$, we define[e]

$$\varphi_\epsilon(x) = \epsilon^{-1}\varphi(\epsilon^{-1}x).$$

Then it is easily checked that

$$\varphi_\epsilon \in C_c^\infty, \quad \operatorname{supp}(\varphi_\epsilon) = [-\epsilon, \epsilon], \quad \varphi_\epsilon \in L^\infty \quad \text{and} \quad \|\varphi_\epsilon\|_1 = \|\varphi\|_1 = 1.$$

We observe that

$$\begin{aligned} f * \varphi_\epsilon(x) - f(x) &= \int_{\mathbb{R}} f(x-y)\varphi_\epsilon(y)\,\mathrm{d}y - \int_{\mathbb{R}} f(x)\varphi_\epsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} [f(x-y) - f(x)]\varphi_\epsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} [f(x-y) - f(x)]\varphi(\epsilon^{-1}y)\epsilon^{-1}\,\mathrm{d}y. \end{aligned} \tag{9.36}$$

If we let $y = \epsilon z$, then the expression (9.36) becomes

$$|f * \varphi_\epsilon(x) - f(x)| = \Big|\int_{\mathbb{R}} [f_{\epsilon z}(x) - f(x)]\varphi(z)\,\mathrm{d}z\Big|. \tag{9.37}$$

Since $f \in L^1$ and $\varphi \in L^\infty$, Theorem 3.8 (with $p = 1$ and $q = \infty$) makes sure that $[f_{\epsilon z}(x) - f(x)]\varphi(z) \in L^1$ (with respect to z). By Theorem 1.33, the expression (9.37) gives

$$\|f * \varphi_\epsilon - f\|_1 = \int_{\mathbb{R}} |f * \varphi_\epsilon(x) - f(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \underbrace{|f_{\epsilon z}(x) - f(x)| \cdot |\varphi(z)|}_{\text{nonnegative integrand}}\,\mathrm{d}z\Big]\,\mathrm{d}x. \tag{9.38}$$

Since the integrand in the expression (9.38) is nonnegative, we may apply Theorem 8.8 (The Fubini Theorem) to change the order of integration in the expression (9.38) to conclude that

$$\|f * \varphi_\epsilon - f\|_1 \leq \int_{\mathbb{R}} |\varphi(z)| \cdot \|f_{\epsilon z} - f\|_1\,\mathrm{d}z. \tag{9.39}$$

By Theorem 9.5, we note that $\|f_{\epsilon z} - f\|_1 \to 0$ as $\epsilon \to 0$ for each $z \in \mathbb{R}$. Furthermore, since $\|f_{\epsilon z} - f\|_1 \leq 2\|f\|_1$, we apply Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) to the inequality (9.39) to obtain

$$\|f * \varphi_\epsilon - f\|_1 \to 0$$

as $\epsilon \to 0$. Now, if we let $\epsilon = \frac{1}{n}$ and $g_n = \varphi_{\frac{1}{n}}$, then we can obtain our desired sequence.

- **Existence of a sequence $\{g_n\} \subseteq C_c^\infty$ such that $(f * g_n)(x) \to f(x)$ a.e. for every $f \in L^1_{\mathrm{loc}}$.** For each *fixed* $n \in \mathbb{N}$, let $K_n = [-\frac{1}{n}, \frac{1}{n}] \subset V_n = (-\frac{1}{n} - 2^{-n}, \frac{1}{n} + 2^{-n})$. By [49, Exercise 6, p. 289], there exists functions $\psi_1, \ldots, \psi_s \in C^\infty$ such that[f]
  - $0 \leq \psi_i \leq 1$ for $1 \leq i \leq s$;
  - $\operatorname{supp}(\psi_i) \subset V_n$;
  - $\psi_1(x) + \psi_2(x) + \cdots + \psi_s(x) = 1$ for every $x \in K_n$.

[e] The collection $\{\varphi_\epsilon\}$ is called an **approximate identity** or **a sequence of mollifiers on** $\mathbb{R}$, see [48, Definition 6.31, p. 173] or [11, p. 108].

[f] For a proof of this, please read [63, Problem 10.6, pp. 266, 267]. Of course, our s depends on n.

If we define $g_n : \mathbb{R} \to \mathbb{R}$ by

$$g_n = \frac{n}{2}(\psi_1 + \psi_2 + \cdots + \psi_s),$$

then the above conditions imply that $g_n \in C_c^\infty$, $g_n(x) = \frac{n}{2}$ on K_n, $g_n(x) = 0$ outside V_n and $0 \leq g_n(x) \leq \frac{n}{2}$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ and $f(x) \geq 0$ on $\mathbb{R}$. On the one hand, we have

$$(f * g_n)(x) \geq \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(x-y)\,\mathrm{d}y = \frac{1}{m\big(B(0, \frac{1}{n})\big)} \int_{B(x,\frac{1}{n})} f(y)\,\mathrm{d}y. \tag{9.40}$$

Since $f \in L^1_{\mathrm{loc}}$, the integral (9.40) is finite. On the other hand, we have

$$\begin{aligned}(f * g_n)(x) &\leq \frac{n}{2} \int_{-\frac{1}{n}-2^{-n}}^{\frac{1}{n}+2^{-n}} f(x-y)\,\mathrm{d}y \\ &= \Big(1 + \frac{n}{2^n}\Big) \times \frac{1}{m\big(B(0, \frac{1}{n} + 2^{-n})\big)} \int_{B(x,\frac{1}{n}+2^{-n})} f(y)\,\mathrm{d}y. \end{aligned} \tag{9.41}$$

Similarly, the integral (9.41) is finite because of the fact that $f \in L^1_{\mathrm{loc}}$. Now if $x \in \mathbb{R}$ is a Lebesgue point of f, then we combine the inequalities (9.40) and (9.41) to get

$$\lim_{n\to\infty} (f * g_n)(x) = \lim_{n\to\infty} \frac{1}{m\big(B(0, \frac{1}{n})\big)} \int_{B(x,\frac{1}{n})} f(y)\,\mathrm{d}y = f(x). \tag{9.42}$$

By Lemma 7.5, we see that almost every $x \in \mathbb{R}$ is a Lebesgue point of f. Hence we conclude from this that $(f * g_n)(x) \to f(x)$ a.e. on $\mathbb{R}$ for every $f \in L^1_{\mathrm{loc}}$ and $f(x) \geq 0$ on $\mathbb{R}$.

For arbitrary $f \in L^1_{\mathrm{loc}}$, we can write $f = f^+ - f^-$ and apply the above analysis to f^+ and f^- individually so that the limit (9.42) also holds when f is replaced by f^+ and f^-. It is clear from Lemma 7.5 that almost every $x \in \mathbb{R}$ is a *common* Lebesgue point of f, f^+ and f^-, so

$$\lim_{n\to\infty} (f * g_n)(x) = \lim_{n\to\infty} (f^+ * g_n)(x) - \lim_{n\to\infty} (f^- * g_n)(x) = f^+(x) - f^-(x) = f(x).$$

Consequently, the result (9.42) is also true in this general case.

- **$(f * h_\lambda)(x) \to f(x)$ a.e. if $f \in L^1$ as $\lambda \to 0$ and $f * h_\lambda \in C^\infty$.** First of all, we notice that the h_λ is in the form of the so-called **Poisson kernel** on $\mathbb{R}$, see [21, p. 228]. Next, it is easy to see that

$$h_\lambda(x) = \lambda^{-1} h_1(\lambda^{-1} x),$$

where $h_1(x) = \frac{1}{1+x^2}$. Furthermore, we obtain from the hypotheses of h_λ that h_1 is an even function such that it is decreasing on $(0, \infty)$ and $h_1 \in L^1$. With the aid of [56, Theorem 1.25, p. 13], we see that

$$\lim_{\lambda\to 0} (f * h_\lambda)(x) = f(x) \int_{\mathbb{R}} h_1(t)\,\mathrm{d}m(t) = f(x)$$

for every Lebesgue point x of f. By Theorem 7.7, we obtain our desired result that $(f * h_\lambda)(x) \to f(x)$ a.e. on $\mathbb{R}$ if $f \in L^1$ as $\lambda \to 0$.

It remains to prove that $f * h_\lambda \in C^\infty$. To see this, the definition of h_λ shows clearly that $h_\lambda \in C^\infty$. In addition, direct computation gives easily that $h_\lambda^{(n)}$ is bounded on $\mathbb{R}$ for every positive integer n. As a consequence of [22, Proposition 8.10, p. 242], we may conclude immediately that $f * h_\lambda \in C^\infty$.

This completes the proof of the problem. ■

9.2 The Poisson Summation Formula and its Applications

Problem 9.11

Rudin Chapter 9 Exercise 11.

Proof. Suppose that f is a Schwartz function. We are going to show that the Poisson summation formula holds in this situation and consider the limiting case as $\alpha \to 0$ under the extra hypothesis that $\widehat{f} \in L^1$.

- **The validity of the Poisson summation formula.** We first assume that $f \in S$. Consider
$$F(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$$
which is a periodic function of period 2π, with Fourier coefficients
$$\widehat{F}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\mathrm{e}^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=-\infty}^{\infty} f(x + 2k\pi)\mathrm{e}^{-inx}\Big)\,\mathrm{d}x. \tag{9.43}$$
Since $f \in S$, there exists a positive constant M such that $|f(x)| \leq M$ and $|x^2 f(x)| \leq M$ for all $x \in \mathbb{R}$. In particular, if $k \geq 1$, then
$$|f(x + 2k\pi)| \leq \frac{M}{(x + 2k\pi)^2} \leq \frac{M}{(2k-1)^2\pi^2}$$
for all $x \in [-\pi, \pi]$. By the Weierstrass M-test [49, Theorem 7.10, p. 148], we see that the series
$$\sum_{k=1}^{\infty} f(x + 2k\pi)\mathrm{e}^{-inx}$$
converges uniformly and absolutely on $[-\pi, \pi]$. Similarly, the series
$$\sum_{k=1}^{\infty} f(x - 2k\pi)\mathrm{e}^{-inx}$$
also converges uniformly and absolutely on $[-\pi, \pi]$. If $k = 0$, then
$$\Big|\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\,\mathrm{d}x\Big| \leq \int_{-\pi}^{\pi} |f(x)|\,\mathrm{d}x \leq 2\pi M.$$
Consequently, the series
$$\sum_{k=-\infty}^{\infty} f(x + 2k\pi)\mathrm{e}^{-inx} \tag{9.44}$$
converges uniformly and absolutely on $[-\pi, \pi]$ so that the integral and summation in the expression (9.43) can be interchanged (by [49, Corollary, p. 152]) and we get
$$\begin{aligned}\widehat{F}(n) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x + 2k\pi)\mathrm{e}^{-inx}\,\mathrm{d}x\\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\pi}^{(2k+1)\pi} f(x)\mathrm{e}^{-inx}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-inx}\,\mathrm{d}x \\
&= \varphi(n)
\end{aligned}
$$

for every $n \in \mathbb{Z}$. In other words, we have

$$
\sum_{k=-\infty}^{\infty} f(x+2k\pi) = F(x) = \sum_{n=-\infty}^{\infty} \varphi(n)\mathrm{e}^{inx}. \tag{9.45}
$$

If we put $x = 0$ in the formula (9.45), then we get

$$
\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \varphi(n). \tag{9.46}
$$

Let $\gamma > 0$ and $g(x) = f(\gamma x)$. Since $g \in S$, the preceding formula (9.46) also gives

$$
\sum_{k=-\infty}^{\infty} g(2k\pi) = \sum_{n=-\infty}^{\infty} \psi(n), \tag{9.47}
$$

where

$$
\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{-itx}\,\mathrm{d}x.
$$

Direct computation shows

$$
\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\gamma x)\mathrm{e}^{-itx}\,\mathrm{d}x = \frac{1}{\gamma}\varphi\Big(\frac{t}{\gamma}\Big)
$$

and after substituting this back into the formula (9.47) and then using the definition of g , we obtain

$$
\sum_{k=-\infty}^{\infty} f(2\gamma k\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{\gamma}\varphi\Big(\frac{n}{\gamma}\Big). \tag{9.48}
$$

Suppose that $\alpha = \frac{1}{\gamma} > 0$ and $\beta = \frac{2\pi}{\alpha} > 0$. Then we have $\alpha\beta = 2\pi$ and the formula (9.48) becomes

$$
\sum_{k=-\infty}^{\infty} f(k\beta) = \alpha \sum_{n=-\infty}^{\infty} \varphi(n\alpha). \tag{9.49}
$$

- **The case when $\alpha \to 0$.** Now we suppose further that the inequality (9.51) in Remark 9.3 below holds. On the one hand, we know that

$$
\lim_{\alpha\to 0} \alpha \sum_{n=-\infty}^{\infty} \varphi(n\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-\infty}^{\infty} \varphi\Big(\frac{n}{N}\Big) = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \widehat{f}(x)\,\mathrm{d}m(x)
$$

because $\varphi(n) = \frac{1}{\sqrt{2\pi}}\widehat{f}(n)$. On the other hand, we have

$$
\lim_{\alpha\to 0} \sum_{k=-\infty}^{\infty} f(k\beta) = \lim_{\alpha\to 0} \sum_{k=-\infty}^{\infty} f\Big(\frac{2\pi k}{\alpha}\Big) = \lim_{N\to\infty} \sum_{k=-\infty}^{\infty} f(2\pi kN). \tag{9.50}
$$

Again the fact $f \in S$ implies that

$$
|f(x)| \leq \frac{M}{|x|^2}
$$

holds for all $x \neq 0$, where M is a positive constant. Thus this shows that

$$\left|\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} f(2\pi kN)\right| \leq \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |f(2\pi kN)| \leq \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{M}{|2\pi kN|^2} \to 0$$

as $N \to \infty$ and then the limit (9.50) becomes

$$\lim_{\alpha\to 0} \sum_{k=-\infty}^{\infty} f(k\beta) = f(0).$$

Hence we arrive at the result

$$f(0) = \int_{-\infty}^{\infty} \widehat{f}(x)\,\mathrm{d}m(x)$$

which is in agreement with Theorem 9.11 (The Inversion Theorem).

We have completed the proof of the problem. ■

Remark 9.3

If there exists a constant M and $p > 1$ such that

$$|f(x)| \leq \frac{M}{|x|^p} \tag{9.51}$$

for all sufficiently large x, then the series (9.44) still converges uniformly and absolutely on $[-\pi, \pi]$ so that the Poisson summation formula (9.49) also holds in this case.

Problem 9.12

Rudin Chapter 9 Exercise 12.

Proof. We note that f is *not* a Schwartz function, but the result [49, Theorem 8.6(f)] ensures that f satisfies the inequality (9.51), so we may apply the first assertion of Problem 9.11. First of all, since $\mathrm{e}^{-|x|}\cos tx$ and $\mathrm{e}^{-|x|}\sin tx$ are even and odd functions in x respectively, we have

$$\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\mathrm{e}^{-itx}\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\infty} \mathrm{e}^{-x}\cos tx\,\mathrm{d}x.$$

By [27, §2.662 Eqn. 2, p. 228], we have

$$\int \mathrm{e}^{-x}\cos tx\,\mathrm{d}x = \frac{(-\cos xt + t\sin xt)\mathrm{e}^{-x}}{1+t^2}$$

so that

$$\varphi(t) = \frac{1}{(1+t^2)\pi}. \tag{9.52}$$

Substituting the expression (9.52) and $f(x) = \mathrm{e}^{-|x|}$ into the formula (9.49), we obtain

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{-|k|\beta} = \alpha \sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2\alpha^2)\pi}$$

$$\frac{1+e^{-\beta}}{1-e^{-\beta}} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\alpha}{1+n^2\alpha^2}$$

$$\frac{e^{\beta}+1}{e^{\beta}-1} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\alpha}{1+n^2\alpha^2}, \tag{9.53}$$

where $\alpha\beta = 2\pi$. If we take $\beta = 2\pi\gamma$, then $\alpha = \frac{1}{\gamma}$. Hence we deduce from the formula (9.53) that

$$\frac{e^{2\pi\gamma}+1}{e^{2\pi\gamma}-1} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\gamma}{\gamma^2+n^2} \tag{9.54}$$

and the desired result follows if we replace γ by α in the formula (9.54). This completes the proof of the problem. ■

Problem 9.13

Rudin Chapter 9 Exercise 13.

Proof.

(a) By Definition 9.1 and similar argument as in the proof of Problem 9.12, we have

$$\widehat{f_c}(t) = \int_{-\infty}^{\infty} f_c(x)e^{-ixt}\,dm(x) = \int_{-\infty}^{\infty} e^{-cx^2}e^{-ixt}\,dm(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-cx^2}\cos tx\,dx. \tag{9.55}$$

By [27, §3.922, Eqn. 4, p. 494], we see that

$$\int_0^{\infty} e^{-cx^2}\cos tx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{c}}\exp\Big(-\frac{t^2}{4c}\Big),$$

so the formula (9.55) becomes

$$\widehat{f_c}(t) = \frac{1}{\sqrt{2c}}\exp\Big(-\frac{t^2}{4c}\Big).$$

(b) By part (a), if $\widehat{f_c} = f_c$, then we have

$$\frac{1}{\sqrt{2c}}\exp\Big(-\frac{x^2}{4c}\Big) = \exp(-cx^2)$$

$$\exp\Big(\frac{4c^2-1}{4c}x^2\Big) = \sqrt{2c} \tag{9.56}$$

for every $x \in \mathbb{R}$. Since the right-hand side of the equation (9.56) is constant, $4c^2-1=0$ and then $c = \frac{1}{2}$. Direct checking immediately shows that $c = \frac{1}{2}$ satisfies the equation (9.56).

(c) Let $a, b \in (0,\infty)$. By the definition, we see that

$$\begin{aligned}(f_a * f_b)(x) &= \int_{-\infty}^{\infty} f_a(x-y)f_b(y)\,dy\\ &= \int_{-\infty}^{\infty}\exp[-a(x-y)^2 - by^2]\,dy\end{aligned}$$

$$= \int_{-\infty}^{\infty} \exp\{-[(a+b)y^2 - 2axy + ax^2]\}\,\mathrm{d}y. \tag{9.57}$$

Using [27, §2.33, Eqn. 1, p. 108] and the facts that[g] $\operatorname{erf}(\infty) = 1$ and $\operatorname{erf}(-\infty) = -1$, we are able to represent the integral (9.57) as

$$\begin{aligned}(f_a * f_b)(x) &= \frac{1}{2}\sqrt{\frac{\pi}{a+b}}\exp\Big(-\frac{ab}{a+b}x^2\Big) \times \operatorname{erf}\Big(\sqrt{a+b}y - \frac{ax}{\sqrt{a+b}}\Big)\Big|_{y=-\infty}^{y=\infty} \\ &= \sqrt{\frac{\pi}{a+b}}\exp\Big(-\frac{ab}{a+b}x^2\Big).\end{aligned}$$

Therefore, we conclude that

$$\gamma = \sqrt{\frac{\pi}{a+b}} \quad \text{and} \quad c = \frac{ab}{a+b}.$$

(d) By Problem 9.7, we know that $f_c \in S$. Thus the definition and the formula [27, §2.33, Eqn. 1, p. 108] together give

$$\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-cx^2}\mathrm{e}^{-itx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[-(cx^2 + itx)]\,\mathrm{d}x = \frac{1}{2\sqrt{\pi c}}\exp\Big(-\frac{t^2}{4c}\Big).$$

Hence we establish from the formula (9.49) that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \mathrm{e}^{-ck^2\beta^2} &= \frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}\frac{1}{2\sqrt{\pi c}}\exp\Big[-\frac{1}{4c}\times\Big(\frac{2n\pi}{\beta}\Big)^2\Big] \\ &= \frac{1}{\beta}\sqrt{\frac{\pi}{c}}\sum_{n=-\infty}^{\infty}\exp\Big(-\frac{n^2\pi^2}{c\beta^2}\Big).\end{aligned}$$

We complete the proof of the problem. ■

9.3 Fourier Transforms on $\mathbb{R}^k$ and its Applications

Problem 9.14

Rudin Chapter 9 Exercise 14.

Proof. Recall that if $f \in L^1(\mathbb{R}^k)$, then

$$\widehat{f}(\mathbf{y}) = \int_{\mathbb{R}^k} f(\mathbf{x})\exp(-i\mathbf{x}\cdot\mathbf{y})\,\mathrm{d}m_k(\mathbf{x}). \tag{9.58}$$

We first show the analogue of Theorem 9.6 that $\widehat{f} \in C_0(\mathbb{R}^k)$ and $\|\widehat{f}\|_\infty \leq \|f\|_1$. To this end, we imitate the proof of Theorem 9.6. The required inequality is obvious from the definition (9.58) and the integrability of f.

To prove $\widehat{f} \in C_0(\mathbb{R}^k)$, we need the analogue of Theorem 9.5 for $\mathbb{R}^k$:[h]

[g] Here $\operatorname{erf}(x)$ is the **error function**.

[h] Its proof is very similar to that of Theorem 9.5, so we omit the details here.

Lemma 9.6

For any function f on $\mathbb{R}^k$ and every $\mathbf{y} \in \mathbb{R}^k$, let $f_{\mathbf{y}}$ be the translate of f defined by

$$f_{\mathbf{y}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{y})$$

for $\mathbf{x} \in \mathbb{R}^k$. If $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^k)$, then the mapping

$$\mathbf{y} \mapsto f_{\mathbf{y}}$$

is a uniformly continuous mapping of $\mathbb{R}^k$ into $L^p(\mathbb{R}^k)$.

Let $\mathbf{t} \in \mathbb{R}^k$. If $\mathbf{t}_n \to \mathbf{t}$, then

$$|\widehat{f}(\mathbf{t}_n) - \widehat{f}(\mathbf{t})| \leq \int_{\mathbb{R}^k} |f(\mathbf{x})| \cdot |\exp(-i\mathbf{t}_n \cdot \mathbf{x}) - \exp(-i\mathbf{t} \cdot \mathbf{x})| \, \mathrm{d}m_k(\mathbf{x}). \tag{9.59}$$

Since the integrand of the integral (9.59) is bounded by $2|f(\mathbf{x})| \in L^1(\mathbb{R}^k)$, Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) implies that

$$\widehat{f}(\mathbf{t}_n) \to \widehat{f}(\mathbf{t})$$

as $n \to \infty$. In other words, $\widehat{f}$ is continuous on $\mathbb{R}^k$.

Next we have to prove that f vanishes at infinity. Suppose that $\mathbf{y} = (\eta_1, \ldots, \eta_k)$ and $\eta_1 \neq 0$. Take

$$\mathbf{y}_1 = \Big(\frac{\pi}{\eta_1}, 0, \ldots, 0\Big).$$

Since $\mathrm{e}^{-\pi i} = -1$, we have

$$\begin{aligned}
\exp\Big[-i\mathbf{y} \cdot \Big(\mathbf{x} + \mathbf{y}_1\Big)\Big] &= \exp\Big\{-i\Big[\eta_1\Big(\xi_1 + \frac{\pi}{\eta_1}\Big) + \sum_{j=2}^{k} \eta_j \xi_j\Big]\Big\} \\
&= \exp\Big(-i\sum_{j=1}^{k} \xi_j \eta_j\Big) \times \mathrm{e}^{-i\pi} \\
&= -\exp(-i\mathbf{x} \cdot \mathbf{y})
\end{aligned}$$

so that the definition (9.58) implies that

$$\widehat{f}(\mathbf{y}) = -\int_{\mathbb{R}^k} f(\mathbf{x}) \exp\Big[-i\mathbf{y} \cdot \Big(\mathbf{x} + \mathbf{y}_1\Big)\Big] \, \mathrm{d}m_k(\mathbf{x}) = -\int_{\mathbb{R}^k} f(\mathbf{x} - \mathbf{y}_1) \exp(-i\mathbf{y} \cdot \mathbf{x}) \, \mathrm{d}m_k(\mathbf{x}).$$

Thus we get from this that

$$\begin{aligned}
2\widehat{f}(\mathbf{y}) &= \int_{\mathbb{R}^k} [f(\mathbf{x}) - f(\mathbf{x} - \mathbf{y}_1)] \exp(-i\mathbf{y} \cdot \mathbf{x}) \, \mathrm{d}m_k(\mathbf{x}) \\
&= \int_{\mathbb{R}^k} [f(\mathbf{x}) - f_{\mathbf{y}_1}(\mathbf{x})] \exp(-i\mathbf{y} \cdot \mathbf{x}) \, \mathrm{d}m_k(\mathbf{x}).
\end{aligned} \tag{9.60}$$

Now we apply Lemma 9.6 to the integral (9.60), we assert that

$$2|\widehat{f}(\mathbf{y})| \leq \|f - f_{\mathbf{y}_1}\|_1 = \|f_{\mathbf{0}} - f_{\mathbf{y}_1}\|_1 \tag{9.61}$$

which tends to 0 as $|\mathbf{y}_1| \to 0$ or equivalently, as $\eta_1 \to \pm\infty$. Or we can say this way: Given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|f_{\mathbf{0}} - f_{\mathbf{y}_1}\|_1 < 2\epsilon$$

for all $\mathbf{y}_1 \in \mathbb{R}^k$ for which $|\mathbf{y}_1| < \delta$ or $|\eta_1| > \frac{\pi}{\delta}$. Notice also that $\eta_1 \to \pm\infty$ means that $|\mathbf{y}| \to \infty$.

By similar argument as the previous part, we note that the result (9.61) also holds if $\mathbf{y}_1$ is replaced by any $\mathbf{y}_j = (0, \ldots, 0, \frac{\pi}{\eta_j}, 0, \ldots, 0)$ with $\eta_j \neq 0$, where $j = 2, 3, \ldots, k$. Write it explicitly, we have

$$\|f_{\mathbf{0}} - f_{\mathbf{y}_j}\|_1 < 2\epsilon \tag{9.62}$$

for all $\mathbf{y}_j \in \mathbb{R}^k$ for which $|\eta_j| > \frac{\pi}{\delta}$, where $j = 1, 2, \ldots, k$. Now for arbitrary $\mathbf{y} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$, there is *at least* one j such that $|\eta_j| \geq \frac{|\mathbf{y}|}{k}$. Otherwise, we have $|\eta_j| < \frac{|\mathbf{y}|}{k}$ for every $j = 1, 2, \ldots, k$, but this means that $k^2 < 1$, a contradiction. Thus if we take

$$K = \left\{\mathbf{y} \in \mathbb{R}^k \,\middle|\, |\mathbf{y}| \leq \frac{k\pi}{\delta}\right\},$$

then K is compact and for $\mathbf{y} \notin K$, we have

$$|\eta_j| \geq \frac{|\mathbf{y}|}{k} > \frac{\pi}{\delta}$$

for some $1 \leq j \leq k$. Hence the inequality (9.62) always holds for this j and then we conclude from the inequality (9.61) that

$$|\widehat{f}(\mathbf{y})| < \epsilon$$

if $\mathbf{y} \notin K$. By Definition 3.16, $\widehat{f}$ vanishes at infinity, i.e., $\widehat{f} \in C_0(\mathbb{R}^k)$.

Our next target is to obtain the analogue of Proposition 9.8. Similar to the work in §9.7, we put

$$H(\mathbf{t}) = \exp[-(|t_1| + |t_2| + \cdots + |t_k|)],$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_k)$. Let $\lambda > 0$. Then $0 < H(\mathbf{t}) \leq 1$ and $H(\lambda\mathbf{t}) \to 1$ as $\lambda \to 0$. Define

$$h_\lambda(\mathbf{x}) = \int_{\mathbb{R}^k} H(\lambda\mathbf{t}) \exp(i\mathbf{t} \cdot \mathbf{x})\, \mathrm{d}m_k(\mathbf{t}).$$

A simple computation with repeated applications of Theorem 8.8 (The Fubini Theorem), it gives

$$h_\lambda(\mathbf{x}) = \prod_{j=1}^{k} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\lambda}{\lambda^2 + \xi_j^2}.$$

By Theorem 8.8 (The Fubini Theorem) and [51, Eqn. (4), p. 183], we get

$$\int_{\mathbb{R}^k} h_\lambda(\mathbf{x})\, \mathrm{d}m_k(\mathbf{x}) = \prod_{j=1}^{k} \left[\int_{\mathbb{R}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\lambda}{\lambda^2 + \xi_j^2}\, \mathrm{d}m(\xi_1)\right] = 1.$$

By this result, we establish the analogue of Proposition 9.8: If $f \in L^1(\mathbb{R}^k)$, then

$$(f * h_\lambda)(\mathbf{x}) = \int_{\mathbb{R}^k} H(\lambda\mathbf{t})\widehat{f}(\mathbf{t}) \exp(i\mathbf{x} \cdot \mathbf{t})\, \mathrm{d}m_k(\mathbf{t}). \tag{9.63}$$

Similarly, we have the following analogue of Theorem 9.10: If $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^k)$, then

$$\lim_{\lambda \to 0} \|f * h_\lambda - f\|_p = 0. \tag{9.64}$$

Now it is time to prove the required results.

- **The Inversion Theorem for $\mathbb{R}^k$.** Suppose that $f, \widehat{f} \in L^1(\mathbb{R}^k)$. The integrand on the right-hand side of the formula (9.63) are bounded by $|\widehat{f}(\mathbf{t})|$. By Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) and the fact that $H(\lambda\mathbf{t}) \to 1$ as $\lambda \to 0$, the right-hand side of the formula (9.63) converges to

$$g(\mathbf{x}) = \int_{\mathbb{R}^k} \widehat{f}(\mathbf{t}) \exp(i\mathbf{x} \cdot \mathbf{t})\, \mathrm{d}m_k(\mathbf{t}).$$

for all $\mathbf{x} \in \mathbb{R}^k$. By the limit (9.64) and Theorem 3.12, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and

$$\lim_{n\to\infty} (f * h_{\lambda_n})(\mathbf{x}) = f(\mathbf{x})$$

a.e. on $\mathbb{R}^k$. Consequently, we have $f(\mathbf{x}) = g(\mathbf{x})$ a.e. on $\mathbb{R}^k$. Since $\widehat{f} \in L^1(\mathbb{R}^k)$ and $g(-\mathbf{x})$ is the Fourier transform of $\widehat{f}$, the analogue of Theorem 9.6 above implies that $g \in C_0(\mathbb{R}^k)$ as desired.

- **The Plancherel Theorem for $\mathbb{R}^k$.** Fix $f \in L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$. Put $\widetilde{f}(\mathbf{x}) = \overline{f(-\mathbf{x})}$ and $g = f * \widetilde{f}$. Then we have

$$g(\mathbf{x}) = \int_{\mathbb{R}^k} f(\mathbf{x}-\mathbf{y})\overline{f(-\mathbf{y})}\, \mathrm{d}m_k(\mathbf{y}) = \int_{\mathbb{R}^k} f(\mathbf{x}+\mathbf{y})\overline{f}(\mathbf{y})\, \mathrm{d}m_k(\mathbf{y}) = \langle f_{-\mathbf{x}}, f\rangle .$$

By the use of Lemma 9.6 and Theorem 4.6, it can be shown that g is a continuous function on $\mathbb{R}^k$ and thus Theorem 4.2 (The Schwarz Inequality) gives

$$|g(\mathbf{x})| \leq \|f_{-\mathbf{x}}\|_2 \cdot \|f\|_2 = \|f\|_2^2$$

so that g is bounded on $\mathbb{R}^k$. Since $f \in L^1(\mathbb{R}^k)$, we have $\widetilde{f} \in L^1(\mathbb{R}^k)$ and then $g \in L^1(\mathbb{R}^k)$ by the analogue of Theorem 8.14 for $\mathbb{R}^k$.[i] By the formula (9.63), we have

$$(g * h_\lambda)(\mathbf{0}) = \int_{\mathbb{R}^k} H(\lambda\mathbf{t})\widehat{g}(\mathbf{t})\, \mathrm{d}m_k(\mathbf{t}). \tag{9.65}$$

Since g is continuous and bounded on $\mathbb{R}^k$, the analogue of Theorem 9.9[j] shows that

$$\lim_{\lambda\to 0} (g * h_\lambda)(\mathbf{0}) = g(\mathbf{0}) = \|f\|_2^2. \tag{9.66}$$

Now the analogues of Theorem 9.2(c) and (d) imply that

$$\widehat{g} = \widehat{f} \times \widehat{\widetilde{f}} = \widehat{f} \times \overline{\widehat{f}} = |\widehat{f}|^2 \geq 0.$$

Since $H(\lambda\mathbf{t})$ is measurable and increases to 1 as $\lambda \to 0$, Theorem 1.26 (Lebesgue's Monotone Convergence Theorem) implies that

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^k} H(\lambda\mathbf{t})\widehat{g}(\mathbf{t})\, \mathrm{d}m_k(\mathbf{t}) = \int_{\mathbb{R}^k} |\widehat{f}(\mathbf{t})|^2\, \mathrm{d}m_k(\mathbf{t}). \tag{9.67}$$

Hence it follows from the expression (9.65) and the two limits (9.66) and (9.67) that

$$\|\widehat{f}\|_2^2 = \lim_{\lambda\to 0} \int_{\mathbb{R}^k} H(\lambda\mathbf{t})\widehat{g}(\mathbf{t})\, \mathrm{d}m_k(\mathbf{t}) = \lim_{\lambda\to 0} (g * h_\lambda)(\mathbf{0}) = \|f\|_2^2 < \infty. \tag{9.68}$$

Suppose that

$$Y = \{\widehat{f} \mid f \in L^1(\mathbb{R}^k) \cap L^1(\mathbb{R}^k)\}.$$

[i] The analogue of Theorem 8.14 is that $f, g \in L^1(\mathbb{R}^k)$ implies $f * g \in L^1(\mathbb{R}^k)$, but we won't give a proof here.
[j] If $g \in L^\infty(\mathbb{R}^k)$ and g is continuous at a point $\mathbf{x} \in \mathbb{R}^k$, then $(g * h_\lambda)(\mathbf{x}) \to g(\mathbf{x})$ as $\lambda \to 0$.

The equality (9.68) implies that Y is a subspace of $L^2(\mathbb{R}^k)$. We claim that Y is dense in $L^2(\mathbb{R}^k)$. To this end, we recall from Problem 4.1 that $(Y^\perp)^\perp = \overline{Y}$. Now we observe that $Y^\perp = \{0\}$ if and only if

$$\overline{Y} = \{0\}^\perp = \{f \in L^2(\mathbb{R}^k) \mid \langle f, 0\rangle = 0\} = L^2(\mathbb{R}^k).$$

Thus it suffices to show that $\overline{Y} = \{0\}^\perp$. Let $\mathbf{a} \in \mathbb{R}^k$ and $\lambda > 0$. Denote the mapping $\mathbf{x} \mapsto \exp(i\mathbf{a}\cdot\mathbf{x})H(\lambda\mathbf{x})$ by F. It is clear that $F(\mathbf{x}) \in L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$ for all $\mathbf{x} \in \mathbb{R}^k$ and

$$\widehat{F}(\mathbf{t}) = \int_{\mathbb{R}^k} F(\mathbf{x}) \exp(-i\mathbf{x}\cdot\mathbf{t})\, \mathrm{d}m_k(\mathbf{t}) = \int_{\mathbb{R}^k} \exp\big(i(\mathbf{a}-\mathbf{t})\cdot\mathbf{x}\big)H(\lambda\mathbf{x})\, \mathrm{d}m_k(\mathbf{x}) = h_\lambda(\mathbf{a}-\mathbf{t})$$

so that $h_\lambda \in Y$. If $w \in Y^\perp = \{f \in L^2(\mathbb{R}^k) \mid \langle f, g\rangle = 0 \text{ for all } g \in Y\}$, then it follows that

$$(h_\lambda * \overline{w})(\mathbf{a}) = \int_{\mathbb{R}^k} h_\lambda(\mathbf{a}-\mathbf{t})\overline{w(\mathbf{t})}\, \mathrm{d}m_k(\mathbf{t}) = 0$$

for all $\mathbf{a} \in \mathbb{R}^k$. By the limit (9.64), we see that $w = 0$, i.e., $Y^\perp = \{0\}$ and thus $\overline{Y} = L^2(\mathbb{R}^k)$.

Now we define $\Phi : L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k) \to Y$ by

$$\Phi(f) = \widehat{f}.$$

Then the equality (9.68) certainly shows that Φ is an $L^2(\mathbb{R}^k)$-isometry. Since both $L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$ and Y are *dense* in $L^2(\mathbb{R}^k)$, as the paragraph before [51, Eqn. (10), p. 187] indicates that Φ can be extended to an isometry

$$\widetilde{\Phi} : L^2(\mathbb{R}^k) \to L^2(\mathbb{R}^k).$$

If we denote $\widehat{f} = \widetilde{\Phi}(f)$ (of course, it is the Plancherel transform), then the equality (9.68) holds for every $f \in L^2(\mathbb{R}^k)$. Using a similar argument as in the proof of Theorem 4.18, we can prove the Parserval formula

$$\langle f, g\rangle = \int_{\mathbb{R}^k} f(\mathbf{x})\overline{g(\mathbf{x})}\, \mathrm{d}m_k(\mathbf{x}) = \int_{\mathbb{R}^k} \widehat{f}(\mathbf{t})\overline{\widehat{g}(\mathbf{t})}\, \mathrm{d}m_k(\mathbf{t}) = \left\langle \widehat{f}, \widehat{g}\right\rangle$$

for every $f, g \in L^2(\mathbb{R}^k)$. Finally, for the symmetric relation in Theorem 9.13(d), the φ_A and ψ_A should be replaced by

$$\widetilde{\varphi_A}(\mathbf{t}) = \int_{[-A,A]^k} f(\mathbf{x}) \exp(-i\mathbf{x}\cdot\mathbf{t})\, \mathrm{d}m_k(\mathbf{x}) \quad \text{and} \quad \widetilde{\psi_A}(\mathbf{x}) = \int_{[-A,A]^k} \widehat{f}(\mathbf{t}) \exp(i\mathbf{x}\cdot\mathbf{t})\, \mathrm{d}m_k(\mathbf{t})$$

respectively. Employing a similar argument as shown in [51, p. 187], it can be verified that

$$\|\widetilde{\varphi_A} - \widehat{f}\|_2 \to 0 \quad \text{and} \quad \|\widetilde{\psi_A} - f\|_2 \to 0$$

as $A \to \infty$.

- **The analogue of Theorem 9.23.** To every complex homomorphism φ on $L^1(\mathbb{R}^k)$ (except for $\varphi = 0$) there corresponds a unique $\mathbf{t} \in \mathbb{R}^k$ such that

$$\varphi(f) = \widehat{f}(\mathbf{t}).$$

Here it is seen that the content up to Eqn. (6) in §9.22 can be extended to $\mathbb{R}^k$ easily. If $\mathbf{x} = \xi\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \cdots + \xi_k\mathbf{e}_k$, where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ is the usual basis of $\mathbb{R}^k$, then we have

$$\beta(\mathbf{x}) = \beta(\xi\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \cdots + \xi_k\mathbf{e}_k) = \beta(\xi_1\mathbf{e}_1) \times \beta(\xi_2\mathbf{e}_2) \times \cdots \times \beta(\xi_1\mathbf{e}_1). \tag{9.69}$$

Let $\beta_j : \mathbb{R} \to \mathbb{R}$ be defined by $\beta_j(\xi_j) = \beta(\xi_j \mathbf{e}_j)$ for $1 \leq j \leq k$. Since β is continuous on $\mathbb{R}^k$, each β_j is continuous on $\mathbb{R}$. Since β is not identity 0, each β_j is not identically 0. Furthermore, each β_j satisfies

$$\beta_j(x+y) = \beta\big((x+y)\mathbf{e}_j\big) = \beta(x\mathbf{e}_j)\beta(y\mathbf{e}_j) = \beta_j(x)\beta_j(y).$$

Thus each β_j satisfies the hypotheses qon [51, p. 192] which implies $\beta_j(\xi_j) = \exp(-it_j\xi_j)$ for a unique $t_j \in \mathbb{R}$. By the formula (9.69), we obtain

$$\beta(\mathbf{x}) = \exp(-it_1\xi_1) \times \exp(-it_2\xi_2) \times \cdots \times \exp(-it_k\xi_k) = \exp(-i\mathbf{t}\cdot\mathbf{x}),$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_k)$ and it is unique. Hence, we get from this that

$$\varphi(f) = \int_{\mathbb{R}^k} f(\mathbf{x})\beta(\mathbf{x})\,\mathrm{d}m_k(\mathbf{x}) = \int_{\mathbb{R}^k} f(\mathbf{x})\exp(-i\mathbf{t}\cdot\mathbf{x})\,\mathrm{d}m_k(\mathbf{x}) = \widehat{f}(\mathbf{t}).$$

We have completed the proof of the problem. ■

Problem 9.15

Rudin Chapter 9 Exercise 15.

Proof. Let $\mathbf{A} : \mathbb{R}^k \to \mathbb{R}^k$ be a linear operator with $\det \mathbf{A} > 0$. Thus $\mathbf{A}^{-1}$ exists. Let $\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x})$. Then we follow from Theorem 7.26 (The Change-of-variables Theorem) (with $X = Y = V = \mathbb{R}^k$, $T = \mathbf{A}^{-1}$ which is one-to-one and differentiable on $\mathbb{R}^k$, $T(\mathbb{R}^k) = \mathbf{A}^{-1}(\mathbb{R}^k) = \mathbb{R}^k$ and $\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x})$) that

$$\begin{aligned}
\widehat{g}(\mathbf{z}) &= \widehat{f\circ A}(\mathbf{z}) \\
&= \underbrace{\int_{\mathbb{R}^k} f\big(\mathbf{A}(\mathbf{y})\big)\exp(-i\mathbf{y}\cdot\mathbf{z})\,\mathrm{d}m_k(\mathbf{y})}_{\text{This is } \int_Y f\,\mathrm{d}m.} \\
&= \int_{\mathbb{R}^k} \underbrace{f(\mathbf{x})\exp[-i\mathbf{A}^{-1}(\mathbf{x})\cdot\mathbf{z}] \times |\det \mathbf{A}^{-1}|}_{\text{This is } (f\circ T)(\mathbf{x}) = (f\circ \mathbf{A}^{-1})(\mathbf{x}).}\,\mathrm{d}m_k(\mathbf{x})
\end{aligned} \tag{9.70}$$

for every $\mathbf{z} \in \mathbb{R}^k$. We need a result from linear algebra:

Lemma 9.7

If $\mathbf{A}$ is a $k \times k$ matrix, then for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^k$, we have $\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle = \big\langle \mathbf{x}, \mathbf{A}^T(\mathbf{y})\big\rangle$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.

Proof of Lemma 9.7. Using [41, p. 109; Example 5.3.1, p. 286], it can be seen easily that

$$\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle = [\mathbf{A}(\mathbf{x})]^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^T(\mathbf{y}) = \big\langle \mathbf{x}, \mathbf{A}^T(\mathbf{y})\big\rangle\,,$$

completing the proof of Lemma 9.7. ■

Thus we apply Lemma 9.7 to the expression (9.70) to deduce that

$$\widehat{g}(\mathbf{z}) = |\det \mathbf{A}| \cdot \int_{\mathbb{R}^k} f(\mathbf{x})\exp[-i\mathbf{x}\cdot\mathbf{A}^{-T}(\mathbf{z})]\,\mathrm{d}m_k(\mathbf{x}) = |\det \mathbf{A}| \cdot \widehat{f}\big(\mathbf{A}^{-T}(\mathbf{z})\big).$$

Consequently, we conclude that

$$\widehat{f\circ \mathbf{A}} = |\det \mathbf{A}|\cdot(\widehat{f}\circ \mathbf{A}^{-T}). \tag{9.71}$$

Particularly, let $\mathbf{A}$ be a rotation matrix of $\mathbb{R}^k$. Then it preserves the distance of $\mathbf{x}$ from the origin, i.e., $\langle \mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{x})\rangle = \langle \mathbf{x}, \mathbf{x}\rangle$. By [4, Proposition 5.1.13, p. 134], $\mathbf{A}$ is orthogonal which means

$$\mathbf{A}^T = \mathbf{A}^{-1}.$$

Furthermore, this implies that $|\det \mathbf{A}| = 1$, so the formula (9.71) can be simplified to

$$\widehat{f\circ \mathbf{A}} = \widehat{f}\circ \mathbf{A}. \tag{9.72}$$

In other words, if f is invariant under rotations, then $f\circ \mathbf{A} = f$ and the expression (9.72) implies

$$\widehat{f}\circ \mathbf{A} = \widehat{f},$$

i.e., $\widehat{f}$ is also invariant under rotations. This completes the proof of the problem. ■

Problem 9.16

Rudin Chapter 9 Exercise 16.

Proof. We make the following assumption: Suppose that $f\in S(\mathbb{R}^k)$, where $S(\mathbb{R}^k)$ is the class of all functions $f:\mathbb{R}^k\to\mathbb{C}$ such that $f\in C^\infty$ and $x^\beta\partial^\alpha f$ is bounded for all multi-indices α and β.[k] Let $1\leq j\leq k$, $\mathbf{x} = (\mathbf{x}_j, x_j)$ and $F(\mathbf{x}) = \dfrac{\partial f}{\partial x_j}$, where $\mathbf{x} = (x_1,\ldots,x_j,\ldots,x_k)$ and $\mathbf{x}_j = (x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k)$. We note that

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ky_k = \mathbf{x}_j\cdot\mathbf{y}_j + x_jy_j.$$

For every $\mathbf{y}\in\mathbb{R}^k$, we follow from this, the definition in Problem 9.14 and Lemma 9.4 that

$$\begin{aligned}
\widehat{F}(\mathbf{y}) &= \int_{\mathbb{R}^k}\frac{\partial f}{\partial x_j}(\mathbf{x})\exp(-i\mathbf{x}\cdot\mathbf{y})\,\mathrm{d}m_k(\mathbf{x})\\
&= \int_{\mathbb{R}^{k-1}}\exp(-i\mathbf{x}_j\cdot\mathbf{y}_j)\,\mathrm{d}m_{k-1}(\mathbf{x}_k)\underbrace{\Big[\int_{\mathbb{R}}\frac{\partial f}{\partial x_j}(\mathbf{x}_j,x_j)\exp(-ix_j\cdot y_j)\,\mathrm{d}m(x_j)\Big]}_{\text{Apply Lemma 9.4 to this.}}\\
&= \int_{\mathbb{R}^{k-1}}\exp(-i\mathbf{x}_j\cdot\mathbf{y}_j)\,\mathrm{d}m_{k-1}(\mathbf{x}_k)\Big[iy_j\int_{\mathbb{R}}f(\mathbf{x})\exp(-ix_j\cdot y_j)\,\mathrm{d}m(x_j)\Big]\\
&= iy_j\int_{\mathbb{R}^k}f(\mathbf{x})\exp(-i\mathbf{x}\cdot\mathbf{y})\,\mathrm{d}m_k(\mathbf{x})\\
&= iy_j\widehat{f}(\mathbf{y})
\end{aligned}$$

which implies that

$$\widehat{\frac{\partial^2 f}{\partial x_j^2}}(\mathbf{y}) = -y_j^2\widehat{f}(\mathbf{y}).$$

Therefore, we have

$$\widehat{g}(\mathbf{x}) = -\sum_{j=1}^k x_j^2\widehat{f}(\mathbf{x}) = -|\mathbf{x}|^2\cdot\widehat{f}(\mathbf{x}) \tag{9.73}$$

[k] In fact, S is the Schwartz space on $\mathbb{R}^k$, see Remark 9.2.

for all $\mathbf{x} \in \mathbb{R}^k$.

Suppose that f has continuous second derivatives so that the Laplacian Δf is well-defined. Suppose further that $\operatorname{supp}(f)$ is compact. In this case, $f \in L^1(\mathbb{R}^k)$ so that $\widehat{f}$ is well-defined on $\mathbb{R}^k$. Furthermore, we suppose that $\mathbf{A}$ is a rotation of $\mathbb{R}^k$ about the origin $\mathbf{0}$ and $g_{\mathbf{A}} = \Delta(f \circ \mathbf{A})$. Then we deduce from the formulas (9.72) and (9.73) that

$$\widehat{g_{\mathbf{A}}}(\mathbf{x}) = -|\mathbf{x}|^2 \cdot \widehat{f \circ \mathbf{A}}(\mathbf{x}) = -|\mathbf{A}\mathbf{x}|^2 \cdot (\widehat{f} \circ \mathbf{A})(\mathbf{x}) = -|\mathbf{A}\mathbf{x}|^2 \cdot \widehat{f}(\mathbf{A}\mathbf{x}) = \widehat{g}(\mathbf{A}\mathbf{x}) = \widehat{g \circ \mathbf{A}}(\mathbf{x}).$$

By the definition, we have $\Delta\widehat{(f \circ \mathbf{A})} = \widehat{(\Delta f) \circ \mathbf{A}}$. To go further, we need the analogue of Theorem 9.12 (The Uniqueness Theorem):

Lemma 9.8

If $f \in L^1(\mathbb{R}^k)$ and $\widehat{f}(\mathbf{t}) = 0$ for all $\mathbf{t} \in \mathbb{R}^k$, then

$$f(\mathbf{x}) = 0 \tag{9.74}$$

a.e. on $\mathbb{R}^k$. Furthermore, if f is continuous on $\mathbb{R}^k$, then the expression (9.74) holds everywhere on $\mathbb{R}^k$.

Proof of Lemma 9.8. Since $\widehat{f} = 0$ on $\mathbb{R}^k$, we have $\widehat{f} \in L^1(\mathbb{R}^k)$ and the result (9.74) follows from the Inversion Theorem for $\mathbb{R}^k$ (Problem 9.14). If f is continuous on $\mathbb{R}^k$, the definition of continuity ensures that the relation (9.74) holds everywhere on $\mathbb{R}^k$. ■

Now we let $F = \Delta(f \circ \mathbf{A}) - (\Delta f) \circ \mathbf{A}$. It is clear that $f \circ \mathbf{A} \in S(\mathbb{R}^k)$ if $f \in S(\mathbb{R}^k)$. Similar to the proof of Problem 9.7, we can show that

$$\partial^\alpha f \in S(\mathbb{R}^k) \quad \text{and} \quad \partial^\alpha f \in L^1(\mathbb{R}^k).$$

Thus we have $\Delta(f \circ \mathbf{A}) \in L^1(\mathbb{R}^k)$ and $(\Delta f) \circ \mathbf{A} \in L^1(\mathbb{R}^k)$ so that $F \in L^1(\mathbb{R}^k)$. Since $\widehat{F}(\mathbf{t}) = 0$ for all $\mathbf{t} \in \mathbb{R}^k$ and F is continuous on $\mathbb{R}^k$, Lemma 9.8 implies that

$$\Delta(f \circ \mathbf{A}) = (\Delta f) \circ \mathbf{A} \tag{9.75}$$

holds everywhere on $\mathbb{R}^k$. It is well-known that any rotation about a point $\mathbf{p}$ is the composition of a rotation about the origin $\mathbf{0}$ and a translation. By this fact, the analysis preceding Lemma 9.8 and the hypothesis that Δ commutes with translations, we conclude that the expression (9.75) is still valid for every rotation about every point $\mathbf{p}$.

It is time to consider the general situation. Fix a $\mathbf{p} \in \mathbb{R}^k$. There exists a $r > 0$ such that $\mathbf{p} \in K_r = \overline{B(\mathbf{0}, r)} \subset V_r = B(\mathbf{0}, 2r)$. By [49, Exercise 6, p. 289], there exists functions $\psi_1, \ldots, \psi_s \in C^\infty(\mathbb{R}^k)$ such that[1]

- $0 \leq \psi_i \leq 1$ for $1 \leq i \leq s$;
- $\operatorname{supp}(\psi_i) \subset V_r$;
- $\psi_1(\mathbf{x}) + \psi_2(\mathbf{x}) + \cdots + \psi_s(\mathbf{x}) = 1$ for every $\mathbf{x} \in K_r$.

If we define $\Psi : \mathbb{R}^k \to \mathbb{R}$ by

$$\Psi = \psi_1 + \psi_2 + \cdots + \psi_s,$$

[1] For a proof of this, please read [63, Problem 10.6, pp. 266, 267].

then the above conditions imply that $\Psi \in C_c^\infty(\mathbb{R}^k)$ and $\Psi(\mathbf{x}) = 1$ on K_r. Thus we have $f \cdot \Psi = f$ on K_r. Particularly, we have

$$(f \cdot \Psi)(\mathbf{p}) = f(\mathbf{p}) \quad \text{and} \quad \frac{\partial \Psi}{\partial x_j}(\mathbf{p}) = \frac{\partial^2 \Psi}{\partial x_j^2}(\mathbf{p}) = 0$$

for every $1 \leq j \leq k$. As a consequence, these imply that

$$\frac{\partial^2 (f \cdot \Psi)}{\partial x_j^2}(\mathbf{p}) = \Psi(\mathbf{p}) \cdot \frac{\partial^2 f}{\partial x_j^2}(\mathbf{p}) + 2\frac{\partial f}{\partial x_j}(\mathbf{p}) \cdot \frac{\partial \Psi}{\partial x_j}(\mathbf{p}) + f \cdot \frac{\partial^2 \Psi}{\partial x_j^2}(\mathbf{p}) = \frac{\partial^2 f}{\partial x_j^2}(\mathbf{p}) \tag{9.76}$$

for $1 \leq j \leq k$. By the definition of the Laplacian and the result (9.76), we yield

$$\big(\Delta(f \cdot \Psi)\big)(\mathbf{p}) = \sum_{j=1}^k \frac{\partial^2 (f \cdot \Psi)}{\partial x_j^2}(\mathbf{p}) = \sum_{j=1}^k \frac{\partial^2 f}{\partial x_j^2}(\mathbf{p}) = (\Delta f)(\mathbf{p}). \tag{9.77}$$

Note that $f \cdot \Psi$ is compactly supported in this case. If $\mathbf{Ap} = \mathbf{q}$, then $\mathbf{q} \in K_r$. On the one hand, the expressions (9.75) (with f replaced by $f \cdot \Psi$) and (9.77) give

$$\big[\Delta\big(f \cdot \Psi) \circ \mathbf{A})\big](\mathbf{p}) = [(\Delta(f \cdot \Psi)) \circ \mathbf{A}](\mathbf{p}) = \big(\Delta(f \cdot \Psi)\big)(\mathbf{q}) = (\Delta f)(\mathbf{q}) = [(\Delta f) \circ \mathbf{A}](\mathbf{p}). \tag{9.78}$$

On the other hand, since

$$[(f \cdot \Psi) \circ \mathbf{A}](\mathbf{p}) = [(f \circ \mathbf{A})(\mathbf{p})] \cdot [(\Psi \circ \mathbf{A})(\mathbf{p})] = (f \cdot \Psi)(\mathbf{q}) = f(\mathbf{q}) = (f \circ \mathbf{A})(\mathbf{p})$$

on K_r, we have $(f \cdot \Psi) \circ \mathbf{A} = f \circ \mathbf{A}$ on K_r so that

$$\big[\Delta\big((f \cdot \Psi) \circ \mathbf{A})\big](\mathbf{p}) = \big[\Delta(f \circ \mathbf{A})\big](\mathbf{p}) \tag{9.79}$$

Finally, by combining the expressions (9.78) and (9.79) and using the fact that $\mathbf{p}$ is arbitrary, we may conclude that our desired formula (9.75) also holds everywhere on $\mathbb{R}^k$ in this general case. This completes the proof of the problem. ■

Problem 9.17

Rudin Chapter 9 Exercise 17.

Proof. We prove the case for $\mathbb{R}^k$ directly. Suppose that $\varphi : \mathbb{R}^k \to \mathbb{C}$ is a Lebesgue measurable character. We define $\Phi : L^1(\mathbb{R}^k) \to \mathbb{C}$ by

$$\Phi(f) = \int_{\mathbb{R}^k} f(\mathbf{x})\varphi(\mathbf{x})\, \mathrm{d}m_k(\mathbf{x}). \tag{9.80}$$

Now direct computation gives

$$\Phi(f * g) = \int_{\mathbb{R}^k} (f * g)(\mathbf{x})\varphi(\mathbf{x})\, \mathrm{d}m_k(\mathbf{x}) = \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\varphi(\mathbf{x})\, \mathrm{d}m_k(\mathbf{y})\, \mathrm{d}m_k(\mathbf{x}). \tag{9.81}$$

Since φ is a character, we have $\varphi(\mathbf{x}) = \varphi(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})$. Substituting this into the expression (9.81), we see that

$$\begin{aligned}\Phi(f * g) &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\varphi(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})\, \mathrm{d}m_k(\mathbf{y})\, \mathrm{d}m_k(\mathbf{x}) \\ &= \int_{\mathbb{R}^k}\Big[\int_{\mathbb{R}^k} f(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x}-\mathbf{y})\, \mathrm{d}m_k(\mathbf{x})\Big] g(\mathbf{y})\varphi(\mathbf{y})\, \mathrm{d}m_k(\mathbf{y})\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^k} \Phi(f) g(\mathbf{y}) \varphi(\mathbf{y}) \, \mathrm{d}m_k(\mathbf{y}) \\
&= \Phi(f) \cdot \Phi(g).
\end{aligned}
$$

Therefore, Φ is a linear functional on $L^1(\mathbb{R}^k)$. In other words, Φ is a complex homomorphism of the Banach algebra $L^1(\mathbb{R}^k)$. Hence the analogue of Theorem 9.23 in Problem 9.14 shows that one may find a $\beta \in L^\infty(\mathbb{R}^k)$ and a unique $\mathbf{y} \in \mathbb{R}^k$ such that

$$\beta(\mathbf{x}) = \exp(-i\mathbf{y} \cdot \mathbf{x}) \quad \text{and} \quad \Phi(f) = \int_{\mathbb{R}^k} f(\mathbf{x}) \exp(-i\mathbf{y} \cdot \mathbf{x}) \, \mathrm{d}m_k(\mathbf{x}) = \widehat{f}(\mathbf{y}). \tag{9.82}$$

Next, we deduce from the formulas (9.80) and (9.82) that

$$\int_{\mathbb{R}^k} f(\mathbf{x}) [\varphi(\mathbf{x}) - \exp(-i\mathbf{y} \cdot \mathbf{x})] \, \mathrm{d}m_k(\mathbf{x}) = 0 \tag{9.83}$$

holds for all $f \in L^1(\mathbb{R}^k)$. Since $f(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|}$ is in $L^1(\mathbb{R}^k)$ and $f(\mathbf{x}) \neq 0$ on $\mathbb{R}^k$, the result (9.83) implies that

$$\varphi(\mathbf{x}) = \exp(-i\mathbf{y} \cdot \mathbf{x}) \tag{9.84}$$

a.e. on $\mathbb{R}^k$. To proceed further, we need the following result which is a generalization of Problem 7.6 and its proof uses the **Steinhaus Theorem** in $\mathbb{R}^k$ (see Remark 7.2), but we don't present it here.

Lemma 9.9

Suppose that G is a subgroup of $\mathbb{R}^k$ (relative to addition), $G \neq \mathbb{R}^k$, and G is Lebesgue measurable. Then $m_k(G) = 0$.

To apply this lemma, we consider $h(\mathbf{x}) = \varphi(\mathbf{x}) - \exp(-i\mathbf{y} \cdot \mathbf{x})$ and $G = \{\mathbf{x} \in \mathbb{R}^k \mid h(\mathbf{x}) = 0\}$. For $\mathbf{a}, \mathbf{b} \in G$, if $\mathbf{a} + \mathbf{b} \notin G$, then we have $h(\mathbf{a} + \mathbf{b}) \neq 0$ and

$$\varphi(\mathbf{a})\varphi(\mathbf{b}) = \varphi(\mathbf{a} + \mathbf{b}) \neq \exp[-i\mathbf{y} \cdot (\mathbf{a} + \mathbf{b})] = \exp(-i\mathbf{y} \cdot \mathbf{a}) \exp(-i\mathbf{y} \cdot \mathbf{b}) = \varphi(\mathbf{a})\varphi(\mathbf{b})$$

which implies $1 \neq 1$, a contradiction. Thus $\mathbf{a} + \mathbf{b} \in G$ which means G is a subgroup of $\mathbb{R}^k$ (relative to vector addition). Besides, since h is obviously Lebesgue measurable by Proposition 1.9(c) and $G = h^{-1}(0)$, G is a Lebesgue measurable set. Since $m_k(G) > 0$, Lemma 9.9 forces that

$$G = \mathbb{R}^k$$

so that the equation (9.84) holds for all $\mathbf{x} \in \mathbb{R}^k$. Now the continuity of the exponential function implies that φ is continuous on $\mathbb{R}^k$. This completes the proof of the problem. ■

9.4 Miscellaneous Problems

Problem 9.18

Rudin Chapter 9 Exercise 18.

Proof. First of all, the equation

$$f(x + y) = f(x) + f(y) \tag{9.85}$$

for all $x, y \in \mathbb{R}$ is called the **Cauchy functional equation**.

- **Existence of real discontinuous functions f satisfying the equation (9.85).** It is well-known that the Hausdorff Maximality Theorem, the Axiom of Choice and Zorn's Lemma are equivalent to each other. To prove this part, we would like to apply Zorn's Lemma which says that

Lemma 9.10 (Zorn's Lemma)

If X is a partially ordered set and every totally ordered subset of X has an upper bound, then X has a maximal element.

As a consequence, Zorn's Lemma can be used to prove that every vector space has a basis. In fact, if A is a linearly independent subset of a vector space V over a field K, then *there is* a basis $\mathcal{B}$ of V that contains A.[m] By this result, there exists a basis of $\mathbb{R}$ over $\mathbb{Q}$ with the usual addition and scalar multiplication.[n]

Let $x \in \mathbb{R}$ and H be a basis of $\mathbb{R}$ over $\mathbb{Q}$. Thus x has a unique representation

$$x = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n,$$

where $b_1, b_2, \ldots, b_n \in H$ and $r_1, r_2, \ldots, r_n \in \mathbb{Q}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = r_1 + r_2 + \cdots + r_n.$$

It is obvious that f satisfies the equation (9.85). Assume that f was continuous on $\mathbb{R}$. Then $f(\mathbb{R})$ must be connected (see [49, Theorem 4.22, p. 93]), but $f(\mathbb{R}) = \mathbb{Q}$ which is a contradiction.

- **If f is Lebesgue measurable and satisfies the equation (9.85), then f is continuous.** We use the following special form of Theorem 2.24 (Lusin's Theorem):[o]

Lemma 9.11

Suppose that $f : [a, b] \to \mathbb{R}$ is Lebesgue measurable. For every $\epsilon > 0$, there is a compact set $K \subseteq [a, b]$ such that $m([a, b] \setminus K) < \epsilon$ and $f|_K$ is continuous.

By Lemma 9.11, there exists a compact set $K \subseteq [0, 1]$ such that

$$m(K) > \frac{2}{3} \tag{9.86}$$

and $f|_K$ is continuous. Since K is compact, f is actually uniformly continuous on K. Given $\epsilon > 0$. There is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every $x, y \in K$ with $|x - y| < \delta$. Without loss of generality, we may assume that $\delta < \frac{1}{3}$. Let $\eta \in (0, \delta)$ and $K - \eta = \{x - \eta \,|\, x \in K\}$. Then $K - \eta \subseteq [-\eta, 1 - \eta]$. Assume that $K \cap (K - \eta) = \varnothing$. Since $K, K - \eta \subseteq [-\eta, 1]$, we have

$$m(K) + m(K - \eta) = m\big(K \cup (K - \eta)\big) \leq m([-\eta, 1]) = 1 + \eta. \tag{9.87}$$

By Theorem 2.20(c), $m(K - \eta) = m(K)$, so we deduce from the inequalities (9.86) and (9.87) that $1 + \eta \geq 2m(K) > \frac{4}{3}$ and thus $\eta > \frac{1}{3}$, a contradiction. Hence we have

$$K \cap (K - \eta) \neq \varnothing.$$

[m]For a hint of this proof, please refer to [42, Exercise 8, p. 72]
[n]This kind of basis is called a **Hamel basis**.
[o]Read [22, Exercise 44, p. 64] or [47, p. 66]

Let $p \in K \cap (K - \eta)$. Then $p, p + \eta \in K$ and $|p + \eta - p| = \eta < \delta$ so that

$$|f(\eta) - f(0)| = |f(p) + f(\eta) - f(p)| = |f(p + \eta) - f(p)| < \epsilon. \tag{9.88}$$

In other words, f is continuous at 0. Now for any $x \in [0, 1]$, the inequality (9.88) implies clearly that

$$|f(x + \eta) - f(x)| = |f(x) + f(\eta) - f(x)| = |f(\eta) - f(0)| < \epsilon.$$

Hence f is continuous on $[0, 1]$.

Next, if $z \in [1, 2]$, then we have $z = x+y$ for some $x, y \in [0, 1]$. Since $f(z) = f(x)+f(y)$, we have

$$|f(z + \eta) - f(z)| = |f(x + \eta) - f(x)| < \epsilon$$

and the above paragraph shows that f is also continuous on $[1, 2]$. Now this kind of argument can be applied repeatedly and we conclude that f is continuous on $\mathbb{R}$.

- **If** graph (f) **is not dense in** $\mathbb{R}^2$ **and satisfies the equation (9.85), then** f **is continuous.** Suppose that f is discontinuous. Then f *cannot* be of the form $f(x) = cx$ for any constant c. We have

$$\text{graph}\,(f) = \{\big(x, f(x)\big) \mid x \in \mathbb{R}\}.$$

Choose a nonzero real number x_1. Then there exists another nonzero real number x_2 such that

$$\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}$$

or equivalently,

$$\begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix} \neq 0.$$

In other words, the vectors $\mathbf{u} = \big(x_1, f(x_1)\big)$ and $\mathbf{v} = \big(x_2, f(x_2)\big)$ are linearly independent and thus span the space $\mathbb{R}^2$. Let $\mathbf{p} \in \mathbb{R}^2$. Given $\epsilon > 0$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, one can find $q_1, q_2 \in \mathbb{Q}$ such that

$$|\mathbf{p} - q_1\mathbf{u} - q_2\mathbf{v}| < \epsilon.$$

Clearly, we have

$$q_1\mathbf{u} + q_2\mathbf{v} = \big(q_1x_1 + q_2x_2, f(q_1x_1 + q_2x_2)\big).$$

Therefore, the set

$$G = \{\big(x, f(x)\big) \in \mathbb{R}^2 \mid x = q_1x_1 + q_2x_2 \text{ and } q_1, q_2 \in \mathbb{Q}\}$$

is dense in $\mathbb{R}^2$. Since $G \subseteq$ graph (f), graph (f) is dense in $\mathbb{R}^2$.

- **The form of all continuous functions satisfying the equation (9.85).** In fact, by a similar argument as in the proof used in [63, Problem 8.6, p. 178], it can be shown that the function f satisfies

$$f(r) = rf(1)$$

for every $r \in \mathbb{Q}$. Now for every $x \in \mathbb{R}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a sequence $\{r_n\} \subseteq \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$. Since f is continuous, we have

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_nf(1) = xf(1).$$

Hence every continuous solution of the equation (9.85) must be in the form

$$f(x) = f(1) \cdot x.$$

We have completed the proof of the problem. ■

Problem 9.19

Rudin Chapter 9 Exercise 19.

Proof. Let $f = \chi_A$ and $g = \chi_B$. Since $f, g \in L^1$, their convolution $h = f * g$ is well-defined by Theorem 8.14. Suppose that p and q are conjugate exponents. Direct computation gives

$$\|f\|_p = \int_{-\infty}^{\infty} |\chi_A(x)|^p \, dm(x) = \frac{1}{\sqrt{2\pi}} \int_A dx = \frac{m(A)}{\sqrt{2\pi}} < \infty$$

and

$$\|g\|_q = \int_{-\infty}^{\infty} |\chi_B(x)|^q \, dm(x) = \frac{1}{\sqrt{2\pi}} \int_B dx = \frac{m(B)}{\sqrt{2\pi}} < \infty.$$

Thus we have $f \in L^p$ and $g \in L^q$. By Problem 9.8, $h = \chi_A * \chi_B$ is (uniformly) continuous on $\mathbb{R}$.

We consider

$$\int_{\mathbb{R}} (f * g)(x) \, dm(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)g(y) \, dm(y) \, dm(x). \tag{9.89}$$

Recall that $\mathbb{R}$ is σ-finite. If we set $F(x,y) = f(x-y)g(y) = \chi_A(x-y)\chi_B(y) \geq 0$ on $\mathbb{R}^2$, then F satisfies the hypotheses of Theorem 8.8 (The Fubini Theorem). We notice that

$$\psi(y) = \int_{\mathbb{R}} F^y \, dm(x) = \int_{\mathbb{R}} f(x-y)g(y) \, dm(x) = g(y) \int_{\mathbb{R}} f(x-y) \, dm(x) = g(y) \int_{\mathbb{R}} f(x) \, dm(x),$$

so the integral on the right-hand side in the expression (9.89) can be simplified to

$$\begin{aligned}
\int_{\mathbb{R}} (f * g)(x) \, dm(x) &= \int_{\mathbb{R}} \psi(y) \, dm(y) \\
&= \int_{\mathbb{R}} \left[g(y) \int_{\mathbb{R}} f(x) \, dm(x) \right] dm(y) \\
&= \left\{ \int_{\mathbb{R}} f(x) \, dm(x) \right\} \times \left\{ \int_{\mathbb{R}} g(y) \, dm(y) \right\} \\
&= \frac{1}{2\pi} m(A)m(B) > 0.
\end{aligned}$$

Hence it is impossible that $\chi_A * \chi_B = 0$.

Recall the definition that

$$(\chi_A * \chi_B)(x) = \int_{-\infty}^{\infty} \chi_A(x-y)\chi_B(y) \, dm(y). \tag{9.90}$$

Since $y \in B$ if and only if $\chi_B(y) = 1$, the integral (9.90) reduces to

$$(\chi_A * \chi_B)(x) = \int_B \chi_A(x-y) \, dm(y). \tag{9.91}$$

Similarly, $x - y \in A$ if and only if $\chi_A(x-y) = 1$. Define $x - A = \{x - a \,|\, a \in A\}$. We notice that $x - y \in A$ implies that $y \in x - A$ and then $y \in B \cap (x - A)$. These facts show that the integral (9.91) can be further simplified to

$$(\chi_A * \chi_B)(x) = \int_{B \cap (x-A)} dm(y) = m\big(B \cap (x - A)\big) \geq 0. \tag{9.92}$$

By the previous paragraph, there exists a $x_0 \in \mathbb{R}$ such that $(\chi_A * \chi_B)(x_0) > 0$, so the expression (9.92) gives

$$m\big(B \cap (x_0 - A)\big) > 0.$$

In other words, we have $B \cap (x_0 - A) \neq \varnothing$ and we can find a $b \in B$ such that $x_0 = a + b$ for some $a \in A$. Consequently, we have

$$x_0 \in A + B. \tag{9.93}$$

Finally, since $\chi_A * \chi_B$ is continuous at x_0, there exists a neighborhood I around x_0 such that $(\chi_A * \chi_B)(x) > 0$ on I. Hence we conclude from the relation (9.93) that

$$I \subseteq A + B,$$

completing the proof of the problem. ■

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A Complete Solution Guide to Real and Complex Analysis II

by Kit-Wing Yu, PhD

kitwing@hotmail.com

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About the author

Dr. Kit-Wing Yu received his B.Sc. (1st Hons), M.Phil. and Ph.D. degrees in Math. at the HKUST, PGDE (Mathematics) at the CUHK. After his graduation, he joined United Christian College (UCC) to serve as a mathematics teacher between 2000 and 2020. From 2002 to 2020, he also took the responsibility of the mathematics panel at UCC. Starting from Sept. 2020, Dr. Yu was promoted to be the Vice Principal (Academic) at Kiangsu-Chekiang College (Kwai Chung). Furthermore, he was appointed as a part-time tutor (2002 – 2005) and then a part-time course coordinator (2006 – 2010) of the Department of Mathematics at the OUHK.

Besides teaching, Dr. Yu has been appointed to be a marker of the HKAL Pure Mathematics and HKDSE Mathematics (Core Part) for over thirteen years. Between 2012 and 2014, Dr. Yu was invited to be a Judge Member by the World Olympic Mathematics Competition (China). In the research aspect, he has published research papers in international mathematical journals, including some well-known journals such as J. Reine Angew. Math., Proc. Roy. Soc. Edinburgh Sect. A and Kodai Math. J.. His research interests are inequalities, special functions and Nevanlinna's value distribution theory. In the area of academic publication, he is the author of the following seven books:

- *A Complete Solution Guide to Complex Analysis*
- *A Complete Solution Guide to Real and Complex Analysis I*
- *A Complete Solution Guide to Principles of Mathematical Analysis*
- *Problems and Solutions for Undergraduate Real Analysis*
- *Problems and Solutions for Undergraduate Real Analysis I*
- *Problems and Solutions for Undergraduate Real Analysis II*
- *Mock Tests for the ACT Mathematics*

Preface

This is the continuum of my book *A Complete Solution Guide to Real and Complex Analysis I*. It covers the "Complex Analysis" part of Rudin's graduate book. In fact, we study all exercises of Chapters 10 to 20.

Same as *A Complete Solution Guide to Real and Complex Analysis I*, the primary aim of this book is to help every mathematics student and instructor to understand the ideas and applications of the theorems in Rudin's book. To accomplish this goal, I have adopted the way I wrote the solution guides of *Baby Rudin* and the first part of *Papa Rudin*. In other words, I intend writing the solutions as comprehensive as I can so that you can understand *every* detailed part of a proof easily. Apart from this, I also keep reminding you what theorems or results I have applied by quoting them *repeatedly* in the proofs. By doing this, I believe that you will become fully aware of the meaning and applications of each theorem.

Before you read this book, I have two gentle reminders for you. Firstly, as a mathematics instructor at a college, I understand that the growth of a mathematics student depends largely on how hard he/she does exercises. When your instructor asks you to do some exercises from Rudin, you are not suggested to read my solutions unless you have tried your best to prove them seriously yourselves. Secondly, when I prepared this book, I found that some exercises require knowledge that Rudin did not cover in his book. To fill this gap, I refer to some other analysis or topology books such as [2], [9], [15], [18], [23], [42] and [65]. Other useful references are [3], [27], [28], [37], [69], [79], [80], [82] and [83]. Of course, we will use the exercises in *Baby Rudin* and the first part of *Papa Rudin* freely and if you want to read proofs of them, you are strongly advised to read my books [77] and [78].

As you will expect, this book always keeps the main features of my previous books [77] and [78]. In fact, its features are as follows:

- It covers all the 221 exercises from Chapters 10 to 20 with *detailed* and *complete* solutions. As a matter of fact, my solutions show every detail, every step and every theorem that I applied.
- There are 29 illustrations for explaining the mathematical concepts or ideas used behind the questions or theorems.
- Sections in each chapter are added so as to increase the readability of the exercises.
- Different colors are used frequently in order to highlight or explain problems, lemmas, remarks, main points/formulas involved, or show the steps of manipulation in some complicated proofs. (ebook only)
- Necessary lemmas with proofs are provided because some questions require additional mathematical concepts which are not covered by Rudin.

- Many useful or relevant references are provided to some questions for your future research.

Since the solutions are written solely by me, you may find typos or mistakes. If you really find such a mistake, please send your valuable comments or opinions to

`kitwing@hotmail.com`.

Then I will post the updated errata on my website

`https://sites.google.com/view/yukitwing/`

irregularly.

Kit Wing Yu
April 2021

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CHAPTER 10

Elementary Properties of Holomorphic Functions

10.1 Basic Properties of Holomorphic Functions

Problem 10.1

Rudin Chapter 10 Exercise 1.

Proof. In fact, this is [61, Exercise 21, p. 101]. For a solution of it, please refer to [77, p. 75]. This completes the proof of the problem. ■

Problem 10.2

Rudin Chapter 10 Exercise 2.

Proof. Let $a \in \mathbb{C}$ and

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n. \tag{10.1}$$

By [62, Eqn. (8), p. 199], we have $n!c_n = f^{(n)}(a)$ for all $n = 0, 1, 2, \ldots$ and so the hypothesis implies that $f^{(n)}(a) = 0$ for some $n \in \mathbb{N} \cup \{0\}$.

For every $n \in \mathbb{N} \cup \{0\}$, let $Z_n = \{z \in \mathbb{C} \,|\, f^{(n)}(z) = 0\} \subseteq \mathbb{C}$. Now the previous paragraph implies that

$$\mathbb{C} = \bigcup_{n=0}^{\infty} Z_n. \tag{10.2}$$

Since f is entire, every $f^{(n)}$ is also entire. If $f^{(n)} \not\equiv 0$ for every $n \in \mathbb{N} \cup \{0\}$, then $Z_n \neq \mathbb{C}$ for every $n \in \mathbb{N} \cup \{0\}$ by Theorem 10.18. Furthermore, each Z_n is at most countable and it deduces from the set relation (10.2) that $\mathbb{C}$ is countable, a contradiction. Thus there exists an $N \in \mathbb{N}$ such that $f^{(n)} \equiv 0$ for all $n > N$ and so the representation (10.1) implies that f is a polynomial of degree at most N, completing the proof of the problem. ■

Problem 10.3

Rudin Chapter 10 Exercise 3.

Proof. We claim that $f(z) = cg(z)$ for some $|c| \leq 1$. If $g \not\equiv 0$ in $\mathbb{C}$, then we follow from Theorem 10.18 that $Z(g)$ has *no* limit point in $\mathbb{C}$. Consider $h : \mathbb{C} \setminus Z(g) \to \mathbb{C}$ given by

$$h(z) = \frac{f(z)}{g(z)}.$$

For each $a \in Z(g)$, we have $h \in H(D'(a; r))$ and $|h(z)| \leq 1$ in $D'(a, r)$ for some $r > 0$. Now we follow from Theorem 10.20 that h has a removable singularity at a and then h can be defined at a so that it is holomorphic in $D(a; r)$. Since it is true for every $a \in Z(g)$, h is in fact entire and $|h(z)| \leq 1$ in $\mathbb{C}$. Therefore, Theorem 10.23 (Liouville's Theorem) asserts that $h(z) = c$ for some constant c such that $|c| \leq 1$. Consequently, we obtain $f(z) = cg(z)$ as required. This proves the claim and we end the proof of the problem. ■

Problem 10.4

Rudin Chapter 10 Exercise 4.

Proof. For every $n = 0, 1, 2, \ldots$, we apply the hint given in Problem 10.2, Theorem 10.26 (The Cauchy's Estimates) and the hypothesis to get

$$|c_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leq \frac{A + R^k}{R^n}, \tag{10.3}$$

where $R > 0$. If $n > k$, then we take $R \to \infty$ to both sides of the inequality (10.3) to conclude $c_n = 0$ for all $n > k$. In other words, f is a polynomial of degree at most k. This completes the proof of the problem. ■

Problem 10.5

Rudin Chapter 10 Exercise 5.

Proof. Since $\{f_n\}$ is uniformly bounded in an open subset $\Omega \subseteq \mathbb{C}$, there exists a $M > 0$ such that $|f_n(z)| \leq M$ for all $z \in \Omega$ and all $n \in \mathbb{N}$. Let $a \in \Omega$ and $f(z) = \lim_{n\to\infty} f_n(z)$. Since Ω is open in $\mathbb{C}$, one can find a $\delta > 0$ such that $D(a; 4\delta) \subseteq \Omega$. Define $\gamma : [0, 2\pi] \to \Omega$ by

$$\gamma(t) = a + 2\delta e^{it} \tag{10.4}$$

which is clearly a circle centered at a with radius 2δ. If $z \in D(a; \delta)$, then we have $|\gamma(t) - z| > \delta$ for all $t \in [0, 2\pi]$. Therefore, we get from this fact and the representation (10.4) that

$$\left|\frac{\gamma'(t)}{\gamma(t) - z}\right| < \frac{2\delta}{\delta} = 2 \tag{10.5}$$

for all $t \in [0, 2\pi]$.

Next, for every $n \in \mathbb{N}$, we define $F_n : [0, 2\pi] \to \mathbb{C}$ and $F : [0, 2\pi] \to \mathbb{C}$ by

$$F_n(t) = f_n(\gamma(t)) \quad \text{and} \quad F(t) = f(\gamma(t))$$

respectively. Clearly, the pointwise convergence of $\{f_n\}$ on Ω implies the pointwise convergence of $\{F_n\}$ on $[0, 2\pi]$. Besides, since all f_n and γ are continuous functions, every F_n is complex measurable on the measurable space $[0, 2\pi]$. Since $2M \in L^1(m)$ and $|F_n(t)| \leq M$ for all $n \in \mathbb{N}$, we deduce from Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) that

$$\lim_{n\to\infty} \int_0^{2\pi} F_n(t)\,\mathrm{d}t = \int_0^{2\pi} F(t)\,\mathrm{d}t. \tag{10.6}$$

Now the bound (10.5) ensures that $\frac{\gamma'(t)}{\gamma(t)-z}$ is continuous on $[0, 2\pi]$ so that *for all* $z \in D(a;\delta)$, we certainly have

$$F_n(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z} \to F(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}$$

pointwisely on $[0, 2\pi]$ and each $F_n(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}$ is complex measurable on $[0, 2\pi]$. Since $2M \in L^1(m)$ and $|F_n(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}| \leq 2M$ for all $n \in \mathbb{N}$, further application of Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) also gives

$$\lim_{n\to\infty} \int_0^{2\pi} F_n(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t = \int_0^{2\pi} F(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t. \tag{10.7}$$

Recall that $z \notin \gamma^*$, so $\mathrm{Ind}_\gamma(z) = 1$. Then we obtain from Theorem 10.15 (The Cauchy's Formula in a Convex Set) and the limit (10.7) that[a]

$$\begin{aligned}
f(z) &= \lim_{n\to\infty} f_n(z) \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_0^{2\pi} f_n(\gamma(t)) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_0^{2\pi} F_n(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t \\
&= \frac{1}{2\pi i} \int_0^{2\pi} F(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t.
\end{aligned} \tag{10.8}$$

Given that $\epsilon > 0$. The result (10.6) guarantees that there is an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\int_0^{2\pi} |F_n(t) - F(t)|\,\mathrm{d}t < \epsilon\pi.$$

In this case, *for every* $z \in D(a, \delta)$, we obtain from the expression (10.8) that

$$\begin{aligned}
|f_n(z) - f(z)| &= \Big|\frac{1}{2\pi i} \int_0^{2\pi} F_n(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t - \frac{1}{2\pi i} \int_0^{2\pi} F(t) \cdot \frac{\gamma'(t)}{\gamma(t)-z}\,\mathrm{d}t\Big| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |F_n(t) - F(t)| \cdot \Big|\frac{\gamma'(t)}{\gamma(t)-z}\Big|\,\mathrm{d}t \\
&< \epsilon
\end{aligned}$$

for all $n \geq N$. In other words, $\{f_n\}$ converges uniformly to f on $D(a;\delta)$.

Finally, every compact $K \subseteq \Omega$ can be covered by $D(a_1, \delta), D(a_2, \delta), \ldots, D(a_m, \delta)$ for some $m \in \mathbb{N}$. Thus the previous analysis shows immediately that for all $z \in D(a_k, \delta)$, the inequality

$$|f_n(z) - f(z)| < \epsilon \tag{10.9}$$

[a] We remark that we cannot apply Theorem 10.15 (The Cauchy's Formula in a Convex Set) directly to f to obtain the representation (10.8) because it is not clear whether $f \in H(\Omega)$ or not.

holds if $n \geq N_k$, where $k = 1, 2, \ldots, m$. Set $N = \max(N_1, N_2, \ldots, N_m)$. Therefore, if $n \geq N$, then the inequality (10.9) remains valid on $D(a_1, \delta) \cup D(a_2, \delta) \cup \cdots \cup D(a_m, \delta)$, so particularly on K. Hence we have shown that the convergence is in fact uniform on every compact subset of Ω. This completes the proof of the problem. ■

Problem 10.6

Rudin Chapter 10 Exercise 6.

Proof. We divide the proof into several parts.

- **The existence of Ω.** Since $f(z) = \frac{1}{z}$ is continuous on $D(1; 1)$, we follow from Theorem 10.14 (The Cauchy's Theorem in a Convex Set) that one can find an $F_0 \in H(D(1; 1))$ such that $F_0'(z) = \frac{1}{z}$ in $D(1; 1)$. If $F(z) = F_0(z) - F_0(1)$, then we also have $F'(z) = F_0'(z) = \frac{1}{z}$ in $D(1; 1)$ and $F(1) = 0$. Next, we define $g : D(1; 1) \to \mathbb{C}$ by
$$g(z) = \frac{\mathrm{e}^{F(z)}}{z}. \tag{10.10}$$
Direct computation gives
$$g'(z) = \frac{z\mathrm{e}^{F(z)}F'(z) - \mathrm{e}^{F(z)}}{z^2} = 0$$
for all $z \in D(1; 1)$. Using [9, Exercise 5, p. 42], we know that g is a constant in $D(1; 1)$. Obviously, we have $g(1) = \mathrm{e}^{F(1)} = 1$, so it is true that $g(z) = 1$ in $D(1; 1)$. Hence it follows from the form (10.10) that
$$\mathrm{e}^{F(z)} = z \tag{10.11}$$
for all $z \in D(1; 1)$.

- **The injection of exp.** Set $\Omega = F(D(1; 1))$. We claim that Ω is a region such that
$$\exp(\Omega) = D(1; 1). \tag{10.12}$$
To this end, since $F'(z) = \frac{1}{z}$, Ω is not a point set. Then Ω is a region follows from the Open Mapping Theorem directly. Finally, the expression (10.11) implies that the set equality (10.12). Furthermore, the expression (10.11) implies immediately that exp is one-to-one in Ω.

- **The number of such region Ω.** For each $n \in \mathbb{Z}$, we define $\Omega_n = \{z + 2n\pi i \,|\, z \in \Omega\}$ which is also a region and satisfies
$$\exp(\Omega_n) = \exp(\Omega + 2n\pi i) = \exp(\Omega) = D(1; 1).$$
Since F is continuous on $D(1; 1)$, Ω must be bounded. Therefore, there exists an $N \in \mathbb{N}$ such that $\Omega_N \neq \Omega$. Hence $\{\Omega_{kN} \,|\, k \in \mathbb{Z}\}$ is a set of distinct regions satisfying the set equality (10.12).

- **The derivative of $\log z$.** Define $\log z$, for $|z - 1| < 1$, such that
$$\mathrm{e}^{\log z} = z. \tag{10.13}$$
By differentiating the equation (10.13), we get $\log'(z)\mathrm{e}^{\log z} = 1$ and so
$$\log'(z) = \frac{1}{\mathrm{e}^{\log z}} = \frac{1}{z}.$$

- **The coefficients a_n and c_n.** It is easy to see that
$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n,$$
so we derive that $a_n = (-1)^n$ for all $n = 0, 1, 2, \ldots$. To find c_n, we first consider the power series
$$h(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n.$$
Since $\limsup_{n\to\infty}\frac{1}{\sqrt[n]{n}} = 1$, the radius of convergence of the power series h is 1, so termwise differentiation can be performed to obtain[b]
$$h'(z) = \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^{n-1} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n = \frac{1}{z}.$$
Since $\log'(z) = \frac{1}{z}$, we have $h(z) = \log z + C$ for some constant C. Since $h(1) = \log 1 = 0$, we get $C = 0$ and
$$\log z = h(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n.$$
Hence we establish
$$c_n = \begin{cases} \dfrac{(-1)^{n-1}}{n}, & \text{if } n \geq 1; \\ \\ 0, & \text{if } n = 0. \end{cases}$$

- **What other discs can this be done?** This can be done in every disc $D(a;|a|)$, where $a \in \mathbb{C}\setminus\{0\}$. In fact, we pick $b \in \mathbb{C}$ such that $e^b = a$. By similar argument as the proof of the first assertion, there exists an $F_1 \in H(D(a;|a|))$ such that $F_1'(z) = \frac{1}{z}$ in $D(a;|a|)$. If we set $F_2(z) = F_1(z) - F_1(a) + b$, then we also have
$$F_2'(z) = F_1'(z) = \frac{1}{z} \quad \text{and} \quad F_2(a) = b$$
in $D(a;|a|)$. Furthermore, the function $G : D(a;|a|) \to \mathbb{C}$ given by
$$G(z) = \frac{e^{F_2(z)}}{z} \tag{10.14}$$
satisfies $G'(z) = 0$ in $D(a;|a|)$. Thus G is a constant in $D(a;|a|)$ and since $G(a) = 1$, we conclude that $G(z) = 1$ in $D(a;|a|)$ and we get from the definition (10.14) that
$$e^{F_2(z)} = z$$
in $D(a;|a|)$. According to this construction, all the above assertions can be proven similarly and we won't repeat the argument here.

We have completed the analysis of the problem. ■

Problem 10.7

Rudin Chapter 10 Exercise 7.

[b] Read [9, Theorem 2.9, pp. 28, 32] for details.

Proof. The conditions should be that Γ is a cycle in the open set Ω and

$$\operatorname{Ind}_\Gamma(\alpha) = \begin{cases} 0, & \text{if } \alpha \notin \Omega; \\ 1, & \text{if } \alpha \in \Omega \setminus \Gamma^*. \end{cases}$$

In fact, with the above hypotheses, we get from Theorem 10.35 (Cauchy's Theorem) that

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for all $z \in \Omega \setminus \Gamma^*$. This proves the formula for the case $n = 0$. Assume that

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta \tag{10.15}$$

on $\Omega \setminus \Gamma^*$ for some $k \in \mathbb{N} \cup \{0\}$. Next, we define $g : \Omega \setminus \{z\} \to \mathbb{C}$ by

$$g(\zeta) = \frac{f(\zeta)}{(\zeta - z)^{k+1}},$$

where $\zeta \in \Omega \setminus \Gamma^*$. Then it is clear that

$$g'(\zeta) = \frac{f'(\zeta)}{(\zeta - z)^{k+1}} - \frac{(k+1)f(\zeta)}{(\zeta - z)^{k+2}}. \tag{10.16}$$

Since $f \in H(\Omega)$, we have $f' \in H(\Omega)$ and thus the formula (10.16) ensures that $g \in H(\Omega \setminus \{z\})$. Notice that $\Omega \setminus \{z\}$ is also an open set in $\mathbb{C}$, so a combined application of Theorem 10.35 (Cauchy's Theorem) and the formula (10.16) implies that

$$\int_\Gamma g(\zeta)\,d\zeta = 0$$

$$\int_\Gamma \frac{f'(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta = (k+1) \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+2}}\,d\zeta. \tag{10.17}$$

Applying the induction step (10.15) to the left-hand side of the formula (10.17) (with f replaced by f'), we see immediately that

$$\frac{2\pi i}{k!} f^{(k+1)}(z) = (k+1) \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+2}}\,d\zeta$$

which gives the desired result for the case $k + 1$. By induction, we have completed the proof of the problem. ■

10.2 Evaluation of Integrals

Problem 10.8

Rudin Chapter 10 Exercise 8.

Proof. We note that

$$\int_{-\infty}^{\infty} R(x)\,dx = \lim_{A \to \infty} \int_{-A}^{A} \frac{P(x)}{Q(x)}\,dx. \tag{10.18}$$

Let Γ_A be the closed contour consisting of the real segment $[-A, A]$ and the upper semi-circle $C_A = \{z \in \mathbb{C} \mid |z| = A \text{ and } \operatorname{Im} z > 0\}$ positively oriented, see Figure 10.1

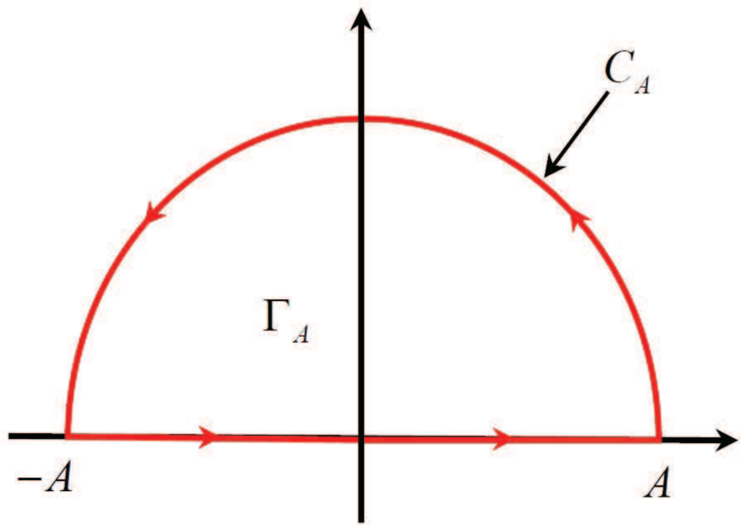


Figure 10.1: The closed contour Γ_A.

Clearly, if A is large enough, then Γ_A will contain all zeros of $Q(z)$ lying in the upper half plane. Hence it follows from Theorem 10.42 (The Residue Theorem) that

$$\int_{\Gamma_A} R(z)\,\mathrm{d}z = 2\pi i \sum_k \mathrm{Res}\,(R; z_k), \tag{10.19}$$

where $\{z_k\}$ is the set of all zeros of Q in the upper half plane. In fact, we can write the expression (10.19) in the form

$$\int_{C_A} R(z)\,\mathrm{d}z + \int_{-A}^{A} R(x)\,\mathrm{d}x = 2\pi i \sum_k \mathrm{Res}\,(R; z_k).$$

Since C_A is a semi-circle of radius A, its length is πA. Using this fact and $\deg Q - \deg P \geq 2$, we obtain from the estimate [62, Eqn. (5), Definition 10.8, p. 202] that

$$\Big|\int_{C_A} R(z)\,\mathrm{d}z\Big| \leq \frac{M}{A^2}\cdot \pi A = \frac{\pi M}{A} \tag{10.20}$$

for some positive constant M. Taking $A \to \infty$ in the inequality (10.20), we get

$$\lim_{A\to\infty} \int_{C_A} R(z)\,\mathrm{d}z = 0. \tag{10.21}$$

Finally, we combine the results (10.18), (10.19) and (10.21) to conclude that

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_k \mathrm{Res}\,(R; z_k). \tag{10.22}$$

For the analogous statement for the lower half plane, the formula (10.22) will be replaced by

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = -2\pi i \sum_k \mathrm{Res}\,(R; z_k),$$

where the set $\{z_k\}$ now consists of all zeros of Q in the lower half plane.[c] To compute the integral, we note from the formula (10.22) and some basic facts of calculating residues[d] that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,\mathrm{d}x = 2\pi i\Big[\mathrm{Res}\Big(\frac{z^2}{1+z^4}; \exp\Big(\frac{\pi i}{4}\Big)\Big) + \mathrm{Res}\Big(\frac{z^2}{1+z^4}; \exp\Big(\frac{3\pi i}{4}\Big)\Big)\Big]$$

[c] Here we have the negative sign in the formula because the corresponding semi-circle in the lower half plane is negatively oriented.

[d] See, for examples, [9, pp. 129, 130] or [65, pp. 75, 76].

$$
\begin{aligned}
&= \frac{\pi i}{2}\Big[\exp\Big(-\frac{\pi i}{4}\Big) + \exp\Big(-\frac{3\pi i}{4}\Big)\Big] \\
&= \frac{\pi i}{2} \times \frac{-2i}{\sqrt{2}} \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}
$$

This completes the analysis of the problem. ■

Problem 10.9

Rudin Chapter 10 Exercise 9.

Proof. Let Γ_A be the closed contour consisting of the real segment $[-A, A]$ and the upper semi-circle $C_A = \{z \in \mathbb{C} \,|\, |z| = A \text{ and } \operatorname{Im} z > 0\}$, see Figure 10.1. Furthermore, we let P and Q be polynomials such that $\deg Q - \deg P \geq 1$, $Q(x) \neq 0$ (except perhaps at zeros of $\cos x$ or $\sin x$) and $R(x) = \frac{P(x)}{Q(x)}$. By the discussion of **Type II** integrals in [9, pp. 144 – 146], we see that

$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{ix}\,\mathrm{d}x = \lim_{A\to\infty}\int_{\Gamma_A} R(z)\mathrm{e}^{iz}\,\mathrm{d}z = 2\pi i\sum_k \operatorname{Res}\big(R(z)\mathrm{e}^{iz}; z_k\big), \tag{10.23}$$

where the points z_k are the poles of $R(z)$ in the upper half plane.

Suppose that $t \geq 0$. By the substitution $y = tx$, we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{itx}}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{t}{t^2+y^2}\mathrm{e}^{iy}\,\mathrm{d}y.$$

Set $R(z) = \frac{t}{t^2+z^2}$. Then it follows from the representation (10.23) that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{itx}}{1+x^2}\,\mathrm{d}x = 2\pi i\operatorname{Res}\Big(\frac{t}{t^2+z^2}\mathrm{e}^{iz}; ti\Big) = \frac{\pi}{\mathrm{e}^t}. \tag{10.24}$$

Next, if $t = -u$ for some $u > 0$, then we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{e}^{itx}}{1+x^2}\,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-iux}}{1+x^2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{u}{u^2+y^2}\mathrm{e}^{iy}\,\mathrm{d}y \\
&= 2\pi i\operatorname{Res}\Big(\frac{u}{u^2+z^2}\mathrm{e}^{iz}; ui\Big) \\
&= \frac{\pi}{\mathrm{e}^u}.
\end{aligned} \tag{10.25}
$$

Combining the two expressions (10.24) and (10.25), we conclude that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{itx}}{1+x^2}\,\mathrm{d}x = \frac{\pi}{\mathrm{e}^{|t|}}. \tag{10.26}$$

Using the theory of Fourier transforms, we notice that if $f(t) = \sqrt{\frac{\pi}{2}}\mathrm{e}^{-|t|}$, then we know from Definition 9.1 that

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-ixt}\,\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|t|}\mathrm{e}^{-ixt}\,\mathrm{d}t \\
&= \frac{1}{2}\Big[\int_{-\infty}^{0} \mathrm{e}^{(1-ix)t}\,\mathrm{d}t + \int_{0}^{\infty} \mathrm{e}^{-(1+ix)t}\,\mathrm{d}t\Big] \\
&= \frac{1}{2}\cdot\Big\{\frac{\exp[(1-ix)t]}{1-ix}\Big|_{-\infty}^{0} + \frac{\exp[-(1+ix)t]}{-1+ix}\Big|_{0}^{\infty}\Big\} \\
&= \frac{1}{2}\Big(\frac{1}{1-ix} + \frac{1}{1+ix}\Big) \\
&= \frac{1}{1+x^2}.
\end{aligned}
$$

Now it is clear that $f, \widehat{f} \in L^1(\mathbb{R})$, so we follow from Theorem 9.11 (The Inversion Theorem) that $g(t) = f(t)$, where

$$g(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(x)\mathrm{e}^{ixt}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ixt}}{1+x^2}\,\mathrm{d}x.$$

Consequently, we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ixt}}{1+x^2}\,\mathrm{d}x = \sqrt{2\pi}\cdot\sqrt{\frac{\pi}{2}}\mathrm{e}^{-|t|} = \frac{\pi}{\mathrm{e}^{|t|}}$$

which is consistent with the result (10.26). Hence we have completed the proof of the problem. ■

Problem 10.10

Rudin Chapter 10 Exercise 10.

Proof. Let $f(z) = (\mathrm{e}^z - \mathrm{e}^{-z})z^{-4}$. Then f is holomorphic in $\mathbb{C}\setminus\{0\}$. By the power series expansion of e^z (see [62, Eqn. (1), p. 1]), we have

$$\frac{\mathrm{e}^z - \mathrm{e}^{-z}}{z^4} = 2\Big(\frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \cdots\Big)$$

so that f has a pole of order 3 at 0. By Theorem 10.21(b), the difference

$$f(z) - 2\Big(\frac{1}{z^3} + \frac{1}{3!z}\Big)$$

has a removable singularity at 0. Thus there exists an entire function g such that[e]

$$f(z) - 2\Big(\frac{1}{z^3} + \frac{1}{3!z}\Big) = g(z)$$

which gives

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \frac{1}{\pi i}\int_\gamma \Big(\frac{1}{z^3} + \frac{1}{3!z}\Big)\,\mathrm{d}z + \frac{1}{2\pi i}\int_\gamma g(z)\,\mathrm{d}z. \tag{10.27}$$

Applying Theorems 10.10 and 10.12 to the right-hand side of the expression (10.27), we establish that

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{z^4}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \frac{1}{6\pi i}\int_\gamma \frac{\mathrm{d}z}{z} = \frac{1}{3}.$$

This ends the proof of the problem. ■

[e]Obviously, we have

$$g(z) = 2\sum_{n=2}^{\infty} \frac{z^{2n-3}}{(2n+1)!}.$$

Problem 10.11

Rudin Chapter 10 Exercise 11.

Proof. Since $|\alpha| \neq 1$, we have either $|\alpha| < 1$ or $|\alpha^{-1}| < 1$. If $z = \mathrm{e}^{i\theta}$, then $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$. Using [9, Eqn. (5), p. 150], we see that

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2\alpha\cos\theta + \alpha^2} &= \int_{|z|=1} \frac{1}{1 - 2\alpha \cdot \frac{1}{2}(z + \frac{1}{z}) + \alpha^2} \cdot \frac{\mathrm{d}z}{iz} \\
&= i \int_{|z|=1} \frac{\mathrm{d}z}{\alpha z^2 - (1+\alpha^2)z + \alpha} \\
&= \begin{cases} -2\pi \mathrm{Res}\left(\dfrac{1}{\alpha z^2 - (1+\alpha^2)z + \alpha}; \alpha\right), & \text{if } |\alpha| < 1; \\ \\ -2\pi \mathrm{Res}\left(\dfrac{1}{\alpha z^2 - (1+\alpha^2)z + \alpha}; \alpha^{-1}\right), & \text{if } |\alpha^{-1}| < 1 \end{cases} \\
&= \begin{cases} -\dfrac{2\pi}{\alpha^2 - 1}, & \text{if } |\alpha| < 1; \\ \\ \dfrac{2\pi}{\alpha^2 - 1}, & \text{if } |\alpha^{-1}| < 1. \end{cases}
\end{aligned}$$

Hence we complete the analysis of the problem. ∎

Problem 10.12

Rudin Chapter 10 Exercise 12.

Proof. Let Γ_A be the path obtained by going from $-A$ to -1 along the real axis, from -1 to 1 along the lower half of the unit circle C and from 1 to A along the real axis, see Figure 10.2 below.

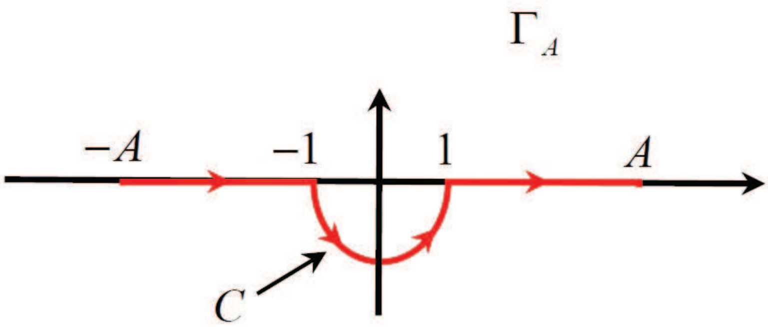


Figure 10.2: The contour Γ_A.

We note that

$$\int_{-A}^{A} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x = \int_{-A}^{-1} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x + \int_{-1}^{1} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x + \int_{1}^{A} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x. \tag{10.28}$$

Since $z^{-2} \cdot \sin^2 z \cdot \mathrm{e}^{itz}$ is entire for every $t \in \mathbb{R}$, it follows from Theorem 10.14 (The Cauchy's Theorem in a Convex Set) that

$$\int_C \frac{\sin^2 z}{z^2}\mathrm{e}^{itz}\,\mathrm{d}z + \int_1^{-1} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x = 0$$

or equivalently,

$$\int_{-1}^{1} \frac{\sin^2 x}{x^2} \mathrm{e}^{itx}\,\mathrm{d}x = \int_C \frac{\sin^2 z}{z^2} \mathrm{e}^{itz}\,\mathrm{d}z. \tag{10.29}$$

Combing the integral relations (10.28) and (10.29), we see immediately that

$$\int_{-A}^{A} \frac{\sin^2 x}{x^2} \mathrm{e}^{itx}\,\mathrm{d}x = \int_{\Gamma_A} \frac{\sin^2 z}{z^2} \mathrm{e}^{itz}\,\mathrm{d}z. \tag{10.30}$$

Next, we write $2i \sin z = \mathrm{e}^{iz} - \mathrm{e}^{-iz}$ so that

$$\int_{\Gamma_A} \frac{\sin^2 z}{z^2} \mathrm{e}^{itz}\,\mathrm{d}z = \int_{\Gamma_A} \frac{\mathrm{e}^{2iz} - 2 + \mathrm{e}^{-2iz}}{-4z^2} \mathrm{e}^{itz}\,\mathrm{d}z. \tag{10.31}$$

Now we define

$$\varphi_A(s) = \int_{\Gamma_A} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z, \tag{10.32}$$

so the expression (10.31) becomes

$$\int_{\Gamma_A} \frac{\sin^2 z}{z^2} \mathrm{e}^{itz}\,\mathrm{d}z = -\frac{1}{4}[\varphi_A(t+2) + \varphi_A(t-2)] + \frac{1}{2}\varphi_A(t). \tag{10.33}$$

If we combine (10.30) and (10.33), then we have

$$\int_{-A}^{A} \frac{\sin^2 x}{x^2} \mathrm{e}^{itx}\,\mathrm{d}x = \frac{1}{2}\varphi_A(t) - \frac{1}{4}[\varphi_A(t+2) + \varphi_A(t-2)]. \tag{10.34}$$

Complete Γ_A to a closed path in two different ways: Firstly, we consider the semi-circle Γ_1 from A to $-Ai$ and then to $-A$; secondly, we consider the semi-circle Γ_2 from A to Ai and then to $-A$, see Figure 10.3.

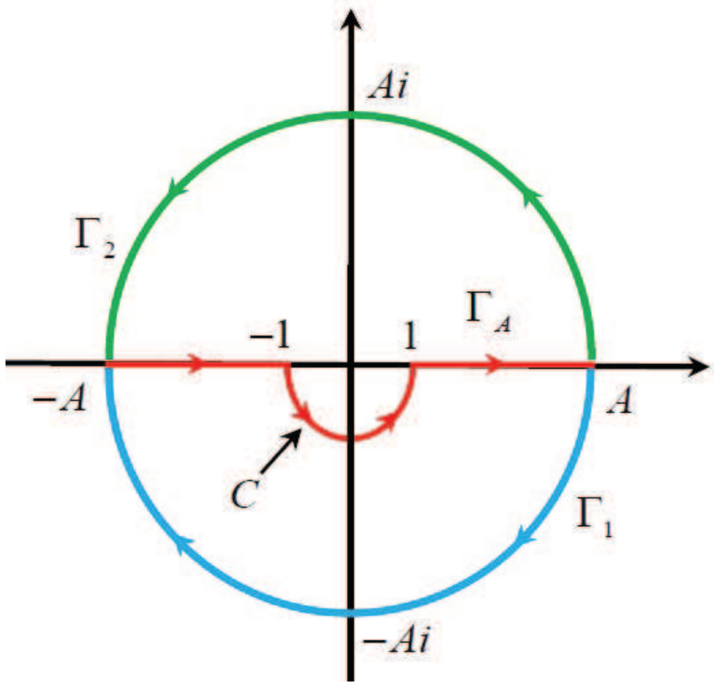


Figure 10.3: The contours Γ_A, Γ_1 and Γ_2.

It is easily checked that the function $\mathrm{e}^{isz} \cdot z^{-2}$ has a pole of order 2 at 0, and so the residue is is. Thus in the first way, we have

$$\int_{\Gamma_A} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z + \int_{\Gamma_1} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z = 0$$

so that if $z = A\mathrm{e}^{i\theta}$, where $\theta \in [-\pi, 0]$, then we deduce from the definition (10.32) that

$$\varphi_A(s) = \int_{\Gamma_A} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z = i\int_{-\pi}^{0} \frac{\exp(isA\mathrm{e}^{i\theta})}{A\mathrm{e}^{i\theta}}\,\mathrm{d}\theta \tag{10.35}$$

and in the second way, Theorem 10.42 (The Residue Theorem) yields

$$\int_{\Gamma_A} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z + \int_{\Gamma_2} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z = 2\pi i \mathrm{Res}\left(\frac{\mathrm{e}^{isz}}{z^2}; 0\right) = -2\pi s$$

which implies that

$$\varphi_A(s) = \int_{\Gamma_A} \frac{\mathrm{e}^{isz}}{z^2}\,\mathrm{d}z = -2\pi s - i\int_{0}^{\pi} \frac{\exp(isA\mathrm{e}^{i\theta})}{A\mathrm{e}^{i\theta}}\,\mathrm{d}\theta. \tag{10.36}$$

Since

$$\left|\frac{\exp(isA\mathrm{e}^{i\theta})}{A\mathrm{e}^{i\theta}}\right| \leq \frac{\exp(-As\sin\theta)}{A} \to 0$$

as $A \to \infty$ if s and $\sin\theta$ have the same sign. Thus it follows from Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) that the integral (10.35) tends to 0 if $s < 0$ and the one in (10.36) tends to 0 if $s > 0$. In other words, we obtain

$$\lim_{A\to\infty} \varphi_A(s) = \begin{cases} -2\pi s, & \text{if } s > 0; \\ 0, & \text{if } s < 0. \end{cases} \tag{10.37}$$

Finally, we apply the result (10.37) to the expression (10.34) to get

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x = \begin{cases} 0, & \text{if } |t| > 2; \\ \pi - \dfrac{\pi t}{2}, & \text{if } -2 < t < 0 \text{ or } 0 < t < 2. \end{cases} \tag{10.38}$$

When $t = 0$, we know from the integral (10.35) that $\varphi_A(0) = -\frac{2}{A}$, so we establish from the expression (10.34) that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x = \begin{cases} \pi, & \text{if } t = 0; \\ 0, & \text{if } t = \pm 2. \end{cases} \tag{10.39}$$

By combining the results (10.38) and (10.39), we achieve

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\mathrm{e}^{itx}\,\mathrm{d}x = \begin{cases} 0, & \text{if } |t| \geq 2; \\ \pi - \dfrac{\pi t}{2}, & \text{if } -2 < t < 0 \text{ or } 0 < t < 2; \\ \pi, & \text{if } t = 0. \end{cases}$$

We have completed the proof of the problem. ■

Problem 10.13

Rudin Chapter 10 Exercise 13.

Proof. This has been solved on [76, p. 143] which completes the proof of the problem. ■

10.3 Composition of Holomorphic Functions and Morera's Theorem

Problem 10.14

Rudin Chapter 10 Exercise 14.

Proof. Both cases can be negative. Let $\Omega_1 = \mathbb{C} \setminus \{0\}$ and $\Omega_2 = \mathbb{C}$. Define $f(z) = z$ in Ω_1 and

$$g(z) = \begin{cases} z, & \text{if } z \neq 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then $f(\Omega_1) \subseteq \Omega_2$ and $h(z) = g(f(z)) = g(z) = z$ in Ω_1. Hence f and h are holomorphic in Ω_1, but g is discontinuous in Ω_2.

Next, we consider $\Omega_1 = \Omega_2 = \mathbb{C}$. Define

$$f(z) = \begin{cases} -1, & \text{if } z \neq 0; \\ 1, & \text{otherwise} \end{cases}$$

and $g(z) = z^2$. Obviously, we have $f(\Omega_1) = \{\pm 1\} \subseteq \Omega_2$ and $h(z) = g(f(z)) = 1$ for all $z \in \Omega_1$. It is clear that both g and h are holomorphic in Ω_2 and Ω_1 respectively, but f is not continuous in Ω_1. This ends the analysis of the proof. ■

Remark 10.1

A problem similar to Problem 10.14 but for (uniform) continuity has been discussed in [61, Exercise 26, p. 102].

Problem 10.15

Rudin Chapter 10 Exercise 15.

Proof. According to Theorem 10.18, we have $f(z) = (z - \omega_0)^m h(z)$, where $h \in H(\varphi(\Omega))$ and $h(\omega_0) \neq 0$. Then we have

$$g(z) = f(\varphi(z)) = [\varphi(z) - \varphi(z_0)]^m h(\varphi(z)). \tag{10.40}$$

Suppose that $n \geq 1$ is the order of the zero of $\varphi(z) - \varphi(z_0)$. Now Theorem 10.18 implies that

$$\varphi(z) - \varphi(z_0) = (z - z_0)^n \phi(z),$$

where $\phi \in H(\Omega)$ and $\phi(z_0) \neq 0$. Assume that $n \geq 2$. We follow from

$$\varphi'(z) = (z - z_0)^{n-1}[(z - z_0)\phi'(z) + \phi(z)] \tag{10.41}$$

that $\varphi'(z_0) = 0$, a contradiction. Consequently, $n = 1$ and we can write the expression (10.40) as

$$g(z) = (z - z_0)^m \cdot \phi^m(z) h(\varphi(z)). \tag{10.42}$$

Finally, since $\phi^m(z_0)h(\varphi(z_0)) = \phi^m(z_0)h(\omega_0) \neq 0$, the representation (10.42) ensures that g has a zero of order m at z_0.

If φ' has a zero of order k at z_0, then the expression (10.41) will imply that $n = k+1$ so the representation (10.40) becomes

$$g(z) = (z - z_0)^{m(k+1)} \cdot \phi^m(z) h(\varphi(z)).$$

In conclusion, g has a zero of order $m(k+1)$ at z_0. This completes the proof of the problem. ∎

Problem 10.16

Rudin Chapter 10 Exercise 16.

Proof. Since φ is bounded on $\Omega \times X$, there exists a $M > 0$ such that $|\varphi(z,t)| \leq M$ for all $(z,t) \in \Omega \times X$. Let $z_0 \in \Omega$. Since Ω is open in $\mathbb{C}$, there exists a $\epsilon > 0$ such that $D(z_0; 3\epsilon) \subseteq \Omega$. Then we have $\overline{D}(z_0; 2\epsilon) \subseteq \Omega$.

We claim that for every pair $z, \omega \in D(z_0; \epsilon)$, $z \neq \omega$ and $p \in X$, we have

$$\Big|\frac{\varphi(z,p) - \varphi(\omega,p)}{z - \omega}\Big| \leq \frac{2M}{\epsilon}. \tag{10.43}$$

To this end, we consider the closed curve $\gamma(t) = z_0 + 2\epsilon e^{it}$, where $t \in [0, 2\pi]$. Obviously, since $\varphi(z,t) \in H(D(z_0; 3\epsilon))$ for each $t \in X$, we establish from Theorem 10.15 (The Cauchy's Formula in a Convex Set) that if $z \in D(z_0; 2\epsilon) \subseteq D(z_0; 3\epsilon)$ and $p \in X$, then $z \notin \gamma^*$ and

$$\varphi(z;p) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta,p)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\varphi(\gamma(t),p)}{\gamma(t) - z} \cdot \gamma'(t)\,dt. \tag{10.44}$$

Note that $z \in D(z_0; \epsilon)$ implies $|\gamma(t) - z| > \epsilon$. Thus we follow from the formula (10.44) that

$$\begin{aligned}
\Big|\frac{\varphi(z,p) - \varphi(\omega,p)}{z - \omega}\Big| &\leq \frac{1}{2\pi|z-\omega|} \cdot \int_0^{2\pi} \Big|\frac{1}{\gamma(t) - z} - \frac{1}{\gamma(t) - \omega}\Big| \cdot |\varphi(\gamma(t),p)| \cdot |\gamma'(t)|\,dt \\
&\leq \frac{1}{2\pi|z-\omega|} \cdot \int_0^{2\pi} \Big|\frac{z - \omega}{\big(\gamma(t) - z\big)\big(\gamma(t) - \omega\big)}\Big| \cdot 2M\epsilon\,dt \\
&\leq \frac{M\epsilon}{\pi} \cdot \int_0^{2\pi} \frac{dt}{|\gamma(t) - z| \cdot |\gamma(t) - \omega|} \\
&\leq \frac{M\epsilon}{\epsilon^2 \pi} \int_0^{2\pi} dt \\
&= \frac{2M}{\epsilon}
\end{aligned}$$

which is exactly the inequality (10.43).

Recall that μ is a complex measure, so Theorem 6.12 tells us that there is a measurable function h such that $|h(x)| = 1$ in X and $d\mu = h\,d|\mu|$. This fact ensures that

$$\frac{2M}{\epsilon}\int_X d|\mu| = \frac{2M|\mu|(X)}{\epsilon} < \infty,$$

i.e., $\frac{2M}{\epsilon} \in L^1(|\mu|)$. Suppose that $\{z_n\} \subseteq D(z_0; \epsilon) \setminus \{z_0\}$ satisfies $z_n \to z_0$. Define

$$g_n(x) = \frac{\varphi(z_n, x) - \varphi(z_0, x)}{z_n - z_0} \cdot h(x).$$

By the hypotheses, we know that each g_n is measurable of x and

$$\lim_{n\to\infty} g_n(x) = \varphi'(z_0, x) \cdot h(x).$$

In other words, the sequence $\{g_n\}$ satisfies the conditions of Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem), so we conclude that $\varphi'(z_0, x) \cdot h(x) \in L^1(|\mu|)$ and furthermore,

$$\begin{aligned}\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} &= \lim_{n\to\infty} \int_X \frac{\varphi(z_n, x) - \varphi(z_0, x)}{z_n - z_0} \cdot h(x)\,\mathrm{d}|\mu| \\ &= \lim_{n\to\infty} \int_X g_n(x)\,\mathrm{d}|\mu| \\ &= \int_X \varphi'(z_0, x) \cdot h(x)\,\mathrm{d}|\mu|.\end{aligned}$$

Consequently, $f'(z_0)$ exists. Since z_0 is arbitrary, we establish that $f \in H(\Omega)$, completing the proof of the problem. ■

Problem 10.17

Rudin Chapter 10 Exercise 17.

Proof. We apply Problem 10.16 to the functions one by one.

- **The function $f(z)$.** We write

$$f(z) = \int_X \varphi(z, t)\,\mathrm{d}m,$$

were $X = [0, 1]$ and $\varphi(z, t) = (1 + tz)^{-1}$. Notice that if $z_0 \in (-\infty, -1]$, then $\varphi(z_0, -\frac{1}{z_0})$ is unbounded. Thus it is reasonable to take $\Omega = \mathbb{C}\setminus(-\infty, -1]$. Now the function $\varphi(z, t)$ satisfies all hypotheses of Problem 10.16 except the boundedness condition because $\varphi(z, 1) \to \infty$ as $z \to -1$ in Ω. Instead, it is really bounded locally. In fact, for every $z \in \Omega$, there exists a $\delta > 0$ such that $D(z; \delta) \subset D(z; 2\delta) \subseteq \Omega$ so that φ is bounded on $\overline{D}(z; \delta) \times X$. Hence Problem 10.16 implies that $f \in H(D(z; \delta))$. Since z is arbitrary, it yields that $f \in H(\Omega)$.

- **The function $g(z)$.** We have

$$g(z) = \int_X \varphi(z, t)\,\mathrm{d}m,$$

where $X = [0, \infty)$ and $\varphi(z, t) = \mathrm{e}^{tz}(1 + t^2)^{-1}$. Since $|\mathrm{e}^{tz}| = \mathrm{e}^{t\mathrm{Re}\, z}$, if $\mathrm{Re}\,(z_0) > 0$, then

$$\lim_{t\to\infty} |\varphi(z_0, t)| = \lim_{t\to\infty} \frac{\mathrm{e}^{t\mathrm{Re}\,(z_0)}}{1 + t^2} = \infty.$$

In other words, $\varphi(z, t)$ is unbounded in any set containing a point of the right half plane. Thus we may take Ω to be the left half plane which is an open set in $\mathbb{C}$. We want to apply Morera's Theorem and Fubini's Theorem.[f]

Let $z_0, z_n \in \Omega$ for all $n \in \mathbb{N}$, where $z_n \to z_0$ as $n \to \infty$. Fix $t \in [0, \infty)$, then it is easy to see that $\varphi(z, t)$ is continuous at z_0. In addition, we know that

$$|\varphi(z, t)| \leq \frac{1}{1 + t^2} \tag{10.45}$$

[f]Problem 10.16 cannot be applied directly in this case because $m(X) = \infty$, i.e., m is *not* a complex measure on X.

for all $z \in \Omega$ and $\frac{1}{1+t^2} \in L^1(m)$. Then we deduce from Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) that

$$\lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} \int_X \varphi(z_n, t)\,\mathrm{d}m = \int_X \varphi(z_0, t)\,\mathrm{d}m = g(z_0),$$

i.e., g is continuous at z_0. Since z_0 is arbitrary, g is continuous in Ω.

Next, we suppose that $\Delta \subseteq \Omega$ is a closed triangle and $\partial\Delta$ is parameterized by a piecewise continuously differentiable curve $\gamma : [a, b] \to \Omega$. We have

$$\int_{\partial\Delta} g(z)\,\mathrm{d}z = \int_a^b g(\gamma(x))\gamma'(x)\,\mathrm{d}x = \int_a^b \int_0^\infty \varphi(\gamma(x), t)\gamma'(x)\,\mathrm{d}t\,\mathrm{d}x. \tag{10.46}$$

Define $\phi(x, t) = \varphi(\gamma(x), t) \cdot \gamma'(x)$ on $[a, b] \times [0, \infty)$. Clearly, both $[a, b]$ and $[0, \infty)$ are σ-finite measure spaces. Since γ is piecewise continuously differentiable on $[a, b]$, there exists a $M > 0$ such that

$$|\gamma'(x)| \leq M$$

on $[a, b]$. Furthermore, $\phi(x, t)$ is piecewise continuous on $[a, b] \times [0, \infty)$ so that ϕ is a measurable function on $[a, b] \times [0, \infty)$. Finally, for each $x \in [a, b]$, we know from the inequality (10.45) that

$$|\phi|_x = |\varphi(\gamma(x), t) \cdot \gamma'(x)| \leq \frac{M}{1+t^2}$$

which gives

$$\phi^*(x) = \int_0^\infty |\phi|_x\,\mathrm{d}t \leq M \int_0^\infty \frac{\mathrm{d}t}{1+t^2} = \frac{M\pi}{2}$$

and

$$\int_a^b \phi^*\,\mathrm{d}x < \infty.$$

Consequently, we may apply Theorem 8.8 (The Fubini Theorem) to change the order of integration in the integral (10.46) and get

$$\int_{\partial\Delta} g(z)\,\mathrm{d}z = \int_0^\infty \Big[\int_a^b \varphi(\gamma(x), t)\gamma'(x)\,\mathrm{d}x\Big]\,\mathrm{d}t = \int_0^\infty \int_{\partial\Delta} \varphi(z, t)\,\mathrm{d}z\,\mathrm{d}t. \tag{10.47}$$

Since $\varphi(z, t)$ is holomorphic in Ω for every $t \in [0, \infty)$, we conclude from Theorem 10.13 (The Cauchy's Theorem for a Triangle) that

$$\int_{\partial\Delta} \varphi(z, t)\,\mathrm{d}z = 0,$$

so the integral (10.47) reduces to

$$\int_{\partial\Delta} g(z)\,\mathrm{d}z = 0.$$

Finally, we apply Theorem 10.17 (Morera's Theorem) to obtain the desired conclusion that $g \in H(\Omega)$.

- **The function** $h(z)$**.** We have

$$h(z) = \int_X \varphi(z, t)\,\mathrm{d}m,$$

where $X = [-1, 1]$ and $\varphi(z, t) = e^{tz}(1 + t^2)^{-1}$. For every $z \in \mathbb{C}$, we have $z \in D(z; 1) \subset \mathbb{C}$. We know that

$$|\varphi(z, t)| \leq e^{|\operatorname{Re} z|} < \infty$$

in $D(z; 1) \times X$. Thus the function φ satisfies all the requirements of Problem 10.16 in $D(z; 1) \times X$, so $h \in H(D(z; 1))$. Since z is arbitrary, we conclude that $h \in H(\mathbb{C})$, i.e., h is entire.

We complete the proof of the problem. ■

10.4 Problems related to Zeros of Holomorphic Functions

Problem 10.18

Rudin Chapter 10 Exercise 18.

Proof. By [76, Problem 10.11, p. 131], we know that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} z^p \, dz = \sum_k m_k z_k^p,$$

where the sum is taken over all the zeros of f inside γ and m_k is the multiplicity of the zero z_k of f. This is the answer of the first assertion.

For the second assertion, let $F(z) = \frac{f'(z)}{f(z)}\varphi(z)$. Then F is a meromorphic function in Ω. Let $A \subseteq \Omega$ be the set of poles of F. Since $f \neq 0$ on γ^*, γ is a cycle in $\Omega \setminus A$ and $\operatorname{Ind}_\gamma(\alpha) = 0$ for all $\alpha \in \Omega$. In other words, our F satisfies Theorem 10.42 (The Residue Theorem) which implies that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\varphi(z) \, dz = \sum_k \operatorname{Res}(F; z_k), \tag{10.48}$$

where the z_k denote the isolated singularities of F inside γ which are *exactly* the zeros of f inside γ. By Theorem 10.18, there exists a $g \in H(\Omega)$ and a unique positive integer m_k such that $f(z) = (z - z_k)^{m_k} g(z)$ and $g(z_k) \neq 0$. Clearly, we have

$$F(z) = \frac{f'(z)}{f(z)}\varphi(z) = \frac{m_k\varphi(z)}{z - z_k} + \frac{g'(z)}{g(z)}\varphi(z).$$

If $\varphi(z_k) = 0$, then F is holomorphic at z_k. Otherwise, F has a simple pole at z_k and it yields from [9, Eqn. (1), p. 129] that

$$\operatorname{Res}(F; z_k) = \lim_{z\to z_k}(z - z_k)F(z) = m_k\varphi(z_k). \tag{10.49}$$

Substituting the residues (10.49) into the formula (10.48), we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\varphi(z) \, dz = \sum_k m_k\varphi(z_k),$$

where the sum is taken over all the zeros of f which are *not* zeros of φ inside γ. This completes the proof of the problem. ■

Problem 10.19

Rudin Chapter 10 Exercise 19.

Proof. We claim that $f = cg$ for some nonzero constant c. To this end, we consider $h = \frac{f}{g}$ in U. Since $f(z) \neq 0$ on U, we know that $h(z) \neq 0$ on U. In addition, since $f, g \in H(U)$ and $g(z) \neq 0$ on U, we conclude that $h \in H(U)$. Direct differentiation gives

$$h'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)} = \frac{f(z)}{g(z)} \cdot \Big[\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\Big]$$

so that $h'(\frac{1}{n}) = 0$ on $\{1, 2, \ldots\}$. By Theorem 10.18, we conclude that $h'(z) = 0$ in U. Next, the Fundamental Theorem of Calculus [9, Proposition 4.12, p. 51] shows that

$$h(z) = h(0) + \int_{[0,z]} h'(\zeta)\,\mathrm{d}\zeta = h(0) \neq 0$$

for every $z \in U$, where $[0, z]$ is a path connecting 0 and z in U. By the definition, we establish that

$$f(z) = h(0)g(z)$$

on U. This ends the proof of the problem. ■

Problem 10.20

Rudin Chapter 10 Exercise 20.

Proof. Suppose that $f(z_0) = 0$ for some $z_0 \in \Omega$. Assume that $f \not\equiv 0$. Then there exists a circle $C(z_0; R)$ for some $R > 0$ such that $f(z) \neq 0$ on $C(z_0; R)$. By Theorem 10.28, we know that $f_n' \to f'$ uniformly on compact subsets of Ω. Thus the convergence

$$\frac{f_n'}{f_n} \to \frac{f'}{f}$$

is also uniform on $C(z_0; R)$. Next, Theorem 10.43(a) gives

$$N_f = \frac{1}{2\pi i}\int_{C(z_0;R)} \frac{f'(z)}{f(z)}\,\mathrm{d}z \quad \text{and} \quad N_{f_n} = \frac{1}{2\pi i}\int_{C(z_0;R)} \frac{f_n'(z)}{f_n(z)}\,\mathrm{d}z.$$

Our hypotheses give $N_f = 1$ but $N_{f_n} = 0$ which is a contradiction, so we establish the result $f \equiv 0$ on Ω.

For the second assertion, suppose that

$$\bigcup_{n=1}^{\infty} f_n(\Omega) \subseteq \Omega'.$$

Choose a point $a \in \mathbb{C} \setminus \Omega'$. Define $F_n : \Omega \to \mathbb{C}$ by $F_n = f_n - a$ for every $n = 1, 2, \ldots$. Then each F_n is holomorphic in Ω, $F_n(z) \neq 0$ for all $z \in \Omega$ and $F_n \to f - a$ uniformly on compact subsets of Ω. Thus the first assertion and the fact that f is nonconstant imply that $f(z) \neq a$ in Ω. In other words, we must have

$$f(\Omega) \subseteq \Omega'$$

which completes the proof of the problem. ■

Remark 10.2

We note that Problem 10.20 is classically called **Hurwitz's Theorem**, see [9, Theorem 10.13, p. 139] or [18, p. 152].

Problem 10.21

Rudin Chapter 10 Exercise 21.

Proof. Let $g(z) = f(z) - z$ and $h(z) = -z$ on Ω. Since $\overline{D}(0;1) \subseteq \Omega$ and $|f(z)| < 1$ on $|z| = 1$, we have

$$|g(z) - h(z)| = |f(z)| < 1 = |h(z)|$$

on $|z| = 1$. By Theorem 10.43(b) (Rouché's Theorem), we conclude that $N_g = N_h = 1$ in the disc $D(0;1)$. This completes the analysis of the proof. ■

Problem 10.22

Rudin Chapter 10 Exercise 22.

Proof. Assume that $f(z) \neq 0$ for all $z \in \Omega$. By the Corollary to Theorem 10.24 (The Maximum Modulus Theorem), we see that

$$1 = |f(0)| \geq \min_{\theta} |f(re^{i\theta})| > 2,$$

a contradiction. Hence f has at least one zero in the unit disc, completing the proof of the problem. ■

Problem 10.23

Rudin Chapter 10 Exercise 23.

Proof. Here we list some observations about the zeros of P_n and Q_n.

- By Theorem 10.25 (The Fundamental Theorem of Algebra), both P_n and Q_n have precisely n zeros in $\mathbb{C}$.
- The definitions of P_n and Q_n guarantee that P_n and Q_n cannot have any common zeros.
- Since $Q_n(0) = 0$, Q_n always has a zero at 0 for every n.
- By [9, Exercise 12, p. 142], we know that for every $R > 0$, if n is large enough, $P_n(z)$ has *no* zeros in $|z| \leq R$, i.e., all zeros ζ of P_n satisfy $|\zeta| > R$ for large n. In fact, it has been shown further in [33] that every zero ζ of P_n lie in the annulus

$$\frac{n}{e^2} < |\zeta| < n$$

  for large enough n.
- Applying [13, Corollary 1.2.3, p. 13] to Q_n, all the zeros $z \neq 0$ of

$$Q_n(z) = z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} = z\Big(1 + \frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\Big)$$

  lie inside the annulus

$$2 = \min_{0 \leq k \leq n-2} (k+2) \leq |z| < \max_{0 \leq k \leq n-2} (k+2) = n.$$

- We imitate the proof of [76, Problem 10.12, pp. 131, 132]. First, we notice that $e^z - 1 = 0$ if and only if $z = \pm 2k\pi i$ for all $k \in \mathbb{N}$, so if we take $2k\pi < R_k < 2(k+1)\pi$, then there is a $\epsilon_k > 0$ such that
$$|e^z - 1| > \epsilon_k. \tag{10.50}$$
Since $P_n(z) \to e^z$ uniformly in $\overline{D(0; R_k)}$, there exists a $M_k \in \mathbb{N}$ such that $n \geq M_k$ implies
$$|e^z - 1 - Q_n| = |e^z - P_n| < \epsilon_k. \tag{10.51}$$
Thus the inequalities (10.50) and (10.51) give
$$|e^z - 1 - Q_n| < |e^z - 1|$$
for all $n \geq M_k$ and for all $z \in C(0; R_k)$. By Theorem 10.43(b) (Rouché's Theorem), we conclude that
$$N_{Q_n} = N_{e^z - 1} = 2k + 1$$
inside $C(0; R_k)$ for all $n \geq M_k$. This also means that
$$N_{Q_n} = n - 2k - 1 \tag{10.52}$$
outside $C(0; R_k)$ for all $n \geq M_k$.
Similarly, since $2(k+1)\pi < R_k + 2\pi < 2(k+2)\pi$, Theorem 10.43(b) (Rouché's Theorem) tells us that
$$N_{Q_n} = N_{e^z - 1} = 2k + 3$$
inside $C(0; R_k + 2\pi)$ for all $n \geq M_{k+1}$, or equivalently,
$$N_{Q_n} = n - 2k - 3 \tag{10.53}$$
outside $C(0; R_k + 2\pi)$ for all $n \geq M_{k+1}$. Now if we combine the results (10.52) and (10.53), there exists a $M'_k \in \mathbb{N}$ such that
$$N_{Q_n} = 2$$
in the annulus $A = \{z \in \mathbb{C} \,|\, R_k < |z| < R_k + 2\pi\}$ for all large enough $n \geq M'_k$, where $k = 1, 2, 3, \ldots$.

We complete the proof of the problem. ■

Remark 10.3

There are some books concerning the location of the zeros of polynomials. For instances, [13, Chap. 1], [39] and [40, Chap. 3].

Problem 10.24

Rudin Chapter 10 Exercise 24.

Proof. We have $\Omega = K^\circ$ which is an open set in $\mathbb{C}$. Put
$$\varphi(z) = |f(z) - g(z)| - |f(z)| - |g(z)|,$$
$E = \{z \in K \,|\, \varphi(z) = 0\}$ and $\{z_n\} \subseteq E$ with $z_n \to z_0$ as $n \to \infty$. We divide the proof into several steps.

- **Step 1: $E \subseteq \Omega$.** Our hypothesis implies that

$$|f(z) - g(z)| < |f(z)| + |g(z)| \tag{10.54}$$

on $\partial\Omega = K \setminus \Omega$ and this ensures that f and g *cannot* have any zero on $\partial\Omega$. In addition, the continuity of f and g imply the continuity of φ and then $z_0 \in E$. Thus E is a closed, hence compact, subset of K by Theorem 2.4. By the inequality (10.54), we know that $E \cap \partial\Omega = \varnothing$ which means that $E \subseteq \Omega$ and

$$|f(z) - g(z)| < |f(z)| + |g(z)| \tag{10.55}$$

in $K \setminus E$.

- **Step 2: The numbers of zeros of f and g are finite.** Suppose that

$$Z(f) = \{a \in \Omega \mid f(a) = 0\}.$$

Assume that $Z(f)$ was infinite. Then [79, Problem 5.25, p. 68] leads to us that $Z(f)$ has a convergent subsequence and Theorem 10.18 says that $Z(f) = \Omega$, but the continuity of f implies immediately that $f \equiv 0$ on K which is impossible. Consequently, $Z(f)$ is finite. Furthermore, since $f(a) = 0$ implies $\varphi(a) = 0$, all zeros of f lie in E. Similarly, $Z(g)$ is also finite and all zeros of g belong to E.

- **Step 3: A lemma and its application.** Here we need the following result whose proof can be found in [63, IX.8 & 9, pp. 115 – 118] or [67, Lemma 5.8, pp. 61, 62]:

Lemma 10.1

Let G be an open subset of $\mathbb{C}$ and K a compact subset of G. Then there exists a cycle Γ in $G \setminus K$ such that $K \subseteq \operatorname{int}\Gamma \subseteq G$ and

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \tag{10.56}$$

for every $f \in H(G)$ and $z \in K$.

By **Step 1**, we see that E and Ω satisfy the roles of K and G of Lemma 10.1 respectively, so it ensures that there exists a cycle Γ in $\Omega \setminus E$ such that $E \subseteq \operatorname{int}\Gamma \subseteq \Omega$ and the formula (10.56) holds for every $f \in H(\Omega)$ and every $z \in E$.

As an application, let $\widetilde{F} \in H(\Omega)$, $z \in E$ and $F(\zeta) = (\zeta - z)\widetilde{F}(\zeta)$. Obviously, we have $F \in H(\Omega)$ and $F(z) = 0$, so the formula (10.56) gives

$$\int_\Gamma \widetilde{F}(\zeta)\,\mathrm{d}\zeta = 0. \tag{10.57}$$

- **Step 4: The calculation of $|Z(f)|$ and $|Z(g)|$.** Recall from the hypotheses and **Step 2** that $f \in H(\Omega)$ and f has only *finitely* many zeros $a_1, a_2, \ldots, a_N$ with multiplicities $p_1, p_2, \ldots, p_N$ respectively in E. Then the function

$$\widetilde{f}(z) = \frac{f'(z)}{f(z)} - \sum_{k=1}^{N} \frac{p_k}{z - a_k}$$

can be shown to have a removable singularity at each a_k by Theorem 10.18 and Definition 10.19. Thus $\widetilde{f}$ is holomorphic in Ω. Hence it follows from the result (10.57) that

$$\int_\Gamma \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \sum_{k=1}^{N} p_k \int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - a_k}. \tag{10.58}$$

Next, we apply Lemma 10.1 with $f \equiv 1$ and $z = a_k$ to get

$$2\pi i = \int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - a_k}. \tag{10.59}$$

By combining the integrals (10.58) and (10.59), we conclude that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \sum_{k=1}^{N} p_k = |Z(f)|. \tag{10.60}$$

Similarly, we obtain

$$\frac{1}{2\pi i}\int_\Gamma \frac{g'(\zeta)}{g(\zeta)}\,\mathrm{d}\zeta = |Z(g)|. \tag{10.61}$$

By the strict inequality (10.55), it is impossible that $f = -N'g$ for some $N' \in \mathbb{N} \cup \{0\}$ on $\Omega \setminus E$. Therefore, the function $h : \Omega \setminus E \to \mathbb{C} \setminus (-\infty, 0]$ given by

$$h(z) = \log \frac{f(z)}{g(z)}$$

is well-defined and holomorphic in the open set $\Omega \setminus E$. Taking differentiation, we have

$$h' = \frac{f'}{f} - \frac{g'}{g}$$

on $\Omega \setminus E$. Since $Z(f), Z(g) \nsubseteq \Omega \setminus E$, h' is continuous in $\Omega \setminus E$. Since Γ is a cycle in $\Omega \setminus E$, Theorem 10.12 guarantees that

$$\int_\Gamma h'(\zeta)\,\mathrm{d}\zeta = 0$$

or equivalently,

$$\int_\Gamma \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \int_\Gamma \frac{g'(\zeta)}{g(\zeta)}\,\mathrm{d}\zeta. \tag{10.62}$$

Finally, by substituting the results (10.60) and (10.61) into the expression (10.62), we have established that

$$|Z(f)| = |Z(g)|.$$

Now we complete the proof of the problem. ■

10.5 Laurent Series and its Applications

Problem 10.25

Rudin Chapter 10 Exercise 25.

Proof. Define $A(r_1, r_2) = \{z \in \mathbb{C} \,|\, r_1 < |z| < r_2\}$. Let $\epsilon > 0$ be such that $r_1 + \epsilon < r_2 - \epsilon$. Furthermore, we define $\rho_r : [0, 2\pi] \to \mathbb{C}$ by

$$\rho_r(t) = re^{it},$$

where $r > 0$. Particularly, we have $\gamma_1 = -\rho_{r_1+\epsilon}$ and $\gamma_2 = \rho_{r_2-\epsilon}$.

(a) Note that γ_1 and γ_2 are negatively oriented and positively oriented circles respectively. By Theorem 10.11, we have

$$\mathrm{Ind}_{\gamma_1}(z) = \begin{cases} -1, & \text{if } z \in D(0; r_1 + \epsilon); \\ 0, & \text{if } z \notin D(0; r_1 + \epsilon) \end{cases}$$

and

$$\mathrm{Ind}_{\gamma_2}(z) = \begin{cases} 1, & \text{if } z \in D(0; r_2 - \epsilon); \\ 0, & \text{if } z \notin D(0; r_2 - \epsilon). \end{cases}$$

Let $\Gamma = \gamma_1 + \gamma_2$ which is the sum of two circles in $A(r_1, r_2)$. Let $\alpha \notin A(r_1, r_2)$. If $|\alpha| \le r_1$, then we have $\alpha \in D(0; r_1 + \epsilon) \subseteq D(0; r_2 - \epsilon)$. Thus we follow from [62, Eqn. (8), p. 218] that

$$\mathrm{Ind}_\Gamma(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha) + \mathrm{Ind}_{\gamma_2}(\alpha) = 0. \tag{10.63}$$

Similarly, if $|\alpha| \ge r_2$, then we still have the result (10.63). Next if $z \in A(r_1 + \epsilon, r_2 - \epsilon)$, then $z \notin D(0; r_1 + \epsilon)$ and $z \in D(0; r_2 - \epsilon)$. Thus it is easy to check that

$$\mathrm{Ind}_\Gamma(z) = \mathrm{Ind}_{\gamma_1}(z) + \mathrm{Ind}_{\gamma_2}(z) = 1.$$

See Figure 10.4 for an illustration below.

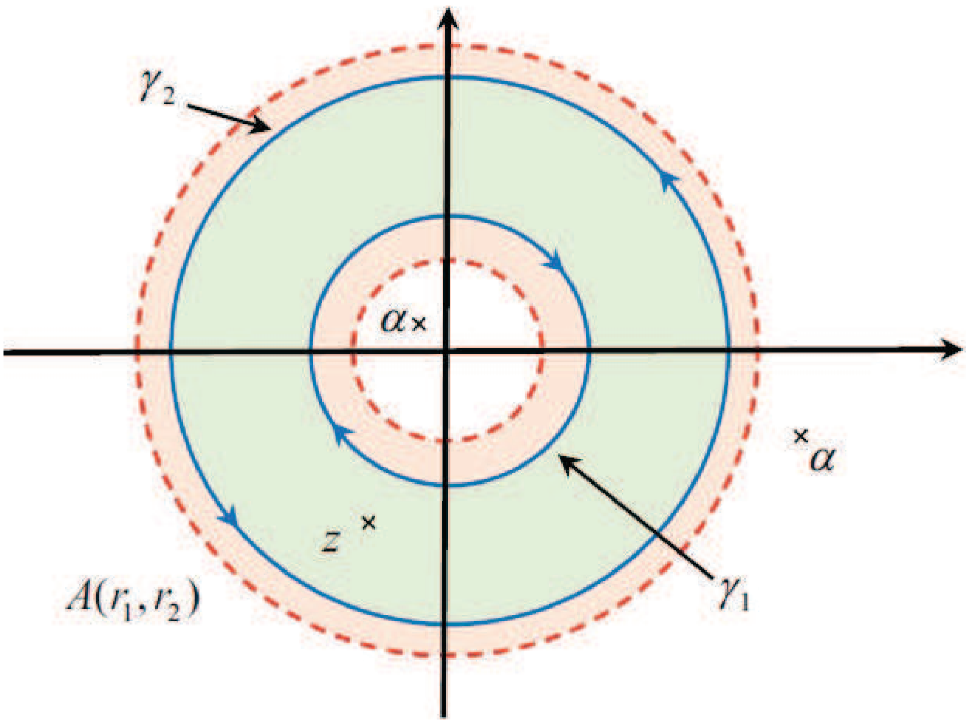


Figure 10.4: The annulus $A(r_1, r_2)$ and the circles γ_1, γ_2.

Hence, Theorem 10.35 (Cauchy's Theorem) and [62, Eqn. (5), p. 217] assert that

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\Big(\int_{\gamma_1} + \int_{\gamma_2}\Big)\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \tag{10.64}$$

for every $z \in A(r_1+\epsilon, r_2-\epsilon)$.

(b) Let $R_1 = r_1+\epsilon$ and $R_2 = r_2-\epsilon$ so that $r_1 < R_1 < |z| < R_2 < r_2$. Define

$$f_1(z) = \frac{1}{2\pi i}\int_{-\rho_{R_1}}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i}\int_{\rho_{R_2}}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{10.65}$$

Then the formula (10.64) simplifies to $f = f_1 + f_2$.

- **Step 1: f_1 and f_2 are well-defined.** To see this, we first recall from the hypothesis that $f \in H(A(r_1, r_2))$. Next, we fix z and take $r_1 < R_1 < R_1' < |z|$ which means that $z \notin D(0; R_1')$. Let $\Gamma_1 = \rho_{R_1} - \rho_{R_1'}$. By Theorem 10.11, we know that

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\rho_{R_1}}(\alpha) - \mathrm{Ind}_{\rho_{R_1'}}(\alpha) = 0$$

for every $\alpha \notin A(r_1, r_2)$. Thus it yields from Theorem 10.35 (Cauchy's Theorem)[g] that $\mathrm{Ind}_{\Gamma_1}(z) = 0 - 0 = 0$ and then

$$\frac{1}{2\pi i}\int_{\rho_{R_1}}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\rho_{R_1'}}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

In other words, $f_1(z)$ is uniquely determined by z, *not* by R_1. Since ϵ is arbitrary, this means that f_1 is actually well-defined in $\mathbb{C} \setminus \overline{D}(0; r_1)$.
Similarly, we take $|z| < R_2 < R_2' < r_2$ so that $z \in D(0; R_2) \subseteq D(0; R_2')$. Let $\Gamma_2 = \rho_{R_2} - \rho_{R_2'}$. By Theorem 10.11 again, we have

$$\mathrm{Ind}_{\Gamma_2}(\alpha) = \mathrm{Ind}_{\rho_{R_2}}(\alpha) - \mathrm{Ind}_{\rho_{R_2'}}(\alpha) = 0$$

for every $\alpha \notin A(r_1, r_2)$. Hence we have $\mathrm{Ind}_{\Gamma_2}(z) = 1 - 1 = 0$ and similar argument shows that $f_2(z)$ is well-defined in $D(0; r_2)$.

- **Step 2: $f_2 \in H(D(0; r_2))$.** As suggested by the proof of Theorem 10.16, we may apply Theorem 10.7 to the integral representation (10.65) of f_2 with $X = [0, 2\pi]$, $\varphi = \rho_{R_2}$ and $\mathrm{d}\mu(t) = f(\rho_{R_2}(t))\rho_{R_2}'(t)\,\mathrm{d}t$ to establish the fact that f_2 is representable by a power series in $D(0; R_2)$. Since ϵ is arbitrary, f_2 is representable by a power series in $D(0; r_2)$ and hence Theorem 10.6 concludes that $f_2 \in H(D(0; r_2))$.

- **Step 3: $f_1 \in H(\mathbb{C} \setminus \overline{D}(0; r_1))$.** For f_1, we consider the function $g(\zeta) = \frac{1}{\zeta}f(\frac{1}{\zeta})$ on $|\zeta| = \frac{1}{R_1}$. Tale $X = [0, 2\pi]$, $\varphi = \rho_{\frac{1}{R_1}}$ and $\mathrm{d}\mu(t) = g\big(\rho_{\frac{1}{R_1}}(t)\big)\rho'_{\frac{1}{R_1}}(t)\,\mathrm{d}t$ in Theorem 10.7. Obviously, we have $\varphi([0, 2\pi]) = \rho_{\frac{1}{R_1}}([0, 2\pi]) = C(0; \frac{1}{R_1})$, so $\varphi([0, 2\pi]) \cap D(0; \frac{1}{R_1}) = \varnothing$ and thus the function

$$g(z) = \frac{1}{2\pi i}\int_{\rho_{\frac{1}{R_1}}}\frac{\mathrm{d}\mu(\zeta)}{\zeta - z}$$

is representable by power series in $D(0; \frac{1}{R_1})$. Furthermore, for $\rho_{\frac{1}{R_1}}(t) = \frac{1}{R_1}\mathrm{e}^{it}$, we have

$$\frac{1}{2\pi i}\int_{\rho_{\frac{1}{R_1}}}\frac{\mathrm{d}\mu(\zeta)}{\zeta - z} = \frac{1}{2\pi i}\int_0^{2\pi} R_1\mathrm{e}^{-it}f(R_1\mathrm{e}^{-it})\frac{i}{R_1}\mathrm{e}^{it}\cdot\frac{\mathrm{d}t}{\frac{1}{R_1}\mathrm{e}^{it} - z}$$

[g] In fact, we have applied the formula [62, Eqn. (2), p. 219] here.

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_0^{2\pi} if(R_1 e^{-it})\cdot \frac{R_1 e^{-it}\,dt}{(\frac{1}{R_1}e^{it}-z)R_1 e^{-it}} \\
&= \frac{1}{z}\cdot\frac{1}{2\pi i}\int_0^{2\pi}\frac{if(R_1 e^{-it})R_1 e^{-it}\,dt}{\frac{1}{z}-R_1 e^{-it}} \\
&= \frac{1}{z}\int_{-\rho_{\frac{1}{R_1}}}\frac{f(\zeta)}{\zeta-\frac{1}{z}}\,d\zeta \\
&= \frac{1}{z}f_1\Big(\frac{1}{z}\Big).
\end{aligned}$$

Therefore, the function $\frac{1}{z}f_1(\frac{1}{z})$ is representable by power series in $D(0;\frac{1}{R_1})$. Again, since ϵ is arbitrary, $\frac{1}{z}f_1(\frac{1}{z})$ can be represented by power series in $D(0;\frac{1}{r_1})$ and then Theorem 10.6 ensures that $\frac{1}{z}f_1(\frac{1}{z})\in H(D(0;\frac{1}{r_1}))$. Hence, this certainly shows that $f_1\in H(\mathbb{C}\setminus\overline{D}(0;r_1))$ as required.

– **Step 4: Uniqueness of the decomposition.** Suppose that $f=g_1+g_2$, where $g_1\in H(\mathbb{C}\setminus\overline{D}(0;r_1))$ and $g_2\in H(D(0;r_2))$. Then we have

$$g_1-f_1=f_2-g_2 \tag{10.66}$$

in $A(r_1,r_2)$. Define $h:\mathbb{C}\to\mathbb{C}$ by

$$h(z)=\begin{cases} f_2(z)-g_2(z), & \text{if } z\in D(0;r_2);\\[2ex] g_1(z)-f_1(z), & \text{if } z\in\mathbb{C}\setminus\overline{D}(0;r_1).\end{cases}$$

Now the equation (10.66) shows that the two definitions of h actually agree in $A(r_1,r_2)$, so h is well-defined in $\mathbb{C}$ and in fact it is entire. Thus it suffices to prove that $h\equiv 0$. Since $f_1(z),g_1(z)\to 0$ as $|z|\to\infty$, we have $h(z)\to 0$ as $|z|\to\infty$ and thus h is bounded. By Theorem 10.23 (Liouville's Theorem), we conclude that $h(z)=0$ for all $z\in\mathbb{C}$ and this means that $f_1(z)=g_1(z)$ and $f_2(z)=g_2(z)$ in $\mathbb{C}\setminus\overline{D}(0;r_1)$ and $D(0;r_2)$ respectively. This proves the uniqueness of the decomposition.

(c) – **Existence of a Laurent series.** On γ_2, we have $|\zeta|>|z|$ so that

$$\frac{1}{\zeta-z}=\frac{1}{\zeta(1-\frac{z}{\zeta})}=\frac{1}{\zeta}+\frac{z}{\zeta^2}+\frac{z^2}{\zeta^2}+\cdots. \tag{10.67}$$

On $-\gamma_1$, since $|\zeta|<|z|$, we have

$$\frac{1}{\zeta-z}=-\frac{1}{z(1-\frac{\zeta}{z})}=-\frac{1}{z}-\frac{\zeta}{z^2}-\frac{\zeta^2}{z^3}-\cdots. \tag{10.68}$$

Substituting the series (10.67) and (10.68) into the formula (10.64), we get

$$f(z)=\frac{1}{2\pi i}\int_{\gamma_2}\sum_{n=0}^{\infty}\frac{f(\zeta)z^n}{\zeta^{n+1}}\,d\zeta+\frac{1}{2\pi i}\int_{-\gamma_1}\sum_{n=-1}^{-\infty}\frac{f(\zeta)z^n}{\zeta^{n+1}}\,d\zeta. \tag{10.69}$$

Since the convergence of the series (10.67) and (10.68) are uniform, we can switch the order of integration and summation in the expression (10.69) to obtain

$$f(z)=\sum_{-\infty}^{\infty}c_n z^n,$$

where

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\,\mathrm{d}\zeta \quad \text{and} \quad \gamma = \begin{cases} \gamma_2, & \text{if } n = 0,1,2,\ldots; \\ \\ -\gamma_1, & \text{if } n = -1,-2,-3,\ldots. \end{cases} \tag{10.70}$$

- **Uniqueness of the Laurent series.** We claim that if $\sum_{-\infty}^{\infty} a_n z^n$ converges to f in A, then $a_n = c_n$ for all $n \in \mathbb{Z}$. To this end, since $\sum_{-\infty}^{\infty} a_n z^n$ converges uniformly to f on γ, we have

$$\int_\gamma \frac{f(z)}{z^{k+1}}\,\mathrm{d}z = \sum_{-\infty}^{\infty}\int_\gamma a_n z^{n-k-1}\,\mathrm{d}z = 2\pi i a_k,$$

where $k \in \mathbb{Z}$. Hence it asserts from the definitions (10.70) that $c_n = a_n$ hold for all $n \in \mathbb{Z}$. This proves the claim and then the uniqueness follows.

- **Uniform convergence on compact subsets of A.** Let K be a compact subset of A. It is easy to check that $\mathbb{C}\setminus A$ is closed in $\mathbb{C}$ and $(\mathbb{C}\setminus A)\cap K = \varnothing$. Thus we deduce from Problem 10.1 that there is a $\delta > 0$ such that $d(K, \mathbb{C}\setminus A) = 2\delta > 0$. In addition, we have

$$K \cap C(0; r_1+\delta) = \varnothing \quad \text{and} \quad K \cap C(0; r_2-\delta) = \varnothing$$

which imply that $K \subseteq A(r_1+\delta, r_2-\delta) \subseteq A(r_1, r_2)$. For $r_1+\delta \leq |z| \leq r_2-\delta$, we know that

$$\begin{aligned}\sum_{-\infty}^{\infty}|c_n z^n| &\leq \sum_{n=-\infty}^{-1}|c_n z^n| + \sum_{n=0}^{\infty}|c_n z^n| \\ &\leq \sum_{n=1}^{\infty}\frac{|c_n|}{|z|^n} + \sum_{n=0}^{\infty}|c_n|(r_2-\delta)^n \\ &\leq \sum_{n=1}^{\infty}\frac{|c_n|}{(r_1+\delta)^n} + \sum_{n=0}^{\infty}|c_n|(r_2-\delta)^n \\ &< \infty.\end{aligned}$$

Hence it follows from the Weierstrass M-Test [9, Theorem 1.9, p. 15] or [61, Theorem 7.10, p. 148] that the series converges to f uniformly on K.

(d) Let $r_1 < s_1 < s_2 < r_2$. Firstly, since $f_2 \in H(D(0; r_2))$ and $\overline{D}(0; s_2)$ is a compact subset of $D(0; r_2)$, f_2 is bounded in $\overline{D}(0; s_2)$. Secondly, since $f_1(z) \to 0$ as $|z| \to \infty$, there exists a $M > s_1$ such that

$$|f_1(z)| < 1 \tag{10.71}$$

on $\mathbb{C}\setminus\overline{D}(0; M)$. The set $\overline{D}(0; M)\setminus D(0; s_1) = \{z \in \mathbb{C} \,|\, s_1 \leq |z| \leq M\}$ is a closed and bounded subset in $\mathbb{C}$, so it is compact. Since $f_1 \in H(\mathbb{C}\setminus\overline{D}(0; r_1))$, f_1 is bounded in $\overline{D}(0; M)\setminus D(0; s_1)$. This fact and the bound (10.71) combine to say that f_1 is bounded in $\mathbb{C}\setminus D(0; s_1)$.

Now we write

$$f_2 = f - f_1$$

on $D(0; r_2)\setminus\overline{D}(0; s_2)$. Since $D(0; r_2)\setminus\overline{D}(0; s_2) \subseteq \mathbb{C}\setminus D(0; s_1)$, f_1 is bounded there. Since f is bounded in A, f_2 is bounded in $D(0; r_2)\setminus\overline{D}(0; s_2)$. Using the first result in the previous paragraph, we conclude that f_2 is bounded in $D(0; r_2)$. Similarly, we write

$f_1 = f - f_2$ on $D(0; s_1) \setminus \overline{D}(0; r_1)$. Clearly, we know that $D(0; s_1) \setminus \overline{D}(0; r_1) \subseteq \overline{D}(0; s_2)$, so f_2 is bounded there and thus f_1 is also bounded there. Using the second assertion in the previous paragraph, we see that f_1 is bounded in $\mathbb{C} \setminus \overline{D}(0; r_1)$.

(e) All the foregoing parts remain valid if $r_1 = 0$, or $r_2 = \infty$ or both. In the case $r_2 = \infty$, we note that f_2 represents an entire function and thus it reduces to 0 by Theorem 10.23 (Liouville's Theorem).

(f) Suppose that $0 < r_1 < r_2 < r_3 < \cdots < r_{n-1} < r_n < \infty$. Each $A(r_k, r_{k+1})$ is an annulus, where $k = 1, 2, \ldots, n-1$. Let $A = \displaystyle\bigcup_{k=1}^{n-1} A(r_k, r_{k+1})$ and $f \in H(A)$. Then the foregoing results can be applied to f in each $A(r_k, r_{k+1})$ and obtain the corresponding Laurent series in each $A(r_k, r_{k+1})$.

Hence we have completed the analysis of the problem. ■

Problem 10.26

Rudin Chapter 10 Exercise 26.

Proof. Let f be the function in the question. The function f is holomorphic in $\mathbb{C} \setminus \{-1, 1, 3\}$. By Problem 10.25, we have to consider the following three regions

$$D(0;1) = \{z \in \mathbb{C} \,|\, |z| < 1\}, \quad A = \{z \in \mathbb{C} \,|\, 1 < |z| < 3\} \quad \text{and} \quad B = \{z \in \mathbb{C} \,|\, |z| > 3\}.$$

Note that if $|z| < 1$, then we have

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}. \tag{10.72}$$

If $|z| > 1$, then $|\frac{1}{z}| < 1$ so that

$$\frac{1}{1-z^2} = -\frac{1}{z^2} \cdot \frac{1}{1-z^{-2}} = -\sum_{n=1}^{\infty} \frac{1}{z^{2n}}. \tag{10.73}$$

Similarly, if $|z| < 3$, then we have $|\frac{z}{3}| < 1$ so that

$$\frac{1}{3-z} = \frac{1}{3} \cdot \frac{1}{1-\frac{z}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{z^n}{3^n} \tag{10.74}$$

and if $|z| > 3$, then $|\frac{3}{z}| < 1$ which gives

$$\frac{1}{3-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{3}{z}} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{3^n}{z^n}. \tag{10.75}$$

Therefore, for every $z \in D(0;1)$, we follow from the series (10.72) and (10.74) that

$$f(z) = \sum_{n=0}^{\infty} z^{2n} + \frac{1}{3} \sum_{n=0}^{\infty} \frac{z^n}{3^n} = \sum_{n=0}^{\infty} \Big[\frac{1+(-1)^n}{2} + \frac{1}{3^{n+1}}\Big] z^n.$$

Thus we have

$$c_n = \begin{cases} \dfrac{1+(-1)^n}{2} + \dfrac{1}{3^{n+1}}, & \text{if } n = 0, 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Next, if $z \in A$, then we deduce from the series (10.73) and (10.74) that

$$f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^{2n}} + \frac{1}{3}\sum_{n=0}^{\infty} \frac{z^n}{3^n} = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where

$$c_n = \begin{cases} \dfrac{1}{3^{n+1}}, & \text{if } n \geq 0; \\[2ex] \dfrac{-1+(-1)^{n+1}}{2}, & \text{otherwise.} \end{cases}$$

Finally, if $z \in B$, then the series (10.73) and (10.75) imply that

$$f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^{2n}} - \frac{1}{z}\sum_{n=0}^{\infty} \frac{3^n}{z^n} = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where

$$c_n = \begin{cases} 0, & \text{if } n \geq 0; \\[2ex] -\Big[\dfrac{1+(-1)^n}{2} + 3^{|n|-1}\Big], & \text{otherwise.} \end{cases}$$

Hence we have completed the analysis of the problem. ■

Problem 10.27

Rudin Chapter 10 Exercise 27.

Proof. Suppose that

$$\zeta = \mathrm{e}^{2\pi i z} \quad \text{or} \quad z = \frac{1}{2\pi}\arg\zeta - \frac{i}{2\pi}\log|\zeta|, \tag{10.76}$$

where $0 < \arg\zeta < 2\pi$. Set $F(\zeta) = f(z)$. We claim that $F(\zeta)$ *does not* depend on the choice of $\arg\zeta$. In fact, it is easy to see from the second equation (10.76) that

$$\frac{\arg\zeta + 2\pi}{2\pi} - \frac{i}{2\pi}\log|\zeta| = \frac{\arg\zeta}{2\pi} + 1 - \frac{i}{2\pi}\log|\zeta| = z + 1.$$

Since f is a function of period 1, we have $F(\zeta) = f(z) = f(z+1) = F(\zeta \mathrm{e}^{2\pi i})$ which proves our claim. Denote the annulus $A = \{\zeta \in \mathbb{C} \,|\, \mathrm{e}^{-2\pi b} < |\zeta| < \mathrm{e}^{-2\pi a}\}$. Then the mapping $G(z) = \mathrm{e}^{2\pi i z}$ clearly maps the horizontal strip $\Omega = \{z \in \mathbb{C} \,|\, a < \operatorname{Im} z < b\}$ onto A. Furthermore, G is holomorphic in Ω and $f = F \circ G$. Here we need a positive result to Problem 10.14:

Lemma 10.2

Suppose that Ω_1 and Ω_2 are two regions, f and g are nonconstant complex functions defined in Ω_1 and Ω_2 respectively. Put $h = g \circ f$. If f and h are holomorphic in Ω_1 and $f(\Omega_1) = \Omega_2$, then g is also holomorphic in Ω_2.

Proof of Lemma 10.2. We consider

$$S = \{z \in \Omega_2 \,|\, \text{every } \omega \in \Omega_1 \text{ with } f(\omega) = z \text{ implies } f'(\omega) = 0\}. \tag{10.77}$$

Let $\beta \in \Omega_2 \setminus S$. Since $f(\Omega_1) = \Omega_2$, there exists an $\alpha \in \Omega_1$ such that $f(\alpha) = \beta$. By the definition (10.77), $f'(\alpha) \neq 0$. Using Theorem 10.30, there exist neighborhoods $V \subseteq \Omega_1$ and $W \subseteq \Omega_2$ of α and β respectively such that $f : V \to W$ is a bijection and a holomorphic inverse $f^{-1} : W \to V$ exists. Thus we have $g = g \circ (f \circ f^{-1}) = h \circ f^{-1}$ on W which guarantees that g is holomorphic in W and thus g is holomorphic in $\Omega_2 \setminus S$.
Next, we want to show that every point b of S is isolated. This means that one can find a neighborhood $W \subseteq \Omega_2$ of b such that $W \cap S = \{b\}$. By the hypothesis, one can find an $a \in \Omega_1$ such that $f(a) = b$. By the definition (10.77), we have $f'(a) = 0$. Since f is nonconstant, f' is nonzero. Since $f' \in H(\Omega_1)$, Theorem 10.18 implies that $Z(f')$ has no limit point in Ω_1. In other words, Ω_1 contains a neighborhood U of a such that

$$f'(z) \neq 0 \tag{10.78}$$

for all $z \in U \setminus \{a\}$. By the Open Mapping Theorem, $f(U) \subseteq \Omega_2$ is our wanted neighborhood because if $f(U)$ contains a $c \in S \setminus \{b\}$, then we have $f(\zeta) = c$ for some $\zeta \in U \setminus \{a\}$ but the definition (10.77) shows that $f'(\zeta) = 0$ which contradicts the result (10.78).
Given $\epsilon > 0$. Since h is continuous at a, there exists a $\delta > 0$ such that $\omega \in D(a;\delta)$ implies

$$h(\omega) \in D(h(a);\epsilon). \tag{10.79}$$

Since f is holomorphic in Ω_1, $f(D(a;\delta))$ is open in $\mathbb{C}$ by the Open Mapping Theorem. Since $b \in f(D(a;\delta))$, there is a $\delta' > 0$ such that $D(b;\delta') \subseteq f(D(a;\delta))$. Obviously, if $z \in D(b;\delta')$, then we have $f(\omega) = z$ for some $\omega \in D(a;\delta)$. Combining this fact with the property (10.79), we conclude that

$$g(z) = g(f(\omega)) = h(\omega) \in D(h(a);\epsilon) = D(g(f(a));\epsilon) = D(g(b);\epsilon).$$

Consequently, g is continuous on S. According to the Riemann's Principle of Removable Singularities [9, Theorem 9.3, p. 118], every point of S is a removable singularity of g. Hence g can be extended to a function holomorphic in Ω_2, as required. ■

Since f and G are holomorphic in Ω and $G(\Omega) = A$, Lemma 10.2 ensures that F is analytic in A. By Problem 10.25(c), F admits the Laurent series

$$F(\zeta) = \sum_{-\infty}^{\infty} c_n \zeta^n \tag{10.80}$$

in A and this implies

$$f(z) = F(\zeta) = \sum_{-\infty}^{\infty} c_n \mathrm{e}^{2n\pi iz}. \tag{10.81}$$

Finally, we denote

$$\Omega' = \{z \in \mathbb{C} \,|\, a + \epsilon \leq \operatorname{Im} z \leq b - \epsilon\} \quad \text{and} \quad A' = \{\zeta \in \mathbb{C} \,|\, \mathrm{e}^{-2\pi(b-\epsilon)} \leq |\zeta| \leq \mathrm{e}^{-2\pi(a+\epsilon)}\}.$$

Then we have $G(\Omega') = A'$ and A' is a compact subset of A. By Problem 10.25(c), the series (10.80) converges uniformly in A' and hence the corresponding series (10.81) converges uniformly in Ω'. This completes the proof of the problem. ■

10.6 Miscellaneous Problems

Problem 10.28

Rudin Chapter 10 Exercise 28.

Proof. If we consider $\gamma = \Gamma - \alpha$, then we observe immediately from [62, Eqn. (2), p. 203] that

$$\mathrm{Ind}_{\Gamma}(\alpha) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Gamma'(s)}{\Gamma(s)-\alpha}\,\mathrm{d}s = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(s)}{\gamma(s)}\,\mathrm{d}s = \mathrm{Ind}_{\gamma}(0).$$

Thus, without loss of generality, we may assume that $\alpha = 0$ in the following discussion. Now we prove the assertions one by one.

- $\mathrm{Ind}_{\Gamma_n}(0) = \mathrm{Ind}_{\Gamma_m}(0)$ **if m and n are sufficiently large.** Since $\Gamma : [0, 2\pi] \to \mathbb{C}$ is a closed curve, it is a continuous function with period 2π. Thus the Stone-Weierstrass Theorem [61, Theorem 8.15, p. 190] asserts that there exists a sequence of trigonometric polynomials $\{\Gamma_n\}$ converges to Γ uniformly in $[0, 2\pi]$. In other words, choose $|\Gamma(\theta)| > \delta > 0$, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies

  $$|\Gamma(\theta) - \Gamma_n(\theta)| < \frac{\delta}{4}$$

  for all $\theta \in [0, 2\pi]$. According to [61, Exercise 26, p. 202][h], we conclude that

  $$\mathrm{Ind}_{\Gamma_n}(0) = \mathrm{Ind}_{\Gamma_m}(0)$$

  for all $m, n \geq N$.

- **The result is independent of the choice of $\{\Gamma_n\}$.** Note that [77, Problem 8.26, pp. 204 – 206] also includes this part.

- **Lemma 10.39 is true for closed curves.** Again, [77, Problem 8.26, pp. 204 – 206] contains this part.

- **Another proof of Theorem 10.40.** By the definition, there exists a continuous map $H : I^2 \to \Omega$ such that

  $$H(s, 0) = \Gamma_0(s), \quad H(s, 1) = \Gamma_1(s) \quad \text{and} \quad H(0, t) = H(1, t).$$

  Put $\Gamma_t(s) = H(s, t)$. According to [62, Eqn. (3) & (4), p. 223], we obtain

  $$|\Gamma_t(s)| > 2\epsilon \tag{10.82}$$

  for all $(s, t) \in I^2$ and there exists a positive integer n such that

  $$|\Gamma_t(s) - \Gamma_{t'}(s)| < \epsilon \tag{10.83}$$

  for all $t, t' \in I$ with $|t - t'| \leq \frac{1}{n}$ and all $s \in I$. Let $t_0 = 0$, $t_n = 1$ and $t_k = t_{k-1} + \frac{1}{n}$, where $k = 1, 2, \ldots, n$. Then the set $\{t_0, t_1, \ldots, t_n\}$ forms a partition of I and $t_k - t_{k-1} = \frac{1}{n}$ for each $k = 1, 2, \ldots, n$. Therefore, it follows from the inequalities (10.82) and (10.83) that

  $$|\Gamma_{t_k}(s) - \Gamma_{t_{k-1}}(s)| < \epsilon < |\Gamma_{t_{k-1}}(s)|$$

[h] The author has proven it in [77, Problem 8.26, pp. 204 – 206].

for every $s \in I$, where $k = 1, 2, \ldots, n$. By the improved version of Lemma 10.39, we see that
$$\mathrm{Ind}\,_{\Gamma_{t_k}}(0) = \mathrm{Ind}\,_{\Gamma_{t_{k-1}}}(0),$$
where $k = 1, 2, \ldots, n$. Consequently, we have the desired result that
$$\mathrm{Ind}\,_{\Gamma_0}(0) = \mathrm{Ind}\,_{\Gamma_{t_0}}(0) = \mathrm{Ind}\,_{\Gamma_{t_1}}(0) = \cdots = \mathrm{Ind}\,_{\Gamma_{t_n}}(0) = \mathrm{Ind}\,_{\Gamma_1}(0).$$

This ends the proof of the problem. ■

Problem 10.29

Rudin Chapter 10 Exercise 29.

Proof. Suppose that $z \neq 0$, so we can write
$$\begin{aligned}
f(z) &= \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \frac{r}{re^{i\theta} + z}\, d\theta\, dr \\
&= \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \frac{re^{-i\theta}}{r + ze^{-i\theta}}\, d\theta\, dr \\
&= \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \frac{\frac{r}{z}e^{-i\theta}}{\frac{r}{z} + e^{-i\theta}}\, d\theta\, dr \\
&= -\frac{1}{\pi i}\int_0^1 \frac{r}{z}\Big[\int_{-\pi}^{\pi} \frac{-ie^{i\theta}}{e^{-i\theta} - (-\frac{r}{z})}\, d\theta\Big]\, dr \\
&= -2\int_0^1 \frac{r}{z}\Big[\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{-ie^{i\theta}}{e^{-i\theta} - (-\frac{r}{z})}\, d\theta\Big]\, dr \\
&= -2\int_0^1 \frac{r}{z} \cdot \mathrm{Ind}\,_{\gamma}\Big(\frac{-r}{z}\Big)\, dr
\end{aligned} \tag{10.84}$$
where $\gamma(\theta) = e^{-i\theta}$ with $-\pi \leq \theta \leq \pi$ which is the negatively oriented circle with center at 0 and radius 1. By Theorem 10.11, we know that
$$\mathrm{Ind}\,_{\gamma}\Big(\frac{-r}{z}\Big) = \begin{cases} -1, & \text{if } \frac{r}{|z|} < 1; \\ 0, & \text{if } \frac{r}{|z|} > 1. \end{cases}$$

If $|z| < 1$, then the expression (10.84) reduces to
$$\begin{aligned}
f(z) &= -2\Big[\int_0^{|z|} \frac{r}{z} \cdot \mathrm{Ind}\,_{\gamma}\Big(\frac{-r}{z}\Big)\, dr + \int_{|z|}^1 \frac{r}{z} \cdot \mathrm{Ind}\,_{\gamma}\Big(\frac{-r}{z}\Big)\, dr\Big] \\
&= 2\int_0^{|z|} \frac{r}{z}\, dr \\
&= \frac{2}{z}\int_0^{|z|} r\, dr \\
&= \frac{|z|^2}{z} \\
&= \overline{z}.
\end{aligned}$$

Next, if $|z| \geq 1$, then we always have $\mathrm{Ind}_\gamma(-\frac{r}{z}) = -1$ so that

$$f(z) = \frac{2}{z}\int_0^1 r\,\mathrm{d}r = \frac{1}{z}.$$

Finally, if $z = 0$, then it is easy to see that

$$f(0) = \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \mathrm{e}^{-i\theta}\,\mathrm{d}\theta\,\mathrm{d}r = 0.$$

We have completed the analysis of the problem. ■

Problem 10.30

Rudin Chapter 10 Exercise 30.

Proof. Without loss of generality, we may assume that $\Omega = \mathbb{C} \setminus \{-1, 1\}$ which is clearly open in $\mathbb{C}$. Consider the boundaries $\gamma_1, \gamma_2, \gamma_3$ and γ_4 of the discs $D(-1; 1), D(1; 1), D(-2; 2)$ and $D(2; 2)$ respectively. Each starts and ends at 0 with the orientation as shown in Figure 10.5. Then

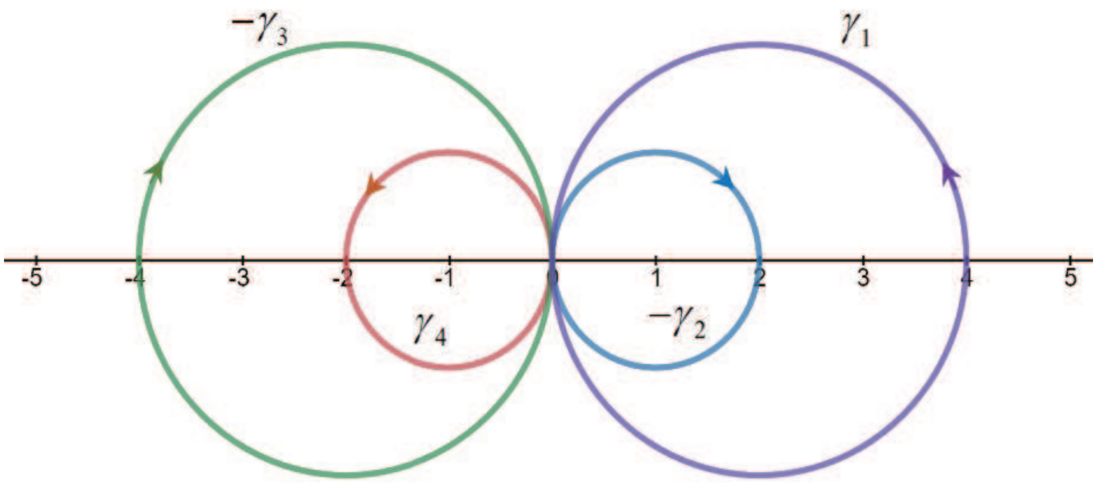


Figure 10.5: A non null-homotopic closed path $\Gamma = \gamma_1 - \gamma_3 - \gamma_2 + \gamma_4$ in Ω.

$\Gamma = \gamma_1 - \gamma_3 - \gamma_2 + \gamma_4$ is a closed curve in Ω and we observe easily that

$$\mathrm{Ind}_\Gamma(\pm 1) = 0.$$

Assume that Γ was null-homotopic to the constant map 0 in Ω. By §10.38, there exists a continuous map $\Gamma_t : [0, 1] \to \Omega$ connecting Γ and 0 such that $\Gamma_0 = 0$ and $\Gamma_1 = \Gamma$. However, the continuity of Γ_t forces that it must pass through one of the omitted points 1 and -1, a contradiction. This completes the proof of the problem.

■

CHAPTER 11

Harmonic Functions

11.1 Basic Properties of Harmonic Functions

Problem 11.1

Rudin Chapter 11 Exercise 1.

Proof. Let $u, v : \Omega \to \mathbb{R}$. We prove the assertions one by one.

- **uv is harmonic if and only if $u + icv \in H(\Omega)$ for some $c \in \mathbb{R}$.** Direct computation gives

$$\begin{aligned}\Delta(uv) &= (uv)_{xx} + (uv)_{yy}\\ &= (uv_{xx} + 2u_x v_x + u_{xx}v) + (uv_{yy} + 2u_y v_y + u_{yy}v)\\ &= 2(u_x v_x + u_y v_y)\\ &= 2(u_x, u_y) \cdot (v_x, v_y).\end{aligned}$$

Thus uv is harmonic if and only if

$$(u_x, u_y) \cdot (v_x, v_y) = 0. \tag{11.1}$$

If $u + icv \in H(\Omega)$, then it yields from the Cauchy-Riemann equations that $u_x = cv_y$ and $u_y = -cv_x$ so that $u_x v_x + u_y v_y = cv_y v_x - cv_x v_y = 0$ in Ω. This means that $\Delta(uv) = 0$. Conversely, suppose that uv is harmonic in Ω. If u is constant, then Theorem 11.2 ensures that $u_x = u_y = 0$ in Ω which gives the equation (11.1). The case for v being constant is similar. Therefore, without loss of generality, we may assume that both u and v are nonconstant. Clearly, the equation (11.1) implies that there exists a function $c : \mathbb{R}^2 \to \mathbb{R}$ such that $(u_x, u_y) = c(x, y)(v_y, -v_x)$ in Ω, or equivalently,

$$u_x = c(x, y)v_y \quad \text{and} \quad u_y = -c(x, y)v_x \tag{11.2}$$

in Ω. On the one hand, since $\Delta u = 0$ in Ω, we obtain

$$c_x v_y - c_y v_x = 0 \tag{11.3}$$

in Ω. On the other hand, $u_{xy} - u_{yx} = 0$ and $\Delta v = 0$ imply that

$$c_y v_y + cv_{yy} - (-c_x v_x - cv_{xx}) = 0$$

and thus

$$c_y v_y - c_x v_x = 0. \tag{11.4}$$

Eliminating v_x and v_y from the equations (11.3) and (11.4), we get

$$(c_x^2 + c_y^2)v_x = 0 \quad \text{and} \quad (c_x^2 + c_y^2)v_y = 0 \tag{11.5}$$

in Ω. Since v is harmonic in Ω, the function $f = v_x - iv_y$ is holomorphic in Ω by Theorem 11.2. If $Z(f) = \Omega$, then $v_x = v_y = 0$ in Ω so that v is constant in Ω, a contradiction. By Theorem 10.18, $Z(f)$ has no limit point in Ω. Pick $z \in \Omega \setminus Z(f)$. Then there exists a $\delta > 0$ such that $D(z;\delta) \cap Z(f) = \varnothing$. By the equations (11.5), we have $(c_x^2 + c_y^2)f = 0$ which implies that $c_x^2 + c_y^2 = 0$ in $D(z;\delta)$. Since z is arbitrary, we conclude that

$$c_x^2 + c_y^2 = 0 \tag{11.6}$$

in $\Omega \setminus Z(f)$. Since $f = v_x - iv_y \in H(\Omega)$, f has continuous derivatives of all orders. Thus both v_x and v_y have continuous partial derivatives of all orders in Ω. Similarly, both u_x and u_y have continuous partial derivatives of all orders in Ω. Therefore, it follows from any one of the equations (11.2) that both c_x and c_y are continuous on Ω. Hence the equation (11.6) holds in Ω so that $c_x = c_y = 0$ in Ω. In conclusion, $c \in \mathbb{R}$ and hence $u + icv$ satisfies the Cauchy-Riemann equations.

- **u^2 cannot be harmonic in Ω unless u is constant.** This part is shown in [76, Problem 16.3, pp. 199, 200].
- **$|f|^2$ is harmonic.** Since $f \in H(\Omega)$, we know from Theorem 11.4 that both u and v are harmonic in Ω. Note that $|f|^2$ is harmonic in Ω if and only if $u^2 + v^2$ is harmonic in Ω. Clearly, we have

$$\begin{aligned}\Delta(u^2 + v^2) &= (u^2 + v^2)_{xx} + (u^2 + v^2)_{yy} \\ &= (2uu_x + 2vv_x)_x + (2uu_y + 2vv_y)_y \\ &= (2uu_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2) + (2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2) \\ &= 2(u_x^2 + v_x^2 + u_y^2 + v_y^2).\end{aligned}$$

Therefore, $\Delta(u^2 + v^2) = 0$ in Ω if and only if $u_x = u_y = v_x = v_y = 0$ in Ω if and only if both $u(x, y)$ and $v(x, y)$ are constant in Ω. In other words, $|f|^2$ is harmonic in Ω if and only if f is constant in Ω.

This completes the proof of the problem. ■

Problem 11.2

Rudin Chapter 11 Exercise 2.

Proof. Let $f = u + iv$. The result is obvious if f is constant, so without loss of generality, we may assume that f is nonconstant. Since f is harmonic in Ω, both u and v are harmonic in Ω. Similarly, since $f^2 = u^2 - v^2 + 2iuv$ is harmonic in Ω, both $u^2 - v^2$ and uv are harmonic in Ω. Thus $\Delta(u^2 - v^2) = 0$ and $\Delta(uv) = 0$ imply that

$$u_x^2 + u_y^2 = v_x^2 + v_y^2 \quad \text{and} \quad u_x v_x + u_y v_y = 0$$

in Ω respectively. Consequently, they show that

$$(u_x + iv_x)^2 + (u_y + iv_y)^2 = 0$$

$$\begin{aligned}(u_x + iv_x)^2 - [i(u_y + iv_y)]^2 &= 0\\ [u_x + iv_x + i(u_y + iv_y)][u_x + iv_x - i(u_y + iv_y)] &= 0\\ (u_x - v_y + iv_x + iu_y)(u_x + v_y + iv_x - iu_y) &= 0\end{aligned}$$

holds in Ω.

Next, the equation $(u_x - iv_x)^2 + (u_y - iv_y)^2 = 0$ gives

$$(u_x - v_y - iv_x - iu_y)(u_x + v_y - iv_x + iu_y) = 0 \tag{11.7}$$

in Ω. We denote

$$g = u_x - v_y - i(v_x + u_y) \quad \text{and} \quad h = u_x + v_y + i(u_y - v_x). \tag{11.8}$$

It is obvious that

$$(u_x - v_y)_x = u_{xx} - v_{yx} = -u_{yy} - v_{xy} = -(u_y + v_x)_y$$

and

$$(u_x - v_y)_y = u_{xy} - v_{yy} = u_{yx} + v_{xx} = -[-(u_y + v_x)]_x$$

hold in Ω. In other words, $g \in H(\Omega)$. Similarly, we have $\overline{h} \in H(\Omega)$.

Assume that $g \not\equiv 0$ and $\overline{h} \not\equiv 0$ in Ω. Then Theorem 10.18 ensures that $Z(g)$ and $Z(\overline{h})$ are at most countable. Therefore, $Z(g\overline{h}) = Z(g)Z(\overline{h})$ is also at most countable. Note that zeros of h and $\overline{h}$ are identical. Therefore, the equation (11.7) says that $Z(g\overline{h}) = \Omega$, a contradiction. Hence we have $g \equiv 0$ or $h \equiv 0$ in Ω. If $g \equiv h \equiv 0$ in Ω, then we deduce from the definition (11.8) that $u_x = u_y = 0$ in Ω, i.e., u is constant in Ω. By this and the fact $f \in H(\Omega)$, we obtain immediately from Theorem 11.2 that v is also constant in Ω. Hence f is also constant in $H(\Omega)$, a contradiction. In other words, we have either $g \equiv 0$ or $h \equiv 0$ in Ω.

- **Case (i): $g \equiv 0$ in Ω.** By the definition, $g \equiv 0$ implies that $u_x - v_y = i(v_x + u_y)$ in Ω. Since u_x, u_y, v_x and v_y are real functions, we get $u_x = v_y$ and $u_y = -v_x$ in Ω. By Theorem 11.2, we conclude that $f \in H(\Omega)$.

- **Case (ii): $h \equiv 0$ in Ω.** Similarly, this shows that $u_x = -v_y$ and $u_y = v_x$ in Ω and these mean that $\overline{f} = u - iv \in H(\Omega)$.

We have completed the proof of the problem. ■

Problem 11.3

Rudin Chapter 11 Exercise 3.

Proof. Let u be a real function. Suppose that

$$V = \{z \in \Omega \,|\, \operatorname{grad} u = 0\} = \{z \in \Omega \,|\, u_x(z) = u_y(z) = 0\} \quad \text{and} \quad f = u_x - iu_y.$$

Since u_x and u_y are continuous in Ω, V is closed in Ω. By Theorem 11.2, $f \in H(\Omega)$. It is trivial that $V = Z(f)$. By Theorem 10.18, either $V = \Omega$ or V has no limit point in Ω.[a] This ends the analysis of the problem. ■

[a]Note that the case $V = \Omega$ implies that u is constant in Ω.

Problem 11.4

Rudin Chapter 11 Exercise 4.

Proof. We prove the assertions one by one.

- **Every partial derivative of a harmonic is harmonic.** See [76, Problem 16.2, p. 199].
- **$P_r(\theta - t)$ is a harmonic function of $re^{i\theta}$ for a fixed t.** It can be shown easily that the Laplacian equation in polar form is given by

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{11.9}$$

Fix t. Write $P = P_r(\theta - t)$ and $A = A(r, \theta) = 1 - 2r\cos(\theta - t) + r^2$ for convenience. Then [62, Eqn. (2), §11.5, p. 233] can be written as $PA = 1 - r^2$. Direct differentiation gives

$$\begin{aligned}
\frac{P_r}{r} &= -\frac{2[r - \cos(\theta - t)]}{rA}P - \frac{2}{A}, \\
P_{rr} &= -\frac{4[r - \cos(\theta - t)]}{A}P_r - \frac{2}{A}P - \frac{2}{A}, \\
\frac{P_{\theta\theta}}{r^2} &= \frac{8\sin^2(\theta - t)}{A^2}P - \frac{2\cos(\theta - t)}{rA}P.
\end{aligned} \tag{11.10}$$

Put these equations (11.10) into the Laplacian equation (11.9) to get

$$\begin{aligned}
\Delta P &= -\frac{4[r - \cos(\theta - t)]}{A}P_r - \frac{2}{A}P - \frac{2}{A} - \frac{2[r - \cos(\theta - t)]}{rA}P - \frac{2}{A} \\
&\quad + \frac{8\sin^2(\theta - t)}{A^2}P - \frac{2\cos(\theta - t)}{rA}P \\
&= \frac{8[r - \cos(\theta - t)]^2}{A^2}P + \frac{8r[r - \cos(\theta - t)]}{A^2} - \frac{4}{A}P - \frac{4}{A} + \frac{8\sin^2(\theta - t)}{A^2}P \\
&= \frac{8[r - \cos(\theta - t)]^2 - 4A + 8\sin^2(\theta - t)}{A^2}P + \frac{8r[r - \cos(\theta - t)] - 4A}{A^2} \\
&= \frac{8[r^2 - 2r\cos(\theta - t) + 1] - 4A}{A^2}P + \frac{4[r^2 + r^2 - 2r\cos(\theta - t)] - 4A}{A^2} \\
&= \frac{4P}{A} + \frac{4(r^2 + A - 1) - 4A}{A^2} \\
&= \frac{4(1 - r^2)}{A^2} + \frac{4(r^2 - 1)}{A^2} \\
&= 0.
\end{aligned}$$

Hence $P_r(\theta - t)$ is a harmonic function of $re^{i\theta}$ for a fixed t.

- **$P[\,\mathrm{d}\mu]$ is harmonic in U.** Suppose that μ is a finite Borel measure on T and $u = P[\,\mathrm{d}\mu]$. It suffices to show that

$$u_r = \frac{\partial}{\partial r}\int_T P_r(\theta - t)\,\mathrm{d}\mu(\mathrm{e}^{it}) = \int_T \frac{\partial}{\partial r}P_r(\theta - t)\,\mathrm{d}\mu(\mathrm{e}^{it}) = \int_T [P_r(\theta - t)]_r\,\mathrm{d}\mu(\mathrm{e}^{it}) \tag{11.11}$$

and

$$u_\theta = \frac{\partial}{\partial \theta}\int_T P_r(\theta - t)\,\mathrm{d}\mu(\mathrm{e}^{it}) = \int_T \frac{\partial}{\partial \theta}P_r(\theta - t)\,\mathrm{d}\mu(\mathrm{e}^{it}) = \int_T [P_r(\theta - t)]_\theta\,\mathrm{d}\mu(\mathrm{e}^{it}) \tag{11.12}$$

because they certainly yield that

$$u_{rr} = \int_T [P_r(\theta - t)]_{rr}\,\mathrm{d}\mu(\mathrm{e}^{it}) \quad \text{and} \quad u_{\theta\theta} = \int_T [P_r(\theta - t)]_{\theta\theta}\,\mathrm{d}\mu(\mathrm{e}^{it})$$

so that

$$\Delta u = \int_T \Delta P_r(\theta - t)\,\mathrm{d}\mu(\mathrm{e}^{it}) = 0.$$

In other words, $u = P[\,\mathrm{d}\mu]$ is harmonic in U.

Fix $r \in [0, 1)$. For very small $h > 0$ such that $1 - r - 2h > 0$, we note that

$$\begin{aligned} \frac{u(r + h, \theta) - u(r, \theta)}{h} &= \int_T \frac{P_{r+h}(\theta - t) - P_r(\theta - t)}{h}\,\mathrm{d}\mu(\mathrm{e}^{it}) \\ u_r = \lim_{h\to 0} &\int_T \frac{P_{r+h}(\theta - t) - P_r(\theta - t)}{h}\,\mathrm{d}\mu(\mathrm{e}^{it}). \end{aligned} \tag{11.13}$$

We observe that

$$\lim_{h\to 0} \frac{P_{r+h}(\theta - t) - P_r(\theta - t)}{h} = \frac{\partial}{\partial r} P_r(\theta - t)$$

and

$$\begin{aligned} &\frac{P_{r+h}(\theta - t) - P_r(\theta - t)}{h} \\ &= \frac{1}{h} \cdot \Big[\frac{1 - (r + h)^2}{1 - 2(r + h)\cos(\theta - t) + (r + h)^2} - \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\Big] \\ &= \frac{[1 - (r + h)^2][1 - 2r\cos(\theta - t) + r^2] - (1 - r^2)[1 - 2(r + h)\cos(\theta - t) + (r + h)^2]}{h[1 - 2(r + h)\cos(\theta - t) + (r + h)^2]\cdot[1 - 2r\cos(\theta - t) + r^2]} \\ &= \frac{2h\cos(\theta - t) + 2r^2 - 2(r + h)^2 + 2hr(r + h)\cos(\theta - t)}{h[1 - 2(r + h)\cos(\theta - t) + (r + h)^2][1 - 2r\cos(\theta - t) + r^2]} \\ &= \frac{2h\cos(\theta - t) - 2h(2r + h) + 2hr(r + h)\cos(\theta - t)}{h[1 - 2(r + h)\cos(\theta - t) + (r + h)^2]\cdot[1 - 2r\cos(\theta - t) + r^2]} \\ &= \frac{2\cos(\theta - t) - 2(2r + h) + 2r(r + h)\cos(\theta - t)}{[1 - 2(r + h)\cos(\theta - t) + (r + h)^2]\cdot[1 - 2r\cos(\theta - t) + r^2]}. \end{aligned} \tag{11.14}$$

Since $1 - r - 2h > 0$ implies $1 - (r + h) > \frac{1-r}{2}$, this and the expression (11.14) give

$$\Big|\frac{P_{r+h}(\theta - t) - P_r(\theta - t)}{h}\Big| \le \frac{2 + 2(2 + h) + 2(1 + h)}{(1 - r)^2[1 - (r + h)]^2} \le \frac{4(8 + 4h)}{(1 - r)^4} \le \frac{40}{(1 - r)^4}$$

for every $\mathrm{e}^{it} \in T$. Since μ is a finite Borel measure on T, we have

$$\int_T \frac{40}{(1 - r)^4}\,\mathrm{d}\mu(\mathrm{e}^{it}) = \frac{40}{(1 - r)^4} \cdot \mu(T) < \infty$$

which means that $\frac{40}{(1-r)^4} \in L^1(T)$. Hence Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) ensures that the order of integration and the limit in (11.13) can be changed and this action shows that the formula (11.11) holds. Similarly, it is easily to check that

$$\begin{aligned} \Big|\frac{P_r(\theta + h - t) - P_r(\theta - t)}{h}\Big| &= \Big|\frac{-4r(1 - r^2)\sin(\theta - t + \frac{h}{2})\sin\frac{h}{2}}{h[1 - 2r\cos(\theta + h - t) + r^2][1 - 2r\cos(\theta - t) + r^2]}\Big| \\ &\le \frac{2r(1 - r^2)}{(1 - r)^4} \cdot \Big|\frac{\sin\frac{h}{2}}{\frac{h}{2}}\Big| \end{aligned}$$

$$\leq \frac{4}{(1-r)^3} \cdot \frac{3}{2}$$
$$= \frac{6}{(1-r)^3}.$$

Again, the finiteness of μ implies that $\frac{6}{(1-r)^3} \in L^1(T)$ and we can apply Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) to conclude that the formula (11.12) holds.

This ends the proof of the problem. ■

Problem 11.5

Rudin Chapter 11 Exercise 5.

Proof. Let $f = u + iv$. Since $f \in H(\Omega)$, both u and v are harmonic in Ω. Since $|f| \neq 0$ in Ω, $\log|f|$ is a well-defined real function and in fact $\log|f| = \log(u^2+v^2)^{\frac{1}{2}}$. Using $\Delta u = \Delta v = 0$ and the Cauchy-Riemann equations, we get

$$\begin{aligned}
\Delta(\log|f|) &= [\log(u^2+v^2)^{\frac{1}{2}}]_{xx} + [\log(u^2+v^2)^{\frac{1}{2}}]_{yy} \\
&= \frac{1}{2}\big[\big(\log(u^2+v^2)\big)_{xx} + \big(\log(u^2+v^2)\big)_{yy}\big] \\
&= \Big(\frac{uu_x+vv_x}{u^2+v^2}\Big)_x + \Big(\frac{uu_y+vv_y}{u^2+v^2}\Big)_y \\
&= \frac{(u^2+v^2)(u\Delta u + u_x^2 + u_y^2 + v\Delta v + v_x^2 + v_y^2) - 2(uu_x+vv_x)^2 - 2(uu_y+vv_y)^2}{(u^2+v^2)^2} \\
&= \frac{(u^2+v^2)(u_x^2+u_y^2+v_x^2+v_y^2) - 2[u^2(u_x^2+u_y^2) + 2uv(u_xv_x+u_yv_y) + v^2(v_x^2+v_y^2)]}{(u^2+v^2)^2} \\
&= \frac{2(u^2+v^2)(v_x^2+v_y^2) - 2(u^2+v^2)(v_x^2+v_y^2)}{(u^2+v^2)^2} \\
&= 0
\end{aligned}$$

in Ω. Hence $\log|f|$ is harmonic in Ω.

If f is holomorphic and non-vanishing in Ω, then $\frac{1}{f}, f' \in H(\Omega)$. Thus we obtain $\frac{f'}{f} \in H(\Omega)$. Since $(\log f)' = \frac{f'}{f}$, we conclude $\log f \in H(\Omega)$ so that its real part, which is $\log|f|$, is harmonic in Ω, completing the proof of the problem. ■

Problem 11.6

Rudin Chapter 11 Exercise 6.

Proof. Let $A(\Omega)$ be the area of the region $\Omega = f(U)$. Referring to [2, §2.4, pp. 75, 76],[b] we know that

$$A(\Omega) = \iint_U |f'(z)|^2\,dx\,dy,$$

[b]See also the discussion on [59, Eqn. (6), p. 11].

where $f(z) = u(x,y) + iv(x,y)$. Put $z = re^{i\theta}$, where $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$. Then the formula of $A(\Omega)$ becomes

$$A(\Omega) = \int_0^{2\pi} \int_0^1 |f'(r\cos\theta, r\sin\theta)|^2 r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{11.15}$$

Since $|f'(z)|^2 = f'(z) \cdot \overline{f'(z)}$, we see that

$$\begin{aligned} |f'(r\cos\theta, r\sin\theta)|^2 &= \Big(\sum_{n=1}^{\infty} nc_n r^{n-1} \mathrm{e}^{i(n-1)\theta}\Big) \times \Big(\sum_{m=1}^{\infty} m\overline{c_m} r^{m-1} \mathrm{e}^{-i(m-1)\theta}\Big) \\ &= \sum_{m,n=1}^{\infty} nmc_n\overline{c_m} r^{n+m-2} \mathrm{e}^{i(n-m)\theta}. \end{aligned}$$

Recall from Definition 4.23 that $\{\mathrm{e}^{in\theta}\}$ forms an orthonormal set, the integral (11.15) becomes

$$A(\Omega) = \int_0^{2\pi} \int_0^1 \Big(\sum_{n=1}^{\infty} n^2|c_n|^2 r^{2n-1}\Big)\, \mathrm{d}r\, \mathrm{d}\theta. \tag{11.16}$$

Since $f'(z) = \sum\limits_{n=1}^{\infty} nc_n z^{n-1} \in H(U)$, we have

$$\limsup_{n\to\infty} \sqrt[n]{n|c_n|} = 1$$

which implies that

$$\limsup_{n\to\infty} \sqrt[n]{n^2|c_n|^2} = 1.$$

Therefore, the radius of convergence of the series $\sum\limits_{n=1}^{\infty} n^2|c_n|^2 r^{2n-1}$ is 1 and then the order of integration in the integral (11.16) can be switched so that

$$A(\Omega) = \int_0^{2\pi} \sum_{n=1}^{\infty} \Big(\int_0^1 n^2|c_n|^2 r^{2n-1}\, \mathrm{d}r\Big)\, \mathrm{d}\theta = 2\pi \sum_{n=1}^{\infty} n^2|c_n|^2 \times \frac{1}{2n} = \pi \sum_{n=1}^{\infty} n|c_n|^2,$$

as required. This completes the proof of the problem. ■

Problem 11.7

Rudin Chapter 11 Exercise 7.

Proof. Suppose that $f = u + iv$.

(a) Let ψ be a twice differentiable function on $(0,\infty)$. Then $f\overline{f} = u^2 + v^2 = |f|^2$. Using the formulas [62, Eqn. (3), p. 231], we see that

$$\begin{aligned} \partial\overline{\partial}[\psi \circ (f\overline{f})] &= \frac{1}{4}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\psi(|f|^2) \\ &= \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)[\psi'(|f|^2)\cdot(uu_x + vv_x) + i\psi'(|f|^2)\cdot(uu_y + vv_y)] \\ &= \frac{1}{2}\Big\{[2\psi''(|f|^2)\cdot(uu_x + vv_x)^2 + \psi'(|f|^2)\cdot(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}) \end{aligned}$$

$$
\begin{aligned}
&+ i\psi''(|f|^2)\cdot(uu_x+vv_x)(uu_y+vv_y)\\
&+ i\psi'(|f|^2)\cdot(u_xu_y+uu_{xy}+v_xv_y+vv_{xy})\big]\\
&+\big[-i\psi''(|f|^2)\cdot(uu_y+vv_y)(uu_x+vv_x)\\
&-i\psi'(|f|^2)(u_yu_x+uu_{xy}+v_xv_y+vv_{xy})\\
&+2\psi''(|f|^2)\cdot(uu_y+vv_y)^2+\psi'(|f|^2)\cdot(u_y^2+uu_{yy}+v_y^2+vv_{yy})\big]\Big\}. \qquad (11.17)
\end{aligned}
$$

Since $f\in H(\Omega)$, both u and v are harmonic in Ω. Furthermore, we note from Theorem 11.2 that $|f'|^2=u_x^2+v_x^2=u_y^2+v_y^2$, so the expression (11.17) becomes

$$
\begin{aligned}
\partial\overline{\partial}[\psi\circ(f\overline{f})] &= \frac{1}{2}\Big\{2\psi''(|f|^2)\cdot\big[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)\big]\\
&\quad+\psi'(|f|^2)\cdot\big[u_x^2+u_y^2+v_x^2+v_y^2+u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})\big]\Big\}\\
&=\frac{1}{2}\big[2\psi''(|f|^2)\cdot|f|^2\cdot|f'|^2+2\psi'(|f|^2)\cdot|f'|^2\big]\\
&=[|f|^2\psi''(|f|^2)+\psi'(|f|^2)]\cdot|f'|^2\\
&=(\varphi\circ|f|^2)\cdot|f'|^2 \qquad (11.18)
\end{aligned}
$$

as required.

Now the function $\psi(t)=t^{\frac{\alpha}{2}}$ is clearly twice differentiable on $(0,\infty)$ with

$$\varphi(t)=\frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)t^{\frac{\alpha}{2}-1}+\frac{\alpha}{2}t^{\frac{\alpha}{2}-1}=\frac{\alpha^2}{4}t^{\frac{\alpha}{2}-1},$$

so we combine the formula (11.18) and [62, Eqn. (3), p. 232] to get

$$\Delta(|f|^\alpha)=4\partial\overline{\partial}(|f|^\alpha)=\alpha^2|f|^{\alpha-2}|f'|^2.$$

(b) Suppose that $\Phi: f(\Omega)\to\mathbb{C}$ is defined by $\Phi(f)=\Phi(u+iv)=\Phi(u(x,y),v(x,y))$. Since $u_{xx}+u_{yy}=v_{xx}+v_{yy}=0$, we have

$$
\begin{aligned}
\Delta[\Phi\circ f] &= \Delta\Phi(f)\\
&=\frac{\partial^2}{\partial x^2}\Phi(u,v)+\frac{\partial^2}{\partial y^2}\Phi(u,v)\\
&=\frac{\partial}{\partial x}(\Phi_u\cdot u_x+\Phi_v\cdot v_x)+\frac{\partial}{\partial y}(\Phi_u\cdot u_y+\Phi_v\cdot v_y)\\
&=u_x\frac{\partial}{\partial x}\Phi_u+\Phi_u\cdot u_{xx}+v_x\frac{\partial}{\partial x}\Phi_v+\Phi_v\cdot v_{xx}\\
&\quad+u_y\frac{\partial}{\partial y}\Phi_u+\Phi_u\cdot u_{yy}+v_y\frac{\partial}{\partial y}\Phi_v+\Phi_v\cdot v_{yy}\\
&=u_x(\Phi_{uu}\cdot u_x+\Phi_{uv}\cdot v_x)+\Phi_u\cdot u_{xx}+v_x(\Phi_{uv}\cdot u_x+\Phi_{vv}\cdot v_x)+\Phi_v\cdot v_{xx}\\
&\quad+u_y(\Phi_{uu}\cdot u_y+\Phi_{uv}\cdot v_y)+\Phi_u\cdot u_{yy}+v_y(\Phi_{uv}\cdot u_y+\Phi_{vv}\cdot v_y)+\Phi_v\cdot v_{yy}\\
&=\Phi_{uu}\cdot(u_x^2+u_y^2)+\Phi_{vv}\cdot(v_x^2+v_y^2)+2\Phi_{uv}\cdot u_xv_x+2\Phi_{uv}\cdot u_yv_y. \qquad (11.19)
\end{aligned}
$$

According to the Cauchy-Riemann equations, we can further reduce the expression (11.19) to

$$
\begin{aligned}
\Delta[\Phi\circ f] &= \Phi_{uu}\cdot(u_x^2+v_x^2)+\Phi_{vv}\cdot(v_x^2+u_x^2)\\
&=(\Phi_{uu}+\Phi_{vv})\cdot|f'|^2\\
&=[(\Delta\Phi)\circ f]\cdot|f'|^2 \qquad (11.20)
\end{aligned}
$$

as desired.

Finally, we take $\Phi(w) = |w|$ so that $\Phi(w) = \Phi(|w|)$. Suppose that $f = u + iv$ and $w = |f|^\alpha = (u^2+v^2)^{\frac{\alpha}{2}}$. Direct differentiation implies that

$$\begin{aligned}\Phi_{uu} + \Phi_{vv} &= 2\alpha(u^2+v^2)^{\frac{\alpha}{2}-1} + \alpha(\alpha-2)u^2(u^2+v^2)^{\frac{\alpha}{2}-2} + \alpha(\alpha-2)v^2(u^2+v^2)^{\frac{\alpha}{2}-2}\\ &= \alpha^2(u^2+v^2)^{\frac{\alpha}{2}-1}\end{aligned}$$

and thus

$$(\Delta\Phi)\circ f = \alpha^2|f|^{\alpha-2}.$$

Substituting this into the formula (11.20), we have established that

$$\Delta(|f|^\alpha) = \alpha^2|f|^{\alpha-2}\cdot|f'|^2.$$

This ends the analysis of the problem. ■

Problem 11.8

Rudin Chapter 11 Exercise 8.

Proof. The proof of this problem will be divided into two steps as follows:

- **Step 1: $\{f_n(z)\}$ converges at every point of Ω.** Suppose that

$$S_1 = \{z \in \Omega \,|\, \{f_n(z)\} \text{ converges}\}$$

Let $a \in \Omega$ and $R > 0$ be such that

$$\overline{D}(a;2R) \subseteq \Omega. \tag{11.21}$$

We consider the functions $\widehat{u_n}(z) = u_n(z+a)$ and $\widehat{f_n}(z) = f_n(z+a)$ which are defined in $\overline{D}(0;2R)$ and in $D(0;2R)$ respectively. By [9, Theorem 16.9, p. 233][c], we have

$$\widehat{f_n}(z) = i\mathrm{Im}\,\widehat{f_n}(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{2R\mathrm{e}^{it}+z}{2R\mathrm{e}^{it}-z}\Big)\widehat{u_n}(2R\mathrm{e}^{it})\,\mathrm{d}t$$

and thus

$$f_n(z+a) = i\mathrm{Im}\,f_n(a) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{2R\mathrm{e}^{it}+z}{2R\mathrm{e}^{it}-z}\Big)u_n(a+2R\mathrm{e}^{it})\,\mathrm{d}t,$$

where $z \in D(0;2R)$. If we take $z = R\mathrm{e}^{i\theta}$, then it is easy to see that

$$\Big|\frac{2R\mathrm{e}^{it}+z}{2R\mathrm{e}^{it}-z}\Big| = \Big|\frac{2R\mathrm{e}^{it}+R\mathrm{e}^{i\theta}}{2R\mathrm{e}^{it}-R\mathrm{e}^{i\theta}}\Big| \leq 3$$

and this implies that

$$\begin{aligned}|f_n(z+a)-f_m(z+a)| &\leq \big|\mathrm{Im}\,[f_n(a)-f_m(a)]\big|\\ &\quad + \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{2R\mathrm{e}^{it}+z}{2R\mathrm{e}^{it}-z}\Big)[u_n(a+2R\mathrm{e}^{it})-u_m(a+2R\mathrm{e}^{it})]\,\mathrm{d}t\Big|\\ &\leq |f_n(a)-f_m(a)| + \frac{3}{2\pi}\int_{-\pi}^{\pi}\big|u_n(a+2R\mathrm{e}^{it})-u_m(a+2R\mathrm{e}^{it})\big|\,\mathrm{d}t\end{aligned}$$

[c]In fact, this is a generalization of Theorem 11.9 and it is called the **Schwarz Integral Formula**. See, for example, [37, p. 408].

$$\leq |f_n(a) - f_m(a)| + 3 \cdot \max_{-\pi \leq t \leq \pi} \left| u_n(a + 2Re^{it}) - u_m(a + 2Re^{it}) \right|.$$

Consequently, we conclude that

$$\sup_{z \in D(0;R)} |f_n(z+a) - f_m(z+a)| \leq 3 \cdot \max_{-\pi \leq t \leq \pi} \left| u_n(a + 2Re^{it}) - u_m(a + 2Re^{it}) \right| + |f_n(a) - f_m(a)|$$

or equivalently,

$$\sup_{z \in D(a;R)} |f_n(z) - f_m(z)| \leq 3 \cdot \max_{\zeta \in C(a;2R)} |u_n(\zeta) - u_m(\zeta)| + |f_n(a) - f_m(a)|. \tag{11.22}$$

Given $\epsilon > 0$. Now our hypothesis asserts that there is an $N_1 \in \mathbb{N}$ such that $n, m \geq N_1$ imply

$$|u_n(\zeta) - u_m(\zeta)| < \frac{\epsilon}{6}$$

for all $\zeta \in \overline{D}(a; 2R)$. Thus it follows from the inequality (11.22) that

$$|f_n(z) - f_m(z)| \leq |f_n(a) - f_m(a)| + \frac{\epsilon}{2} \tag{11.23}$$

for all $z \in D(a; R)$.

Particularly, if $a \in S_1$, then there exists an $N_2 \in \mathbb{N}$ such that $n, m \geq N_2$ imply

$$|f_n(a) - f_m(a)| < \frac{\epsilon}{2}$$

so that the inequality (11.23) gives

$$|f_n(z) - f_m(z)| < \epsilon \tag{11.24}$$

for all $z \in D(a; R)$ and all $n, m \geq N = \max(N_1, N_2)$. By [61, Theorem 7.8, p. 147], the sequence $\{f_n\}$ converges uniformly in $D(a; R)$ and this definitely implies

$$D(a; R) \subseteq S_1. \tag{11.25}$$

To finish the proof that $S_1 = \Omega$, we need a result from [15, Theorem 1.28, p. 28]:

Lemma 11.1 (The Basic Connectedness Lemma)

Suppose that Ω is open in $\mathbb{C}$ and A is a nonempty subset of Ω. If there exists a $\theta > 0$ such that for every $a \in A$ and $\overline{D}(a; 2R) \subseteq \Omega$ for some $R > 0$, we have $D(a; 2\theta R) \subseteq A$, then A is relatively open and closed in Ω. Furthermore, if Ω is a region, then we have

$$A = \Omega.$$

Put $A = S_1$, the above set relations (11.21) and (11.25) mean that we can take $\theta = \frac{1}{2}$ in Lemma 11.1 (The Basic Connectedness Lemma). Hence we conclude immediately that $S_1 = \Omega$, as desired.

- **Step 2: $\{f_n\}$ converges uniformly on compact subsets of Ω.** Let K be a compact subset of Ω. Clearly,

$$K \subseteq \bigcup_{a \in K} D(a; R_a),$$

where $R_a > 0$ is a number satisfying the set relation (11.21). Therefore, it follows from **Step 1** that the inequality (11.24) holds in $D(a; R_a)$ and all $n, m \geq N(a, R_a)$ (of course, the positive integer $N(a, R_a)$ depends on a and R_a) for every $a \in K$. Since K is compact, there exists a finite set $\{a_1, a_2, \ldots, a_p\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^{p} D(a_j; R_j) \subseteq \Omega,$$

where $R_j \in \{R_a \,|\, a \in K\}$ for $1 \leq j \leq p$. In particular, if we take $N = \max\limits_{1 \leq j \leq p} N(a_j, R_j)$, then it is true that $n, m \geq N$ imply

$$|f_n(z) - f_m(z)| < \epsilon$$

for all $z \in K$. Using [61, Theorem 7.8, p. 147] again, the sequence $\{f_n\}$ converges uniformly on K.

This completes the analysis of the problem. ■

Problem 11.9

Rudin Chapter 11 Exercise 9.

Proof. Let $\overline{D}(a; r) \subseteq \Omega$. Since u is locally in L^1, the double integral considered in the question is well-defined. If u is harmonic in Ω, then Definition 11.12 says that it satisfies

$$u(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta}) \, d\theta \tag{11.26}$$

for every $r > 0$ with $\overline{D}(a; r) \subseteq \Omega$. Multiplying both sides of the expression (11.26) by ρ and integrating from 0 to r, we obtain

$$\begin{aligned} \frac{r^2}{2} u(a) &= \int_0^r u(a) \rho \, d\rho \\ &= \int_0^r \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + \rho e^{i\theta}) \, d\theta \Big) \rho \, d\rho \\ &= \frac{1}{2\pi} \int_0^r \int_{-\pi}^{\pi} u(a + \rho e^{i\theta}) \rho \, d\theta \, d\rho \end{aligned}$$

which gives exactly

$$u(a) = \frac{1}{\pi r^2} \int_0^r \int_{-\pi}^{\pi} u(a + \rho e^{i\theta}) \rho \, d\theta \, d\rho = \frac{1}{\pi r^2} \iint\limits_{D(a;r)} u(x, y) \, dx \, dy. \tag{11.27}$$

Conversely, suppose that the formula (11.27) holds for any $\overline{D}(a; r) \subseteq \Omega$. On the one hand, applying the polar coordinates, we can write

$$\iint\limits_{D(a;r)} u(x, y) \, dx \, dy = \int_0^r \int_{-\pi}^{\pi} u(a + \rho e^{i\theta}) \rho \, d\theta \, d\rho. \tag{11.28}$$

On the other hand, we have

$$r^2 u(a) = 2 \int_0^r \rho u(a) \, d\rho. \tag{11.29}$$

Substituting the expressions (11.28) and (11.29) into the formula (11.27), we get

$$u(a)\int_0^r \rho\,\mathrm{d}\rho = \int_0^r \rho u(a)\,\mathrm{d}\rho \tag{11.30}$$
$$= \frac{1}{2\pi}\int_0^r\int_{-\pi}^{\pi} u(a+\rho\mathrm{e}^{i\theta})\rho\,\mathrm{d}\theta\,\mathrm{d}\rho$$
$$= \frac{1}{2\pi}\int_0^r U(a,\rho,\theta)\rho\,\mathrm{d}\rho, \tag{11.31}$$

where

$$U(a,\rho,\theta) = \int_{-\pi}^{\pi} u(a+\rho\mathrm{e}^{i\theta})\,\mathrm{d}\theta.$$

As the integral on the left-hand side in the formula (11.30) is differentiable with respect to r, so is the integral (11.31). Consequently, we derive from the First Fundamental Theorem of Calculus [79, p. 161] that

$$ru(a) = \frac{1}{2\pi}\cdot U(a,r,\theta)r = \frac{r}{2\pi}\int_{-\pi}^{\pi} u(a+r\mathrm{e}^{i\theta})\,\mathrm{d}\theta. \tag{11.32}$$

After cancelling the r in the expression (11.32), we find that

$$u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+r\mathrm{e}^{i\theta})\,\mathrm{d}\theta$$

whenever $\overline{D}(a;r)\subseteq\Omega$.

To finish the proof, we have to show that u is continuous in Ω. To this end, fix $z\in\Omega$. Given that $\epsilon>0$. Since $u\in L^1_{\mathrm{loc}}(\Omega)$,[d] we must have $u\in L^1(D(z;2r'))$ for some $r'>0$ such that $D(z;2r')\subseteq\Omega$. We fix this r'. By Problem 1.12, there exists a $\delta_z>0$ such that

$$\int_E |u|\,\mathrm{d}m < \pi(r')^2\epsilon \tag{11.33}$$

whenever $m(E)<\delta_z$ and $E\subseteq D(z;2r')$. Clearly, we may assume that $\delta_z<r'$. For every $\omega\in D(z;r')$, it is always true that $D(\omega;r')\subseteq D(z;2r')$ and

$$m\big(D(\omega;r')\setminus D(z;r')\big) = m\big(D(z;r')\setminus D(\omega;r')\big) < \delta_z$$

if ω is very close to z. Therefore, we follow from the formula (11.27) and the inequality (11.33) that

$$|u(z)-u(\omega)| = \frac{1}{\pi(r')^2}\left|\iint\limits_{D(z;r')} u(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint\limits_{D(\omega;r')} u(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|$$
$$= \frac{1}{\pi(r')^2}\left|\int_{D(z;r')} u\,\mathrm{d}m - \int_{D(\omega;r')} u\,\mathrm{d}m\right|$$
$$= \frac{1}{2\pi(r')^2}\left|\int_{D(z;r')\setminus D(\omega;r')} u\,\mathrm{d}m - \int_{D(\omega;r')\setminus D(z;r')} u\,\mathrm{d}m\right|$$
$$\leq \frac{1}{2\pi(r')^2}\left[\int_{D(z;r')\setminus D(\omega;r')} |u|\,\mathrm{d}m + \int_{D(\omega;r')\setminus D(z;r')} |u|\,\mathrm{d}m\right]$$
$$< \frac{1}{2\pi(r')^2}\left[\pi(r')^2\epsilon + \pi(r')^2\epsilon\right]$$

[d] The notation $L^1_{\mathrm{loc}}(\Omega)$ is the set of all locally integrable functions on Ω.

$$= \epsilon.$$

By the definition, u is continuous at z. Since z is arbitrary, u is actually continuous in Ω. Finally, Theorem 11.13 asserts that u is harmonic in Ω, completing the proof of the problem. ■

Problem 11.10

Rudin Chapter 11 Exercise 10.

Proof.

- By the definition, we have

$$\begin{aligned} f(x+i\epsilon) - f(x-i\epsilon) &= \frac{1}{2\pi i}\int_a^b \varphi(t)\Big(\frac{1}{t-x-i\epsilon} - \frac{1}{t-x+i\epsilon}\Big)\,\mathrm{d}t \\ &= \int_a^b \frac{\epsilon}{\pi}\cdot\frac{\varphi(t)}{(x-t)^2+\epsilon^2}\,\mathrm{d}t \\ &= \int_a^b P_\epsilon(x-t)\varphi(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \varphi(x-t)P_\epsilon(t)\,\mathrm{d}t, \end{aligned} \tag{11.34}$$

where

$$P_\epsilon(t) = \frac{1}{\pi}\cdot\frac{\epsilon}{t^2+\epsilon^2}$$

relates to the formula [62, Eqn. (3), §9.7, p. 183] (in fact, it is the Poisson kernel for the upper half-plane, see [66, p. 149]) and $\varphi(t) = 0$ if $t \in \mathbb{R}\setminus I$. Using the convolution notation introduced in Theorem 8.14, the expression (11.34) becomes

$$f(x+i\epsilon) - f(x-i\epsilon) = (\varphi * P_\epsilon)(x). \tag{11.35}$$

Since $|\varphi(x)| \leq \max_{t\in I}|\varphi(t)|$ for every $x \in \mathbb{R}$, we get $\varphi \in L^\infty(\mathbb{R})$.

To proceed, we need to modify Theorem 9.9 and gives[e]

Lemma 11.2

If $g \in L^\infty$, then we have

$$\lim_{\lambda\to 0}(g\circ h_\lambda)(x) = \frac{g(x+)+g(x-)}{2},$$

where h_λ is the formula [62, Eqn. (3), §9.7, p. 183].

[e] If g is continuous at x, then $g(x+) = g(x-)$ and it is exactly Theorem 9.9.

Proof of Lemma 11.2. Following the proof of Theorem 9.9, we have

$$
\begin{aligned}
(g * h_\lambda)(x) - \frac{g(x+) + g(x-)}{2} &= \int_{-\infty}^{0} \Big[g(x-\lambda s) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&\quad + \int_{0}^{\infty} \Big[g(x-\lambda s) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s).
\end{aligned}
$$

Since the integrands are dominated by $2\|g\|_\infty h_1(s)$ and the integrals converge pointwise for every s as $\lambda \to 0$, we deduce from Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) that

$$
\begin{aligned}
&\lim_{\lambda\to 0} \Big[(g * h_\lambda)(x) - \frac{g(x+) + g(x-)}{2}\Big] \\
&= \lim_{\lambda\to 0} \int_{-\infty}^{0} \Big[g(x-\lambda s) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&\quad + \lim_{\lambda\to 0} \int_{0}^{\infty} \Big[g(x-\lambda s) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&= \int_{-\infty}^{0} \lim_{\lambda\to 0} \Big[g(x-\lambda s) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&\quad + \int_{0}^{\infty} \lim_{\lambda\to 0} \Big[g(x-\lambda s) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&= \int_{-\infty}^{0} \Big[g(x+) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&\quad + \int_{0}^{\infty} \Big[g(x-) - \frac{g(x+) + g(x-)}{2}\Big] h_1(s)\,\mathrm{d}m(s) \\
&= \int_{-\infty}^{0} \frac{g(x+) - g(x-)}{2} h_1(s)\,\mathrm{d}m(s) + \int_{0}^{\infty} \frac{g(x-) - g(x+)}{2} h_1(s)\,\mathrm{d}m(s) \\
&= \frac{g(x+) - g(x-)}{2} \int_{0}^{\infty} h_1(s)\,\mathrm{d}m(s) + \frac{g(x-) - g(x+)}{2} \int_{0}^{\infty} h_1(s)\,\mathrm{d}m(s) \\
&= 0
\end{aligned}
$$

which implies the desired result. We complete the proof of Lemma 11.2. ■

By Lemma 11.2, we conclude at once that

$$
\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} [f(x+i\epsilon) - f(x-i\epsilon)] = \frac{\varphi(x+) + \varphi(x-)}{2} \tag{11.36}
$$

for every $x \in \mathbb{R}$. Hence the formula (11.36) asserts that

$$
\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} [f(x+i\epsilon) - f(x-i\epsilon)] = \begin{cases} \varphi(x), & \text{if } x \in (a,b); \\ 0, & \text{if } x \in \mathbb{R} \setminus I; \\ \dfrac{\varphi(a+)}{2}, & \text{if } x = a; \\ \dfrac{\varphi(b-)}{2}, & \text{if } x = b. \end{cases}
$$

- **The case when $\varphi \in L^1$.** In this case, we apply Theorem 9.10 to the expression (11.35) to get

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\int_{-\infty}^{\infty}|f(x+i\epsilon)-f(x-i\epsilon)-\varphi(x)|\,dm(x)=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\|\varphi*P_\epsilon-\varphi\|_1=0. \tag{11.37}$$

Denote $f(x+i\epsilon)=f_\epsilon^+(x)$ and $f(x-i\epsilon)=f_\epsilon^-(x)$. Then the result (11.37) means that

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\|f_\epsilon^+-f_\epsilon^-\|_1=\|\varphi\|_1.$$

- **The case when $\varphi(x+)$ and $\varphi(x-)$ exist at x.** This case has been settled already in the formula (11.36).

We have completed the proof of the problem. ■

Remark 11.1

The integral considered in Problem 11.10 is an example of the so-called **Cauchy type integrals**. For more details of this subject, please refer to Muskhelishvili's book [44].

Problem 11.11

Rudin Chapter 11 Exercise 11.

Proof. We note that the following proof uses only the techniques from Chapter 10, not from Chapter 11. Actually, the author admits that he is not able to apply the theory of harmonic functions to prove this problem.

- By Theorem 10.17 (Morera's Theorem), it suffices to prove that

$$\int_{\partial\Delta} f(z)\,dz=0 \tag{11.38}$$

for every closed triangle $\Delta\subseteq\Omega$. There are three cases.

  - **Case (i): $\Delta\cap I=\varnothing$.** Then $\Delta\subseteq\Omega\setminus I$. Since $\mathrm{Ind}_{\partial\Delta}(x)=0$ for every $x\in I$, Theorem 10.35 (Cauchy's Theorem) implies that the result (11.38) holds trivially.
  - **Case (ii): $\partial\Delta$ has a side lying on I.** Since Δ is compact, $\mathbb{C}\setminus\Omega$ is closed in $\mathbb{C}$ and $\Delta\cap(\mathbb{C}\setminus\Omega)=\varnothing$, Problem 10.1 ensures that there exists a $\delta_1>0$ such that $d(\Delta,\mathbb{C}\setminus\Omega)=\delta_1>0$. Therefore, it is true that

$$\Delta_n=\Delta+\frac{i}{n}\subseteq\Omega \tag{11.39}$$

for all $n>\frac{1}{\delta_1}$. Thus we have $\Delta_n\cap I=\varnothing$ so that the result (11.38) holds for every $\partial\Delta_n$ with $n>\frac{1}{\delta_1}$. Suppose that $\partial\Delta=[a,b]+[b,c]+[c,a]$, where a,b and c are vertices of the triangle Δ. Then we have

$$\partial\Delta_n=\Big([a,b]+\frac{i}{n}\Big)+\Big([b,c]+\frac{i}{n}\Big)+\Big([c,a]+\frac{i}{n}\Big).$$

Using [62, Eqn. (4), p. 202], we know that

$$\int_{[a,b]+\frac{i}{n}} f(z)\,dz=(b-a)\int_0^1 f\Big(a+\frac{i}{n}+(b-a)t\Big)\,dt$$

so that

$$\begin{aligned}
&\Big|\int_{[a,b]} f(z)\,\mathrm{d}z - \int_{[a,b]+\frac{i}{n}} f(z)\,\mathrm{d}z\Big| \\
&= |b-a|\cdot\Big|\int_0^1 \Big[f\Big(a+\frac{i}{n}+(b-a)t\Big) - f\big(a+(b-a)t\big)\Big]\,\mathrm{d}t\Big| \\
&\leq |b-a|\cdot\int_0^1 \Big|f\Big(a+\frac{i}{n}+(b-a)t\Big) - f\big(a+(b-a)t\big)\Big|\,\mathrm{d}t.
\end{aligned}$$

Since f is continuous on Ω, it is uniformly continuous on any compact subset of Ω. Given $\epsilon > 0$, there exists a $\delta_2 > 0$ such that $n > \frac{1}{\delta_2}$ implies

$$\Big|f\Big(a+\frac{i}{n}+(b-a)t\Big) - f\big(a+(b-a)t\big)\Big| < \frac{\epsilon}{3|b-a|}$$

and then

$$\Big|\int_{[a,b]} f(z)\,\mathrm{d}z - \int_{[a,b]+\frac{i}{n}} f(z)\,\mathrm{d}z\Big| < \frac{\epsilon}{3}. \tag{11.40}$$

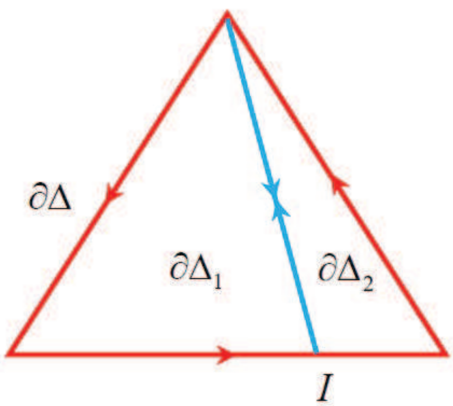
Thus, if $n > \max(\frac{1}{\delta_1}, \frac{1}{\delta_2})$, then both the conditions (11.39) and (11.40) hold simultaneously. Since the inequality (11.40) also holds for $[b,c]$ and $[c,a]$ for large enough n, we obtain immediately that

$$\Big|\int_{\partial\Delta_n} f(z)\,\mathrm{d}z\Big| < \epsilon$$

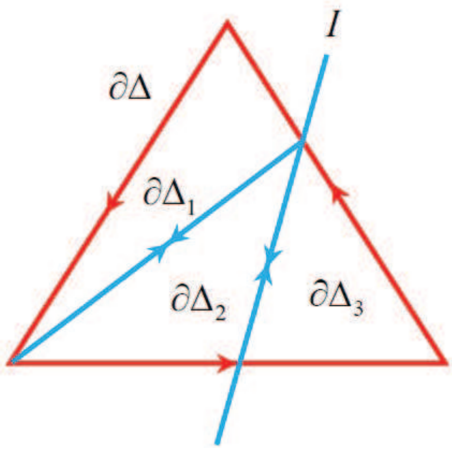
which means

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = \lim_{n\to\infty}\int_{\partial\Delta_n} f(z)\,\mathrm{d}z = 0.$$

– **Case (iii): Δ intersects with I at *only* two points.** Then I divides Δ into a triangle and a quadrangle or another triangle. If it is a quadrangle, then it can be further divided into two triangles, see Figure 11.1 for an illustration. Since $\partial\Delta$ is a sum of two or three boundaries of triangles, it follows from **Case (i)** and **Case (ii)** that our result (11.38) remains true in this case.



(a) $\partial\Delta = \partial\Delta_1 + \partial\Delta_2$.



(b) $\partial\Delta = \partial\Delta_1 + \partial\Delta_2 + \partial\Delta_3$.

Figure 11.1: The I divides $\partial\Delta$ into several triangles.

- **Removable sets for holomorphic functions of class $\mathscr{C}(\Omega)$.** Let $N \in \mathbb{N}$. It is clear that the above argument can be applied to the compact set in the form
$$K = I_1 \cup I_2 \cup \cdots \cup I_N$$
and $f \in H(\Omega \setminus K)$, where each $I_j = [a_j, b_j]$ is a subset of Ω for $1 \leq j \leq N$.

We have completed the analysis of the problem.[f] ■

Remark 11.2

(a) Classically, Problem 11.11 is a topic of the so-called **removable sets for holomorphic functions**. To say it more precisely, let $\mathscr{F}$ be a class of functions from Ω to $\mathbb{C}$. Then a compact set $K \subseteq \Omega$ is said to be removable for holomorphic functions of class $\mathscr{F}$ if every $f \in \mathscr{F}$ such that $f \in H(\Omega \setminus E)$ can be extended to a holomorphic function in Ω. Examples of $\mathscr{F}$ are $L^\infty(\Omega)$, $\mathscr{C}(\Omega)$ and $\mathrm{Lip}_\alpha(\Omega)$, where $0 < \alpha \leq 1$.

(b) It is clear that Theorem 10.20 is a positive result for bounded and holomorphic functions. Because of this, Painlevé was motivated and studied some more general problems: "Which subsets of $\mathbb{C}$ are removable? What geometric characterization(s) must these subsets satisfy?" In fact, Painlevé proved a sufficient condition that if a compact set $K \subseteq \Omega$ is of one-dimensional **Hausdorff measure** [12, pp. 215, 216], then it is removable for bounded and holomorphic functions in $\Omega \setminus K$.

(c) For the class $\mathrm{Lip}_\alpha(\Omega)$ with $0 < \alpha < 1$, Dolženko [19] has shown in 1963 that a compact set K is removable for holomorphic functions of this class if and only if the $(1+\alpha)$-dimensional Hausdorff measure is zero. For the remaining case $\alpha = 1$, Uy [74] verified in 1979 that K is removable if and only if $m(K) = 0$.

(d) Besides the approach of Hausdorff measure, Ahlfors [1] introduced the **analytic capacity** (a purely complex-analytic concept) of a compact set K to study removable compact sets for holomorphic functions of class L^∞. In fact, he proved that K is removable for bounded analytic functions if and only if its analytic capacity vanishes.

11.2 Harnack's Inequalities and Positive Harmonic Functions

Problem 11.12

Rudin Chapter 11 Exercise 12.

Proof.

- **Proof of Harnack's Inequalities.** We first prove the special case that if u is harmonic in $D(a; R)$ and $u > 0$ in $D(a; R)$, then for every $0 \leq r < R$ and $z = a + re^{i\theta} \in D(a; R)$, we have
$$\frac{R-r}{R+r} u(a) \leq u(z) \leq \frac{R+r}{R-r} u(a). \tag{11.41}$$

[f] See also Problem 16.10.

To see this, choose $\rho > 0$ such that $r < \rho < R$. Using the second set of inequalities on [62, p. 236], we know that

$$\frac{\rho - r}{\rho + r} u(a) \leq u(z) = u(a + re^{i\theta}) \leq \frac{\rho + r}{\rho - r} u(a) \tag{11.42}$$

for every $\theta \in \mathbb{R}$. Letting $\rho \to R$ in the inequalities (11.42), we obtain the desired results (11.41).

Since Ω is a region and $K \subseteq \Omega$, $\mathbb{C} \setminus \Omega$ is closed in $\mathbb{C}$ and $(\mathbb{C} \setminus \Omega) \cap K = \varnothing$. By Problem 10.1, there exists a $\delta > 0$ (depending on K and Ω) such that $d(\mathbb{C} \setminus \Omega, K) = 2\delta > 0$. Clearly, we have

$$K \subseteq \bigcup_{z \in K} D(z; \delta_z) \subseteq \bigcup_{z \in K} D(z; \delta) \subseteq \Omega,$$

where $0 < \delta_z < \delta$. Since K is compact, one can find a finite set $\{z_1, z_2, \ldots, z_m\}$ with positive numbers $\delta_1, \delta_2, \ldots, \delta_m$ such that

$$K \subseteq \bigcup_{k=1}^{m} D(z_k; \delta_k) \subseteq \bigcup_{k=1}^{m} D(z_k; \delta) \subseteq \Omega.$$

Now we apply the special case (11.41) to each disc $D(z_k; \delta)$ and consider only points $z \in D(z_k; \delta_k)$ to get

$$\frac{\delta - \delta_k}{\delta + \delta_k} u(z_k) \leq \frac{\delta - r}{\delta + r} u(z_k) \leq u(z) \leq \frac{\delta + r}{\delta - r} u(z_k) \leq \frac{\delta + \delta_k}{\delta - \delta_k} u(z_k).$$

Take positive numbers α and β such that[g]

$$\alpha = \frac{1}{u(z_0)} \cdot \min_{1 \leq k \leq m} \frac{\delta - \delta_k}{\delta + \delta_k} u(z_k) \quad \text{and} \quad \beta = \frac{1}{u(z_0)} \cdot \max_{1 \leq k \leq m} \frac{\delta + \delta_k}{\delta - \delta_k} u(z_k).$$

Then we establish

$$\alpha u(z_0) \leq u(z) \leq \beta u(z_0) \tag{11.43}$$

for every $z \in \bigcup_{k=1}^{m} D(z_k; \delta_k)$. In particular, the inequalities (11.43) are true for all $z \in K$.

- **The behavior of $\{u_n\}$ in $\Omega \setminus \{z_0\}$ if $u_n(z_0) \to 0$.** Let $u_n(z) \to u(z)$ for every $z \in \Omega$ and $a \in \Omega \setminus \{z_0\}$. Since $\Omega \setminus \{z_0\}$ is open in $\mathbb{C}$, there exists a $R > 0$ such that $\overline{D}(a; R) \subseteq \Omega \setminus \{z_0\}$. Obviously, $\overline{D}(a; R)$ is a compact subset of Ω. By the inequalities (11.43), we have

$$\alpha u_n(z_0) \leq u_n(z) \leq \beta u_n(z_0) \tag{11.44}$$

for every $z \in \overline{D}(a; R)$ and $n = 1, 2, \ldots$, where α and β depend on z_0, K and Ω only. Take $n \to \infty$ in the inequalities (11.44) and then use the hypothesis, we get

$$u(z) = 0 \tag{11.45}$$

for all $z \in \overline{D}(a; R)$. Particularly, $u(a) = 0$. Since a is arbitrary, we conclude that $u \equiv 0$ in $\Omega \setminus \{z_0\}$. Finally, the continuity of u ensures that $u(z_0) = 0$ and then $u \equiv 0$ in Ω.

- **The behavior of $\{u_n\}$ in $\Omega \setminus \{z_0\}$ if $u_n(z_0) \to \infty$.** Instead of the result (11.45), we obtain

$$u(z) = \infty$$

for all $z \in \overline{D}(a; R)$. Hence, using similar argument as the previous assertion, we conclude that $u(z) = \infty$ in Ω.

[g] Since δ depends on K and Ω, α and β trivially depend on z_0, K and Ω.

- **The positivity of $\{u_n\}$ is essential.** For each $n \in \mathbb{N}$, we consider $u(x,y) = nx - \frac{y}{n}$ in $\mathbb{C}$. Then it is easily checked that $\Delta u_n = 0$ so that u_n is harmonic in $\mathbb{C}$. As $u_n(0,1) = -\frac{1}{n} < 0$ and $u_n(1,0) = n > 0$, each u_n is neither positive nor negative. Furthermore, $u_n(0,1) \to 0$ but $u_n(1,0) \to \infty$ as $n \to \infty$. Hence this counterexample shows that the positivity of $\{u_n\}$ cannot be omitted for these results.

We have completed the analysis of the problem. ■

Problem 11.13

Rudin Chapter 11 Exercise 13.

Proof. By the Poisson formula, we have

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} u(\mathrm{e}^{it})\,\mathrm{d}t, \tag{11.46}$$

where $0 \leq r < 1$ and $-\pi \leq \theta \leq \pi$. Pick $r = \frac{1}{2}$ and $\theta = 0$ in the formula (11.46) to get

$$u(\tfrac{1}{2}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \frac{1}{4}}{1 - \cos t + \frac{1}{4}} u(\mathrm{e}^{it})\,\mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{3}{5 - 4\cos t} u(\mathrm{e}^{it})\,\mathrm{d}t. \tag{11.47}$$

Since $\frac{3}{5+4} \leq \frac{3}{5-4\cos t} \leq \frac{3}{5-4}$ on $[-\pi, \pi]$, the integral (11.47) gives

$$\frac{1}{3} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\mathrm{e}^{it})\,\mathrm{d}t \leq u(\tfrac{1}{2}) \leq 3 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\mathrm{e}^{it})\,\mathrm{d}t. \tag{11.48}$$

Put $r = 0$ in the Poisson formula (11.46), we obtain

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\mathrm{e}^{it})\,\mathrm{d}t,$$

therefore, we conclude from the hypothesis $u(0) = 1$ and the inequalities (11.48) that

$$\frac{1}{3} \leq u(\tfrac{1}{2}) \leq 3.$$

This completes the proof of the problem. ■

Problem 11.14

Rudin Chapter 11 Exercise 14.

Proof. By translation and/or rotation, we may assume that L_1 is the real axis (i.e., $y = 0$) and if L_1 and L_2 intersect, the intersection point is the origin.

- **Case (i): L_1 and L_2 are parallel.** Suppose that the equation of L_2 is $y = A$ for some $A \in \mathbb{R} \setminus \{0\}$. Then we consider the function

$$u(x,y) = \mathrm{e}^{\frac{\pi x}{A}} \sin \frac{\pi y}{A}.$$

  It is obvious that $u(x,0) = u(x,A) = 0$ for all $x \in \mathbb{R}$. Furthermore, direct computation gives $u_{xx} + u_{yy} = 0$ in $\mathbb{R}^2$ so that u is a harmonic function in $\mathbb{R}^2$.

- **Case (ii): L_1 and L_2 are perpendicular.** Suppose that $L_2 : x = 0$. Then it is easy to check that the function $u(x, y) = xy$ satisfies all the requirements.

- **Case (iii): L_1 and L_2 are non-parallel and non-perpendicular.** Suppose that the angle between L_1 and L_2 is a rational multiple of π, say $\frac{m\pi}{n}$, where $m, n \in \mathbb{N}$ and $\frac{m}{n}$ is not a multiple of $\frac{1}{2}$. Since $z^n = u_n(x, y) + iv_n(x, y)$ is entire, its imaginary part $v_n(x, y)$ is harmonic in $\mathbb{R}^2$ by Theorem 11.4. On L_1, we have

$$x^n = (x + i \cdot 0)^2 = u_n(x, 0) + iv_n(x, 0)$$

for all $x \in \mathbb{R}$. This implies that $v_n(x, 0) = 0$ on $\mathbb{R}$. Similarly, since points on L_2 are in the form $z = \cos \frac{m\pi}{n} + i \sin \frac{m\pi}{n}$, so we have

$$(-1)^m = \Big(\cos \frac{m\pi}{n} + i \sin \frac{m\pi}{n}\Big)^n = u_n\Big(\cos \frac{m\pi}{n}, \sin \frac{m\pi}{n}\Big) + iv_n\Big(\cos \frac{m\pi}{n}, \sin \frac{m\pi}{n}\Big)$$

which implies that $v_n(x, y) = 0$ on L_2.

Now suppose that the angle between them is an irrational multiple of π, say $\alpha\pi$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $u(x, y)$ was a harmonic function in $\mathbb{R}^2$ vanishing on $L_1 \cup L_2$. We need to develop a harmonic version of Theorem 11.14 (The Schwarz reflection principle). To this end, we recall the following concept: Let L be a straight line passing through the origin. We say that a pair of points are *symmetric* with respect to L if L is the perpendicular bisector of the line segment joining these points. For each $z = (x, y) \in \mathbb{C}$, it is easy to see that there exists a unique $z_L = (x_L, y_L) \in \mathbb{C}$ such that z and z_L are symmetric with respect to L. Next, the following result is taken from [7, Theorem 4.12, p. 68]:

Lemma 11.3

Let $z_0 = (a, b) \in \mathbb{C}$ and consider the line $L = \{(x, y) \in \mathbb{C} \,|\, (x, y) \cdot (a, b) = c\}$ for some $a, b, c \in \mathbb{R}$. Define $L^+ = \{(x, y) \in \mathbb{C} \,|\, (x, y) \cdot (a, b) > c\}$. Suppose that $\Omega \subseteq \mathbb{C}$ is a region symmetric with respect to L. If u is continuous on $\Omega \cap \overline{L^+}$, u is harmonic on $\Omega \cap L^+$ and $u = 0$ on $\Omega \cap L$, then the function

$$U(x, y) = \begin{cases} u(x, y), & \text{if } (x, y) \in \Omega \cap \overline{L^+}; \\ -u(x_L, y_L), & \text{if } (x, y) \in \Omega \cap L^- \end{cases}$$

is harmonic in Ω.

Applying Lemma 11.3 to our u with $L = L_2$, we see immediately that u vanishes on the line $L_3 : y = (\tan \alpha\pi)x$. In fact, repeated applications of Lemma 11.3 show that u vanishes on lines in the form

$$y = (\tan n\alpha\pi)x \tag{11.49}$$

for every $n \in \mathbb{Z}$.

To finish the proof, we have to show that the collection of the straight lines (11.49), denoted by $\mathcal{L}$, is dense in $\mathbb{R}^2$. To see this, let $\alpha > 0$. Given that $0 < \theta < \frac{1}{2}$, $\delta > 0$ and $\epsilon > 0$. By the Kronecker's Approximation Theorem[h], we find that there exist $m, n \in \mathbb{N}$ such that

$$|n\alpha\pi - m\pi - \theta\pi| < \delta.$$

[h]See, for example, [5, §7.4, pp. 148, 149].

Note that $\tan(n\alpha\pi - m\pi) = (-1)^m \tan(n\alpha\pi)$. The continuity of $\tan x$ implies that

$$|(-1)^m \tan(n\alpha\pi) - \tan(\theta\pi)| < \epsilon. \tag{11.50}$$

If m is odd, then we can replace $(-1)^m \tan(n\alpha\pi)$ by $\tan(-n\alpha\pi)$ in the estimation (11.50). If $-\frac{1}{2} < \theta < 0$, then a similar argument gives the following estimation

$$|(-1)^{m+1} \tan(n\alpha\pi) - \tan(\theta\pi)| < \epsilon.$$

In this case, if m is even, then $(-1)^{m+1} \tan(n\alpha\pi)$ will be replaced by $\tan(-n\alpha\pi)$. In other words, for every $\theta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, we always have

$$|\tan(n\alpha\pi) - \tan(\theta\pi)| < \epsilon \tag{11.51}$$

for some $n \in \mathbb{Z}$. If $\theta = 0$, then we follow from the Dirichlet's Approximation Theorem[i] that for a positive integer N with $\frac{\pi}{N} < \delta$, there exist $m, n \in \mathbb{N}$ with $0 < n \leq N$ such that

$$|n\alpha\pi - m\pi| < \frac{\pi}{N} < \delta$$

and again the continuity of $\tan x$ implies that

$$|\tan(n\alpha\pi)| < \epsilon.$$

Consequently, the inequality (11.51) actually holds for all $-\frac{1}{2} < \theta < \frac{1}{2}$. Since the range of $\tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ is $\mathbb{R}$, our inequality (11.51) means that the collection $\mathcal{L}$ is dense in $\mathbb{R}^2$ for the case $\alpha > 0$. For the case $\alpha < 0$, we just consider $-\alpha > 0$ and then the inequality (11.51) remains valid with the integer $-n$.

Finally, since u is continuous on $\mathbb{R}^2$ and vanishes on $\mathcal{L}$, we conclude that $u \equiv 0$ on $\mathbb{R}^2$.

This completes the proof of the problem. ∎

Problem 11.15

Rudin Chapter 11 Exercise 15.

Proof. We first prove the following lemma:

Lemma 11.4

Given $\epsilon \in (0, 1)$. There exists a constant $M > 0$ such that

$$P_{1-\epsilon}(t) \geq \frac{M}{\epsilon} \tag{11.52}$$

for all $t \in [-\epsilon, \epsilon]$.

[i] Read [79, Problem 2.4, p. 12].

Proof of Lemma 11.4. Recall from the series expansion of the Poisson kernel that $P_r(t) = P_r(-t)$. We know from [62, Eqn. (4), p. 233] that $P_r(t)$ is decreasing on $[0,\pi]$ and $[-\pi,0]$. Therefore, it suffices to prove that the inequality (11.52) holds for $t = \epsilon$. In fact, we have $1-(1-\epsilon)^2 = 2\epsilon - \epsilon^2 \geq \epsilon$ and

$$\begin{aligned}1-2(1-\epsilon)\cos\epsilon + (1-\epsilon)^2 &= 2(1-\epsilon) - 2(1-\epsilon)\cos\epsilon + \epsilon^2\\ &= 2(1-\epsilon)(1-\cos\epsilon) + \epsilon^2\\ &= 2(1-\epsilon)\Big(\frac{\epsilon^2}{2!} - \frac{\epsilon^4}{4!} + \cdots\Big) + \epsilon^2\\ &\leq \frac{\epsilon^2}{M}\end{aligned}$$

for some constant $M > 0$. Thus they imply that

$$\epsilon P_{1-\epsilon}(\epsilon) = \epsilon\cdot\frac{1-(1-\epsilon)^2}{1-2(1-\epsilon)\cos\epsilon + (1-\epsilon)^2} \geq \epsilon\cdot\frac{M\epsilon}{\epsilon^2} = M$$

which gives the desired result. ■

Let's return to the proof of the problem. When u is considered in $D(0;r)$ for $0 < r < 1$, [9, Theorem 16.5, p. 227] implies that u attains its maximum on $C(0;r)$. This fact and the positivity of u imply that

$$u(re^{i\theta}) \leq u(se^{i\theta})$$

for every $\theta \in [-\pi,\pi]$ if $0 \leq r \leq s < 1$. Using this result and the hypothesis, if $\theta \neq 0$, then we obtain

$$\sup_{0<r<1}\|u_r\|_1 = \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus\{0\}}\Big(\sup_{0<r<1} u(re^{i\theta})\Big)\,d\theta \leq \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus\{0\}}\Big(\lim_{r\to 1} u(re^{i\theta})\Big)\,d\theta = 0.$$

Hence it follows from Theorem 11.30 that one can find a unique positive Borel measure on T such that $u = P[\,d\mu]$. Next, it is easy to see that

$$u\big((1-\epsilon)e^{i\theta}\big) = \frac{1}{2\pi}\int_T P_{1-\epsilon}(\theta - t)\,d\mu(e^{it}) = \frac{1}{2\pi}\int_{I(e^{i\theta};2\pi)} P_{1-\epsilon}(t)\,d\mu\big(e^{i(\theta-t)}\big), \tag{11.53}$$

where $\theta \neq 0$ and $I(e^{i\theta};2\pi) \subseteq T$ denotes the open arc centred at $e^{i\theta} \neq 1$ with arc length 2π. We notice that $[-\epsilon,\epsilon] \subseteq I(e^{i\theta};2\pi)$, so we apply Lemma 11.4 to the representation (11.53) and get

$$u\big((1-\epsilon)e^{i\theta}\big) \geq \frac{1}{2\pi}\int_{[-\epsilon,\epsilon]}\frac{M}{\epsilon}\,d\mu\big(e^{i(\theta-t)}\big) \geq \frac{M}{\epsilon}\int_{[-\epsilon,\epsilon]} d\mu\big(e^{i(\theta-t)}\big) \geq \frac{M}{\epsilon}\mu\big(I(e^{i\theta};2\epsilon)\big). \tag{11.54}$$

Take $\epsilon \to 0$ in the inequality (11.54), the condition $u(re^{i\theta}) \to 0$ as $r \to 1$ implies that

$$\lim_{\epsilon\to 0}\frac{\mu\big(I(e^{i\theta};2\epsilon)\big)}{\epsilon} = 0$$

for every $e^{i\theta} \neq 1$. Consequently, we have $\mu(T\setminus\{1\}) = 0$ so that μ is a positive point mass at 1 which means that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_T P_r(\theta - t)\,d\mu(e^{it}) = \mu(1)P_r(\theta),$$

completing the proof of the problem. ■

Problem 11.16

Rudin Chapter 11 Exercise 16.

Proof. Assume that u was the Poisson integral of a complex measure μ on T, i.e., $u = P[\,\mathrm{d}\mu]$. We know from [62, Eqn. (2), p. 244] that

$$\|u_r\|_1 \leq \|\mu\| = |\mu|(T) < \infty$$

for every $0 < r < 1$. Thus it deduces from Theorem 11.30(a) that μ is a (unique) complex Borel measure. Next, we express u as

$$u(z) = -4 \cdot \operatorname{Im}(z) \cdot \frac{1 - |z|^2}{|1 + z|^4}. \tag{11.55}$$

On the one hand, as the first inequality of Theorem 11.20 only requires that μ is a Borel measure on T, so we may apply it with a fixed $0 < \alpha < 1$ and one can find a constant $c_\alpha > 0$ such that

$$0 \leq c_\alpha (N_\alpha u)(-1) \leq (M_{\text{rad}}\, u)(-1) = \sup\{|u(-r)| \,|\, 0 \leq r < 1\} \leq \lim_{r \to 1} |u(-r)|. \tag{11.56}$$

If $z \in (-1, 0]$, then we have $\operatorname{Im}(z) = 0$ and we see from the representation (11.55) that $u(z) = 0$. Consequently, the inequalities (11.56) force that

$$(N_\alpha u)(-1) = 0. \tag{11.57}$$

On the other hand, given $0 < \epsilon < \sin^{-1}\alpha$, if $z = (-1 + \epsilon) + i\epsilon^2$, then it is easily checked that $z \in -\Omega_\alpha$ and the expression (11.55) becomes

$$u(z) = u\big((-1 + \epsilon) + i\epsilon^2\big) = -\frac{4}{\epsilon} \cdot \frac{2 - \epsilon - \epsilon^3}{(1 + \epsilon^2)^2}.$$

Therefore, we obtain

$$\lim_{\substack{z \to -1 \\ z \in -\Omega_\alpha}} u(z) = \lim_{\epsilon \to 0} -\frac{4}{\epsilon} \cdot \frac{2 - \epsilon - \epsilon^3}{(1 + \epsilon^2)^2} = -\infty$$

which implies $(N_\alpha u)(-1) = \infty$, a contradiction to the result (11.57). This proves the first assertion that u cannot be a Poisson integral of any measure on T.

For the second assertion, if $u = v - w$, where both v and w are positive harmonic functions in U, then we have $|u(z)| \leq |v(z)| + |w(z)| = v(z) + w(z)$ for all $z \in U$ so that

$$\|u_r\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})| \,\mathrm{d}\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} v(re^{i\theta}) \,\mathrm{d}\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} w(re^{i\theta}) \,\mathrm{d}\theta \tag{11.58}$$

for every $0 < r < 1$. Since v and w are harmonic in U, they satisfy the mean value property, so the inequality (11.58) gives

$$\|u_r\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})| \,\mathrm{d}\theta \leq v(0) + w(0)$$

for every $0 < r < 1$. In other words, $\sup\limits_{0<r<1} \|u_r\|_1$ is bounded and Theorem 11.30(a) shows that $u = P[\,\mathrm{d}\mu]$ for a unique complex Borel measure on T. However, it is impossible by the first assertion and hence u is not the difference of two positive harmonic functions in U. We have completed the analysis of the problem. ■

Problem 11.17

Rudin Chapter 11 Exercise 17.

Proof. Set $\Phi = \{u : U \to \mathbb{R} \,|\, u$ is positive, harmonic and $u(0) = 1\}$ and let C be the set whose members are the positive Borel measures μ on T of $|\mu|(T) = 1$. We divide the proof into several steps:

- **Step 1: There is an isomorphism between Φ and C.** On the one hand, for each $\mu \in C$, we know from §11.17 that
$$u(z) = \int_T P(z, \mathrm{e}^{it})\,\mathrm{d}\mu(\mathrm{e}^{it}) \tag{11.59}$$
is harmonic and also positive in U. In addition, it is easy to see that
$$u(0) = \int_T P(0, \mathrm{e}^{it})\,\mathrm{d}\mu(\mathrm{e}^{it}) = \int_T \mathrm{d}\mu(\mathrm{e}^{it}) = |\mu|(T) \tag{11.60}$$
which implies $u(0) = 1$, i.e., $u \in \Phi$. On the other hand, if $u \in \Phi$, then the mean value property shows that
$$\|u_r\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r\mathrm{e}^{i\theta})|\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r\mathrm{e}^{i\theta})\,\mathrm{d}\theta = u(0) < \infty$$
so that $u = P[\,\mathrm{d}\mu]$ for a *unique* positive Borel measure on T by Theorem 11.30(a). Besides, the expression (11.60) gives $|\mu|(T) = 1$, so $\mu \in C$. Consequently, the mapping $f : \Phi \to C$ defined by
$$f(u) = \mu$$
is bijective. In fact, f is a homemorphism because for every $u, v \in \Phi$, if their corresponding positive Borel measures are μ and ν respectively, then
$$\begin{aligned}\alpha u(z) + \beta v(z) &= \alpha \int_T P(z, \mathrm{e}^{it})\,\mathrm{d}\mu(\mathrm{e}^{it}) + \beta \int_T P(z, \mathrm{e}^{it})\,\mathrm{d}\nu(\mathrm{e}^{it}) \\ &= \int_T P(z, \mathrm{e}^{it})\,\mathrm{d}[\alpha\mu + \beta\nu](\mathrm{e}^{it}),\end{aligned}$$
i.e., $f(\alpha u + \beta v) = \alpha\mu + \beta\nu = \alpha f(u) + \beta f(v)$.

- **Step 2: Both Φ and C are convex.** Suppose that $u_1, u_2 \in \Phi$ and $u = \lambda u_1 + (1-\lambda)u_2$, where $0 \leq \lambda \leq 1$. Since u_1 and u_2 are positive and harmonic in U, u is also positive and harmonic in U. Furthermore, we have $u(0) = \lambda u_1(0) + (1-\lambda)u_2(0) = 1$. In other words, $u \in \Phi$ and thus Φ is a convex set. By **Step 1**, C is also convex.

- **Step 3: Extreme points of C.** Denote $\operatorname{ext} C$ to be the set of extreme points of C.[j] Let $\mu \in \operatorname{ext} C$. Since u is positive in U, it follows from **Step 1** that $\mu(E) > 0$ for every Borel subset E of T. Now we claim that $\operatorname{supp}\mu = \overline{\{\mathrm{e}^{it} \in T \,|\, \mu(\mathrm{e}^{it}) \neq 0\}}$ is a **unit mass concentrated at** e^{it} (see [62, Example 1.20(b), p. 17] for the definition). Otherwise, there was a measurable set $E \subseteq T$ such that $0 < \mu(E) < 1$. Then we have $0 < \mu(T \setminus E) < 1$. We define
$$\mu_E(F) = \frac{\mu(F\cap E)}{\mu(E)} \quad \text{and} \quad \mu_{T\setminus E}(F) = \frac{\mu(F\setminus E)}{\mu(T\setminus E)}. \tag{11.61}$$
Since μ is a positive Borel measure on T, both μ_E and $\mu_{T\setminus E}$ are also positive Borel measures on T. Furthermore, it is clear that
$$|\mu_E|(T) = \frac{|\mu|(T\cap E)}{|\mu|(E)} = 1 \quad \text{and} \quad |\mu_{T\setminus E}|(T) = \frac{|\mu|(T\setminus E)}{|\mu|(T\setminus E)} = 1.$$

[j] A point x of a convex set X is an **extreme point** of X if x cannot be written as a proper convex combination $x = \lambda x_1 + (1-\lambda)x_2$, where $0 < \lambda < 1$, $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Thus μ_E and $\mu_{T\setminus E}$ belong to C. If we set $\lambda = \mu(E)$, then for every measurable subset $F \subseteq T$, we get from the definitions (11.61) that

$$\begin{aligned}\lambda\mu_E(F) + (1-\lambda)\mu_{T\setminus E}(F) &= \mu(E)\cdot\frac{\mu(F\cap E)}{\mu(E)} + [1-\mu(E)]\cdot\frac{\mu(F\setminus E)}{\mu(T\setminus E)}\\ &= \mu(F\cap E) + \mu(F\setminus E)\\ &= \mu(F)\end{aligned}$$

which means $\mu \notin \operatorname{ext} C$, a contradiction. Hence, we obtain

$$\operatorname{ext} C = \{\delta_{e^{it}} \,|\, e^{it} \in T\},$$

where δ_x is the **Dirac delta function at** x.

- **Step 4: Extreme points of Φ.** By **Step 2**, the restriction $f_{\operatorname{ext}\Phi}$ is certainly an isomorphism between $\operatorname{ext}\Phi$ and $\operatorname{ext} C$. Using **Step 3** and the definition (11.59), we see that
$$\operatorname{ext}\Phi = \{P(z, e^{it}) \,|\, e^{it} \in T\}.$$

This completes the proof of the problem.

■

11.3 The Weak* Convergence and Radial Limits of Holomorphic Functions

Problem 11.18

Rudin Chapter 11 Exercise 18.

Proof. Let X be a Banach space and X^* be its dual space.[k] Recall Definition 5.3 that

$$\|\Lambda_n\| = \sup\{|\Lambda_n(x)| \,|\, x \in X \text{ and } \|x\| = 1\}.$$

Suppose that $\{\Lambda_n\} \subseteq X^*$ converges weakly to $\Lambda \in X^*$, so $\Lambda_n(x) \to \Lambda(x)$ for every $x \in X$ and then

$$\sup_{n\in\mathbb{N}} |\Lambda_n(x)| = M_x < \infty$$

for each $x \in X$. By the definition, $\{\Lambda_n\}$ is a collection of bounded linear functionals. Hence Theorem 5.8 (The Banach-Steinhaus Theorem) asserts the existence of a positive constant M such that

$$\sup_{n\in\mathbb{N}} \|\Lambda_n\| \leq M,$$

completing the proof of the problem. ■

Remark 11.3

We note that the convergence in Problem 11.18 is called the **weak* convergence**. In fact, there are two important convergences in functional analysis. They are called **strong convergence** and **weak convergence**. To be more precisely, let X be a Banach space and $f_n, f \in X$ for every $n \in \mathbb{N}$. If $\|f_n - f\| \to 0$, then $\{f_n\}$ is said to **converge strongly** to f. Similarly, we say $\{f_n\}$ **converges weakly** to f if we have $g(f_n) \to g(f)$ for every $g \in X^{**}$. Read, for example, [7, p. 115, 116].

[k]Recall from Remark 5.21 that X^* is the collection of all bounded linear functionals on X.

Problem 11.19

Rudin Chapter 11 Exercise 19.

Proof.

(a) Since $0 < r < 1$, $1 - r > 0$. By the power series of the cosine function, we have
$$\cos(1-r) > 1 - \frac{(1-r)^2}{2}$$
which implies that
$$(1-r)P_r(1-r) = \frac{(1-r)(1-r^2)}{1 - 2r\cos(1-r) + r^2} > \frac{1 - r - r^2 + r^3}{1 + r(1-r)^2 - 2r + r^2} = 1.$$

(b) Clearly, we have
$$\delta u(1-\delta) = \int_T \delta P(1-\delta, e^{it})\,d\mu(e^{it}) = \int_T \delta P_{1-\delta}(t)\,d\mu(e^{it}). \tag{11.62}$$
We apply part (a) and the fact that $P_r(t)$ is an even function of t to the integral (11.62) to get
$$\delta u(1-\delta) \geq \int_{I_\delta} d\mu(e^{i\theta}) = \mu(I_\delta).$$
Since $\mu \geq 0$, we have $u(1-\delta) \geq 0$. By Definition 11.19, we have
$$\frac{\mu(I_\delta)}{\pi\sigma(I_\delta)} = \frac{\mu(I_\delta)}{\delta} \leq u(1-\delta) \leq (M_{\mathrm{rad}}\, u)(1) \tag{11.63}$$
for every $\delta > 0$. Recall the definition [62, Eqn. (3), §11.19, p. 241], we conclude that
$$(M\mu)(1) \leq \pi(M_{\mathrm{rad}}\, u)(1)$$
as desired.

(c) In fact, the results of part (b) are valid for every point $e^{i\theta}$ on T if we apply the special case to the rotated measure $\mu_\theta(E) = \mu(e^{i\theta}E)$. In particular, we deduce from the inequalities (11.63) that
$$\frac{\mu(I_\delta)}{\pi\sigma(I_\delta)} \leq u\big((1-\delta)e^{i\theta}\big) \leq (M_{\mathrm{rad}}\, u)(e^{i\theta}), \tag{11.64}$$
where $e^{i\theta} \in T$ and I_δ is the open arc with center $e^{i\theta}$ and length 2δ.

Next, we know from [62, Eqn. (2), §11.17 p. 240] that
$$\|u_r\|_1 \leq \|\mu\| = |\mu|(T) < \infty$$
for every $0 < r < 1$. Therefore, we follow from Theorem 11.30(A) that μ is a Borel measure on T. By the hypothesis, μ is positive. Since $\mu \perp m$, it follows from Theorem 7.15 that
$$(D\mu)(e^{i\theta}) = \infty \quad \text{a.e. } [\mu]. \tag{11.65}$$
Finally, we observe from the inequality (11.64), the result (11.65) and [62, Eqn. (4), p. 241] that
$$\lim_{\delta\to 0} u\big((1-\delta)e^{i\theta}\big) = \infty \quad \text{a.e. } [\mu]$$
which is exactly the required result.

This ends the proof of the problem. ∎

Problem 11.20

Rudin Chapter 11 Exercise 20.

Proof. Since $m(E) = 0$, for each $n \in \mathbb{N}$, we know from [61, Remark 11.11(b), p. 309] that there exists an open set $V_n \subseteq T$ containing E such that $m(V_n) \leq \frac{1}{2^n}$. Define

$$\chi_{V_n}(\mathrm{e}^{i\theta}) = \begin{cases} 1, & \text{if } \mathrm{e}^{i\theta} \in V_n; \\ 0, & \text{otherwise} \end{cases}$$

and $\varphi : T \to \mathbb{R}$ by

$$\varphi(\mathrm{e}^{i\theta}) = \sum_{n=1}^{\infty} \chi_{V_n}(\mathrm{e}^{i\theta}).$$

Obviously, we have

$$\sum_{n=1}^{\infty} \int_T \chi_{V_n}(\mathrm{e}^{i\theta})\,\mathrm{d}m = \sum_{n=1}^{\infty} m(V_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

so Theorem 1.38 implies that

$$\int_T |\varphi(\mathrm{e}^{i\theta})|\,\mathrm{d}m = \sum_{n=1}^{\infty} \int_T \chi_{V_n}(\mathrm{e}^{i\theta})\,\mathrm{d}m = 1.$$

In other words, $\varphi \in L_1(T)$ and $\|\varphi\|_1 = 1$.

Define $u : U \to \mathbb{R}$ by $u = P[\varphi]$ which is harmonic in U by Theorem 11.7. Now, for each $\mathrm{e}^{i\theta} \in E$, we have $\mathrm{e}^{i\theta} \in V_n$ for every $n \in \mathbb{N}$, so it follows from the definition that

$$\lim_{r\to 1} u(r\mathrm{e}^{i\theta}) = \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r\mathrm{e}^{i\theta}, \mathrm{e}^{it})\varphi(\mathrm{e}^{it})\,\mathrm{d}t = \infty.$$

Since U is simply connected, u is the real part of a holomorphic function g in U, see [9, Theorem 16.3, p. 226]. Let $f = \mathrm{e}^{-g}$. Then we have $f \in H(U)$ and $|f| = \mathrm{e}^{-\mathrm{Re}\, g} = \mathrm{e}^{-u}$ so that

$$\lim_{r\to 1} f(r\mathrm{e}^{i\theta}) = \lim_{r\to 1} \mathrm{e}^{-u} = 0$$

for every $\mathrm{e}^{i\theta} \in E$. By the definition of f, $f(z) \neq 0$ for every $z \in U$. Thus f is nonconstant. Since $\varphi \geq 0$, we have $u \geq 0$ and consequently, $f \in H^\infty$. If $f(0) \neq 1$, then we can replace f by $\widetilde{f}(z) = \frac{f(z)}{f(0)}$ so that $\widetilde{f}(0) = 1$. This completes the proof of the problem. ∎

Remark 11.4

For further information, the reader can refer to [53, p. 295] and [84, pp. 105, 276].

Problem 11.21

Rudin Chapter 11 Exercise 21.

Proof. We first show that $g \notin H^\infty$. In fact, fix $s \in \mathbb{R}$, consider $z_t = \frac{t+is-1}{t+is+1}$ for $0 < t < \infty$. Then it is easy to check that $|z_t| < 1$ and $\frac{1+z_t}{1-z_t} = t + is$. Therefore, we have

$$g(z_t) = \frac{2}{(t+1)+is} \exp(-\mathrm{e}^{t+is})$$

which gives

$$|g(z_t)| = \frac{2}{\sqrt{(t+1)^2 + s^2}} \exp(-\mathrm{e}^t \cos s). \tag{11.66}$$

Now we put $s = \pi$ in the expression (11.66) to get $|g(z_t)| \to \infty$ as $t \to \infty$. Consequently, $g \notin H^\infty$.

Next, if $\mathrm{e}^{i\theta} \in T$ and $0 \le r < 1$, then we have

$$\frac{1 + r\mathrm{e}^{i\theta}}{1 - r\mathrm{e}^{i\theta}} = \frac{1 - r^2 + 2i \sin\theta}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad f(r\mathrm{e}^{i\theta}) = \exp\Big(\frac{1 - r^2 + 2i\sin\theta}{1 - 2r\cos\theta + r^2}\Big).$$

By the definition of g, we have

$$g(r\mathrm{e}^{i\theta}) = (1 - r\mathrm{e}^{i\theta}) \cdot \exp\Big[-\exp\Big(\frac{1 - r^2 + 2i\sin\theta}{1 - 2r\cos\theta + r^2}\Big)\Big]$$

and so

$$g^*(\mathrm{e}^{i\theta}) = \lim_{r\to 1} g(r\mathrm{e}^{i\theta}) = \begin{cases} (1 - \mathrm{e}^{i\theta}) \exp\Big[-\exp\Big(\dfrac{i\sin\theta}{1 - \cos\theta}\Big)\Big], & \text{if } \theta \neq 0; \\ 0, & \text{if } \theta = 0. \end{cases} \tag{11.67}$$

Hence $g^*(\mathrm{e}^{i\theta})$ exists for every $\mathrm{e}^{i\theta} \in T$. By the representation (11.67), we know that $g^*(\mathrm{e}^{i\theta})$ is continuous at every $\theta \in (0, 2\pi)$. Thus it suffices to show that $g^*(\mathrm{e}^{i\theta})$ is continuous at $\theta = 0$. To see this, since

$$\lim_{\theta\to 0} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{-\sin\theta}{\cos\theta} = 0,$$

the representation (11.67) gives

$$\begin{aligned} \lim_{\theta\to 0} g^*(\mathrm{e}^{i\theta}) &= \lim_{\theta\to 0} (1 - \mathrm{e}^{i\theta}) \exp\Big[-\exp\Big(\frac{i\sin\theta}{1 - \cos\theta}\Big)\Big] \\ &= \lim_{\theta\to 0} (1 - \mathrm{e}^{i\theta}) \lim_{\theta\to 0} \exp\Big[-\exp\Big(\frac{i\sin\theta}{1 - \cos\theta}\Big)\Big] \\ &= 0 \cdot \mathrm{e}^{-1} \\ &= 0. \end{aligned}$$

As a consequence, we establish the fact that $g^* \in C(T)$ and this completes the proof of the problem. ■

11.4 Miscellaneous Problems

Problem 11.22

Rudin Chapter 11 Exercise 22.

Proof. For each $0 \leq r < 1$, the function $u_r : T \to \mathbb{C}$ is continuous, so $u_r^{-1}(\mathbb{R})$ is measurable. Suppose that $u_r = u_r^+ - u_r^-$, where u_r^+ and u_r^- are the positive and negative parts of u_r, see Definition 1.15. Since $\{u_r\}$ is uniformly integrable, there exists a $\delta > 0$ such that

$$\Big|\int_E u_r \,\mathrm{d}m\Big| < 1 \tag{11.68}$$

whenever $0 \leq r < 1$ and $m(E) < 2\delta$. Let $E_r^+ = \{\theta \in [0, 2\pi] \,|\, u_r^+(\mathrm{e}^{i\theta}) > 0\}$. Then E_r^+ is measurable and

$$\int_{-\pi}^{\pi} u_r^+ \,\mathrm{d}m = \int_{E_r^+} u_r \,\mathrm{d}m. \tag{11.69}$$

Let N be the least positive integer such that $\frac{2\pi}{N} \leq \delta$.[1] Define I_k to be the arc from $\mathrm{e}^{(k-1)\delta i}$ to $\mathrm{e}^{k\delta i}$ including $\mathrm{e}^{(k-1)\delta i}$ but not $\mathrm{e}^{k\delta i}$, where $k = 1, 2, \ldots, N-1$. Similarly, we define I_N to be the arc from $\mathrm{e}^{(N-1)\delta i}$ to $\mathrm{e}^{2\pi i} = 1$ including $\mathrm{e}^{(N-1)\delta i}$ but not $\mathrm{e}^{2\pi i} = 1$. Clearly, the central angles of $I_1, I_2, \ldots, I_{N-1}$ are exactly δ and the central angle of I_N is less than or equal to δ. Next, we denote

$$E_{r,k}^+ = E_r^+ \cap I_k$$

so that $\{E_{r,1}^+, E_{r,2}^+, \ldots, E_{r,N}^+\}$ forms a *disjoint* measurable subsets of E_r^+ and

$$m(E_{r,k}^+) \leq m(I_k) \leq \delta < 2\delta$$

for $k = 1, 2, \ldots, N$. Hence we follow from the inequality (11.68) and the result (11.69) that

$$\Big|\int_{-\pi}^{\pi} u_r^+ \,\mathrm{d}m\Big| = \Big|\sum_{k=1}^{N} \int_{E_{r,k}^+} u_r \,\mathrm{d}m\Big| \leq \sum_{k=1}^{N} \Big|\int_{E_{r,k}^+} u_r \,\mathrm{d}m\Big| \leq N$$

for every $0 \leq r < 1$. Similarly, it can be shown that

$$\Big|\int_{-\pi}^{\pi} u_r^- \,\mathrm{d}m\Big| \leq N$$

for every $0 \leq r < 1$. Thus they imply that

$$\|u_r\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u_r(\mathrm{e}^{i\theta})| \,\mathrm{d}\theta \leq \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} u_r^+(\mathrm{e}^{i\theta}) \,\mathrm{d}\theta + \int_{-\pi}^{\pi} u_r^-(\mathrm{e}^{i\theta}) \,\mathrm{d}\theta\Big) \leq \frac{N}{\pi}$$

for every $0 \leq r < 1$.

Hence we apply Theorem 11.30(a) to obtain a unique complex Borel measure μ such that $u = P[\,\mathrm{d}\mu]$. Next, we know form Theorem 11.24 that there exists a $f \in L^1(T)$ such that

$$\lim_{j\to\infty} u(z_j) = f(\mathrm{e}^{i\theta})$$

for almost all points $\mathrm{e}^{i\theta} \in T$, where $z_j \to \mathrm{e}^{i\theta}$ and $z_j \in \mathrm{e}^{i\theta}\Omega_\alpha$ for $\alpha < 1$. Particularly, this implies that

$$u_{r_j}(\mathrm{e}^{i\theta}) = u(r_j \mathrm{e}^{i\theta}) \to f(\mathrm{e}^{i\theta})$$

as $j \to \infty$ a.e. on T, where $\{r_j\} \subseteq [0, 1)$ and $r_j \to 1$ as $j \to \infty$. Furthermore, since $\sigma(T) < \infty$ and $\{u_r\} \subseteq L^1(T)$ is uniformly integrable, Problem 6.10(d) ensures that

$$\|u_{r_j} - f\|_1 = \int_T |u_{r_j}(\mathrm{e}^{i\theta}) - f(\mathrm{e}^{i\theta})| \,\mathrm{d}\sigma(\mathrm{e}^{i\theta}) \to 0$$

[1] It is obvious that N is independent of r.

as $j \to \infty$ or equivalently, we have

$$\lim_{j\to\infty} \int_T u_{r_j}(\mathrm{e}^{i\theta})\,\mathrm{d}\sigma(\mathrm{e}^{i\theta}) = \int_T f(\mathrm{e}^{i\theta})\,\mathrm{d}\sigma(\mathrm{e}^{i\theta}). \tag{11.70}$$

Finally, we take $g = 1$ in [62, Eqn. (3), p. 247] to get

$$\lim_{j\to\infty} \int_T u_{r_j}(\mathrm{e}^{i\theta})\,\mathrm{d}\sigma(\mathrm{e}^{i\theta}) = \int_T \mathrm{d}\mu(\mathrm{e}^{i\theta}). \tag{11.71}$$

Combining the results (11.70) and (11.71), we establish

$$\mathrm{d}\mu = f\,\mathrm{d}\sigma$$

and hence $u = P[f]$ for some $f \in L^1(T)$ which completes the proof of the problem. ∎

Remark 11.5

We can also apply Theorem 17.13 (F. and M. Riesz Theorem) on [62, p. 341] to prove that $\mu \ll \sigma$.

Problem 11.23

Rudin Chapter 11 Exercise 23.

Proof. Given $z \in U$, since $|\mathrm{e}^{i\theta} - z|^2 \geq \big||\mathrm{e}^{i\theta}| - |z|\big|^2 = (1-|z|)^2$, we have

$$|P(z, \mathrm{e}^{i\theta})| = \left|\frac{1-|z|^2}{|\mathrm{e}^{i\theta} - z|^2}\right| \leq \frac{1-|z|^2}{(1-|z|)^2} = \frac{1+|z|}{1-|z|} < \infty.$$

Thus the series

$$v(z) = \sum_{n=1}^{\infty} n^{-2} P(z, \mathrm{e}^{i\theta_n}) \quad \text{and} \quad w(z) = \sum_{n=1}^{\infty} n^{-2} P(z, \mathrm{e}^{-i\theta_n})$$

converge absolutely so that we may split the representation of u into the difference of v and w.

For each $n \in \mathbb{N}$, we define

$$v_n(z) = \sum_{k=1}^{n} k^{-2} P(z, \mathrm{e}^{i\theta_k}) \quad \text{and} \quad w_n(z) = \sum_{k=1}^{n} k^{-2} P(z, \mathrm{e}^{-i\theta_k}).$$

Using Problem 11.4, we know that $P(z, \mathrm{e}^{i\theta_k})$ and $P(z, \mathrm{e}^{-i\theta_k})$ are harmonic in U for $k = 1, 2, \ldots, n$ so that v_n and w_n are also harmonic in U. Fix $0 \leq r < 1$. If $z \in D(0, r)$, then we have $1 - |z| > 1 - r$ and thus

$$\sum_{n=1}^{\infty} \left|\frac{P(z, \mathrm{e}^{i\theta_n})}{n^2}\right| \leq \frac{1+r}{1-r} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By the Weierstrass M-test, we see that $\{v_n\}$ converges uniformly to v in $D(0; r)$. Let K be a compact subset of U. Then there exists a $0 < r < 1$ such that $K \subseteq D(0, r)$, so it follows from Theorem 11.11 (Harnack's Theorem) that v is harmonic in U. By a similar argument, we are able to show that w is also harmonic in U.

Since $P(z, \mathrm{e}^{it}) > 0$ for every $z \in U$ and $\mathrm{e}^{it} \in T$, we have $v > 0$ and $w > 0$. Now, for every $0 \le r < 1$, we observe that

$$\begin{aligned}
\|v_r\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r\mathrm{e}^{i\theta})\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\sum_{n=1}^{\infty} n^{-2}P(r\mathrm{e}^{i\theta}, \mathrm{e}^{i\theta_n})\Big]\,\mathrm{d}\theta \\
&= \sum_{n=1}^{\infty} n^{-2}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r\mathrm{e}^{i\theta}, \mathrm{e}^{i\theta_n})\,\mathrm{d}\theta\Big] \\
&= \sum_{n=1}^{\infty} n^{-2} \\
&< \infty.
\end{aligned}$$

Similarly, we have $\sup\limits_{0<r<1} \|w_r\|_1 < \infty$ and hence v and w satisfy the requirements of Theorem 11.30. Then there exist unique positive Borel measures μ and ν on T such that

$$v = P[\,\mathrm{d}\mu] \quad \text{and} \quad w = P[\,\mathrm{d}\nu].$$

Since $u = v - w$, we establish

$$u = P[\,\mathrm{d}(\mu - \nu)],$$

where $\mu - \nu$ is clearly a measure on T.

Next, if $x \in (-1, 1)$, then we get from [62, Eqn. (2), p. 233] that

$$P(x, \mathrm{e}^{i\theta_n}) = P(x, \mathrm{e}^{-i\theta_n}) = \frac{1 - x^2}{1 - 2x\cos\theta_n + x^2}.$$

Consequently, we conclude that $u(x) = 0$ if $-1 < x < 1$. Finally, suppose that $z = 1 - \epsilon + i\epsilon$, where $\epsilon = \sin\theta > 0$. Then it is easy to check that

$$\begin{aligned}
P(z, \mathrm{e}^{i\theta}) - P(z, \mathrm{e}^{-i\theta}) &= P(1 - \sin\theta + i\sin\theta, \mathrm{e}^{i\theta}) - P(1 - \sin\theta + i\sin\theta, \mathrm{e}^{-i\theta}) \\
&= \frac{1 - [(1 - \sin\theta)^2 + \sin^2\theta]}{|\cos\theta - 1 + \sin\theta|^2} + \frac{1 - [(1 - \sin\theta)^2 + \sin^2\theta]}{|(\cos\theta - 1 + \sin\theta) - 2i\sin\theta|^2} \\
&= 2\sin\theta(1 - \sin\theta)\cdot\Big[\frac{1}{(\cos\theta + \sin\theta - 1)^2} - \frac{1}{(\cos\theta + \sin\theta - 1)^2 + 4\sin^2\theta}\Big] \\
&= 2\sin\theta(1 - \sin\theta)\cdot\Big[\frac{1}{4\sin^2\frac{\theta}{2}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2} \\
&\qquad - \frac{1}{4\sin^2\frac{\theta}{2}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2 + 4\sin^2\theta}\Big].
\end{aligned} \tag{11.72}$$

Since $(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2 = 1 - \sin\theta$, the expression (11.72) reduces to

$$\begin{aligned}
P(z, \mathrm{e}^{i\theta}) - P(z, \mathrm{e}^{-i\theta}) &= 2\sin\theta\cdot\frac{4\sin^2\frac{\theta}{2}(1 - \sin\theta) + 4\sin^2\theta - 4\sin^2\frac{\theta}{2}(1 - \sin\theta)}{4\sin^2\frac{\theta}{2}[4\sin^2\frac{\theta}{2}(1 - \sin\theta) + 4\sin^2\theta]} \\
&= \frac{\sin^3\theta}{2\sin^2\frac{\theta}{2}[\sin^2\frac{\theta}{2}(1 - \sin\theta) + \sin^2\theta]} \\
&\ge \frac{\sin^3\theta}{2\sin^2\frac{\theta}{2}(\sin^2\frac{\theta}{2} + \sin^2\theta)}
\end{aligned}$$

$$\geq \frac{\sin\theta}{2\sin^2\frac{\theta}{2}}. \tag{11.73}$$

By the definition, ϵ is small if and only if $\theta > 0$ is small. In this case, $\cos^2\frac{\theta}{2} > \frac{1}{2}$, so the inequality (11.73) becomes

$$P(z, \mathrm{e}^{i\theta}) - P(z, \mathrm{e}^{-i\theta}) \geq \frac{\sin^2\theta}{2\sin^2\frac{\theta}{2}} \cdot \frac{1}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{\sin\theta} > \frac{1}{\sin\theta} = \frac{1}{\epsilon}. \tag{11.74}$$

If $z = x + iy \in U$ with $x > 0$ and $y > 0$, then it is easy to see from [62, Eqn. (6), p. 233] that

$$P(x + iy, \mathrm{e}^{i\theta}) - P(x + iy, \mathrm{e}^{-i\theta}) > 0 \tag{11.75}$$

for every $\theta \in (0, \frac{\pi}{2})$. By the definition of u, if we put $z = 1 - \epsilon + i\epsilon$, then we may apply the estimate (11.75) to get

$$u(1 - \epsilon + i\epsilon) > n^{-2}[P(1 - \epsilon + i\epsilon, \mathrm{e}^{i\theta_n}) - P(1 - \epsilon + i\epsilon, \mathrm{e}^{-i\theta_n})] \tag{11.76}$$

for every $n \in \mathbb{N}$. Now we take $\epsilon_n = \sin\theta_n = \sin\frac{1}{2^n}$ in the inequality (11.76) and then apply the estimate (11.74) as well as the fact that $\sin\theta_n \leq \theta_n$ for every $n \in \mathbb{N}$ to obtain

$$u(1 - \epsilon_n + i\epsilon_n) > \frac{1}{n^2\epsilon_n} = \frac{1}{n^2\sin\frac{1}{2^n}} \geq \frac{2^n}{n^2} \tag{11.77}$$

for every $n \in \mathbb{N}$. Since $\frac{2^n}{n^2} \to \infty$ as $n \to \infty$, we conclude immediately from the inequality (11.77) that

$$\lim_{n\to\infty} u(1 - \epsilon_n + i\epsilon_n) = \infty,$$

completing the proof of the problem. ■

Problem 11.24

Rudin Chapter 11 Exercise 24.

Proof. Most assertions of this problem come from [61, Exercise 15, p. 199] and their solutions are shown by the author in [77, Problem 8.15, pp. 187 – 190], so we just prove the necessary assertions here.

- **Proof of $K_{N-1}(t) \leq L_N(t)$.** Since $\sin 2\theta = \frac{1-\cos\theta}{2}$, we know that

$$K_{N-1}(t) = \frac{1}{N} \cdot \left(\frac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2. \tag{11.78}$$

By [61, Exercise 8, p. 197], we always have $|\sin n\theta| \leq n|\sin\theta|$ for $n = 0, 1, 2, \ldots$ so that

$$\left(\frac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2 = \frac{|\sin\frac{Nt}{2}|^2}{|\sin\frac{t}{2}|^2} \leq N^2 \tag{11.79}$$

for $\frac{N}{2}|t| \leq \frac{\pi}{2}$. If $0 \leq t \leq \pi$, then the power series of $\sin x$ implies that $\sin\frac{t}{2} \geq \frac{t}{\pi} \geq 0$ so that

$$\left(\frac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2 \leq \frac{1}{\sin^2\frac{t}{2}} \leq \frac{\pi^2}{t^2}. \tag{11.80}$$

If $-\pi \leq t \leq 0$, then we let $t = -s$ so that $0 \leq s \leq \pi$. In this case, we have

$$\left(\frac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2 = \left(\frac{\sin\frac{Ns}{2}}{\sin\frac{s}{2}}\right)^2 \leq \frac{1}{\sin^2\frac{s}{2}} \leq \frac{\pi^2}{s^2} = \frac{\pi}{t^2}. \tag{11.81}$$

Hence, by putting the inequalities (11.79), (11.80) and (11.81) into the expression (11.78), we conclude that if $|t| \leq \frac{\pi}{N}$, then

$$K_{N-1}(t) \leq \frac{1}{N} \cdot N^2 = N; \tag{11.82}$$

if $\frac{\pi}{N} \leq |t| \leq \pi$, then

$$K_{N-1}(t) \leq \frac{\pi^2}{Nt^2}. \tag{11.83}$$

By the definition of L_N, we see that $K_{N-1}(t) \leq L_N(t)$ for every $t \in [-\pi, \pi]$.

- **Proof of $\int_T L_N \, d\sigma \leq 2$.** Using the estimates (11.82) and (11.83), we have

$$\begin{aligned}
\int_T L_N \, d\sigma &= \frac{1}{2\pi}\int_{-\pi}^{\pi} L_n(t)\, dt \\
&= \frac{1}{2\pi}\int_{|t|\leq\frac{\pi}{N}} N\, dt + \frac{\pi}{2N}\int_{\frac{\pi}{N}\leq|t|\leq\pi} \frac{dt}{t^2} \\
&= 2 - \frac{1}{N} \\
&\leq 2
\end{aligned}$$

for every $N \in \mathbb{N}$.

- **Proof of Fejér's Theorem.** The result about the convergence of the arithmetic means is called **Fejér's Theorem.**[m] Recall from [62, Eqn. (1), p. 101] that

$$s_n(f;\theta) = \int_T f(e^{it})D_n(\theta - t)\, d\sigma = \int_T f\big(e^{i(\theta-t)}\big)D_n(t)\, d\sigma,$$

so it is easy to check that

$$\sigma_N(f;\theta) = \int_T f\big(e^{i(\theta-t)}\big)\Big[\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\Big]\, d\sigma = \int_T f\big(e^{i(\theta-t)}\big)K_N(t)\, d\sigma. \tag{11.84}$$

Suppose that $e^{i\theta}$ is a Lebesgue point of $f \in L^1(T)$. Imitating the proof of Theorem 11.23 (Fatou's Theorem), we may assume without loss of generality that $f(e^{i\theta}) = 0$. Thus it suffices to prove that

$$\lim_{N\to\infty} \sigma_N(f;\theta) = 0.$$

It is easy to check that the K_N satisfies $K_N(t) \geq 0$, $K_N(t)$ is even and

$$\int_T K_N(t)\, d\sigma = 1$$

for every $N \in \mathbb{N}$. Thus it follows from the expression (11.84) that

$$|\sigma_N(f;\theta)| = \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(e^{i(\theta-t)}\big)K_N(t)\, dt\Big|$$

[m] See, for instances, [61, Exercises 15, 16, p. 199] or [84, Theorem 3.4, p. 89].

$$= \frac{1}{2\pi}\Big|\int_0^{\pi} f\big(e^{i(\theta-t)}\big)K_N(t)\,dt + \int_{-\pi}^{0} f\big(e^{i(\theta-t)}\big)K_N(t)\,dt\Big|$$
$$\leq \frac{1}{2\pi}\int_0^{\pi} \big|f\big(e^{i(\theta-t)}\big) + f\big(e^{i(\theta+t)}\big)\big| \cdot K_N(t)\,dt.$$

Put $g(t) = \big|f\big(e^{i(\theta-t)}\big) + f\big(e^{i(\theta+t)}\big)\big|$ and

$$G(x) = \int_0^x g(t)\,dt,$$

where $0 \leq x \leq \pi$. We have

$$0 < \frac{G(x)}{x} \leq \frac{1}{x}\int_0^x |f\big(e^{i(\theta-t)}\big)|\,dt + \frac{1}{x}\int_0^x |f\big(e^{i(\theta+t)}\big)|\,dt$$

and since $e^{i\theta}$ is a Lebesgue point of f, we know from the definition [62, Eqn. (5), p. 241] that

$$\lim_{x\to 0}\frac{G(x)}{x} = 0. \tag{11.85}$$

Given $\epsilon > 0$. Firstly, the result (11.85) means that we may choose a $\delta > 0$ such that $G(x) < \epsilon x$ for all $0 < x < \delta$. If $N > \frac{1}{\delta}$, then we deduce from (11.82) that

$$\int_0^{\frac{1}{N}} g(t)K_N(t)\,dt \leq (N+1)\int_0^{\frac{1}{N}} g(t)\,dt < \frac{N+1}{N}\epsilon < 2\epsilon. \tag{11.86}$$

Secondly, the property (11.83) actually holds for all $0 < |t| \leq \pi$ so that

$$\begin{aligned}
\int_{\frac{1}{N}}^{\delta} g(t)K_N(t)\,dt &< \frac{\pi^2}{N}\int_{\frac{1}{N}}^{\delta}\frac{g(t)}{t^2}\,dt \\
&= \frac{\pi^2}{N}\int_{\frac{1}{N}}^{\delta}\frac{dG(t)}{t^2} \\
&= \frac{\pi^2}{N}\Big[\frac{G(\delta)}{\delta^2} - N^2 G\Big(\frac{1}{N}\Big) + 2\int_{\frac{1}{N}}^{\delta}\frac{G(t)}{t^3}\,dt\Big] \\
&< \frac{\pi^2}{N}\Big(\frac{\epsilon}{\delta} + 2\epsilon\int_{\frac{1}{N}}^{\delta}\frac{dt}{t^2}\Big) \\
&< \frac{\pi^2}{N}\Big(\frac{\epsilon}{\delta} + 2\epsilon N\Big) \\
&< 3\pi^2\epsilon.
\end{aligned} \tag{11.87}$$

Thirdly, we note that

$$\begin{aligned}
\int_{\delta}^{\pi} g(t)K_N(t)\,dt &\leq \frac{\pi^2}{N\delta^2}\int_{\delta}^{\pi} g(t)\,dt \\
&\leq \frac{\pi^2}{N\delta^2}\Big[\int_T |f\big(e^{i(\theta-t)}\big)|\,dt + \int_T |f\big(e^{i(\theta+t)}\big)|\,dt\Big] \\
&= \frac{\pi^2}{N\delta^2}\cdot 2\|f\|_1 \\
&< \epsilon
\end{aligned} \tag{11.88}$$

for large enough N. Finally, by combining the inequalities (11.86), (11.87) and (11.88), we conclude immediately that

$$|\sigma_N(f;\theta)| \leq \frac{1}{2\pi}\int_0^{\pi} g(t)K_N(t)\,dt < \frac{1}{2\pi}(3\epsilon + 3\pi^2\epsilon) = \frac{3(1+\pi^2)}{2\pi}\epsilon$$

for sufficiently large N. Since ϵ is arbitrary, we have obtained the desired result that $\sigma_N(f;\theta) \to 0$ as $N \to \infty$.

We have completed the proof of the problem. ■

Problem 11.25

Rudin Chapter 11 Exercise 25.

Proof. Let $z = x + i\lambda$, where $\lambda > 0$. Using [62, Eqn. (3), §9.7, p. 183], we find that

$$u(z) = u(x,\lambda) = (f * h_\lambda)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)h_\lambda(y)\,\mathrm{d}y = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda f(x-y)}{y^2+\lambda^2}\,\mathrm{d}y. \tag{11.89}$$

We prove the problem by showing the following steps:

- **Step 1: $\varphi(z,t)$ is harmonic in Π^+ for every $t \in \mathbb{R}$.** By the change of variable $t = x - y$, the expression (11.89) can be written as
$$u(z) = u(x,\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda f(t)}{(x-t)^2+\lambda^2}\,\mathrm{d}t = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(x+i\lambda,t)f(t)\,\mathrm{d}t,$$
where $\varphi : \Pi^+ \times \mathbb{R}$ is given by
$$\varphi(x+i\lambda,t) = \frac{\lambda}{(x-t)^2+\lambda^2} = \operatorname{Im}\Big(\frac{1}{t-(x+i\lambda)}\Big). \tag{11.90}$$
Since $\frac{1}{t-(x+i\lambda)}$ is holomorphic in Π^+ for every $t \in \mathbb{R}$, Theorem 11.4 ensures that φ is harmonic in Π^+.[n] Hence $\varphi(z,t)$ is continuous on Π^+ and then it satisfies the mean value property for every $t \in \mathbb{R}$:
$$\varphi(z,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(z+re^{i\theta},t)\,\mathrm{d}\theta,$$
where r is any positive number such that $\overline{D}(z,r) \subseteq \Pi^+$.

- **Step 2: $\varphi(z,t) \leq \frac{M_z}{1+t^2}$ for some $M_z > 0$.** We claim that there exists a constant $M_z > 0$ such that
$$0 < \varphi(z,t) = \varphi(x+i\lambda,t) \leq \frac{M_z}{1+t^2} \tag{11.91}$$
for all $t \in \mathbb{R}$. To see this, simple calculation shows that
$$\frac{\lambda}{(x-t)^2+\lambda^2} \leq \frac{M_z}{1+t^2}$$
holds for all $t \in \mathbb{R}$ if and only if
$$(\lambda - M_z)t^2 + 2M_z xt + (\lambda - M_z x^2 - M_z\lambda^2) \leq 0$$
for all $t \in \mathbb{R}$ if and only if
$$(2M_z x)^2 - 4(\lambda - M_z)(\lambda - M_z x^2 - M_z\lambda^2) \leq 0$$
$$\lambda M_z^2 - (x^2+\lambda^2+1)M_z + \lambda \geq 0 \tag{11.92}$$
and the inequality (11.92) is true for large $M_z > 0$. This proves the claim.

[n] In fact, it can be shown directly from the definition (11.90) that, for each $t \in \mathbb{R}$, we have
$$\varphi_{xx} = \frac{6\lambda(x-t)^2 - 2\lambda^3}{[(x-t)^2+\lambda^2]^3} \quad \text{and} \quad \varphi_{\lambda\lambda} = \frac{[(x-t)^2+\lambda^2](-2\lambda) - 4\lambda[(x-t)^2-\lambda^2]}{[(x-t)^2+\lambda^2]^3}$$
which give $\Delta\varphi = 0$.

- **Step 3:** $\varphi(z,t) \in L^q(\mathbb{R})$ **for every** $1 \leq q \leq \infty$**.** By **Step 2**, if $1 \leq q < \infty$, then we have $1 \leq 1+t^2 \leq (1+t^2)^q$ so that
$$\int_{-\infty}^{\infty} \varphi^q(z,t)\,\mathrm{d}t \leq \int_{-\infty}^{\infty} \frac{M_z^q}{(1+t^2)^q}\,\mathrm{d}t \leq M_z^q \cdot \int_{-\infty}^{\infty} \frac{1}{1+t^2}\,\mathrm{d}t < \infty. \tag{11.93}$$
In other words, we have $\varphi(z,t) \in L^q(\mathbb{R}^1)$. By the inequality (11.91), since $\varphi(z,t)$ is obviously bounded by M_z, we know that $\varphi(z,t) \in L^\infty(\mathbb{R}^1)$.

- **Step 4:** u **satisfies the mean value property.** Suppose that $z \in \Pi^+$ and r is a positive number such that $\overline{D}(z;r) \subseteq \Pi^+$. If q is the conjugate exponent of p, then we obtain from Theorem 3.8 and **Step 3** that
$$\|\varphi(z+r\mathrm{e}^{i\theta},\cdot)f(\cdot)\|_1 = \int_{-\infty}^{\infty} |\varphi(z+r\mathrm{e}^{i\theta},t)f(t)|\,\mathrm{d}t \leq \|f\|_p \times \|\varphi(z+r\mathrm{e}^{i\theta},\cdot)\|_q < \infty.$$
Next, we observe from the inequality (11.92) that we can pick M_ζ in such a way that the set $\{M_\zeta \,|\, \zeta = z + r\mathrm{e}^{i\theta}$ and $-\pi \leq \theta \leq \pi\}$ is bounded. Therefore, we get
$$\int_{-\pi}^{\pi} \|\varphi(z+r\mathrm{e}^{i\theta},\cdot)f(\cdot)\|_1\,\mathrm{d}\theta < \infty.$$
Consequently, we apply Theorem 8.8 (The Fubini Theorem) and **Step 1** to conclude that
$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} u(z+r\mathrm{e}^{i\theta})\,\mathrm{d}\theta &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \varphi(z+r\mathrm{e}^{i\theta},t)f(t)\,\mathrm{d}t\,\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(z+r\mathrm{e}^{i\theta},t)\,\mathrm{d}\theta\Big)f(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(z,t)f(t)\,\mathrm{d}t \\ &= u(z).\end{aligned}$$

- **Step 5:** u **is continuous on** Π^+**.** Let $\{z_n\}$ be a sequence of Π^+ converging to $z_0 \in \Pi^+$. Then we must have $\varphi^q(z_n,t) \to \varphi^q(z_0,t)$ as $n \to \infty$. Now the inequality (11.93) guarantees that we may use Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) to conclude that
$$\begin{aligned}\lim_{n\to\infty} u(z_n) &= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(z_n,t)f(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\Big[\lim_{n\to\infty}\varphi(z_n,t)\Big]f(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(z_0,t)f(t)\,\mathrm{d}t \\ &= u(z_0).\end{aligned}$$
Thus u is continuous on Π^+.

- **Step 6:** u **is harmonic in** Π^+**.** In fact, this follows immediately from **Steps 4**, **5** and Theorem 11.13.

Hence we have completed the analysis of the proof. ■

CHAPTER 12

The Maximum Modulus Principle

12.1 Applications of the Maximum Modulus Principle

Problem 12.1

Rudin Chapter 12 Exercise 1.

Proof. Let $f(z) = (z-a)(z-b)(z-c)$. Then f is nonconstant and entire. By Theorem 10.24 (The Maximum Modulus Theorem), we know that

$$\max_{z\in\Delta} |f(z)| = \max_{z\in\partial\Delta} |f(z)|.$$

Let $L = |a-b| = |b-c| = |c-a|$ and $t = t(z) = |z-a| \in [0, L]$, see Figure 12.1 below.

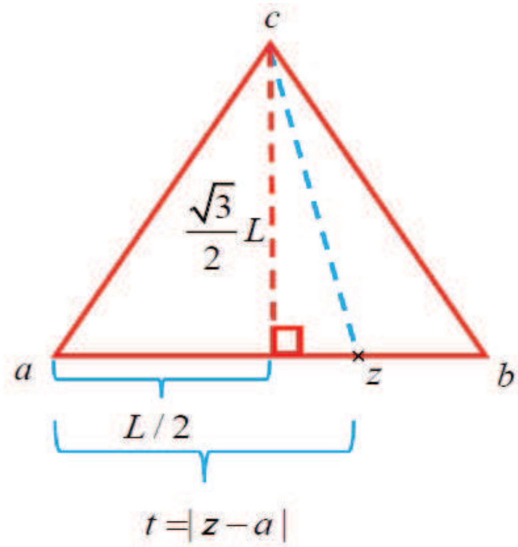


Figure 12.1: The boundary $\partial\Delta$.

If $z \in [a, b]$, then we have

$$\begin{aligned}|f(z)|^2 &= |z-a|^2 \cdot |z-b|^2 \cdot |z-c|^2 \\ &= t^2(L-t)^2 \cdot \left[\left(t-\frac{L}{2}\right)^2 + \left(\frac{\sqrt{3}L}{2}\right)^2\right]\end{aligned}$$

$$= t^2(L-t)^2(t^2 - Lt + L^2).$$

Define the function $F : [0, L] \to \mathbb{R}$ by $F(t) = t^2(L-t)^2(t^2 - Lt + L^2)$. Elementary differentiation shows that

$$\begin{aligned} F'(t) &= 6t^5 - 15Lt^4 + 16L^2t^3 - 9L^3t^2 + 2L^4t \\ &= t(2t - L)(3t^3 - 6Lt^2 + 5L^2t - 2L^3) \\ &= t(2t - L)(t - L)(3t^2 - 3Lt + 2L^2). \end{aligned}$$

Thus $F'(t) = 0$ if and only if $t = 0, \frac{L}{2}, L$. By the First Derivative Test, it is easily seen that F attains its maximum $\frac{3}{64}L^6$ at $t = \frac{L}{2}$. Hence we conclude that

$$\max_{z\in\Delta} |f(z)| = \max_{z\in\partial\Delta} |f(z)| = \max_{t\in[0,L]} \sqrt{F(t)} = \sqrt{F\Big(\frac{L}{2}\Big)} = \frac{\sqrt{3}}{8}L^3.$$

This completes the analysis of the problem. ■

Problem 12.2

Rudin Chapter 12 Exercise 2.

Proof. Suppose that $f(i) = \alpha$. If $|\alpha| = 1$, then Theorem 10.24 (The Maximum Modulus Theorem) forces that $f(z) = \alpha$ in Π^+. In this case, we have $|f'(i)| = 0$. Suppose that $\alpha \in U$. By Definition 12.3, $\varphi_\alpha(\alpha) = 0$. Furthermore, we know from [9, Theorem 13.16, p. 183] that the mapping $h : \Pi^+ \to U$ given by

$$h(z) = e^{i\theta}\Big(\frac{z-\beta}{z-\overline{\beta}}\Big), \tag{12.1}$$

where $\operatorname{Im}\beta > 0$ and $\theta \in \mathbb{R}$, is a bijection. Particularly, we take $\beta = i$ in the definition (12.1). Clearly, we have $h(i) = 0$. Next, we consider the mapping $F = \varphi_\alpha \circ f \circ h^{-1} : U \to U$. Then we have $F \in H^\infty$, $\|F\|_\infty \leq 1$ and

$$F(\alpha) = \varphi_\alpha\big(f\big(h^{-1}(0)\big)\big) = \varphi_\alpha\big(f(i)\big) = \varphi_\alpha(\alpha) = 0.$$

Hence it follows from Theorem 12.2 (Schwarz's Lemma) that

$$|F'(0)| \leq 1. \tag{12.2}$$

Since $h^{-1}(z) = i \cdot \frac{z+e^{i\theta}}{e^{i\theta}-z}$, we have $(h^{-1})'(0) = 2ie^{-i\theta}$. Consequently, we see from Theorem 12.4 that

$$F'(0) = \varphi'_\alpha(\alpha) \times f'(i) \times (h^{-1})'(0) = \frac{2ie^{-i\theta}f'(i)}{1-|\alpha|^2}, \tag{12.3}$$

so the inequality (12.2) implies that

$$|f'(i)| \leq \frac{1-|\alpha|^2}{2}.$$

Thus $|f'(i)|$ attains the maximum $\frac{1}{2}$ when $\alpha = 0$. In this case, we observe from the expression (12.3) that $|F'(0)| = 1$, so Theorem 12.2 (Schwarz's Lemma) implies that $F(z) = \lambda z$ for some constant λ with $|\lambda| = 1$. Since $\varphi_0(z) = z$, we conclude that $f\big(h^{-1}(z)\big) = \lambda z$. Now if we put $z = h(\zeta)$, then it asserts that

$$f(\zeta) = \lambda h(\zeta) = \lambda e^{i\theta}\Big(\frac{\zeta - i}{\zeta + i}\Big). \tag{12.4}$$

Since $\lambda = e^{i\phi}$ for some $\phi \in \mathbb{R}$, we may simply replace $\lambda e^{i\theta}$ by $e^{i\theta}$ in the representation (12.4) which gives all extremal functions with $|f'(i)| = \frac{1}{2}$. This completes the proof of the problem. ■

Problem 12.3

Rudin Chapter 12 Exercise 3.

Proof. If f is constant, then there is nothing to prove. Thus, without loss of generality, we may assume that f is nonconstant. In this case, f has a local minimum in Ω if and only if f has a zero in Ω. Assume that f was a non-vanishing function in Ω. Then $\frac{1}{f} \in H(\Omega)$ and $|f|$ has a local minimum at $z_0 \in \Omega$ if and only if $\frac{1}{|f|}$ has a local maximum at z_0. By Theorem 10.24 (The Maximum Modulus Theorem), $\frac{1}{f}$ is forced to be constant which is impossible by our hypothesis. Hence f has a zero in Ω, completing the proof of the problem. ■

Problem 12.4

Rudin Chapter 12 Exercise 4.

Proof.

(a) Assume that f was non-vanishing in D. By Theorem 10.24 (The Maximum Modulus Theorem), $f \neq 0$ on ∂D. Thus it is true that $f \neq 0$ in $\overline{D}$. Denote $M = |f(z)|$ on ∂D. On the one hand, Theorem 10.24 (The Maximum Modulus Theorem) again implies that

$$|f(z)| \leq M \tag{12.5}$$

for all $z \in D$. On the other hand, since $\frac{1}{f} \in H(D)$ and $\frac{1}{f} \in C(\overline{D})$, we follow from Theorem 10.24 (The Maximum Modulus Theorem) that

$$\frac{1}{|f(z)|} \leq \frac{1}{M} \tag{12.6}$$

for all $z \in D$. Combining the inequalities (12.5) and (12.6), we conclude that $|f(z)| = M$ in D, or equivalently, $f(D) \in \{-M, M\}$ which contradicts the Open Mapping Theorem. Hence f has at least one zero in D.

(b) This part is proven in [76, Problem 7.5, pp. 85 – 87].

We end the proof of the problem. ■

Problem 12.5

Rudin Chapter 12 Exercise 5.

Proof. Given that $\epsilon > 0$. Since $f_n \to f$ uniformly on $\partial\Omega$, there exists an $N \in \mathbb{N}$ such that $m, n \geq N$ imply that

$$|f_n(z) - f_m(z)| < \epsilon \tag{12.7}$$

for every $z \in \partial\Omega$. Since Ω is bounded, we may apply Theorem 10.24 (The Maximum Modulus Theorem) to assert that the inequality (12.7) holds for every $z \in \overline{\Omega}$, i.e.,

$$\max_{z\in\overline{\Omega}} |f_n(z) - f_m(z)| = \max_{z\in\partial\Omega} |f_n(z) - f_m(z)| < \epsilon \tag{12.8}$$

for $m, n \geq N$. According to the Cauchy Criterion for Uniform Convergence [61, Theorem 7.8, p. 147], the inequality (12.8) ensures that $\{f_n\}$ converges uniformly on $\overline{\Omega}$ which ends the proof of the problem. ■

Problem 12.6

Rudin Chapter 12 Exercise 6.

Proof. By the definitions, Γ^* is closed in $\mathbb{C}$ and then $\mathbb{C} \setminus \Gamma^*$ is open in $\mathbb{C}$. By Definition 10.1, $\mathbb{C} \setminus \Gamma^*$ is a union of disjoint open connected sets, and hence components. Suppose that V is a component of $\mathbb{C} \setminus \Gamma^*$ such that $\mathrm{Ind}_\Gamma(z) \neq 0$ for every $z \in V$. Let $\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$, where each γ_j is a closed path in Γ. Assume that V was unbounded. If $z \in V$, then it follows from Theorem 10.10 that $\mathrm{Ind}_{\gamma_j}(z) = 0$ for every $1 \leq j \leq n$. Since

$$\mathrm{Ind}_\Gamma(z) = \sum_{j=1}^{n} \mathrm{Ind}_{\gamma_j}(z),$$

we have $\mathrm{Ind}_\Gamma(z) = 0$, a contradiction. Therefore, V must be bounded. Since $\mathrm{Ind}_\Gamma(\alpha) = 0$ for all $\alpha \notin \Omega$, we get

$$V \subseteq \Omega. \tag{12.9}$$

Let $\zeta \in \partial V$. Then $\zeta \notin V$. If ζ lies in another component U, then there exists a $\delta > 0$ such that $D(\zeta; \delta) \cap V \neq \varnothing$ and $D(\zeta; \delta) \cap U \neq \varnothing$, but this is a contradiction by [42, Theorem 25.1, p. 159]. Hence $\zeta \notin \mathbb{C} \setminus \Gamma^*$, i.e., $\zeta \in \Gamma^*$ and then $\partial V \subseteq \Gamma^*$. Since Γ is a cycle in Ω, we have

$$\Gamma^* \subseteq \Omega. \tag{12.10}$$

Combining the set relations (12.9) and (12.10), we conclude that $\overline{V} \subseteq \Omega$. Recall that V is bounded, $\overline{V}$ is compact. By the hypothesis, we have $|f(\zeta)| \leq 1$ for every $\zeta \in \partial V \subseteq \Gamma^*$. By Theorem 10.24 (The Maximum Modulus Theorem), we observe that

$$|f(z)| \leq 1$$

for every $z \in V$ with $\mathrm{Ind}_\Gamma(z) \neq 0$. This completes the proof of the problem. ■

Problem 12.7

Rudin Chapter 12 Exercise 7.

Proof. Suppose that $\Omega = \{x + iy \,|\, a < x < b,\, y \in \mathbb{R}\}$ and $M(a) = 0$. We have to show that $M(x) = 0$ for all $x \in (a, b)$. Consider the function $g(x + iy) = f(y + a + ix)$ which is defined in the horizontal strip $\Omega_g^+ = \{x + iy \,|\, -\infty < x < \infty \text{ and } 0 < y < b - a\}$. By the definition, we know that

$$|g(x)| = |f(a + ix)| = 0$$

for all $x \in \mathbb{R}$. In particular, g is real on the segment $L = (0, 1)$. By Theorem 11.14 (The Schwarz Reflection Principle), there exists a function G holomorphic in $\Pi = \Omega_g^+ \cup L \cup \Omega_g^-$ such that $G(x) = g(x) = 0$ for every $x \in L$. By Theorem 10.18, we have $G(z) = 0$ in Π. Since $G(z) = g(z) = 0$ in Ω_g^+, we obtain

$$f(z) = 0$$

in Ω which implies the required result that $M(x) = 0$ for all $x \in (a, b)$ and hence Theorem 12.8 (The Hadamard's Three-Line Theorem) is also true if $M(a) = 0$. This completes the proof of the problem. ■

Problem 12.8

Rudin Chapter 12 Exercise 8.

Proof. Suppose that $\Omega = \{z = x + iy \,|\, c < x < d \text{ and } y \in \mathbb{R}\}$ for some $-\infty < c < d < \infty$. By the hypothesis, the exponential function $\zeta = e^z$ maps Ω onto $A(R_1, R_2)$. We are given that $R_1 < a < r < b < R_2$. Then there exist $c < \alpha < \beta < d$ such that $\zeta = e^z$ maps the closed strip $\overline{\Omega'} = \{z = x + iy \,|\, \alpha \leq x \leq \beta \text{ and } y \in \mathbb{R}\}$ onto the closed annulus $\overline{A}(a, b)$. Thus we have

$$e^{\alpha} = a \quad \text{and} \quad e^{\beta} = b. \tag{12.11}$$

We define $F : \overline{\Omega'} \to \mathbb{C}$ by

$$F(z) = f(e^z) = f(\zeta)$$

which is continuous on $\overline{\Omega'}$ and $F \in H(\Omega')$. By Theorem 10.24 (The Maximum Modulus Theorem) and the Extreme Value Theorem, there is a positive constant $B > 0$ such that $|F(z)| < B$ for all $z \in \Omega'$. In other words, we may apply Theorem 12.8 (The Hadamard's Three-Line Theorem) to our function F to get

$$M(x)^{\beta-\alpha} \leq M(\alpha)^{\beta-x} \times M(\beta)^{x-\alpha}, \tag{12.12}$$

where the two expressions (12.11) give

$$\begin{aligned}
M(x) &= \sup\{|F(x + iy)| \,|\, \alpha \leq x \leq \beta \text{ and } y \in \mathbb{R}\} \\
&= \sup\{|f(e^x \cdot e^{iy})| \,|\, \alpha \leq x \leq \beta \text{ and } y \in \mathbb{R}\} \\
&= \sup\{|f(re^{iy})| \,|\, a \leq r \leq b \text{ and } y \in \mathbb{R}\} \\
&= M(r).
\end{aligned}$$

Therefore, it deduces from the inequality (12.12) that

$$\begin{aligned}
(\beta - \alpha) \log M(r) &\leq (\beta - x) \log M(a) + (x - \alpha) \log M(b) \\
\log M(r) &\leq \frac{\log \frac{b}{r}}{\log \frac{b}{a}} \log M(a) + \frac{\log \frac{r}{a}}{\log \frac{b}{a}} \log M(b)
\end{aligned} \tag{12.13}$$

which is the required result.

We claim that the equality (12.13) holds if and only if $f(\zeta) = A\zeta^{\lambda}$ for some $A \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$. Obviously, the equality holds if f is of this form. Conversely, the we can rewrite the equality (12.13) as

$$\begin{aligned}
\log \frac{b}{a} \log M(r) &= \log \frac{b}{r} \log M(a) + \log \frac{r}{a} \log M(b) \\
&= \Big(\log \frac{b}{a} - \log \frac{r}{a}\Big) \cdot [\log M(a)] + \log \frac{r}{a} \log M(b) \\
&= \log \frac{b}{a} \log M(a) + \log \frac{r}{a} \log \frac{M(b)}{M(a)} \\
\log M(r) &= \log M(a) + \frac{\log \frac{r}{a}}{\log \frac{b}{a}} \log \frac{M(b)}{M(a)}.
\end{aligned} \tag{12.14}$$

If we denote $\lambda = (\log \frac{b}{a})^{-1} \log \frac{M(a)}{M(b)} \in \mathbb{R}$, then the expression (12.14) can be further simplified to

$$\log M(r) = \log M(a) - \lambda \log \frac{r}{a}$$

$$M(r) = \left(\frac{a}{r}\right)^{\lambda} M(a). \tag{12.15}$$

If we combine the Extreme Value Theorem and the expression (12.15), then one can show that there exists an $|\zeta_0| = r \in (a,b)$ such that $|f(\zeta_0)| = M(r) = (\frac{a}{r})^{\lambda} M(a)$ which can be rewritten as

$$|\zeta_0^{\lambda} f(\zeta_0)| = a^{\lambda} M(a). \tag{12.16}$$

Notice that $\zeta^{\lambda} f(\zeta)$ may *not* be holomorphic because λ may not be an integer. However, remember that $\zeta = \mathrm{e}^z$, so we can express the expression (12.16) as

$$|\mathrm{e}^{\lambda z_0} f(\mathrm{e}^{z_0})| = a^{\lambda} M(a) = \mathrm{e}^{\lambda\alpha} M(\alpha),$$

where $\zeta_0 = \mathrm{e}^{z_0} \in \Omega'$. Now $\mathrm{e}^{\lambda z} f(\mathrm{e}^z) \in H(\Omega')$ and continuous on $\overline{\Omega'}$, so we may apply Theorem 12.4 (The Maximum Modulus Theorem) to this function to conclude that

$$\mathrm{e}^{\lambda z} f(\mathrm{e}^z) = c \tag{12.17}$$

in Ω', where c is some constant. Since Ω' is a region, Theorem 10.18 asserts that the result (12.17) also holds in Ω. Using the substitution $\zeta = \mathrm{e}^z$ again, the result (12.17) becomes

$$f(\zeta) = c\zeta^{-\lambda}$$

in $A(R_1, R_2)$. Recall that $f \in H\big(A(R_1, R_2)\big)$, it forces that $\lambda \in \mathbb{Z}$. This ends the analysis of the problem. ■

Problem 12.9

Rudin Chapter 12 Exercise 9.

Proof. We prove the assertions one by one.

- **$|f(z)| \leq 1$ in the open right half plane Π.** If $\alpha = 0$, then $|f(z)| \leq A\mathrm{e}$ in Π. Since Π is unbounded and $\partial\Pi$ is exactly the imaginary axis, we may apply Problem 12.11 to conclude that $|f(z)| \leq 1$ in Π. Next, if $\alpha < 0$, then there exists a $R > 0$ such that $|z| \geq R$ implies
$$|f(z)| \leq 1. \tag{12.18}$$
Since $\overline{D}(0;R) \cap \Pi$ is compact, it is bounded. Therefore, Theorem 10.24 (The Maximum Modulus Theorem), the Extreme Value Theorem and the hypothesis $|f(iy)| \leq 1$ for all $y \in \mathbb{R}$ ensure that the inequality (12.18) is also valid for all $z \in \Pi$. Thus we may assume that $0 < \alpha < 1$ in the following discussion.

  Denote $\Omega = \{x + iy \,|\, x \in \mathbb{R},\, |y| < \frac{\pi}{2}\}$. Consider the mapping $\varphi : \Omega \to \Pi$ defined by
$$\varphi(z) = \mathrm{e}^z \tag{12.19}$$
which is clearly an isomorphism and $\varphi \in H(\Omega)$. Next, we define $g : \Omega \to \mathbb{C}$ by $g = f \circ \varphi$. By Problem 10.14, we know that $g \in H(\Omega)$. By the definition of (12.19), we see that $\varphi(\partial\Omega) = \partial\Pi$. Since φ and f are continuous on $\overline{\Omega}$ and $\overline{\Pi}$ respectively, g is continuous on $\overline{\Omega}$. Furthermore, we have
$$\left|g\Big(x \pm \frac{\pi i}{2}\Big)\right| = \left|f\Big(\exp\Big(x \pm \frac{\pi i}{2}\Big)\Big)\right| = |f(\pm i\mathrm{e}^x)| \leq 1$$

for all $x \in \mathbb{R}$. Finally, if $z \in \Omega$, then we have

$$|g(z)| = |f(\varphi(z))| < A\exp(|\varphi(z)|^\alpha) < A\exp(|e^z|^\alpha) = A\exp(e^{\alpha x}) \leq A\exp(e^{\alpha|x|}). \quad (12.20)$$

Choose B such that $B - 1 \geq \log A$. Note that $e^{\alpha|x|} \geq 1$ for all $x \in \mathbb{R}$. Then it is easy to see that

$$\begin{aligned} \log A &\leq (B-1)e^{\alpha|x|} \\ \log A + e^{\alpha|x|} &\leq Be^{\alpha|x|} \\ A\exp(e^{\alpha|x|}) &\leq \exp(Be^{\alpha|x|}). \end{aligned} \quad (12.21)$$

Combining the inequalities (12.20) and (12.21), the inequality

$$|g(z)| < \exp(Be^{\alpha|x|})$$

holds for all $z = x + iy \in \Omega$. Hence Theorem 12.9 (The Phragmen-Lindelöf Theorem) asserts that $|g(z)| \leq 1$ in Ω which means that the inequality (12.18) holds in Π.

- **The conclusion is false for $\alpha = 1$.** It is easy to check that the function $f(z) = e^{e^z}$ gives a counterexample to the result.

- **The modified result.** Suppose that Δ is an open sector between two rays from the origin with sectoral angle $\frac{\pi}{\beta} < \pi$ for some $\beta > 1$. Suppose that f is continuous on $\overline{\Delta}$, $f \in H(\Delta)$ and there are constants $A < \infty$ and $\alpha \in (0, \beta)$ such that

$$|f(z)| < A\exp(|z|^\alpha) \quad (12.22)$$

for all $z \in \Delta$. Furthermore, if $|f(z)| \leq 1$ on $\partial\Delta$, then we have $|f(z)| \leq 1$ in Δ.

To see this, let θ be the angle between the real axis and the ray nearest to it. Then we see that the mapping $\phi : \Delta \to \Pi$ defined by

$$\phi(z) = -i(e^{-i\theta}z)^\beta = -ie^{-i\beta\theta}\exp(\beta\log z)$$

is clearly an isomorphism such that ϕ maps the boundary of Δ onto the boundary of Π. Since $0 \notin \Delta$, we can define a branch for $\log z$ so that $\phi \in H(\Delta)$. By Theorem 10.33, we have $\phi^{-1} \in H(\Pi)$. Next, the map $F : \Pi \to \mathbb{C}$ defined by

$$F = f \circ \phi^{-1}$$

is continuous on $\overline{\Pi}$ and $F \in H(\Pi)$. Since $\phi^{-1}(\zeta) = e^{i\theta}i^{\frac{1}{\beta}}\zeta^{\frac{1}{\beta}}$, we have $|\phi^{-1}(\zeta)| = |\zeta|^{\frac{1}{\beta}}$ and thus the hypothesis (12.22) implies

$$|F(\zeta)| = |f(\phi^{-1}(\zeta))| < A\exp(|\phi^{-1}(\zeta)|^\alpha) = A\exp(|\zeta|^{\frac{\alpha}{\beta}}),$$

where $\frac{\alpha}{\beta} < 1$. If $\zeta = iy$ for some $y \in \mathbb{R}$, then since $\phi^{-1}(iy)$ lies on $\partial\Delta$, we get

$$|F(iy)| = |f(\phi^{-1}(\zeta))| \leq 1.$$

Hence we establish from our first assertion that $|F(\zeta)| \leq 1$ in Π or equivalently, $|f(z)| \leq 1$ in Δ.

We have completed the analysis of the problem. ■

Problem 12.10

Rudin Chapter 12 Exercise 10.

Proof. For each $n = 1, 2, 3, \ldots$, we define $g_n(z) = f(z)e^{nz}$. Suppose that $L_\alpha = \{z = re^{i\alpha} \,|\, r \geq 0\}$,

$$\Delta_1 = \Big\{z \in \Pi \,\Big|\, -\frac{\pi}{2} < \arg z < \alpha\Big\} \quad \text{and} \quad \Delta_2 = \Big\{z \in \Pi \,\Big|\, \alpha < \arg z < \frac{\pi}{2}\Big\}.$$

Certainly, Δ_1 and Δ_2 have the sectoral angles $\alpha + \frac{\pi}{2} < \pi$ and $\frac{\pi}{2} - \alpha < \pi$ respectively. Refer to Figure 12.2:

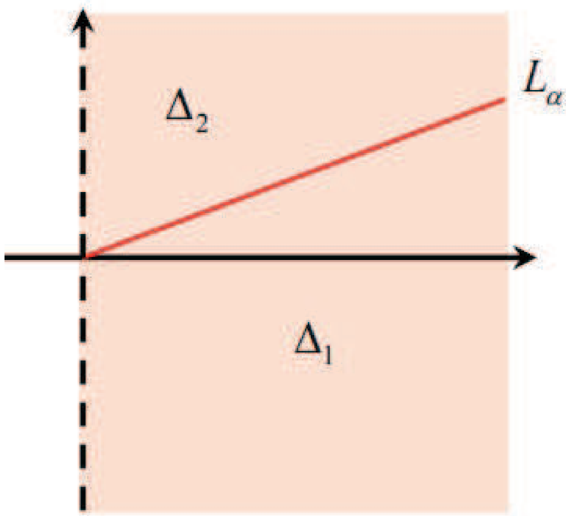


Figure 12.2: The sectors Δ_1, Δ_2 and the ray L_α.

In order to use Problem 12.9 as stated in the hint, it is necessary to *assume* that f is continuous on the closure of Π. Since $f \in H(\Pi)$ and Δ_1, Δ_2 are proper subsets of Π, each g_n is continuous on $\overline{\Delta_j}$ and $g_n \in H(\Delta_j)$ for $j = 1, 2$. Choose an $\gamma \in (0, \frac{2\pi}{2\alpha+\pi})$ and let $F_n(x) = nx - x^\gamma$ for $x \geq 0$. By elementary calculus, the F_n attains its maximum value

$$M_n = n\Big(\frac{n}{\gamma}\Big)^{\frac{1}{\gamma-1}}\Big(\frac{\gamma-1}{\gamma}\Big)$$

at $x = (n\gamma^{-1})^{\frac{1}{\gamma-1}} > 0$. Therefore, if we pick $A_n = 1 + e^{M_n}$, then we have

$$\exp(n|z| - |z|^\gamma) = e^{F_n(|z|)} \leq e^{M_n} < A_n \tag{12.23}$$

for all $z \in \Delta_1$. Since $|f(z)| < 1$ for all $z \in \Pi$, for each *fixed* $n \in \mathbb{N}$, we note from the inequality (12.23) that

$$|g_n(z)| < \exp(n|z|) < A_n \exp(|z|^\gamma)$$

for all $z \in \Delta_1$. Consequently, each g_n satisfies the inequality (12.22). Furthermore, we observe from the definition of g_n that

$$\frac{\log|g_n(re^{i\alpha})|}{r} = \frac{\log|f(re^{i\alpha})|}{r} + n\cos\alpha \to -\infty$$

as $r \to \infty$, so $|g_n(z)|$ is bounded on L_α. Without loss of generality, we may assume that the bound is 1. Obviously, we know from the additional assumption that

$$|g_n(re^{\pm i\frac{\pi}{2}})| = |f(re^{\pm i\frac{\pi}{2}})| < 1 \tag{12.24}$$

for all $r \geq 0$. In other words, g_n is bounded by 1 on $\partial\Delta_1$. By the modified result of Problem 12.9, we conclude that each g_n is bounded by 1 in Δ_1. Similarly, each g_n is also bounded by 1 on $\partial\Delta_2$ and then in Δ_2. Since $\Pi = \Delta_1 \cup L_\alpha \cup \Delta_2$, what we have shown is that each g_n is bounded by 1 in Π, so for every $z \in \Pi$, this implies that

$$|f(z)| < \mathrm{e}^{-nr\cos\theta}$$

for all $n \in \mathbb{N}$ which means $f(z) = 0$ because $\cos\theta > 0$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, i.e., $f = 0$ as required. This completes the proof of the problem.

■

Problem 12.11

Rudin Chapter 12 Exercise 11.

Proof. Since the result is trivial if f is constant, we assume that f is nonconstant in the following discussion. Besides, without loss of generality, we may assume that $|f(z)| \leq 1$ on $\Gamma = \partial\Omega$. It suffices to prove that

$$|f(\omega)| \leq 1 \tag{12.25}$$

for every $\omega \in \Omega$. We choose $a \in \Omega$ and consider

$$\widetilde{f}(z) = \frac{f(z) - f(a)}{z - a}.$$

Using Theorem 10.16 and then Theorem 10.6, we see that $\widetilde{f} \in H(\Omega)$. Furthermore, the continuity of f on $\Omega \cup \Gamma$ certainly implies that $\widehat{f}$ is also continuous on $\Omega \cup \Gamma$. Now the boundedness of f guarantees that $\widetilde{f}(z) \to 0$ as $z \to \infty$. In other words, there is a positive constant C such that

$$|\widetilde{f}(z)| \leq C \tag{12.26}$$

in $\Omega \cup \Gamma$. Next, let $\Omega_R = \overline{D}(0; R) \cap \Omega \subseteq \Omega$ for $R > 0$ and $\widetilde{F}(z) = f^N(z)\widetilde{f}(z)$ for some $N \in \mathbb{N}$. Clearly, we have

$$\widetilde{F} \in H(\Omega).$$

By the boundedness of f and the fact $\widetilde{f}(z) \to 0$ as $z \to \infty$, we can find R large enough such that $\omega \in \Omega_R$ and $|\widetilde{F}(z)| \leq C$ for all $z \in \partial\Omega_R = C(0; R) \cap \Gamma \subseteq \Gamma$ from the bound (12.26). By Theorem 10.24 (The Maximum Modulus Theorem) and the Extreme Value Theorem, we assert that

$$|\widetilde{F}(\omega)| \leq C. \tag{12.27}$$

If $\widetilde{f}(\omega) \neq 0$, then we deduce from the inequality (12.27) that

$$|f(\omega)| \leq \frac{C^{\frac{1}{N}}}{|\widetilde{f}(\omega)|^{\frac{1}{N}}}. \tag{12.28}$$

Taking $N \to \infty$ in the inequality (12.28) which yields the required result (12.25) immediately. Consequently, the inequality

$$|f(\omega)| \leq 1$$

holds for all $\omega \in \Omega \setminus Z_{\widetilde{f}}$.

Assume that $Z_{\widetilde{f}}$ had a limit point in Ω. Now Theorem 10.18 ensures that $\widetilde{f} \equiv 0$ in Ω and then $f(z) = f(a)$ for all $z \in \Omega$, a contradiction to our hypothesis that f is nonconstant. Therefore, $Z_{\widetilde{f}}$ is discrete and hence the continuity of f forces definitely that the inequality (12.25) remains valid in Ω which completes the proof of the problem. ■

12.2 Asymptotic Values of Entire Functions

Problem 12.12

Rudin Chapter 12 Exercise 12.

Proof. Let $E_1 = \{z \in \mathbb{C} \,|\, |f(z)| > 1\}$. Since f is nonconstant, $E_1 \neq \varnothing$. Let F_1 be a component of E_1. By Definition 10.1, F_1 is an open set. By the continuity of f, it is true that $|f(z)| \geq 1$ on ∂F_1. Assume that $|f(z_0)| > 1$ for some $z_0 \in \partial F_1$. Obviously, $z_0 \in E_1$. Furthermore, we also have $|f(\omega)| > 1$ for all $\omega \in D(z_0;\delta)$ for some $\delta > 0$ by the Sign-preserving Property [79, Problem 7.15, p. 112]. Therefore, $D(z_0;\delta) \subseteq E_1$. By [42, Theorem 25.1, p. 159], we know that $D(z_0;\delta)$ intersects only F_1 which is impossible by the definition of a boundary point of a set [4, Definition 3.40, p. 64]. Hence we have $|f(z)| = 1$ on ∂F_1. Assume that F_1 was bounded. Since $\overline{F_1}$ is compact, it follows from Theorem 10.24 (The Maximum Modulus Theorem) and the Extreme Value Theorem that f attains its maximum on ∂F_1. Thus $|f(z)| \leq 1$ on F_1, contradicting to the fact that $F_1 \subseteq E_1$. Consequently, every component of E_1 is unbounded. Now for every $n = 2, 3, \ldots$, we define

$$E_n = \{z \in \mathbb{C} \,|\, |f(z)| > n\}.$$

By similar argument, it can be shown easily that every component of E_n is unbounded.

Since f is unbounded on F_1, $E_n \neq \varnothing$ for every $n = 2, 3, \ldots$. Clearly, we have $E_{n+1} \subseteq E_n$ for every $n \in \mathbb{N}$. Let F_{n+1} be a component of E_{n+1}. By similar argument of the previous paragraph, F_{n+1} is open in $\mathbb{C}$ so that it is a region. By the definition, we have $F_{n+1} \subseteq E_n$ and then it must lie entirely in a component of E_n, namely F_n. Hence, we obtain a sequence of regions

$$F_1 \supseteq F_2 \supseteq \cdots. \tag{12.29}$$

For each $k = 1, 2, \ldots$, pick $z_k \in F_k$. Now the sequence (12.29) ensures that $z_n \in F_k$ for all $n \geq k$. Using [9, Proposition 1.7, p. 14], one can find a continuous mapping $\gamma_k : [k-1, k] \to F_k$ connecting z_k and z_{k+1}. We note that the definition of E_k asserts that $|f(z)| > k$ for all $z \in F_k$, so particularly, it is also true on γ_k, i.e., $|f\big(\gamma(t)\big)| > k$ for every $t \in [k-1, k]$. Define $\gamma : [0, \infty) \to F_1$ by

$$\gamma = \gamma_1 \dotplus \gamma_2 \dotplus \cdots.$$

Then it is easy to see that $f(\gamma(t)) \to \infty$ as $t \to \infty$. Finally, if we set $\widetilde{\gamma}(t) = \gamma(\frac{t}{1-t})$, then $\widetilde{\gamma}$ is a well-defined continuous function on $[0, 1)$ and satisfies

$$\lim_{t\to 1} f(\widetilde{\gamma}(t)) = \infty,$$

completing the proof of the problem. ∎

Problem 12.13

Rudin Chapter 12 Exercise 13.

Proof. If $z = x < \infty$, then $|e^z| = e^{-x} \to 0$ as $|z| \to \infty$. By the definition, 0 is an asymptotic value of e^z. Since e^z is nonconstant entire, ∞ is also one of its asymptotic value by Problem 12.12. Hence it remains to show that if α is an asymptotic value of e^z, then α is either 0 or ∞. Let $\gamma : [0, 1) \to \mathbb{C}$ be a continuous curve such that $\gamma(t) \to \infty$ and $\exp(\gamma(t)) \to \alpha$ as $t \to 1$. Let $\gamma(t) = a(t) + ib(t)$, where a and b are continuous real-valued functions. Then we note that $a^2(t) + b^2(t) \to \infty$ as $t \to 1$.

If $a(t) \to +\infty$ as $t \to 1$, then it is clear that $|\exp(\gamma(t))| = \exp(a(t)) \to \infty$ as $t \to 1$ so that $\alpha = \infty$. Next, if $a(t) \to -\infty$ as $t \to 1$, then we $|\exp(\gamma(t)) = \exp(a(t)) \to 0$ as $t \to 1$ and so $\alpha = 0$ in this case. Finally, if $a(t) \to A$ as $t \to 1$ for some finite A, then $b(t) \to \infty$ as $t \to 1$. Since b is continuous, there exist sequences $\{t_n\}$ and $\{t'_n\}$ in $[0,1)$ such that $b(t_n) = 2n\pi$ and $b(t'_n) = (2n+1)\pi$ respectively. Thus we have

$$\mathrm{e}^{\gamma(t_n)} = \mathrm{e}^{a(t_n)} \times \mathrm{e}^{ib(t_n)} \to A \quad \text{and} \quad \mathrm{e}^{\gamma(t'_n)} = \mathrm{e}^{a(t'_n)} \times \mathrm{e}^{ib(t'_n)} \to -A$$

as $n \to \infty$. These imply that $\alpha = A = -A$ and so $\alpha = A = 0$. In conclusion, we have shown that exp has *exactly* two asymptotic values: 0 and ∞.

For the entire functions $\sin z$ and $\cos z$, we notice that

$$\sin z = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}.$$

Therefore, ∞ is the only asymptotic value of $\sin z$ and $\cos z$, completing the proof of the problem. ■

Problem 12.14

Rudin Chapter 12 Exercise 14.

Proof. Suppose that f is nonconstant. Since $f(z) \neq \alpha$ for all $z \in \mathbb{C}$, the function

$$F(z) = \frac{1}{f(z) - \alpha}$$

is nonconstant entire. Now Problem 12.12 implies that F has ∞ as an asymptotic value which means that α is an asymptotic value of f. This completes the proof of the problem. ■

12.3 Further Applications of the Maximum Modulus Principle

Problem 12.15

Rudin Chapter 12 Exercise 15.

Proof. The case is trivial if f is constant. So we may assume that f is nonconstant. Suppose first that $Z(f) = \varnothing$. Then we have $\frac{1}{f} \in H(U)$. For every $n \in \mathbb{N}$, we have $0 < 1 - \frac{1}{n} < 1$. By combining Theorem 10.24 (The Maximum Modulus Theorem) and the Extreme Value Theorem, we see that

$$\frac{1}{|f(0)|} \leq \max_{z \in C(0;1-\frac{1}{n})} \frac{1}{|f(z)|}$$

or equivalently,

$$|f(0)| \geq \min_{z \in C(0;1-\frac{1}{n})} |f(z)|.$$

We simply take $z_n = 1 - \frac{1}{n}$ which gives $|f(z_n)| \leq |f(0)|$ for all $n \in \mathbb{N}$. Obviously, $|z_n| \to 1$ as $n \to \infty$, so our assertion is true in this case.

Next, we suppose that $Z(f) \neq \varnothing$. Then there are two cases.

- **Case (i): $Z(f)$ is infinite.** Since $Z(f) \subseteq U$ which is bounded, the Bolzano-Weierstrass Theorem [79, Problem 5.25, pp. 68, 69] ensures that $Z(f)$ has a convergent subsequence $\{\zeta_k\}$. Now Theorem 10.18 forces that $\zeta_k \to \zeta \in C(0,1)$. Since $f(\zeta_k) = 0$ for all $k \in \mathbb{N}$, the assertion remains true in this case.

- **Case (ii): $Z(f)$ is finite.** Suppose that $Z(f) = \{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ for some $N \in \mathbb{N}$. Suppose further that m_k is the order of zero of f at ζ_k, where $1 \leq k \leq N$. Consider

$$g(z) = \frac{f(z)}{\prod_{k=1}^{N}(z - \zeta_k)^{m_k}}. \tag{12.30}$$

Therefore, we know that $g \in H(U)$ and $Z(g) = \varnothing$. Thus the special case implies that there is a sequence $\{z_n\} \subseteq U$ and a positive constant M such that $|z_n| \to 1$ and $|g(z_n)| \leq M$ for all $n \in \mathbb{N}$. Hence it follows from the representation (12.30) that

$$|f(z_n)| \leq M \prod_{k=1}^{N} |z_n - \zeta_k|^{m_k} \leq M \prod_{k=1}^{N}(1 + |\zeta_k|) < \infty$$

for all $n \in \mathbb{N}$.

Consequently, we have completed the proof of the problem. ■

Problem 12.16

Rudin Chapter 12 Exercise 16.

Proof. The result is always true if f is a constant function. Without loss of generality, we may assume that f is nonconstant. Let $\alpha = \sup\{|f(z)| \,|\, z \in \Omega\}$. If $|f(\zeta)| = \alpha$ *for some* $\zeta \in \Omega$, then f is constant by Theorem 10.24 (The Maximum Modulus Theorem), a contradiction. Thus we always have

$$|f(z)| < \alpha \tag{12.31}$$

for all $z \in \Omega$. By the definition, there is a sequence $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to \alpha$ as $n \to \infty$. Since Ω is bounded, the Bolzano-Weierstrass Theorem ensures that $\{z_n\}$ contains a convergent subsequence $\{z_{n_k}\}$. Let $z_{n_k} \to z_0$. If $z_0 \in \Omega$, then $\alpha = |f(z_0)|$ which is impossible. Therefore, we must have $z_0 \in \partial\Omega$ and our hypothesis gives $\alpha \leq M$. By the inequality (12.31), we conclude that

$$|f(z)| < \alpha \leq M$$

for all $z \in \Omega$. This ends the proof of the analysis.

■

Problem 12.17

Rudin Chapter 12 Exercise 17.

Proof. By the definitions, we have

$$\Phi = \{f \in H(U) \,|\, 0 < |f(z)| < 1 \text{ for all } z \in U\} \quad \text{and} \quad \Phi_c = \{f \in \Phi \,|\, f(0) = c\},$$

where $0 < c < 1$.

- **The value of M.** Without loss of generality, we may assume that $f(0) > 0$. Otherwise, we can consider the function $\widehat{f} = e^{i\theta} f$, where $\theta = -\arg f(0)$. Then $\widehat{f}(0) = |f(0)| > 0$ and $\widehat{f} \in \Phi$.

  Let $\Omega^- = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z < 0\}$. Since $0 < |f(z)| < 1$ for $z \in U$, the mapping $f_1 = \log f$ maps U into Ω^-. Next, the mapping $f_2(z) = -iz$ clearly maps Ω^- onto the upper half plane Π^+. Finally, for $\mathrm{Im}\, \alpha > 0$, we know that the mapping

  $$f_3(z) = \frac{z - \alpha}{z - \overline{\alpha}}$$

  maps Π^+ into U. Hence the mapping $F = f_3 \circ f_2 \circ f_1$ maps U into U. Since $f_1 \in H(U)$, $f_2 \in H(\Omega^-)$ and $f_3 \in H(\Pi^+)$, it is true that $F \in H(U)$. Clearly, the definition of F implies that $F \in H^\infty$, $\|F\|_\infty \leq 1$ and

  $$F(0) = \frac{-i \log f(0) - \alpha}{-i \log f(0) - \overline{\alpha}}. \tag{12.32}$$

  If we take $\alpha = -i \log f(0)$, then $\mathrm{Im}\, \alpha = -\log f(0) > 0$ because $0 < f(0) < 1$. Thus the expression (12.32) gives $F(0) = 0$ in this case. By Theorem 12.2 (Schwarz's Lemma), we have $|F(z)| \leq |z|$ for all $z \in U$ and $|F'(0)| \leq 1$.

  Now the explicit formula of F is given by

  $$F(z) = \frac{-i \log f(z) + i \log f(0)}{-i \log f(z) - i \log f(0)} = \frac{\log \frac{f(z)}{f(0)}}{\log[f(0)f(z)]}$$

  so that

  $$F'(z) = \frac{f'(z)}{f(z)} \cdot \frac{\log f(0)^2}{\{\log[f(0)f(z)]\}^2}. \tag{12.33}$$

  Since $|F'(0)| \leq 1$, we see from the formula (12.33) that

  $$|f'(0)| \leq 2|f(0) \log f(0)|. \tag{12.34}$$

  Elementary calculus shows that the function $g : (0,1) \to \mathbb{R}$ defined by $g(x) = x \log x$ attains its absolute minimum $-e^{-1}$ at $x = e^{-1}$. Therefore, we have $M \leq 2e^{-1}$.

  Next, we claim that $M = 2e^{-1}$. To see this, we consider the function $f(z) = e^{-2z-1}$ which is holomorphic in U. Since $e^{-2} < |e^{2z}| = e^{2r\cos\theta} < e^2$ for all $z = re^{i\theta} \in U$, it is easy to see that

  $$0 < e^{-3} < |f(z)| < e^{-1} < 1$$

  in U so that $f \in \Phi$. As $f'(z) = -2e^{-2z-1}$, we have $|f'(0)| = 2e^{-1} = M$ as required.

- **The value of $M(c)$.** Since $f(0) = c$, it follows from the inequality (12.34) that

  $$M(c) = \begin{cases} 2c|\log c|, & \text{if } c < e^{-1}; \\ 2e^{-1}, & \text{if } e^{-1} \leq c < 1. \end{cases}$$

Hence we complete the analysis of the problem. ■

Remark 12.1

The first assertion of Problem 12.17 is called **Rogosinski's Theorem**. See, for instances, [57] and [15, Exercise 6.36, pp. 213, 214] for a different proof.

CHAPTER 13

Approximations by Rational Functions

13.1 Meromorphic Functions on S^2 and Applications of Runge's Theorem

Problem 13.1

Rudin Chapter 13 Exercise 1.

Proof. Suppose that f is meromorphic on S^2 and $A \subseteq S^2$ is the set of poles of f. If A is infinite, then the compactness of S^2 implies that A has a limit point in S^2. However, this contradicts the note following Definition 10.41 and so A must be finite. Let $A = \{a_1, a_2, \ldots, a_N\}$ for some $N \in \mathbb{N}$ and m_k be the order of a_k for $1 \leq k \leq N$. If we define

$$P(z) = f(z) \cdot \prod_{k=1}^{N} (z - a_k)^{m_k},$$

then this expression implies that $P(z)$ is entire and has *at most* a pole at ∞. If ∞ is not a pole of f, then P is a constant by Theorem 10.23 (Liouville's Theorem). Otherwise, the function $P(\frac{1}{z})$ has a pole at 0. If $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, then we have $P(\frac{1}{z}) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$ and the nature of its singularity at 0 implies that $c_p = c_{p+1} = \cdots = 0$ for some $p \in \mathbb{N}$. In other words, P must be a polynomial. Since

$$f(z) = \frac{P(z)}{\displaystyle\prod_{k=1}^{N} (z - a_k)^{m_k}},$$

f must be rational which completes the proof of the problem. ■

Problem 13.2

Rudin Chapter 13 Exercise 2.

Proof.

(a) It is clear that Ω is simply connected, so $S^2 \setminus \Omega$ is connected by Theorem 13.11. Hence Theorem 13.9 (Runge's Theorem) implies that there exists a sequence $\{P_n\}$ of polynomials such that $P_n \to f$ uniformly on compact subsets of Ω. See Figure 13.1 for details.

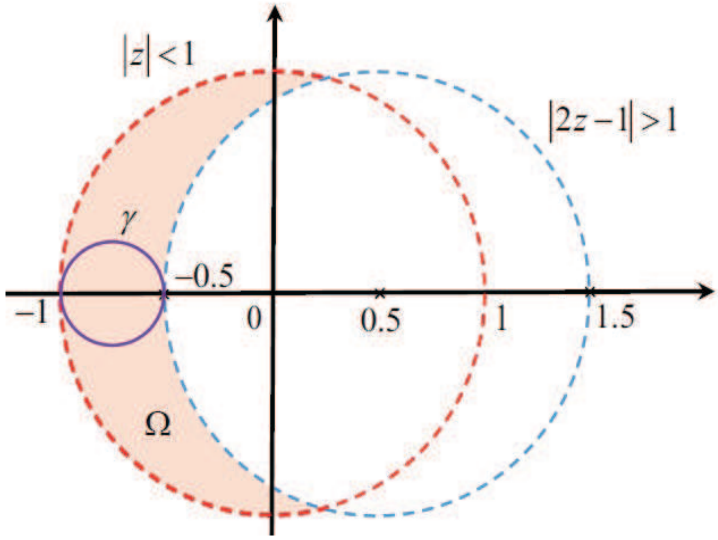


Figure 13.1: The simply connected set Ω.

(b) The answer is negative. Consider the function

$$f(z) = \frac{1}{z - \frac{1}{2}} \tag{13.1}$$

which belongs to $H(\Omega)$. Assume that there was a sequence $\{P_n\}$ of polynomials such that $P_n \to f$ uniformly in Ω. Take $\gamma = C(-\frac{3}{4}; \frac{1}{4})$. Then we have $\gamma \subseteq \Omega$, see Figure 13.1 again. On the one hand, we have

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \cdot \mathrm{Ind}_\gamma\Big(\frac{1}{2}\Big) = 2\pi i.$$

Furthermore, Theorem 10.12 gives

$$\int_\gamma P_n(z)\,\mathrm{d}z = 0$$

for every $n \in \mathbb{N}$. On the other hand, the uniform convergence shows that

$$0 = \lim_{n\to\infty} \int_\gamma P_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z = 2\pi i,$$

a contradiction. Hence no such sequence exists.

(c) The answer is still negative. The function (13.1) considered in part (b) is in fact holomorphic in $\mathbb{C} \setminus \{\frac{1}{2}\}$ which is open in $\mathbb{C}$ and it contains $\overline{\Omega}$.

We complete the proof of the problem. ∎

Problem 13.3

Rudin Chapter 13 Exercise 3.

Proof. For every $n \in \mathbb{N}$, let $D_n = \{z \in \overline{D}(0; n) \mid |\mathrm{Im}\, z| \geq \frac{1}{n}\}$ and $E_n = [\frac{1}{n}, n] \times \{0\}$, see Figure 13.2. It is clear that both D_n and E_n are compact, so the set $K_n = D_n \cup E_n \cup \{0\}$ is compact too. Furthermore, we note that $S^2 \setminus K_n$ is connected. Take $0 < \delta_n < \frac{1}{2n}$. Then the sets

$$D_n' = \Big\{z \in D(0; n + \delta_n) \,\Big|\, |\mathrm{Im}\, z| \geq \frac{1}{n} - \delta_n\Big\} \quad \text{and} \quad E_n' = \Big(\frac{1}{n} - \delta_n, n + \delta_n\Big) \times (-\delta_n, \delta_n)$$

are open sets containing D_n and E_n respectively. Obviously, the set $D_n' \cup E_n'$ is open in $\mathbb{C}$ and is disjoint from $U_n = (-\delta_n, \delta_n) \times (-\delta_n, \delta_n)$. Define the function $f_n : \Omega_n = D_n' \cup E_n' \cup U_n \to \mathbb{C}$ by

$$f_n(z) = \begin{cases} 1, & \text{if } z \in U_n; \\ \\ 0, & \text{if } z \in D_n' \cup E_n'. \end{cases}$$

Then we have $f_n \in H(\Omega_n)$ and Ω_n is an open set containing K_n. According to Theorem 13.7, one can find a polynomial Q_n such that $|Q_n(z) - f_n(z)| < \frac{1}{n}$ for all $z \in K_n$. In fact, we get

$$|Q_n(z) - f_n(z)| = \begin{cases} |Q_n(z) - 1|, & \text{if } z \in U_n; \\ \\ |Q_n(z)|, & \text{if } z \in D_n' \cup E_n'. \end{cases} \tag{13.2}$$

If we define $P_n(z) = Q_n(z) - Q_n(0) + 1$, then the definition (13.2) implies immediately that $P_n(0) = Q_n(0) - Q_n(0) + 1 = 1$ for $n = 1, 2, \ldots$. Since we have

$$\mathbb{C} \setminus \{0\} = \bigcup_{n=1}^{\infty} (D_n' \cup E_n'),$$

if $z \neq 0$, then there exists an $N \in \mathbb{N}$ such that $z \in D_n' \cup E_n'$ for all $n \geq N$ and thus the definition (13.2) implies that

$$|P_n(z)| = |Q_n(z) - Q_n(0) + 1| \leq |Q_n(z)| + |Q_n(0) - 1| < \frac{2}{n}$$

for all $n \geq N$. Consequently, we conclude from this that $P_n(z) \to 0$ as $n \to \infty$, completing the proof of the problem.

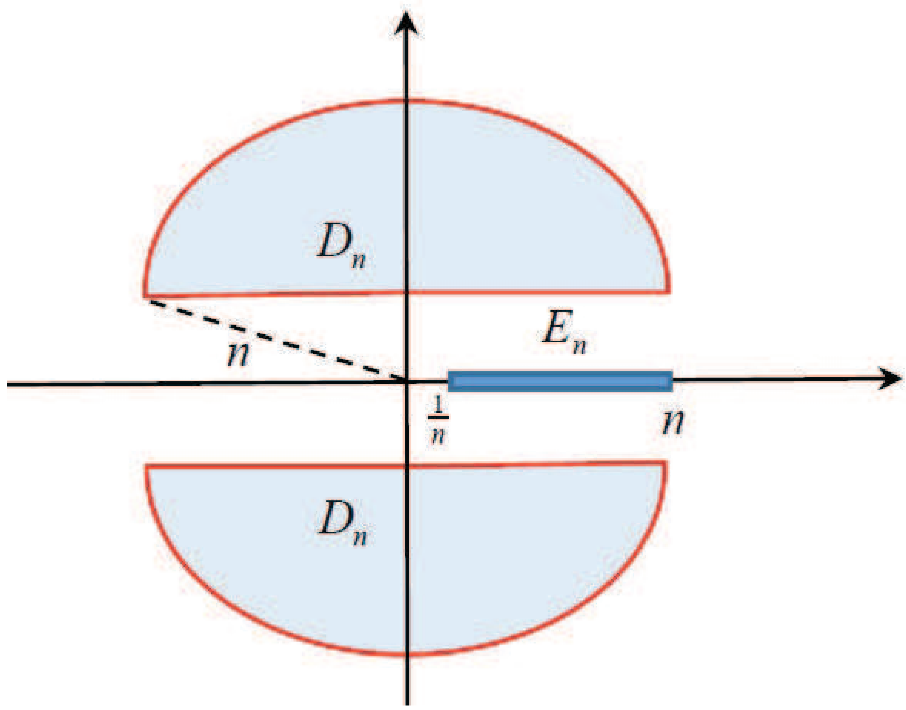


Figure 13.2: The compact sets D_n and E_n.

■

Problem 13.4

Rudin Chapter 13 Exercise 4.

Proof. For every $n \in \mathbb{N}$, we consider the three sets $A_n = [-n, n] \times [\frac{1}{n}, n]$, $B_n = [-n, n] \times \{0\}$ and $C_n = [-n, n] \times [-\frac{1}{n}, -n]$, see Figure 13.3 for an illustration. Let

$$K_n = A_n \cup B_n \cup C_n.$$

Obviously, each K_n is compact and the set $S^2 \setminus K_n$ is connected. Choose $0 < \delta_n < \frac{1}{2n}$. Then the three sets $A'_n = (-n - \delta_n, n + \delta_n) \times (\frac{1}{n} - \delta_n, n + \delta_n)$, $B'_n = (-n - \delta_n, n + \delta_n) \times (-\delta_n, \delta_n)$ and $C'_n = (-n - \delta_n, n + \delta_n) \times (-\frac{1}{n} - \delta_n, n + \delta_n)$ are open sets containing A_n, B_n and C_n respectively. Besides, they are disjoint and $K_n \subseteq \Omega_n = A'_n \cup B'_n \cup C'_n$. Define

$$f_n(z) = \begin{cases} 1, & \text{if } z \in A'_n; \\ 0, & \text{if } z \in B'_n; \\ -1, & \text{if } z \in C'_n. \end{cases}$$

Since $f_n \in H(\Omega_n)$, it follows from Theorem 13.7 that there exists a polynomial P_n such that $|P_n(z) - f_n(z)| < \frac{1}{n}$ for all $z \in K_n$. In fact, we have

$$|P_n(z) - f_n(z)| = \begin{cases} |P_n(z) - 1|, & \text{if } z \in A_n; \\ |P_n(z)|, & \text{if } z \in B_n; \\ |P_n(z) + 1|, & \text{if } z \in C_n. \end{cases}$$

Therefore, such sequence $\{P_n\}$ of polynomials satisfy the requirements of the problem. This completes the proof of the problem.

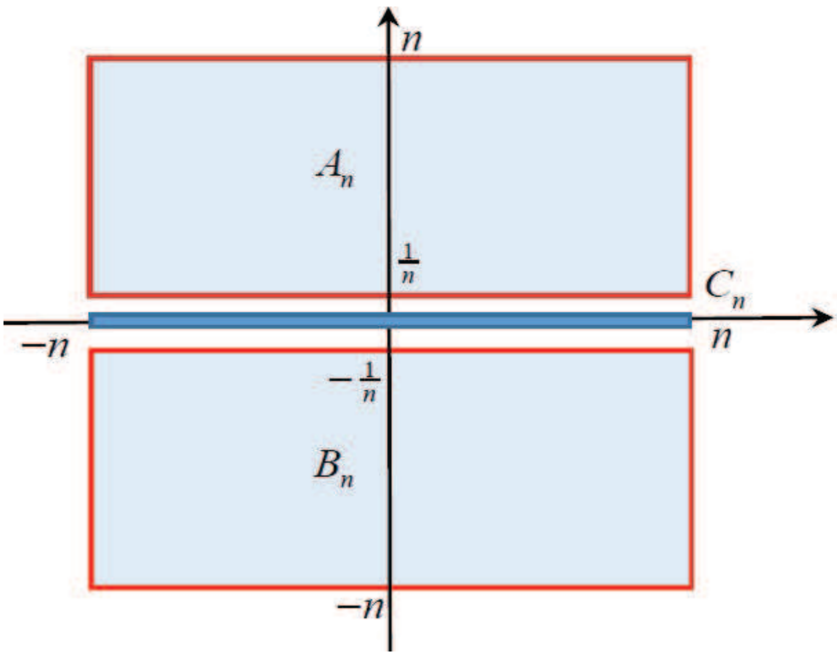


Figure 13.3: The compact sets A_n, B_n and C_n.

■

13.2 Holomorphic Functions in the Unit Disc without Radial Limits

Problem 13.5

Rudin Chapter 13 Exercise 5.

Proof. For each $n \in \mathbb{N}$, suppose that $\Delta_n = \overline{D(0; 1 - \frac{1}{2n})}$, $C_n = \{(1 - \frac{1}{2n+1})e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq 2\pi\}$, $D_n = [1 - \frac{1}{2n+1}, 1 - \frac{1}{2n+2}]$ and $E_n = \{(1 - \frac{1}{2n+2})e^{i\theta} \mid 0 \leq \theta \leq \frac{\pi}{2}\}$. Then the union

$$L_n = C_n \cup D_n \cup E_n$$

is an arc in $U \setminus \Delta_n$ with the property that each L_n intersects every radius of U. Finally, we suppose that Ω_n is a tubular region of L_n with width $\frac{1}{2(2n+2)^4}$. The construction ensures that $\Omega_n \cap \Delta_n = \varnothing$, see Figure 13.4 for an illustration. Similar to Problem 13.4, we can select very small $\delta_n > 0$ such that the modified tubes Ω_n' of L_n with widths $\frac{1}{2(2n+2)^4} + \delta_n$ and $\Pi_n = \overline{D(0; 1 - \frac{1}{2n} + \delta_n)}$ also satisfy $\Omega_n' \cap \Pi_n = \varnothing$. Notice that $\Omega_n' \subseteq \Omega_n$.

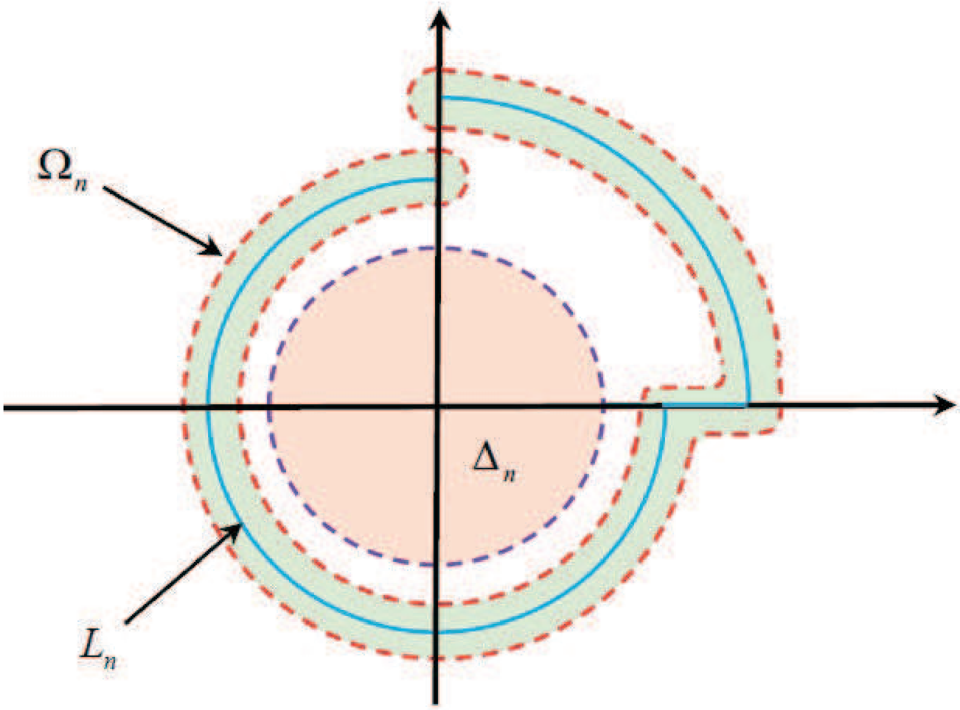


Figure 13.4: The disc Δ_n, the arc L_n and its neighborhood Ω_n.

We apply induction to construct the sequence of polynomials $\{Q_n\}$ and a holomorphic function f such that $Q_n \to f$ uniformly on U: Consider $Q_0 \equiv 0$ and

$$f_1(z) = \begin{cases} Q_0(z), & \text{if } z \in \Pi_1^\circ; \\ 1, & \text{if } z \in \Omega_1'. \end{cases}$$

Obviously, we have $f_1 \in H(\Pi_1^\circ \cup \Omega_1')$. Since $\mathbb{C} \setminus (\Pi_1^\circ \cup \Omega_1')$ is connected, it follows from Theorem 13.9 (Runge's Theorem) that one can find a polynomial Q_1 such that

$$|Q_1(z) - f_1(z)| < \frac{1}{2} \tag{13.3}$$

on compact subsets of $\Pi_1^\circ \cup \Omega_1'$. Particularly, the estimate (13.3) holds for all $z \in \Delta_1 \cup \Omega_1$ because $\Delta_1 \subseteq \Pi_1^\circ$ and $\Omega_1 \subseteq \Omega_1'$.

Assume that the polynomial Q_n is chosen with the property that

$$|Q_n(z) - f_n(z)| < \frac{1}{2^n} \tag{13.4}$$

on $\Delta_n \cup \Omega_n$. We consider the function defined by

$$f_{n+1}(z) = \begin{cases} Q_n(z), & \text{if } z \in \Pi_{n+1}^\circ; \\ \dfrac{1+(-1)^n}{2}, & \text{if } z \in \Omega_{n+1}'. \end{cases} \tag{13.5}$$

Then we have $f_{n+1} \in H(\Pi_{n+1}^\circ \cap \Omega_{n+1}')$. Since $\mathbb{C} \setminus (\Pi_{n+1}^\circ \cap \Omega_{n+1}')$ is connected, Theorem 13.9 (Runge's Theorem) guarantees that one can find a polynomial Q_{n+1} such that

$$|Q_{n+1}(z) - f_{n+1}(z)| < \frac{1}{2^{n+1}} \tag{13.6}$$

on compact subsets of $\Pi_{n+1}^\circ \cap \Omega_{n+1}'$ and thus on $\Delta_{n+1} \cup \Omega_{n+1}$ because $\Delta_{n+1} \subseteq \Pi_{n+1}^\circ$ and $\Omega_{n+1} \subseteq \Omega_{n+1}'$. Consequently, we have constructed the polynomial Q_{n+1} and then a sequence of polynomials $\{Q_n\}$ satisfying the property (13.4).

Next, let N be a positive integer. Then we have $\Delta_N \subseteq \Delta_n$ for all $n \geq N$. Therefore, it deduces from the definition (13.5) and the inequality (13.6) that

$$|Q_{n+1}(z) - Q_n(z)| < \frac{1}{2^{n+1}} \tag{13.7}$$

holds for all $z \in \Delta_N \subseteq \Delta_n \subseteq \Delta_{n+1}$ and all $n \geq N$. Given $\epsilon > 0$. We pick the N large enough so that $\frac{1}{2^N} < \epsilon$. For all positive integers p and $n \geq N$, we observe from the inequality (13.7) that

$$\begin{aligned} |Q_{n+p}(z) - Q_n(z)| &\leq |Q_{n+p}(z) - Q_{n+p-1}(z)| + \cdots + |Q_{n+1}(z) - Q_n(z)| \\ &< \frac{1}{2^{n+p}} + \frac{1}{2^{n+p-1}} + \cdots + \frac{1}{2^{n+1}} \\ &< \frac{1}{2^n} \\ &< \epsilon \end{aligned}$$

for all $z \in \Delta_N$. Hence it follows from the Cauchy Criterion for Uniform Convergence (see [61, Theorem 7.8, p. 147]) that the sequence of polynomials $\{Q_n\}$ converges uniformly to a function f on Δ_N. By Theorem 10.28, we have $f \in H(\Delta_N^\circ)$. Since it is true for every large N and

$$U = \bigcup_{N=1}^{\infty} \Delta_N,$$

we obtain $f \in H(U)$ and $Q_n \to f$ uniformly on compact subsets of U. Finally, we define $P_n = Q_n - Q_{n-1}$ for every $n = 1, 2, \ldots$. Each P_n is also a polynomial and

$$\sum_{k=1}^{n} P_k = Q_n.$$

Therefore, we have constructed a sequence of polynomials $\{P_n\}$ such that the series $f = \displaystyle\sum_{n=1}^{\infty} P_n$ is holomorphic in U.

The above construction of $\{Q_n\}$ starts with the inequality (13.3), where $\frac{1}{2}$ can be actually replaced by *any* small $\delta > 0$ so that

$$|P_1(z)| = |Q_1(z) - Q_0(z)| = |Q_1(z)| < \delta$$

on Δ_1. By the inequality (13.7) again, we obtain

$$|P_n(z)| = |Q_n(z) - Q_{n-1}(z)| < \delta^n$$

on Δ_n for every $n = 2, 3, \ldots$. This means that the polynomials P_n are very small on Δ_n. Furthermore, we can also replace the values 0 and 1 by another $Q_n + g \in H(\Omega'_{n+1})$ in the definition (13.5), and hence $Q_n + g \in H(\Omega_{n+1})$. As a result, we obtain from the inequality (13.6) that

$$|P_{n+1}(z) - g(z)| < \frac{1}{2^{n+1}}$$

in Ω_{n+1}. In other words, it means that the sequence $\{P_n\}$ can be chosen more or less arbitrary on L_n, i.e., *any* holomorphic g can be approximated by the sequence of polynomials $\{P_n\}$ in a neighborhood of L_n.

Now it remains to show that f has *no* radial limit at any point of T. To this end, by the definitions of Δ_n and Ω_n, we see easily that $\Omega_{2n} \subseteq \Delta_p$ for all $p \geq 2n + 2$. On the one hand, suppose that $z \in \Omega_{2n} \subseteq \Omega'_{2n}$. Then we recall from the definition (13.5) that $|Q_{2n}(z)| < \frac{1}{2^{2n}}$. Combining this fact, $Q_n \to f$ in U and the inequality (13.7), we know that

$$\begin{aligned}
|f(z)| &\leq |f(z) - Q_{2n}(z)| + |Q_{2n}(z)| \\
&\leq |Q_{2n+1}(z) - Q_{2n}(z)| + |f(z) - Q_{2n+1}(z)| + |Q_{2n}(z)| \\
&\leq |Q_{2n+1}(z) - Q_{2n}(z)| + |Q_{2n+2}(z) - Q_{2n+1}(z)| + |f(z) - Q_{2n+2}(z)| + |Q_{2n}(z)| \\
&\leq \sum_{k=2n}^{\infty} |Q_{k+1}(z) - Q_k(z)| + |Q_{2n}(z)| \\
&< \sum_{k=2n}^{\infty} \frac{1}{2^{k+1}} + \frac{1}{2^{2n}} \\
&\leq \frac{1}{2^{2n-1}}.
\end{aligned} \tag{13.8}$$

On the other hand, if $z \in \Omega_{2n+1} \subseteq \Omega'_{2n+1}$, then we have $|Q_{2n+1}(z) - 1| < \frac{1}{2^{2n+1}}$ so that

$$\begin{aligned}
|f(z) - 1| &\leq \sum_{k=2n+1}^{\infty} |Q_{k+1}(z) - Q_k(z)| + |Q_{2n+1}(z) - 1| \\
&< \sum_{k=2n+1}^{\infty} \frac{1}{2^{k+1}} + \frac{1}{2^{2n+1}} \\
&\leq \frac{1}{2^{2n}}.
\end{aligned} \tag{13.9}$$

Assume that $\lim_{r \to 1} f(re^{i\theta_0})$ existed for some $\theta_0 \in [0, 2\pi)$. Since every L_n intersects the radius $\ell_0 = \{re^{i\theta_0} \,|\, 0 \leq r < 1\}$, every Ω_n also intersects ℓ_0. In other words, we can choose a sequence $\{z_n = r_n e^{i\theta_0}\}$ tending to $e^{i\theta_0} \in T$ with the property that $z_{2n+1} \in \Omega_{2n+1}$ and $z_{2n} \in \Omega_{2n}$. Then the above estimates (13.8) and (13.9) show that

$$\lim_{n \to \infty} f(z_{2n}) = 0 \quad \text{and} \quad \lim_{n \to \infty} f(z_{2n+1}) = 1$$

which are contrary. Hence this contradiction means that our f has no radial limit on T, completing the analysis of the problem. ∎

Problem 13.6

Rudin Chapter 13 Exercise 6.

Proof. Let's prove the assertions one by one.

- **The series converges if $|z| < 1$.** It is easily seen that

$$n_k \geq \begin{cases} 2^{2^{k-1}} & \text{if } 1 \leq k \leq 3; \\ (k-1)! \cdot 2^k & \text{if } k \geq 4. \end{cases}$$

For large k, since

$$\frac{k}{n_k} \leq \frac{k}{(k-1)! \cdot 2^k},$$

we have $\frac{k}{n_k} \to 0$ as $k \to \infty$ and then

$$\limsup_{k\to\infty} 5^{\frac{k}{n_k}} = 1.$$

Hence it asserts from §10.5 that the radius of convergence of the series of h is 1, as required.

- **There is a constant $c > 0$ such that $|h(z_m)| > c \cdot 5^m$ for all z with $|z_m| = 1 - \frac{1}{n_m}$ and $m \geq 3$.** We remark that the hint is not true for the cases $m = 1, 2$. For example, take $n_1 = 4$ and $n_2 = 9$, so we have

$$h\Big(\frac{3}{4}e^{i\theta}\Big) = 5(0.75)^4 e^{4i\theta} + 25(0.75)^9 e^{9i\theta} + \sum_{k=3}^{\infty} 5^k (0.75)^{n_k} e^{i\theta n_k}.$$

Since $0.75^4 < 0.32$ and $5(0.75)^9 > 0.375$, the first term is *not* the dominant term in the series defining $h(z)$ which means the hint is not correct anymore. Examples for the case $m = 2$ can be found similarly. However, we can show affirmative result if $m \geq 3$. Suppose that $z_m = (1 - \frac{1}{n_m})e^{i\theta}$.

$$\begin{aligned}
\frac{h\big((1-\frac{1}{n_m})e^{i\theta}\big)}{5^m} &= \sum_{k=1}^{m-1} \frac{5^k}{5^m}\Big(1 - \frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k} + \Big(1 - \frac{1}{n_m}\Big)^{n_m} e^{i\theta n_m} \\
&\quad + \sum_{k=m+1}^{\infty} \frac{5^k}{5^m}\Big(1 - \frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k} \\
&= \Big[\sum_{k=1}^{m-1} \frac{5^k}{5^m}\Big(1 - \frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k} - S_{m-1}\Big] + (S_{m-1} + a_m) \\
&\quad + \sum_{k=m+1}^{\infty} \frac{5^k}{5^m}\Big(1 - \frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k},
\end{aligned}$$

where

$$S_{m-1} = \sum_{k=1}^{m-1} \frac{5^k}{5^m} e^{i\theta n_k} \quad \text{and} \quad a_m = \Big(1 - \frac{1}{n_m}\Big)^{n_m} e^{i\theta n_m}$$

for every $m = 3, 4, \ldots$. By elementary calculus, we can show easily the following result:

Lemma 13.1

If $\alpha \geq 1$, then the function

$$f_\alpha(x) = \left(1 - \frac{1}{x}\right)^{\alpha x}$$

is strictly increasing and $0 < f_\alpha(x) \leq \mathrm{e}^{-\alpha}$ on $[1, \infty)$. In addition, $f_\alpha(x) \to \mathrm{e}^{-\alpha}$ as $x \to \infty$.

Particularly, Lemma 13.1 says that $\{|a_m|\}$ is a strictly increasing sequence of positive numbers such that

$$\frac{1}{4} \leq |a_m| \leq \frac{1}{\mathrm{e}} \tag{13.10}$$

for all $m = 1, 2, \ldots$ and $|a_m| \to \mathrm{e}^{-1}$ as $m \to \infty$. In fact, if $m \geq 3$, then $|a_m| > 0.35$. Simple algebra gives

$$|S_{m-1}| \leq \frac{1}{4} \times \left(1 - \frac{1}{5^{m-1}}\right)$$

for all $m = 3, 4, \ldots$. Thus it is true that

$$|S_{m-1} + a_m| \geq |a_m| - |S_m| > 0.11 \tag{13.11}$$

for all $m = 3, 4, \ldots$.

Next, we know that

$$\left|\sum_{k=1}^{m-1} \frac{5^k}{5^m}\left(1 - \frac{1}{n_m}\right)^{n_k} \mathrm{e}^{i\theta n_k} - S_{m-1}\right| \leq \sum_{k=1}^{m-1} \frac{5^k}{5^m}\left[1 - \left(1 - \frac{1}{n_m}\right)^{n_k}\right]. \tag{13.12}$$

Using differentiation, we always have $1 - (1-x)^n \leq nx$ for every $x \in [0, 1]$ and $n \in \mathbb{N}$. By the definition of n_m, we see that

$$n_m > 2^{j+1}(m-1)(m-2)\cdots(m-j-1)n_{m-j-1}$$

for $j = 0, 1, \ldots, m-2$. Applying these to the inequality (13.12) to obtain

$$\begin{aligned}
\left|\sum_{k=1}^{m-1} \frac{5^k}{5^m}\left(1 - \frac{1}{n_m}\right)^{n_k} \mathrm{e}^{i\theta n_k} - S_{m-1}\right| &\leq \frac{1}{5^m}\sum_{k=1}^{m-1} 5^k \cdot \frac{n_k}{n_m} \\
&\leq \frac{1}{5^m}\sum_{k=1}^{m-1} \frac{5^k}{2^{m-k}(m-1)(m-2)\cdots k} \\
&\leq \frac{1}{m-1}\sum_{k=1}^{m-1} \frac{1}{2^{m-k}} \cdot 5^{k-m} \\
&\leq \frac{1}{8(m-1)} \cdot \left(1 - \frac{1}{5^{m-1}}\right) \\
&< 0.06 \qquad (13.13)
\end{aligned}$$

for every $m = 3, 4, \ldots$.

Clearly, we deduce from the upper bound (13.10) that

$$\left|\sum_{k=m+1}^{\infty} \frac{5^k}{5^m}\left(1 - \frac{1}{n_m}\right)^{n_k} \mathrm{e}^{i\theta n_k}\right| = \left|\sum_{k=m+1}^{\infty} \frac{5^k}{5^m} \cdot a_k^{\frac{n_k}{n_m}}\right| \leq \frac{5}{\mathrm{e}^{\frac{n_{m+1}}{n_m}}} + \frac{1}{5^m}\sum_{k=m+2}^{\infty} \frac{5^k}{\mathrm{e}^{\frac{n_k}{n_m}}}. \tag{13.14}$$

As $m \geq 3$, we get $\frac{n_{m+1}}{n_m} \geq 6$ which implies that

$$\frac{5}{e^{\frac{n_{m+1}}{n_m}}} \leq \frac{5}{e^6} < 0.0124. \tag{13.15}$$

Since $n_k > 2(k-1)n_m$ for $k = m+2, m+3, \ldots$, we have

$$e^{\frac{n_k}{n_m}} > e^{2(k-1)}$$

and then the inequality (13.14) becomes

$$\Big|\sum_{k=m+2}^{\infty} \frac{5^k}{5^m}\Big(1-\frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k}\Big| \leq \frac{e^2}{5^m}\sum_{k=m+2}^{\infty}\Big(\frac{5}{e^2}\Big)^k = \frac{25}{(e^2-5)e^{2m}} < 0.026 \tag{13.16}$$

for every $m \geq 3$. Finally, by combining the bounds (13.11), (13.13), (13.15) and (13.16), we conclude immediately that

$$\begin{aligned}\Big|\frac{h\big((1-\frac{1}{n_m})e^{i\theta}\big)}{5^m}\Big| &\geq |S_{m-1}+a_m| - \Big|\sum_{k=1}^{m-1}\frac{5^k}{5^m}\Big(1-\frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k} - S_{m-1}\Big| \\ &\quad - \Big|\sum_{k=m+1}^{\infty}\frac{5^k}{5^m}\Big(1-\frac{1}{n_m}\Big)^{n_k} e^{i\theta n_k}\Big| \\ &\geq 0.11 - 0.06 - 0.0124 - 0.026 \\ &> 0.\end{aligned}$$

Consequently, there exists a constant $C > 0$ such that

$$|h(z)| > C \cdot 5^m$$

holds for every z with $|z| = 1 - \frac{1}{n_m}$ and $m \geq 3$.

- **Proof that h has no finite radial limits.** If $z_m = (1-\frac{1}{n_m})e^{i\theta}$, then the previous assertion shows that

$$\Big|h\Big(\Big(1-\frac{1}{n_m}\Big)e^{i\theta}\Big)\Big| > C \cdot 5^m$$

for every $m \geq 3$ and $\theta \in [0, 2\pi]$. Therefore, it guarantees that

$$\lim_{m\to\infty}\Big|h\Big(\Big(1-\frac{1}{n_m}\Big)e^{i\theta}\Big)\Big| = \infty.$$

In other words, h has no finite radial limits.

- **The h has infinitely many zeros in U.** Assume that h had only finitely many zeros $\alpha_1, \alpha_2, \ldots, \alpha_p$ in U. Then the function

$$\varphi(z) = z^m \prod_{k=1}^{p} \varphi_{\alpha_k}(z) = z^m \prod_{k=1}^{p} \frac{z-\alpha_k}{1-\overline{\alpha_k}z} \tag{13.17}$$

has exactly the same zeros as h counted with multiplicity, where m is the order of zero of h at the origin. If h has no zero in U, then we let $\varphi = 1$. Now the function $f = \frac{h}{\varphi}$ satisfies $f \in H(U)$. By the proof of Theorem 12.4, we know that $|\varphi_{\alpha_k}(z)| < 1$ if $|z| < 1$, so the definition (13.17) implies that

$$|f(z)| > C \cdot 5^m \tag{13.18}$$

for every z with $|z| = 1 - \frac{1}{n_m}$ and $m \geq 3$. Furthermore, f has no zero in U so that $\frac{1}{f} \in H(U)$. Combining this fact, the inequality (13.18) and Theorem 10.24 (The Maximum Modulus Theorem), we see immediately that

$$\left|\frac{1}{f(z)}\right| < \frac{1}{C \cdot 5^m}$$

for all $z \in \overline{D(0; 1 - \frac{1}{n_m})}$ and $m \geq 3$. Since $n_m \to \infty$ as $m \to \infty$, we conclude from this that $\frac{1}{f(z)} = 0$ in U which is impossible.

- **The function h assumes every complex number α infinitely many times in U.** Define $\widehat{h}(z) = f(z) - \alpha$. Then $\widehat{h} \in H(U)$. For large enough m, $5^m > \frac{2}{C}|\alpha|$ so that

$$|\widehat{h}(z)| = |h(z) - \alpha| \geq |h(z)| - |\alpha| > C \cdot 5^m - |\alpha| > \frac{C}{2} \cdot 5^m$$

for $|z| = 1 - \frac{1}{n_m}$. Therefore, we can apply similar argument as above to obtain the desired result.

This completes the analysis of the problem. ∎

Remark 13.1

(a) A sequence $\{n_k\}$ of positive integers is said to be **lacunary** if there is a constant $c > 1$ such that $n_{k+1} > cn_k$ for all $k \in \mathbb{N}$. A power series

$$\sum_{k=1}^{\infty} a_k z^{n_k} \tag{13.19}$$

is called a **lacunary power series** or a power series with **Hadamard gaps**. Thus our h is an example of this type of power series with $c = 2$. See, for instance, [84, Chap. V].

(b) In [29, Problem 5.36 & Update 5.36, p. 113], it is pointed out that the best known result concerning the number of zeros of a lacunary power series inside U is due to Chang [16], who proved that if

$$\sum_{k=0}^{\infty} |a_k|^{2+\epsilon} = \infty$$

for some $\epsilon > 0$, then the series (13.19) has infinitely many zeros in any sector. See also [26], [43] and [75]

13.3 Simply Connectedness and Miscellaneous Problems

Problem 13.7

Rudin Chapter 13 Exercise 7.

Proof. Suppose that $\overline{A}$ intersects each component of $S^2 \setminus \Omega$. Choose a sequence of compact sets K_n in Ω with the properties specified in Theorem 13.3. Fix a positive integer n. Let V be a component of $S^2 \setminus K_n$. By the proof of Theorem 13.9 (Runge's Theorem), it suffices to prove that $V \cap A \neq \varnothing$.

Since every component of $S^2 \setminus K_n$ contains a component of $S^2 \setminus \Omega$, we have

$$U \subseteq V \tag{13.20}$$

for *at least* one component U of $S^2 \setminus \Omega$. Since $\overline{A}$ intersects *every* component of $S^2 \setminus \Omega$, we have

$$\overline{A} \cap U \neq \varnothing. \tag{13.21}$$

Now the set relations (13.20) and (13.21) together imply that $V \cap \overline{A} \neq \varnothing$. If $V \cap A \neq \varnothing$, then we are done. Otherwise, $p \in V \cap A'$, where A' is the set of limit points of A. By the openness of V and the definition of limit points, there exists a $\delta > 0$ such that $q \in A \cap D'(p;\delta)$ and $D(p;\delta) \subseteq V$. This means that $V \cap A \neq \varnothing$. Hence we have obtained the requirement and this completes the analysis of the problem. ∎

Problem 13.8

Rudin Chapter 13 Exercise 8.

Proof. Let $\Omega = \mathbb{C}$. For every $n = 1, 2, \ldots$, we denote $K_n = \overline{D(0;n)}$ which is compact. Put $A_1 = A \cap K_1$ and $A_n = A \cap (K_n \setminus K_{n-1})$ for $n = 2, 3, \ldots$. Since $A_n \subseteq K_n$ and A has no limit point in $\mathbb{C}$ (hence none in K_n), every A_n is a finite set. Put

$$Q_n(z) = \sum_{\alpha \in A_n} P_\alpha(z),$$

where $n = 1, 2, \ldots$. Since A_n is finite, each Q_n is a rational function and the poles of Q_n lie in A_n for $n \geq 2$. In particular, Q_n is holomorphic in an open set V containing K_{n-1}. By the known fact given in Definition 10.5, the power series of Q_n at 0 converges uniformly to Q_n in K_{n-1}. This means that for each $n = 2, 3, \ldots$, there exists a polynomial R_n such that

$$|R_n(z) - Q_n(z)| < \frac{1}{2^n} \tag{13.22}$$

for all $z \in K_{n-1}$.

By imitating the remaining part of the proof of Theorem 13.10 (The Mittag-Leffler Theorem), it can be seen that the function

$$f(z) = Q_1(z) + \sum_{n=2}^{\infty} [Q_n(z) - R_n(z)],$$

where $z \in \mathbb{C}$, has the desired properties. In fact, we fix a positive integer N first. On K_N, we have

$$f = Q_1 + \sum_{n=2}^{N} (Q_n - R_n) + \sum_{n=N+1}^{\infty} (Q_n - R_n). \tag{13.23}$$

Using the inequality (13.22), each term in the last sum in the expression (13.23) is less than 2^{-n} on K_N, hence this last series converges uniformly on K_N to a function g_{N+1} which is holomorphic in K_N°. Since each R_n is a polynomial, the function

$$f - (Q_1 + Q_2 + \cdots + Q_N) = g_{N+1} - \sum_{n=2}^{N} R_n$$

is holomorphic in K_N° and therefore, f has precisely the prescribed principal parts in K_N°. Since N is arbitrary, it is actually true in $\mathbb{C}$. This completes the analysis of the problem. ∎

Problem 13.9

Rudin Chapter 13 Exercise 9.

Proof. As $f(z) \neq 0$ for all $z \in \Omega$, we have $\frac{1}{f} \in H(\Omega)$. Since Ω is simply connected, Theorem 13.11 says that there exists an $h \in H(\Omega)$ such that

$$f = \mathrm{e}^h.$$

Take $g = \exp(\frac{h}{n})$ which is obviously holomorphic in Ω. Hence we have

$$g^n = \mathrm{e}^h = f$$

which ends the proof of the problem. ■

Problem 13.10

Rudin Chapter 13 Exercise 10.

Proof. We want to show that there exists a $g \in H(\Omega)$ such that $f = \exp g$ if and only if there exists a $\varphi_n \in H(\Omega)$ such that $f = \varphi_n^n$ for every positive integer n.

Suppose that there exists a $g \in H(\Omega)$ such that $f = \exp g$. Then the function $\varphi_n = \exp(\frac{g}{n})$ satisfies $\varphi_n \in H(\Omega)$ and $\varphi_n^n = \exp g = f$ for every positive integer n. Conversely, suppose that there exists a $\varphi_n \in H(\Omega)$ such that $f = \varphi_n^n$ for every $n = 1, 2, \ldots$. We claim that $f(z) \neq 0$ for all $z \in \Omega$. Assume that there was an $a \in \Omega$ such that $f(a) = 0$. Since $f \not\equiv 0$, Theorem 10.18 asserts that there is a unique positive integer m such that

$$f(z) = (z-a)^m g(z) \tag{13.24}$$

for some $g \in H(\Omega)$ and $g(a) \neq 0$. Particularly, we take $n = m+1$. Since $\varphi_{m+1}(a) = 0$, Theorem 10.18 ensures that

$$\varphi_{m+1}(z) = (z-a)\phi_{m+1}(z) \tag{13.25}$$

for some $\phi \in H(\Omega)$. If we combine the representations (13.24) and (13.25), then we conclude that

$$g(z) = (z-a)\phi_{m+1}^{m+1}(z)$$

for all $z \in \Omega$, but this implies that $g(a) = 0$, a contradiction which proves our claim. In other words, we have $\frac{1}{f} \in H(\Omega)$. Next, we may take $n = 2$ so that $f = \varphi_2^2$ for some $\varphi_2 \in H(\Omega)$. Hence it follows from Theorem 13.11 that f has a holomorphic logarithm g in Ω and thus we complete the analysis of the problem. ■

Problem 13.11

Rudin Chapter 13 Exercise 11.

Proof. Put $\varphi(z) = \sup\limits_{n\in\mathbb{N}} |f_n(z)|$ in Ω. Now φ is well-defined because $f_n \to f$ pointwise in Ω. Suppose that U is a nonempty open subset of Ω with $\overline{U} \subseteq \Omega$. Such a set exists because Ω is an open set. For $k = 1, 2, \ldots$, we define

$$V_k = \{z \in \overline{U} \mid |f_n(z)| \leq k \text{ for all } n \in \mathbb{N}\} \subseteq \overline{U}. \tag{13.26}$$

By the hypothesis, we know that $f_n(z) \to f(z)$ for every $z \in \overline{U}$, the set $\{f_n(z) \,|\, n \in \mathbb{N}\}$ must be bounded for each *fixed* z. Thus each $z \in \overline{U}$ lies in *some* V_k, i.e.,

$$\overline{U} = \bigcup_{k=1}^{\infty} V_k.$$

Since $\overline{U}$ is a compact subset of the metric space $\mathbb{C}$, it is a complete metric space. By Theorem 5.6 (Baire's Theorem) (see also §5.7), it is impossible that all V_k are nowhere dense sets. Thus there exists an $N \in \mathbb{N}$ such that the closure $\overline{V_N}$ contains a nonempty disc D_U of $\overline{U}$, so it yields from the definition (13.26) that

$$|f_n(z)| \leq N$$

for all $n \in \mathbb{N}$ and $z \in D_U$. Equivalently, it means that $\varphi(z)$ is bounded on D_U and we may apply Problem 10.5 to conclude that $f_n \to f$ uniformly on every compact subset of D_U. According to Theorem 10.28, we have $f \in H(D_U)$.

Now we let

$$V = \bigcup_{U} D_U,$$

where U is *any* arbitrary nonempty open subset of Ω with $\overline{U} \subseteq \Omega$. Obviously, we observe that $f \in H(V)$. If $D(z_0;\delta) \subset \Omega$ and $D(z_0;\delta) \cap V = \varnothing$ for some $z_0 \in \Omega$ and $\delta > 0$,[a] then the above argument shows that there exists a nonempty open subset D of $D(z_0;\delta)$ such that $f \in H(D)$. Therefore, $D \subseteq V$ which implies the contradiction $D(z_0;\delta) \cap V \neq \varnothing$. As a result, it means that V is a dense open subset of Ω which ends the proof of the problem. ■

Remark 13.2

The result of Problem 13.11 is sometimes called **Osgood's Theorem** [48]. In fact, Problem 10.5 is the well-known **Vitali Convergence Theorem**, see [72, p. 168].

Problem 13.12

Rudin Chapter 13 Exercise 12.

Proof. We regard $\mathbb{C}$ as $\mathbb{R}^2$ and consider the two-dimensional Lebesgue measure m_2. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is measurable. Let $f = u + iv$ and $R_N = [-N, N] \times [-N, N]$. Consider the measurable function

$$u|_{R_N} : R_N \to \mathbb{R},$$

where $N \in \mathbb{N}$. By [67, Theorem 4.3, p. 32], there exists a sequence $\{\psi_{N,k}\}$ of step functions converging to $u|_{R_N}$ for almost every $z \in R_N$, i,e., there corresponds an $p_N \in \mathbb{N}$ such that $k \geq p_N$ implies

$$\big|u|_{R_N}(z) - \psi_{N,k}(z)\big| < \frac{1}{2^{N+1}} \tag{13.27}$$

for almost every $z \in R_N$.

Note that each $\psi_{N,k}$ has the form

$$\psi_{N,k}(z) = \sum_{j=1}^{m_k} \alpha_{N,j} \chi_{R_{N,j}}(z),$$

[a]Without loss of generality, we may assume further that $\overline{D(z_0;\delta)} \subseteq \Omega$.

where $\{R_{N,j}\}$ forms a set of disjoint open rectangles and

$$m_2\Big(R_N \Delta \bigcup_{j=1}^{m_k} R_{N,j}\Big) = m_2\bigg(\Big(R_N \setminus \bigcup_{j=1}^{m_k} R_{N,j}\Big) \cup \Big(\bigcup_{j=1}^{m_k} R_{N,j} \setminus R_N\Big)\bigg) < \frac{1}{2^{N+1}}. \tag{13.28}$$

Suppose that

$$\Omega_N = \bigcup_{j=1}^{m_k} R_{N,j}.$$

Then we can represent the inequality (13.28) as

$$m_2(R_N \Delta \Omega_N) = m_2\big((R_N \setminus \Omega_N) \cup (\Omega_N \setminus R_N)\big) < \frac{1}{2^{N+1}}.$$

By [67, Theorem 3.4, p. 21], we know that one can find a compact subset K_N of Ω_N such that

$$m_2(\Omega_N \setminus K_N) < \frac{1}{2^{N+1}}.$$

Since we have $R_N \setminus K_N \subseteq (R_N \setminus \Omega_N) \cup (\Omega_N \setminus K_N)$ and $K_N \setminus R_N \subseteq \Omega_N \setminus R_N$, we get

$$\begin{aligned} m_2(R_N \Delta K_N) &= m_2\big((R_N \setminus K_N) \cup (K_N \setminus R_N)\big) \\ &\leq m_2\big([(R_N \setminus \Omega_N) \cup (\Omega_N \setminus K_N)] \cup (\Omega_N \setminus R_N)\big) \\ &\leq m_2(R_N \Delta \Omega_N) + m_2(\Omega_N \setminus K_N) \\ &< \frac{1}{2^N}. \end{aligned} \tag{13.29}$$

Obviously, each Ω_N is open in $\mathbb{C}$ because each $R_{N,j}$ is an open set. Besides, as the complement $\mathbb{C} \setminus \Omega_N$ is path-connected, it is also connected.[b] Since $\psi_{N,k}$ is constant on each $R_{N,j}$, it is holomorphic in $R_{N,j}$ and hence in Ω_N, i.e., $\psi_{N,k} \in H(\Omega_N)$. By Theorem 13.9 (Runge's Theorem), for each pair of *fixed* N and k, there is a polynomial sequence $\{Q_{N,k,n}\}$ such that $Q_{N,k,n} \to \psi_{N,k}$ as $n \to \infty$ uniformly on compact subsets of Ω_N. Fix the K_N as above, there is an $q_N \in \mathbb{N}$ such that $n \geq q_N$ implies that

$$|Q_{N,k,n}(z) - \psi_{N,k}(z)| < \frac{1}{2^{N+1}} \tag{13.30}$$

for all $z \in K_N$. Combining the estimates (13.27) and (13.30), for $k \geq p_N$ and $n \geq q_N$, we can establish that

$$\big|u|_{R_N}(z) - Q_{N,k,n}(z)\big| \leq \big|u|_{R_N}(z) - \psi_{N,k}(z)\big| + \big|\psi_{N,k}(z) - Q_{N,k,n}(z)\big| < \frac{1}{2^N} \tag{13.31}$$

for almost every $R_N \cap K_N$. Recall that

$$\mathbb{C} = \bigcup_{N=1}^{\infty} R_N.$$

By this fact and the inequality (13.29), we obtain

$$\lim_{N\to\infty} K_N = \mathbb{C}.$$

If we pick $U_N(z) = Q_{N,p_N,q_N}(z)$, then it can be established from the inequality (13.31) that

$$\lim_{N\to\infty} U_N(z) = u(z) \tag{13.32}$$

[b] Refer to the paragraph following the definition in [42, p. 155].

for almost every $z \in \mathbb{C}$. Now the same result also holds for the imaginary part, i.e., there exists a sequence of polynomials $\{V_N\}$ such that

$$\lim_{N\to\infty} V_N(z) = v(z) \tag{13.33}$$

for almost every $z \in \mathbb{C}$. If we let $P_N = U_N + iV_N$, then each P_N is a polynomial and therefore, our desired result follows immediately from the limits (13.32) and (13.33). Hence we have ended the proof of the problem. ∎

CHAPTER 14

Conformal Mapping

14.1 Basic Properties of Conformal Mappings

Problem 14.1

Rudin Chapter 14 Exercise 1.

Proof. The linear fractional transformation

$$f(z) = \frac{az+b}{cz+d} \tag{14.1}$$

maps Π^+ onto itself if and only if a, b, c and d are real numbers such that $ad - bc > 0$.

Suppose that a, b, c and d are real numbers such that $ad - bc > 0$. By §14.3, the f in the form (14.1) is already a one-to-one mapping of S^2 onto S^2. Since $a, b, c, d \in \mathbb{R}$, f must map the real axis onto itself. Furthermore, we have

$$\operatorname{Im} f(i) = \frac{ad-bc}{c^2+d^2} > 0,$$

so i is mapped into Π^+ which proves that the transformation f maps Π^+ onto itself. The converse part can be found in [76, Problem 13.16, p. 181], completing the proof of the problem. ∎

Problem 14.2

Rudin Chapter 14 Exercise 2.

Proof. Denote Π^+ to be the upper half plane. Let $z \in U$. We have to make clear the meaning of the reflection, namely z^*, of z with respect to the arc L. In fact, by the discussion in [9, pp. 102, 103][a], we know that z^* lies on the same ray as z and $|z^*| = |z|^{-1}$. In other words, we have

$$z^* = \frac{1}{\bar{z}}.$$

[a] See also [18, pp. 50, 51]

(a) We have the following analogous reflection theorem for this part:

Lemma 14.1

Suppose that $\Omega \subseteq \Pi^+$, $L = \mathbb{R}$ and every point $t \in L$ is the center of an open disc D_t such that $\Pi^+ \cap D_t \subseteq \Omega$. Let Ω^- be the reflection of Ω, i.e.,

$$\Omega^- = \{z \in \mathbb{C} \,|\, \overline{z} \in \Omega\}.$$

If $f \in H(\Omega)$ and $|f(z_n)| \to 1$ for every $\{z_n\}$ in Ω which converges to a point of L, then there exists a function F, holomorphic in $\Omega \cup L \cup \Omega^-$, such that

$$F(z) = \begin{cases} f(z), & \text{if } z \in \Omega \cup L; \\ \dfrac{1}{\overline{f(\overline{z})}}, & \text{if } z \in \Omega^-. \end{cases} \tag{14.2}$$

Proof of Lemma 14.1. Fix a point $t \in L$. By the hypothesis $|f(z)| \to 1$ as $z \to t \in L$, it is legitimate to select a disc D_t so small that $f(z) \neq 0$ in $\Pi^+ \cap D_t$. Then the function $g(z) = i \log f(z)$ is well-defined and holomorphic in $\Pi^+ \cap D_t$. Furthermore, we know that

$$\operatorname{Im} g(z_n) = \log |f(z_n)| \to 0$$

for every $\{z_n\}$ in Ω converging to a point of L. By the application of Theorem 11.14 (The Schwarz Reflection Principle), we see that one can find a function G, holomorphic in $\Omega \cup L \cup \Omega^-$, such that $G(z) = g(z)$ in Ω and satisfies

$$G(\overline{z}) = \overline{G(z)} \tag{14.3}$$

for all $z \in \Omega \cup L \cup \Omega^-$. Define $F(z) = \mathrm{e}^{-iG(z)}$. Since $f(z) = \mathrm{e}^{-ig(z)}$, if $z \in \Omega$, then we have

$$F(z) = \mathrm{e}^{-iG(z)} = \mathrm{e}^{-ig(z)} = f(z).$$

Next, if $z \in \Omega^-$, then $\overline{z} \in \Omega$ and we deduce from the equation (14.3) and the definition of F that

$$F(z) = \mathrm{e}^{-iG(z)} = \mathrm{e}^{-i\overline{G(\overline{z})}} = \overline{\mathrm{e}^{iG(\overline{z})}} = \frac{1}{\overline{F(\overline{z})}} = \frac{1}{\overline{f(\overline{z})}}$$

which is exactly the equation (14.2). ■

(b) Recall from [62, Eqn. (1), p. 281] that

$$\psi(z) = \frac{z - i}{z + i}$$

is a conformal one-to-one mapping of Π^+ onto U and $\psi(\mathbb{R}) \subseteq T$. So the inverse

$$\psi^{-1}(\zeta) = \frac{i(1 + \zeta)}{1 - \zeta}$$

is a conformal one-to-one mapping of U onto Π^+. For every $\theta \in [0, 2\pi]$, we know that

$$\psi^{-1}(\mathrm{e}^{i\theta}) = \frac{i(1 + \mathrm{e}^{i\theta})}{1 - \mathrm{e}^{i\theta}} = -\frac{\sin\theta}{1 - \cos\theta} \in \mathbb{R}.$$

Thus $\widehat{L} = \psi^{-1}(L)$ and $\widehat{\Omega} = \psi^{-1}(\Omega)$ are a segment of $\mathbb{R}$ and a region in Π^+ respectively. Define the map

$$\widehat{f} = f \circ \psi_{\widehat{\Omega}} : \widehat{\Omega} \subseteq \Pi^+ \to \mathbb{C}. \tag{14.4}$$

Then for every $\{z_n\} \in \widehat{\Omega}$ converging to a $z_0 \in \widehat{L}$, the points $\zeta_n = \psi(z_n) \in \Omega \subseteq U$ converging to $\zeta_0 = \psi(z_0) \in L \subseteq T$, so the hypothesis guarantees that

$$|\widehat{f}(z_n)| = \big|f\big(\psi(z_n)\big)\big| = |f(\zeta_n)| \to 1$$

as $n \to \infty$. Hence it follows from Lemma 14.1 that there exists a function $\widehat{F}$, holomorphic in $\widehat{\Omega} \cup \widehat{L} \cup \widehat{\Omega}^-$, such that

$$\begin{aligned}
\widehat{F}(z) &= \begin{cases} \widehat{f}(z), & \text{if } z \in \widehat{\Omega} \cup \widehat{L}; \\[2ex] \dfrac{1}{\overline{\widehat{f}(\overline{z})}}, & \text{if } z \in \widehat{\Omega}^-, \end{cases} \\
&= \begin{cases} f\big(\psi(z)\big), & \text{if } z \in \widehat{\Omega} \cup \widehat{L}; \\[2ex] \dfrac{1}{\overline{f\big(\psi(\overline{z})\big)}}, & \text{if } z \in \widehat{\Omega}^-. \end{cases}
\end{aligned} \tag{14.5}$$

Here $\widehat{\Omega}^- = \{z \in \mathbb{C} \,|\, \overline{z} \in \widehat{\Omega}\}$. If $z \in \widehat{\Omega}^-$, then $\overline{z} \in \widehat{\Omega}$ and so we note from the definition of ψ that

$$\psi(\overline{z}) = \frac{1}{\overline{\psi(z)}}. \tag{14.6}$$

Hence the formula (14.5) becomes

$$F(\zeta) = \begin{cases} f(\zeta), & \text{if } \zeta \in \Omega \cup L; \\[2ex] \dfrac{1}{\overline{f\big(\overline{\zeta}^{-1}\big)}}, & \text{if } \zeta \in \Omega^*, \end{cases} \tag{14.7}$$

where $\Omega^* = \{\zeta \in \mathbb{C} \,|\, \overline{\zeta}^{-1} \in \Omega\}$.

(c) We have the following analogous reflection theorem for U:

Lemma 14.2 (The Schwarz Reflection Principle for U)

Every $e^{it} \in L \subseteq T$ is the center of an open disc D_t such that $D_t \cap U$ lies in Ω. Denote Ω^* to be the reflection of Ω, i.e.,

$$\Omega^* = \Big\{z \in \mathbb{C} \,\Big|\, z^* = \frac{1}{\overline{z}} \in \Omega\Big\}.$$

Suppose that $f \in H(\Omega)$ and $\operatorname{Im} f(z_n) \to 0$ for every sequence $\{z_n\}$ in Ω which converges to a point of L. Then there exists a function F, holomorphic in the set $\Omega \cup L \cup \Omega^*$, such that

$$F(z) = \begin{cases} f(z), & \text{if } z \in \Omega \cup L; \\[2ex] \overline{f\Big(\dfrac{1}{\overline{z}}\Big)}, & \text{if } z \in \Omega^*. \end{cases} \tag{14.8}$$

Proof of Lemma 14.2. With the same function $\psi : \Pi^+ \to U$ as in part (b), we see that "$z \in \widehat{\Omega}$ and $z \to \widehat{L}$" is equivalent to saying that "$\psi(z) \in \Omega$ and $\psi(z) \to L$". This property implies that the function (14.4) satisfies

$$\operatorname{Im} \widehat{f}(z) = \operatorname{Im} f\big(\psi(z)\big) \to 0$$

as $z \in \widehat{\Omega}$ and $z \to \overline{L}$. According to Theorem 11.14 (The Schwarz Reflection Principle), there is a function $\widehat{F}$, holomorphic in $\widehat{\Omega} \cup \widehat{L} \cup \widehat{\Omega}^-$, such that

$$\widehat{F}(z) = \begin{cases} f\big(\psi(z)\big), & \text{if } z \in \widehat{\Omega} \cup \widehat{L}; \\ \\ \overline{f\big(\psi(\overline{z})\big)}, & \text{if } z \in \widehat{\Omega}^-. \end{cases} \tag{14.9}$$

Using the formula (14.6), the function (14.9) can be expressed in the following form:

$$F(\zeta) = \begin{cases} f(\zeta), & \text{if } \zeta \in \Omega \cup L; \\ \\ \overline{f\Big(\dfrac{1}{\overline{\zeta}}\Big)}, & \text{if } \zeta \in \Omega^*. \end{cases}$$

This completes the proof of Lemma 14.2. ■

Now, since $(\overline{\frac{1}{\overline{\alpha}}})^{-1} = \alpha$, it is easy to conclude from the expression (14.7) that if $f(\alpha) = 0$ for some $\alpha \in \Omega$, then $\frac{1}{\overline{\alpha}} \in \Omega^*$ so that

$$F\Big(\frac{1}{\overline{\alpha}}\Big) = \frac{1}{\overline{f(\alpha)}} = \infty,$$

i.e., F has a pole at $\frac{1}{\overline{\alpha}}$. By the expression (14.8), the analogue of part (c) is that $F(\frac{1}{\overline{\alpha}}) = \overline{f(\alpha)} = 0$. For part (a), if $f(\alpha) = 0$ for some $\alpha \in \Omega$, then the expression (14.2) implies that

$$F(\overline{\alpha}) = \frac{1}{\overline{f(\alpha)}} = \infty,$$

i.e., F has a pole at $\overline{\alpha}$. This finishes the proof of the problem. ■

Problem 14.3

Rudin Chapter 14 Exercise 3.

Proof. If $|z| = 1$, then it is easy to see that $z = \frac{1}{\overline{z}}$ and the rational function $\widehat{R}(z) = R(z) \cdot \overline{R(\frac{1}{\overline{z}})}$ satisfies

$$\widehat{R}(z) = R(z) \cdot \overline{R\Big(\frac{1}{\overline{z}}\Big)} = R(z) \cdot \overline{R(z)} = |R(z)|^2 = 1$$

if $|z| = 1$. Since $\widehat{R} = \frac{P}{Q}$, where P and Q are polynomials, we have $P(z) = Q(z)$ on the unit circle. Now the Corollary following Theorem 10.18 guarantees that $P(z) \equiv Q(z)$ in $\mathbb{C}$ and this implies that $\overline{R}(z) = 1$ for every $z \in \mathbb{C}$, i.e.,

$$R\Big(\frac{1}{z}\Big) = \frac{1}{\overline{R(\overline{z})}}$$

for every $z \in \mathbb{C}$. Therefore, ω is a zero of order m of R if and only if $\frac{1}{\overline{\omega}}$ is a pole of order m of R. This fact shows that the zeros and poles of R inside U completely determines *all* zeros and poles of R in $\mathbb{C}$.

Next, suppose that R has a zero at $z = 0$ of order m. Suppose, further, that $\{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}$ are the *distinct* zeros and poles of R inside U. We consider the product

$$B(z) = z^m \cdot \prod_{n=1}^{k} \frac{z - \alpha_n}{1 - \overline{\alpha_n} z}$$

which is a rational function having the *same* zeros and poles of the *same* order as R. Recall from the proof of Theorem 12.4, we know that $|\frac{z-\alpha_n}{1-\overline{\alpha_n}z}| = 1$ for $|z| = 1$ and thus $|B(z)| = 1$ for $|z| = 1$. Consequently, the quotient

$$f(z) = \frac{R(z)}{B(z)}$$

is a rational function *without* zeros or poles in $D(0; r)$ for some $r > 1$. Since $|f(z)| = \frac{|R(z)|}{|B(z)|} = 1$ on $|z| = 1$, we get from the Corollary following Theorem 10.18 that $f(z) = c$ for some constant c with $|c| = 1$ in $D(0; r)$ and hence in $\mathbb{C} \setminus \{\frac{1}{\overline{\alpha_1}}, \frac{1}{\overline{\alpha_2}}, \ldots, \frac{1}{\overline{\alpha_n}}\}$, i.e.,

$$R(z) = cB(z) = cz^m \cdot \prod_{n=1}^{k} \frac{z - \alpha_n}{1 - \overline{\alpha_n} z}$$

as desired. This ends the proof of the problem. ■

Problem 14.4

Rudin Chapter 14 Exercise 4.

Proof. We have $R(z) > 0$ on $|z| = 1$. By the hint, R must have the same number of zeros as poles in U. Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ be the zeros and poles of R inside U, where N is a positive integer. Next, we consider the rational function

$$f(\alpha, \beta, z) = \frac{(z - \alpha)(1 - \overline{\alpha} z)}{(z - \beta)(1 - \overline{\beta} z)}, \tag{14.10}$$

where $\alpha, \beta \in U$. Obviously, if $|z| = 1$, then $z \cdot \overline{z} = |z|^2 = 1$ and $1 - \alpha\overline{z} \neq 0$ which imply that

$$f(\alpha, \beta, z) = \frac{\overline{z}}{\overline{z}} \cdot \frac{(z - \alpha)(1 - \overline{\alpha} z)}{(z - \beta)(1 - \overline{\beta} z)} = \frac{(1 - \alpha\overline{z})(1 - \overline{\alpha} z)}{(1 - \beta\overline{z})(1 - \overline{\beta} z)} = \frac{|1 - \alpha\overline{z}|^2}{|1 - \beta\overline{z}|^2} > 0.$$

Now the representation (14.10) indicates that α and β are the *only* zeros and poles of $f(\alpha, \beta, z)$ inside U respectively. Therefore, the rational function

$$Q(z) = \prod_{n=1}^{N} f(\alpha_n, \beta_n, z)$$

has the same numbers of zeros and poles as those of R inside U.

Consequently, the quotient

$$F(z) = \frac{R(z)}{Q(z)}$$

is a rational function *without* zeros or poles in U, i.e., $F \in H(U)$. Since $f(\alpha_n, \beta_n, z) > 0$ on $|z| = 1$ for every $n = 1, 2, \ldots, N$, we also have $Q(z) > 0$ on $|z| = 1$ and hence $F(z) > 0$ on $|z| = 1$. In other words, $\operatorname{Im} F \equiv 0$ on $|z| = 1$. Recall that $\operatorname{Im} F$ is a continuous real-valued function on $\overline{U}$ and is harmonic in U, so we follow from [7, Corollary 1.9, p. 7] that $\operatorname{Im} F \equiv 0$ in U. Finally, using [9, Proposition 3.6, p. 39], there is a positive constant c such that

$$F(z) = c \tag{14.11}$$

in U. By the Corollary following Theorem 10.18, we conclude that the result (14.11) holds in $\Omega = \mathbb{C} \setminus \{\beta_1, \beta_2, \ldots, \beta_N, \frac{1}{\overline{\beta_1}}, \frac{1}{\overline{\beta_2}}, \ldots, \frac{1}{\overline{\beta_N}}\}$ which means that

$$R(z) = cQ(z) = c \prod_{n=1}^{N} \frac{(z-\alpha_n)(1-\overline{\alpha_n}z)}{(z-\beta_n)(1-\overline{\beta_n}z)} \tag{14.12}$$

in Ω, completing the analysis of the problem. ■

Problem 14.5

Rudin Chapter 14 Exercise 5.

Proof. Suppose that $g(\zeta) = \sum_{k=-n}^{n} a_k \zeta^n$ which is a rational function. Then $g(\mathrm{e}^{i\theta}) = f(\theta)$, so g is positive on T. By the representation (14.12), one can find a positive constant c such that

$$g(\zeta) = c \prod_{j=1}^{n} \frac{(\zeta-\alpha_j)(1-\overline{\alpha_j}\zeta)}{(\zeta-\beta_j)(1-\overline{\beta_j}\zeta)} \tag{14.13}$$

in $\Omega = \mathbb{C} \setminus \{\beta_1, \beta_2, \ldots, \beta_n, \frac{1}{\overline{\beta_1}}, \frac{1}{\overline{\beta_2}}, \ldots, \frac{1}{\overline{\beta_n}}\}$. Here $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ are sets of zeros and poles of g inside U respectively. Since $\zeta^n g(\zeta)$ is a polynomial, the expression (14.13) implies that $\beta_1 = \beta_2 = \cdots = \beta_n = 0$. Hence we obtain

$$\begin{aligned}
f(\theta) &= g(\mathrm{e}^{i\theta}) \\
&= c \prod_{j=1}^{n} \frac{(\mathrm{e}^{i\theta}-\alpha_j)(1-\overline{\alpha_j}\mathrm{e}^{i\theta})}{\mathrm{e}^{i\theta}} \\
&= c \prod_{j=1}^{n} (\mathrm{e}^{i\theta}-\alpha_j)(\mathrm{e}^{-i\theta}-\overline{\alpha_j}) \\
&= c \prod_{j=1}^{n} (\mathrm{e}^{i\theta}-\alpha_j)\overline{(\mathrm{e}^{i\theta}-\alpha_j)} \\
&= c \prod_{j=1}^{n} |\mathrm{e}^{i\theta}-\alpha_j|^2 \\
&= \left| \sqrt{c} \prod_{j=1}^{n} (\mathrm{e}^{i\theta}-\alpha_j) \right|^2 \\
&= |P(\mathrm{e}^{i\theta})|^2,
\end{aligned}$$

where $P(z) = \sqrt{c}(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$. We have completed the proof of the problem. ■

Problem 14.6

Rudin Chapter 14 Exercise 6.

Proof. If $\alpha = 0$, then $\varphi_0(z) = z$, so the fixed points of φ_0 are U. Next, we suppose that $\alpha \neq 0$. By Definition 12.3, $\varphi_\alpha(z) = z$ if and only if $z^2 = \frac{\alpha}{\overline{\alpha}}$ if and only if

$$z = \pm\frac{\alpha}{|\alpha|}.$$

This gives our first assertion.

For the second assertion, we note that since φ_α is a special case of the linear fractional transformation $\varphi(z) = \frac{az+b}{cz+d}$, we consider the general case for φ. By a suitable rotation, we may assume that the straight line is the real axis. We claim that φ maps $\mathbb{R}\cup\{\infty\}$ into $\mathbb{R}\cup\{\infty\}$ if and only if a, b, c and d are real. It is easy to see that if a, b, c and d are real, then we have $\varphi(\mathbb{R}\cup\{\infty\}) \subseteq \mathbb{R}\cup\{\infty\}$. Conversely, suppose that $\varphi(\mathbb{R}\cup\{\infty\}) \subseteq \mathbb{R}\cup\{\infty\}$. Since $\varphi(0) \in \mathbb{R}\cup\{\infty\}$, we have either $d = 0$ or $\frac{b}{d} \in \mathbb{R}$.

- **Case (i):** $d = 0$**.** Since $\varphi(z) \to \frac{a}{c}$ as $z \to \infty$ along $\mathbb{R}$, we have $\frac{a}{c} \in \mathbb{R}$ or $c = 0$. In the latter case, the transformation is $\varphi(z) = \infty$ for all $z \in \mathbb{R}\cup\{\infty\}$ and we can rewrite it as $\varphi(z) = \frac{1}{0\cdot z+0}$. For the case $\frac{a}{c} \in \mathbb{R}$, since $\varphi(1) = \frac{a}{c} + \frac{b}{c} \in \mathbb{R}$ with $c \neq 0$, we have $\frac{b}{c} \in \mathbb{R}$. Therefore, we obtain
$$\varphi(z) = \frac{az+b}{cz} = \frac{\frac{a}{c}z + \frac{b}{c}}{1\cdot z + 0}.$$
- **Case (ii):** $\frac{b}{d} \in \mathbb{R}$**.** Note that $d \neq 0$. Let $B = \frac{b}{d}$. Then we have
$$\varphi(z) = \frac{az + Bd}{cz+d}. \tag{14.14}$$
Using $\varphi(z) \to \frac{a}{c}$ as $z \to \infty$ along $\mathbb{R}$ again, we know that $A = \frac{a}{c} \in \mathbb{R}$ or $c = 0$. In the latter case, we have $\varphi(z) = \frac{a}{d}z + \frac{b}{d}$. Since $\varphi(1) = \frac{a}{d} + \frac{b}{d} \in \mathbb{R}$ and $\frac{b}{d} \in \mathbb{R}$, we have $\frac{a}{d} \in \mathbb{R}$ so that
$$\varphi(z) = \frac{\frac{a}{d}z + \frac{b}{d}}{0z+1}.$$
For the case $A = \frac{a}{c} \in \mathbb{R}$, the representation (14.14) becomes
$$\varphi(z) = \frac{Acz + Bd}{cz+d}. \tag{14.15}$$
If $c + d = 0$, then $c = -d \neq 0$ so that the representation (14.15) reduces to
$$\varphi(z) = \frac{-Adz + Bd}{-dz+d} = \frac{-Az+B}{-z+1}.$$
Otherwise, $\varphi(1) = \frac{Ac+Bd}{c+d} = A + (B - A)\cdot\frac{d}{c+d} \in \mathbb{R}$ if and only if $\frac{d}{c+d} \in \mathbb{R}$. Since $d \neq 0$, $\frac{d}{c+d} \in \mathbb{R}$ if and only if $\frac{c}{d} \in \mathbb{R}$. Let $c = Cd$ for some $C \in \mathbb{R}$. Now the representation (14.15) reduces to
$$\varphi(z) = \frac{ACdz + Bd}{Cdz+d} = \frac{ACz+B}{Cz+1}.$$

This proves our claim.

Return to our original problem. Suppose that φ_α maps a straight line L into itself. It is clear that $e^{i\theta}L = \mathbb{R}$ for some $\theta \in [0, 2\pi]$, so we assume that $e^{i\theta}\varphi_\alpha$ maps $\mathbb{R}$ into $\mathbb{R}$. Write

$$e^{i\theta}\varphi_\alpha(z) = \frac{e^{i\theta}z - e^{i\theta}\alpha}{(e^{i\theta}\overline{\alpha})z + e^{i\theta}}.$$

Then the above claim says that $e^{i\theta}, e^{i\theta}\alpha$ and $e^{i\theta}\overline{\alpha}$ are real, or equivalently, both $e^{i\theta}$ and α are real. Hence we end the analysis of the problem. ∎

Problem 14.7

Rudin Chapter 14 Exercise 7.

Proof. Suppose that $z, \omega \in U$. By the definition, $f_\alpha(z) = f_\alpha(0) = 0$ so that $z = 0$ and α is arbitrary in this case. Thus, without loss of generality, we may assume that $z, \omega \neq 0$. Now $f_\alpha(z) = f_\alpha(\omega)$ if and only if $(z-\omega)(1-\alpha z\omega) = 0$ if and only if

$$z = \omega \quad \text{or} \quad \alpha = \frac{1}{z\omega}. \tag{14.16}$$

Since $|z| < 1$ and $|\omega| < 1$, we have $\frac{1}{|z\omega|} > 1$. Suppose that $|\alpha| \leq 1$. Then the result (14.16) leads to us that $z = \omega$, i.e., f_α is one-to-one in U.

Let $|\alpha| \leq 1$. It is easy to check that $f_\alpha(0) = 0$ and $f'_\alpha(0) = 1$. Besides, we have $f \in H(U)$. Combining these facts and the previous result, we know that $f_\alpha \in \mathscr{L}$ and

$$f(z) = z - \alpha z^3 + \alpha^2 z^5 + \cdots .$$

Therefore, it deduces from Theorem 14.14 that $D(0; \frac{1}{4}) \subseteq f_\alpha(U)$. This ends the analysis of the problem. ∎

Problem 14.8

Rudin Chapter 14 Exercise 8.

Proof. Let $z = r\mathrm{e}^{i\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$. Then we have

$$f(r\mathrm{e}^{i\theta}) = r\mathrm{e}^{i\theta} + \frac{\mathrm{e}^{-i\theta}}{r} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Suppose that

$$x = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad y = \left(r - \frac{1}{r}\right)\sin\theta. \tag{14.17}$$

If $r = 1$, then $x = 2\cos\theta$ and $y = 0$ so that $f\big(C(0;1)\big) = [-2, 2]$. Suppose, otherwise, that $r \neq 1$, so we obtain

$$\frac{x^2}{(r + \frac{1}{r})^2} + \frac{y^2}{(r - \frac{1}{r})^2} = 1$$

which is an ellipse.

Next, suppose that θ is a fixed number. Denote $L_\theta = \{r\mathrm{e}^{i\theta} \,|\, 0 \leq r < \infty\}$. If $\theta = 0$, then $\cos\theta = 1$ and $\sin\theta = 0$ and thus it easily follows from the representations (14.17) that

$$f(L_0) = (0, \infty).$$

Similarly, if $\theta = \pi$, then $\cos\theta = -1$ and $\sin\theta = 0$ so that

$$f(L_\pi) = (-\infty, 0).$$

If $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, then $\cos\theta = 0$ and $\sin\theta = \pm 1$. Simple computation verifies that

$$f\big(L_{\frac{\pi}{2}}\big) = f\big(L_{\frac{3\pi}{2}}\big) = i\mathbb{R}.$$

Finally, if $\theta \in [0, 2\pi) \setminus \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, then we have $\cos\theta \cdot \sin\theta \neq 0$ and we deduce from the representations (14.17) that

$$\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 4$$

which is trivially a hyperbola. This completes the analysis of the problem. ∎

Problem 14.9

Rudin Chapter 14 Exercise 9.

Proof. Define $\Omega_1 = \{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < \pi\}$ and $\Pi^+ = \{z \in \mathbb{C} \,|\, \operatorname{Im} z > 0\}$.

(a) Notice that the map $f_1 : \Omega \to \Omega_1$ defined by $f_1(z) = \frac{\pi i}{2}(z+1)$ satisfies $f'(z) = \frac{\pi i}{2} \neq 0$ in Ω. By Theorem 14.2, it is a one-to-one conformal mapping of Ω onto Ω_1. Next, we know from [9, p. 176] that $f_2(z) = e^z$ is a one-to-one conformal mapping of Ω_1 onto the upper half plane Π^+. Recall from [62, Eqn. (1), p. 281] that $f_3(z) = \frac{z-i}{z+i}$ is a one-to-one conformal mapping of Π^+ onto U. Hence the mapping $f = f_3 \circ f_2 \circ f_1 : \Omega \to U$ is the required mapping. Explicitly, we have

$$f(z) = \frac{\exp(\frac{i\pi}{2}z) - 1}{\exp(\frac{i\pi}{2}z) + 1}.$$

(b) The inverse $f^{-1} : U \to \Omega$ is given by

$$f^{-1}(z) = -\frac{2i}{\pi}\log\frac{1+z}{1-z} = \frac{2}{\pi}\arg\Big(\frac{1+z}{1-z}\Big) - \frac{2i}{\pi}\log\Big|\frac{1+z}{1-z}\Big|. \tag{14.18}$$

If $f^{-1} = u + iv$, then we have

$$u(z) = \frac{2}{\pi}\arg\Big(\frac{1+z}{1-z}\Big) \quad \text{and} \quad v(z) = -\frac{2}{\pi}\log\Big|\frac{1+z}{1-z}\Big|.$$

Since $f^{-1} \in H(U)$, u must be bounded and harmonic in U. Its harmonic conjugate v is unbounded in U because $|v(z)| \to \infty$ as $z \to \pm 1$. It remains to prove that u can be extended continuously to $\overline{U}$. The definition of f shows that it can be extended to a continuous function of $\overline{\Omega}$ onto $\overline{U}$. Hence its inverse, which is u, can also be extended to a continuous function on $\overline{U}$.

(c) Since our f^{-1} and g satisfy the hypotheses of Problem 14.10, we establish immediately that

$$g\big(D(0;r)\big) \subseteq f^{-1}\big(D(0;r)\big) \tag{14.19}$$

for all $0 < r < 1$. Of course, we observe from the definition (14.18) that

$$-\frac{\pi}{2}\operatorname{Im} f^{-1}(z) = \log\Big|\frac{1+z}{1-z}\Big| \tag{14.20}$$

for all $z \in U$. Let $f^{-1}(z) = \sum_{n=0}^{\infty} c_n z^n$, where $z \in U$. Since $n!c_n = (f^{-1})^{(n)}(0)$ for every $n = 0, 1, 2, \ldots$, it is easy to see that f^{-1} has the form

$$f^{-1}(z) = -\frac{4i}{\pi}\sum_{n=1}^{\infty}\frac{z^{2n-1}}{2n-1}$$

for all $z \in U$. Thus for every $z \in D(0;r)$, we deduce from the expression (14.20) that

$$|f^{-1}(z)| \leq \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{|z|^{2n-1}}{2n-1} = -\operatorname{Im} f^{-1}(|z|) = \frac{2}{\pi}\log\Big|\frac{1+|z|}{1-|z|}\Big| = \frac{2}{\pi}\log\frac{1+r}{1-r}.$$

Finally, the set relation (14.19) asserts that

$$|g(z)| \leq \frac{2}{\pi}\log\frac{1+r}{1-r}$$

for all $z \in D(0;r)$.

(d) Now suppose that $\Omega = \{x + iy \,|\, -\frac{\pi}{2} < y < \frac{\pi}{2}\}$ and $h : \Omega \to \Omega$ is a conformal bijective mapping such that $h(a + i\beta) = 0$. It is known from Problem 14.32 that the mapping

$$\psi(z) = \log \frac{1+z}{1-z}$$

sends U conformally, one-to-one and onto the horizontal strip Ω with $\psi(0) = 0$. Thus its inverse

$$\psi^{-1}(z) = \frac{e^z - 1}{e^z + 1}$$

is a conformal one-to-one mapping of Ω onto U and $\psi^{-1}(0) = 0$. Denote $\psi^{-1}(\alpha + i\beta) = A$ so that

$$\frac{e^{\alpha+i\beta} - 1}{e^{\alpha+i\beta} + 1} = A. \tag{14.21}$$

It is well-known [9, Theorem 13.15, p. 183] that the conformal mapping φ of U onto itself and $\varphi(A) = 0$ is represented by

$$\varphi(z) = e^{i\theta} \cdot \frac{z - A}{1 - \overline{A}z}$$

for some $\theta \in \mathbb{R}$. Therefore, the composition $h = \psi \circ \varphi \circ \psi^{-1}$ is a conformal one-to-one mapping from Ω onto itself and

$$h\big(\alpha + i\beta\big) = \psi\big(\varphi\big(\psi^{-1}(\alpha + i\beta)\big)\big) = \psi\big(\varphi(A)\big) = \psi(0) = 0.$$

Notice that

$$\psi'(z) = \frac{2}{1 - z^2}, \quad (\psi^{-1})'(z) = \frac{2e^z}{(e^z + 1)^2} \quad \text{and} \quad \varphi'(z) = e^{i\theta} \cdot \frac{1 - |A|^2}{(1 - \overline{A}z)^2}.$$

By the Chain Rule, we have

$$\begin{aligned} h'(\alpha + i\beta) &= \psi'\big(\varphi\big(\psi^{-1}(\alpha + i\beta)\big)\big) \cdot \varphi'\big(\psi^{-1}(\alpha + i\beta)\big) \cdot (\psi^{-1})'(\alpha + i\beta) \\ &= \psi'(0) \times \varphi'(A) \times \frac{2e^{\alpha+i\beta}}{(e^{\alpha+i\beta} + 1)^2} \\ &= 2 \times \frac{e^{i\theta}}{1 - |A|^2} \times \frac{2e^{\alpha+i\beta}}{(e^{\alpha+i\beta} + 1)^2}. \end{aligned} \tag{14.22}$$

According to the value (14.21) and the expression (14.22), we see that

$$\begin{aligned} |h'(\alpha + i\beta)| &= 2\Big(1 - \frac{|e^{\alpha+i\beta} - 1|^2}{|e^{\alpha+i\beta} + 1|^2}\Big)^{-1} \times \frac{2e^{\alpha}}{|e^{\alpha+i\beta} + 1|^2} \\ &= \frac{4e^{\alpha}}{|e^{\alpha+i\beta} + 1|^2 - |e^{\alpha+i\beta} - 1|^2} \\ &= \frac{1}{\cos\beta} \end{aligned}$$

which is the desired result.

Consequently, we have completed the proof of the problem. ■

Remark 14.1

Problem 14.9 contributes to the theory of the **Principle of Subordination**, read [15, Chap. VI, §5, pp. 207 - 215], [21, §1.5, pp. 10 – 13] or [47, Chap V, §9, pp. 226 – 236].

Problem 14.10

Rudin Chapter 14 Exercise 10.

Proof. Since f is one-to-one, $f^{-1} : \Omega \to U$ exists. Let $h = f^{-1} \circ g$. Then it is clear that $h \in H(U)$, $h(U) = f^{-1}\big(g(U)\big) \subseteq U$ and $h(0) = 0$. Thus Theorem 12.2 (Schwarz's Lemma) ensures that

$$|h(z)| \leq |z| \tag{14.23}$$

for all $z \in U$. For every $0 < r < 1$, if $z \in D(0; r)$, then the inequality (14.23) implies that $h(z) \in D(0; r)$. Equivalently, this means that $g(z) \in f\big(D(0; r)\big)$ and hence[b]

$$g(D(0; r)) \subseteq f(D(0; r)),$$

completing the proof of the problem. ∎

Problem 14.11

Rudin Chapter 14 Exercise 11.

Proof. Now we have $\Omega = \{z \in U \mid \operatorname{Im} z > 0\}$. Using [9, Example 1, p. 180; Theorem 13.16, p. 183], if $f : \Omega \to U$ is a conformal bijective mapping, then f has the representation

$$f(z) = e^{i\theta} \cdot \frac{(z-1)^2 + 4\alpha(z+1)^2}{(z-1)^2 + 4\overline{\alpha}(z+1)^2}, \tag{14.24}$$

where $\theta \in [0, 2\pi]$ and $\operatorname{Im} \alpha > 0$. Putting $f(-1) = -1$, $f(0) = -i$ and $f(1) = 1$ into the formula (14.24), we obtain that $\theta = \pi$ and $\alpha = \frac{i}{4}$, so

$$f(z) = -\frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2} \tag{14.25}$$

is the desired conformal mapping. By the representation (14.25), it is easily seen that if $z \in \Omega$ satisfies $f(z) = 0$, then $z = (-1 + \sqrt{2})i$. Furthermore, simple algebra gives $f(\frac{i}{2}) = \frac{i}{7}$ which ends the proof of the problem. ∎

Problem 14.12

Rudin Chapter 14 Exercise 12.

Proof. For convenience, we let $u(z) = \operatorname{Re} f'(z) : \mathbb{C} \to \mathbb{R}$. We prove the assertions as follows:

- **f is one-to-one in Ω when $u(z) > 0$ for all $z \in \Omega$.** Choose $a, b \in \Omega$ and $a \neq b$. Since Ω is convex, the path $\gamma(t) = a + (b-a)t$ for all $t \in [0, 1]$ is in Ω. Then we know from the Fundamental Theorem of Calculus that

$$\int_\gamma f'(z)\,dz = (b-a)\int_0^1 f'\big(a + (b-a)t\big)\,dt = f(b) - f(a)$$

so that

$$\operatorname{Re}\left[\frac{f(b) - f(a)}{b-a}\right] = \int_0^1 u\big(a + (b-a)t\big)\,dt > 0.$$

Consequently, $\operatorname{Re} f(a) \neq \operatorname{Re} f(b)$ which implies that $f(a) \neq f(b)$. Since the pair of points $\{a, b\}$ is arbitrary, we assert that f is one-to-one in Ω.

[b] Rudin used the notation "$\subset$" to mean "$\subseteq$", see [61, Definition 1.3, p. 3].

- **f is either one-to-one or constant in Ω when $u(z) \geq 0$ for all $z \in \Omega$.** Since $f \in H(\Omega)$, u has continuous derivative of all orders. Thus the set

$$S = \{z \in \Omega \,|\, u(z) = 0\} = u^{-1}(0) \tag{14.26}$$

is closed in $\mathbb{C}$. Let $p \in S$. Since u is harmonic in Ω, it follows from the mean value property that

$$0 = u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p + Re^{i\theta})\,\mathrm{d}\theta \tag{14.27}$$

for some $R > 0$ such that $\overline{D(p;R)} \subseteq \Omega$. If there exists a measurable set $E \subseteq [0, 2\pi]$ such that $m(E) > 0$ and $u(p + Re^{i\theta}) > 0$ for every $\theta \in E$, then we have

$$\int_0^{2\pi} u(p + Re^{i\theta})\,\mathrm{d}\theta = \int_E u(p + Re^{i\theta})\,\mathrm{d}m + \int_{[0,2\pi]\setminus E} u(p + Re^{i\theta})\,\mathrm{d}m > 0$$

which contradicts the result (14.27). Therefore, no such E exists and then $u(p+Re^{i\theta}) = 0$ a.e. on $[0, 2\pi]$. Now the continuity of u on Ω forces that $u(z) = 0$ for every $z \in \overline{D(p;R)}$. In other words, $D(p;R) \subseteq S$ which means that S is open in $\mathbb{C}$.

Since S is both open and closed in $\mathbb{C}$, we have either $S = \varnothing$ or $S = \mathbb{C}$. Suppose that $S = \varnothing$, then $\operatorname{Re} f'(z) > 0$ in Ω so that f is one-to-one in Ω by the previous assertion. Next, we suppose that $S = \mathbb{C}$, then the definition (14.26) implies $f'(\Omega)$ is purely imaginary. Since $f' \in H(\Omega)$, the Open Mapping Theorem ensures that $f'(\Omega) = A$ for some constant A. Finally, if we consider $g(z) = f(z) - Az$, then $g \in H(\Omega)$ and $g' \equiv 0$ there. Thus it follows from [9, Exercise 5, p. 42] that g is a constant B. Consequently, we have $f(z) = Az + B$. If $A \neq 0$, then f is linear and so it is one-to-one. Otherwise, $f \equiv B$ in Ω.

- **The condition "convex" cannot be replaced by "simply connected".** The following example can be found in [30].[c] Take $\beta = \frac{\pi}{2} + \delta$ for very small $\delta > 0$. Define $\Omega = \{z \in \mathbb{C} \,|\, -\beta < \arg z < \beta\}$ and $f(z) = z^{1+\frac{\pi}{2\beta}}$. Then it is clear that $f \in H(\Omega)$ and $f'(z) = (1 + \frac{\pi}{2\beta})z^{\frac{\pi}{2\beta}}$ so that

$$\operatorname{Re} f'(z) = \Big(1 + \frac{\pi}{2\beta}\Big)|z|^{\frac{\pi}{2\beta}} \cos\Big(\frac{\pi \arg z}{2\beta}\Big). \tag{14.28}$$

Since $-\beta < \arg z < \beta$, we have $-\frac{\pi}{2} < \frac{\pi \arg z}{2\beta} < \frac{\pi}{2}$ and it follows from the expression (14.28) that $\operatorname{Re} f'(z) > 0$ for every $z \in \Omega$. Suppose that $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ are points of Ω, where $\theta_1 \neq \theta_2$. Clearly,

$$z_1^{1+\frac{\pi}{2\beta}} = z_2^{1+\frac{\pi}{2\beta}}. \tag{14.29}$$

if and only if $\exp\big(i(\theta_1 - \theta_2)(1 + \frac{\pi}{2\beta})\big) = 1$ if and only if

$$\theta_1 - \theta_2 = \frac{4\pi\beta}{2\beta + \pi} = \frac{\pi}{\pi + \delta}\cdot(\pi + 2\delta). \tag{14.30}$$

If $-\frac{\pi}{2} - \delta < \theta_2 < -\frac{\pi}{2} + \frac{\delta^2}{\pi+\delta} < \frac{\pi}{2} + \delta$, then the inequality (14.30) yields

$$\theta_1 = \theta_2 + \frac{\pi}{\pi+\delta}\cdot(\pi + 2\delta) < -\frac{\pi}{2} + \frac{\delta^2}{\pi+\delta} + \frac{\pi}{\pi+\delta}\cdot(\pi+2\delta) = \frac{\pi}{2} + \delta.$$

Now we have found distinct $\theta_1, \theta_2 \in (-\beta, \beta)$, and hence distinct z_1 and z_2 of Ω such that the equation (14.29) holds. Consequently, our f is *not* one-to-one in Ω.

[c] In fact, it was shown by Tims [71, Theorem 2] that for any simply connected non-convex region Ω whose boundary contains more than one point, there exists distinct points $\alpha, \beta \in \Omega$ and an $f \in H(\Omega)$ such that $\operatorname{Re} f'(z)$ is nonzero in Ω and $f(\alpha) = f(\beta)$.

This completes the proof of the problem. ■

Problem 14.13

Rudin Chapter 14 Exercise 13.

Proof. Assume that $z_1 \neq z_2$ but $f(z_1) = f(z_2)$. Pick a $\delta > 0$ such that $z_2 \notin D(z_1;\delta)$ and $D(z_1;\delta) \subseteq \Omega$. Since every f_n is one-to-one in $D(z_1;\delta)$, none of the functions $f_n(z) - f_n(z_2)$ has a zero in $D(z_1;\delta)$. By the hypotheses, $f_n(z) - f_n(z_2) \to f(z) - f(z_2)$ uniformly on every compact subset in $D(z_1;\delta)$, so we deduces from Problem 10.20 (Hurwitz's Theorem) that either $f(z) - f(z_2) \neq 0$ for all $z \in D(z_1;\delta)$ or $f(z) - f(z_2) = 0$ in $D(z_1;\delta)$. Since $f(z_1) = f(z_2)$, we have $f(z) = f(z_2)$ in $D(z_1;\delta)$ and this means that

$$D(z_1;\delta) \subseteq Z\big(f - f(z_2)\big).$$

By Theorem 10.18, we conclude immediately that f is constant in Ω.

If $f_n(z) = \frac{1}{n}\mathrm{e}^z$ for every $n \in \mathbb{N}$, then each f_n is entire and one-to-one in $\mathbb{C}$. Since $f_n(z) \to 0$ pointwise in $\mathbb{C}$, we have $f \equiv 0$. On each compact set K of $\mathbb{C}$, since e^z is bounded on K, $f_n \to f$ uniformly on K. In this case, our limit function f is a constant. Next, we consider $f_n(z) = \mathrm{e}^{z+\frac{1}{n}}$ in $\mathbb{C}$, then each f_n is entire and one-to-one in $\mathbb{C}$. Besides, it is easily checked that $f_n \to f = \mathrm{e}^z$ uniformly on every compact subset of $\mathbb{C}$. In this case, we have $f(z) = \mathrm{e}^z$ which is one-to-one in $\mathbb{C}$. This completes the proof of the problem. ■

Problem 14.14

Rudin Chapter 14 Exercise 14.

Proof. Let's answer the questions step by step.

- **$f(x+iy) \to 0$ as $x \to \infty$ for all $y \in (-1,1)$.** Assume that one could find an $y' \in (-1,1) \setminus \{0\}$ such that the limit $\lim\limits_{x\to\infty} f(x+iy')$ is nonzero. Since $|f| < 1$, the Bolzano-Weierstrass Theorem ensures the existence of a sequence $\{x_n\}$ and a nonzero complex number L such that $x_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} f(x_n + iy') = L. \tag{14.31}$$

  Consider the family $\mathscr{F} = \{f_n\} \subseteq H(\Omega)$, where $f_n(z) = f(z + x_n)$. Now the boundedness of f implies that $\mathscr{F}$ is uniformly bounded on each compact subset of Ω. By Theorem 14.6 (Montel's Theorem), it is a normal family and then there exists a subsequence $\{n_k\}$ and an $F \in H(\Omega)$ such that $f_{n_k} \to F$ uniformly on every compact subset of Ω. It is clear that $\{x, iy'\}$ is compact. On the one hand, the hypothesis gives

$$F(x) = \lim_{k\to\infty} f_{n_k}(x) = \lim_{k\to\infty} f(x + x_{n_k}) = 0.$$

  On the other hand, it follows from the limit (14.31) that

$$F(iy') = \lim_{k\to\infty} f_{n_k}(iy') = \lim_{k\to\infty} f(x_{n_k} + iy') = L.$$

  Thus it is a contradiction and we obtain our desired result.

- **The passage to the limit is uniform if y is confined to $[-\alpha, \alpha]$, where $\alpha < 1$.** Assume that the limit was not uniform in $K_\alpha = \{x + iy \,|\, x \in \mathbb{R} \text{ and } y \in [-\alpha, \alpha]\}$ for some $\alpha < 1$. Then there exists some $\epsilon > 0$ so that for all $N \in \mathbb{N}$, one can find $x_N \geq N$ and $y_N \in [-\alpha, \alpha]$ such that
$$|f_N(iy_N)| = |f(x_N + iy_N)| > \epsilon. \tag{14.32}$$
As we have shown above that $\{f_N\}$ has a subsequence $\{f_{N_k}\}$ which converges uniformly on compact subsets of Ω to a holomorphic function g, and $g \not\equiv 0$ in view of the inequality (14.32). By the previous assertion, we have
$$f_{N_k}(x + iy) = f\big(x_{N_k} + (x + iy)\big) \to 0$$
as $k \to \infty$ for *every* $(x, y) \in [-\alpha, \alpha]^2 \subseteq K_\alpha \subseteq \Omega$, so this means that $g(z) = 0$ for all $z \in [-\alpha, \alpha]^2$ and then the Corollary following Theorem 10.18 implies that $g(z) = 0$ in Ω, a contradiction. Hence the limit must be uniform in K_α.

- **Boundary behavior of a function $g \in H^\infty$ with a radial limit.**[d] Let $g \in H^\infty$. Without loss of generality, we may assume that $|g(z)| < C_1$ for all $z \in U$ and $g(re^{i\theta}) \to C_2$ as $r \to 1$ for some θ, where C_1 and C_2 are some constants. Using the mapping (14.124), we know that
$$\kappa(z) = \frac{e^{\frac{\pi}{2}z} - 1}{e^{\frac{\pi}{2}z} + 1}$$
is a conformal one-to-one mapping of Ω onto U. Since $\kappa(x) \to 1$ as $x \to \infty$, the composite
$$h(z) = \frac{g\big(\kappa(z)e^{i\theta}\big) - C_2}{C_1 + C_2}$$
is a mapping from Ω into U satisfying $h \in H(\Omega)$, $|h(z)| < 1$ for all $z \in \Omega$ and
$$h(x) = \frac{g\big(\kappa(x)e^{i\theta}\big) - C_2}{C_1 + C_2} \to 0$$
as $x \to \infty$. Hence the first assertion implies that
$$\lim_{x\to\infty} h(x + iy) = 0 \quad \text{or} \quad \lim_{x\to\infty} g\big(\kappa(x + iy)e^{i\theta}\big) = C_2 \tag{14.33}$$
for every $y \in (-1, 1)$. We observe from the definition that $|\kappa(x+iy)| < 1$ and $\kappa(x+iy) \to 1$ as $x \to \infty$. By §11.21, the limit (14.33) means that g has non-tangential limit C_2 at $e^{i\theta}$. The analogue of the second assertion can be stated similarly and we omit the details here.

We complete the proof of the problem. ∎

14.2 Problems on Normal Families and the Class $\mathscr{S}$

Problem 14.15

Rudin Chapter 14 Exercise 15.

Proof. Let Π be the right half plane. Recall that the φ given by [62, Eqn. (6), p. 281] is a conformal one-to-one mapping of U onto Π. Thus $\varphi^{-1} : \Pi \to U$ is conformal and bijective. Since $\varphi^{-1}(z) = \frac{z-1}{z+1}$ and $f : U \to \Pi$ is holomorphic, we have $g = \varphi^{-1} \circ f : U \to U$ is holomorphic and
$$g(0) = \varphi^{-1}(f(0)) = \varphi^{-1}(1) = 0.$$

[d] Recall from Theorem 11.32 (Fatou's Theorem) that our g has radial limits almost everywhere on T.

According to Theorem 12.2 (Schwarz's Lemma), we always have

$$|g(z)| \leq |z| \tag{14.34}$$

for all $z \in U$. Thus the auxiliary family $\mathscr{G} = \{g = \varphi^{-1} \circ f \,|\, f \in \mathscr{F}\}$ is uniformly bounded on each compact subset of U. Particularly, Theorem 14.6 (Montel's Theorem) implies that $\mathscr{G}$ is a normal family.

Let K be a compact subset of U. Then there exists a constant $0 < R < 1$ such that $K \subseteq \overline{D(0;R)}$, so the inequality (14.34) gives $|g(z)| \leq R$ for every $g \in \mathscr{G}$ and all $z \in K$. Since φ is conformal, it is continuous on U. Therefore, $\varphi\big(\overline{D(0;R)}\big)$ is bounded by a positive constant M and then

$$f(z) = \varphi\big(g(z)\big) \in \varphi\big(\overline{D(0;R)}\big) \subseteq D(0;M) \tag{14.35}$$

for all $z \in K$. As g runs through $\mathscr{G}$, f runs through $\mathscr{F}$. Hence, it yields from the result (14.35) that $\mathscr{F}$ is uniformly bounded on K. Again, Theorem 14.6 (Montel's Theorem) implies that $\mathscr{F}$ is normal.

The condition "$f(0) = 1$" can be omitted or replaced by "$|f(0)| \leq 1$". In fact, we suppose that $\mathscr{F}' = \{f \in H(U) \,|\, \mathrm{Re}\, f > 0\}$ and the auxiliary family $\mathscr{G}' = \{g = \mathrm{e}^{-f} \,|\, f \in \mathscr{F}'\}$. It is evident that

$$|g(z)| = |\mathrm{e}^{-f(z)}| = \frac{1}{\mathrm{e}^{\mathrm{Re}\, f(z)}} \leq 1$$

for all $z \in U$, the family $\mathscr{G}'$ is uniformly bounded on U. Using similar argument as the previous paragraph, it can be shown that the family $\mathscr{F}'$ is also normal. This completes the proof of the problem. ∎

Problem 14.16

Rudin Chapter 14 Exercise 16.

Proof. Let $p \in U$. Then there exists a $R > 0$ such that $D(p; 2R) \subseteq U$. For every $z \in D(p;R)$, we have $\overline{D(p;r)} \subseteq U$ for all $0 < r \leq R$. Since $f \in H(U)$, it is harmonic in U by Theorem 11.4. By the mean value property, we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + r\mathrm{e}^{it})\,\mathrm{d}t$$

which implies

$$\begin{aligned} rf(z) &= \frac{1}{2\pi}\int_0^{2\pi} rf(z + r\mathrm{e}^{it})\,\mathrm{d}t \\ \int_0^R rf(z)\,\mathrm{d}r &= \frac{1}{2\pi}\int_0^R\int_0^{2\pi} rf(z + r\mathrm{e}^{it})\,\mathrm{d}t\,\mathrm{d}r \\ \frac{R^2}{2} f(z) &= \frac{1}{2\pi}\int_0^R\int_0^{2\pi} rf(z + r\mathrm{e}^{it})\,\mathrm{d}t\,\mathrm{d}r. \end{aligned} \tag{14.36}$$

Applying Theorem 3.5 (Hölder's Inequality) to the expression (14.36), we obtain

$$\begin{aligned} |f(z)| &= \frac{1}{\pi R^2}\Big|\int_0^R\int_0^{2\pi} rf(z + r\mathrm{e}^{it})\,\mathrm{d}t\,\mathrm{d}r\Big| \\ &\leq \frac{1}{\pi R^2}\Big\{\int_0^R\int_0^{2\pi} (\sqrt{r})^2\,\mathrm{d}r\,\mathrm{d}t\Big\}^{\frac{1}{2}} \cdot \Big\{\int_0^R\int_0^{2\pi} \big(\sqrt{r}|f(z + r\mathrm{e}^{it})|\big)^2\,\mathrm{d}r\,\mathrm{d}t\Big\}^{\frac{1}{2}} \end{aligned}$$

$$= \frac{1}{\pi R^2} \cdot \sqrt{\pi} R \cdot \left\{ \iint\limits_U |f(z)|^2 \,\mathrm{d}x\,\mathrm{d}y \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{\pi} R} \tag{14.37}$$

for all $z \in D(p; R)$. Let $K \subseteq \Omega$ be compact and $\{D(p; 2R)\}$ be an open cover of K, where $D(p; 2R) \subseteq U$. Then there exist finitely many points $p_1, p_2, \ldots, p_N$ and positive numbers $R_1, R_2, \ldots, R_N$ such that

$$K \subseteq D(p_1; R_1) \cup D(p_2; R_2) \cup \cdots \cup D(p_N; R_N).$$

If $R = \min(R_1, R_2, \ldots, R_N)$, then we conclude from the inequality (14.37) that $\mathscr{F}$ is uniformly bounded by $\frac{1}{\sqrt{\pi}R}$ on K. Hence Theorem 14.6 (Montel's Theorem) shows that $\mathscr{F}$ is a normal family and we complete the proof of the problem. ∎

Problem 14.17

Rudin Chapter 14 Exercise 17.

Proof. The conclusion is affirmative. To this end, we need the following version of Hurwitz's Theorem [18, p. 152]:

Lemma 14.3 (Hurwitz's Theorem)

Let Ω be a region and $f_n \in H(\Omega)$ for $n = 1, 2, \ldots$. If $\{f_n\}$ converges to f uniformly on compact subsets of Ω, $f \not\equiv 0$, $\overline{D(a; R)} \subseteq \Omega$ and $f(z) \neq 0$ on $C(a; R)$, then there is an $N \in \mathbb{N}$ such that f and f_n have the same number of zeros in $D(a; R)$ for all $n \geq N$.

Fix an $z_0 \in \Omega$ and let $a \in \Omega$. Then the functions $g_n = f_n - f_n(z_0)$ converge uniformly to $g = f - f(z_0)$ on compact subsets of Ω. Since f is one-to-one in Ω, $g(z) \neq 0$ on $C(a; r)$ for every $r > 0$ such that $\overline{D(a; r)} \subseteq \Omega$. By Lemma 14.3 (Hurwitz's Theorem), there corresponds an $N(a, r) \in \mathbb{N}$ such that g_n and g have the same number of zeros in $D(a; r)$ for all $n \geq N(a, r)$. In other words, we obtain

$$f_n(z) \neq f_n(z_0) \tag{14.38}$$

in $D(a; r) \setminus \{z_0\}$ and for all $n \geq N(a, r)$. Suppose that $K \subseteq \Omega$ is compact, $p \in K$ and $\{D(a; r)\}$ is an open covering of K, where each $\overline{D(a; r)}$ is a subset of Ω. Then we have

$$K \subseteq D(a_1; r_1) \cup D(a_2; r_2) \cup \cdots \cup D(a_m; r_m)$$

for some positive integer m. Let $N(K) = \max\{N(a_1, r_1), N(a_2, r_2), \ldots, N(a_m, r_m)\}$. It follows from the result (14.38) that

$$f_n(z) \neq f_n(p) \tag{14.39}$$

in $K \setminus \{p\}$ for all $n \geq N(K)$. Since p is an arbitrary point of K, the result (14.39) means that f_n is one-to-one in K for all $n \geq N(K)$, completing the proof of the problem.

∎

Problem 14.18

Rudin Chapter 14 Exercise 18.

Proof. Suppose that $f, g : \Omega \to U$ and $f(z_0) = g(z_0) = 0$. Then $F = g \circ f^{-1} : U \to U$ is a bijective conformal mapping such that

$$F(0) = g\big(f^{-1}(0)\big) = g(z_0) = 0 \quad \text{and} \quad F'(0) = \frac{g'(z_0)}{f'(z_0)}. \tag{14.40}$$

By [9, Theorem 13.15, p. 183], we know that

$$F(z) = e^{i\theta} \cdot \Big(\frac{z-\alpha}{1-\overline{\alpha}z}\Big) \tag{14.41}$$

for some $|\alpha| < 1$ and $\theta \in [0, 2\pi]$. By the conditions (14.40), we have $\alpha = 0$ and $e^{i\theta} = \frac{g'(z_0)}{f'(z_0)}$ which imply that

$$g(z) = \frac{g'(z_0)}{f'(z_0)} f(z) = \frac{g'(z_0)}{f'(z_0)} \varphi_0\big(f(z)\big)$$

for all $z \in \Omega$.

For the case that $f(z_0) = g(z_0) = a$, the conditions (14.40) are replaced by

$$F(a) = a \quad \text{and} \quad F'(a) = \frac{g'(z_0)}{f'(z_0)}.$$

By the form (14.41) again, we can show that $e^{i\theta} = \frac{g'(z_0)}{f'(z_0)} \cdot \frac{(1-\overline{\alpha}a)^2}{1-|\alpha|^2}$ which gives

$$g(z) = \frac{g'(z_0)}{f'(z_0)} \cdot \frac{(1-\overline{\alpha}a)^2}{1-|\alpha|^2} \varphi_\alpha\big(f(z)\big)$$

for all $z \in \Omega$. This ends the proof of the problem. ∎

Problem 14.19

Rudin Chapter 14 Exercise 19.

Proof. We claim that the mapping given by

$$f(z) = z \exp\Big(\frac{i}{1-|z|}\Big) \tag{14.42}$$

is a homeomorphism of U onto U. If we consider $z = re^{i\theta}$ and represent the function (14.42) as

$$f(r, \theta) = \Big(r, \theta + \frac{1}{1-r}\Big), \tag{14.43}$$

then the inverse function f^{-1} is given by

$$f^{-1}(r, \theta) = \Big(r, \theta - \frac{1}{1-r}\Big).$$

Now it is easy to see that both f and f^{-1} are continuous and bijective. In other words, f is a homeomorphism. However, this homeomorphism cannot be extended to $\overline{U}$ continuously. Otherwise, we assume that $F : \overline{U} \to \overline{U}$ was a continuous extension to f, i.e., $F = f$ on U. Therefore, we have $F\big(C(0;1)\big) = C(0;1)$. Take $F(e^{i\theta'}) = (1, 0)$ for some $\theta' \in [0, 2\pi]$ and V *any* neighborhood of $(1, 0)$. Suppose that $i \notin V$. Then there exists a neighborhood W of $e^{i\theta'}$ such that $F(W) \subseteq V$.[e] Indeed, we can find a sequence $\{r_n e^{i\theta'}\}$ in W such that $r_n < 1$ for all $n \in \mathbb{N}$,

$$r_n \to 1 \quad \text{and} \quad F(r_n e^{i\theta'}) = f(r_n, \theta') \to (1, 0)$$

[e] See, for example, [42, Theorem 18.1, p. 104].

as $n \to \infty$. Using the formula (14.43), we have

$$f(r_n, \theta') = \left(r_n, \theta' + \frac{1}{1-r_n}\right) = r_n \exp\left[i\left(\theta' + \frac{1}{1-r_n}\right)\right]. \tag{14.44}$$

Since the function $s(r) = \theta' + \frac{1}{1-r}$ is continuous on $[0,1)$ and $s\big([0,1)\big) = [\theta'+1, \infty)$, if we take the points $\theta' + \frac{1}{1-r_n} = (2n+1)\pi$ for all large n, then we deduce from the formula (14.44) that

$$f(r_n, \theta') = i\Big[1 - \frac{1}{(2n+1)\pi - \theta'}\Big]$$

which means that $f(r_n, \theta')$ cannot be contained in the neighborhood V of $(1,0)$ for all large n. Hence no such continuous extension exists and we complete the proof of the problem. ∎

Problem 14.20

Rudin Chapter 14 Exercise 20.

Proof. Write $f(z) = z\varphi(z)$. Then $\varphi \in H(U)$, $\varphi(0) = 1$ and φ has no zero in U. By Problem 13.9, there exists an $h \in H(U)$ such that $h^n(z) = \varphi(z)$ and $h(0) = 1$. Put

$$g(z) = zh(z^n) \tag{14.45}$$

in U. Then we know that

$$g^n(z) = z^n h^n(z^n) = z^n \varphi(z^n) = f(z^n) \tag{14.46}$$

for every $z \in U$. It is clear that $g(0) = 0$ and $g'(0) = h(0) = 1$.

To prove $g \in \mathscr{L}$, it suffices to show that g is one-to-one in U. Suppose that $z, \omega \in U$ and $g(z) = g(\omega)$. Since f is one-to-one in U, the formula (14.46) ensures that $z^n = \omega^n$ or equivalently,

$$z = \mathrm{e}^{\frac{2k\pi i}{n}}\omega,$$

where $k = 0, 1, \ldots, n-1$. Put this into the equation (14.45) to get

$$g(z) = \mathrm{e}^{\frac{2k\pi i}{n}}\omega h(\mathrm{e}^{2k\pi i}\omega^n) = \mathrm{e}^{\frac{2k\pi i}{n}} \cdot \big[\omega h(\omega^n)\big] = \mathrm{e}^{\frac{2k\pi i}{n}} g(\omega). \tag{14.47}$$

Recall that $g(z) = g(\omega)$, so it follows from the equation (14.47) that we have $g(z) = g(\omega) = 0$ or $\mathrm{e}^{\frac{2k\pi i}{n}} = 1$. In the latter case, we have $z = w$ Otherwise, since $g(z) = 0$ if and only if $z = 0$ in U, we have $z = w = 0$, completing the proof of the problem. ∎

Problem 14.21

Rudin Chapter 14 Exercise 21.

Proof. By Definition 14.10, we have

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

(a) Since $f \in \mathscr{S}$, it is one-to-one. By Theorems 10.33 and 14.2, f is conformal and its inverse $f^{-1} : f(U) \to U$ exists and is conformal.[f] As $U \subseteq f(U)$, we consider $g = f^{-1}|_U : U \to U$. Clearly, we have

$$g(0) = f^{-1}(0) = 0 \quad \text{and} \quad g'(0) = \frac{1}{f'\big(f^{-1}(0)\big)} = \frac{1}{f'(0)} = 1.$$

By Theorem 12.2 (Schwarz's Lemma), we conclude that $g(z) = z$ and consequently, $f(z) = z$ in U.

(b) We claim that there is *no* element $f \in \mathscr{S}$ with $\overline{U} \subseteq f(U)$. Assume that f was such a function. By part (a), we know that $f(z) = z$ so that $f(U) = U$ which contradicts our assumption that $\overline{U} \subseteq f(U)$.

(c) Consider the function F given in [62, Eqn. (1), p. 286]. If $|\alpha_1| = 1$, then Theorem 14.13 (The Area Theorem) implies that $\alpha_n = 0$ for all $n = 2, 3, \ldots$. In this case, we have

$$F(z) = \frac{1}{z} + \alpha_0 + e^{i\theta} z.$$

Now we know from the proof of Theorem 14.14 that $|a_2| = 2$ is equivalent to $|\alpha_1| = 1$, so we have

$$\frac{1}{g(z)} = G(z) = \frac{1}{z} + \alpha_0 + e^{i\theta} z.$$

By Theorem 14.12, we have $f(z^2) = g^2(z)$ which implies definitely that $\alpha_0 = 0$ and then

$$f(z^2) = \frac{z^2}{(1 + e^{i\theta} z^2)^2}.$$

Consequently, f must be in the form

$$f(z) = \frac{z}{(1 + e^{i\theta} z)^2}.$$

We complete the proof of the problem. ■

Problem 14.22

Rudin Chapter 14 Exercise 22.

Proof. Let $f : U \to S$ be a one-to-one conformal mapping, where S is a square with center at 0. Now it is clear that both $if : U \to S$ and $f^{-1} : S \to U$ are conformal. We consider

$$g = f^{-1} \circ (if) : U \to U$$

which is also a one-to-one conformal mapping of U onto itself. Clearly, $g(0) = f^{-1}\big(if(0)\big) = 0$. By Remark 10.3, we have

$$g'(z) = \frac{if'(z)}{f'\big(if(z)\big)}$$

which gives $|g'(0)| = |i| = 1$. Hence Theorem 12.2 (Schwarz's Lemma) implies that the formula

$$if(z) = f(\lambda z) \tag{14.48}$$

[f] Or it can be seen directly from [9, Theorem 13.8, p. 174].

holds in U for some constant λ with $|\lambda| = 1$. By the note following Theorem 14.2, we have $f'(0) \neq 0$. Now we observe from this fact and the expression (14.48) that $if'(z) = \lambda f'(\lambda z)$ so that $\lambda = i$ if we put $z = 0$ into this equation. Thus we get what we want

$$if(z) = f(iz) \tag{14.49}$$

for every $z \in U$. Suppose that $f(z) = \sum c_n z^n$. By the expression (14.49), we have

$$\sum_{n=1}^{\infty} ic_n(1 - i^{n-1})z^n = 0$$

for all $z \in U$. If $n-1$ is not a multiple of 4, then $i^{n-1} \neq 1$, so it yields from Theorem 10.18 that $c_n = 0$ for such n.

A generalization is as follows: Let S be a simply connected region with rotational symmetry of order N and center at 0, i.e., $\exp(\frac{2\pi i}{N})S = S$. Let $f : U \to S$ be a one-to-one conformal mapping with $f(0) = 0$. Thus both $\exp(\frac{2\pi i}{N})f : U \to S$ and $f^{-1} : S \to U$ are conformal. By considering $g = f^{-1} \circ [\exp(\frac{2\pi i}{N})f] : U \to U$ which is clearly a one-to-one conformal mapping of U onto itself. Obviously, we have

$$g(0) = f^{-1}\Big(\exp\Big(\frac{2\pi i}{N}\Big)f(0)\Big) = 0.$$

We observe from Remark 10.3 that

$$g'(z) = \frac{\exp(\frac{2\pi i}{N})f'(z)}{f'\big(\exp(\frac{2\pi i}{N})f(z)\big)},$$

so $|g'(0)| = 1$. Hence Theorem 12.2 (Schwarz's Lemma) implies that

$$e^{\frac{2\pi i}{N}} f(z) = f(\lambda z) \tag{14.50}$$

holds for some constant λ with $|\lambda| = 1$. Since $f'(0) \neq 0$, we take differentiation to both sides of the formula (14.50) to conclude that $\lambda = \exp(\frac{2\pi i}{N})$ and hence

$$e^{\frac{2\pi i}{N}} f(z) = f\big(e^{\frac{2\pi i}{N}} z\big) \tag{14.51}$$

for every $z \in U$. Next, if $n-1$ is not a multiple of N and $f(z) = \sum c_n z^n$, then the formula (14.51) gives

$$\sum_{n=1}^{\infty} e^{\frac{2\pi i}{N}} c_n \big[1 - e^{\frac{2\pi(n-1)i}{N}}\big] z^n = 0$$

for all $z \in U$. Since $e^{\frac{2\pi(n-1)i}{N}} \neq 1$ if $n-1$ is not a multiple of N, Theorem 10.18 implies that $c_n = 0$ for such n. This completes the analysis of the problem. ∎

14.3 Proofs of Conformal Equivalence between Annuli

Problem 14.23

Rudin Chapter 14 Exercise 23.

Proof. Suppose that $\partial\Omega = C_1 \cup C_2$, where C_1 lies in the inside of C_2. By appropriate translation, rotation and homothety, we assume that $C_2 = T$ (the unit circle) and the center of C_1 lies on the real axis with x-intercepts a and b, where $|a| < b < 1$. See Figure 14.1 below:

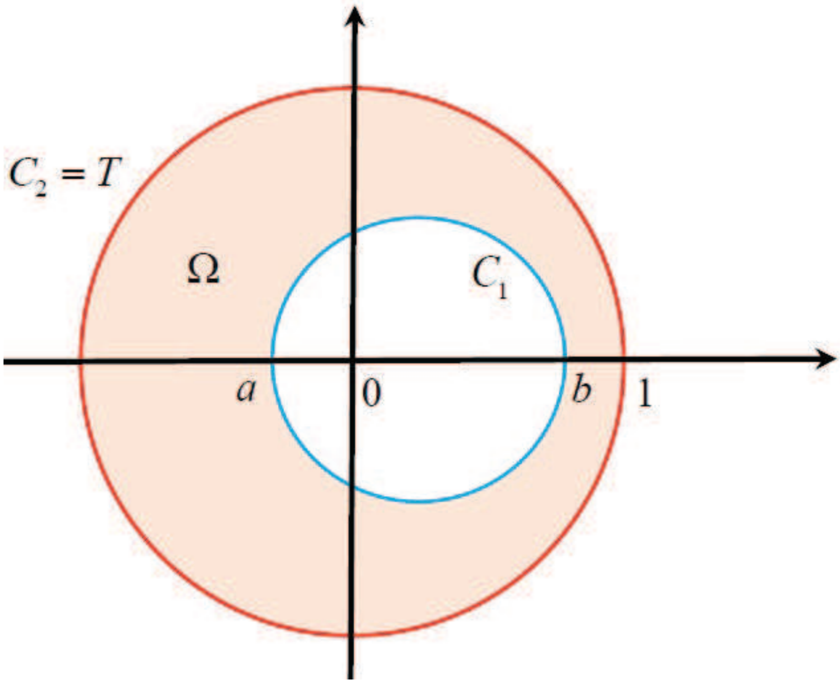


Figure 14.1: The region Ω bounded by C_1 and C_2.

By Theorem 12.4, φ_α carries T onto itself and U onto U, where $|\alpha| < 1$. Since φ_α is a linear fractional transformation, §14.3 ensures that it will send $C_1 \subset U$ onto a circle or a line. Now the condition $\varphi_\alpha(U) = U$ implies that $\varphi_\alpha(C_1)$ must be a circle.

Suppose that $\alpha \in \mathbb{R}$. Since C_1 intersects the real axis at a and b *perpendicularly* and φ_α is a conformal map, $\varphi_\alpha(a)$ and $\varphi_\alpha(b)$ are the end-points of a diameter of $\varphi_\alpha(C_1)$. Furthermore, since $\varphi_\alpha(\mathbb{R}) = \mathbb{R}$, both $\varphi_\alpha(a)$ and $\varphi_\alpha(b)$ are real. Thus if $\varphi_\alpha(C_1)$ is a circle centered at 0, then we must have

$$\varphi_\alpha(a) = -\varphi_\alpha(b)$$

which gives

$$\begin{aligned} \frac{a-\alpha}{1-\alpha a} &= -\frac{b-\alpha}{1-\alpha b} \\ \alpha^2 - \frac{2(1+ab)}{a+b}\alpha + 1 &= 0. \end{aligned}$$

Solving this equation to get

$$\alpha_\pm = \frac{1+ab}{a+b} \pm \sqrt{\left(\frac{1+ab}{a+b}\right)^2 - 1}. \tag{14.52}$$

Since $|a| < b < 1$, we always have $a(1-b) < 1-b$ or equivalently, $1+ab > a+b > 0$. Combining this and the formulas (14.52), it follows that

$$\alpha_+ = \frac{1+ab}{a+b} + \sqrt{\left(\frac{1+ab}{a+b}\right)^2 - 1} > 1.$$

Since $\alpha_+ \cdot \alpha_- = 1$, we conclude that $0 < \alpha_- < 1$. Take this α_-. Then the conformal map φ_{α_-} carries U onto U, T onto T and C_1 onto $C\big(0; \varphi_{\alpha_-}(a)\big)$. In other words, it is a one-to-one conformal mapping of Ω onto $A\big(\varphi_{\alpha_-}(a), 1\big)$. This completes the proof of the problem. ■

Remark 14.2

An example of this kind of conformal mappings can be found in [76, Problem 13.20, pp. 182, 183].

Problem 14.24

Rudin Chapter 14 Exercise 24.

Proof. Since $1 < R_2 < R_1$, we have $A(1, R_2) \subseteq A(1, R_1)$. Assume that $f : A(1, R_1) \to A(1, R_2)$ was a bijective conformal mapping. By the first half of the proof of Theorem 14.22, we may assume without loss of generality that $|f(z)| \to 1$ as $|z| \to 1$ and $|f(z)| \to R_2$ as $|z| \to R_1$. Consider the family of holomorphic functions $\mathscr{F} = \{f_n\}$, where

$$f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n : A(1; R_1) \to A(1; R_2)$$

for all $n \in \mathbb{N}$. Thus each f_n is bijective.

By the definition, we have

$$f\big(A(1, R_1)\big) = A(1, R_2) \subseteq A(1, R_1) \quad \text{and} \quad f_n\big(A(1, R_1)\big) \subseteq A(1, R_1) \tag{14.53}$$

for every $n = 1, 2, \ldots$, so the family $\mathscr{F}$ is uniformly bounded on compact subsets of $A(1, R_1)$. Therefore, it follows from Theorem 14.6 (Montel's Theorem) that $\mathscr{F}$ is normal and then there exists a subsequence $\{f_{n_k}\}$ converging uniformly on compact subsets of $A(1, R_1)$ to a holomorphic function

$$g : A(1, R_1) \to A(1, R_2).$$

Denote $\Omega = A(R_2, R_1)$. We claim that

$$f_n(\Omega) \cap f_m(\Omega) = \varnothing, \tag{14.54}$$

where $n, m \in \mathbb{N}$ and $n \neq m$. Fix m. Assume that $\omega \in f_m(\Omega) \cap f_n(\Omega)$. Then we can find $p, q \in \Omega$ such that $f_n(p) = \omega$ and $f_m(q) = \omega$. If $n < m$, then we have

$$f_n(p) = f_m(q) = f_n\big(f_{m-n}(q)\big)$$

which gives $f_{m-n}(q) = p \in \Omega$, but it contradicts the fact that $f_{m-n}\big(A(1, R_1)\big) \cap \Omega = \varnothing$. If $n > m$, then we have

$$f_m(q) = f_n(p) = f_m\big(f_{n-m}(p)\big)$$

which gives $f_{n-m}(p) = q \in \Omega$, a contradiction again. Consequently, we prove the claim (14.54).

Assume that the range of g contained a nonempty open set. This means that g is not constant and so we pick an $p \in \Omega \subseteq A(1, R_1)$ such that $g(p) = \omega$. By the Open Mapping Theorem, $g\big(A(1, R_1)\big)$ is an open set so that there exists a $\delta > 0$ such that $0 \notin g\big(D(p; \delta)\big)$. Suppose that

$$h_k(z) = f_{n_k}(z) - \omega \quad \text{and} \quad h(z) = g(z) - \omega.$$

We observe that $h_k, h \in H\big(A(1, R_1)\big)$, $\{h_k\}$ converges to h uniformly on compact subsets of $A(1, R_1)$ and $h \not\equiv 0$. Furthermore, we can select δ if necessary so that $h(z) \neq 0$ on $C(p; \delta)$.[g] Hence it follows from Lemma 14.3 (Hurwitz's Theorem) that there corresponds an $N \in \mathbb{N}$ such

[g] Otherwise, $h(z) = 0$ for all $z \in A(1, R_1)$ by Theorem 10.18 which means that g is constant.

that if $k \geq N$, then h_k and h have the same number of zeros in $D(p;\delta)$. Since $h(p) = g(p) - \omega = 0$, there exists an $z_k \in D(p;\delta) \subseteq \Omega$ such that $k \geq N$ implies

$$f_{n_k}(z_k) = \omega$$

but this contradicts the fact (14.54). Therefore, the range of g cannot contain any nonempty open set and the Open Mapping Theorem ensures that g is constant.

On the other hand, g cannot be constant on the circle $C(0;\sqrt{R_1})$. Otherwise, Theorem 10.24 (The Maximum Modulus Theorem) and its Corollary establish that g is constant in $A(1,\sqrt{R_1})$. Let K be a compact subset of $A(1,\sqrt{R_1})$. Since $f_{n_k} \to g$ uniformly on K, f_{n_k} is also constant in K for large enough k. However, this contradicts the fact that f_n is injective for every $n = 1, 2, \ldots$. Therefore, g is not constant on $C(0;\sqrt{R_1})$.

Now the above two results are contrary, so they force that no such f exists and we have completed the proof of the problem. ■

Problem 14.25

Rudin Chapter 14 Exercise 25.

Proof. We have $f : A(1,R_1) \to A(1,R_2)$, where $1 < R_2 < R_1$. By the first half of the proof of Theorem 14.22, we may assume without loss of generality that

$$\lim_{|z|\to 1} |f(z)| = 1 \quad \text{and} \quad \lim_{|z|\to R_1} |f(z)| = R_2.$$

Applying Problem 14.2(b), the reflection across the inner circle (i.e., the unit circle) extends f to a conformal mapping

$$f_1 : A(R_1^{-1}, R_1) \to A(R_2^{-1}, R_2)$$

and f_1 satisfies

$$\lim_{|z|\to R_1^{-1}} |f_1(z)| = R_2^{-1}, \quad \lim_{|z|\to 1} |f_1(z)| = 1 \quad \text{and} \quad \lim_{|z|\to R_1} |f_1(z)| = R_2.$$

Next, by considering the function

$$g_1(z) = \frac{f_1(R_1 z)}{R_2},$$

we see that g_1 is holomorphic in the region $\Omega = \{z \in \mathbb{C} \mid R_1^{-2} < |z| < 1\}$ such that $|g_1(z)| \to 1$ as $|z| \to 1$ and $|g(z)| \to R_2^{-2}$ as $|z| \to R_1^{-2}$. Thus Problem 14.2(b) may be applied to extend g_1 to a conformal mapping $f_2 : A(R_1^{-2}, R_1^2) \to A(R_2^{-2}, R_2^2)$ and f_2 satisfies

$$\lim_{|z|\to R_1^{-2}} |f_2(z)| = R_2^{-2}, \quad \lim_{|z|\to 1} |f_2(z)| = 1 \quad \text{and} \quad \lim_{|z|\to R_1^2} |f_2(z)| = R_2^2.$$

This process can be repeated infinitely many times and finally we obtain a conformal mapping F of the punctured plane $\mathbb{C} \setminus \{0\}$. If F has a pole of order m at the origin, then $z^m F(z)$ is entire and $|F(z)| = 1$ whenever $|z| = 1$. Now Problem 12.4(b) asserts that $F(z) = \alpha z^n$ for some $|\alpha| = 1$ and some $n \in \mathbb{Z} \setminus \{0\}$. Assume that $n \geq 2$. Let $a \in A(1,R_1)$. Take $\zeta \neq 1$ to be an n-root of unity. Then we have

$$F(a) = \alpha a^n = \alpha(\zeta a)^n = F(\zeta a), \tag{14.55}$$

but $a \neq \zeta a$. This contradicts the fact that F is one-to-one in $A(1,R_1)$. Assume that $n \leq -1$. The equation (14.55) also holds in this case which in turn contradicts the injectivity of F again. In other words, $F(z) = \alpha z$ which implies that $R_1 = R_2$, a contradiction. Next, if F has a removable singularity at the origin, then F is actually entire and we use the same argument as above to show that no such F exists. Hence no such mapping f exists which completes the proof of the problem. ■

Remark 14.3

(a) Theorem 14.22 is sometimes called **Schottky's Theorem**. Besides the proofs given in the text (Problems 14.24 and 25), you can also find a simple and elegant proof ofs this theorem in [6].

(b) Besides the analytical proofs provided in the text and the problems, one can find a pure algebraic proof in [58].

14.4 Constructive Proof of the Riemann Mapping Theorem

Problem 14.26

Rudin Chapter 14 Exercise 26.

Proof.

(a) Suppose that the regions $\Omega_0, \Omega_1, \ldots, \Omega_{n-1}$ and functions $f_1, f_2, \ldots, f_n$ are constructed such that $\Omega_k = f_j(\Omega_{k-1})$, where $k = 1, 2, \ldots, n$. Define

$$r_n = \inf\{|z| \mid z \in \mathbb{C} \setminus \Omega_{n-1}\}. \tag{14.56}$$

Then r_n is the largest number such that $D(0; r_n) \subseteq \Omega_{n-1}$ and the definition shows that there is an $\alpha_n \in \partial\Omega_{n-1}$ with $|\alpha_n| = r_n$.[h] See Figure 14.2 below.

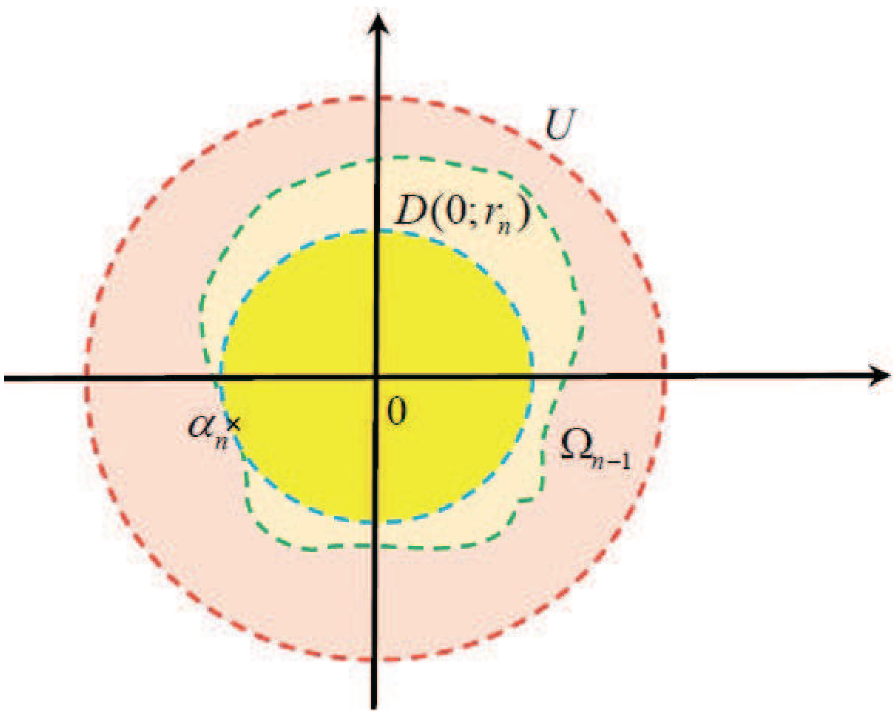


Figure 14.2: The constructions of Ω_{n-1}, $D(0; r_n)$ and α_n.

[h]Geometrically, α_n is a point on $\partial\Omega_{n-1}$ nearest the origin.

Choose $\beta_n^2 = -\alpha_n$ and put

$$F_n = \varphi_{-\alpha_n} \circ s \circ \varphi_{-\beta_n} : U \to U, \tag{14.57}$$

where

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z} \quad \text{and} \quad s(\omega) = \omega^2.$$

By the Chain Rule, we have

$$\begin{aligned} F_n'(z) &= \varphi_{-\alpha_n}'\big(s(\varphi_{-\beta_n}(z))\big) \times s'\big(\varphi_{-\beta_n}(z)\big) \times \varphi_{-\beta_n}'(z) \\ &= 2\varphi_{-\alpha_n}'\big(s(\varphi_{-\beta_n}(z))\big) \times \varphi_{-\beta_n}'(z) \times \varphi_{-\beta_n}(z). \end{aligned}$$

By Theorems 10.33 and 12.4, $\varphi_{-\alpha_n}'(z) \neq 0$ and $\varphi_{-\beta_n}'(z) \neq 0$ for every $z \in \Omega_{n-1}$. Thus $F_n'(z) \neq 0$ for all $z \in \Omega_{n-1}$ and it follows from Theorem 10.30(c) that F_n has a holomorphic inverse G_n in Ω_{n-1}.

(b) By the definition and Theorem 12.4, we may write

$$G_n = \varphi_{-\beta_n}^{-1} \circ s^{-1} \circ \varphi_{-\alpha_n}^{-1} = \varphi_{\beta_n} \circ s^{-1} \circ \varphi_{\alpha_n} : \Omega_{n-1} \to U, \tag{14.58}$$

where $s^{-1}(z) = \sqrt{z}$. Combining the Chain Rule and Theorem 12.4, we get

$$\begin{aligned} G_n'(0) &= \varphi_{\beta_n}'\big(s^{-1}(\varphi_{\alpha_n}(0))\big) \times (s^{-1})'\big(\varphi_{\alpha_n}(0)\big) \times \varphi_{\alpha_n}'(0) \\ &= \varphi_{\beta_n}'\big(s^{-1}(-\alpha_n)\big) \times \frac{1}{2\sqrt{-\alpha_n}} \times (1-|\alpha_n|^2) \\ &= \frac{1-r_n^2}{2\beta_n} \times \varphi_{\beta_n}'(\beta_n) \\ &= \frac{1-r_n^2}{2(1-|\beta_n|^2)\beta_n}. \end{aligned} \tag{14.59}$$

Put $f_n = \lambda_n G_n : \Omega_{n-1} \to U$, where $\lambda_n = \frac{|G_n'(0)|}{G_n'(0)}$. Now the formula (14.59) implies that

$$f_n'(0) = \lambda_n G_n'(0) = |G_n'(0)| = \frac{1-r_n^2}{2(1-|\beta_n|^2)\cdot|\beta_n|} = \frac{1-r_n^2}{2(1-r_n)\sqrt{r_n}} = \frac{1+r_n}{2\sqrt{r_n}}.$$

Using the A.M. $\geq$ G.M., it is easy to see that $f_n'(0) > 1$.

(c) We prove the assertions one by one.

- **Each $\psi_n : \Omega \to \Omega_n$ is bijective.** By the definition, we have $\psi_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$. Since $\Omega = \Omega_0$ and $f_n(\Omega_{n-1}) = \Omega_n \subseteq U$, we have

$$\psi_n(\Omega) = f_n(f_{n-1}(\cdots f_1(\Omega_0))) = f_n(f_{n-1}(\cdots f_2(\Omega_1))) = f_n(\Omega_{n-1}) = \Omega_n \subseteq U.$$

By the representation (14.58) and Theorem 12.4, each G_n and hence each f_n is injective on Ω_{n-1}. Consequently, each

$$\psi_n : \Omega \to \Omega_n \subseteq U$$

is injective on Ω.

- **$\{\psi_n'(0)\}$ is bounded.** Since $\varphi_{-\alpha_n}, \varphi_{-\beta_n}, s \in H(U)$, the definition (14.57) gives $F_n \in H(U)$. Furthermore, it is clear that

$$F_n(0) = \varphi_{-\alpha_n}\big(\varphi_{-\beta_n}^2(0)\big) = \varphi_{-\alpha_n}(\beta_n^2) = \varphi_{-\alpha_n}(-\alpha_n) = 0, \tag{14.60}$$

so we know from Theorem 12.2 (Schwarz's Lemma) that

$$|F_n(\omega)| \leq |\omega| \quad \text{and} \quad |F_n'(0)| \leq 1 \tag{14.61}$$

for all $\omega \in U$. Put $\omega = G_n(z)$ into the inequality (14.61), we get that

$$|G_n(z)| \geq |z| \quad \text{and} \quad |G_n'(0)| \geq 1 \tag{14.62}$$

hold for all $z \in \Omega_{n-1}$.

Lemma 14.4

For every $n = 1, 2, \ldots$, we have $0 < r_1 \leq r_2 \leq \cdots \leq 1$.

Proof of Lemma 14.4. Notice that we have $f_n(\Omega_{n-1}) = \Omega_n$, $D(0; r_n) \subseteq \Omega_{n-1}$ and $D(0; r_{n+1}) \subseteq \Omega_n$. By the definition (14.56), there exists a boundary point $\alpha \in \partial\Omega_n$ such that $|\alpha| = r_{n+1}$. Select $\{z_{n-1,k}\} \subseteq \Omega_{n-1}$ such that $f_n(z_{n-1,k}) \to \alpha$ as $k \to \infty$. Assume that $\{z_{n-1,k}\}$ had a limit point β in Ω_{n-1}. Since f_n is obviously continuous on Ω_{n-1}, we have $\alpha = f_n(\beta)$ which means that $\Omega_n \cap \partial\Omega_n \neq \varnothing$, a contradiction. Therefore, the sequence $\{z_{n-1,k}\}$ cannot have a limit point in Ω_{n-1}. Since $D(0; r_n) \subseteq \Omega_{n-1}$, we must have

$$\limsup_{k\to\infty} |z_{n-1,k}| \geq r_n.$$

Since $|f_n(z)| = |G_n(z)|$, we derive from the first inequality (14.62) that

$$r_{n+1} = |\alpha| = \lim_{k\to\infty} |f_n(z_{n-1,k})| = \lim_{k\to\infty} |G_n(z_{n-1,k})| \geq \limsup_{k\to\infty} |z_{n-1,k}| \geq r_n$$

as desired. This ends the proof of the lemma. ■

Now Lemma 14.4 ensures that $D(0; r_1)$ lies in every Ω_n. Since $\phi(z) = \frac{z}{r_1}$ maps $D(0; r_1)$ one-to-one and onto U, the map $\Psi_n = \psi_n \circ \phi^{-1} : U \to U$ satisfies the hypotheses of Theorem 12.2 (Schwarz's Lemma) so that $|\Psi_n'(0)| \leq 1$ which means that

$$|\psi_n'(0)| \leq \frac{1}{r_1} \tag{14.63}$$

for every $n = 1, 2, \ldots$. In other words, $\{\psi_n'(0)\}$ is bounded.

– **A formula of $\psi_n'(0)$.** By the aid of the Chain Rule and part (b), we establish easily that

$$\psi_n'(0) = f_n'(0) \times f_{n-1}'(0) \times \cdots \times f_1'(0) = \prod_{k=1}^{n} \frac{1 + r_k}{2\sqrt{r_k}}. \tag{14.64}$$

– **The sequence $\{r_n\}$ converges to 1.** For $m > n \geq 1$, we define

$$\psi_{m,n} = f_m \circ f_{m-1} \circ \cdots \circ f_{n+1}$$

which is holomorphic in Ω_n and hence on $D(0; r_{n+1})$. In view of the value (14.60), we know that $G_n(0) = 0$ and then $f_n(0) = 0$ for every $n \in \mathbb{N}$. Recalling the fact $\psi_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$, we therefore have $\psi_n(0) = 0$ for every $n \in \mathbb{N}$ which implies $\psi_{m,n}(0) = 0$. Using similar attack as in proving the inequality (14.63),[i] we can obtain

$$|\psi_{m,n}'(0)| \leq \frac{1}{r_{n+1}}.$$

[i]That is, $D(0; r_{n+1})$ lies in every Ω_m for all $m \geq n+1$, the map $\Psi_{m,n} = \psi_{m,n} \circ \phi^{-1} : U \to U$ satisfies the hypotheses of Theorem 12.2 (Schwarz's Lemma), where $\phi(z) = r_{n+1}^{-1} z$ maps $D(0; r_{n+1})$ one-to-one and onto U.

Now the Chain Rule and part (b) assert that

$$|\psi'_{m,n}(0)| = \prod_{k=n+1}^{m} |f'_k(0)| = \prod_{k=n+1}^{m} \frac{1+r_k}{2\sqrt{r_k}}$$

so that

$$1 < \prod_{k=n+1}^{m} \frac{1+r_k}{2\sqrt{r_k}} \leq \frac{1}{r_{n+1}} \tag{14.65}$$

which implies the convergence of

$$\prod_{n=1}^{\infty} \frac{1+r_n}{2\sqrt{r_n}}$$

by taking $m \to \infty$. By a basic fact about the convergence theory of infinite products [9, Note 1, p. 242], we see that

$$\lim_{n\to\infty} \frac{1+r_n}{2\sqrt{r_n}} = 1.$$

Combining this fact and the identity

$$\frac{1+r_n}{2\sqrt{r_n}} - 1 = \frac{(1-\sqrt{r_n})^2}{2\sqrt{r_n}},$$

we conclude immediately that $r_n \to 1$ as $n \to \infty$.

(d) We verify the parts one by one.

- $|h_n| \leq |h_{n+1}|$ **for each** $n \in \mathbb{N}$**.** By the definition of ψ_n, we have $\psi_{n+1}(z) = f_{n+1}\big(\psi_n(z)\big)$ so that
$$zh_{n+1}(z) = f_{n+1}\big(zh_n(z)\big). \tag{14.66}$$
Applying the first inequality (14.62) to the right-hand side of the expression (14.66), we assert that
$$|zh_{n+1}(z)| \geq |zh_n(z)|$$
which implies our expected result immediately.
- $\psi_n \to \psi$ **converges uniformly on compact subsets of** Ω**.** Recall from part (c) that each ψ_n is injective on Ω, so every h_n is zero-free in Ω and we can define $g_n = \log h_n$ as a holomorphic function in Ω. Define $u_n = \operatorname{Re} g_n = \log|h_n|$ which is harmonic in Ω. Since each $\psi_n(\Omega)$ is a subset of U, $|h_n(z_0)| < \infty$ for some $z_0 \in \Omega\setminus\{0\}$ and all $n \in \mathbb{N}$. Now the inequality $|h_n| \leq |h_{n+1}|$ guarantees that
$$u_1 \leq u_2 \leq \cdots .$$
According to Theorem 11.11 (Harnack's Theorem), we see immediately that $\{u_n\}$ converges uniformly on compact subsets of Ω. Since $\psi'_n(z) = h_n(z) + zh'_n(z)$, the second fact in part (c) assures us that $\{h_n(0)\}$ is bounded and therefore $\{h_n(0)\}$ converges. Consequently, $\{g_n(0)\}$ converges and we conclude from Problem 11.8 that $\{g_n\}$ converges uniformly on compact subsets of Ω to g. By the definition, we know that
$$\psi_n(z) = z\mathrm{e}^{g_n(z)},$$
so $\{\psi_n\}$ also converges uniformly on compact subsets of Ω to ψ. In view of Theorem 10.28, it is true that $\psi \in H(\Omega)$.

- **The map $\psi : \Omega \to U$ is surjective.** Let $\omega \in U$. By Lemma 14.4 and the fact $r_n \to 1$ as $n \to \infty$, we may select an $N \in \mathbb{N}$ such that

$$|\omega| < r_n$$

for all $n \geq N$. Therefore, for $p \geq 1$, we have

$$\begin{aligned}\omega &\in D(0, r_{N+p}) \\ &\subseteq \Omega_{N+p-1} \\ &= f_{N+p-1}(\Omega_{N+p-2}) \\ &= \cdots \\ &= f_{N+p-1} \circ f_{N+p-2} \circ \cdots \circ f_{N+1} \circ f_N(\Omega_{N-1}) \\ &= f_{N+p-1} \circ f_{N+p-2} \circ \cdots \circ f_{N+1} \circ \psi_N(\Omega)\end{aligned}$$

which means that there is a $z_p \in \Omega$ such that

$$\omega = f_{N+p-1} \circ f_{N+p-2} \circ \cdots \circ f_{N+1} \circ \psi_N(z_p) \tag{14.67}$$
$$= \psi_{N+p-1}(z_p). \tag{14.68}$$

Applying the inequality (14.62) repeatedly to the expression (14.67), we see that

$$|\omega| \geq |\psi_N(z_p)| \tag{14.69}$$

for all $p \geq 1$. Clearly, $\overline{D(0; |\omega|)}$ is a compact subset of Ω_N, so $\psi_N^{-1}\big(\overline{D(0; |\omega|)}\big)$ is also a compact subset of Ω. Denote this set by K. By the result (14.69), we establish that $\{z_p\} \subseteq K$. Then the Bolzano-Weierstrass Theorem [79, Problem 5.25, p. 68] suggests that there corresponds a subsequence $\{z_{p_j}\}$ such that $z_{p_j} \to z \in K$. Since $\psi_n \to \psi$ uniformly on K, it observes that

$$\lim_{j\to\infty} \psi_{N+p_j-1}(z_{p_j}) = \psi(z). \tag{14.70}$$

Combining the expression (14.68) and the limit (14.70), we conclude at once that

$$\omega = \psi(z)$$

which means that ψ is surjective.

- **The map $\psi : \Omega \to U$ is injective.** Since ψ is surjective, it is not constant. Recall that each ψ_n is injective, so Problem 14.13 indicates easily that ψ is also injective.

Hence we have completed the analysis of the problem. ■

Problem 14.27

Rudin Chapter 14 Exercise 27.

Proof. Taking logarithms in the inequality (14.64) with $m = 2n$ and then using the hint to get

$$\sum_{k=n+1}^{2n} \log\Big[1 + \frac{(1-\sqrt{r_k})^2}{2\sqrt{r_k}}\Big] = \sum_{k=n+1}^{2n} \log\frac{1+r_k}{2\sqrt{r_k}} \leq \log\frac{1}{r_{n+1}} = -\log r_{n+1}. \tag{14.71}$$

Using the Mean Value Theorem for Derivatives, one can show that $\log(1+x) > \frac{x}{1+x}$ for $x > 0$. Since $0 < \frac{(1-\sqrt{r_k})^2}{2} < \frac{1}{2}$ for all $k \in \mathbb{N}$, the inequality (14.71) reduces to

$$-\log r_{n+1} > \sum_{k=n+1}^{2n} \log\left[1 + \frac{(1-\sqrt{r_k})^2}{2\sqrt{r_k}}\right] > \sum_{k=n+1}^{2n} \frac{(1-\sqrt{r_k})^2}{3} > 0. \tag{14.72}$$

By elementary calculus again, we know that $\frac{\log(1+x)}{x}$ is strictly decreasing for $x > -1$. Combining this fact and Lemma 14.4, it is true that for all $n \geq 1$,

$$\frac{\log r_{n+1}}{r_{n+1}-1} = \frac{\log[1+(r_{n+1}-1)]}{r_{n+1}-1} \leq \frac{\log[1+(r_1-1)]}{r_1-1} = \frac{\log r_1}{r_1-1}$$

which implies

$$-\log r_{n+1} \leq (1-r_{n+1}) \cdot \frac{\log r_1}{r_1-1} \leq (1-r_n) \cdot \frac{\log r_1^{-1}}{1-r_1}. \tag{14.73}$$

As $(1+\sqrt{r_k})^2 \leq 4$, we have

$$(1-\sqrt{r_k})^2 = \frac{(1-r_k)^2}{(1+\sqrt{r_k})^2} \geq \frac{(1-r_k)^2}{4}. \tag{14.74}$$

Let $A = \frac{\log r_1^{-1}}{1-r_1}$. Now we observe by substituting the inequalities (14.73) and (14.74) into the inequality (14.72) that

$$0 < \frac{1}{3}\sum_{k=n+1}^{2n}\Big(\frac{1-r_k}{2}\Big)^2 \leq \sum_{k=n+1}^{2n}\frac{(1-\sqrt{r_k})^2}{3} < -\log r_{n+1} < A(1-r_n)$$

or equivalently,

$$0 < \sum_{k=n+1}^{2n}\Big(\frac{1-r_k}{2B}\Big)^2 < \frac{1-r_n}{B}, \tag{14.75}$$

where $B = 3A$. Using Lemma 14.4, the inequality (14.75) gives

$$n\Big(\frac{1-r_{2n}}{2B}\Big)^2 < \frac{1-r_n}{B}.$$

After some algebra, we can show that it is actually equivalent to

$$\Big[2n\Big(\frac{1-r_{2n}}{16B}\Big)\Big]^2 < n\Big(\frac{1-r_n}{16B}\Big).$$

Particularly, for $p = 1, 2, \ldots$, we obtain

$$\begin{aligned}
\Big[2^p\Big(\frac{1-r_{2^p}}{16B}\Big)\Big]^2 &< 2^{p-1}\Big(\frac{1-r_{2^{p-1}}}{16B}\Big)\\
\Big[2^{p-1}\Big(\frac{1-r_{2^{p-1}}}{16B}\Big)\Big]^2 &< 2^{p-2}\Big(\frac{1-r_{2^{p-2}}}{16B}\Big)\\
&\vdots\\
\Big[2^1\Big(\frac{1-r_{2^1}}{16B}\Big)\Big]^2 &< \frac{1-r_1}{16B}
\end{aligned}$$

and they certainly imply that

$$\Big[2^p\Big(\frac{1-r_{2^p}}{16B}\Big)\Big]^{2^p} < \frac{1-r_1}{16B}.$$

Since $\frac{1-r_1}{16B}$ is constant and $a^{\frac{1}{n}} \to 1$ as $n \to \infty$ if $a > 0$, there exists a positive constant M such that

$$0 < 1 - r_{2^p} < \frac{M}{2^p}$$

for every $p = 1, 2, \ldots$. Thus it gives

$$0 < 2^p(1 - r_{2^p})^2 < \frac{M^2}{2^p}$$

for every $p = 1, 2, \ldots$. Obviously, the series $\displaystyle\sum_{p=1}^{\infty} \frac{M^2}{2^p}$ converges, so we deduce from the results [79, Theorems 6.5, 6.6, p. 76] that

$$\sum_{n=1}^{\infty}(1 - r_n)^2 < \infty.$$

This completes the proof of the problem. ■

Remark 14.4

(a) The proofs of Problems 14.26 and 14.27 are more or less the same as Ostrowski's paper [49].

(b) For more properties of the so-called **Koebe mapping** and details of the constructive proof of the Riemann Mapping Theorem, please refer to [52, §3, pp. 15, 16, 183 – 186] and [10, §1.7, pp. 180 – 214] respectively.

Problem 14.28

Rudin Chapter 14 Exercise 28.

Proof. Since $\alpha_n \in U \setminus \Omega_{n-1}$, we have $|\alpha_n| \geq r_n$, see Figure 14.2 again. By Problem 14.26, we may assume further that

$$r_n < |\alpha_n| \leq \frac{1 + r_n}{2}. \tag{14.76}$$

In this case, we also have $\beta^2 = -\alpha_n$. Furthermore, recall from the definition (14.57) that $F_n'(z) \neq 0$ for all $z \in \Omega_{n-1}$ so that F_n has a holomorphic inverse G_n in Ω_{n-1}. Next, we observe from the formula (14.59) and the A.M. $\geq$ G.M. that

$$|G_n'(0)| = \frac{1 - |\alpha_n|^2}{2(1 - |\beta_n|^2)|\beta_n|} = \frac{1 + |\beta_n|^2}{2|\beta_n|} = \frac{1 + |\alpha_n|}{2\sqrt{|\alpha_n|}} > 1.$$

In conclusion, we always have $|f_n'(0)| > 1$ for $n = 1, 2, \ldots$. Now it is easy to see that Lemma 14.4 remains valid, each $\psi_n : \Omega \to \Omega_n$ is bijective and $\{\psi_n'(0)\}$ is bounded by $\frac{1}{r_1}$. It remains to prove that $r_n \to 1$ as $n \to \infty$.

Instead of the formula (14.64) and the inequalities (14.65), we obtain

$$\psi_n'(0) = f_n'(0) \times f_{n-1}'(0) \times \cdots \times f_1'(0) = \prod_{k=1}^{n} \frac{1 + |\alpha_k|}{2\sqrt{|\alpha_k|}}$$

and

$$1 < \prod_{k=n+1}^{m} \frac{1 + |\alpha_k|}{2\sqrt{|\alpha_k|}} \leq \frac{1}{r_{n+1}},$$

where $m > n \geq 1$. By the right-most inequality (14.76), we may write

$$\frac{1+r_n}{2} = |\alpha_n| + \delta_n,$$

where $0 \leq \delta_n < \frac{1-r_n}{2}$. Since we have

$$\frac{1+|\alpha_k|}{2\sqrt{|\alpha_k|}} = \frac{1+r_k+\delta_k}{2\sqrt{r_k+\delta_k}},$$

we can show that the infinite product

$$\prod_{n=1}^{\infty} \frac{1+r_n+\delta_n}{2\sqrt{r_n+\delta_n}}$$

converges and so

$$1 = \lim_{n\to\infty} \frac{1+r_n+\delta_n}{2\sqrt{r_n+\delta_n}}. \tag{14.77}$$

By Lemma 14.4, $\{r_n\}$ converges to a positive number r. Suppose that $\delta_n \to \delta$ as $n \to \infty$, where $0 \leq \delta \leq \frac{1-r}{2}$. Then it follows from the limit (14.77) that

$$1 = \frac{1+r+\delta}{2\sqrt{r+\delta}}$$

which implies $r + \delta = 1$. If $r \neq 1$, then $\delta \neq 0$. Now the expression $\delta = 1 - r$ contradicts the fact $0 < \delta \leq \frac{1-r}{2}$. Hence we have $r = 1$ and we complete the proof of the problem.

■

Problem 14.29

Rudin Chapter 14 Exercise 29.

Proof.

(a) By translating Ω by $-a$, we may assume without loss of generality that $a = 0$. We claim that

$$f_n'(0) = [f'(0)]^n \tag{14.78}$$

for all $n = 1, 2, \ldots$. We use induction and the case $n = 1$ is trivial. Assume that

$$f_k'(0) = [f'(0)]^k \tag{14.79}$$

for some $k \in \mathbb{N}$. By the definition, we have

$$f_{k+1}'(0) = f'\big(f_k(0)\big) \cdot f_k'(0). \tag{14.80}$$

The hypothesis $f(0) = 0$ implies that $f_n(0) = 0$ for all $n \in \mathbb{N}$, so the expression (14.80) reduces to $f_{k+1}'(0) = f'(0) \cdot f_k'(0)$. By the inductive step (14.79), we conclude that

$$f_{k+1}(0) = [f'(0)]^{k+1}.$$

Hence the claim follows from induction.

Since Ω is bounded, there exists a positive constant M such that $|z| \leq M$ for all $z \in \Omega$. Since Ω is a region, one can find a $r > 0$ such that $\overline{D(0;r)} \subseteq \Omega$. By the hypothesis

$f(\Omega) \subseteq \Omega$, it is true that $f_n(\Omega) \subseteq \Omega$ for every $n = 1, 2, \ldots$. Hence we have $|f_n(z)| \leq M$ for all $z \in \overline{D(0; r)}$ and $n \in \mathbb{N}$, and then we yield from Theorem 10.26 (Cauchy's Estimate) that

$$|f_n^{(k)}(0)| \leq \frac{k!M}{r^k} \tag{14.81}$$

for $k = 1, 2, \ldots$. Combining the expression (14.78) and the estimate (14.81), we get

$$|f'(0)| = [f_n'(0)]^{\frac{1}{n}} \leq \left(\frac{M}{r}\right)^{\frac{1}{n}}$$

for every $n = 1, 2, \ldots$. By taking $n \to \infty$, we establish that $|f'(0)| \leq 1$ as required.

(b) If $f^{(k)}(0) = 0$ for all $k \geq 2$, then since $f \in H(\Omega)$, Theorem 10.16 implies that $f(z) = a + bz$ for some $a, b \in \mathbb{C}$. Since $f(0) = 0$ and $f'(0) = 1$, we have $a = 0$ and $b = 1$. Thus $f(z) = z$ as desired. Suppose that $N \geq 2$ is the smallest positive integer such that $f^{(N)}(0) \neq 0$ and let $c_k = \frac{f^{(k)}(0)}{k!}$ for $k \geq N$. In particular, we have $c_N \neq 0$. Since $f'(0) = 1$, we have

$$f(z) = z + \sum_{k=N}^{\infty} c_k z^k = z + c_N z^N g(z), \tag{14.82}$$

where $g(0) = 1$.

We claim that for each $n \in \mathbb{N}$, there is a $r_n > 0$ and an $g_n \in H\big(D(0; r_n)\big)$ such that for all $z \in D(0; r_n)$, we have

$$f_n(z) = z + c_N z^N g_n(z) \quad \text{and} \quad g_n(0) = n. \tag{14.83}$$

The expression (14.82) is just the case $n = 1$, so we assume that the result (14.83) holds for some positive integer n. Since $f_n(0) = 0$, we can pick $r_{n+1} \in (0, r_n)$ such that $f_n\big(D(0; r_{n+1})\big) \subseteq D(0; r_n)$. Then it follows from the expressions (14.82) and (14.83) that if $z \in D(0; r_{n+1})$, then

$$\begin{aligned}
f_{n+1}(z) &= f\big(f_n(z)\big) \\
&= f_n(z) + c_N[f_n(z)]^N g\big(f_n(z)\big) \\
&= z + c_N z^N g_n(z) + c_N\big[z + c_N z^N g_n(z)\big]^N g\big(f_n(z)\big) \\
&= z + c_N z^N\big\{g_n(z) + \big[1 + c_N z^{N-1} g_n(z)\big]^N g\big(f_n(z)\big)\big\}.
\end{aligned}$$

Suppose that $g_{n+1}(z) = g_n(z) + \big[1 + c_N z^{N-1} g_n(z)\big]^N g\big(f_n(z)\big)$. Recall that $N - 1 \geq 1$, so we obtain

$$g_{n+1}(0) = g_n(0) + g\big(f_n(0)\big) = n + 1.$$

By induction, our claim follows. Next, we differentiate the expression (14.83) N times and put $z = 0$, we have

$$f_n^{(N)}(0) = N! c_N g_n(0) = nN! c_N$$

for every $n = 1, 2, \ldots$. Since $c_N \neq 0$, we have $\big|f_n^{(N)}(0)\big| \to \infty$ as $n \to \infty$ which contradicts the fact (14.81). Consequently, this means that $f^{(k)}(0) = 0$ for all $k \geq 2$ which implies $f(z) = z$ in $D(0; r)$. By the Corollary following Theorem 10.18, it is actually true in Ω.

(c) By the hypothesis, we know that $f'(0) = e^{i\theta}$ for some $\theta \in [0, 2\pi]$. If $e^{i\theta}$ is an N-root of unity, then the integer $n_k = kN$ satisfies

$$[f'(0)]^{n_k} = (e^{iN\theta})^k = 1.$$

Otherwise, we need the following form of the Kronecker's Approximation Theorem [5, Theorem 7.8, p. 149]:

Lemma 14.5 (The Kronecker's Approximation Theorem)

Given any real α, any irrational β and any $\epsilon > 0$, there exist integers m and n with $n > 0$ such that
$$|n\beta - m - \alpha| < \epsilon.$$

Take $\alpha = 0$ and $\beta = \frac{\theta}{2\pi}$. For every $k \in \mathbb{N}$, we obtain from Lemma 14.5 that there exist integers n_k and m_k with $n_k > 0$ such that
$$\Big|\frac{n_k\theta}{2\pi} - m_k\Big| < \frac{1}{2k\pi}$$
or equivalently
$$|n_k\theta - 2m_k\pi| < \frac{1}{k}$$
which implies that $|n_k\theta - 2m_k\pi| \to 0$ as $k \to \infty$. Consequently, we have $e^{in_k\theta} \to 1$ as $k \to \infty$.

Next, we consider the family $\mathscr{F} = \{f_{n_k}\}$. Recall the fact that $f_n(\Omega) \subseteq \Omega$, so $\mathscr{F}$ is bounded on Ω and Theorem 14.6 (Montel's Theorem) ensures that $\mathscr{F}$ is normal and thus it has a convergent subsequence which we also call $\{f_{n_k}\}$ for convenience. Let g be its limit function. By Theorem 10.28, $g \in H(\Omega)$ and $f'_{n_k}(0) \to g'(0)$ as $k \to \infty$. By the formula (14.78) and the hypothesis $f'(0)]^{n_k} \to 1$ as $k \to \infty$, we get $f'_{n_k}(0) \to 1$ as $k \to \infty$ and thus $g'(0) = 1$. In other words, g is not constant and it follows from Problem 10.20 that $g(\Omega) \subseteq \Omega$. By part (b), we see that
$$g(z) = z \tag{14.84}$$
in Ω.

Finally, if $z, \omega \in \Omega$ and $f(z) = f(\omega)$, then we have
$$z = g(z) = \lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} f_{n_k}(\omega) = g(\omega) = \omega.$$

Thus f is one-to-one. If $p \in \Omega \setminus f(\Omega)$, then $p \in \Omega \setminus f_n(\Omega)$ for every $n \in \mathbb{N}$ because we always have $f_n(\Omega) \subseteq f(\Omega)$. Particularly, the function $f_{n_k}(z) - p \neq 0$ in Ω. However, $f_{n_k}(p) - p \to g(p) - p = 0$ as $k \to \infty$. By Problem 10.20 again, we establish that $g(z) = p$ for all $z \in \Omega$ which certainly contradicts the result (14.84). Hence we have $f(\Omega) = \Omega$, i.e., f is onto.

We end the proof of the problem. ∎

Problem 14.30

Rudin Chapter 14 Exercise 30.

Proof. We have
$$\Lambda = \Big\{\varphi(z) = \frac{az+b}{cz+d} \,\Big|\, a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0\Big\}.$$
The number $ad - bc$ is called the **determinant of** φ.

(a) If α, β and γ are distinct, then the definition gives

$$\varphi(z) = [z, \alpha, \beta, \gamma] = \frac{(z-\alpha)(\beta-\gamma)}{(z-\gamma)(\beta-\alpha)} = \frac{(\beta-\gamma)z - \alpha(\beta-\gamma)}{(\beta-\alpha)z - \gamma(\beta-\alpha)}. \tag{14.85}$$

By direct computation, we see that

$$-\gamma(\beta-\gamma)(\beta-\alpha) + \alpha(\beta-\gamma)(\beta-\alpha) = (\beta-\gamma)(\beta-\alpha)(\alpha-\gamma) \neq 0$$

which means $\varphi \in \Lambda$ in this case. Next, if $\alpha = \infty$ and $\beta \neq \gamma$, then we have

$$\varphi(z) = [z, \infty, \beta, \gamma] = \frac{0 \cdot z + (\beta-\gamma)}{z-\gamma} \tag{14.86}$$

so that $0 \cdot (-\gamma) - 1 \cdot (\beta-\gamma) \neq 0$. Thus we also have $\varphi \in \Lambda$ in this case.[j]

Furthermore, the representations (14.85) and (14.86) imply easily that φ maps $\{\alpha, \beta, \gamma\}$ to $\{0, 1, \infty\}$.

(b) Suppose that $\{\alpha, \beta, \gamma\}$ and $\{a, b, c\}$ are two groups of distinct complex numbers. By the definition (14.85), we have

$$\frac{(b-c)\omega - a(b-c)}{(b-a)\omega - c(b-a)} = \frac{(\beta-\gamma)z - \alpha(\beta-\gamma)}{(\beta-\alpha)z - \gamma(\beta-\alpha)}$$

which implies

$$\begin{aligned}
\omega &= \frac{a(b-c)(\beta-\alpha)(z-\gamma) - c(b-a)(\beta-\gamma)(z-\alpha)}{(b-c)(\beta-\alpha)(z-\gamma) - (b-a)(\beta-\gamma)(z-\alpha)} \\
&= \frac{\big[a(b-c)(\beta-\alpha) - c(b-a)(\beta-\gamma)\big]z + c\alpha(b-a)(\beta-\gamma) - a\gamma(b-c)(\beta-\alpha)}{\big[(b-c)(\beta-\alpha) - (b-a)(\beta-\gamma)\big]z + \alpha(b-a)(\beta-\gamma) - \gamma(b-c)(\beta-\alpha)} \\
&= \frac{\big[ab(\beta-\alpha) + ac(\alpha-\gamma) + bc(\gamma-\beta)\big]z + bc\alpha(\beta-\gamma) + ac\beta(\gamma-\alpha) + ab\gamma(\alpha-\beta)}{\big[b(\gamma-\alpha) + a(\beta-\gamma) + c(\alpha-\beta)\big]z + \big[b\beta(\alpha-\gamma) + a\alpha(\gamma-\beta) + c\gamma(\beta-\alpha)\big]} \\
&= \varphi(z).
\end{aligned} \tag{14.87}$$

Direct checking gives the determinant of φ as follows

$$\begin{aligned}
&\big[ab(\beta-\alpha) + ac(\alpha-\gamma) + bc(\gamma-\beta)\big] \cdot \big[b\beta(\alpha-\gamma) + a\alpha(\gamma-\beta) + c\gamma(\beta-\alpha)\big] \\
&\quad - \big[bc\alpha(\beta-\gamma) + ac\beta(\gamma-\alpha) + ab\gamma(\alpha-\beta)\big] \cdot \big[b(\gamma-\alpha) + a(\beta-\gamma) + c(\alpha-\beta)\big] \\
&= ab^2\beta(\alpha-\gamma)(\beta-\alpha) + abc\beta(\alpha-\gamma)^2 + b^2c\beta(\gamma-\beta)(\alpha-\gamma) + a^2b\alpha(\gamma-\beta)(\beta-\alpha) \\
&\quad + ac^2\alpha(\gamma-\beta)(\alpha-\gamma) + abc\alpha(\gamma-\beta)^2 + abc\gamma(\beta-\alpha)^2 + ac^2\gamma(\beta-\alpha)(\alpha-\gamma) \\
&\quad + bc^2\gamma(\gamma-\beta)(\beta-\alpha) - b^2c\alpha(\gamma-\alpha)(\beta-\gamma) - abc\beta(\gamma-\alpha)^2 ab^2\gamma(\gamma-\alpha)(\alpha-\beta) \\
&\quad - abc\alpha(\beta-\gamma)^2 - a^2c\beta(\beta-\gamma)(\gamma-\alpha) - a^2b\gamma(\alpha-\beta)(\beta-\gamma) - bc^2\alpha(\beta-\gamma)(\alpha-\beta) \\
&\quad - ac^2\beta(\alpha-\beta)(\gamma-\alpha) - abc\gamma(\alpha-\beta)^2 \\
&= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(ab^2 - b^2c - a^2b + a^2c - ac^2 + bc^2) \\
&= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(a-b)(b-c)(c-a) \\
&\neq 0
\end{aligned}$$

so that $\varphi \in \Lambda$ in this case.

[j] The cases for $\beta = \infty$ or $\gamma = \infty$ can be done similarly, so we omit the details here.

Next, if $\{\infty,\beta,\gamma\}$ and $\{a,b,c\}$ are two groups of distinct complex numbers[k], then we follow from the definition (14.86) (or by taking $\alpha\to\infty$ in the definition (14.87) that

$$\frac{(b-c)\omega-a(b-c)}{(b-a)\omega-c(b-a)}=\frac{\beta-\gamma}{z-\gamma}$$

which gives

$$\omega=\frac{a(b-c)z-a\gamma(b-c)-c(b-a)(\beta-\gamma)}{(b-c)z-\gamma(b-c)-(\beta-\gamma)(b-a)}=\varphi(z). \tag{14.88}$$

Now simple algebra shows that

$$\begin{aligned}&a(b-c)[-\gamma(b-c)-(\beta-\gamma)(b-a)]-(b-c)[-a\gamma(b-c)-c(b-a)(\beta-\gamma)]\\&=-a\gamma(b-c)^2-a(b-a)(b-c)(\beta-\gamma)+a\gamma(b-c)^2+c(b-a)(b-c)(\beta-\gamma)\\&=(b-a)(b-c)(c-a)(\beta-\gamma)\\&\neq 0\end{aligned}$$

which means $\varphi\in\Lambda$ in this case.

Furthermore, if $\{\infty,\beta,\gamma\}$ and $\{\infty,b,c\}$ are two groups of distinct complex numbers[l], then we have

$$\frac{b-c}{\omega-c}=\frac{\beta-\gamma}{z-\gamma}$$

and so

$$\omega=\frac{(b-c)z+c\beta-b\gamma}{\beta-\gamma}=\varphi(z). \tag{14.89}$$

Since $(b-c)(\beta-\gamma)\neq 0$, we also have $\varphi\in\Lambda$.

Finally, by the formulas (14.87), (14.88) and (14.89), it is easy to see that φ maps $\{\alpha,\beta,\gamma\}$ to $\{a,b,c\}$.

(c) Suppose that $\phi(z)=[z,\beta,\gamma,\delta]$ which sends $\{\beta,\gamma,\delta\}$ to $\{0,1,\infty\}$ by part (a). Then the mapping $\phi\circ\varphi^{-1}$ carries $\{\varphi(\beta),\varphi(\gamma),\varphi(\delta)\}$ to $\{0,1,\infty\}$. By §14.3, this map $\phi\circ\varphi^{-1}$ is unique, so $(\phi\circ\varphi^{-1})(z)=[z,\varphi(\beta),\varphi(\gamma),\varphi(\delta)]$ and then we have

$$[\varphi(\alpha),\varphi(\beta),\varphi(\gamma),\varphi(\delta)]=(\phi\circ\varphi^{-1})\big(\varphi(\alpha)\big)=\phi(\alpha)=[\alpha,\beta,\gamma,\delta].$$

(d) This part has been solved in [76, Problem 13.18, p. 182].

(e) Now we have

$$[z^*,\alpha,\beta,\gamma]=\overline{[z,\alpha,\beta,\gamma]}. \tag{14.90}$$

If C is a straight line, then we choose $\gamma=\infty$ in the equation (14.90) to get

$$\frac{z^*-\beta}{\alpha-\beta}=\frac{\overline{z}-\overline{\beta}}{\overline{\alpha}-\overline{\beta}}. \tag{14.91}$$

This certainly gives $|z^*-\beta|=|z-\beta|$. Since β is an *arbitrary* point on C, z and z^* are in fact equidistance from each point on C. Furthermore, the equation (14.91) implies that

$$\operatorname{Im}\frac{z^*-\beta}{\alpha-\beta}=\operatorname{Im}\frac{\overline{z}-\overline{\beta}}{\overline{\alpha}-\overline{\beta}}=-\operatorname{Im}\frac{z-\beta}{\alpha-\beta}$$

which means that z and z^* lie in different half planes determined by C.

[k] The other cases can be done similarly, so we only consider this case and omit the others.
[l] Again, we have omitted other similar cases.

Next, we suppose that $C = \{z \in \mathbb{C} \,|\, |z| = 1\}$. By the equation (14.90) and applications of the invariance property of $\varphi \in \Lambda$ as verified in part (c), we see that

$$[z^*, \alpha, \beta, \gamma] = \overline{[z, \alpha, \beta, \gamma]} = [\overline{z}, \overline{\alpha}, \overline{\beta}, \overline{\gamma}] = \left[\overline{z}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right] = \left[\frac{1}{\overline{z}}, \alpha, \beta, \gamma\right]$$

which implies $z^* = \frac{1}{\overline{z}}$ or $z^* \cdot \overline{z} = 1$. Thus we have $|z^*| = |z|^{-1}$ and furthermore, it deduces from the ratio

$$\frac{z^*}{z} = \frac{1}{|z|^2} > 0$$

that z^* lies on the ray $L = \{tz \,|\, t \in \mathbb{R}\}$. Geometrically, see Figure 14.3 for the construction of the point z^*.

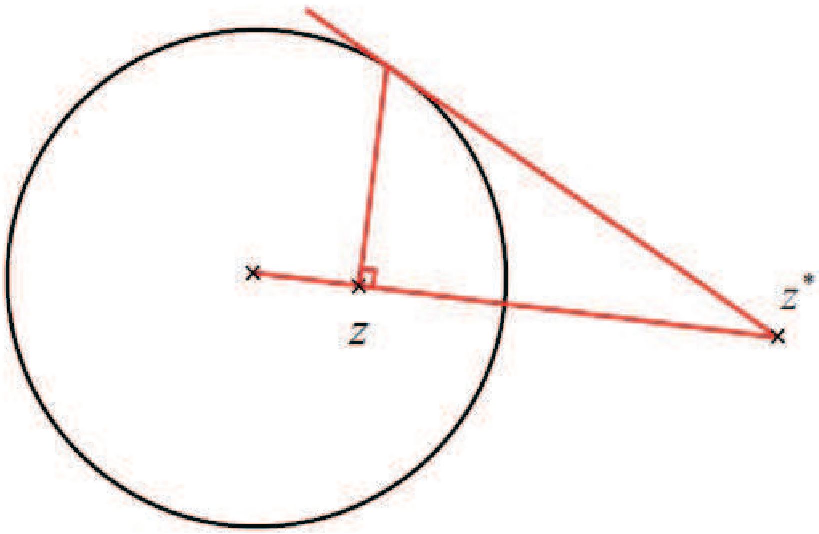


Figure 14.3: The construction of the symmetric point z^* of z.

(f) Let $\alpha, \beta, \gamma \in C$. Then it follows from part (c) and the definition that

$$[\varphi(z^*), \varphi(\alpha), \varphi(\beta), \varphi(\gamma)] = [z^*, \alpha, \beta, \gamma] = \overline{[z, \alpha, \beta, \gamma]} = \overline{[\varphi(z), \varphi(\alpha), \varphi(\beta), \varphi(\gamma)]}.$$

Hence $\varphi(z^*)$ and $\varphi(z)$ are symmetric with respect to $\varphi(C)$.

This finishes the analysis of the problem. ■

Problem 14.31

Rudin Chapter 14 Exercise 31.

Proof.

(a) Given $\varphi, \psi, \phi \in \Lambda$ by

$$\varphi(z) = \frac{az+b}{cz+d}, \quad \psi(z) = \frac{\alpha z+\beta}{\gamma z+\delta} \quad \text{and} \quad \phi(z) = \frac{Az+B}{Cz+D}.$$

– **Composition as group operation.** It is easy to see that

$$\psi\big(\varphi(z)\big) = \frac{(\alpha a+\beta c)z + \alpha b + \beta d}{(\gamma a + \delta c)z + \gamma b + \delta d}$$

and its determinant is

$$(\alpha a+\beta c)(\gamma b+\delta d)-(\gamma a+\delta c)(\alpha b+\beta d)=(ad-bc)(\alpha\delta-\beta\gamma)\neq 0$$

so that $\psi\circ\varphi\in\Lambda$.

- **Associativity.** Simple algebra verifies

$$[\varphi(z)+\phi(z)]+\psi(z)=\varphi(z)+[\phi(z)+\psi(z)].$$

- **The identity element.** Now the usual identity map $\mathrm{id}:\mathbb{C}\to\mathbb{C}$ is the identity element of Λ because its determinant is 1 and

$$\mathrm{id}\circ\varphi=\varphi\circ\mathrm{id}=\varphi.$$

- **The inverse of φ.** The equation $\omega=\varphi(z)$ has *exactly* one solution and indeed, it is

$$z=\varphi^{-1}(\omega)=\frac{d\omega-b}{c\omega-a}.$$

Since the determinant of φ^{-1} is $-ad+bc\neq 0$, we have φ^{-1} belongs to Λ. Clearly, we know that $\varphi^{-1}\circ\varphi=\varphi\circ\varphi^{-1}=\mathrm{id}$.

By the definition (see [25, Definition 4.1, pp. 37, 38]), Λ is indeed a group. If we take $\varphi(z)=z+1$ and $\psi(z)=\frac{1}{z}$, then we see that $\varphi\big(\psi(z)\big)=\frac{1}{z}+1$ and $\psi\big(\varphi(z)\big)=\frac{1}{z+1}$ which imply that Λ is not commutative.

(b) Let $\varphi\in\Lambda$ be given by

$$\varphi(z)=\frac{az+b}{cz+d}$$

and $\varphi\neq\mathrm{id}$. Define $\Delta=ad-bc\neq 0$. Since we may write

$$\varphi(z)=\frac{\frac{a}{\sqrt{\Delta}}z+\frac{b}{\sqrt{\Delta}}}{\frac{c}{\sqrt{\Delta}}z+\frac{d}{\sqrt{\Delta}}},$$

we may assume without loss of generality that $ad-bc=1$. Now the equation $z=\varphi(z)$ is equivalent to saying that

$$cz^2+(d-a)z-b=0. \tag{14.92}$$

- **Case (i): $c=0$.** Thus we have $ad\neq 0$ and $\varphi(\infty)=\infty$ so that ∞ is a fixed point of φ. If $a\neq d$, then by solving the equation (14.92), we get one more (finite) fixed point which is

$$z=\frac{b}{d-a}.$$

Otherwise, $a=d$ implies that

$$\varphi(z)=z+\frac{b}{d} \tag{14.93}$$

whose fixed point is also ∞. Since $\varphi\neq\mathrm{id}$, $b\neq 0$ so that φ has only a unique (infinite) fixed point in this case.

- **Case (ii): $c\neq 0$.** Then the equation (14.92) has two roots

$$z=\frac{a-d\pm\sqrt{(d-a)^2+4bc}}{2c}=\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}. \tag{14.94}$$

Since $\varphi(\infty)=\frac{a}{c}$, ∞ is *not* transformed into itself. This means that φ has either one or two finite fixed points on S^2 depending on whether $a+d=\pm 2$ or not. In the case of the unique finite fixed point, it is given by

$$z=\frac{a-d}{2c}.$$

In conclusion, φ has either one or two fixed points on S^2.

(c) We consider two cases.

- **Case (i): φ has a unique fixed point.** Given $\varphi_1(z) = z + 1$ which is obviously an element of Λ.

  If $c = 0$, then we follow from part (b) that φ has a unique (infinite) fixed point if and only if it takes the form (14.93). Define $\psi(z) = \frac{b}{d}z$. Recall that $bd \neq 0$, so $\psi \in \Lambda$ and $\psi^{-1}(z) = \frac{d}{b}z$. Furthermore, it is clear that

$$\psi^{-1}\big(\varphi(\psi(z))\big) = \frac{d}{b} \cdot \Big(\frac{b}{d}z + \frac{b}{d}\Big) = z + 1 = \varphi_1(z).$$

  Hence we have shown that φ is conjugate to φ_1 in this case.

  Next, if $c \neq 0$, then we observe from the roots (14.94) that φ has a unique fixed point $z_1 = \frac{a-d}{2c}$ if and only if $a + d = \pm 2$. Define the linear fractional transformation

$$S(z) = \frac{1}{z - z_1}$$

  which carries z_1 to ∞. Therefore, the linear fractional transformation

$$T = S \circ \varphi \circ S^{-1} \tag{14.95}$$

  has ∞ as its *only* fixed point because if p is a fixed point of T, then $S^{-1}(p)$ will be a fixed point of φ so that

$$p = S(z_1) = \infty.$$

  Hence it follows from part (b) that $T(z) = z + B$ for some $B \in \mathbb{C} \setminus \{0\}$. If we take $P(z) = Bz$, then $P^{-1}(z) = \frac{z}{B}$ and so

$$\varphi_1 = P^{-1} \circ T \circ P. \tag{14.96}$$

  By combining the expressions (14.95) and (14.96), we conclude that

$$\varphi_1 = (P^{-1} \circ S) \circ \varphi \circ (S^{-1} \circ P) = (S^{-1} \circ P)^{-1} \circ \varphi \circ (S^{-1} \circ P).$$

  Hence we have $\psi = S^{-1} \circ P$. Since $P, S \in \Lambda$, we have $\psi \in \Lambda$.

- **Case (ii): φ has two distinct fixed points.** Consider the linear fractional transformation

$$\phi_\alpha(z) = \alpha z,$$

  where α is a non-zero complex number which will be determined soon. By part (b), we have either "$c = 0$ and $a \neq d$" or "$c \neq 0$ and $a + d \neq \pm 2$".

  * **Subcase (i): $c = 0$ and $a \neq d$.** We notice that

$$\varphi(z) = \frac{az + b}{d}.$$

    Take $\psi(z) = z + \frac{b}{d-a}$ which is obviously a linear fractional transformation and $\psi^{-1}(z) = z - \frac{b}{d-a}$. Direct computation gives

$$\varphi\big(\psi(z)\big) = \frac{a}{d}\Big(z + \frac{b}{d-a}\Big) + \frac{b}{d} = \frac{a}{d}z + \frac{b}{d-a}$$

and then

$$\psi^{-1}\big(\varphi(\psi(z))\big) = \frac{a}{d}z + \frac{b}{d-a} - \frac{b}{d-a} = \frac{a}{d}z = \phi_{\frac{a}{d}}(z).$$

Consequently, φ is conjugate to ϕ_α with

$$\alpha = \frac{a}{d}. \tag{14.97}$$

Particularly, 0 is the finite fixed point if and only if $b = 0$, so we have $\varphi(z) = \frac{a}{d}z$ and $\psi(z) = z$.

* **Subcase (ii): $c \neq 0$ and $a + d \neq \pm 2$.** The two distinct finite fixed points are given by (14.94). Let z_1 and z_2 be the roots corresponding to the negative square root and the positive square root respectively. Now the linear fractional transformation

$$S(z) = \frac{z - z_1}{z - z_2}$$

maps the ordered pair $\{z_1, z_2\}$ into $\{0, \infty\}$. Then the linear fractional transformation $T = S \circ \varphi \circ S^{-1}$ fixes 0 and ∞. By the particular case of **Subcase (i)**, we know that

$$\phi_\alpha = \psi^{-1} \circ T \circ \psi = T$$

for some complex α. Since $S^{-1}(z) = \frac{z_2 z - z_1}{z-1}$, we obtain from the definition that

$$\varphi\big(S^{-1}(z)\big) = \frac{(az_2 + b)z - (az_1 + b)}{(cz_2 + d)z - (cz_1 + d)}.$$

and

$$T(z) = \frac{(az_2 - dz_1 + 2b)z + cz_1^2 + (d-a)z_1 - b}{-\big[cz_2^2 + (d-a)z_2 - b\big]z - (az_1 - dz_2 + 2b)}. \tag{14.98}$$

Since z_1, z_2 are roots of the equation (14.92), the formula (14.98) simplifies to

$$T(z) = -\frac{(az_2 - dz_1 + 2b)}{(az_1 - dz_2 + 2b)}z. \tag{14.99}$$

Using the formula (14.94), the expression (14.99) can further reduce to

$$T(z) = -\frac{\sqrt{(a+d)^2 - 4} + (a+d)}{\sqrt{(a+d)^2 - 4} - (a+d)}z.$$

Hence we obtain the formula

$$\alpha = -\frac{\sqrt{(a+d)^2 - 4} + (a+d)}{\sqrt{(a+d)^2 - 4} - (a+d)}. \tag{14.100}$$

Finally, α is determined by either (14.97) or (14.100).

(d) We prove the assertions one by one.

- **The existence of β.** Since φ has only a unique finite fixed point, the analysis of part (b) leads us to the result that $c \neq 0$ and $a + d = \pm 2$. In this case, we have $\alpha = \frac{a-d}{2c}$. Then we have

$$d = a - 2\alpha c. \tag{14.101}$$

Since α is a root of the equation (14.92), we have

$$b - d\alpha = c\alpha^2 - a\alpha = \alpha(c\alpha - a). \tag{14.102}$$

Obviously, $c\alpha - a \neq 0$. Otherwise, put $c\alpha = a$ into the equation (14.101) will give $a + d = 0$ which is impossible. Now we note that

$$\frac{1}{\varphi(z)-\alpha} = \frac{1}{\frac{az+b}{cz+d}-\alpha} = \frac{cz+d}{az+b-c\alpha z-d\alpha} = \frac{cz+d}{(a-c\alpha)z+(b-d\alpha)} \tag{14.103}$$

Substituting the values (14.101) and (14.102) into the expression (14.103) to get

$$\frac{1}{\varphi(z)-\alpha} = \frac{cz+a-2\alpha c}{(a-c\alpha)z+\alpha(c\alpha-a)} = \frac{c(z-\alpha)+a-\alpha c}{(z-\alpha)(a-\alpha c)} = \frac{1}{z-\alpha}+\frac{c}{a-\alpha c} \tag{14.104}$$

which means that

$$\beta = \frac{c}{a-\alpha c} \in \mathbb{C}.$$

With the aid of $\alpha = \frac{a-d}{2c}$ and $a + d = \pm 2$, we can further show that

$$\beta = c. \tag{14.105}$$

– G_α **is a subgroup of** Λ**.** Let $G_\alpha = \{\varphi \in \Lambda \,|\, \varphi(\alpha) = \alpha\} \cup \{\mathrm{id}\} \subseteq \Lambda$. Let φ and ϕ be elements of G_α with the corresponding constant β_φ and β_ϕ respectively. Therefore, we see that

$$\frac{1}{\varphi\big(\phi(z)\big)-\alpha} = \frac{1}{\phi(z)-\alpha}+\beta_\varphi = \frac{1}{z-\alpha}+\beta_\varphi+\beta_\phi \tag{14.106}$$

so that $\varphi \circ \phi \in G_\alpha$. By the definition, we have $\mathrm{id} \in G_\alpha$. Furthermore, since φ has α as its only finite fixed point, so is φ^{-1}. Thus we have

$$\frac{1}{\varphi^{-1}(z)-\alpha} = \frac{1}{z-\alpha}-\beta.$$

In other words, it means that $\varphi^{-1} \in G_\alpha$. By [25, Theorem 5.14, p. 52], G_α is a subgroup of Λ.

– G_α **is isomorphic to** $(\mathbb{C}, +)$**.** Define $f : G_\alpha \to \mathbb{C}$ by $f(\varphi) = \beta_\varphi$, where β_φ is the complex number satisfying the equation

$$\frac{1}{\varphi(z)-\alpha} = \frac{1}{z-\alpha}+\beta_\varphi. \tag{14.107}$$

As the expression (14.106) shows definitely that

$$f(\varphi \circ \phi) = \beta_\varphi + \beta_\phi,$$

so f is a homomorphism. Next, suppose that $\beta = 0$. Then $\frac{1}{\varphi(z)-\alpha} = \frac{1}{z-\alpha}$ if and only if $\varphi = \mathrm{id}$. In other words, the kernel of f is $\{\mathrm{id}\}$. Finally, given $\beta \in \mathbb{C} \setminus \{0\}$. we consider $a = 1 + \alpha\beta$, $b = -\alpha^2\beta$, $c = \beta$ and $d = 1 - \alpha\beta$. Direct computation gives $ad - bc = 1$. Besides, the linear fractional transformation

$$\varphi_\beta(z) = \frac{(1+\alpha\beta)z-\alpha^2\beta}{\beta z+(1-\alpha\beta)} \tag{14.108}$$

fixes α only and satisfies the equation (14.107).[m] Consequently, we have $\varphi_\beta \in G_\alpha$ and $f(\varphi_\beta) = \beta$, i.e., f is surjective. Hence f is in fact an isomorphism[n].

[m]In fact, we establish from the value (14.105) that the representation (14.108) becomes

$$\varphi(z) = \frac{(1+c\alpha)z-c\alpha^2}{cz+(1-c\alpha)}.$$

[n]See, for instance, [25, p. 132].

(e) Now we have $G_{\alpha,\beta} = \{\varphi \in \Lambda \,|\, \varphi(\alpha) = \alpha \text{ and } \varphi(\beta) = \beta\}$. This refers to the case $c \neq 0$ and $a + d \neq \pm 2$.

- **Every** $\varphi \in G_{\alpha,\beta}$ **satisfies the required equation.** Since α and β are roots of the equation (14.92), the formula (14.102) also holds for β. Obviously, $a - \beta c \neq 0$. Otherwise, it implies the contradiction that $a + d = \pm\sqrt{(a+d)^2 - 4}$. We observe that

$$\begin{aligned}\frac{\varphi(z)-\alpha}{\varphi(z)-\beta} &= \frac{\frac{az+b}{cz+d}-\alpha}{\frac{az+b}{cz+d}-\beta}\\ &= \frac{(a-\alpha c)z + (b - d\alpha)}{(a-\beta c)z + (b - d\beta)}\\ &= \frac{(a-\alpha c)z + \alpha(c\alpha - a)}{(a-\beta c)z + \beta(c\beta - a)}\\ &= \gamma \cdot \frac{z-\alpha}{z-\beta},\end{aligned}$$

where

$$\gamma = \frac{a - \alpha c}{a - \beta c} \in \mathbb{C}.^{\text{o}} \tag{14.109}$$

- $G_{\alpha,\beta}$ **is a subgroup of** Λ**.** Since id fixes α and β, we have $\mathrm{id} \in G_{\alpha,\beta}$. For every $\varphi, \phi \in G_{\alpha,\beta}$, let γ_φ and γ_ϕ be their corresponding complex numbers respectively. Since

$$\frac{\varphi(\phi(z)) - \alpha}{\varphi(\phi(z)) - \beta} = \gamma_\varphi \cdot \frac{\phi(z)-\alpha}{\phi(z)-\beta} = \gamma_\varphi \cdot \gamma_\phi \cdot \frac{z-\alpha}{z-\beta}, \tag{14.110}$$

we have $\varphi \circ \phi \in G_{\alpha,\beta}$. Assume that $\varphi \in G_{\alpha,\beta}$ was a constant map. Then it implies that $\alpha = \beta$, a contradiction. In addition, $\gamma_\varphi \neq 0$. Otherwise, $\varphi(z) = \alpha$ for all $z \in S^2$ which is impossible. Next, if $\varphi \in G_{\alpha,\beta}$, then φ^{-1} also fixes α and β, and we have

$$\frac{\varphi^{-1}(z) - \alpha}{\varphi^{-1}(z) - \beta} = \frac{1}{\gamma_\varphi} \cdot \frac{z-\alpha}{z-\beta}.$$

Consequently, these imply that $\varphi^{-1} \in G_{\alpha,\beta}$. Hence $G_{\alpha,\beta}$ is a subgroup of Λ.

- $G_{\alpha,\beta}$ **is isomorphic to** $(\mathbb{C} \setminus \{0\}, \times)$**.** Define $g : G_{\alpha,\beta} \to \mathbb{C} \setminus \{0\}$ by $g(\varphi) = \gamma_\varphi$, where γ_φ is the complex number satisfying the equation

$$\frac{\varphi(z)-\alpha}{\varphi(z)-\beta} = \gamma_\varphi \cdot \frac{z-\alpha}{z-\beta}. \tag{14.111}$$

The equation (14.110) implies that $g(\varphi \circ \phi) = \gamma_\varphi \times \gamma_\phi$ so that g is a homomorphism. If $g(\varphi) = 1$, then we have $a - \alpha c = a - \beta c$ so that $c = 0$ and φ takes the form

$$\varphi(z) = \frac{az+b}{d}.$$

Put this into the equation (14.111) with $\gamma_\varphi = 1$ and after simplification, we conclude that $\varphi(z) = z$, i.e., the kernel of g is $\{\mathrm{id}\}$. Let $\gamma \in \mathbb{C} \setminus \{0\}$. By changing the subject of the formula (14.109) to c and using the formula $\alpha + \beta = \frac{a-d}{c}$, we can show that

$$d = \frac{\alpha\gamma - \beta}{\alpha - \beta\gamma} a.$$

[o] This number is called the **multipler of the transformation** φ, read [24, pp. 15, 16].

Next, we apply the fact

$$\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}} = a + d$$

to represent a, c and d in terms of α, β and γ as follows:

$$\begin{aligned} a &= \frac{\alpha - \beta\gamma}{(1+\gamma)(\alpha-\beta)}\Big(\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}}\Big), \\ d &= \frac{\alpha\gamma - \beta}{(1+\gamma)(\alpha-\beta)}\Big(\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}}\Big), \\ c &= \frac{1 - \gamma}{(1+\gamma)(\alpha-\beta)}\Big(\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}}\Big). \end{aligned} \tag{14.112}$$

Finally, we employ the formula $\alpha\beta = -\frac{b}{c}$ to obtain

$$b = -\frac{\alpha\beta(1-\gamma)}{(1+\gamma)(\alpha-\beta)}\Big(\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}}\Big). \tag{14.113}$$

Now it is a routine task to check that the linear fractional transformation φ with the coefficients given by the formulas (14.112) and (14.113) has α and β as its fixed points, $ad - bc = 1$ and satisfies

$$g(\varphi) = \gamma.$$

In other words, the map g is surjective and hence, an isomorphism.

(f) By the hypothesis, the fixed points are finite. The following proof is due to Drazin [20] who verified that the linear fractional transformation $\varphi(z) = \frac{az+b}{cz+d}$ has invariant circles if and only if its determinant $\Delta \neq 0$ and $\frac{(a+d)^2}{\Delta}$ is real.

- **Case (i): φ has a unique finite fixed point.** Recall from the explicit form (14.108) that

$$\varphi(z) = \frac{(1+\beta\alpha)z - \beta\alpha^2}{\beta z + (1-\beta\alpha)}, \tag{14.114}$$

where $\beta = c \neq 0$. Simple algebra gives

$$\beta\varphi(z) = 1 + \beta\alpha - \frac{1}{\beta z + (1-\beta\alpha)}. \tag{14.115}$$

Define

$$\varphi^* = \beta\varphi + (1-\beta\alpha) \quad \text{and} \quad \zeta = \beta z + (1-\beta\alpha). \tag{14.116}$$

Then the equation (14.115) becomes

$$\varphi^*(\zeta) = 2 - \frac{1}{\zeta}. \tag{14.117}$$

Consequently, this change of variables establishes a one-to-one correspondence between the invariant circles of the linear fractional transformations (14.114) and (14.117). Let $C^* = \{\zeta \in \mathbb{C} \,|\, |\zeta - \rho e^{i\theta}| = R\}$ be an invariant circle of φ^*, where ρ, θ, R are real and $R > 0$. Consider the two points $(\rho - R)e^{i\theta}$ and $(\rho + R)e^{i\theta}$ which are the endpoints of a diameter of C^*. Since φ^* is conformal, the points $P = \varphi^*\big((\rho-R)e^{i\theta}\big)$ and $Q = \varphi^*\big((\rho+R)e^{i\theta}\big)$ are also endpoints of a diameter of C^* so that $|P - Q| = 2R$ and $\frac{1}{2}(P+Q) = \rho e^{i\theta}$. Notice that

$$P = 2 - \frac{1}{(\rho-R)e^{i\theta}} \quad \text{and} \quad Q = 2 - \frac{1}{(\rho+R)e^{i\theta}}, \tag{14.118}$$

so we have

$$\rho^2 - R^2 = \pm 1 \quad \text{and} \quad 2 = \rho e^{i\theta} + \frac{\rho e^{-i\theta}}{\rho^2 - R^2}.$$

If $\rho^2 - R^2 = -1$, then we have $1 = i\rho \sin\theta$ which is impossible. Therefore, we must have

$$R^2 = \rho^2 - 1 \quad \text{and} \quad 1 = \rho\cos\theta.$$

In this case, C^* are circles with centers $1 \pm iR$. By the transformation (14.116), the invariant circles of φ satisfy the equations

$$\left| z - \left(\alpha \pm i\frac{R}{\beta}\right)\right| = \frac{R}{|\beta|},$$

where $R > 0$.

– **Case (ii): φ has two finite fixed points.** By the expressions (14.112) and (14.113), the explicit form of φ (after the cancellation of the common coefficient) is given by

$$\varphi(z) = \frac{(\alpha - \beta\gamma)z - \alpha\beta(1-\gamma)}{(1-\gamma)z + (\alpha\gamma - \beta)} \tag{14.119}$$

and $\Delta = (\alpha - \beta\gamma)(\alpha\gamma - \beta) + \alpha\beta(1-\gamma)^2 = \gamma(\alpha - \beta)^2 \neq 0$. Now we have

$$(1-\gamma)\varphi(z) = \frac{(\alpha - \beta\gamma)[(1-\gamma)z + (\alpha\gamma - \beta)] - \Delta}{(1-\gamma)z + (\alpha\gamma - \beta)} = (\alpha - \beta\gamma) - \frac{\sqrt{\Delta}}{\zeta},$$

where

$$\zeta = \frac{(1-\gamma)z + (\alpha\gamma - \beta)}{\sqrt{\Delta}}. \tag{14.120}$$

Define $\varphi^* = \Delta^{-\frac{1}{2}}[(1-\gamma)\varphi + (\alpha\gamma - \beta)]$. Then we have

$$\varphi^*(\zeta) = \Delta^{-\frac{1}{2}}\left[(\alpha - \beta)(1+\gamma) - \frac{\sqrt{\Delta}}{\zeta}\right] = \frac{1+\gamma}{\sqrt{\gamma}} - \frac{1}{\zeta}. \tag{14.121}$$

Similar to **Case (i)**, this change of variables establishes a one-to-one correspondence between the invariant circles of the linear fractional transformations (14.119) and (14.121). Instead of the expressions (14.118), we have

$$P = \frac{1+\gamma}{\sqrt{\gamma}} - \frac{1}{(\rho - R)e^{i\theta}} \quad \text{and} \quad Q = \frac{1+\gamma}{\sqrt{\gamma}} - \frac{1}{(\rho + R)e^{i\theta}}$$

so that

$$\rho^2 - R^2 = \pm 1 \quad \text{and} \quad \frac{1+\gamma}{\sqrt{\gamma}} = \rho e^{i\theta} + \frac{\rho e^{-i\theta}}{\rho^2 - R^2}.$$

Denote $\chi = \frac{1+\gamma}{2\sqrt{\gamma}}$. Thus we have either

$$R^2 = \rho^2 + 1 \quad \text{and} \quad \chi = i\rho\sin\theta \tag{14.122}$$

or

$$R^2 = \rho^2 - 1 \quad \text{and} \quad \chi = \rho\cos\theta. \tag{14.123}$$

Since ρ and θ are real, the expressions involving χ in (14.122) and (14.123) show that it is either purely real or purely imaginary.

* **Subcase (i): $\chi^2 < 0$.** Here $\chi = it$, where $t = |\frac{1+\gamma}{2\sqrt{\gamma}}|$. By the equations (14.122), the invariant circle of φ^* has the form
$$\big|\zeta - \big(\pm\sqrt{R^2 - t^2 - 1} + it\big)\big| = R,$$
where $R^2 \geq 1 - \chi^2 = 1 + t^2$. Transforming back to the original system (using (14.120)), we get
$$\begin{aligned}\Big|\frac{(1-\gamma)z + (\alpha\gamma - \beta)}{\sqrt{\Delta}} - \big(\pm\sqrt{R^2 - t^2 - 1} + it\big)\Big| &= R \\ \Big|z - \frac{(\beta - \alpha\gamma) + \sqrt{\Delta}\big(\pm\sqrt{R^2 - t^2 - 1} + it\big)}{1-\gamma}\Big| &= \frac{R|\alpha - \beta|\sqrt{|\gamma|}}{|1-\gamma|} \\ \left|z - \frac{(\beta - \alpha\gamma) + \sqrt{\Delta}\Big(\pm\sqrt{R^2 - |\frac{1+\gamma}{2\sqrt{\gamma}}|^2 - 1} + i|\frac{1+\gamma}{2\sqrt{\gamma}}|\Big)}{1-\gamma}\right| &= \frac{R|\alpha - \beta|\sqrt{|\gamma|}}{|1-\gamma|}.\end{aligned}$$

* **Subcase (ii): $\chi^2 = 0$.** In this subcase, we know that $\gamma = -1$. Furthermore, it can be seen from the expressions involving χ in (14.122) and (14.123) that $\rho = 0$ and then $R = 1$. Consequently, we have $|\zeta| = 1$ which gives
$$\Big|z - \frac{\alpha + \beta}{2}\Big| = \frac{|\alpha - \beta|}{2}.$$

* **Subcase (iii): $\chi^2 > 0$.** Then $\chi = \frac{1+\gamma}{2\sqrt{\gamma}}$ is real and we get from the expression (14.123) that
$$\big|\zeta - \big(\chi \pm i\sqrt{R^2 + 1 - \chi^2}\big)\big| = R,$$
where $R^2 \geq \chi^2 - 1$. Hence, after transforming back to the original system, we assert that
$$\left|z - \frac{(\beta - \alpha\gamma) + \sqrt{\Delta}\Big(\frac{1+\gamma}{2\sqrt{\gamma}} \pm i\sqrt{R^2 + 1 - (\frac{1+\gamma}{2\sqrt{\gamma}})^2}\Big)}{1-\gamma}\right| = \frac{R|\alpha - \beta|\sqrt{|\gamma|}}{|1-\gamma|},$$
where $R^2 \geq \frac{(1+\gamma)^2}{4\gamma} - 1 = \frac{(1-\gamma)^2}{4\gamma}$.

Now we have completed the analysis of the problem. ■

Remark 14.5

(a) The expressions (14.107) and (14.111) are called the **normal forms** of the linear fractional transformation φ.

(b) The number of finite fixed points can be used to classify the linear fractional transformations φ. In fact, we rewrite the multiplier (14.109) as $\rho e^{i\theta}$. If $\rho \neq 1$ and $\theta = 2n\pi$, then φ is called a **hyperbolic transformation**. If $\rho = 1$ and $\theta \neq 2n\pi$, then it is called an **elliptic transformation**. If $\rho > 0$ but $\rho \neq 1$ and $\theta \neq 2n\pi$, then it is called a **loxodromic transformation**. The case for one finite fixed point is called a **parabolic transformation** and it can be thought as corresponding to $\rho = 1$ and $\theta = 2n\pi$. See [1, §3.5, pp. 84 – 89] and [24, pp. 15 – 23] for further details.

Problem 14.32

Rudin Chapter 14 Exercise 32.

Proof. We notice from §14.3 that the linear fractional transformation

$$\omega = \varphi(z) = \frac{1+z}{1-z}$$

is a conformal one-to-one mapping of U onto the open right half plane

$$\Pi = \{\omega = X + iY \in \mathbb{C} \,|\, X > 0\}.$$

The images of the upper semi-circle and the lower semi-circle under φ are the positive Y-axis and the negative Y-axis respectively. Next, the mapping

$$\zeta = \phi(\omega) = \log\omega = \log\frac{1+z}{1-z}$$

maps Π conformally onto the horizontal strip

$$S = \{\zeta = u + iv \in \mathbb{C} \,|\, u \in \mathbb{R} \text{ and } -\tfrac{\pi}{2} < v < \tfrac{\pi}{2}\}.$$

Furthermore, the positive Y-axis is mapped onto the line $v = \frac{\pi}{2}$ and the negative Y-axis is mapped onto the line $v = -\frac{\pi}{2}$. Consequently, the mapping

$$\psi(z) = \phi\big(\varphi(z)\big) = \log\frac{1+z}{1-z} \tag{14.124}$$

sends U conformally onto the horizontal strip S, the images of the upper semi-circle and the lower semi-circle under ψ are the lines $v = \frac{\pi}{2}$ and $v = -\frac{\pi}{2}$ respectively. See Figure 14.4 for the illustration.

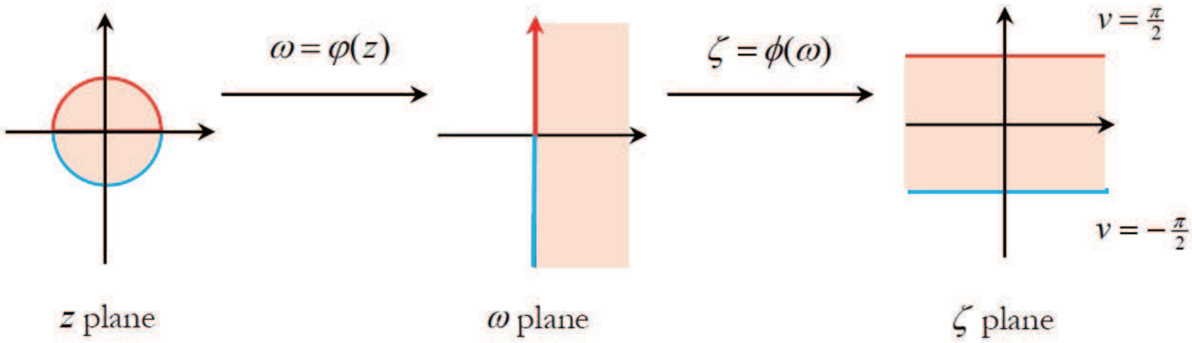


Figure 14.4: The conformal mapping $\psi(z) = \phi\big(\varphi(z)\big)$

Finally, it is easy to see that the mapping $\zeta \mapsto i\zeta$ carries the horizontal strip S conformally onto the vertical strip

$$H = \{\varpi = \overline{X} + i\overline{Y} \in \mathbb{C} \,|\, -\tfrac{\pi}{2} < \overline{X} < \tfrac{\pi}{2} \text{ and } \overline{Y} \in \mathbb{R}\}$$

and the lines $v = \pm\frac{\pi}{2}$ onto the lines $\overline{X} = \mp\frac{\pi}{2}$ respectively.

(a) Since e^z maps the horizontal strip $\{z \in \mathbb{C} \,|\, a < \operatorname{Re} z < b\}$ conformally onto the annulus $A(e^a, e^b)$, our conformal mapping f carries U onto the annulus $A(e^{-\frac{\pi}{2}}, e^{\frac{\pi}{2}})$.[p] Furthermore, the images of the upper semi-circle and the lower semi-circle under f are the circles $C\big(0, e^{-\frac{\pi}{2}}\big)$ and $C\big(0, e^{\frac{\pi}{2}}\big)$ respectively, see Figure 14.5 for details below.

[p]By direct differentiation, we see that

$$f'(z) = \frac{2i}{1-z^2}\exp\left\{i\log\frac{1+z}{1-z}\right\} \neq 0$$

in U, so f is conformal in U by Theorem 14.2.

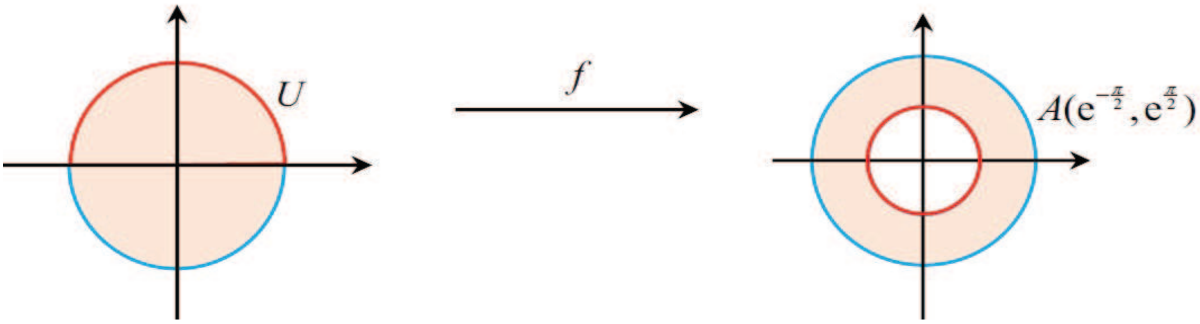


Figure 14.5: The conformal mapping $f : U \to A(e^{-\frac{\pi}{2}}, e^{\frac{\pi}{2}})$.

(b) Refer to part (a).

(c) Refer to part (a).

(d) The circular arc in U from -1 to 1 can be parametrized as

$$x = \cos t \quad \text{and} \quad y = b\sin t,$$

where $b \in (-1, 0) \cup (0, 1)$ and $t \in (0, \pi) \cup (\pi, 2\pi)$. Straightforward computation gives

$$\begin{aligned}\frac{1+z}{1-z} &= \frac{1+\cos t + ib\sin t}{1-\cos t - ib\sin t} \times \frac{1-\cos t + ib\sin t}{1-\cos t + ib\sin t}\\ &= \frac{\sin^2 t - b^2\sin^2 t + 2ib\sin t}{(1-\cos t)^2 + b^2\sin^2 t}\\ &= \frac{(1-b^2)\cos^2\frac{t}{2}}{\sin^2\frac{t}{2} + b^2\cos^2\frac{t}{2}} + i\cdot\frac{b\cos\frac{t}{2}}{\sin\frac{t}{2}(\sin^2\frac{t}{2} + b^2\cos^2\frac{t}{2})}.\end{aligned}$$

which implies

$$\begin{aligned}\log\frac{1+z}{1-z} &= \frac{1}{2}\log\left|\frac{1+z}{1-z}\right|^2 + i\arg\frac{1+z}{1-z}\\ &= \frac{1}{2}\log\frac{\left[(1-b^2)^2\sin^2\frac{t}{2}\cos^2\frac{t}{2} + b^2\right]\cos^2\frac{t}{2}}{\sin^2\frac{t}{2}(\sin^2\frac{t}{2} + b^2\cos^2\frac{t}{2})^2} + i\tan^{-1}\frac{2b}{(1-b^2)\sin t}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}f(z) &= \exp\left[-\tan^{-1}\frac{2b}{(1-b^2)\sin t}\right]\\ &\quad\times\exp\left\{\frac{i}{2}\log\frac{\left[(1-b^2)^2\sin^2\frac{t}{2}\cos^2\frac{t}{2} + b^2\right]\cos^2\frac{t}{2}}{\sin^2\frac{t}{2}(\sin^2\frac{t}{2} + b^2\cos^2\frac{t}{2})^2}\right\}\end{aligned} \tag{14.125}$$

so that

$$|f(\cos t, b\sin t)| = \exp\left[-\tan^{-1}\frac{2b}{(1-b^2)\sin t}\right].$$

Furthermore, suppose that

$$F(b,t) = \frac{\left[(1-b^2)^2\sin^2\frac{t}{2}\cos^2\frac{t}{2} + b^2\right]\cos^2\frac{t}{2}}{\sin^2\frac{t}{2}(\sin^2\frac{t}{2} + b^2\cos^2\frac{t}{2})^2}$$

which is continuous on $[(-1, 0) \cup (0, 1)] \times [(0, \pi) \cup (\pi, 2\pi)]$. Now for any fixed b, we have $F(b, t) \to \infty$ as $t \to 0$. This implies that $\arg f(z)$ takes any value in $[0, 2\pi]$.

For example, put $b = \frac{1}{2}$ into the equation (14.125), we see that

$$f(z) = \exp\Big(-\tan^{-1}\frac{4}{3\sin t}\Big) \times \exp\Big[\frac{i}{2}\log\frac{(9\sin^2\frac{t}{2}\cos^2\frac{t}{2}+4)\cos^2\frac{t}{2}}{\sin^2\frac{t}{2}(4\sin^2\frac{t}{2}+\cos^2\frac{t}{2})^2}\Big],$$

where $t \in (0,\pi)\cup(\pi,2\pi)$. For $t \in (0,\pi)$, the locus of $f(z)$ is pictured in Figure 14.6:

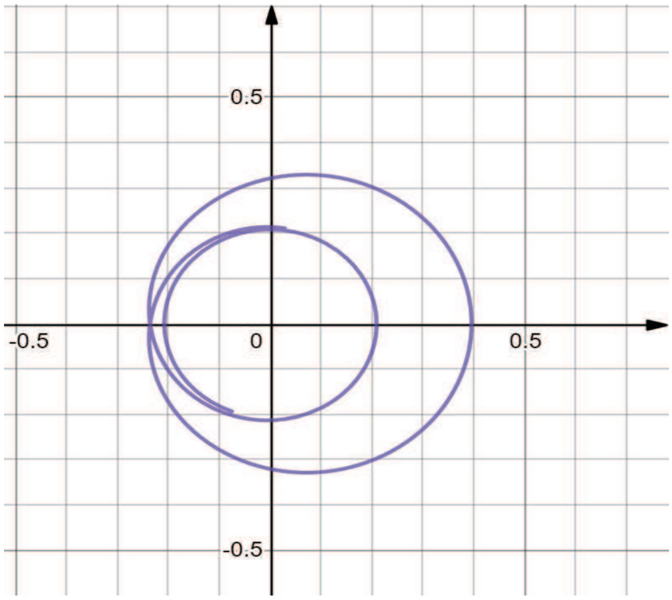


Figure 14.6: The locus of $f(z)$ for $t \in (0,\pi)$.

Similarly, Figure 14.7 shows the locus of $f(z)$ for $t \in (\pi,2\pi)$:

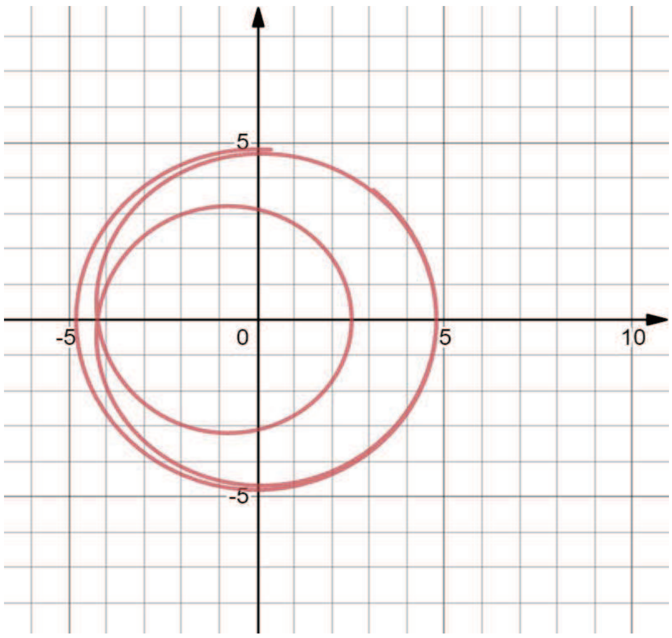


Figure 14.7: The locus of $f(z)$ for $t \in (\pi,2\pi)$.

(e) On the radius $[0,1)$, z is real so that $\log\frac{1+z}{1-z}$ is also real. In this case, $|f(z)| = 1$ for every $z \in [0,1)$, so $f\big([0,1)\big)$ starts at $z = 1$ and runs through the unit circle T anticlockwise infinitely many times.

(f) Suppose that $z = r + (1-r)e^{i\theta}$, where $0 < r < 1$ and $-\pi \le \theta \le \pi$. Then it is easy to check that

$$\omega = \varphi(z) = \frac{1+z}{1-z} = \frac{r}{1-r} + i\frac{\cot\frac{\theta}{2}}{1-r}$$

so that φ maps the circle $C(r;1-r)$ onto the vertical line $\operatorname{Re}\omega = \frac{r}{1-r}$. Hence φ maps the disc $E = D(r;1-r)$ conformally onto the vertical strip $\{\omega \,|\, 0 < \operatorname{Re}\omega < \frac{r}{1-r}\}$. Next, we have

$$\log\varphi(z) = \frac{1}{2}\log\frac{r^2+\cot^2\frac{\theta}{2}}{(1-r)^2} + i\arg\frac{\cot\frac{\theta}{2}}{r}$$

so that

$$f(z) = \exp\Big(-\arg\frac{\cot\frac{\theta}{2}}{r}\Big) \times \exp\Big[\frac{i}{2}\log\frac{r^2+\cot^2\frac{\theta}{2}}{(1-r)^2}\Big].$$

Fix an r. As θ runs through $[-\pi,\pi]$, $\frac{\theta}{2}$ will run through $[-\frac{\pi}{2},\frac{\pi}{2}]$ so that $\arg\frac{\cot\frac{\theta}{2}}{r}$ runs through $[-\frac{\pi}{2},\frac{\pi}{2}]$. Since $|f(z)| = \exp(-\arg\frac{\cot\frac{\theta}{2}}{r})$, this means that $f\big(C(r;1-r)\big)$ is a curve connecting the two circles $C(0;e^{-\frac{\pi}{2}})$ and $C(0;e^{\frac{\pi}{2}})$, and thus f maps E onto the annulus $A(e^{-\frac{\pi}{2}},e^{\frac{\pi}{2}})$, see Figure 14.8 when $r = 0.25$.

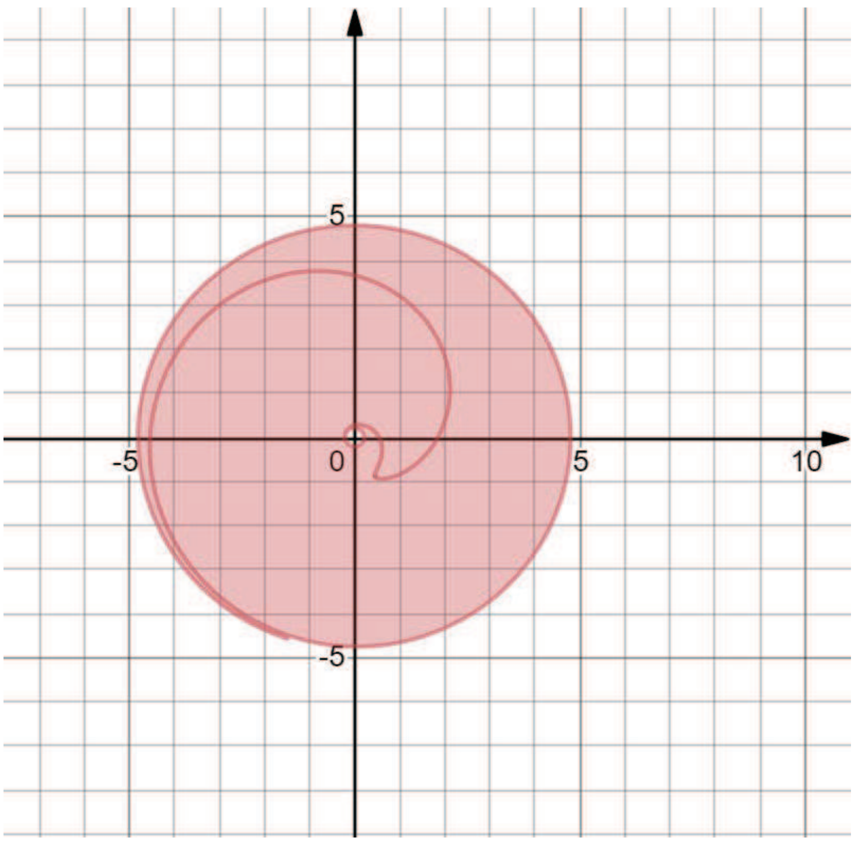


Figure 14.8: The image of $f(E)$ when $r = 0.25$.

(g) Suppose that $\gamma : [0,1] \to U$ is a curve in U such that $\gamma(t) \to 1$ as $t \to 1$. Generally speaking, $f\big(\gamma(t)\big)$ is also a curve in $\mathbb{C}$ and we have to study the behaviour of $f\big(\gamma(t)\big)$ as $t \to 1$. The fact $\gamma(t) \to 1$ as $t \to 1$ implies that $\gamma(t) \in D(r;1-r)$ for any $0 < r < 1$ and for all t sufficiently close to 1. Furthermore, $\gamma(t) \to 1$ as $t \to 1$ also implies that $\theta(t) \to 0$ so that

$$\exp\Big(-\arg\frac{\cot\frac{\theta(t)}{2}}{r}\Big) \to e^{-\frac{\pi}{2}}$$

as $t \to 1$. Therefore, it follows from part (f) that

$$\lim_{t\to 1} \left|f\big(\gamma(t)\big)\right| = \mathrm{e}^{-\frac{\pi}{2}}.$$

We have completed the analysis of the problem. ■

Remark 14.6

In the literature, there are two ways to define conformal maps. The first definition says that a holomorphic function f to be conformal at all points with $f'(z) \neq 0$. This is the only Rudin uses in his book, see also [2, p. 73]. The second way to define a conformal map f is that it is one-to-one and holomorphic on an open set in the plane, see for example [65, p. 206].

Problem 14.33

Rudin Chapter 14 Exercise 33.

Proof.

(a) We may write $\varphi_\alpha = u + iv = (u, v)$, so we may think of $\varphi_\alpha : U \to U$ is given by

$$\varphi_\alpha(x, y) = \big(u(x, y), v(x, y)\big)$$

which is an invertible C^2 mapping. Now the Jacobian of φ_α is

$$J_{\varphi_\alpha}(x, y) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

Therefore, it follows from this and the Cauchy-Riemann equations that

$$\det J_{\varphi_\alpha}(x, y) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y} = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2 = \left|\varphi'_\alpha(x, y)\right|^2 \neq 0$$

for all $(x, y) \in U$. Finally, we follow from the Change of Variables Theorem [61, Theorem 10.9, pp. 252, 253] that

$$\pi = \int_U 1\,\mathrm{d}x\,\mathrm{d}y = \int_U \det J_{\varphi_\alpha}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_U \left|\varphi'_\alpha(x, y)\right|^2 \mathrm{d}x\,\mathrm{d}y = \int_U |\varphi'_\alpha|^2\,\mathrm{d}m$$

which is exactly our required result.

(b) An easy computation gives

$$\begin{aligned} \left|\varphi'_\alpha(z)\right| &= \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}z|^2} \\ &= \frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)(1 - \alpha\overline{z})} \\ &= (1 - |\alpha|^2)\cdot\Big\{\sum_{n=0}^{\infty}(\overline{\alpha}z)^n\Big\}\cdot\Big\{\sum_{k=0}^{\infty}(\alpha\overline{z})^k\Big\} \end{aligned} \tag{14.126}$$

holds for all $z\in U$. Since $|\alpha|<1$, the radii of convergence of the power series in the expression (14.126) are $\frac{1}{|\alpha|}>1$. Consequently, they converge absolutely and uniformly in U so that an interchange of integration and summation is legitimate. In terms of polar coordinates, this means that

$$\begin{aligned}
\frac{1}{\pi}\int_U |\varphi'_\alpha|\,\mathrm{d}m &= \frac{1-|\alpha|^2}{\pi}\int_U \sum_{n,k=0}^{\infty}(\overline{\alpha}z)^n(\alpha\overline{z})^k\,\mathrm{d}m \\
&= \frac{1-|\alpha|^2}{\pi}\sum_{n,k=0}^{\infty}\int_U (\overline{\alpha}z)^n(\alpha\overline{z})^k\,\mathrm{d}m \\
&= \frac{1-|\alpha|^2}{\pi}\sum_{n,k=0}^{\infty}\int_0^{2\pi}\int_0^1 (\overline{\alpha}r\mathrm{e}^{i\theta})^n(\alpha r\mathrm{e}^{-i\theta})^k r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1-|\alpha|^2}{\pi}\cdot\sum_{n,k=0}^{\infty}(\overline{\alpha}^n\alpha^k)\int_0^{2\pi}\int_0^1 r^{n+k+1}\mathrm{e}^{i(n-k)\theta}\,\mathrm{d}r\,\mathrm{d}\theta. \qquad (14.127)
\end{aligned}$$

If $n\neq k$, then

$$\int_0^{2\pi}\int_0^1 r^{n+k+1}\mathrm{e}^{i(n-k)\theta}\,\mathrm{d}r\,\mathrm{d}\theta = 0.$$

Thus the expression becomes

$$\begin{aligned}
\frac{1}{\pi}\int_U |\varphi'_\alpha|\,\mathrm{d}m &= \frac{1-|\alpha|^2}{\pi}\cdot\sum_{n=0}^{\infty}|\alpha|^{2n}\int_0^{2\pi}\int_0^1 r^{2n+1}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1-|\alpha|^2}{\pi}\cdot\sum_{n=0}^{\infty}|\alpha|^{2n}\frac{\pi}{n+1} \\
&= (1-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n+1}. \qquad (14.128)
\end{aligned}$$

Consider the power series expansion of $\log(1+z)$ about $z=0$, we know that

$$\log(1+z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{n+1}}{n+1}$$

which has radius of convergence 1. Substituting this with $z=-|\alpha|^2$ into the right-hand side of the formula (14.128), we obtain finally that

$$\frac{1}{\pi}\int_U |\varphi'_\alpha|\,\mathrm{d}m = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.$$

We have ended the proof of the problem. ■

Remark 14.7

Problem 14.33(a) is a special case of the classical result **Lusin Area Integral**, see [34, p. 150].

CHAPTER 15

Zeros of Holomorphic Functions

15.1 Infinite Products and the Order of Growth of an Entire Function

Problem 15.1

Rudin Chapter 15 Exercise 1.

Proof. Let S be the set in which the infinite product converges uniformly. Define $u_n(z) = \frac{b_n - a_n}{z - b_n}$. Note that

$$\frac{z - a_n}{z - b_n} = 1 + \frac{b_n - a_n}{z - b_n} = 1 + u_n(z).$$

Suppose that

$$\delta = d\big(S, \{b_n\}\big) = \inf\{|z - \omega| \mid z \in S \text{ and } \omega \in \{b_n\}\} > 0.$$

Then it is easy to see that

$$|u_n(z)| \leq \frac{1}{\delta}|b_n - a_n| < \infty$$

for every $n \in \mathbb{N}$ and $z \in S$. Furthermore, we know that

$$\sum_{n=1}^{\infty} |u_n(z)| = \sum_{n=1}^{\infty} \frac{|b_n - a_n|}{|z - b_n|} \leq \frac{1}{\delta} \sum_{n=1}^{\infty} |b_n - a_n| < \infty$$

for every $z \in S$. By Theorem 15.4, the infinite product

$$f(z) = \prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n} \tag{15.1}$$

converges uniformly on S.

Clearly, S° is an open set in $\mathbb{C}$, $f_n(z) = \frac{z - a_n}{z - b_n} \in H(S^\circ)$ for $n = 1, 2, \ldots$ and $f_n \not\equiv 0$ in any component of S°. By the above paragraph,

$$\sum_{n=1}^{\infty} |1 - f_n(z))| = \sum_{n=1}^{\infty} |u_n(z)|$$

converges uniformly on every compact subset of S°. By Theorem 15.6, the infinite product (15.1) is holomorphic in S°. This ends the proof of the problem. ∎

Problem 15.2

Rudin Chapter 15 Exercise 2.

Proof. Denote λ to be the order of the entire function f. By the definition, we have

$$\lambda = \inf\{\rho \,|\, |f(z)| < \exp(|z|^\rho) \text{ holds for all large enough } |z|\}.$$

Using the fact $a_n = \frac{f^{(n)}(0)}{n!}$ and Theorem 10.26 (Cauchy's Estimates), we have

$$|a_n| \leq \frac{e^{r^\lambda}}{r^n} \tag{15.2}$$

for large enough r. Let $g(r) = r^{-n}\exp(r^\lambda)$, where $r > 0$. Applying elementary differentiation, we can show that g attains its minimum

$$\left(\frac{\lambda}{n}\right)^{\frac{n}{\lambda}} \exp\left(\frac{n}{\lambda}\right)$$

at $r = n^{\frac{1}{\lambda}}\lambda^{-\frac{1}{\lambda}}$. Note that r is large if and only if n is large. Thus it follows from the inequality (15.2) that

$$|a_n| \leq \left(\frac{\lambda}{n}\right)^{\frac{n}{\lambda}} \exp\left(\frac{n}{\lambda}\right) = \left(\frac{e\lambda}{n}\right)^{\frac{n}{\lambda}}$$

holds for all large enough n.

Consider the entire functions $f(z) = \exp(z^k)$, where $k = 1, 2, \ldots$. It is clear that $\lambda = k$. By the power series expansion of e^{z^k}, we have $a_{nk} = \frac{1}{n!}$. By induction, we obtain

$$|a_{nk}| = \frac{1}{n!} < \left(\frac{e}{n}\right)^n$$

for every large enough n. Consequently, the above bound is not close to best possible. This completes the analysis of the proof. ∎

Problem 15.3

Rudin Chapter 15 Exercise 3.

Proof. The part of finding solutions of $e^{e^z} = 1$ has been solved in [76, Problem 3.19, pp. 44 – 45]. In fact, they are given by

$$z = \begin{cases} \ln(2k\pi) + i\Big(2n\pi + \dfrac{\pi}{2}\Big), & \text{if } k > 0 \text{ and } n \in \mathbb{Z}; \\[2ex] \ln(-2k\pi) + i\Big(2n\pi + \dfrac{3\pi}{2}\Big), & \text{if } k < 0 \text{ and } n \in \mathbb{Z}. \end{cases} \tag{15.3}$$

Denote the zeros (15.3) by $z_{k,n}$, where $k, n \in \mathbb{Z}$ and $k \neq 0$, see Figure 15.1.

Assume that f was an entire function of finite order having a zero at each (15.3). Suppose further that $f \not\equiv 0$. Consider the disc $\overline{D(0, R_N)}$, where $R_N = N\pi$ and N is a sufficiently large positive integer. Then we have $z_{k,n} \in \overline{D(0, R_N)}$, where

$$\frac{\exp(N\pi)}{2\pi} - 1 \leq k \leq \frac{\exp(N\pi)}{2\pi}.$$

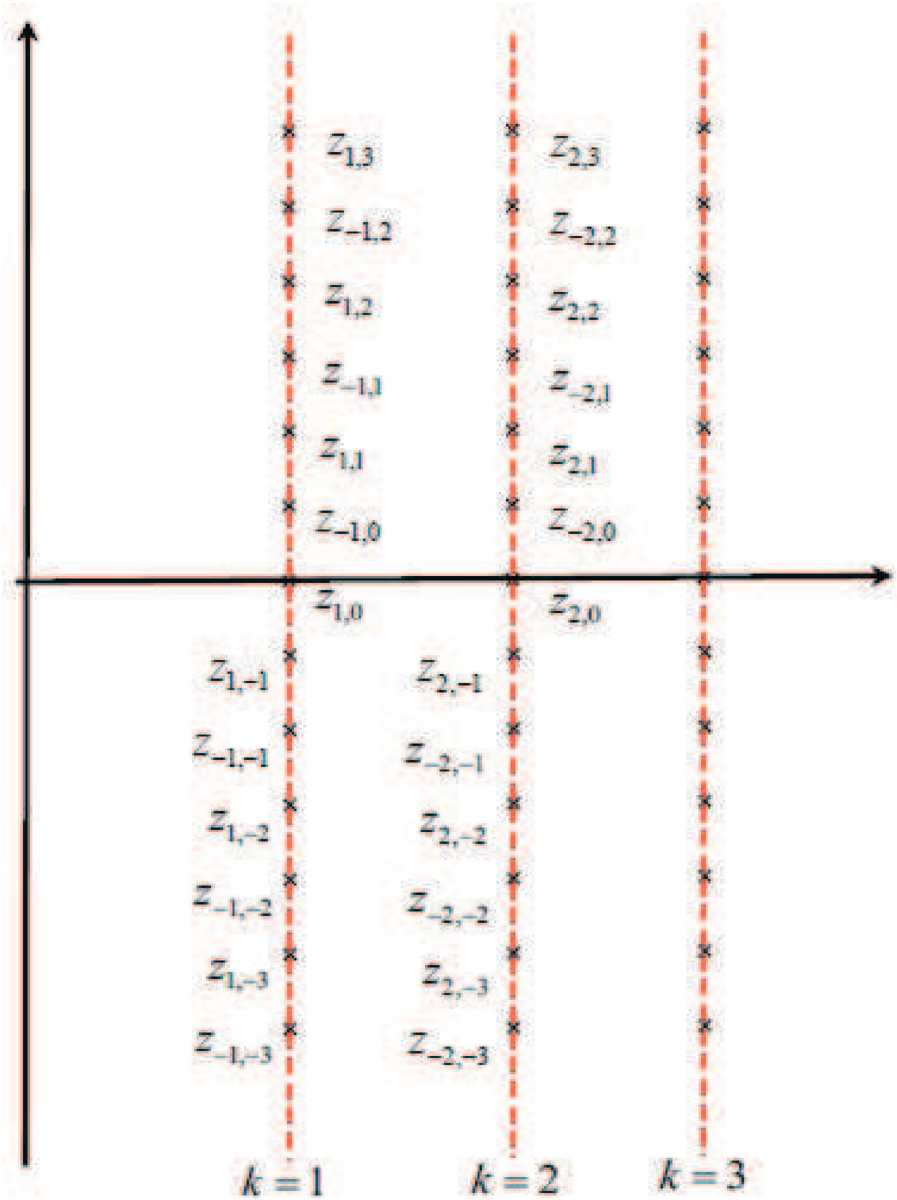


Figure 15.1: The distribution of the zeros $z_{k,n}$ of $\exp(\exp(z))$.

Therefore, we gain

$$n(R_N) \geq \frac{\exp(N\pi)}{4\pi}$$

so that

$$\frac{\log n(R_N)}{\log R_N} \geq \frac{N\pi - \log 4\pi}{\log N\pi} \to \infty$$

as $N \to \infty$, but it means that f is of infinite order, a contradiction. Hence no such entire function exists and we finish the proof of the problem. ■

Problem 15.4

Rudin Chapter 15 Exercise 4.

Proof. We prove the assertions one by one.

- **Both functions have a simple pole with residue 1 at each integer.** Since $\sin \pi z$ has a simple zero at every integer N, we have

$$\operatorname{Res}(\pi \cot \pi z; N) = \operatorname{Res}\Big(\frac{\pi \cos \pi z}{\sin \pi z}; N\Big) = \frac{\pi \cos \pi N}{\pi \cos \pi N} = 1$$

which shows that $\pi \cot \pi z$ has a simple pole with residue 1 at each integer.

We claim that

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{z}{n(z-n)} \tag{15.4}$$

converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$. To see this, let $|z| \leq R$ with $R > 0$. Then we have

$$\sum_{|n|\geq 2R} \frac{|z|}{|n|\cdot|n-z|} \leq \sum_{|n|\geq 2R} \frac{R}{|n|\cdot(|n|-R)} \leq \sum_{|n|\geq 2R} \frac{R}{|n|\cdot|\frac{n}{2}|} \leq 2R \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^2} < \infty$$

which gives the desired claim. Since

$$\frac{z}{n(z-n)} + \frac{z}{(-n)(z+n)} = \frac{2z}{z^2-n^2},$$

the function f can be expressed as

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{z}{n(z-n)} = \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big).$$

By Theorem 10.28, we see that $f \in H(\mathbb{C} \setminus \mathbb{Z})$. Furthermore, for each $N \in \mathbb{N}$, we see that

$$\lim_{z\to N} (z-N)f(z) = \lim_{z\to N} \Big[\frac{z-N}{z} + 1 + \frac{z-N}{N} + (z-N) \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,N}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big)\Big] = 1$$

and

$$\lim_{z\to N} (z-N)^2 f(z) = \lim_{z\to N} \Big[\frac{(z-N)^2}{z} + (z-N) + \frac{(z-N)^2}{N} + (z-N)^2 \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,N}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big)\Big] = 0,$$

so it follows from [9, Theorem 9.5, p. 118] that f also has a simple pole with residue 1 at each integer N. Simple computation gives

$$\sum_{\substack{n=-N\\ n\neq 0}}^{N} \Big(\frac{1}{z-n} + \frac{1}{n}\Big) = -\frac{1}{z} + \sum_{n=-N}^{N} \frac{1}{z-n},$$

so the function f also has the form

$$f(z) = \frac{1}{z} + \lim_{N\to\infty} \sum_{\substack{n=-N\\ n\neq 0}}^{N} \Big(\frac{1}{z-n} + \frac{1}{n}\Big) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n}.$$

- **Both functions are periodic.** If $z \notin \mathbb{Z}$, then it is obvious to check that
$$\pi \cot \pi(z+1) = \pi i \cdot \frac{e^{\pi i(z+1)} + e^{-\pi i(z+1)}}{e^{\pi i(z+1)} - e^{-\pi i(z+1)}} = \pi i \cdot \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} = \pi \cot \pi z.$$
By the representation (15.4), we have
$$\begin{aligned} f(z+1) &= \frac{1}{z+1} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Big[\frac{1}{z-(n-1)} + \frac{1}{n}\Big] \\ &= \frac{1}{z+1} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq -1}} \Big(\frac{1}{z-n} + \frac{1}{n+1}\Big) \\ &= \frac{1}{z+1} + 1 + \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq -1,0}} \Big(\frac{1}{z-n} + \frac{1}{n} - \frac{1}{n} + \frac{1}{n+1}\Big) \\ &= \frac{1}{z+1} + 1 + \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq -1,0}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big) - \sum_{\substack{n\in\mathbb{Z}\\ n\neq -1,0}} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) \\ &= \frac{1}{z+1} - 1 + \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq -1,0}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big) \\ &= \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big) \\ &= f(z). \end{aligned}$$
Hence both functions are periodic.

- **Their difference is a constant.** Now the function $\Delta(z) = \pi \cot \pi z - f(z)$ must be entire and periodic (i.e., $\Delta(z+1) = \Delta(z)$). To show that it is bounded, it is enough to show that $\Delta(z)$ is bounded in the strip $|\mathrm{Re}\, z| \leq \frac{1}{2}$. By Theorem 10.24 (The Maximum Modulus Theorem), Δ is bounded in the rectangle $\{z \in \mathbb{C} \mid |\mathrm{Re}\, z| \leq \frac{1}{2}$ and $|\mathrm{Im}\, z| \leq 1\}$. Let $z = x + iy$, where $|x| \leq \frac{1}{2}$ and $|y| > 1$. On the one hand, we have
$$\cot \pi z = i \cdot \frac{e^{-2\pi y} + e^{-2\pi ix}}{e^{-2\pi y} - e^{-2\pi ix}}$$
so that
$$|\cot \pi z| \leq \frac{|e^{-2\pi y}| + 1}{\big||e^{-2\pi y}| - 1\big|} < \infty.$$
On the other hand, we write
$$f(z) = \frac{1}{x+iy} + \sum_{n=1}^{\infty} \frac{2(x+iy)}{x^2 - y^2 - n^2 + 2ixy}.$$
If $y > 1$ and $|x| \leq \frac{1}{2}$, we have $|x + iy| \leq \sqrt{2}y$ and
$$|x^2 - y^2 - n^2 + 2ixy| = \sqrt{[x^2 - (y^2+n^2)]^2 + 4x^2y^2} \geq \Big|(y^2+n^2) - \frac{1}{4}\Big| \geq \frac{y^2+n^2}{2}.$$
Thus they imply that
$$|f(z)| \leq 1 + \sum_{n=1}^{\infty} \frac{2|x+iy|}{|x^2 - y^2 - n^2 + 2ixy|}$$

$$
\begin{aligned}
&\leq 1 + 4\sqrt{2} \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} \\
&\leq 1 + 4\sqrt{2} \int_0^{\infty} \frac{y\,\mathrm{d}x}{y^2 + x^2}.
\end{aligned} \tag{15.5}
$$

By the change of variable $x = yt$, it is easily checked that the integral in the inequality (15.5) becomes

$$
\int_0^{\infty} \frac{y\,\mathrm{d}x}{y^2 + x^2} = \int_0^{\infty} \frac{\mathrm{d}t}{1 + t^2}
$$

which implies that $|f(z)| \leq M$ for some $M > 0$ and for all $|x| \leq \frac{1}{2}$ and $y > 1$. Similarly, $f(z)$ is also bounded for all $|x| \leq \frac{1}{2}$ and $y < -1$. Consequently, we have shown that $\Delta(z)$ is a bounded entire function and Theorem 10.23 (Liouville's Theorem) says that it is in fact a constant.

To find this constant, we note that

$$
\begin{aligned}
\lim_{y\to\infty} f(iy) &= \lim_{y\to\infty} \Big[\frac{1}{iy} + \sum_{n=1}^{\infty} \frac{2iy}{-y^2 - n^2}\Big] \\
&= -2i \lim_{y\to\infty} \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} \\
&= -2i \int_0^{\infty} \frac{\mathrm{d}t}{1 + t^2} \\
&= -\pi i
\end{aligned}
$$

and

$$
\lim_{y\to\infty} \pi \cot i\pi y = -\pi i.
$$

Thus $\Delta(z) \equiv 0$ and then

$$
\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{15.6}
$$

- **The product representation of $\dfrac{\sin \pi z}{\pi z}$.** If $g(z) = \sin \pi z$, then it is clear that

$$
\frac{g'(z)}{g(z)} = \pi \cot \pi z.
$$

Consequently, we observe from the representation (15.6) that

$$
\frac{g'(z)}{g(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{15.7}
$$

Next, we consider the infinite product

$$
P(z) = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big). \tag{15.8}
$$

Now each $P_n(z) = 1 - \frac{z^2}{n^2} \in H(\mathbb{C} \setminus \mathbb{Z})$ and $P_n \not\equiv 0$ in $\mathbb{C} \setminus \mathbb{Z}$. In addition, if K is a compact subset of $\mathbb{C} \setminus \mathbb{Z}$, then the Weierstrass M-test [61, Theorem 7.10, p. 148] guarantees that

$$
\sum_{n=1}^{\infty} |1 - P_n(z)| = \sum_{n=1}^{\infty} \frac{|z|^2}{n^2}
$$

converges uniformly on K. By Theorem 15.6, the product (15.8) is holomorphic in $\mathbb{C} \setminus \mathbb{Z}$. Using [18, Exercise 10, p. 174], we have

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{P_n'(z)}{P_n(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \tag{15.9}$$

for every $z \in \mathbb{C} \setminus \mathbb{Z}$. Substituting the result (15.9) into the formula (15.7), we see that

$$\frac{g'(z)}{g(z)} = \frac{P'(z)}{P(z)}$$

and then it gives

$$\left[\frac{P(z)}{g(z)}\right]' = 0.$$

Therefore, we have $P(z) = cg(z)$ for some constant c in $\mathbb{C} \setminus \mathbb{Z}$. Since $\frac{P(z)}{z} \to 1$ as $z \to 0$, we have $c = 1$ and eventually,

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{15.10}$$

in $\mathbb{C} \setminus \mathbb{Z}$. Since $\sin \pi z$ and $\pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$ agree on $\mathbb{Z}$, the formula (15.10) holds in the whole plane.

Hence we have completed the analysis of the problem. ■

Remark 15.1

Another way to prove Problem 15.4 is to consider the square C_N with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$ for $N \in \mathbb{N}$. According to Theorem 10.42 (The Residue Theorem), we have

$$\frac{1}{2\pi i} \int_{C_N} \frac{\cot \pi z}{z - \zeta} \, \mathrm{d}z = \sum_k \operatorname{Res}\left(\frac{\cot \pi z}{z - \zeta}; z_k\right), \tag{15.11}$$

where $\zeta \in \mathbb{C}$ and the z_k denotes a pole of $g(z) = \frac{\cot \pi z}{z - \zeta}$ inside C_N. Take $\zeta \notin \mathbb{Z}$ and ζ to be a point inside C_N. Then it is clear that the poles of the function $g(z)$ occur at $z = \zeta$ and at $z = n \in \{-N, -N+1, \ldots, 0, 1, \ldots, N\}$, and they are all simple. By the basic method of evaluating residue [9, p. 129], we know that

$$\operatorname{Res}\left(\frac{\cot \pi z}{z - \zeta}; \zeta\right) = \frac{\cot \pi \zeta}{1} = \cot \pi \zeta \quad \text{and} \quad \operatorname{Res}\left(\frac{\cot \pi z}{z - \zeta}; n\right) = \frac{1}{\pi(n - \zeta)}.$$

By putting these values into the equation (15.11), we get

$$\frac{1}{2i} \int_{C_N} \frac{\cot \pi z}{z - \zeta} \, \mathrm{d}z = \pi \cot \pi \zeta - \sum_{n=-N}^{N} \frac{1}{\zeta - n}.$$

It can be shown that

$$\lim_{N \to \infty} \int_{C_N} \frac{\cot \pi z}{z - \zeta} \, \mathrm{d}z = 0$$

which implies the desired result. For details, please refer to [54, Lemma 7.22, pp. 181, 182].

Problem 15.5

Rudin Chapter 15 Exercise 5.

Proof. By Theorem 15.9, our f is an entire function having a zero at each point z_n. We claim that

$$M(r) < \exp(|z|^{k+1})$$

for sufficiently large enough $|z|$. If $|z| < \frac{1}{2}$, then

$$\begin{aligned}\log|E_k(z)| &= \operatorname{Re}\Big[\log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\Big] \\ &= \operatorname{Re}\Big(-\frac{1}{k+1}z^{k+1} - \frac{1}{k+1}z^{k+2} - \cdots\Big) \\ &\leq |z|^{k+1}\Big(\frac{1}{k+1} + \frac{|z|}{k+2} + \frac{|z|^2}{k+3} + \cdots\Big) \\ &\leq |z|^{k+1}\Big(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\Big) \\ &\leq 2|z|^{k+1}. \end{aligned} \tag{15.12}$$

Since $|E_k(z)| \leq \big(1+|z|\big)\exp\big(|z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^k}{k}\big)$, we have

$$\log|E_k(z)| \leq \log\big(1+|z|\big) + |z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^k}{k}$$

which gives

$$\lim_{z\to\infty}\frac{\log|E_k(z)|}{|z|^{k+1}} = 0$$

so that if $M_1 > 0$, then there exists a $R > 0$ such that

$$\log|E_k(z)| < M_1|z|^{k+1} \tag{15.13}$$

for all $|z| > R$. On the set $S = \{z \in \mathbb{C} \,|\, \frac{1}{2} \leq |z| \leq R\}$, the function $g(z) = |z|^{-(k+1)}\log|E_k(z)|$ is continuous except at $z = 1$, where $g(z) \to -\infty$ as $z \to 1$. Thus there is a constant $M_2 > 0$ such that

$$\log|E_k(z)| \leq M_2|z|^{k+1} \tag{15.14}$$

for all $z \in S$.

Let $M = \max\{2, M_1, M_2\}$. Now we combine the inequalities (15.12), (15.13) and (15.14) to get

$$\log|E_k(z)| \leq M|z|^{k+1} \tag{15.15}$$

for every $z \in \mathbb{C}$. By the hypothesis, one can find an $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty}\frac{1}{|z_n|^{k+1}} < \frac{1}{2M} \quad\text{and}\quad \sum_{n=1}^{N}\frac{1}{|z_n|^{k+1}} \leq N.$$

Using the inequality (15.15), we obtain

$$\sum_{n=N+1}^{\infty}\log\Big|E_k\Big(\frac{z}{z_n}\Big)\Big| \leq M\sum_{n=N+1}^{\infty}\Big|\frac{z}{z_n}\Big|^{k+1} < \frac{|z|^{k+1}}{2}. \tag{15.16}$$

Since M_1 can be chosen arbitrary, we deduce from the inequality (15.13) that there exists a $R_1 > 0$ such that

$$\log|E_k(z)| \leq \frac{|z|^{k+1}}{2N}$$

for all $|z| > R_1$. Let $R_2 = \max\{|z_1|R_1, |z_2|R_1, \ldots, |z_N|R_1\}$. Then it is obvious that

$$\sum_{n=1}^{N} \log\left|E_k\left(\frac{z}{z_n}\right)\right| \leq \frac{|z|^{k+1}}{2} \tag{15.17}$$

for all $|z| > R_2$. Finally, the inequalities (15.16) and (15.17) give

$$\log|f(z)| = \sum_{n=1}^{\infty} \log\left|E_k\left(\frac{z}{z_n}\right)\right| \leq |z|^{k+1}$$

for all $|z| > R_2$, and it is equivalent to saying that

$$|f(z)| < \exp(|z|^{k+1})$$

for all $|z| > R_2$. This proves our claim and thus f is of finite order. This completes the analysis of the problem. ■

Problem 15.6

Rudin Chapter 15 Exercise 6.

Proof. Given $\epsilon > 0$. Notice that

$$\sum_{|z_n|\geq 1} |z_n|^{-p-\epsilon} = \sum_{k=0}^{\infty}\Big(\sum_{2^k\leq|z_n|<2^{k+1}} |z_n|^{-p-\epsilon}\Big) \leq \sum_{k=0}^{\infty} 2^{-k(p+\epsilon)} n(2^{k+1}). \tag{15.18}$$

Since $|f(z)| < \exp(|z|^p)$, it follows from [62, Eqn. (2) & (3), p. 309] that

$$n(r) \leq Cr^p \tag{15.19}$$

for some constant $C > 0$ and all sufficiently large enough r. Combining the inequalities (15.18) and (15.19), we see that

$$\sum_{|z_n|\geq 1} |z_n|^{-p-\epsilon} < C\sum_{k=0}^{\infty} 2^{-k(p+\epsilon)} \cdot 2^{(k+1)p} = C\cdot 2^p \sum_{k=0}^{\infty}\left(\frac{1}{2^\epsilon}\right)^k < \infty.$$

Hence we have

$$\sum_{n=1}^{\infty} |z_n|^{-p-\epsilon} < \infty$$

for every $\epsilon > 0$, completing the proof of the problem. ■

Problem 15.7

Rudin Chapter 15 Exercise 7.

Proof. Without loss of generality, we may assume that $f \not\equiv 0$. By the definition (see Problem 15.2), f is of finite order. In the disc $D\big(0; N+\frac{1}{2}\big)$ for large enough positive integer N, the number of zeros of f inside $D\big(0; N+\frac{1}{2}\big)$ is *at least* N^2, i.e., $n(N+\frac{1}{2}) \geq N^2$. By the hypothesis, we know that $M(r) < \exp(|z|^\alpha)$, so it follows from [62, Eqn. (4), p. 309] that

$$2 \leq \limsup_{N\to\infty} \frac{\log n(N+\frac{1}{2})}{\log N} \leq \alpha.$$

Therefore, if $0 < \alpha < 2$, then no entire function can satisfy the hypotheses of the problem. In other words, $f(z) = 0$ for all $z \in \mathbb{C}$ if $0 < \alpha < 2$, completing the proof of the problem. ■

15.2 Some Examples

Problem 15.8

Rudin Chapter 15 Exercise 8.

Proof. Let $A = \{z_n\}$. We are going to verify the results one by one.

- **f is independent of the choice of $\gamma(z)$.** Suppose that $\gamma, \eta : [0,1] \to \mathbb{C}$ are paths from 0 to z and they pass through none of the points z_n. Then $\Gamma = \gamma - \eta$ is a simple closed path. Let $\{z_1, z_2, \ldots, z_N\}$ be the set of points which are surrounded by Γ for some $N \in \mathbb{N}$. Now we follow from Theorem 10.42 (The Residue Theorem) that

$$\frac{1}{2\pi i}\int_{\Gamma(z)} g(\zeta)\,\mathrm{d}\zeta = \sum_{n=1}^{N} \mathrm{Res}\,(g; z_n) = \sum_{n=1}^{N} m_n$$

which gives

$$\int_{\gamma(z)} g(\zeta)\,\mathrm{d}\zeta = 2\pi i \sum_{n=1}^{N} m_n + \int_{\eta(z)} g(\zeta)\,\mathrm{d}\zeta.$$

Since the summation is a positive integer, it is true that

$$\exp\Big\{\int_{\gamma(z)} g(\zeta)\,\mathrm{d}\zeta\Big\} = \exp\Big\{2\pi i\sum_{n=1}^{N} m_n + \int_{\eta(z)} g(\zeta)\,\mathrm{d}\zeta\Big\} = \exp\Big\{\int_{\eta(z)} g(\zeta)\,\mathrm{d}\zeta\Big\}$$

and this means that f is independent of the choice of $\gamma(z)$.

- **$f \in H(\mathbb{C}\setminus A)$.** The definition of f shows that the holomorphicity of f depends on the holomorphicity of the integral. Let $z \in \mathbb{C}\setminus A$ and denote

$$G(z) = \int_{\gamma(z)} g(\zeta)\,\mathrm{d}\zeta.$$

Now there exists a disc $D(0; R)$ containing z. Let h be so small that $z + h \in D(0; R)$ and $[z, z+h] \cap A = \varnothing$. By Definition 10.41, $D(0; R)$ contains only finitely many points of A. Without loss of generality, we may assume that $\{z_1, z_2, \ldots, z_N\} \subseteq D(0; R)$ for some positive integer N. Then both $\gamma(z)$ and $\gamma(z+h)$ must lie in $\Omega = D(0; R)\setminus\{z_1, z_2, \ldots, z_N\}$. Suppose that $\gamma(z)$ consists of *only* horizontal or vertical line segments in Ω and $\gamma(z+h)$ shares the *same* path with $\gamma(z)$ until the point z. Since $[z, z+h]\cap A = \varnothing$, we can connect z and $z+h$ by another set of horizontal or vertical line segments in a way that only triangles are produced. Figure 15.2 illustrates this setting.

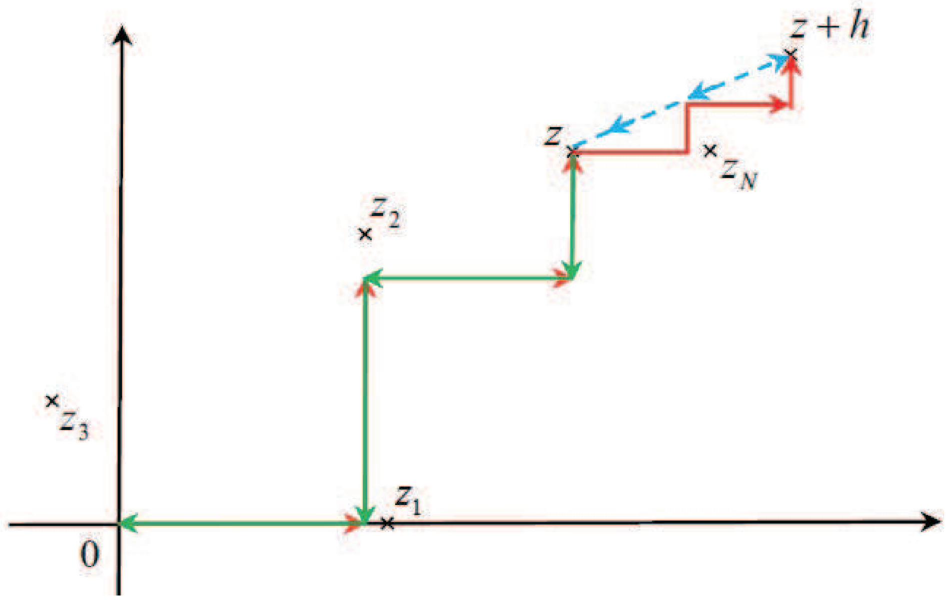


Figure 15.2: The paths $\gamma(z+h)$ and $-\gamma(z)$.

Now we consider the difference

$$\begin{aligned} G(z+h)-G(z) &= \int_{\gamma(z+h)} g(\zeta)\,\mathrm{d}\zeta - \int_{\gamma(z)} g(\zeta)\,\mathrm{d}\zeta \\ &= \sum_{k=1}^{m}\int_{\partial\Delta_k} g(\zeta)\,\mathrm{d}\zeta + \int_{[z,z+h]} g(\zeta)\,\mathrm{d}\zeta \end{aligned} \tag{15.20}$$

where $\Delta_1, \Delta_2, \ldots, \Delta_m$ are the triangles produced. Obviously, the compactness of the set $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$ ensures that there corresponds an open set Ω' in Ω such that $\Delta_k \subseteq \Omega'$ for every $k = 1, 2, \ldots, m$. Since $g \in H(\Omega)$, Theorem 1013 (The Cauchy's Theorem for a Triangle) implies that each integral in the summation (15.20) is 0. As g is continuous at z, we can write $g(\zeta) = g(z) + \epsilon(\zeta)$, where $\epsilon(\zeta) \to 0$ as $\zeta \to z$. Therefore, the expression (15.20) becomes

$$G(z+h)-G(z) = \int_{[z,z+h]} g(z)\,\mathrm{d}\zeta + \int_{[z,z+h]} \epsilon(\zeta)\,\mathrm{d}\zeta = hg(z) + \int_{[z,z+h]} \epsilon(\zeta)\,\mathrm{d}\zeta. \tag{15.21}$$

Since

$$\left|\int_{[z,z+h]} \epsilon(\zeta)\,\mathrm{d}\zeta\right| \leq \sup_{\zeta\in[z,z+h]} |\epsilon(\zeta)| \cdot |h|,$$

the last integral actually tends to 0 as $h \to 0$. Consequently, we have proved that

$$\lim_{h\to\infty} \frac{G(z+h)-G(z)}{h} = g(z),$$

i.e., G is holomorphic at z. Since z is arbitrary, $G \in H(\mathbb{C} \setminus A)$ which implies the desired result that $f \in H(\mathbb{C} \setminus A)$.

- **f has a removable singularity at each z_n.** It suffices to verify that

$$\lim_{z\to z_n} (z - z_n)f(z) = 0. \tag{15.22}$$

Since z_n is a simple pole of g with residue m_n, there exists a $\delta_n > 0$ such that

$$g(z) = \frac{m_n}{z - z_n} + h(z) \tag{15.23}$$

for all $|z - z_n| < 2\delta_n$ and $h \in H\big(D(z_n; 2\delta_n)\big)$.[a] In fact, we can choose δ_n small enough such that $D(z_n; 2\delta_n)$ contains only the pole z_n. Suppose that z lies on the line segment joining 0 and z_n, $z \in \overline{D(z_n; \delta_n)}$ and $\theta_n = \arg z_n$. We split the path $\gamma(z)$ into two paths $\gamma_1(z)$ and $\gamma_2(z)$ as follows: The path $\gamma_1(z)$ is the line segment from 0 to $z_n - \delta_n \mathrm{e}^{i\theta_n}$. If it passes through a pole of g, then we can make a small circular arc around that pole. The path $\gamma_2(z)$ is the line segment from $z_n - \delta_n \mathrm{e}^{i\theta_n}$ to z.

On $\gamma_1(z)$, since δ_n is *fixed* and g is continuous on $\gamma_1(z)$, there exists a positive constant M_1 such that

$$\Big| \int_{\gamma_1(z)} g(\zeta)\,\mathrm{d}\zeta \Big| \leq M_1. \tag{15.24}$$

By the definition, we parameterize $\gamma_2(z) : [0,1] \to \mathbb{C}$ as

$$\gamma_2(z;t) = z + (1-t)(z_n - \delta_n \mathrm{e}^{i\theta_n} - z)$$

so that $\gamma_2(z;0) = z_n - \delta_n \mathrm{e}^{i\theta_n}$ and $\gamma_2(z;1) = z$. Substitute $\gamma_2(z;t)$ into the Laurent series (15.23) to get[b]

$$\begin{aligned}
\int_{\gamma_2(z)} g(\zeta)\,\mathrm{d}\zeta &= m_n \int_0^1 \Big\{ \frac{\big[z - \big(z_n - \delta_n \mathrm{e}^{i\theta_n}\big)\big]\,\mathrm{d}t}{t\big[z - \big(z_n - \delta_n \mathrm{e}^{i\theta_n}\big)\big] - \delta_n \mathrm{e}^{i\theta_n}} \Big\} + \int_{\gamma_2(z)} h(\zeta)\,\mathrm{d}\zeta \\
&= m_n \ln\{t[z - (z_n - \delta_n \mathrm{e}^{i\theta_n})] - \delta_n \mathrm{e}^{i\theta_n}\}\Big|_0^1 + \int_{\gamma_2(z)} h(\zeta)\,\mathrm{d}\zeta \\
&= m_n \big[\ln(z - z_n) - i(\theta_n + \pi) - \ln \delta_n\big] + \int_{\gamma_2(z)} h(\zeta)\,\mathrm{d}\zeta.
\end{aligned} \tag{15.25}$$

Since $h \in H\big(D(z_n; 2\delta_n)\big)$, there exists a positive constant M_2 such that

$$\Big| \int_{\gamma_2(z)} h(\zeta)\,\mathrm{d}\zeta \Big| \leq M_2$$

which gives

$$\Big| \exp\Big\{ \int_{\gamma_2(z)} g(\zeta)\,\mathrm{d}\zeta \Big\} \Big| \leq |z - z_n|^{m_n} \mathrm{e}^{M_2} \times \big| \exp(-m_n \ln \delta_n) \big|. \tag{15.26}$$

Combining the estimate (15.24) and the expression (15.26), we establish

$$\begin{aligned}
|f(z)| &= \Big| \exp\Big\{ \int_{\gamma_1(z)} g(\zeta)\,\mathrm{d}\zeta \Big\} \Big| \times \Big| \exp\Big\{ \int_{\gamma_2(z)} g(\zeta)\,\mathrm{d}\zeta \Big\} \Big| \\
&\leq \mathrm{e}^{M_1 + M_2} \cdot \big| \exp(-m_n \ln \delta_n) \big| \times |z - z_n|^{m_n}
\end{aligned}$$

which implies the limit (15.22).

- **The extension of f has a zero of order m_n at z_n.** Since f has a removable singularity at each z_n, it follows from Definition 10.19 that f can be extended to be holomorphic at z_n. The analysis in the previous part further shows that

$$\begin{aligned}
f(z) &= \exp\Big\{ \int_{\gamma_1(z)} g(\zeta)\,\mathrm{d}\zeta \Big\} \times \exp\Big\{ \int_{\gamma_2(z)} g(\zeta)\,\mathrm{d}\zeta \Big\} \\
&= \exp\Big\{ \int_{\gamma_1(z)} g(\zeta)\,\mathrm{d}\zeta \Big\} \times \exp\Big\{ \int_{\gamma_2(z)} h(\zeta)\,\mathrm{d}\zeta \Big\} \times \delta_n^{-m_n} \mathrm{e}^{-im_n(\theta_n + \pi)} (z - z_n)^{m_n}.
\end{aligned}$$

In other words, f has a zero of order m_n at z_n.

[a] See, for example, [9, Corollary 9.11, p. 124].

[b] If z_n is real, then $\theta_n = 0$ and z is also real. In this case, the integrated result (15.25) becomes $\ln|z - z_n| - \ln \delta_n$.

Thus we have completed the analysis of the problem. ■

Problem 15.9

Rudin Chapter 15 Exercise 9.

Proof. Suppose that $z_1, z_2, \ldots, z_n$ are the zeros of f, listed according to their multiplicities, such that $z_1, z_2, \ldots, z_n \in \overline{D(0;\beta)}$. Define

$$g(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}.$$

Then g is clearly holomorphic in a neighbourhood V containing $\overline{U}$ and $|g(z)| = 1$ on T by Theorem 12.4. Now the function

$$h(z) = \frac{f(z)}{g(z)}$$

must be holomorphic in U. Since $f(U) \subseteq U$, we deduce from Theorem 10.24 (The Maximum Modulus Theorem) that

$$|h(z)| \leq 1$$

for all $z \in U$. If $z = 0$, then we get

$$\frac{\alpha}{|z_1| \times |z_2| \times \cdots \times |z_n|} \leq 1$$

which implies $0 < \alpha \leq \beta^n$. Consequently, we obtain

$$n \leq \frac{\log \alpha}{\log \beta}. \tag{15.27}$$

(a) Put $\alpha = \beta = \frac{1}{2}$ into the inequality (15.27), we conclude that $n \leq 1$.

(b) Similarly, by the inequality (15.27) again, we know that $n \leq 2$.

(c) In this case, we have $n = 0$.

(d) In this case, we have $n \leq 3$.

This completes the analysis of the problem. ■

Problem 15.10

Rudin Chapter 15 Exercise 10.

Proof. Let I be the ideal generated by the set $\{g_N \mid N \in \mathbb{N}\}$. By Definition 15.14, every element of I is of the form

$$f_{N_1} g_{N_1} + f_{N_2} g_{N_2} + \cdots + f_{N_k} g_{N_k} \tag{15.28}$$

for some increasing sequence $\{N_k\}$ of positive integers, where $f_{N_1}, f_{N_2}, \ldots, f_{N_k}$ are entire. By the definition of g_N, if $N < M$, then there exists an entire function h_M such that $g_N = h_M g_M$. Consequently, the element (15.28) can be expressed as

$$f_{N_1} g_{N_1} + f_{N_2} h_{N_2} g_{N_1} + \cdots + f_{N_k} h_{N_k} g_{N_1} = f g_{N_1},$$

where f is entire. Thus we have

$$I = \{fg_N \,|\, f \text{ is entire and } N \geq 1\}. \tag{15.29}$$

Assume that I was principal. One can find an entire function $g \in I$ such that $I = [g]$. By the representation (15.29), we know that $g = fg_N$ for some entire f and some $N \in \mathbb{N}$. Thus we may assume that there exists some positive integer N such that $I = \{fg_N \,|\, f \text{ is entire}\}$. Then we have

$$g_{N+1} = fg_N \tag{15.30}$$

for some entire f. However, since $g_N(N) = 0$ but $g_{N+1}(N) \neq 0$, the equation (15.30) is a contradiction. Hence I is not principal and we have completed the proof of the problem. ■

Problem 15.11

Rudin Chapter 15 Exercise 11.

Proof. Recall from [62, Eqn. (6), p. 281] that $\varphi^{-1}(z) = \frac{z-1}{z+1}$ is a conformal one-to-one mapping of $\Pi = \{z \in \mathbb{C} \,|\, \operatorname{Re} z > 0\}$ onto U. Therefore, we can reduce the problem to the existence of a bounded holomorphic function f in U which is not identically zero and its zeros are precisely at

$$\alpha_n = \frac{iy_n}{2 + iy_n},$$

where $n = 1, 2, \ldots$. In this case, §15.22 shows that $\{\alpha_n\}$ satisfies

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$

(a) We have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(1 - \left|\frac{i \log n}{2 + i \log n}\right|\right) &= \sum_{n=1}^{\infty} \frac{\sqrt{4 + (\log n)^2} - \log n}{\sqrt{4 + (\log n)^2}} \\ &= \sum_{n=1}^{\infty} \frac{4}{\sqrt{4 + (\log n)^2} \cdot \left[\sqrt{4 + (\log n)^2} + \log n\right]}. \end{aligned} \tag{15.31}$$

For $n \geq 3$, we have $\sqrt{4 + (\log n)^2} \leq 2 \log n$ so that

$$\sum_{n=3}^{\infty} \left(1 - \left|\frac{i \log n}{2 + i \log n}\right|\right) \geq \sum_{n=3}^{\infty} \frac{2}{3(\log n)^2} \geq \frac{2}{3} \sum_{n=3}^{\infty} \frac{1}{n}.$$

Hence the series (15.31) diverges and thus no such bounded holomorphic function in Π.

(b) In this case, we know from the A.M. $\geq$ G.M. that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(1 - \left|\frac{i\sqrt{n}}{2 + i\sqrt{n}}\right|\right) &= \sum_{n=1}^{\infty} \frac{\sqrt{4 + n} - \sqrt{n}}{\sqrt{4 + n}} \\ &= \sum_{n=1}^{\infty} \frac{4}{\sqrt{4 + n} \cdot (\sqrt{4 + n} + \sqrt{n})} \\ &\leq \sum_{n=1}^{\infty} \frac{2}{\sqrt{4 + n} \cdot \sqrt[4]{(4 + n)n}} \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{n=1}^{\infty} \frac{1}{(n+4)^{\frac{5}{4}} \cdot n^{\frac{1}{4}}} \\
&< 2\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \\
&< \infty,
\end{aligned}
$$

so the answer is affirmative.

(c) The answer to this part is affirmative because

$$
\begin{aligned}
\sum_{n=1}^{\infty} \Big(1 - \Big|\frac{in}{2+in}\Big|\Big) &= \sum_{n=1}^{\infty} \frac{\sqrt{4+n^2} - n}{\sqrt{4+n^2}} \\
&= \sum_{n=1}^{\infty} \frac{4}{\sqrt{4+n^2} \cdot (\sqrt{4+n^2} + n)} \\
&< \sum_{n=1}^{\infty} \frac{2}{n^2} \\
&< \infty.
\end{aligned}
$$

(d) The answer to this part is affirmative because

$$
\sum_{n=1}^{\infty} \Big(1 - \Big|\frac{in^2}{2+in^2}\Big|\Big) = \sum_{n=1}^{\infty} \frac{|2+in^2| - n^2}{|2+in^2|} \leq \sum_{n=1}^{\infty} \frac{2+n^2-n^2}{n^2} = \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty
$$

We complete the analysis of the problem. ■

Problem 15.12

Rudin Chapter 15 Exercise 12.

Proof. Let $E = \{\frac{1}{\overline{\alpha_n}}\}$ and $\Omega = \mathbb{C} \setminus \overline{E}$. The Blaschke condition implies that $\alpha_n \to e^{i\theta}$ as $n \to \infty$ for some real θ, so $e^{i\theta} \in \overline{E}$. Let K be a compact subset of Ω. Since $\overline{E}$ is closed and $K \cap \overline{E} = \varnothing$, Problem 10.1 shows that $\delta = d(K, \overline{E}) > 0$. In particular, we have $|e^{i\theta} - z| \geq \delta$ for every $z \in K$ or equivalently

$$
|1 - e^{-i\theta} z| \geq \delta \tag{15.32}
$$

for all $z \in K$. It is easy to see from the representation

$$
B(z) = \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \cdot \frac{|\alpha_n|}{\alpha_n}
$$

that B has a pole at each point $\frac{1}{\overline{\alpha_n}}$. Next, the nth term of the series

$$
\sum_{n=1}^{\infty} \Big|1 - \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \cdot \frac{|\alpha_n|}{\alpha_n}\Big| \tag{15.33}
$$

is given by

$$
\Big|\frac{\alpha_n + |\alpha_n| z}{(1 - \overline{\alpha_n} z)\alpha_n}\Big| \cdot (1 - |\alpha_n|) \leq (1 - |\alpha_n|) \cdot \frac{1 + |z|}{|1 - \overline{\alpha_n} z|}. \tag{15.34}
$$

As K is compact, it is bounded by a positive constant M so that $|z| \leq M$ for all $z \in K$. Now there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|\overline{\alpha_n} - \mathrm{e}^{-i\theta}| \leq \frac{\delta}{2M}. \tag{15.35}$$

Combining the estimates (15.32) and (15.35), if $n \geq N$, then we gain

$$\delta \leq |1 - \mathrm{e}^{-i\theta} z| \leq |1 - \overline{\alpha_n} z| + |\overline{\alpha_n} - \mathrm{e}^{-i\theta}| \cdot |z| \leq |1 - \overline{\alpha_n} z| + \frac{\delta}{2}$$

so that

$$|1 - \overline{\alpha_n} z| \geq \frac{\delta}{2} > 0$$

for all $z \in K$. Thus the inequality (15.34) reduces to

$$\Big|\frac{\alpha_n + |\alpha_n| z}{(1 - \overline{\alpha_n} z)\alpha_n}\Big| \cdot \big(1 - |\alpha_n|\big) \leq \big(1 - |\alpha_n|\big) \cdot \frac{2(1 + M)}{\delta}.$$

for all $z \in K$ and all $n \geq N$. Using the Blaschke condition and the Weierstrass M-test, we conclude immediately that the series (15.33) converges uniformly on K. Eventually, Theorem 10.28 ensures that $B \in H(\Omega)$, and we end the analysis of the problem. ■

15.3 Problems on Blaschke Products

Problem 15.13

Rudin Chapter 15 Exercise 13.

Proof. Since α_n are real, we have

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \alpha_n z}$$

for some $k \in \mathbb{N}$. Suppose that $\alpha_{N-1} < r < \alpha_N$. Since $0 < \alpha_n < 1$ and $\alpha_n < \alpha_{n+1}$ for all positive integers n, we obtain $r^k < 1$ and

$$\Big|\frac{\alpha_n - r}{1 - \alpha_n r}\Big| = \frac{\alpha_n - r}{1 - \alpha_n r} < 1$$

for every $n = N, N+1, \ldots$. Consequently, we have

$$|B(r)| = \Big|r^k \prod_{n=1}^{\infty} \frac{\alpha_n - r}{1 - \alpha_n r}\Big| < \Big|\prod_{n=1}^{N-1} \frac{\alpha_n - r}{1 - \alpha_n r}\Big| \times \Big|\prod_{n=N}^{\infty} \frac{\alpha_n - r}{1 - \alpha_n r}\Big| < \prod_{n=1}^{N-1} \frac{r - \alpha_n}{1 - \alpha_n r}. \tag{15.36}$$

Simple algebra shows that $\alpha_N - \alpha_n > r - \alpha_n > 0$ and $1 - \alpha_n r > 1 - \alpha_n$ for each $n = 1, 2, \ldots, N-1$, the inequality (15.36) reduces to

$$|B(r)| < \prod_{n=1}^{N-1} \frac{\alpha_N - \alpha_n}{1 - \alpha_n}. \tag{15.37}$$

Since $\frac{\alpha_N - \alpha_n}{1 - \alpha_n} = 1 - \frac{n^2}{N^2}$ and $\log x \leq x - 1$ for $x > 0$, the inequality (15.37) becomes

$$|B(r)| < \prod_{n=1}^{N-1} \Big(1 - \frac{n^2}{N^2}\Big)$$

$$
\begin{aligned}
&\leq \exp\left[\log \prod_{n=1}^{N-1}\left(1-\frac{n^2}{N^2}\right)\right] \\
&= \exp\left[\sum_{n=1}^{N-1}\log\left(1-\frac{n^2}{N^2}\right)\right] \\
&\leq \exp\left(-\sum_{n=1}^{N-1}\frac{n^2}{N^2}\right) \\
&\leq \exp\left[-\frac{(N-1)(2N-1)}{6N}\right] \\
&< \exp\left[-\frac{(N-1)^2}{3N}\right] \\
&< e^{\frac{2}{3}}e^{-\frac{N}{3}} \\
&< 2e^{-\frac{N}{3}}. \qquad (15.38)
\end{aligned}
$$

Hence, by combining the fact $r \to 1$ if and only if $N \to \infty$ and the estimate (15.38), we have established that $B(r) \to 0$ as $r \to 1$ and $r \in (0,1)$, as required. This completes the proof of the problem. ■

Problem 15.14

Rudin Chapter 15 Exercise 14.

Proof. Consider $\alpha_n = 1 - e^{-n}$ and $x_n = 1 - \frac{1}{2}e^{-n}$ for $n = 1, 2, \ldots$. Now we are going to modify the sequence $\{\alpha_n\}$ and select a subsequence of $\{x_n\}$ so that a Blaschke product with zeros at the modified sequence satisfies the requirement.

We note from [8, Eqn. (15), p. 12] that we can write

$$
\begin{aligned}
|B(x_{2p})| &= \prod_{n=1}^{p}\frac{x_{2p}-\alpha_n}{1-\alpha_n x_{2p}} \times \prod_{n=p+1}^{4p-1}\left|\frac{\alpha_n - x_{2p}}{1-\alpha_n x_{2p}}\right| \times \prod_{n=4p}^{\infty}\frac{\alpha_n - x_{2p}}{1-\alpha_n x_{2p}} \\
&= T_1(p)\cdot T_2(p)\cdot T_3(p). \qquad (15.39)
\end{aligned}
$$

For $n = 1, 2, \ldots, p$, since

$$\frac{x_{2p}-\alpha_n}{1-\alpha_n x_{2p}} \geq \frac{x_{2p}-\alpha_p}{1-\alpha_p x_{2p}} \geq 1 - e^{-p},$$

we obtain $(1-e^{-p})^p < T_1(p) < 1$ which implies that

$$\lim_{p\to\infty} T_1(p) = 1. \qquad (15.40)$$

Furthermore, for sufficiently large p, the inequalities

$$\exp\left(-8\sum_{n=4p}^{\infty} e^{-\frac{n}{2}}\right) < T_3(p) < 1$$

give

$$\lim_{n\to\infty} T_3(p) = 1. \qquad (15.41)$$

However, the function $T_2(p)$ *only* satisfies

$$\lim_{p\to\infty} T_2(p) > 0.$$

Now here is the trick: For a positive integer p, we first replace $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{4p-1}$ by α_{4p}. Then we have

$$1 > S_2(p) = \prod_{n=p+1}^{4p-1} \frac{\alpha_{4p} - x_{2p}}{1 - \alpha_{4p} x_{2p}} = \left(\frac{\alpha_{4p} - x_{2p}}{1 - \alpha_{4p} x_{2p}}\right)^{3p-1} \geq \left(1 - \frac{4}{\mathrm{e}^{2p} + 2}\right)^{3p-1}$$

so that

$$\lim_{p\to\infty} S_2(p) = 1. \tag{15.42}$$

Next, we select a subsequence $\{p_k\}$ of positive integers such that $4p_k - 1 < p_{k+1} + 1$ for each $k = 1, 2, \ldots$. This makes sure that

$$\{\alpha_{p_k+1}, \alpha_{p_k+2}, \ldots, \alpha_{4p_k-1}\} \cap \{\alpha_{p_{k+1}+1}, \alpha_{p_{k+1}+2}, \ldots, \alpha_{4p_{k+1}-1}\} = \varnothing$$

for each $k = 1, 2, \ldots$. Then the modified sequence $\{\alpha_n'\}$ will be the one that replaces *only* the terms $\alpha_{p_k+1}, \alpha_{p_k+2}, \ldots, \alpha_{4p_k-1}$ by α_{4p_k} from the original sequence $\{\alpha_n\}$ for $k = 1, 2, \ldots$. Since

$$\sum_{n=1}^{\infty}(1 - |\alpha_n'|) \leq \sum_{n=1}^{\infty}(1 - |\alpha_n|) < \infty,$$

Thus it follows from Theorem 15.21 that

$$B(z) = \prod_{k=1}^{\infty} \frac{\alpha_k' - z}{1 - \alpha_k' z} \tag{15.43}$$

is an element of H^∞ and has no zeros except at α_k'. It is no doubt that $|B(\alpha_n')| \to 0$ as $n \to \infty$, so

$$\liminf_{r\to 1} |B(r)| = 0. \tag{15.44}$$

Besides, we apply the representation (15.39) to our Blaschke product (15.43) to obtain

$$\begin{aligned} |B(x_{2p_k})| &= \prod_{n=1}^{p_k} \frac{x_{2p_k} - \alpha_n'}{1 - \alpha_n' x_{2p_k}} \times \prod_{n=p_k+1}^{4p_k-1} \left|\frac{\alpha_{4p_k} - x_{2p_k}}{1 - \alpha_{4p_k} x_{2p_k}}\right| \times \prod_{n=4p_k}^{\infty} \frac{\alpha_n' - x_{2p_k}}{1 - \alpha_n' x_{2p_k}} \\ &= T_1(p_k) \cdot S_2(p_k) \cdot T_3(p_k). \end{aligned}$$

Combining the limits (15.40), (15.41) and (15.42), we conclude immediately that

$$\lim_{k\to\infty} |B(x_{2p_k})| = 1$$

which means

$$\limsup_{r\to 1} |B(r)| = 1. \tag{15.45}$$

Hence the two results (15.44) and (15.45) imply that the function (15.43) has no radial limit at $z = 1$, completing the analysis of the problem. ■

Problem 15.15

Rudin Chapter 15 Exercise 15.

Proof. As a linear fractional transformation, φ is one-to-one and $\varphi \in H(U)$. Since $\varphi(U) = U$, it follows from Theorem 12.6 that

$$\varphi(z) = \lambda \cdot \frac{z - \alpha}{1 - \overline{\alpha} z} \tag{15.46}$$

for some $\alpha \in U$ and $|\lambda| = 1$.

(a) If $\alpha = 0$, then $\lambda \neq 1$ because φ is not the identity function. In this case, 0 is the unique fixed point and $\varphi(z) = \lambda z$. However, it implies that

$$\sum_{n=1}^{\infty} 1 < \infty$$

which is impossible. Thus $\alpha \neq 0$ and we deduce from the expression (15.46) that $\varphi(z) = z$ is equivalent to

$$\overline{\alpha} z^2 - (1-\lambda)z - \lambda\alpha = 0. \tag{15.47}$$

It is clear from the equation (15.47) that *any* fixed point must be of modulus 1 because if z is a root of the equation (15.47), then the fact $\overline{\lambda} = \frac{1}{\lambda}$ implies that $z \neq 0$ and $\frac{1}{\overline{z}}$ is also one of its roots.

- **Case (i):** Suppose that φ has a unique fixed point on T. Let it be b. Consider the linear fractional transformation $\mu^{-1}(z) = \frac{1+bz}{z}$. Clearly, $\mu^{-1}(\infty) = b$, so the conjugate $\psi = \mu \circ \varphi \circ \mu^{-1}$ fixes ∞ and then it can be expressed as[c]

$$\psi(z) = z + A$$

for some nonzero constant A. Now $\psi_n(z) = z + nA$ for every $z \in U$, so $\psi_n(z) \to \infty$ as $n \to \infty$. Since

$$\varphi_n = \mu^{-1} \circ \psi_n \circ \mu,$$

it establishes that

$$\varphi_n(z) = \frac{(1+nAb)z - nAb^2}{nAz + (1-nAb)}$$

for every $z \in U$. Put $z = 0$ to get

$$\varphi_n(0) = \frac{-nAb^2}{1-nAb} = b - \frac{b}{1-nAb},$$

so for sufficiently large n, we have

$$|\varphi_n(0)| \approx 1 - \frac{1}{n|A|-1},$$

but it implies that

$$1 - |\varphi_n(0)| \approx \frac{1}{n|A|-1}.$$

Therefore, such φ cannot satisfy the Blaschke condition by [79, Theorem 6.10, p. 77].

- **Case (ii):** Suppose that φ has two distinct fixed points on T. Call them b_1 and b_2 respectively. Using Problem 14.31(c), we have $\psi = \nu \circ \varphi \circ \nu^{-1}$, where

$$\nu(z) = \frac{z-b_1}{z-b_2}.$$

Since ψ fixes 0 and ∞, it can be expressed as $\psi(z) = kz$ for some nonzero complex constant k. Since $\psi_n(z) = k^n z$, we have

$$\varphi_n = \nu^{-1} \circ \psi_n \circ \nu.$$

Note that $\nu^{-1}(z) = \frac{b_2 z - b_1}{z-1}$, so

$$\varphi_n(z) = \nu^{-1}\Big(k^n \cdot \frac{z-b_1}{z-b_2}\Big)$$

[c]Or we may apply Problem 14.31(c) directly here.

$$\begin{aligned} &= \frac{b_2 k^n \cdot \frac{z-b_1}{z-b_2} - b_1}{k^n \cdot \frac{z-b_1}{z-b_2} - 1} \\ &= \frac{(b_2 k^n - b_1)z + (1-k^n)b_1 b_2}{(k^n-1)z + (b_2 - b_1 k^n)}. \end{aligned} \tag{15.48}$$

- **Subcase (i): $|k| < 1$.** Then for sufficiently large enough n, the expression (15.48) gives

$$|\varphi_n(z)| \approx \Big|\frac{(k^n - b_1)z + (b_1 - k^n)b_2}{-z + b_2}\Big| \approx 1 - |k|^n$$

so that $1 - |\varphi_n(z)| \approx |k|^n$. In other words, φ satisfies the Blaschke condition by [79, Theorem 6.9, p. 77].

- **Subcase (ii): $|k| > 1$.** In this case, for sufficiently large enough n, the expression (15.48) implies that

$$\begin{aligned} |\varphi_n(z)| &= \Big|\frac{(b_2 - \frac{b_1}{k^n})z + (\frac{1}{k^n} - 1)b_1 b_2}{(1 - \frac{1}{k^n})z + (\frac{b_2}{k^n} - b_1)}\Big| \\ &\approx \Big|\frac{(b_2 - \frac{1}{k^n})z + (\frac{1}{k^n} - b_2)b_1}{z - b_1}\Big| \\ &\approx 1 - \frac{1}{|k|^n} \end{aligned}$$

which means $1 - |\varphi_n(z)| \approx \frac{1}{|k|^n}$. Hence the φ also satisfies the Blaschke condition in this case.

- **Subcase (iii): $|k| = 1$ and k is an Nth root of unity for some N.** Put $z = \frac{1}{2}$ into the expression (15.48), we have

$$\Big|\varphi_N\Big(\frac{1}{2}\Big)\Big| = \frac{1}{2}$$

so that

$$\sum_{n=1}^{\infty}\Big[1 - \Big|\varphi_n\Big(\frac{1}{2}\Big)\Big|\Big] \geq \sum_{p=1}^{\infty}\Big[1 - \Big|\varphi_{pN}\Big(\frac{1}{2}\Big)\Big|\Big] = \infty.$$

Thus φ *does not* satisfy the Blaschke condition in this case.

- **Subcase (iv): $|k| = 1$ and k is not an nth root of unity for all n.** We claim that the set $S = \{k^n \,|\, n \geq 0\}$ is dense on T. To see this, we write $k = \mathrm{e}^{i\theta}$, where θ is an *irrational* multiple of 2π.[d] Therefore, we obtain

$$S = \{\mathrm{e}^{in\theta} \,|\, n \in \mathbb{N}\}.$$

Given $\epsilon > 0$. Let ℓ be an arc on T with angle ϵ. Choose $N \in \mathbb{N}$ such that $\frac{2\pi}{N} < \epsilon$. By the hypothesis, the $(N+1)$ points $1, \mathrm{e}^{i\theta}, \mathrm{e}^{2i\theta}, \ldots, \mathrm{e}^{Ni\theta}$ are all distinct. As a result, two of them must have a counterclockwise angle less than $\frac{2\pi}{N}$, i.e.,

$$\mathrm{e}^{(p-q)i\theta} < \frac{2\pi}{N} < \epsilon$$

for some $p < q$. This means that the points $\mathrm{e}^{n(p-q)i\theta}$ with $n \geq 0$ are distributed on T at successive angles less then ϵ. Consequently, we have $\mathrm{e}^{n(p-q)i\theta} \in \ell$ for some $n \geq 0$.

[d] Otherwise, we have $\theta = \frac{2\pi p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. This implies that $k^q = 1$, a contradiction.

Put $z = 0$ into the expression (15.48) to get

$$|\varphi_n(0)| = \frac{|1 - k^n|}{|b_2 - b_1 k^n|}.$$

Now the claim ensures that there exists a sequence $\{n_m\}$ such that $|b_1 - b_2 k^{n_m}| < \frac{1}{m}$ and then

$$|\varphi_{n_m}(0)| > m \cdot |1 - k^{n_m}| > \frac{m}{2}$$

for large enough m. This shows that φ does not satisfy the Blaschke condition.

(b) We claim that φ satisfies

$$\varphi_n(z) = z \tag{15.49}$$

for some $n \in \mathbb{N}$. To this end, if $f \circ \varphi = f$ for some nonconstant $f \in H^\infty$, then we must have

$$f(\varphi_n(z)) = f(\varphi(\varphi_{n-1}(z))) = f(\varphi_{n-1}(z)) = \cdots = f(z)$$

for every $z \in U$ and every $n = 1, 2, \ldots$.

- **Case (i): φ has a unique fixed point in U.** Recall from the equation (15.46) that φ has a unique fixed point in U if and only if $\alpha = 0$. In this case, we have $\varphi(z) = \lambda z$ for some λ such that $|\lambda| = 1$. Obviously, we know that

$$\varphi_n(z) = \lambda^n z$$

for every $n \in \mathbb{N}$ and $z \in U$. If λ is an Nth root of unity, then the expected result (15.49) holds immediately for N. Otherwise, we fix $z = z_0 \in U \setminus \{0\}$ so that

$$f(\lambda^n z_0) = f(z_0)$$

for every $n \geq 0$. By **Subcase (iv)**, the set $S' = \{\lambda^n \,|\, n \geq 0\}$ is dense on T which implies that

$$f(z) = f(z_0) \tag{15.50}$$

on $C(0; |z_0|)$. By Theorem 10.18, the result (15.50) shows that $f(z) = f(z_0)$ for every $z \in U$ and this contradicts the Open Mapping Theorem.

- **Case (ii): φ has a unique fixed point b on T.** In this case, it follows from the equation (15.48) that $\varphi_n(z) \to b$ as $n \to \infty$ for every $z \in U$. This means that

$$f(z) = \lim_{n\to\infty} f(\varphi_n(z)) = f(b)$$

for every $z \in U$, but it also contradicts the Open Mapping Theorem.

- **Case (iii): φ has two distinct fixed points b_1 and b_2 on T.** Then it yields from the equation (15.48) that for every $z \in U$, we have

$$\lim_{n\to\infty} \varphi_n(z) = \begin{cases} b_1, & \text{if } |k| < 1; \\ \\ b_2, & \text{if } |k| > 1. \end{cases}$$

By similar argument to **Case (ii)**, we can show that it is impossible for these two cases so that $|k| = 1$.

Suppose that k is an Nth root of unity for some $N \in \mathbb{N}$. Then we see from the equation (15.48) again that $\varphi_N(z) = z$. Otherwise, **Subcase (iv)** ensures that the set $S = \{k^n \,|\, n \geq 0\}$ is dense on T. Using similar argument as **Case (i)**, we conclude that this is impossible.

Consequently, the linear fractional transformation φ must satisfy the claim (15.49).

Hence we have completed the analysis of the problem. ■

Problem 15.16

Rudin Chapter 15 Exercise 16.

Proof. Note that

$$1-|\alpha_j|=\int_{|\alpha_j|}^{1}\mathrm{d}r,$$

so we have

$$\sum_{j=1}^{\infty}(1-|\alpha_j|)=\sum_{j=1}^{\infty}\int_{|\alpha_j|}^{1}\mathrm{d}r. \tag{15.51}$$

Define the characteristic function

$$\chi_j(r)=\begin{cases}1, & \text{if } r\geq|\alpha_j|;\\ 0, & \text{otherwise.}\end{cases}$$

Then formula (15.51) becomes

$$\sum_{j=1}^{\infty}(1-|\alpha_j|)=\sum_{j=1}^{\infty}\int_{0}^{1}\chi_j(r)\,\mathrm{d}r. \tag{15.52}$$

Observe that

$$\sum_{j=1}^{\infty}\chi_j(r)=n(r). \tag{15.53}$$

Since each $\chi_j:[0,1]\to[0,\infty]$ is measurable for every $j=1,2,\ldots$, we apply Theorem 1.27 to interchange the summation and the integration in the equation (15.52) and then use the formula (15.53) to gain

$$\sum_{j=1}^{\infty}(1-|\alpha_j|)=\int_0^1\Big(\sum_{j=1}^{\infty}\chi_j(r)\Big)\,\mathrm{d}r=\int_0^1 n(r)\,\mathrm{d}r.$$

This ends the proof of the problem. ■

Problem 15.17

Rudin Chapter 15 Exercise 17.

Proof. Assume that $B(z)=\sum_{k=0}^{\infty}c_kz^k$ was a Blaschke product with $c_k\geq 0$ for all $k=0,1,2,\ldots$ and $B(\alpha)=0$ for some $\alpha\in U\setminus\{0\}$. Combining Theorem 15.24 and the fact[e] that $|f(x)|\leq\lambda$ holds for almost all x if and only if $\|f\|_{L^\infty}\leq\lambda$, we have

$$\|B^*\|_{L^\infty(T)}\leq 1 \quad\text{or}\quad B(\mathrm{e}^{i\theta})\in L^\infty(T).$$

[e] Refer to [62, p. 66]

Furthermore, we also have $A_n(e^{i\theta}) = e^{-in\theta}B(e^{i\theta}) \in L^2(T)$ for every $n = 1, 2, \ldots$. Clearly, we have $L^\infty(T) \subseteq L^2(T)$.

Recall that $L^2(T)$ is an inner product space. On the one hand, we may apply [62, Eqn. (7) & (8), p. 89] to get

$$\begin{aligned}\langle B, A_n\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} B(e^{i\theta})\cdot\overline{A_n(e^{i\theta})}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=0}^{\infty} c_k e^{ik\theta}\Big)\times\Big(\sum_{k=0}^{\infty} c_k e^{i(n-k)\theta}\Big)\,d\theta \\ &= \sum_{k=0}^{\infty} c_k c_{n+k}. \end{aligned} \tag{15.54}$$

On the other hand, the fact $|B(e^{i\theta})| = 1$ a.e. on T gives

$$\langle B, A_n\rangle = \Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}|B(e^{i\theta})|^2\,d\theta\Big]^2 = \Big(\frac{1}{2\pi}\int_{T\setminus N} e^{in\theta}\,d\theta\Big)^2, \tag{15.55}$$

where $|B(e^{i\theta})| \neq 1$ on N with $m(N) = 0$. Recall from Remark 3.10 that $L^2(T)$ is a space whose elements are equivalence classes of functions, so we can express (15.55) by

$$\langle B, A_n\rangle = \Big(\frac{1}{2\pi}\int_T e^{in\theta}\,d\theta\Big)^2 = 0. \tag{15.56}$$

Combining the two results (15.54) and (15.56), we conclude that $c_k = 0$ for all $k \in \mathbb{N}$, which is impossible. Hence no such Blaschke product exists and we complete the proof of the problem. ∎

Problem 15.18

Rudin Chapter 15 Exercise 18.

Proof. Obviously, we have

$$f'(z) = 2(z-1)B(z) + (z-1)^2 B'(z).$$

Thus it suffices to show that $(z-1)B'(z)$ is bounded in U. Suppose that $\{\alpha_n\}$ is the sequence of zeros of B. Then we know that $\{\alpha_n\} \subseteq (0,1)$ and it satisfies the Blaschke condition

$$\sum_{n=1}^{\infty}(1-\alpha_n^2) < \infty. \tag{15.57}$$

Without loss of generality, we may assume that

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots < 1. \tag{15.58}$$

Furthermore, suppose that

$$B(z) = \prod_{n=1}^{\infty}\frac{\alpha_n - z}{1-\alpha_n z} \quad\text{and}\quad B_n(z) = \prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{\alpha_k - z}{1-\alpha_k z}.$$

We yield from [18, Exercise 10, p. 174] that

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty}\Big(\frac{\alpha_n - z}{1-\alpha_n z}\Big)'\Big(\frac{\alpha_n - z}{1-\alpha_n z}\Big)^{-1}$$

$$B'(z) = -\sum_{n=1}^{\infty} \frac{1-\alpha_n^2}{(1-\alpha_n z)^2} B_n(z).$$

It is well-known that $|B_n(z)| < 1$ for every $z \in U$. Now we observe from the assumption (15.58) that

$$\delta = \min(\alpha_1, \alpha_2, \ldots) > 0.$$

Geometrically, if $z \in U$ and $p > 1$, then

$$|p - z| > |1 - z|. \tag{15.59}$$

Put $p = \frac{1}{\alpha_n}$ into the estimate (15.59) to get $|1 - \alpha_n z|^2 \geq \delta^2 |1 - z|^2$ for every $z \in U$. Hence, the condition (15.57) implies

$$|B'(z)| \leq \sum_{n=1}^{\infty} \left| \frac{1-\alpha_n^2}{(1-\alpha_n z)^2} \right| \cdot |B_n(z)| \leq \frac{1}{\delta^2 |1-z|^2} \sum_{n=1}^{\infty} (1 - \alpha_n^2) < \infty$$

holds for all $z \in U$. Consequently, the modulus $|(z-1)^2 B'(z)|$ is bounded in U which shows the boundedness of f' in U, completing the proof of the problem. ■

Remark 15.2

Blaschke products serve as an important subclass of $H(U)$. If you are interested in the literature of Blaschke products, you are suggested to read the book by Colwell [17].

15.4 Miscellaneous Problems and the Müntz-Szasz Theorem

Problem 15.19

Rudin Chapter 15 Exercise 19.

Proof. Let $0 < r < 1$. On the one hand, we notice that

$$\log |f(re^{i\theta})| = \log \exp \left(\operatorname{Re} \frac{re^{i\theta} + 1}{re^{i\theta} - 1} \right) = \frac{r^2 - 1}{r^2 - 2r\cos\theta + 1}.$$

Thus [62, Eqn, (3), §11.5, p. 233] asserts that

$$\mu_r(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r^2 - 1}{r^2 - 2r\cos\theta + 1} \, d\theta = -\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) \, d\theta = -1. \tag{15.60}$$

On the other hand, we have

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) = \lim_{r \to 1} \exp \left(\frac{re^{i\theta} + 1}{re^{i\theta} - 1} \right) = \exp \left(\frac{e^{i\theta} + 1}{e^{i\theta} - 1} \right)$$

so that

$$\log |f^*(e^{i\theta})| = \operatorname{Re} \left(\frac{e^{i\theta} + 1}{e^{i\theta} - 1} \right) = 0.$$

Therefore, we obtain

$$\mu^*(f) = 0. \tag{15.61}$$

Hence it follows from the results (15.60) and (15.61) that

$$\lim_{r\to 1} \mu_r(f) < \mu^*(f),$$

completing the proof of the problem. ■

Problem 15.20

Rudin Chapter 15 Exercise 20.

Proof. If $\lambda_N < 0$ for some $N \in \mathbb{N}$, then there exists a $\delta > 0$ such that $\lambda_N < \delta < 0$. By the hypothesis, we actually have $\lambda_n < \delta < 0$ for all $n \geq N$, but this contradicts another hypothesis that $\lambda_n \to 0$ as $n \to \infty$. Therefore, we have $\lambda_1 > \lambda_2 > \cdots > 0$ which implies that

$$0 < \frac{1}{\lambda_1} < \frac{1}{\lambda_2} < \cdots.$$

Suppose that X is the closure of the in $C(I)$ of the set of all finite linear combinations of the functions

$$1, t^{\frac{1}{\lambda_1}}, t^{\frac{1}{\lambda_2}}, t^{\frac{1}{\lambda_3}}, \ldots.$$

By Theorem 15.26 (The Müntz-Szasz Theorem), we have the following results:

- If $\displaystyle\sum_{n=1}^{\infty} \lambda_n = \infty$, then $X = C(I)$.
- If $\displaystyle\sum_{n=1}^{\infty} \lambda_n < \infty$, $\lambda \notin \{\lambda_n\}$ and $\lambda \neq \infty$, then X does not contain the function $t^{\frac{1}{\lambda}}$.

This ends the analysis of the problem. ■

Problem 15.21

Rudin Chapter 15 Exercise 21.

Proof. Let $\{\lambda_n\}$ be a sequence of distinct real numbers and $\lambda_n > -\frac{1}{2}$. Then the set of all finite linear combinations of the functions

$$t^{\lambda_0}, t^{\lambda_1}, t^{\lambda_2}, \ldots$$

if dense in $L^2(I)$ if and only if

$$\sum_{n=0}^{\infty} \frac{2\lambda_n + 1}{(2\lambda_n + 1)^2 + 1} = \infty.$$

To this end, let $m \in \mathbb{N}$ and $m \notin \{\lambda_n\}$. By [13, p. 173], we know that

$$\min_{a_k \in \mathbb{C}} \left\| t^m - \sum_{k=0}^{n-1} a_k t^{\lambda_k} \right\|_2 = \frac{1}{\sqrt{1+2m}} \prod_{k=0}^{n-1} \left| \frac{m - \lambda_k}{m + \lambda_k + 1} \right|.$$

Thus we see that

$$t^m \in \overline{\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}} \tag{15.62}$$

if and only if

$$\limsup_{n\to\infty}\prod_{k=0}^{n-1}\left|\frac{m-\lambda_k}{m+\lambda_k+1}\right|=0$$

if and only if

$$\limsup_{n\to\infty}\prod_{\substack{k=0\\ \lambda_k>m}}^{n-1}\left|1-\frac{2m+1}{m+\lambda_k+1}\right|\times\prod_{\substack{k=0\\ -\frac{1}{2}<\lambda_k\leq m}}^{n-1}\left|1-\frac{2\lambda_k+1}{m+\lambda_k+1}\right|=0.$$

Hence the set relation (15.62) is true if and only if

$$\limsup_{n\to\infty}\prod_{\substack{k=0\\ \lambda_k>m}}^{n-1}\left|1-\frac{2m+1}{m+\lambda_k+1}\right|=0\quad\text{or}\quad\limsup_{n\to\infty}\prod_{\substack{k=0\\ -\frac{1}{2}<\lambda_k\leq m}}^{n-1}\left|1-\frac{2\lambda_k+1}{m+\lambda_k+1}\right|=0.\tag{15.63}$$

Since $m+\lambda_k+1>0$ and $\lambda_k>-\frac{1}{2}$, we have $\frac{2m+1}{m+\lambda_k+1},\frac{2\lambda_k+1}{m+\lambda_k+1}\in(0,1)$. By Theorem 15.5, the results (15.63) hold if and only if

$$\sum_{\substack{k=0\\ \lambda_k>m}}^{\infty}\frac{2m+1}{m+\lambda_k+1}=\infty\quad\text{or}\quad\sum_{\substack{k=0\\ -\frac{1}{2}<\lambda_k\leq m}}^{\infty}\frac{2\lambda_k+1}{m+\lambda_k+1}=\infty.\tag{15.64}$$

Since $\lambda_k+\frac{1}{2}<m+\lambda_k+1<2\lambda_k+1$, we get

$$\frac{1}{2\lambda_k+1}<\frac{1}{m+\lambda_k+1}<\frac{2}{2\lambda_k+1}$$

which means that the first summation (15.64) is equivalent to

$$\sum_{\substack{k=0\\ \lambda_k>m}}^{\infty}\frac{1}{2\lambda_k+1}=\infty.\tag{15.65}$$

Since $1<m+\lambda_k+1\leq 2m+1$, the second summation (15.64) is equivalent to

$$\sum_{\substack{k=0\\ -\frac{1}{2}<\lambda_k\leq m}}^{\infty}(2\lambda_k+1)=\infty.\tag{15.66}$$

If $\lambda_k>m$, then it is clear that

$$\frac{2m+1}{2\lambda_k+1}<\frac{2m+1}{(2\lambda_k+1)^2+1}<\frac{2\lambda_k+1}{(2\lambda_k+1)^2+1}<2\lambda_k+1.$$

Similarly, if $-\frac{1}{2}<\lambda_k\leq m$, then it is easy to see that

$$\frac{2\lambda_k+1}{(2m+1)^2+1}\leq\frac{2\lambda_k+1}{(2\lambda_k+1)^2+1}\leq\frac{2m+1}{(2\lambda_k+1)^2+1}<\frac{2m+1}{2\lambda_k+1}.$$

Therefore, we establish that one of the summations (15.65) and (15.66) holds if and only if

$$\sum_{k=0}^{\infty}\frac{2\lambda_k+1}{(2\lambda_k+1)^2+1}=\infty$$

holds. Finally, our desired result is proven if we combine Theorem 3.14 and the Weierstrass Approximation Theorem [61, Theorem 7.26, p. 159]. This completes the proof of the problem. ■

Remark 15.3

The proof of Problem 15.21 follows basically the proof of Theorem 2.2 in [14]. For the version of a complex sequence $\{\lambda_n\}$, please refer to [50, §12, pp. 32 – 36].

Problem 15.22

Rudin Chapter 15 Exercise 22.

Proof. Let M be the set of all finite linear combinations of the functions f_n. It is clear that M is a subspace of $L^2(0,\infty)$. Let $g \in L^2(0,\infty)$ be orthogonal to each f_n, i.e.,

$$\langle f_n, g\rangle = \int_0^\infty f_n(t)\overline{g(t)}\,\mathrm{d}t = 0$$

for each $n = 1, 2, \ldots$. Define

$$F(z) = \int_0^\infty \mathrm{e}^{-tz}\overline{g(t)}\,\mathrm{d}t$$

on $\Pi = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0\}$. For every *fixed* $z \in \Pi$, since

$$\int_0^\infty |\mathrm{e}^{-tz}|^2\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-2t\mathrm{Re}\, z}\,\mathrm{d}t = \frac{1}{2\sqrt{\mathrm{Re}\, z}} < \infty,$$

we have $\mathrm{e}^{-tz} \in L^2(0,\infty)$. Using Theorem 3.8, we see that

$$|F(z)| = \|\mathrm{e}^{-tz}\overline{g}\|_1 \leq \|\mathrm{e}^{-tz}\|_2 \cdot \|\overline{g}\|_2 < \infty.$$

In other words, F is well-defined in Π.

Next, we have to compute $F'(z)$ in Π. To this end, we consider

$$\frac{F(\omega) - F(z)}{\omega - z} = \int_0^\infty \frac{\mathrm{e}^{-t\omega} - \mathrm{e}^{-tz}}{\omega - z} \cdot \overline{g(t)}\,\mathrm{d}t. \tag{15.67}$$

Given $\epsilon > 0$. Let $t > 0$ and

$$\omega = z - \zeta, \tag{15.68}$$

where $|\zeta| < \epsilon$. Then we have

$$\frac{\mathrm{e}^{-t(z-\zeta)} - \mathrm{e}^{-tz}}{z - \zeta - z} = -\mathrm{e}^{-tz} \cdot \frac{\mathrm{e}^{t\zeta} - 1}{\zeta}.$$

Suppose that $h(\zeta) = \frac{\mathrm{e}^{t\zeta}-1}{\zeta}$. The power series expansion of h is given by

$$h(\zeta) = \sum_{n=1}^\infty \frac{t^n}{n!}\zeta^{n-1}, \tag{15.69}$$

so we have $h \in H(D(0;\epsilon))$ by Theorem 10.6. Since h is continuous on $\overline{D(0;\epsilon)}$, we obtain from Theorem 10.24 (The Maximum Modulus Theorem) and the Extreme Value Theorem that the maximum of $|h(\zeta)|$ occurs on the boundary $|\zeta| = \epsilon$. By the expansion (15.69), we see that

$$\max_{|\zeta|=\epsilon} |h(\zeta)| \leq \sum_{n=1}^\infty \frac{t^n}{n!} \cdot |\zeta|^{n-1} = \frac{\mathrm{e}^{t\epsilon} - 1}{\epsilon}. \tag{15.70}$$

By the Mean Value Theorem, we know that $e^{t\epsilon}-1=\epsilon t e^{t\xi}$ for some $\xi\in(0,\epsilon)$, so we induce from the estimate (15.70) that

$$\Big|\frac{e^{t\zeta}-1}{\zeta}\Big|\leq te^{t\xi}<te^{t\epsilon},$$

where $|\zeta|<\epsilon$ and $t>0$. If $z\in\Pi$, then $\mathrm{Re}\,z>2\epsilon$ for some $\epsilon>0$. With this $\epsilon>0$, we may pick $\omega\in\Pi$ satisfying the condition (15.68). Therefore, we have

$$\Big|\frac{e^{-t\omega}-e^{-tz}}{\omega-z}\Big|=\Big|e^{-tz}\cdot\frac{e^{t\zeta}-1}{\zeta}\Big|<te^{-t\mathrm{Re}\,z}\cdot e^{t\epsilon}=te^{-t(\mathrm{Re}\,z-\epsilon)}\leq te^{-t\epsilon}$$

for $t>0$. Since $te^{-t\epsilon},\overline{g}\in L^2(0,\infty)$, Theorem 3.8 implies that $te^{-t\epsilon}\overline{g}\in L^1(0,\infty)$. Consequently, Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) can be applied to show the limit and the integral can be interchanged in the following deduction:

$$\begin{aligned}F'(z)&=\lim_{\omega\to z}\frac{F(\omega)-F(z)}{\omega-z}\\&=\lim_{\omega\to z}\int_0^\infty\frac{e^{-\omega t}-e^{-tz}}{\omega-z}\cdot\overline{g(t)}\,dt\\&=\int_0^\infty\lim_{\omega\to z}\frac{e^{-\omega t}-e^{-tz}}{\omega-z}\cdot\overline{g(t)}\,dt\\&=\int_0^\infty -te^{tz}\overline{g(t)}\,dt.\end{aligned}$$

Since z is arbitrary, we have proven that $F\in H(\Pi)$.

In fact, this process can be repeated to get

$$F^{(n)}(z)=(-1)^n\int_0^\infty t^ne^{-tz}\overline{g(t)}\,dt, \tag{15.71}$$

where $n\in\mathbb{N}$ and $z\in\Pi$. By the hypothesis and the formula (15.71), we have

$$F^{(n)}(1)=(-1)^n\int_0^\infty t^ne^{-t}\overline{g(t)}\,dt=(-1)^n\langle f_n,g\rangle=0$$

so that $F\equiv 0$. Particularly, if we denote $G(t)=\chi_{(0,\infty)}(t)e^{-t}\overline{g(t)}$, then we see that

$$\widehat{G}(y)=\sqrt{2\pi}\int_{-\infty}^\infty G(t)e^{-ity}\,dt=\sqrt{2\pi}\int_0^\infty e^{-t}\overline{g(t)}e^{-ity}\,dt=\sqrt{2\pi}F(1+iy)=0,$$

where $y\in\mathbb{R}$. As $G\in L^1(0,\infty)$, we observe from Theorem 9.12 (The Uniqueness Theorem) that $G(t)=0$ a.e. on $\mathbb{R}$ which implies that

$$g(t)=0 \tag{15.72}$$

a.e. on $(0,\infty)$. Recall that $L^2(0,\infty)$ is Hilbert and $\overline{M}$ is a subspace of $L^2(0,\infty)$. If we have $\overline{M}\neq L^2(0,\infty)$, according to the Corollary of Theorem 4.11, there corresponds a $g\in L^2(0,\infty)$ such that $g\neq 0$ and $\langle f_n,g\rangle=0$ for every $n=1,2,\ldots$. However, this definitely contradicts the above conclusion (15.72). Hence $\overline{M}=L^2(0,\infty)$, as desired. This completes the analysis of the problem. ■

Problem 15.23

Rudin Chapter 15 Exercise 23.

Proof. Since $f(0) = 0$, $\lambda_n \neq 0$ for all $n = 1, 2, \ldots, N$. Suppose that

$$B(z) = \prod_{n=1}^{N} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z} \cdot \frac{|\lambda_n|}{\lambda_n}.$$

Since $\displaystyle\sum_{n=1}^{N}(1 - |\lambda_n|) < \infty$, Theorem 15.21 implies that $B \in H^\infty$ and B has no zeros except at the points λ_n. Consider the function $g = \frac{B}{1-f}$. Then we have $g \in H(U)$. By Theorem 12.4, g is continuous on $\overline{U}$ and

$$\left|\frac{\lambda_n - \mathrm{e}^{i\theta}}{1 - \overline{\lambda_n}\mathrm{e}^{i\theta}}\right| = 1$$

for every $n = 1, 2, \ldots, N$. Therefore, we see that

$$|g(\mathrm{e}^{i\theta})| = \frac{|B(\mathrm{e}^{i\theta})|}{|1 - f(\mathrm{e}^{i\theta})|} \leq \frac{1}{|f(\mathrm{e}^{i\theta})| - 1} \leq \frac{1}{2}.$$

Consequently, it yields from Theorem 10.24 (The Maximum Modulus Theorem) that

$$|\lambda_1 \lambda_2 \cdots \lambda_N| = |g(0)| < \frac{1}{2}.$$

We end the proof of the problem. ∎

CHAPTER 16

Analytic Continuation

16.1 Singular Points and Continuation along Curves

Problem 16.1

Rudin Chapter 16 Exercise 1.

Proof. By Theorem 16.2, f has a singularity at some point $e^{i\theta}$. If we consider the power series for f about the point $\frac{1}{2}$, then the representation

$$f(z)=\sum_{k=0}^{\infty}b_k\Big(z-\frac{1}{2}\Big)^k=\sum_{k=0}^{\infty}\frac{f^{(k)}(\frac{1}{2})}{k!}\Big(z-\frac{1}{2}\Big)^k \tag{16.1}$$

holds in $D(\frac{1}{2};\frac{1}{2})$. Thus the radius of convergence of the power series (16.1) *must be* $\frac{1}{2}$. Otherwise, the power series would define a holomorphic extension of f beyond $e^{i\theta}$, a contradiction.

Assume that f was regular at $z=1$. Then the series

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(\frac{1}{2})}{k!}\Big(x-\frac{1}{2}\Big)^k$$

converges at some $x>1$. For every $k\geq 1$, we derive from the representation $f(z)=\sum a_nz^n$ that

$$f^{(k)}\Big(\frac{1}{2}\Big)=\sum_{n=k}^{\infty}\frac{n(n-1)\cdots(n-k+1)a_n}{2^{n-k}}.$$

Now we deduce from the binomial theorem that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{f^{(k)}(\frac{1}{2})}{k!}\Big(x-\frac{1}{2}\Big)^k&=\sum_{k=0}^{\infty}\Big[\sum_{n=k}^{\infty}\frac{n(n-1)\cdots(n-k+1)}{k!}\times\frac{a_n}{2^{n-k}}\Big]\times\Big(x-\frac{1}{2}\Big)^k\\&=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}C_k^n\cdot\frac{a_n}{2^{n-k}}\cdot\Big(x-\frac{1}{2}\Big)^k.\end{aligned} \tag{16.2}$$

Since $a_n\geq 0$, $C_k^n\cdot\frac{a_n}{2^{n-k}}\cdot(x-\frac{1}{2})^k\geq 0$. Consequently, the order of the summation in the expression (16.2) can be switched (see [61, Exercise 3, p. 196]) so that

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(\frac{1}{2})}{k!}\Big(x-\frac{1}{2}\Big)^k=\sum_{n=0}^{\infty}a_n\Big[\sum_{k=0}^{n}C_k^n\Big(\frac{1}{2}-0\Big)^{n-k}\cdot\Big(x-\frac{1}{2}\Big)^k\Big]=\sum_{n=0}^{\infty}a_nx^n$$

which means that the radius of convergence of $f(z) = \sum a_n z^n$ is greater than 1. This is a contradiction to our hypothesis and we have completed the analysis of the problem. ■

Problem 16.2

Rudin Chapter 16 Exercise 2.

Proof. Suppose that (f, D) and (g, D) can be analytically continued along γ to (f_n, D_n) and (g_m, D_m) respectively. According to Definition 16.9, there exist chains $\mathscr{C}_f = \{D_0, D_1, \ldots, D_n\}$ and $\mathscr{C}_g = \{D'_0, D'_1, \ldots, D'_m\}$, where $D_0 = D'_0 = D$. Furthermore, there are numbers

$$0 = s_0 < s_1 < \cdots < s_n = 1 \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma(0)$ is the center of $D_0 = D'_0$, $\gamma(1)$ is the center of D_n and D'_m,

$$\gamma\big([s_j, s_{j+1}]\big) \subseteq D_j \quad \text{and} \quad \gamma\big([t_k, t_{k+1}]\big) \subseteq D'_k$$

for $j = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, m-1$. We notice that there are function elements $(f_j, D_j) \sim (f_{j+1}, D_{j+1})$ and $(g_k, D'_k) \sim (g_{k+1}, D'_{k+1})$ for $j = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, m-1$, where $f_0 = f$ and $g_0 = g$.

Since $P\big(f_0(z), g_0(\zeta)\big) = 0$ for all $z, \zeta \in D_0$, we have $P\big(f_1(z), g_0(\zeta)\big) = 0$ for all $z \in D_0 \cap D_1$ and $\zeta \in D_0$. For each *fixed* $\zeta \in D_0$, P is a polynomial in z so that $P \in H(D_1)$. By the Corollary to Theorem 10.18, we have

$$P\big(f_1(z), g_0(\zeta)\big) = 0 \tag{16.3}$$

for all $z \in D_1$. Since ζ is arbitrary, the equation (16.3) is actually true for all $\zeta \in D_0$. Repeat this process, we conclude that $P(f_n(z), g_0(\zeta)) = 0$ for all $z \in D_n$ and $\zeta \in D_0$. Next, we fix a $z \in D_n$ and now P is a polynomial in ζ so that $P \in H(D'_1)$. Similar argument shows that

$$P\big(f_n(z), g_1(\zeta)\big) = 0$$

holds for all $\zeta \in D'_1$ and hence also for every $z \in D_n$. Repeat the process also implies that the equation

$$P\big(f_n(z), g_m(\zeta)\big) = 0 \tag{16.4}$$

holds in D_n and D_m. Finally, we can establish the required equation of the problem if we replace f_n and g_m by f_1 and g_1 in the equation (16.4).

Obviously, this can be extended to n function elements $(f_1, D), (f_2, D), \ldots, (f_n, D)$ provided that $P(z_1, z_2, \ldots, z_n)$ is a polynomial in n variables, $f_1, f_2, \ldots, f_n$ can be analytically continued along a curve γ to $g_1, g_2, \ldots, g_n$ and $P(f_1, f_2, \ldots, f_n) = 0$ in D. In fact, our above proof only uses the *holomorphicity* of the polynomial P, so similar results can be established if we only require that P is a function of n variables such that $P(\ldots, z_j, \ldots)$ is holomorphic in each variable z_j for $j = 1, 2, \ldots, n$. This completes the analysis of the problem. ■

Problem 16.3

Rudin Chapter 16 Exercise 3.

Proof. By Theorem 11.2, the function $f = u_x - iu_y$ is holomorphic in Ω. Since Ω is simply connected, Theorem 13.11 ensures that there corresponds an $F \in H(\Omega)$ such that $F' = f$. If $F = A + iB$, then we have

$$F'(z) = A_x + iB_x = A_x - iA_y = u_x - iu_y$$

so that $A(x,y) = u(x,y) + C$ for some constant C. Hence u is the real part of $F - C \in H(\Omega)$.

Next, we suppose that Ω is a region, but not simply connected. Let $f \in H(\Omega)$ and $f(z) \neq 0$ for every $z \in \Omega$. Then $\log|f|$ is harmonic in Ω by Problem 11.5. If it has harmonic conjugate, then there corresponds an $F \in H(\Omega)$ such that

$$|\mathrm{e}^{F(z)}| = \mathrm{e}^{\mathrm{Re}\,F(z)} = \mathrm{e}^{\log|f(z)|} = |f(z)|$$

which means that

$$|f\mathrm{e}^{-F}| = 1$$

in Ω. According to [9, Proposition 3.7, p. 39], $f(z)\mathrm{e}^{-F(z)} = \mathrm{e}^{i\theta}$ for some constant θ in Ω. Now we can write

$$f(z) = \mathrm{e}^{g(z)}$$

for some $g \in H(\Omega)$. By Theorem 13.11, Ω is simply connected, a contradiction. Hence this shows that the statement of the problem fails in every region that is not simply connected, completing the proof of the problem. ■

Problem 16.4

Rudin Chapter 16 Exercise 4.

Proof. Denote $I = [0,1]$. Let $\alpha : I \to \mathbb{C} \setminus \{0\}$ be an *arbitrary* path from 1 to $f(0)$. By Definition 10.8, without loss of generality, we may assume further that α' is continuous on I. Define

$$g(0) = \int_0^1 \frac{\alpha'(t)\,\mathrm{d}t}{\alpha(t)}. \tag{16.5}$$

We note that

$$g(0) = \int_0^1 \mathrm{d}\big(\ln\alpha(t)\big) = \log f(0)$$

and it means that $f(0) = \mathrm{e}^{g(0)}$. Let $\zeta \in X \setminus \{0\}$ and $\beta_\zeta : I \to X$ be the line segment joining 0 and ζ. Next, we define $\gamma_\zeta : I \to \mathbb{C} \setminus \{0\}$ by

$$\gamma_\zeta(t) = \begin{cases} \alpha(2t), & \text{if } t \in [0, \frac{1}{2}];\\[2ex] f\big(\beta_\zeta(2t-1)\big), & \text{otherwise.}\end{cases} \tag{16.6}$$

Finally, we define

$$g(\zeta) = \int_0^1 \frac{\gamma_\zeta'(t)\,\mathrm{d}t}{\gamma_\zeta(t)}. \tag{16.7}$$

Clearly, we know that

$$g(\zeta) = \int_0^1 \mathrm{d}\big[\log\gamma_\zeta(t)\big] = \log\gamma_\zeta(1) - \log\gamma_\zeta(0) = \log f\big(\beta_\zeta(1)\big) - \log\alpha(0) = \log f(\zeta)$$

so that $f(\zeta) = \mathrm{e}^{g(\zeta)}$. See Figure 16.1 for the paths $\beta_\zeta(I)$ and γ_ζ.

Now it suffices to prove that the function $g : X \to \mathbb{C}$ defined by the equations (16.5) and (16.7) is continuous on X. To this end, let $z, \omega \in X$ and suppose that $\beta_{\omega,z} : I \to X$ is the line segment from ω to z. We also define $\gamma_{\omega,z} : I \to \mathbb{C} \setminus \{0\}$ by

$$\gamma_{\omega,z}(t) = f\big(\beta_{\omega,z}(t)\big).$$

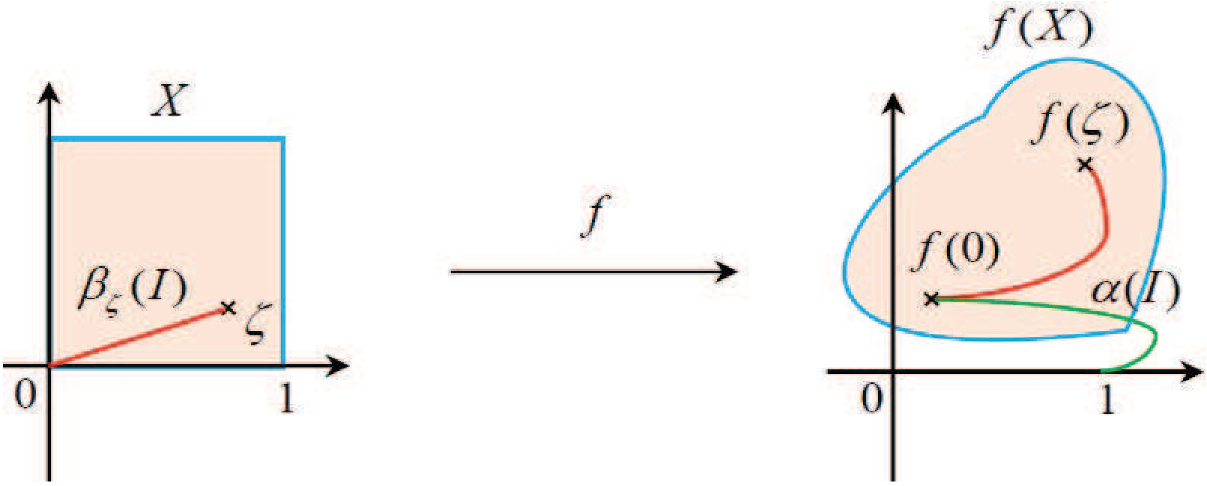


Figure 16.1: The paths $\beta_\zeta(I)$ and γ_ζ.

If γ is a path in X from x_0 to x_1, and if λ is a path in X from x_1 to x_2, then we define the product $\gamma * \lambda$ of γ and λ to be the path η given by the equations

$$\eta(t) = \begin{cases} \gamma(2t), & \text{if } t \in [0, \frac{1}{2}]; \\ \lambda(2t-1), & \text{otherwise.} \end{cases}$$

Thus both β_z and $\beta_\omega * \beta_{\omega,z}$ are paths in X from 0 to z. Since X is simply connected, we follow from Theorem 16.14 or [42, p. 323] that β_z and $\beta_\omega * \beta_{\omega,z}$ are (path) homotopic in X, i.e., $\beta_z \simeq_p \beta_\omega * \beta_{\omega,z}$ in X. Since f is continuous on X, we must have $f(\beta_z) \simeq_p f\big(\beta_\omega * \beta_{\omega,z}\big)$ in $f(X)$. Recall that α is arbitrary in the definition (16.6), it establishes that

$$\gamma_z \simeq_p \gamma_\omega * \gamma_{\omega,z}$$

in $f(X) \cup \alpha(I)$, see Figure 16.2 for the paths $\gamma_z(I), \gamma_\omega(I)$ and $\gamma_{\omega,z}(I)$

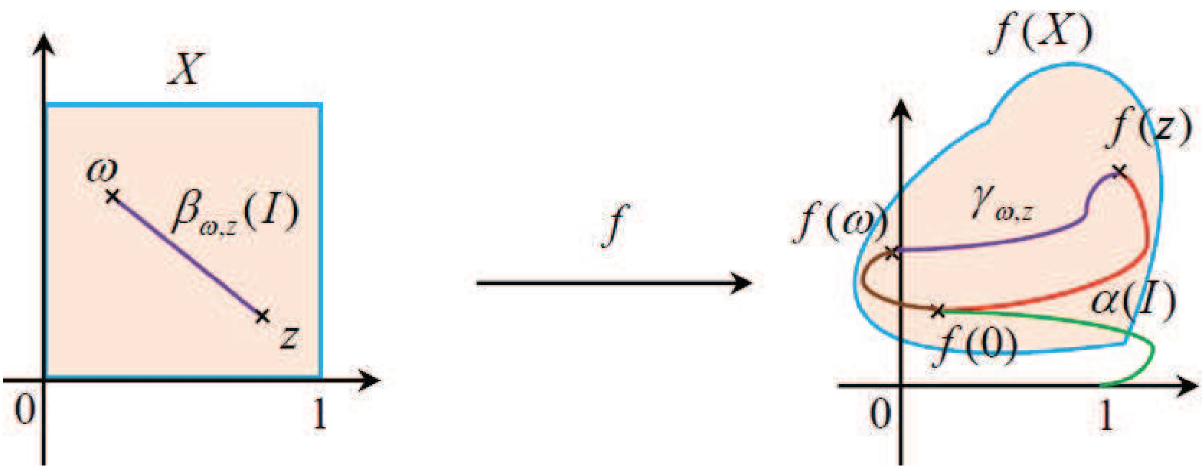


Figure 16.2: The paths $\gamma_z(I), \gamma_\omega(I)$ and $\gamma_{\omega,\zeta}(I)$.

Since X is compact and $f(z) \neq 0$ for all $z \in X$, there exists a constant $m > 0$ such that $m = \min_{z\in X} |f(z)|$. Given $\epsilon > 0$. Choose $\kappa > 0$ such that $\frac{\kappa}{m} < \epsilon$. By the continuity of f, there corresponds a $\delta > 0$ such that for $z, \omega \in X$ and $|z - \omega| < \delta$, the length of $\gamma_{\omega,z}$ is less than κ. Let

$\ell(z,\omega)$ be the length of $\gamma_{\omega,z}$. Hence, if $|z-\omega|<\delta$, then we obtain

$$\begin{aligned}
|g(z)-g(\omega)| &= \Big|\int_0^1 \frac{\gamma_z'(t)\,\mathrm{d}t}{\gamma_z(t)} - \int_0^1 \frac{\gamma_\omega(t)\,\mathrm{d}t}{\gamma_\omega(t)}\Big| \\
&= \Big|\int_0^1 \frac{(\gamma_\omega * \gamma_{\omega,z})'(t)\,\mathrm{d}t}{(\gamma_\omega * \gamma_{\omega,z})(t)} - \int_0^1 \frac{\gamma_\omega'(t)\,\mathrm{d}t}{\gamma_\omega(t)}\Big| \\
&= \Big|\int_{\gamma_\omega * \gamma_{\omega,z}} \frac{\mathrm{d}t}{t} - \int_{\gamma_\omega} \frac{\mathrm{d}t}{t}\Big| \\
&= \Big|\int_{\gamma_\omega} \frac{\mathrm{d}t}{t} + \int_{\gamma_{\omega,z}} \frac{\mathrm{d}t}{t} - \int_{\gamma_\omega} \frac{\mathrm{d}t}{t}\Big| \\
&= \Big|\int_{\gamma_{\omega,z}} \frac{\mathrm{d}t}{t}\Big|. \qquad (16.8)
\end{aligned}$$

Applying the definition of m to the integral (16.8), we get

$$|g(z)-g(\omega)| \leq \frac{\ell(z,\omega)}{m} < \frac{\kappa}{m} < \epsilon.$$

In other words, g is continuous at ω so that it is actually continuous on X, as required. This completes the proof of the problem. ∎

16.2 Problems on the Modular Group and Removable Sets

Problem 16.5

Rudin Chapter 16 Exercise 5.

Proof. Let $\tau(z)=z+1$, $\sigma(z)=-\frac{1}{z}$ and

$$\varphi(z)=\frac{az+b}{cz+d}\in G.$$

Then we have $a,b,c,d\in\mathbb{Z}$ and $ad-bc=1$.[a] Furthermore, we notice that $\tau^{-1}(z)=z-1$ and $\sigma^{-1}(z)=-\frac{1}{z}$.

- **Case (i):** $a=0$. We have $bc=-1$ so that $b=-c$. Obviously, $b=\pm1$ if and only if $c=\mp1$. If $b=1$, then $c=-1$ and we have

  $$\varphi(z)=\frac{1}{-z+d}=-\frac{1}{z-d}=-\frac{1}{\tau^{-d}(z)}=\sigma\big(\tau^{-d}(z)\big),$$

  i.e., $\varphi=\sigma\circ\tau^{-d}$. Similarly, we have $\varphi=\sigma\circ\tau^{d}$ if $b=-1$ and $c=1$.

- **Case (ii):** $a=\pm1$. Since $\frac{-az-b}{-cz-d}=\frac{az+b}{cz+d}$, we may only consider the case that $a=1$. Thus we have $d-bc=1$ and $\varphi(z)=\frac{z+b}{cz+d}$. We note that

  $$\sigma\big(\varphi(z)\big)=\frac{-cz-d}{z+b}\quad\text{and}\quad \tau\Big(\frac{-cz-d}{z+b}\Big)=\frac{-cz-d}{z+b}+1=\frac{-(c-1)z-(d-b)}{z+b}$$

  which imply that

  $$\tau^c\big(\sigma\big(\varphi(z)\big)\big)=\frac{-(d-bc)}{z+b}=-\frac{1}{z+b}=\sigma\big(\tau^b(z)\big).$$

  Hence we have $\varphi=\sigma^{-1}\circ\tau^{-c}\circ\sigma\circ\tau^{b}$.

[a] With the aid of Problem 16.7, we remark that $G=\mathrm{SL}_2(\mathbb{Z})$.

- **Case (iii):** $|a| > 1$. We may take $|a| > |c|$. Otherwise, we consider $\sigma\big(\varphi(z)\big) = -\frac{cz+d}{az+b}$ instead of $\varphi(z) = \frac{az+b}{cz+d}$. Now it is easy to see that one can find an $N \in \mathbb{Z}$ satisfying $0 \leq |a - Nc| < |c| < |a|$. Since

$$\tau^{-1}\big(\varphi(z)\big) = \frac{az+b}{cz+d} - 1 = \frac{(a-c)z + (b-d)}{cz+d},$$

we establish that

$$\varphi_1(z) = \sigma\big(\tau^{-N}\big(\varphi(z)\big)\big) = \frac{-cz-d}{(a-Nc)z + (b-Nd)} = \frac{a_1 z + b_1}{c_1 z + d_1}.$$

Simple algebra shows that $\varphi_1 \in G$, $|a_1| < |a|$ and $0 \leq |c_1| < |c|$. If $|c_1| = 0$, then $a_1 d_1 = 1$ so that $a_1 = \pm 1$ which goes back to **Case (ii)**. Otherwise, we can repeat the above process finitely many times, say m times, to get

$$\varphi_m(z) = \frac{a_m z + b_m}{c_m z + d_m} \in G,$$

where either $a_m = 0$ or $a_m = \pm 1$. Therefore, the φ_m, and hence φ, is generated by τ and σ.

Consequently, this proves the first assertion that τ and σ generate the modular group G.

For the second assertion, suppose that

$$\begin{aligned} R_1 &= \{z = x + iy \,|\, |x| < \tfrac{1}{2},\ y > 0 \text{ and } |z| > 1\}, \\ R_2 &= \{z = x + iy \,|\, -\tfrac{1}{2} \leq x \leq 0 \text{ and } |z| = 1\}, \\ R_3 &= \Big\{z = -\frac{1}{2} + iy \,\Big|\, y > 0 \text{ and } |z| \geq 1\Big\}. \end{aligned} \tag{16.9}$$

Then we have $R = R_1 \cup R_2 \cup R_3$, see Figure 16.3.

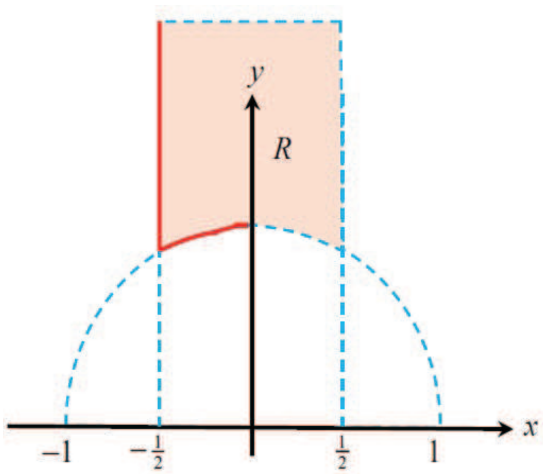


Figure 16.3: The fundamental domain R of G.

We check Theorem 16.19(a) and (b). Based on Apostol's description [5, p. 30], two points $\omega, \omega' \in \Pi^+ = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0\}$ are said to be **equivalent under** G if $\omega' = \varphi(\omega)$ for some $\varphi \in G$. With this terminology, property (a) means that *no* two distinct points of R are equivalent under G and property (b) implies that for every $\omega \in \Pi^+$, there exists a $z \in R$ such that z is equivalent to ω.

Lemma 16.1

Suppose that $z_1, z_2 \in \overline{R}$, $z_1 \neq z_2$ and $z_2 = \varphi(z_1)$ for some $\varphi \in G$. Then we have

$$\operatorname{Re} z_1 = \pm\tfrac{1}{2} \text{ and } z_2 = z_1 \mp 1 \quad \text{or} \quad |z_1| = 1 \text{ and } z_2 = -\tfrac{1}{z_1}.$$

Proof of Lemma 16.1. Without loss of generality, we may assume that $\operatorname{Im} z_2 \geq \operatorname{Im} z_1$ by symmetry. Let $\varphi(z) = \frac{az+b}{cz+d}$. Combing the assumption and the relation

$$\operatorname{Im} \varphi(z) = \frac{\operatorname{Im} z}{|cz+d|^2} \tag{16.10}$$

to get the condition

$$|cz_1 + d|^2 \leq 1. \tag{16.11}$$

Since $z_1 \in \overline{R}$, it is easy to see that $\operatorname{Im} z_1 \geq \frac{\sqrt{3}}{2}$. Thus $|c| \cdot \frac{\sqrt{3}}{2} \leq |c| \operatorname{Im} z_1 \leq |cz_1 + d| \leq 1$. As $c \in \mathbb{Z}$, this forces that either $c = 0$ or $|c| = 1$.

- **Case (i): $c = 0$.** Then $ad = 1$ and since $a, d \in \mathbb{Z}$, we have $a = d = \pm 1$. In this case, the relation (16.10) shows that $\operatorname{Im} z_2 = \operatorname{Im} z_1$. Furthermore, $\varphi(z) = z \pm b$ so that $\operatorname{Re} z_2 = \operatorname{Re} z_1 \pm b$. Since b is an integer, the definition (16.9) shows $b = 1$ which implies that $\operatorname{Re} z_1 = \pm\frac{1}{2}$ and hence $z_2 = z_1 \mp 1$.
- **Case (ii): $|c| = 1$.** Then the condition (16.11) becomes $|z_1 \pm d|^2 \leq 1$ or equivalently,
$$(\operatorname{Re} z_1 \pm d)^2 + (\operatorname{Im} z_1)^2 \leq 1. \tag{16.12}$$
Further reduction implies that
$$(\operatorname{Re} z_1 \pm d)^2 \leq 1 - (\operatorname{Im} z_1)^2 \leq 1 - \frac{3}{4} = \frac{1}{4},$$
so that
$$|\operatorname{Re} z_1 \pm d| \leq \frac{1}{2}. \tag{16.13}$$
Since $-\frac{1}{2} \leq \operatorname{Re} z_1 \leq \frac{1}{2}$, we have $|d| \leq 1$ which means either $d = 0$ or $|d| = 1$.
  - **Subcase (i): $|d| = 1$.** Using the inequality (16.13), we have $|\operatorname{Re} z_1 \pm 1| = \frac{1}{2}$ and then $\operatorname{Re} z_1 = \pm\frac{1}{2}$. Next, it follows from the inequality (16.12) that $0 \leq \operatorname{Im} z_1 \leq \frac{\sqrt{3}}{2}$, so actually we have $\operatorname{Im} z_1 = \frac{\sqrt{3}}{2}$. Consequently, we have $z_1 = \pm\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and then both our results hold in this case.
  - **Subcase (ii): $d = 0$.** Now the condition (16.11) implies that $|z_1| \leq 1$. Since $z_1 \in \overline{R}$ by the definition (16.9), we actually have $|z_1| = 1$. Simple calculation gives
  $$z_2 = \pm a - \frac{1}{z_1},$$
  where $a \in \mathbb{Z}$. Let $z_1 = x + iy$. Thus $z_2 = \pm a - x + iy$. If $a \neq 0$, then since $z_1, z_2 \in \overline{R}$ and $z_1 \neq z_2$, we get $z_1 = -\frac{1}{2} + iy$ and $z_2 = \frac{1}{2} + iy$. Otherwise, $a = 0$ so that $z_1 = x + iy$ and $z_2 = -x + iy$ for every $0 < x \leq \frac{1}{2}$. Obviously, this case satisfies $|z_1| = 1$ and $z_2 = -\frac{1}{z_1}$.

This completes the proof of the lemma. ∎

Combining the definition (16.9) of R and Lemma 16.1, we see immediately that no two distinct points of R are equivalent under G which is property (a). For proving property (b), we need the following result whose proof can be found in [5, Lemma 1, pp. 31, 32]:

Lemma 16.2

Given $\omega_1', \omega_2' \in \mathbb{C}$ with $\frac{\omega_2'}{\omega_1'}$ not real. Let $\Omega = \{m\omega_1' + n\omega_2' \,|\, m, n \in \mathbb{Z}\}$. Then there exist $\omega_1, \omega_2 \in \mathbb{C}$ such that $\omega_2 = a\omega_2' + b\omega_1'$ and $\omega_1 = c\omega_2' + d\omega_1'$, where $ad - bc = 1$, $|\omega_2| \geq |\omega_1|$ and $|\omega_1 \pm \omega_2| \geq |\omega_2|$.

Now we go back to the proof of our problem. If $\omega_1' = 1$ and $\omega_2' = \omega \in \Pi^+$, then it is easy to see that $\frac{\omega_2'}{\omega_1'} \notin \mathbb{R}$. By Lemma 16.2, there exist ω_1 and ω_2 with $|\omega_2| \geq |\omega_1|$ and $|\omega_1 \pm \omega_2| \geq |\omega_2|$ such that

$$\omega_2 = a\omega + b \quad \text{and} \quad \omega_1 = c\omega + d.$$

Let $z = \frac{\omega_2}{\omega_1}$. These relations give

$$z = \frac{a\omega + b}{c\omega + d} = \varphi(\omega) \tag{16.14}$$

with $ad - bc = 1$, $|z| \geq 1$ and $|z \pm 1| \geq |z|$. The relation (16.14) means that there exists a point $z \in R$ equivalent to $\omega \in \Pi^+$ under G which is exactly property (b). Hence we obtain the result that R is a fundamental domain of G and we end the proof of the problem. ■

Problem 16.6

Rudin Chapter 16 Exercise 6.

Proof. Since $\psi\big(\varphi(z)\big) = z + 1$, it follows from Problem 16.5 that G is also generated by φ and ψ. It is easy to see that

$$\varphi^2(z) = \varphi\big(\varphi(z)\big) = z$$

and

$$\psi^2(z) = -\frac{1}{z-1} \quad \text{and} \quad \psi^3(z) = z.$$

Hence φ has period 2 and ψ has period 3. This completes the proof of the problem. ■

Problem 16.7

Rudin Chapter 16 Exercise 7.

Proof. For each linear fractional transformation $\varphi(z) = \frac{az+b}{cz+d}$, we associate the 2×2 matrix

$$\mathbf{M}_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here we identify each matrix with its negative because $\mathbf{M}_\varphi$ and $-\mathbf{M}_\varphi$ represent the *same* transformation. If $\mathbf{M}_\varphi$ and $\mathbf{M}_\psi$ are the matrices associated with the linear fractional transformations φ and ψ respectively, then it is easy to see that the matrix product $\mathbf{M}_\varphi\mathbf{M}_\psi$ is associated with the function composition $\varphi \circ \psi$.

- **An algebraic proof of Theorem 16.19(c).** Now the group Γ is generated by the matrices
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$
If $\mathbf{M} \in \mathbf{\Gamma}$, then we have
$$\mathbf{M} = \mathbf{A}^{n_1}\mathbf{B}^{m_1}\mathbf{A}^{n_2}\mathbf{B}^{m_2}\cdots\mathbf{A}^{n_p}\mathbf{B}^{m_p}, \tag{16.15}$$
where the n_k, m_k are integers. Direct computation gives
$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$
so that
$$\mathbf{A}^{n_k} = \begin{pmatrix} 1 & 0 \\ (-2)^{n_k} & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}^{m_k} = \begin{pmatrix} 1 & (-2)^{m_k} \\ 0 & 1 \end{pmatrix}$$
and then
$$\mathbf{A}^{n_k}\mathbf{B}^{m_k} = \begin{pmatrix} 1 & (-2)^{m_k} \\ (-2)^{n_k} & (-2)^{m_k+n_k}+1 \end{pmatrix} = \begin{pmatrix} 1 & 2N_k \\ 2M_k & 2L_k+1 \end{pmatrix},$$
where N_k, M_k and L_k are integers. Thus we obtain
$$\mathbf{A}^{n_k}\mathbf{B}^{m_k}\mathbf{A}^{n_j}\mathbf{B}^{m_j} = \begin{pmatrix} 1+2N_{k,j} & 2M_{k,j} \\ 2P_{k,j} & 1+2L_{k,j} \end{pmatrix},$$
where $N_{k,j}, M_{k,j}, L_{k,j}$ and $P_{k,j}$ are integers. Hence we apply this to the expression (16.15) to conclude immediately that if
$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
then a and d are odd, b and c are even.

- **Proof of the first part of Problem 16.5.** Note that the transformations $z \mapsto z+1$ and $z \mapsto -\frac{1}{z}$ correspond to the matrices
$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
respectively. We claim that if $\mathbf{M} \in G$, then it has the form
$$\mathbf{M} = \mathbf{T}^{n_1}\mathbf{S}\mathbf{T}^{n_2}\mathbf{S}\cdots\mathbf{T}^{n_p}\mathbf{S}, \tag{16.16}$$
where the n_k are integers. To this end, we first notice that $\mathbf{S}^2 = \mathbf{I}$,[b] so this explains why only the $\mathbf{S}$ appears in the form (16.16). Next, it suffices to prove those matrices
$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
with $c \geq 0$. If $c = 0$, then $ad = 1$ or equivalently, $a = d = \pm 1$ so that
$$\mathbf{M} = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} = \mathbf{T}^{\pm b}.$$

[b] Remember that we have identified $\mathbf{I} = -\mathbf{I}$.

Next, if $c = 1$, then $ad - b = 1$ so that $b = ad - 1$ and

$$\mathbf{M} = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \mathbf{T}^d\mathbf{S}\mathbf{T}^d.$$

Assume that the form (16.16) is true for all matrices with lower left-hand element less than c for some $c \geq 1$. Since $ad - bc = 1$, c and d must be coprime so that $d = cq + r$ for some $q \in \mathbb{Z}$ and $0 < r < c$. Since

$$\mathbf{M}\mathbf{T}^{-q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -aq+b \\ c & r \end{pmatrix},$$

we have

$$\mathbf{M}\mathbf{T}^{-q}\mathbf{S} = \begin{pmatrix} -aq+b & -a \\ r & -c \end{pmatrix}. \tag{16.17}$$

By the hypothesis, the matrix (16.17) has the form (16.16) which implies that $\mathbf{M}$ can be expressed in the form (16.17).

This completes the proof of the problem. ■

Problem 16.8

Rudin Chapter 16 Exercise 8.

Proof. Since $E \subseteq \mathbb{R}$ is compact, we can define

$$A = \max_{x\in E} x \quad\text{and}\quad B = \min_{x\in E} x.$$

Notice that $\Omega = \mathbb{C} \setminus E$. Denote $R = \max(|A|, |B|)$.

(a) Let $x, y \in \mathbb{R}$ such that $x < y < A$. Then $x \neq y$ and

$$f(x) - f(y) = \int_E \Big(\frac{1}{t-x} - \frac{1}{t-y}\Big)\,\mathrm{d}t = (x-y)\int_E \frac{\mathrm{d}t}{(t-x)(t-y)}. \tag{16.18}$$

Since $x, y \in \mathbb{R}$ and $E \subset \mathbb{R}$, the integrals in the equation (16.18) are real integrals. Clearly, $\frac{1}{t-x} \geq \frac{1}{A-x} > 0$ and $\frac{1}{t-y} \geq \frac{1}{A-y} > 0$ for all $t \in E$. Let $\delta_x = \frac{1}{A-x}$ and $\delta_y = \frac{1}{A-y}$. Then it follows from the expression (16.18) that

$$f(x) - f(y) \geq (x-y)\int_E \delta_x\delta_y\,\mathrm{d}t = (x-y)\delta_x\delta_y m(E) > 0,$$

i.e., $f(x) \neq f(y)$. Consequently, f is nonconstant.

(b) The answer is negative. Assume that f could be extended to an entire function. Since $|t| \leq R$ on E, we have $|\frac{1}{t-z}| \to 0$ as $|z| \to \infty$ for every $t \in E$. In other words, we see that $f(z) \to 0$ as $|z| \to \infty$ which means f is bounded in $\mathbb{C}$. By Theorem 10.23 (Liouville's Theorem), f is constant which contradicts part (a). Hence we conclude that f cannot be extended to an entire function.

(c) Given $\epsilon > 0$ and $z \in \mathbb{C} \setminus D(0; R + \frac{R}{\epsilon})$. Define $g(z,t) = -\frac{z}{z-t}$ on E. We claim that $g(z,t) \to -1$ uniformly on E. In fact, we see that

$$|g(z,t) + 1| = \frac{|t|}{|z-t|}$$

on E. Since $|z| > R$, we have $|z-t| \geq |z| - |t| \geq |z| - R > 0$ so that

$$|g(z,t)+1| = \frac{|t|}{|z-t|} \leq \frac{R}{|z|-R} \tag{16.19}$$

on E. Since $|z| > R + \frac{R}{\epsilon}$, the inequality (16.19) implies that

$$|g(z,t)+1| < \epsilon$$

for all $t \in E$. This proves our claim which asserts that

$$\begin{aligned} |zf(z) + m(E)| &= \Big|\int_E g(z,t)\,\mathrm{d}t + m(E)\Big| \\ &= \Big|\int_E [g(z,t)+1]\,\mathrm{d}t\Big| \\ &\leq \int_E |g(z,t)+1|\,\mathrm{d}t \\ &< \epsilon m(E) \end{aligned} \tag{16.20}$$

for every $|z| \geq R + \frac{R}{\epsilon}$. Since ϵ is arbitrary, we conclude from the estimate (16.20) that

$$\lim_{z\to\infty} zf(z) = -m(E).$$

(d) The compactness of E implies that Ω is open in $\mathbb{C}$. We have to show that Ω is connected. Since $E \subset \mathbb{R}$, we have $\mathbb{C} \setminus \mathbb{R} \subseteq \Omega$. Thus the upper half plane Π^+ lies in a component of Ω. Similarly, the lower half plane Π^- must lie in a component of Ω. Since $E \neq \mathbb{R}$, one can have a *real* number a lying in Ω. Since Π^+ is connected, it follows from [42, Theorem 23.4, p. 150] that $\Pi^+ \cup \{a\}$ is also connected. Similarly, the set $\Pi^- \cup \{a\}$ is also connected. According to [42, Theorem 23.3, p. 150], the union $\Pi^+ \cup \{a\} \cup \Pi^- = (\mathbb{C} \setminus \mathbb{R}) \cup \{a\}$ is connected. Finally, since

$$(\mathbb{C} \setminus \mathbb{R}) \cup \{a\} \subseteq \Omega \subseteq \mathbb{C},$$

the connectedness of Ω can be deduced again from [42, Theorem 23.4, p. 150].

Assume that f had a holomorphic square root in Ω. By the definition, Ω is a region. Furthermore, we observe from Theorem 13.11 that Ω is simply connected. However, the closed curve $C(0; 2R)$ is *not* null-homotopic in Ω because E lies inside $C(0; 2R)$. By the definition, Ω is not simply connected and hence f has no holomorphic square root in Ω.

(e) Assume that $\operatorname{Re} f$ was bounded in Ω. We use part (f) in advance that f will be bounded in Ω. This implies that f can be extended to a bounded entire function because E is compact. Hence it contradicts part (a) and then $\operatorname{Re} f$ is unbounded in Ω.

(f) Suppose that $z = a + ib \in \Omega$. Then it is easy to see that

$$f(z) = \int_E \frac{\mathrm{d}t}{(t-a)-ib} = \int_E \frac{(t-a)\,\mathrm{d}t}{(t-a)^2+b^2} + i\int_E \frac{b\,\mathrm{d}t}{(t-a)^2+b^2}. \tag{16.21}$$

For every $z \in \Omega$, we deduce from the expression (16.21) that

$$\begin{aligned} |\operatorname{Im} f(z)| &= \Big|\int_E \frac{b\,\mathrm{d}t}{(t-a)^2+b^2}\Big| \\ &\leq \Big|\int_{-\infty}^{\infty} \frac{b\,\mathrm{d}t}{(t-a)^2+b^2}\Big| \end{aligned}$$

$$
\begin{aligned}
&= \left| \tan^{-1}\Big(\frac{t-a}{b}\Big)\Big|_{-\infty}^{\infty} \right| \\
&= \frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big) \\
&= \pi.
\end{aligned}
$$

(g) Suppose that γ is a positively oriented circle which has E in its interior. Since γ^* (the range of γ) is closed in $\mathbb{C}$ and $\gamma^* \cap E = \varnothing$, we have $\delta = \inf_{\substack{z\in\gamma^* \\ t\in E}} |t-z| > 0$ so that

$$\Big|\int_\gamma \Big(\int_E \frac{\mathrm{d}t}{t-z}\Big)\,\mathrm{d}z\Big| \leq \Big|\int_\gamma \int_E \frac{1}{\delta}\,\mathrm{d}t\,\mathrm{d}z\Big| = \frac{\ell(\gamma)m(E)}{\delta} < \infty,$$

where $\ell(\gamma)$ is the circumference of the circle γ. Hence Theorem 8.8 (The Fubini Theorem) and Theorem 10.11 together assert that

$$\int_\gamma \int_E \frac{1}{t-z}\,\mathrm{d}t\,\mathrm{d}z = \int_E \Big(\int_\gamma \frac{1}{t-z}\,\mathrm{d}z\Big)\,\mathrm{d}t = \int_E 2\pi i \mathrm{Ind}_\gamma(t)\,\mathrm{d}t = 2\pi i \int_E \mathrm{d}t = 2\pi m(E) i.$$

(h) By parts (e) and (f), we see that

$$f(\Omega) \subseteq \{z = x+iy \,|\, x\in\mathbb{R} \text{ and } -\pi \leq y \leq \pi\}.$$

Define $g(z) = \mathrm{e}^{iz}$ and $\varphi = g\circ f : \Omega \to \mathbb{C}$. Now part (a) ensures that φ is not constant. Furthermore, it is clear that

$$\varphi(\Omega) = g\big(f(\Omega)\big) \subseteq \{r\mathrm{e}^{i\theta} \,|\, \mathrm{e}^{-\pi} \leq r \leq \mathrm{e}^{\pi} \text{ and } \theta \in [0, 2\pi]\}$$

so that φ is bounded on Ω. Thus it remains to show that $\varphi \in H(\Omega)$ and this follows from the result $f \in H(\Omega)$. To see this, we write

$$f(z) = \int_E \psi(z,t)\,\mathrm{d}t,$$

where $\psi(z,t) = \frac{1}{t-z}$. For each fixed $z\in\Omega$, $\psi(z,t)$ is measurable. For each fixed $t\in E$, we have $\psi(z,t)\in H(\Omega)$. Furthermore, for each $z_0\in\Omega$, we have $\inf_{t\in E}|t-z_0| > 0$. Let this number be 2δ. Then we have

$$\delta = \inf_{\substack{z\in\overline{D}(z_0;\delta) \\ t\in E}} |t-z|$$

which implies that $|t-z| \geq \delta$ for every $z \in \overline{D}(z_0;\delta)$ and $t\in E$. Therefore, we get

$$\sup_{z\in\overline{D}(z_0;\delta)} \int_E \frac{1}{|t-z|}\,\mathrm{d}t \leq \int_E \frac{\mathrm{d}t}{\delta} = \frac{m(E)}{\delta} < \infty.$$

In other words, $\displaystyle\int_E |\psi(z,t)|\,\mathrm{d}t$ is locally bounded. Hence we conclude that $f\in H(\Omega)$.[c]

We end the proof of the problem. ■

Problem 16.9

Rudin Chapter 16 Exercise 9.

[c] See the online paper `http://www.nieuwarchief.nl/serie5/pdf/naw5-2001-02-1-032.pdf` or [22].

Proof.

(a) It is easy to see that

$$f(-2) = \int_{-1}^{1} \frac{\mathrm{d}t}{t+2} = \log(t+2)\Big|_{-1}^{1} = \log 3 \quad \text{and} \quad f(-4) = \log(t+4)\Big|_{-1}^{1} = \log\frac{5}{3}.$$

Thus f is not constant in Ω.

(b) Similar to Problem 16.8(b), f cannot be extended to an entire function.

(c) According to Problem 16.8(c), the value of the limit is -2 because $m(E) = 2$.

(d) Similar to Problem 16.8(d), f has no holomorphic square root in Ω.

(e) Similar to Problem 16.8(e), $\operatorname{Re} f$ is unbounded in Ω.

(f) Since $E = [-1, 1]$, we see that

$$\operatorname{Im} f = \int_{-1}^{1} \frac{b}{(t-a)^2 + b^2}\,\mathrm{d}t = \tan^{-1}\Big(\frac{t-a}{b}\Big)\Big|_{-1}^{1} = \tan^{-1}\Big(\frac{1-a}{b}\Big) - \tan^{-1}\Big(\frac{-1-a}{b}\Big).$$

(g) We have the exact result

$$\int_{\gamma}\int_{-1}^{1} \frac{1}{t-z}\,\mathrm{d}t\,\mathrm{d}z = 2\pi m(E) i = 4\pi i.$$

(h) By Problem 16.18(h), the nonconstant bounded holomorphic function φ in Ω is given by

$$\varphi(z) = \exp\Big(i\int_{-1}^{1} \frac{\mathrm{d}t}{t-z}\Big) = \exp\Big(i\log\frac{z-1}{z+1}\Big).$$

This completes the proof of the problem. ■

Problem 16.10

Rudin Chapter 16 Exercise 10.

Proof.

(a) We first need the following lemma:

Lemma 16.3

Suppose that E is compact and has no interior, and K satisfies the following two conditions:

- **Condition (1):** $K \subseteq E$ is compact (K can possibly be empty) and
- **Condition (2):** Each $f \in H(\mathbb{C}\setminus E)$ can be extended to an $f_K \in H(\mathbb{C}\setminus K)$.

Let E' be the intersection of *all* such compact subsets K of E. Then E' also satisfies the conditions.

Proof of Lemma 16.3. Obviously, we have $E' \subseteq K$ and E' is compact. This means that E' satisfies **Condition (1)**. To show that E' also satisfies **Condition (2)**, let $f \in H(\mathbb{C} \setminus E)$ and $z \in \mathbb{C} \setminus E'$. Then $z \notin K$ (or equivalently $z \in \mathbb{C} \setminus K$) *for some* compact $K \subseteq E$ satisfying **Condition (2)**. Thus our $f \in H(\mathbb{C} \setminus E)$ can be extended to an $f_K \in H(\mathbb{C} \setminus K)$. Since E has no interior, there exists a sequence $\{z_n\} \subseteq \mathbb{C} \setminus E$ such that $z_n \to z$ and

$$f_K(z) = \lim_{n\to\infty} f(z_n)$$

which implies that the value $f_K(z)$ is *uniquely determined* by the limit, i.e., all the values $f_K(z)$ must be equal for all compact $K \subseteq E$ satisfying $z \notin K$ and the conditions. Hence we may define $\widehat{f} : \mathbb{C} \setminus E' \to \mathbb{C}$ by

$$\widehat{f}(z) = f_K(z) \tag{16.22}$$

for *any* such compact K. Recall that $f_K \in H(\mathbb{C} \setminus K)$ and $E' \subseteq K$, so the expression (16.22) ensures that $\widehat{f} \in H(\mathbb{C} \setminus E')$, completing the proof of Lemma 16.3 ■

Suppose that E is countable compact, i.e., $E = \{z_1, z_2, \ldots\}$. Since $\{z_n\}$ has no interior, it is nowhere dense and we observe from the Baire Category Theorem (see §5.7) that E has no interior. Let E' be the set in Lemma 16.3. Assume that $E' \neq \varnothing$. Notice that

$$E' = \bigcup_{n=1}^{\infty} \big(E' \cap \{z_n\}\big).$$

Since E' is compact, E' is closed in $\mathbb{C}$ so that it is a complete metric space by [61, Theorem 3.11, p. 53]. If each $E' \cap \{z_n\}$ has no interior, then it is nowhere dense and the Baire Category Theorem shows that E' is of the first category, a contradiction. Thus $\{z_N\} = E' \cap \{z_N\}$ has a nonempty interior for some $N \in \mathbb{N}$ which is impossible. Hence we have $E' = \varnothing$ and we deduce from Lemma 16.3 that every $f \in H(C \setminus E)$ can be extended to an entire function and we denote it by the same notation f. Particularly, if f is bounded, then the corresponding entire function f is also bounded and Theorem 10.23 (Liouville's Theorem) forces that it is a constant. By the definition, E is removable.

(b) Let $E \subseteq \mathbb{R}$ be compact and $m(E) = 0$. Let $f \in H(\mathbb{C} \setminus E)$ be bounded by a positive constant M. By the proof of Theorem 13.5, there exists a cycle Γ in $\mathbb{C} \setminus E$ such that

$$\operatorname{Ind}_{\Gamma}(z) = 1$$

for every $z \in E$. Suppose that V is the union of the collection of those components of $\mathbb{C} \setminus \Gamma$ intersecting E. Thus we have $E \subseteq V$. Define $g : V \to \mathbb{C}$ by

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

for every $z \in V$. By an argument similar to part (c) below, we see that $F \in H(V)$.

Fix $\alpha \in V \setminus E$. Since the set $(\mathbb{C} \setminus V) \cup \{\alpha\}$ is closed in $\mathbb{C}$ and $[(\mathbb{C} \setminus V) \cup \{\alpha\}] \cap E = \varnothing$, $d\big(E, (\mathbb{C}\setminus V)\cup\{\alpha\}\big) > 0$. Let $0 < \epsilon < d\big(E, (\mathbb{C}\setminus V)\cup\{\alpha\}\big)$. Since E is compact and $m(E) = 0$, E can be covered by a finite number of open intervals $I_1 = (a_1, b_1), I_2 = (a_2, b_2), \ldots,$ $I_n = (a_n, b_n)$ whose total length is less than ϵ. Without loss of generality, we may assume that $I_1, I_2, \ldots, I_n$ are pairwise disjoint and intersect E. Let γ_k denote the counterclockwise circle having I_k as its diameter and

$$\Gamma_\epsilon = \bigcup_{k=1}^{n} \gamma_k.$$

We notice that the length of Γ_ϵ is less than $\pi\epsilon$. By applying Theorem 10.35 (Cauchy's Theorem) to the cycle $\Gamma - \Gamma_\epsilon$ in $\mathbb{C} \setminus E$, we obtain

$$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma - \Gamma_\epsilon} \frac{f(\zeta)}{\zeta - \alpha} \, d\zeta = F(\alpha) - \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{f(\zeta)}{\zeta - \alpha} \, d\zeta. \tag{16.23}$$

It is easy to see that

$$\Big| \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{f(\zeta)}{\zeta - \alpha} \, d\zeta \Big| \leq \frac{1}{2\pi} \times \frac{M\pi\epsilon}{d(\alpha, E)}.$$

Since ϵ is arbitrary small, the second integral in the equation (16.23) is actually zero, we get

$$f(\alpha) = F(\alpha)$$

for every $\alpha \in V \setminus E$. Since $F \in H(V)$, we conclude immediately that $f \in H(\mathbb{C})$ which implies that it is a constant. By the definition, E is removable.

(c) Suppose that $f : \Omega \setminus E \to \mathbb{C}$ is bounded by M. We *fix* $z_0 \in \Omega \setminus E$. Let Γ_1 be a cycle in $\Omega \setminus \big(E \cup \{z_0\}\big)$ with winding number 1 around $E \cup \{z_0\}$ and zero around $\mathbb{C} \setminus \Omega$. Similarly, suppose that Γ_2 is a cycle in $\Omega \setminus \big(E \cup \{z_0\}\big)$ with winding number 1 around E and zero around $(\mathbb{C} \setminus \Omega) \cup \{z_0\}$. Since $\operatorname{Ind}_{\Gamma_1}(\alpha) = \operatorname{Ind}_{\Gamma_2}(\alpha) = 0$ for every $\alpha \notin \Omega$, Theorem 10.35 (Cauchy's Theorem) asserts that our construction guarantees

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta.$$

Define $f_1, f_2 : \Omega \setminus E \to \mathbb{C}$ by

$$f_1(z) = \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{and} \quad f_2(z) = \int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

We claim that $f_1 \in H(\Omega)$ and $f_2 \in H(\mathbb{C} \setminus E)$. To this end, we first note from Theorem 10.35 (Cauchy's Theorem) that f_1 is *independent of* Γ_1. Next, we take $z \in \Omega$ and fix the cycle Γ_1 as constructed above. Denote the length of Γ_1 to be $\ell(\Gamma_1)$. Since $E \cup \{z\}$ lies entirely inside Γ_1, we have $\Gamma_1^* \cap \big(E \cup \{z\}\big) = \varnothing$. Recall that E is compact, so is $E \cup \{z\}$ and then $d\big(\Gamma_1^*, E \cup \{z\}\big) > 0$. Let this number be 2δ. If h is very small such that $z + h \in D(z; \delta)$, then we have

$$\frac{f_1(z+h) - f_1(z)}{h} = \int_{\Gamma_1} \frac{f(\zeta)}{(\zeta - z)(\zeta - z - h)} \, d\zeta. \tag{16.24}$$

Clearly, for every $\zeta \in \Gamma_1$, we have

$$\Big| \frac{f(\zeta)}{(\zeta - z)(\zeta - z - h)} \Big| \leq \frac{M}{2\delta^2}.$$

Using this and the fact that $\ell(\Gamma_1) < \infty$, we may apply Theorem 1.34 (Lebesgue's Dominated Convergence Theorem) to the expression (16.24) to conclude that

$$f_1'(z) = \int_{\Gamma_1} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta.$$

Since $z \in \Omega$ is arbitrary, we get the desired result that $f_1 \in H(\Omega)$. Using a similar argument, we can show that $f_2 \in H(\mathbb{C} \setminus E)$ which proves the desired claim.

Therefore, we have $f = f_1 - f_2$ on $\Omega \setminus E$. Now the boundedness of f certainly implies the boundedness of f_2. Since E is removable, f_2 is a constant. Consequently, we obtain $f \in H(\Omega)$.

(d) Suppose that $E \subset \mathbb{C}$ is compact and $m_2(E) = 0$.[d] Then E is removable. Here we need the following lemma to prove this result.

Lemma 16.4

$E \subseteq \mathbb{C}$ is removable if and only if every bounded holomorphic function f on $\mathbb{C} \setminus E$ satisfies $f'(\infty) = 0$.

Proof of Lemma 16.4. By the definition, we know that E is removable if and only if every bounded holomorphic function f on $\mathbb{C}\setminus E$ is constant. Obviously, if $f \in H(\mathbb{C}\setminus E)$ is constant, then $f'(\infty) = 0$. Conversely, let $g : \mathbb{C} \setminus E \to \mathbb{C}$ be nonconstant and bounded. Then there exists an $z_0 \in \mathbb{C} \setminus E$ such that $g(z_0) \neq g(\infty)$. Define

$$f(z) = \frac{g(z) - g(z_0)}{z - z_0}$$

on $\mathbb{C} \setminus E$. Obviously, f is also a bounded and nonconstant function on $\mathbb{C} \setminus E$ and

$$f(\infty) = \lim_{z\to\infty} f(z) = 0.$$

Consequently, we establish

$$\begin{aligned} f'(\infty) &= \lim_{z\to\infty} z[f(z) - f(\infty)] \\ &= \lim_{z\to\infty} \frac{z[g(z) - g(z_0)]}{z - z_0} \\ &= \lim_{z\to\infty} \frac{z}{z - z_0} g(z) - g(z_0) \lim_{z\to\infty} \frac{z}{z - z_0} \\ &= g(\infty) - g(z_0) \neq 0, \end{aligned}$$

completing the proof of Lemma 16.4. ∎

We return to the proof of the problem. Let f be a bounded holomorphic function on $\mathbb{C}\setminus E$, i.e., $|f(z)| \leq M$ on $\mathbb{C} \setminus E$ for some positive constant M. Given $\epsilon > 0$. Then E can be covered by open discs $D_1, D_2, \ldots, D_n$ of radii $r_1, r_2, \ldots, r_n$ respectively such that

$$\sum_{k=1}^{n} r_k < \epsilon.$$

Let $\Gamma = \partial D_1 \cup \partial D_2 \cup \cdots \cup \partial D_n$. Using [73, Eqn. (1.2), p. 16], we have

$$|f'(\infty)| = \left|\frac{1}{2\pi i}\int_\Gamma f(z)\,\mathrm{d}z\right| \leq M \sum_{k=1}^{n} r_k < M\epsilon. \tag{16.25}$$

Since ϵ is arbitrary, the inequality (16.25) guarantees that $f'(\infty) = 0$. Now we conclude from Lemma 16.4 that E is in fact removable.

(e) Suppose first that $E \subset \mathbb{C}$ is compact and removable. If $F \subseteq E$ is a connected component of E containing more than one point, then it follows from Theorem 14.8 (The Riemann Mapping Theorem) that there exists a conformal mapping $f : \mathbb{C} \setminus F \to U$ which is non-constant. Thus $f|_{\mathbb{C}\setminus E} \in H(\mathbb{C} \setminus F)$ and $f|_{\mathbb{C}\setminus E}$ must be bounded. By the definition, E is

[d] Here m_2 denotes the Lebesgue measure in two dimensional space $\mathbb{C}$.

non-removable, a contradiction. Hence connected components of E are one-point sets, i.e., E is **totally disconnected**, see [42, Exercise 5, p. 152].

Now if $E \subset \mathbb{C}$ is a connected subset with more than one point, then the above paragraph ensures that E must be non-removable.[e]

We have completed the analysis of the proof of the problem. ■

Remark 16.1

Recall that we have studied the special case of removable sets in Problem 11.11. See also Remark 11.2.

16.3 Miscellaneous Problems

Problem 16.11

Rudin Chapter 16 Exercise 11.

Proof. By the definition, we have $\Omega_\alpha \subset \Omega_\beta$ if $\alpha < \beta$. In Figure 16.4, Ω_β is the union of Ω_α and the region shaded by straight lines.

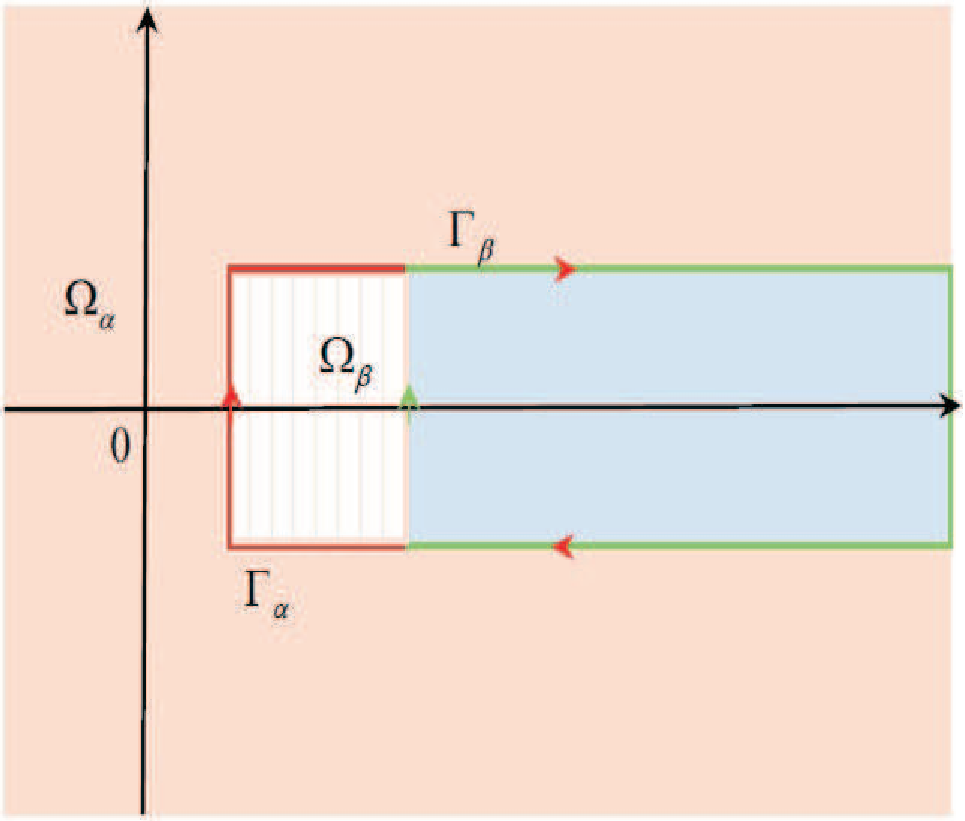


Figure 16.4: The regions Ω_α and Ω_β if $\alpha < \beta$.

[e]Recall from point-set topology [45, p. 3] that a nonempty compact connected metric space is a **continuum**.

- **f_β is an analytic continuation of f_α if $\alpha < \beta$.** Let $\zeta \in \Omega_\alpha$. Then there exists a $\delta > 0$ such that $D(\zeta;\delta) \subseteq \Omega_\alpha$ and $D(\zeta;\delta) \cap \Gamma_\alpha^* = \varnothing$. Define $\psi_\alpha : D(\zeta;\delta) \times \Gamma_\alpha^* \to \mathbb{C}$ by
$$\psi_\alpha(z,\omega) = \frac{\exp(\mathrm{e}^\omega)}{\omega - z}.$$
If $\omega = t + \pi i$ for $t \in [\alpha,\infty)$, then we have $\mathrm{e}^\omega = -\mathrm{e}^t$ so that
$$|\exp(\mathrm{e}^\omega)| = \exp(-\mathrm{e}^t) \leq \frac{1}{\mathrm{e}^{\mathrm{e}^\alpha}} < \infty \tag{16.26}$$
for all $t \in [\alpha,\infty)$. In fact, this bound (16.26) remains true on $-t-\pi i$ for all $t \in (-\infty,-\alpha]$. If $\omega = \alpha + \frac{\pi i t}{\alpha}$ for $t \in [-\alpha,\alpha]$, then we see that
$$|\exp(\mathrm{e}^\omega)| = \exp\Big(\mathrm{e}^\alpha \cos\frac{\pi t}{\alpha}\Big) \leq \exp(\mathrm{e}^\alpha) < \infty.$$
In other words, the function $\exp(\mathrm{e}^\omega)$ is bounded on Γ_α^*. Next, it is obvious that $\psi_\alpha(z,\omega)$ is a measurable function of ω, for each *fixed* $z \in D(\zeta;\delta)$, and $\psi_\alpha(z,\omega)$ is holomorphic in $D(\zeta;\delta)$, for each *fixed* $\omega \in \Gamma_\alpha^*$. By Problem 10.18, we conclude immediately that f_α is holomorphic at ζ. Since ζ is arbitrary, we obtain $f_\alpha \in H(\Omega_\alpha)$.

  Recall that $\Omega_\alpha \subset \Omega_\beta$ if $\alpha < \beta$, so f_β is an analytic continuation of f_α if $\alpha < \beta$.

- **Existence of an entire function f such that $f = f_\alpha$ on Ω_α.** We *fix* an $\alpha > 0$ and an $\zeta \in \Omega_\alpha$. Since Ω_α is open in $\mathbb{C}$, one can find a $\delta_\zeta > 0$ such that
$$D(\zeta;\delta_\zeta) \subseteq \Omega_\alpha.$$
Let γ be a curve in $\mathbb{C}$ with parameter interval $[0,1]$ that starts at the center of $D(\zeta;\delta_\zeta)$. Note that this may happen that $\gamma([0,1]) \nsubseteq \Omega_\alpha$. However, the compactness of $\gamma([0,1])$ ensures that there corresponds an $\beta > \alpha$ such that $\gamma([0,1]) \subseteq \Omega_\beta$. Hence the first assertion guarantees that $\big(f_\alpha, D(\zeta;\delta)\big)$ can be analytically continued along the curve γ in $\mathbb{C}$. By Theorem 16.15 (The Monodromy Theorem), there exists an entire function f such that
$$f(z) = f_\alpha(z)$$
for all $z \in D(\zeta;\delta_\zeta)$. By the Corollary to Theorem 10.18, we have $f = f_\alpha$ on Ω_α.

- **$f(r\mathrm{e}^{i\theta}) \to 0$ as $r \to \infty$ for every $\mathrm{e}^{i\theta} \neq 1$.** Suppose that $r > 0$ and θ is real. By the second assertion, we know that $f(z) = f_1(z)$ on Ω_1. By the assumption, we have $r\mathrm{e}^{i\theta} \in \Omega_1$ for large enough $r > 0$. Write $\Gamma_\alpha = \gamma_\alpha^- + L_\alpha + \gamma_\alpha^+$, where $\gamma_\alpha^- = -t - \pi i$ for $t \leq -\alpha$, $\gamma_\alpha^+ = t + \pi i$ for $t \geq \alpha$ and $L_\alpha = \alpha + \frac{\pi i t}{\alpha}$ for $t \in [-\alpha,\alpha]$. Therefore, we see that
$$\begin{aligned}
|f(r\mathrm{e}^{i\theta})| &= \frac{1}{2\pi}\Big|\int_{\Gamma_1} \frac{\exp(\mathrm{e}^\omega)}{\omega - z}\,\mathrm{d}\omega\Big| \\
&\leq \frac{1}{2\pi}\Big|\int_{\gamma_1^-} \frac{\exp(\mathrm{e}^\omega)}{\omega - z}\,\mathrm{d}\omega\Big| + \frac{1}{2\pi}\Big|\int_{L_1} \frac{\exp(\mathrm{e}^\omega)}{\omega - z}\,\mathrm{d}\omega\Big| + \frac{1}{2\pi}\Big|\int_{\gamma_1^+} \frac{\exp(\mathrm{e}^\omega)}{\omega - z}\,\mathrm{d}\omega\Big| \\
&= \frac{1}{2\pi}\Big|\int_{-\infty}^{-1} \frac{\exp(-\mathrm{e}^{-t})}{-t - \pi i - r\mathrm{e}^{i\theta}}\,\mathrm{d}t\Big| + \frac{1}{2\pi}\Big|\int_{-1}^{1} \frac{\exp(\mathrm{e}\cos\pi t)\cdot\exp(i\mathrm{e}\sin\pi t)}{1 + \pi i t - r\mathrm{e}^{i\theta}}\pi i\,\mathrm{d}t\Big| \\
&\quad + \frac{1}{2\pi}\Big|\int_{1}^{\infty} \frac{\exp(-\mathrm{e}^t)}{t + \pi i - r\mathrm{e}^{i\theta}}\,\mathrm{d}t\Big|.
\end{aligned} \tag{16.27}$$
Since $|\omega - r\mathrm{e}^{i\theta}| \geq r\sin\theta - 1$ for large enough $r > 0$ and for every $\omega \in \Gamma_1^*$, the inequality (16.27) reduces to
$$|f(r\mathrm{e}^{i\theta})| \leq \frac{1}{2\pi(r\sin\theta - 1)}\Big[\int_{-\infty}^{-1} \exp(-\mathrm{e}^{-t})\,\mathrm{d}t + \int_{-1}^{1} \exp(\mathrm{e}\cos\pi t)\,\mathrm{d}t + \int_{1}^{\infty} \exp(-\mathrm{e}^t)\,\mathrm{d}t\Big]$$

$$\leq \frac{1}{2\pi(r\sin\theta - 1)}\Big[2\int_1^\infty \exp(-\mathrm{e}^t)\,\mathrm{d}t + 2\mathrm{e}^{\mathrm{e}}\Big]. \tag{16.28}$$

Since $\mathrm{e}^t \leq \exp(\mathrm{e}^t)$ for every $t \geq 0$, the inequality (16.28) further reduces to

$$|f(r\mathrm{e}^{i\theta})| \leq \frac{1}{\pi(r\sin\theta - 1)}(\mathrm{e}^{\mathrm{e}} + \mathrm{e}^{-1}).$$

Since $\mathrm{e}^{i\theta} \neq 1$, $\sin\theta \neq 0$ which implies that

$$\lim_{r\to\infty} f(r\mathrm{e}^{i\theta}) = 0.$$

- **f is not constant.** Fix $r > 0$. Let $0 < r < \alpha < R$ and $\Gamma = \Gamma_\alpha \cup L_R$, where $L_R = R + \frac{\pi i t}{R}$ for $t \in [-R, R]$. Assume that f was constant. Since f is entire, the third assertion forces that $f(z) = 0$ in $\mathbb{C}$. In particular, we have

$$0 = f(r) = \frac{1}{2\pi i}\int_{\Gamma_\alpha} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega \tag{16.29}$$

for every $\alpha > r$. It is clear that Γ is closed and $\mathrm{Ind}_\Gamma(r) = 0$. Using Theorem 10.35 (Cauchy's Theorem), we know that

$$\frac{1}{2\pi i}\int_{\Gamma_\alpha} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega + \frac{1}{2\pi i}\int_{L_R} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega = \frac{1}{2\pi i}\int_{\Gamma} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega = 0$$

which implies

$$\int_{L_R} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega = 0 \tag{16.30}$$

for every $R > r$. Write

$$\begin{aligned}
f(r) &= \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi i}\int_{\gamma_R^+} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega + \frac{1}{2\pi i}\int_{\gamma_R^-} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega + \frac{1}{2\pi i}\int_{L_R} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi i}\int_R^\infty \frac{\exp(-\mathrm{e}^t)}{t + \pi i - r}\,\mathrm{d}t - \frac{1}{2\pi i}\int_R^\infty \frac{\exp(-\mathrm{e}^t)}{t - \pi i - r}\,\mathrm{d}t + \frac{1}{2\pi i}\int_{L_R} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega \\
&= -\int_R^\infty \frac{\exp(-\mathrm{e}^t)}{(t-r)^2 + \pi^2}\,\mathrm{d}t + \frac{1}{2\pi i}\int_{L_R} \frac{\exp(\mathrm{e}^\omega)}{\omega - r}\,\mathrm{d}\omega.
\end{aligned}$$

Using the results (16.29) and (16.30), we immediately see that

$$\int_R^\infty \frac{-\exp(-\mathrm{e}^t)}{(t-r)^2 + \pi^2}\,\mathrm{d}t = 0 \tag{16.31}$$

for every $R > r$. Since $\exp(-\mathrm{e}^t) \leq \mathrm{e}^{-t}$ and $\frac{1}{(t-r)^2+\pi^2} \leq \frac{1}{(R-r)^2+\pi^2}$, we get

$$\int_R^\infty \frac{-\exp(-\mathrm{e}^t)}{(t-r)^2 + \pi^2}\,\mathrm{d}t \geq -\frac{1}{(R-r)^2 + \pi^2}\int_R^\infty \mathrm{e}^{-t}\,\mathrm{d}t = \frac{1}{\mathrm{e}^R[(R-r)^2 + \pi^2]} > 0$$

which contradicts the result (16.31). Hence $f(r) \neq 0$.

- **$g(re^{i\theta}) \to 0$ as $r \to \infty$ for every $e^{i\theta}$.** If $e^{i\theta} \neq 1$, then the third assertion implies that

$$\lim_{r\to\infty} g(re^{i\theta}) = \lim_{r\to\infty} f(re^{i\theta}) \exp[-f(re^{i\theta})] = 0 \cdot 1 = 0.$$

Next, suppose that $e^{i\theta} = 1$. Since $f(r) \to \infty$ as $r \to \infty$, we see immediately that

$$\lim_{r\to\infty} g(r) = \lim_{r\to\infty} \frac{f(r)}{\exp[f(r)]} = 0.$$

This gives the fifth assertion.

- **Existence of an entire function h with the required properties.** By the fourth and the fifth assertions, we know that g is a nonconstant entire function such that $g(re^{i\theta}) \to 0$ as $r \to \infty$ for every $e^{i\theta}$. If g has a zero of order N at $z = 0$, then we write $g(z) = z^N G(z)$. Thus G is nonconstant entire, $G(0) \neq 0$ and

$$\lim_{r\to\infty} G(re^{i\theta}) = 0 \tag{16.32}$$

for every $e^{i\theta}$. Define $h(z) = \frac{G(z)}{G(0)}$. Therefore, h is nonconstant entire and $h(0) = 1$. Furthermore, if $z \neq 0$, then we write $z = re^{i\theta}$ for some $r > 0$. Since

$$h(nz) = h(nre^{i\theta}) = \frac{G(nre^{i\theta})}{G(0)},$$

we follow from the limit (16.32) that $h(nz) \to 0$ as $n \to \infty$. If $g(0) \neq 0$, then we consider the nonconstant entire function $h(z) = \frac{g(z)}{g(0)}$ which satisfies $h(0) = 1$ and $h(nz) \to 0$ as $n \to \infty$. In conclusion, there exists an entire function h such that

$$\lim_{n\to\infty} h(nz) = \begin{cases} 1, & \text{if } z = 0; \\ 0, & \text{if } z \neq 0. \end{cases}$$

We have ended the analysis of the problem. ■

Problem 16.12

Rudin Chapter 16 Exercise 12.

Proof. Suppose that f is represented by the series

$$\sum_{k=1}^{\infty} \left(\frac{z - z^2}{2}\right)^{3^k}. \tag{16.33}$$

Evidently, if $|z - z^2| < 2$, then $|\frac{z-z^2}{2}| < 1$ so that the series (16.33) converges by [61, Theorem 3.26, p. 61]. Furthermore, if $|z - z^2| > 2$, then the series (16.33) diverges. The red shaded part in Figure 16.5 indicates the region[f] of convergence of the power series (16.33).

Next, suppose that $P_k(z) = [z(1-z)]^{3^k}$, so

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{2^{3^k}} P_k(z).$$

[f]This is $(x - x^2 + y)^2 + (y - 2xy)^2 < 4$.

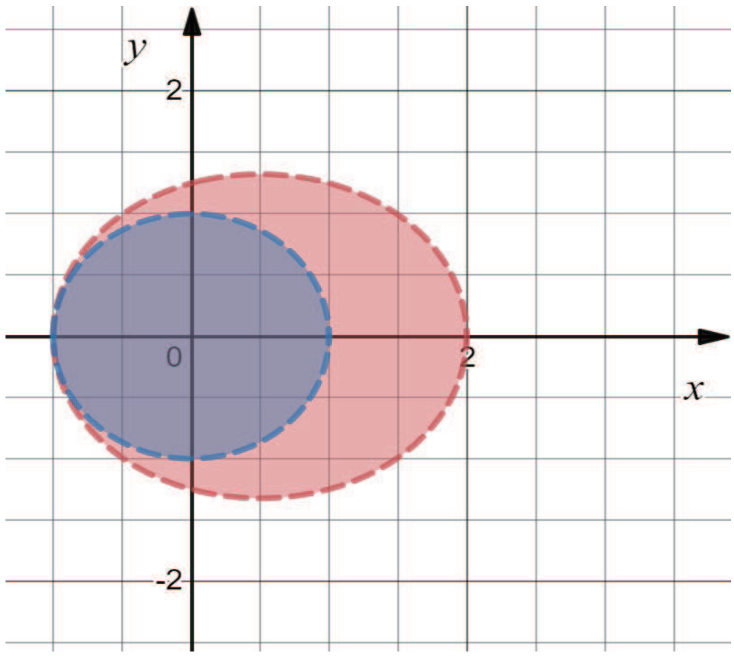


Figure 16.5: The regions of convergence of the two series.

Note that the highest power and the lowest power of z in $P_k(z)$ and in $P_{k+1}(z)$ are $2 \cdot 3^k$ and 3^{k+1} respectively. Since $3^{k+1} - 2 \cdot 3^k = 3^k > 0$ for every $k \geq 1$, the polynomial $P_k(z)$ contains *no* power of z that appear in any other $P_j(z)$ for all $j \neq k$. If we replace every $P_k(z)$ by its expansion in powers of z, then we get the power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{16.34}$$

with the property that $a_1 = a_2 = 0$ and for each positive integer k, we have

$$a_n = \begin{cases} \dfrac{(-1)^{n+1} C_{n-3^k}^{3^k}}{2^{3^k}}, & \text{if } n = 3^k, 3^k+1, \ldots, 2 \cdot 3^k; \\[2ex] 0, & \text{if } 2 \cdot 3^k < n < 3^{k+1}. \end{cases} \tag{16.35}$$

If both n and r tend to infinity, then it follows from Stirling's formula [61, Eq. (103), p. 194] that

$$C_r^n \sim \sqrt{\frac{n}{2\pi r(n-r)}} \cdot \frac{n^n}{r^r (n-r)^{n-r}}.$$

Recall that $C_{n-3^k}^{3^k}$ takes its maximum value when $n_k = \frac{3^{k+1}+1}{2}$, so it is true that

$$\begin{aligned} \frac{3^k}{2\pi(n_k - 3^k)[3^k - (n_k - 3^k)]} &= \frac{3^k}{2\pi(n_k - 3^k)(2 \cdot 3^k - n_k)} \\ &= \frac{3^k}{2\pi \cdot \frac{3^k+1}{2} \cdot \frac{3^k-1}{2}} \\ &= \frac{2}{\pi} \cdot \frac{3^k}{3^{2k}-1} \\ &\sim \frac{2}{\pi} \cdot \frac{1}{3^k} \end{aligned} \tag{16.36}$$

and

$$\frac{(3^k)^{3^k}}{(\frac{3^k+1}{2})^{\frac{3^k+1}{2}}(\frac{3^k-1}{2})^{\frac{3^k-1}{2}}} \sim 2^{3^k}. \tag{16.37}$$

Since $g(x) = x^{\frac{1}{x}}$ is decreasing for $x > \mathrm{e}$ and $x^{\frac{1}{x}} \to 1$ as $x \to \infty$ and $n_k \sim 1.5 \times 3^k$ for large k, it yields from the estimates (16.36) and (16.37) that

$$\begin{aligned}
\lim_{k\to\infty} |a_{n_k}|^{\frac{1}{n_k}} &= \lim_{k\to\infty} \left(\frac{C^{3^k}_{\frac{3^k+1}{2}}}{2^{3^k}}\right)^{\frac{1}{1.5\times 3^k}} \\
&= \lim_{k\to\infty} \frac{1}{2^{\frac{2}{3}}} \times \big(C^{3^k}_{\frac{3^k+1}{2}}\big)^{\frac{1}{1.5\times 3^k}} \\
&= \lim_{k\to\infty} \frac{1}{2^{\frac{2}{3}}} \cdot \left(\frac{2}{\pi}\right)^{\frac{1}{3^{k+1}}} \cdot \left(\frac{1}{3^k}\right)^{\frac{1}{3^{k+1}}} \cdot 2^{\frac{2}{3}} \\
&= 1.
\end{aligned}$$

In other words, the radius of convergence of the power series is 1. Check the blue shaded part in Figure 16.5.

Let $\lambda = 3$. Let $p_k = 2 \cdot 3^k$ and $q_k = 3^{k+1}$ for $k = 1, 2, \ldots$. Then they satisfy $\lambda q_k > (\lambda+1)p_k$ and $a_n = 0$ for $p_k < n < q_k$ for all positive integers k. Now the power series (16.33) and (16.34) assert that there exists a $\delta > 0$ such that

$$\sum_{k=1}^{\infty} \left(\frac{z - z^2}{2}\right)^{3^k} = \sum_{n=1}^{\infty} a_n z^n$$

for all $z \in D(0;1) \cap D(1;\delta)$. By Definition 16.1, it means that 1 is a regular point of f. Finally, it concludes from Theorem 16.5 that the sequence $\{s_{p_k}(z)\}$ converges in a neighborhood of 1, where $s_p(z)$ is the pth partial sum of the power series (16.34).

By Figure 16.5 again, we know that all boundary points of T, except $z = -1$, are regular points of f. Observe from the representation (16.34) and the definition (16.35) that

$$-f(-z) = \sum_{n=1}^{\infty} b_n z^n,$$

where $b_n \geq 0$ for every $n \geq 1$. Thus Problem 16.1 ensures that $-f(-z)$ has a singularity at $-z = 1$. Hence $z = -1$ is the singular point of f which is nearest to the origin, so we have completed the proof of the problem. ■

Problem 16.13

Rudin Chapter 16 Exercise 13.

Proof. For each positive integer n, we have

$$X_n = \{f \in H(\Omega) \,|\, f = g^{(n)} \text{ for some } g \in H(\Omega)\}.$$

(a) If $f \in X_1$, then $f = g'$ for some $g \in H(\Omega)$. Since γ is a closed path lying in Ω, Theorem 10.12 implies that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma g'(z)\,\mathrm{d}z = 0. \tag{16.38}$$

Conversely, suppose that the integral (16.38) holds. Since $f \in H(\Omega)$ and Ω is an annulus, Problem 10.25 shows that f admits the Laurent series

$$f(z) = f_1(z) + f_2(z) = \sum_{-\infty}^{-1} c_n z^n + f_2(z),$$

where $f_1 \in H\big(\mathbb{C} \setminus \overline{D}(0; \frac{1}{2})\big)$ and $f_2 \in H\big(D(0; 2)\big)$. Since

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\,\mathrm{d}z,$$

the integral (16.38) implies that $c_{-1} = 0$. Thus we obtain

$$f_1(z) = \sum_{n=2}^{\infty} c_{-n} z^{-n}.$$

If we let $\omega = \frac{1}{z}$, then the function

$$F_1(\omega) = f_1\Big(\frac{1}{\omega}\Big) = \sum_{n=2}^{\infty} c_{-n}\omega^n$$

is holomorphic in $\{\omega \in \mathbb{C} \,|\, |\omega| < \frac{1}{2}\}$. Therefore, it is true that $\limsup\limits_{n\to\infty} \sqrt[n]{c_{-n}} \geq \dfrac{1}{2}$ which implies that the radius of convergence of the series

$$G(\omega) = \sum_{n=2}^{\infty} \frac{c_{-n}}{1-n}\omega^{n-1}$$

is at least $\frac{1}{2}$ too. Next, it is clear from Theorem 10.6 that $G'(\omega) = -\omega^{-2}F_1(\omega)$ in the disc$\{\omega \in \mathbb{C} \,|\, |\omega| < \frac{1}{2}\}$. By transforming back to the variable z, we see that

$$f_1(z) = \frac{\mathrm{d}}{\mathrm{d}z}\Big(\sum_{n=2}^{\infty} \frac{c_{-n}}{1-n} z^{1-n}\Big) \tag{16.39}$$

holds in $\mathbb{C} \setminus \overline{D}(0; \frac{1}{2})$. Similarly, we can show that

$$f_2(z) = \frac{\mathrm{d}}{\mathrm{d}z}\Big(\sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}\Big) \tag{16.40}$$

holds in $D(0; 2)$. Finally, if we define

$$g(z) = \sum_{n=2}^{\infty} \frac{c_{-n}}{1-n} z^{1-n} + \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}$$

for all $z \in \Omega$, then the facts (16.39) and (16.40) combine to imply immediately that $f(z) = g'(z)$ in Ω, i.e., $f \in X_1$.

(b) Since $f \in H(\Omega)$ and Ω is an annulus, Problem 10.25 shows that f admits a representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{16.41}$$

If $f \in X_m$, then $f = g_m^{(m)}$ for some $g_m \in H(\Omega)$. Again, g_m has the Laurent series in Ω, i.e.,

$$g_m(z) = \sum_{n=-\infty}^{\infty} b_{n,m} z^n$$

which gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty} a_n z^n &= g_m^{(m)}(z) \\ &= \sum_{n=-\infty}^{\infty} n(n-1)\cdots(n-m+1) b_{n,m} z^{n-m} \\ &= \sum_{n=-\infty}^{\infty} (n+m)(n+m-1)\cdots(n+1) b_{n+m,m} z^n.\end{aligned}$$

Therefore, it means that $a_{-1} = a_{-2} = \cdots = a_{-m} = 0$. As m runs through all positive integers, the Laurent series (16.41) reduces to

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

which implies that $f \in H\big(D(0;2)\big)$.

Conversely, suppose that there exists an $g \in H\big(D(0;2)\big)$ such that $f(z) = g(z)$ for all $z \in \Omega$. By Theorem 13.11, the simply connectedness of $D(0;2)$ ensures that one can find an $g_1 \in H\big(D(0;2)\big)$ such that $g_1' = g$. In fact, this argument can be repeated to achieve the existence of an $g_n \in H\big(D(0;2)\big)$ with $g_n^{(n)} = g$ for each positive integer n. Hence we obtain

$$f(z) = g_n^{(n)}(z)$$

for all $z \in \Omega$ and this means that $f \in X_n$ for every positive integer n.

We have completed the proof of the problem. ■

Problem 16.14

Rudin Chapter 16 Exercise 14.

Proof. Our proof here basically follows that in [64, §2.7, pp. 54 – 56]. Since normality is a local property, we may assume that Ω is the unit disc U. Suppose that

$$\mathscr{F} = \{f \in H(U) \,|\, |f(p)| \leq R \text{ and } 0, 1 \notin f(U)\}.$$

Recall from Theorem 16.20 that the modular function λ is invariant under Γ (i.e., $\lambda \circ \varphi = \lambda$ for all $\varphi \in \Gamma$) and maps Π^+ onto $\mathbb{C} \setminus \{0, 1\}$. Since $f(U) \subseteq \mathbb{C} \setminus \{0, 1\}$, the function $\lambda^{-1} \circ f$ has a local branch defined in a sufficiently small neighbourhood of $f(0)$. Then this function element may be analytically continued in U, so we assert from Theorem 16.15 (The Monodromy Theorem) that there exists a holomorphic function $\widehat{f} : U \to \Pi^+$ such that

$$\lambda \circ \widehat{f} = f. \tag{16.42}$$

Let $\{f_n\} \subseteq \mathscr{F}$. Since $|f_n(p)| \leq R$ for all $n \in \mathbb{N}$, the Bolzano-Weierstrass Theorem [79, Problem 5.25, pp. 68, 69] ensures that there is a convergent subsequence $\{f_{n_k}(p)\}$. Let this limit be ℓ.

- **Case (i): $\ell \neq 0$ and $\ell \neq 1$.** Then we can fix a branch of λ^{-1} in a neighborhood of ℓ and use this to define the functions $\widehat{f_{n_k}}$ by the equation (16.42). Since $\operatorname{Im}\widehat{f_{n_k}} > 0$, we have $\operatorname{Re}(-i\widehat{f_{n_k}}) = \operatorname{Im}\widehat{f_{n_k}} > 0$ so that the family $\{-i\widehat{f_{n_k}} \,|\, k \in \mathbb{N}\}$ is normal by Problem 14.15. Consequently, the family
$$\widehat{\mathscr{F}} = \{\widehat{f_{n_k}} \,|\, k \in \mathbb{N}\}$$
is also normal. For simplicity, we may assume that $\widehat{f_{n_k}}$ converges normally to $g \in H(U)$. Clearly, we have $g(U) \subseteq \overline{\Pi^+}$. By the equation (16.42) again, we conclude that
$$g(p) = \lim_{k\to\infty} \widehat{f_{n_k}}(p) = \lim_{k\to\infty} \lambda^{-1}\big(f_{n_k}(p)\big) = \lambda^{-1}(\ell).$$
Recall that the domain of λ is Π^+, so the Open Mapping Theorem implies that $g(U) \subseteq \Pi^+$ and $\lambda \circ g : U \to \mathbb{C}$ is well-defined such that
$$\lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} \lambda\big(\widehat{f_{n_k}}(z)\big) = \lambda\big(g(z)\big)$$
for all $z \in U$. Hence $\{f_{n_k}\}$ is the required subsequence.

- **Case (ii): $\ell = 1$.** Since $f_{n_k} \in H(U)$ and $0 \notin f_{n_k}(U)$, we have $\frac{1}{f_{n_k}} \in H(U)$. Since U is simply connected, we deduce from Theorem 13.11 that each f_{n_k} has a holomorphic square root h_k in U. We choose the branch such that
$$\lim_{k\to\infty} h_k(p) = -1. \tag{16.43}$$
Since $f_{n_k} = h_k^2$, we have $0, 1 \notin h_k(U)$ and $|h_k(p)| \leq \sqrt{R}$. Consider the family
$$\mathscr{H} = \{h_k \,|\, k \in \mathbb{N}\}.$$
Now the limit (16.43) guarantees that we can apply **Case (i)** to $\mathscr{H}$ to obtain a convergent subsequence h_{k_j} in U. In conclusion, the limit
$$\lim_{j\to\infty} f_{n_{k_j}}(z)$$
exists for all $z \in U$.

- **Case (iii): $\ell = 0$.** In this case, we may apply **Case (ii)** to the sequence $\{1 - f_{n_k} \,|\, k \in \mathbb{N}\}$.

Hence we have completed the proof of the problem. ■

Remark 16.2

Problem 16.14 is classically called the **Fundamental Normality Test**.

Problem 16.15

Rudin Chapter 16 Exercise 15.

Proof. Without loss of generality, we may assume that D is the unit disc and $D \subseteq \Omega = D(0; R)$ for some $R > 1$. Let (f, D) be analytically continued along every curve in Ω that starts at the origin 0. Since $f \in H(D)$, f has a power series expansion at 0, i.e.,
$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Suppose that r is the radius of convergence of this power series. Clearly, we have $1 \leq r$. If $r = 1$, then we know from Theorem 16.2 that f has *at least* one singular point on the unit circle T. Let ω be a singular point of f on T and γ be a curve in Ω starting at 0 and passing through ω. Now the assumption ensures that (f, D) can be analytically continued along γ in Ω, so ω is a regular point of f, a contradiction. As a result, $1 < r \leq R$. Next, if $r < R$, then similar argument can be applied to conclude that *no* point on $C(0; r)$ is a singular point, but this contradicts Theorem 16.2. Hence we must have $r \geq R$ and it means that Theorem 16.5 holds in this special case. This proves the first assertion.

Let Ω be any simply connected region (other than the complex plane itself), $D \subseteq \Omega$ and (f, D) be analytically continued along every curve in Ω that starts at the center of D. Let $z_0 \in D$. By Theorem 14.8 (The Riemann Mapping Theorem) or [9, §14.2, pp. 200 − 204], one can find a (unique) conformal mapping $F : \Omega \to U$ such that $F(z_0) = 0$ and $F'(z_0) > 0$. Let $S = F(D)$. Obviously, we have $G = F^{-1}|_S : S \subset U \to D$ so that $G(0) = z_0$. We consider the mapping

$$g = f \circ G : S \subset U \to \mathbb{C}. \tag{16.44}$$

Since S is an open set containing the origin, we may assume that it is an open disc centered at 0 so that S and U are concentric. Since $f \in H(D)$, we conclude that $g \in H(S)$. Furthermore, since (f, D) can be analytically continued along every curve in Ω, the function element (g, S) can also be analytically continued along every curve in U. Therefore, the first assertion guarantees that there corresponds a $h \in H(U)$ such that $h(z) = g(z)$ for all $z \in S$. Using the definition (16.44), we have $h \circ F \in H(\Omega)$ and for all $z \in D$,

$$h\big(F(z)\big) = h\big(G^{-1}(z)\big) = h\big(g^{-1}\big(f(z)\big)\big) = f(z).$$

This proves the second assertion and we end the analysis of the problem. ■

CHAPTER 17

H^p-Spaces

17.1 Problems on Subharmonicity and Harmonic Majoriants

Problem 17.1

Rudin Chapter 17 Exercise 1.

Proof. Let $u : \Omega \to \mathbb{R}$ be an upper semicontinuous subharmonic function. By Definition 2.8, for every real α, the set $\{z \in \mathbb{C} \,|\, u(z) < \alpha\}$ is open in $\mathbb{C}$.

- Let $K \subset \Omega$ be compact and $h : K \to \mathbb{R}$ be continuous such that h is harmonic in $V = K^\circ$ and $u(z) \leq h(z)$ for all boundary points of K. Put $u_1 = u - h$. Assume that $u_1(\zeta) > 0$ for some $\zeta \in V$. Since h is continuous on K, $-h$ is upper semicontinuous on K. Thus u_1 is also upper semicontinuous on K by [78, Problem 2.1, pp. 17, 18]. Since K is compact, [78, Problem 2.21, p. 60] ensures that u_1 attains its maximum m on K. Since $u_1 \leq 0$ on the boundary of K, the set $E = \{z \in K \,|\, u_1(z) = m\}$ is a nonempty compact subset of V. Let z_0 be a boundary point of E. Since E is compact, V is open in $\mathbb{C}$ and $E \subset V$, there exists an $r > 0$ such that $\overline{D(z_0; r)} \subset V$. Now some subarc of the boundary of $\overline{D(z_0; r)}$ lies in $V \setminus E$. Hence we have
$$u_1(z_0) = m > \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(z_0 + re^{i\theta})\, d\theta$$
which means that u_1 is *not* subharmonic in V. However, since u and h are subharmonic and harmonic in V respectively, it follows from the mean value property that u_1 is also subharmonic in V, a contradiction. This proves that *no* such ζ exists and then $u(z) \leq h(z)$ for all $z \in K$.

- By Definition 17.1, it suffices to prove that Theorem 17.5 is true for subharmonic function in U. Let $0 \leq r < 1$. Then $K = \overline{D(0; r)} \subset U$ and $u : K \to \mathbb{R}$ is subharmonic. In particular, u is an upper semicontinuous function. We need the following result:[a]

  Lemma 17.1 (Baire's Theorem on Semicontinuous Functions)

  Let $K \subset \mathbb{C}$ be compact and $-\infty \leq u(z) < \infty$ on K. If u is upper semicontinuous, then it is the limit of a monotone decreasing sequence of continuous functions $\{u_n\}$ on K.

[a] Read https://encyclopediaofmath.org/wiki/Baire_theorem#Baire.27s_theorem_on_semi-continuous_functions.

Let $0 \leq r_1 < r_2 < 1$. Using this lemma, we know that $u(z) \leq u_n(z)$ on $C(0; r_2)$. By Theorem 11.8, $Hu_n \in C(K)$, Hu_n is harmonic in $D(0; r_2)$ and $(Hu_n)|_{C(0;r_2)} = u_n$. Clearly, we have $u(z) \leq u_n(z) = (Hu_n)(z)$ on $C(0; r_2)$, so the first assertion implies that

$$u(z) \leq (Hu_n)(z) \tag{17.1}$$

for all $z \in K$. Furthermore, the mean value property gives

$$(Hu_n)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (Hu_n)(r_2 e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(r_2 e^{it})\,dt. \tag{17.2}$$

Combining the inequality (17.1) and the formula (17.2) we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_1 e^{it})\,dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} (Hu_n)(r_1 e^{it})\,dt = (Hu_n)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(r_2 e^{it})\,dt. \tag{17.3}$$

Finally, we apply Problem 1.7[b] to the inequality (17.3) to get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_1 e^{it})\,dt \leq \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(r_2 e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_2 e^{it})\,dt.$$

This completes the proof of the problem. ■

Problem 17.2

Rudin Chapter 17 Exercise 2.

Proof. Let $u(z) = \log\big(1 + |f(z)|\big) = \log\big(1 + e^{\log|f(z)|}\big)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = \log(1 + e^x)$. Then we can write

$$u(z) = \varphi\big(\log|f(z)|\big). \tag{17.4}$$

Since $f \in H(\Omega)$, it follows from Theorem 17.3 that $\log|f(z)|$ is subharmonic in Ω. Evidently, $\varphi'(x) = \frac{e^x}{1+e^x} > 0$ for all $x \in \mathbb{R}$ and $\varphi'(s) < \varphi'(t)$ if $s < t$, so φ is a monotonically increasing convex function on $\mathbb{R}$. By applying Theorem 17.2 to the function (17.4), we conclude that u is subharmonic in Ω, as required. This completes the analysis of the problem. ■

Problem 17.3

Rudin Chapter 17 Exercise 3.

Proof. It seems that the hypothesis $0 < p \leq \infty$ should be replaced by $0 < p < \infty$ because if $p = \infty$, then the function $f(z) \equiv 2$ belongs to H^∞. In this case, we have $|f(z)|^\infty = \infty$ for any $z \in U$.

- **$f \in H^p$ if and only if $|f(z)|^p \leq u(z)$ for some harmonic function u in U.** Suppose that there exists a harmonic function u in U such that $|f(z)|^p \leq u(z)$ for all $z \in U$. Combining this and the mean value property, we get

$$\|f_r\|_p = \Big\{\int_T |f(re^{i\theta})|^p\,d\sigma\Big\}^{\frac{1}{p}} \leq \Big\{\int_T u(re^{i\theta})\,d\sigma\Big\}^{\frac{1}{p}} = [u(0)]^{\frac{1}{p}} < \infty$$

for every $0 \leq r < 1$. By Definition 17.7, we have $f \in H^p$.

[b] In fact, it is

Conversely, suppose that $f \in H^p$. Now it is easy to see that

$$\|f_r^p\|_1 = \int_T |f_r^p|\,\mathrm{d}\sigma = \int_T |f_r|^p\,\mathrm{d}\sigma = \|f_r\|_p^p$$

which means $\|f^p\|_1 = \|f\|_p^p < \infty$. As a consequence, we have $f^p \in H^1$. Applying [58, Eqn. (5), p. 344] directly to f^p, we have

$$\begin{aligned}
|f(z)|^p &\leq |Q_{f^p}(z)| \\
&= \left|\exp\left\{\frac{1}{2\pi}\int_T \frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z}\log|(f^p)^*(\mathrm{e}^{it})|\,\mathrm{d}t\right\}\right| \\
&= \exp\left\{\mathrm{Re}\left[\frac{1}{2\pi}\int_T \frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z}\log|(f^p)^*(\mathrm{e}^{it})|\,\mathrm{d}t\right]\right\} \\
&= \exp\left\{\frac{1}{2\pi}\int_T P_r(\theta-t)\log|(f^p)^*(\mathrm{e}^{it})|\,\mathrm{d}t\right\} \qquad (17.5)
\end{aligned}$$

for all $z \in U$. Using the same argument as in proving the inequality in the proof of Theorem 17.16(c), we obtain from the inequality (17.5) that

$$|f(z)|^p \leq \frac{1}{2\pi}\int_T P_r(\theta-t)|(f^p)^*(\mathrm{e}^{it})|\,\mathrm{d}t = P[(f^p)^*](z)$$

in U.[c] By Theorem 17.11(b), $(f^p)^* \in L^1(T)$, so Theorem 11.7 ensures that $P[(f^p)^*]$ is harmonic in U.

- **The existence of a least harmonic majorant.** Let $u_f = P[(f^p)^*]$. In fact, this is a least harmonic majorant. To see this, let u be a harmonic majorant in U, i.e., u is harmonic in U and $|f(z)|^p \leq u(z)$ for every $z \in U$. For any $\rho < 1$, we have

$$\frac{1}{2\pi}\int_T P_r(\theta-t)|f(\rho\mathrm{e}^{it})|^p\,\mathrm{d}t \leq \frac{1}{2\pi}\int_T P_r(\theta-t)u(\rho\mathrm{e}^{it})\,\mathrm{d}t = u(\rho z),$$

where $0 \leq r < 1$. As $\rho \to 1$, we get[d]

$$\begin{aligned}
u_f(z) &= \frac{1}{2\pi}\int_T P_r(\theta-t)|(f^p)^*(\mathrm{e}^{it})|\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_T P_r(\theta-t)\lim_{\rho\to 1}|f^p(\rho\mathrm{e}^{it})|\,\mathrm{d}t \\
&= \lim_{\rho\to 1}\left[\frac{1}{2\pi}\int_T P_r(\theta-t)|f(\rho\mathrm{e}^{it})|^p\,\mathrm{d}t\right] \\
&\leq u(z)
\end{aligned}$$

for every $z \in U$.

- **$\|f\|_p = u_f(0)^{\frac{1}{p}}$.** Let $0 < p < \infty$. By the previous assertion, we know that

$$u_f(z) = \int_T P(z,\mathrm{e}^{it})|f(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma.$$

When $z = 0$, we have $P(0,\mathrm{e}^{it}) = 1$ so that $u_f(0) = \|f\|_p^p$, i.e., $\|f\|_p = u_f(0)^{\frac{1}{p}}$.

[c]Recall that $P[f]$ is the Poisson integral of f.

[d]We can interchange the limit and the integral because Theorem 10.24 (The Maximum Modulus Theorem) asserts that $F_\rho(t) = P_r(\theta-t)|f(\rho\mathrm{e}^{it})|^p$ is increasing with respect to ρ, so we may apply Theorem 1.26 (The Lebesgue's Monotone Convergence Theorem).

We have completed the proof of the problem. ■

Problem 17.4

Rudin Chapter 17 Exercise 4.

Proof. On the one hand, if $\log^+|f|$ has a harmonic majorant in U, then there exists a harmonic function u in U such that $0 \leq \log^+|f(z)| = \big|\log^+|f(z)|\big| \leq u(z)$. Combining this and the mean value property, we obtain

$$0 \leq \|f_r\|_0 = \exp\Big(\int_T \log^+|f_r(e^{it})|\,d\sigma\Big) \leq \int_T u(re^{it})\,d\sigma = u(0) < \infty$$

for all $z = re^{it} \in U$ and all $0 \leq r < 1$. By Definition 17.7, we get $\|f\|_0 < \infty$ so that $f \in N$.

On the other hand, suppose that $f \in N$. If $f \equiv 0$ so that $\log^+|f(z)| = 0$ in U, then there is nothing to prove. With the aid of the result in §17.19, there correspond two functions $b_1, b_2 \in H^\infty$ such that b_2 has no zero in U and

$$f = \frac{b_1}{b_2}.$$

Without loss of generality, we may assume that $\|b_1\|_\infty \leq 1$ and $\|b_2\|_\infty \leq 1$. Since U is simply connected and $\frac{1}{b_2} \in H(U)$, we deduce from Theorem 13.11 that there exists an $g \in H(U)$ such that $b_2 = e^g$. Since $\|b_2\|_\infty = e^{\operatorname{Re} g} \leq 1$, the function $u(z) = \operatorname{Re} g(z)$ is less than or equal to zero in U and this implies

$$\log|f(z)| = \log|b_1(z)| - \log|b_2(z)| \leq -\log e^{u(z)} = -u(z) \tag{17.6}$$

for all $z \in U$. Since $-u(z) \geq 0$, the inequality (17.6) and the definition in §15.22 yield

$$\log^+|f(z)| \leq -u(z)$$

in U. By Theorem 11.4 and $g \in H(U)$, $-u$ is harmonic in U. Hence, $\log^+$ has a harmonic majorant in U and this completes the proof of the problem. ■

17.2 Basic Properties of H^p

Problem 17.5

Rudin Chapter 17 Exercise 5.

Proof. Since $f \in H(U)$, it is true that $f \circ \varphi \in H(U)$. Since $f \in H^p$, Problem 17.3 ensures that there is a harmonic function u in U such that $|f(z)|^p \leq u(z)$ for all $u \in U$. Thus this shows that

$$\big|f\big(\varphi(z)\big)\big|^p \leq u\big(\varphi(z)\big)$$

for all $z \in U$. Applying Problem 11.7(b) with $\Phi = u$ and $f = \varphi$ there, we see that

$$\Delta[u \circ \varphi] = [(\Delta u) \circ \varphi] \times |\varphi'|^2 = 0.$$

By the definition, $u \circ \varphi$ is harmonic in U and then Problem 17.3 asserts that $f \circ \varphi \in H^p$.

The assertion is also true when we replace H^p by N. To see this, Problem 17.4 ensures that there exists a harmonic function u in U such that $\log^+ |f(z)| \leq u(z)$ holds for all $z \in U$. Then we have

$$\log^+ \big|f\big(\varphi(z)\big)\big| \leq u\big(\varphi(z)\big) \tag{17.7}$$

in U. Applying Problem 11.7 to $u \circ \varphi$ and then using the fact that $\Delta u = 0$ in U, we see immediately that

$$\Delta[u \circ \varphi] = [(\Delta u) \circ \varphi] \cdot |\varphi'|^2 = 0.$$

In other words, $u \circ \varphi$ is harmonic in U. Combining the inequality (17.7) and Problem 17.4, we conclude that $f \circ \varphi \in N$, completing the proof of the problem. ■

Problem 17.6

Rudin Chapter 17 Exercise 6.

Proof. Let $\alpha > 0$ and consider the function[e]

$$f_\alpha(z) = \frac{1}{(1-z)^\alpha}.$$

Put $z = re^{it}$, we have

$$I_\alpha(r) = \int_T |f_\alpha(re^{it})|\,dt = \int_{-\pi}^{\pi} \frac{1}{|1-re^{it}|^\alpha}\,dt.$$

We want to estimate $I_\alpha(r)$ as $r \to \infty$. Of course, it depends on the value of α.

Since $r \to 1$, we may assume that $r > \frac{1}{2}$. By considering the triangle formed by $1, r$ and re^{it}. Clearly, this is an obtuse triangle. If $t \in [-\pi, \pi]$, then we have

$$\begin{aligned}
|1-re^{it}| &> \max\{1-r, r|1-e^{it}|\} \\
&\geq \frac{1}{2}\big(1-r+r|1-e^{it}|\big) \\
&> \frac{1}{2}\Big(1-r+\frac{1}{2}|1-e^{it}|\Big) \\
&= \frac{1}{2}\Big(1-r+\Big|\sin\frac{t}{2}\Big|\Big).
\end{aligned} \tag{17.8}$$

Recall the fact [61, Exercise 7, p. 197] that $|\sin x| \geq \frac{2}{\pi}|x|$ for every $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so the inequality (17.8) can further reduce to

$$|1-re^{it}| \geq \frac{1}{2}\Big(1-r+\frac{|t|}{\pi}\Big) \geq \frac{1}{2\pi}(1-r+|t|). \tag{17.9}$$

On the other hand, the triangle inequality gives

$$|1-re^{it}| \leq |1-e^{it}| + |e^{it}-re^{it}| = |1-e^{it}| + 1 - r \leq |t| + 1 - r. \tag{17.10}$$

Combining the inequalities (17.9) and (17.10), we obtain

$$\frac{1}{\big(|t|+1-r\big)^\alpha} \leq \frac{1}{|1-re^{it}|^\alpha} \leq \frac{(2\pi)^\alpha}{\big(|t|+1-r\big)^\alpha}$$

[e] Here we note that $1-z \neq 0$ in U, so we may take the branch such that $(1-z)^\alpha = \exp\big(\alpha\log(1-z)\big)$, where $-\frac{\pi}{2} < \arg(1-z) < \frac{\pi}{2}$.

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}t}{\big(|t|+1-r\big)^{\alpha}}\leq I_{\alpha}(r)\leq(2\pi)^{\alpha}\int_{-\pi}^{\pi}\frac{\mathrm{d}t}{\big(|t|+1-r\big)^{\alpha}}. \tag{17.11}$$

Since $|t|$ is an even function in t, we get

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{\mathrm{d}t}{\big(|t|+1-r\big)^{\alpha}}&=2\int_{0}^{\pi}\frac{\mathrm{d}t}{(t+1-r)^{\alpha}}\\&=\begin{cases}2\big[\log(1-r+\pi)-\log(1-r)\big], & \text{if } \alpha=1;\\ \frac{2}{1-\alpha}\big[(1-r+\pi)^{1-\alpha}-(1-r)^{1-\alpha}\big], & \text{otherwise.}\end{cases}\end{aligned}$$

If $\alpha<1$, then the integral in the inequalities (17.11) tends to $\frac{2\pi^{1-\alpha}}{1-\alpha}$ as $r\to 1$ so that $I_\alpha(r)$ is bounded as $r\to 1$. If $\alpha=1$, then we observe from the inequalities (17.11) that

$$0<m_1\leq\frac{I_\alpha(r)}{\log(1-r)}\leq M_1$$

as $r\to 1$ for some positive constants m_1 and M_1. Similarly, if $\alpha>1$, then the inequalities (17.11) tells us that

$$0<m_2\leq\frac{I_\alpha(r)}{(1-r)^{1-\alpha}}\leq M_2$$

as $r\to 1$ for some positive constants m_2 and M_2.

We notice that

$$\|(f_\alpha)_r\|_p=\Big\{\frac{1}{2\pi}\int_T|f_\alpha(r\mathrm{e}^{it})|^p\,\mathrm{d}t\Big\}^{\frac{1}{p}}=\Big(\frac{I_{\alpha p}(r)}{2\pi}\Big)^{\frac{1}{p}},$$

so the previous paragraph indicates that $f_\alpha\in H^p$ if and only if $\alpha p<1$. Thus, for $0<r<s<\infty$, if we take $\alpha\in(\frac{1}{s},\frac{1}{r})$, then it is easy to see that $f_\alpha\in H^r$ but $f_\alpha\notin H^s$, i.e., $H^s\subset H^r$. The case for $s=\infty$ is obvious because we have $f_\alpha\notin H^\infty$ for every $\alpha>0$. In particular, if we take $\alpha=\frac{1}{2r}$, then we have $f_{\frac{1}{2r}}\in H^r$ but $f_{\frac{1}{2r}}\notin H^\infty$. This completes the analysis of the problem. ■

Problem 17.7

Rudin Chapter 17 Exercise 7.

Proof. Now Problem 17.6 guarantees that $H^\infty\subset H^p$ for every $0<p<\infty$. By Definition 17.7, $H^p\subseteq N$ holds for every $0<p<\infty$, so we have $H^\infty\subset N$. Hence we conclude easily that

$$H^\infty\subset N\cap\bigcap_{0<p<\infty}H^p\subseteq\bigcap_{0<p<\infty}H^p.$$

This completes the proof of the problem. ■

Problem 17.8

Rudin Chapter 17 Exercise 8.

Proof. If $p=1$, then there is nothing to prove. If $0<p<1$, then Problem 17.6 implies that $H^1\subset H^p$ so that the result is trivial. Thus we may assume that $1<p<\infty$. Since $f\in H^1$, Theorem 17.11 guarantees that f is the Poisson integral of f^*, i.e.,

$$f(z)=\int_T P(z,\mathrm{e}^{it})f^*(\mathrm{e}^{it})\,\mathrm{d}\sigma$$

which shows
$$f_r(\mathrm{e}^{i\theta}) = f(r\mathrm{e}^{i\theta}) = \int_T P_r(\theta - t) f^*(\mathrm{e}^{it})\,\mathrm{d}\sigma,$$
where $0 \leq r < 1$. Since $P_r(t) > 0$ for all $t \in T$, we may apply Theorem 3.5 (Hölder's Inequality) to obtain
$$\begin{aligned}
|f_r(\mathrm{e}^{i\theta})|^p &= \Big|\int_T P_r(\theta - t) f^*(\mathrm{e}^{it})\,\mathrm{d}\sigma\Big|^p \\
&\leq \Big|\int_T P_r(\theta - t) |f^*(\mathrm{e}^{it})|\,\mathrm{d}\sigma\Big|^p \\
&= \Big|\int_T \Big[P_r^{\frac{p-1}{p}}(\theta - t)\Big] \times \Big[P_r^{\frac{1}{p}}(\theta - t)|f^*(\mathrm{e}^{it})|\Big]\,\mathrm{d}\sigma\Big|^p \\
&\leq \Big\{\int_T P_r(\theta - t)|f^*(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma\Big\} \times \Big\{\int_T \Big[P_r^{\frac{p-1}{p}}(\theta - t)\Big]^{\frac{p}{p-1}}\,\mathrm{d}\sigma\Big\}^{p-1} \\
&= \Big\{\int_T P_r(\theta - t)|f^*(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma\Big\} \times \Big\{\int_T P_r(\theta - t)\,\mathrm{d}\sigma\Big\}^{p-1} \\
&= \int_T P_r(\theta - t)|f^*(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma
\end{aligned}$$
which implies
$$\begin{aligned}
\|f_r\|_p^p &= \frac{1}{2\pi}\int_T |f_r|^p\,\mathrm{d}t \\
&\leq \frac{1}{2\pi}\int_T \int_T P_r(\theta - t)|f^*(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma\,\mathrm{d}\theta \\
&\leq \int_T \Big[\frac{1}{2\pi}\int_T P_r(\theta - t)\,\mathrm{d}\theta\Big]|f^*(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma \\
&= \int_T |f^*(\mathrm{e}^{it})|^p\,\mathrm{d}\sigma \\
&= \|f^*\|_p,
\end{aligned}$$
where $0 \leq r < 1$. By the hypothesis, $\|f^*\|_p < \infty$ so that $f \in H^p$ as required. We complete the analysis of the problem. ■

Problem 17.9

Rudin Chapter 17 Exercise 9.

Proof. Since $f(U)$ is not dense in the complex plane, there exists a point $\alpha \in \mathbb{C}$ and $r > 0$ such that $D(\alpha; r) \subseteq \mathbb{C} \setminus f(U)$, i.e., $|f(z) - \alpha| > r$ for all $z \in U$. Let $F(z) = \frac{1}{f(z)-\alpha}$. Then F is a bounded holomorphic function in U so that $F \in H^\infty$. Thus it follows from Theorem 11.32 (Fatou's Theorem) that
$$F^*(\mathrm{e}^{i\theta}) = \lim_{r\to 1} F(r\mathrm{e}^{i\theta})$$
exists at almost all points of T. This implies that the same conclusion also holds for the function f, completing the proof of the problem. ■

Problem 17.10

Rudin Chapter 17 Exercise 10.

Proof. Define $\Phi_\alpha : H^2 \to \mathbb{C}$ by

$$\Phi_\alpha(f) = f(\alpha).$$

It is obvious that Φ_α is a linear functional on H^2. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The Cauchy-Schwarz inequality and Theorem 17.12 combine to yield

$$|f(\alpha)| = \Big|\sum_{n=0}^{\infty} a_n\alpha^n\Big| \leq \Big(\sum_{n=0}^{\infty}|a_n|^2\Big)^{\frac{1}{2}} \times \Big(\sum_{n=0}^{\infty}|\alpha|^{2n}\Big)^{\frac{1}{2}} = \Big(\sum_{n=0}^{\infty}|\alpha|^{2n}\Big)^{\frac{1}{2}} \cdot \|f\|_2 < \infty.$$

By Definition 5.3, Φ_α is a bounded linear functional. By Theorem 5.4, Φ_α is continuous and since H^2 is a Hilbert space, Theorem 4.12 (The Riesz Representation Theorem) ensures the existence of an $g \in H^2$ such that[f]

$$f(\alpha) = \Phi_\alpha(f) = \langle f, g\rangle = \lim_{r\to 1}\int_T f(re^{i\theta})\overline{g(re^{i\theta})}\,d\sigma = \int_T f^*(re^{i\theta})g^*(re^{i\theta})\,d\sigma, \tag{17.12}$$

where $f^*, g^* \in L^2(T)$ by Theorem 17.11. If we let

$$g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then Theorem 17.12 implies $\sum_{n=0}^{\infty}|b_n|^2 < \infty$. By the Parseval Theorem [62, Eqn. (6), p. 91], the formula (17.12) becomes

$$\sum_{n=0}^{\infty} a_n\alpha^n = f(\alpha) = \sum_{n=0}^{\infty} a_n\overline{b_n}.$$

As a result, we have $b_n = \overline{\alpha}^n$ for all $n = 0, 1, 2, \ldots$. Hence g has the representation

$$g(z) = \sum_{n=0}^{\infty}\overline{\alpha}^n z^n.$$

This ends the proof of the problem. ■

Problem 17.11

Rudin Chapter 17 Exercise 11.

Proof. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in U. By the Parseval Theorem [62, Eqn. (6), p. 91], we have

$$\int_T |f(re^{it})|^2\,d\sigma = \sum_{n=0}^{\infty}|a_n|^2 r^{2n},$$

[f] Or you may apply [62, Eqn. (6), p. 347] directly.

where $0 \le r < 1$, so that

$$\Big(\sum_{n=0}^{\infty} |a_n|^2\Big)^{\frac{1}{2}} = \sup_{r<1} \Big\{ \int_T |f(re^{it})|^2 \,d\sigma \Big\}^{\frac{1}{2}} = \|f\|_2 \le 1.$$

By the Cauchy-Schwarz inequality, we can establish

$$\begin{aligned} |f'(\alpha)|^2 &= \Big| \sum_{n=1}^{\infty} n a_n \alpha^{n-1} \Big|^2 \\ &\le \Big(\sum_{n=1}^{\infty} |a_n|^2\Big) \times \Big(\sum_{n=1}^{\infty} n^2 |\alpha|^{2(n-1)}\Big) \\ &\le \sum_{n=1}^{\infty} n^2 |\alpha|^{2(n-1)}. \end{aligned} \tag{17.13}$$

The identity

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3} \tag{17.14}$$

is valid for $|x| < 1$, so we may apply this to the inequality (17.13) to obtain

$$|f'(\alpha)| \le \sqrt{\frac{1+|\alpha|^2}{(1-|\alpha|^2)^3}}. \tag{17.15}$$

We claim that the estimation (17.15) is sharp. To see this, let $\alpha \neq 0$ and C_α be the number in the estimation (17.15). Consider the function

$$F_\alpha(z) = \frac{e^{i\theta}}{C_\alpha} \sum_{n=1}^{\infty} n|\alpha|^{n-1} z^n, \tag{17.16}$$

where $z \in U$. Direct computation indicates that the radius of convergence of the series (17.16) is $\frac{1}{|\alpha|} > 1$ so that $F_\alpha \in H(U)$. Next, using the identity (17.14), we see that

$$\begin{aligned} \|F_\alpha\|_2^2 &= \sum_{n=0}^{\infty} \Big|\frac{1}{C_\alpha}\Big|^2 \cdot n^2 |\alpha|^{2(n-1)} \\ &= \frac{(1-|\alpha|^2)^3}{1+|\alpha|^2} \cdot \sum_{n=0}^{\infty} n^2 |\alpha|^{2(n-1)} \\ &= \frac{(1-|\alpha|^2)^3}{1+|\alpha|^2} \times \frac{1+|\alpha|^2}{(1-|\alpha|^2)^3} \\ &= 1. \end{aligned}$$

Finally, we know that

$$|F'_\alpha(\alpha)| = \frac{1}{|C_\alpha|} \sum_{n=1}^{\infty} n^2 |\alpha|^{2(n-1)} = \sqrt{\frac{(1-|\alpha|)^3}{1+|\alpha|^2}} \times \frac{1+|\alpha|^2}{(1-|\alpha|)^3} = \sqrt{\frac{1+|\alpha|^2}{(1-|\alpha|)^3}}.$$

If $\alpha = 0$, then instead of the function (17.16), we consider

$$F_0(z) = e^{i\theta} z$$

which satisfies $\|F_0\|_2^2 = 1$ and $|F_0'(0)| = 1$. Hence this proves the claim and the functions (17.16) are extremal.

Next, we consider the general n. Firstly, we have

$$f^{(n)}(\alpha) = \sum_{m=n}^{\infty} m(m-1)\cdots(m-n+1)a_m\alpha^{m-n}$$

so that

$$\begin{aligned}|f^{(n)}(\alpha)|^2 &= \Big|\sum_{m=n}^{\infty} m(m-1)\cdots(m-n+1)a_m\alpha^{m-n}\Big|^2\\ &\leq \Big(\sum_{m=n}^{\infty}|a_m|^2\Big)^2 \times \Big\{\sum_{m=n}^{\infty}[m(m-1)\cdots(m-n+1)]^2|\alpha|^{2(m-n)}\Big\}^2\\ &\leq \Big\{\sum_{m=n}^{\infty}|m(m-1)\cdots(m-n+1)\alpha^{m-n}|^2\Big\}^2\end{aligned}$$

which implies

$$|f^{(n)}(\alpha)| \leq \sum_{m=n}^{\infty}|m(m-1)\cdots(m-n+1)\alpha^{m-n}|^2.$$

We denote

$$C_{\alpha,n} = \sqrt{\sum_{m=n}^{\infty}|m(m-1)\cdots(m-n+1)\alpha^{m-n}|^2} > 0.$$

Then we claim that

$$F_\alpha(z) = \frac{e^{i\theta}}{C_{\alpha,n}}\sum_{m=n}^{\infty} m(m-1)\cdots(m-n+1)|\alpha|^{m-n}z^m$$

are the extremal functions for $\alpha \neq 0$. In fact, we know that

$$\|F_\alpha\|_2^2 = \frac{1}{|C_{\alpha,n}|^2}\sum_{m=n}^{\infty}|m(m-1)\cdots(m-n+1)\alpha^{m-n}|^2 = \frac{1}{|C_{\alpha,n}|^2}\times|C_{\alpha,n}|^2 = 1.$$

Furthermore, direct computation gives

$$|F_\alpha^{(n)}(\alpha)| = \frac{1}{C_{\alpha,n}}\times\sum_{m=n}^{\infty}|m(m-1)\cdots(m-n+1)\alpha^{m-n}|^2 = C_{\alpha,n}$$

and we prove the claim. If $\alpha = 0$, then it is easily seen that

$$F_0(z) = e^{i\theta}n!z^n$$

are the extremal functions in this case because $|F_0^{(n)}(\alpha)| = (n!)^2$. Therefore, we have completed the proof of the problem. ■

Problem 17.12

Rudin Chapter 17 Exercise 12.

Proof. Since $p \geq 1$, we know that $f \in H^1$ and Theorem 17.11 tells us that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(z, \mathrm{e}^{it}) f^*(\mathrm{e}^{it})\,\mathrm{d}t$$

for all $z \in U$. Now the hypothesis guarantees that f is real a.e. in U. By the Open Mapping Theorem, f must be constant.

Consider $f(z) = i\frac{1+z}{1-z}$ in U. It is easy to see that $f \in H^p$ for every $0 < p < 1$ and

$$f^*(\mathrm{e}^{it}) = \lim_{r\to 1} f(r\mathrm{e}^{it}) = -\cot\frac{t}{2}$$

is real a.e. on T, but f is not constant. Hence we have completed the proof of the problem. ■

Problem 17.13

Rudin Chapter 17 Exercise 13.

Proof. Since $|f(r\mathrm{e}^{it})| = |\gamma_r(t)| \leq M$ for every $0 \leq r < 1$ and $t \in [-\pi, \pi]$, f is bounded in U. Thus $f \in H^\infty$ so that $f \in H^1$. Furthermore, we also have the fact that f^* is bounded on T. Let $f^*(\mathrm{e}^{it}) = \mu(t)$. Since $f(r\mathrm{e}^{\pi i}) = f(r\mathrm{e}^{-\pi i})$, it is easy to see that

$$\mu(\pi) = f^*(\mathrm{e}^{\pi i}) = \lim_{r\to 1} f(r\mathrm{e}^{\pi i}) = \lim_{r\to 1} f(r\mathrm{e}^{-\pi i}) = f^*(\mathrm{e}^{-\pi i}) = \mu(-\pi). \tag{17.17}$$

On the one hand, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then we have

$$a_n = \frac{1}{2\pi i}\int_{C(0;r)} \frac{f(\zeta)}{\zeta^{n+1}}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{-n}\mathrm{e}^{-int} f(r\mathrm{e}^{it})\,\mathrm{d}t \tag{17.18}$$

for every $n \geq 0$ and $0 < r < 1$. On the other hand, it follows from Theorem 17.11 that $f^* \in L^1(T)$, so the Fourier coefficients of f^* are given by

$$\widehat{f^*}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-int} f^*(\mathrm{e}^{it})\,\mathrm{d}t \tag{17.19}$$

for every $n \in \mathbb{Z}$. By observing the coefficients (17.18) and (17.19), we know that

$$|r^n a_n - \widehat{f^*}(n)| = \Big|\int_T \mathrm{e}^{-int}[f(r\mathrm{e}^{it}) - f^*(\mathrm{e}^{it})]\,\mathrm{d}\sigma\Big| \leq \int_T |f(r\mathrm{e}^{it}) - f(\mathrm{e}^{it})|\,\mathrm{d}\sigma = \|f_r - f^*\|_1 \to 0$$

as $r \to 1$ by Theorem 17.11. In other words, we get

$$\widehat{f^*}(n) = \begin{cases} a_n, & \text{if } n \geq 0; \\ 0, & \text{if } n < \infty. \end{cases}$$

This implies that

$$\int_{-\pi}^{\pi} \mathrm{e}^{int}\mu(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} \mathrm{e}^{int} f^*(\mathrm{e}^{it})\,\mathrm{d}t = 0 \tag{17.20}$$

for every $n = 1, 2, \ldots$.

Using the two facts (17.17) and (17.20), we find

$$\int_{-\pi}^{\pi} \mathrm{e}^{int}\,\mathrm{d}\mu(t) = [\mathrm{e}^{int}\mu(t)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \mu(t)\,\mathrm{d}(\mathrm{e}^{int}) = -in\int_{-\pi}^{\pi} \mathrm{e}^{int}\mu(t)\,\mathrm{d}t = 0$$

holds for every $n = 1, 2, \ldots$. Next, Theorem 17.13 (The F. and M. Riesz Theorem) shows that μ is absolutely continuous with m, i.e., $\mu(E) = 0$ if $m(E) = 0$. Recall that $f^*(\mathrm{e}^{it}) = \mu(t)$, so $f^* \in C(T)$. In fact, f^* is AC on T because of Theorem 7.18 because f maps sets of measure 0 to sets of measure 0. Finally, according to Theorem 17.11, we have

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(z, \mathrm{e}^{it})\mu(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(z, \mathrm{e}^{it})f^*(\mathrm{e}^{it})\,\mathrm{d}t = P[f^*](z)$$

for all $z \in U$. Thus if we define the function $Hf^* : \overline{U} \to \mathbb{C}$ as [62, Eqn. (1), p. 234], then Theorem 11.8 shows that $Hf^* \in C(\overline{U})$ and the restriction of Hf^* to T is exactly f^*. This is a required extension of f and we end the analysis of the problem. ■

Problem 17.14

Rudin Chapter 17 Exercise 14.

Proof. Recall from Problem 2.11 that the support of a measure μ is the *smallest* closed set $K \subseteq T$ such that $\mu(T \setminus K) = 0$. Without loss of generality, we may assume that $\mu \not\equiv 0$. Given that K is a *proper* closed subset of T. Note that $T \setminus K$ is nonempty open in T, so $\sigma(T \setminus K) > 0$. Our goal is to show that

$$\mu(T \setminus K) > 0. \tag{17.21}$$

Now we observe from the proof of Theorem 17.13 (The F. and M. Riesz Theorem) that $\mathrm{d}\mu = f^*(\mathrm{e}^{it})\,\mathrm{d}\sigma$, i.e.,

$$\mu(E) = \frac{1}{2\pi}\int_E f^*(\mathrm{e}^{it})\,\mathrm{d}t$$

holds *for every* measurable subset E of T, where $f = P[f^*] \in H^1$ and $f^* \in L^1(T)$. Thusit suffices to show that f^* doesn't vanish on $T \setminus K$. Let $g(t) = f^*(\mathrm{e}^{it})$. Assume that $g(T \setminus K) = 0$. Then it means that $\log|g| = \infty$ on $T \setminus K$, but it contradicts Theorem 17.17 (The Canonical Factorization Theorem) that

$$\log|g| = \log|f^*| \in L^1(T).$$

Hence f^* does not vanish on $T \setminus K$ with $\sigma(T \setminus K) > 0$ which implies the result (17.21). This completes the analysis of the problem. ■

Problem 17.15

Rudin Chapter 17 Exercise 15.

Proof. Denote $\mathscr{C}_K$ to be the set of all continuous functions on K.[g] Then the problem is equivalent to show that the set of polynomials $\mathscr{P}_K$ on K is dense in $\mathscr{C}_K$ with respect to the norm

$$\|f\|_\infty = \sup\{|f(z)| \mid z \in K\}.$$

This is well-defined because f is continuous on the compact set K. Assume that $\mathscr{P}_K$ was not dense in $\mathscr{C}_K$. In other words, there exists an $g \in \mathscr{C}_K$ such that $g \notin \overline{\mathscr{P}_K}$. It is clear that $\mathscr{P}_K$

[g] See Definition 3.16, p. 70.

is a linear subspace of the normed linear space $\mathscr{C}_K$. Then Theorem 5.19 implies that there is a bounded linear functional Φ on $\mathscr{C}_K$ such that

$$\Phi(P) = 0$$

for all $P \in \mathscr{P}_K$ and $\Phi(g) \neq 0$. According to Theorem 6.19, there exists a unique regular complex Borel measure ν on K such that

$$\Phi(f) = \int_K f \, \mathrm{d}\nu \tag{17.22}$$

for every $f \in \mathscr{C}_K$. This measure ν must be nonzero because of $\Phi(g) \neq 0$.

Define $\mu(E) = \nu(E \cap K)$ for $E \in \mathfrak{M}_T$. Then it is easily checked that μ is also a complex Borel measure on T. Combining this and the representation (17.22), we get

$$\int_T P \, \mathrm{d}\mu = \int_K P \, \mathrm{d}\nu + \int_{T \setminus K} P \, \mathrm{d}\mu = \Phi(P) = 0$$

for every $P \in \mathscr{P}_K$. In particular, we have

$$\int_T \mathrm{e}^{-int} \, \mathrm{d}\mu = 0$$

for every $n = 1, 2, \ldots$. Since $\mu \not\equiv 0$, Problem 17.14 asserts that the support of μ is exactly all of T so that $\mu(T \setminus K) > 0$ because K is a proper compact (hence closed) subset of T. However, the definition of μ implies that $\mu(T \setminus K) = \nu(\varnothing) = 0$, a contradiction. This completes the analysis of the problem. ■

17.3 Factorization of $f \in H^p$

Problem 17.16

Rudin Chapter 17 Exercise 16.

Proof. Suppose that $0 < p < 1$. Recall from the first paragraph of the proof of Theorem 7.17 that we may assume that f has no zeros in U. Therefore, we deduce from Theorem 17.10 that one can find a zero-free function $h \in H^2$ such that $f = h^{\frac{2}{p}}$. Note that $h = M_h Q_h$ by Theorem 17.17, so we have

$$f = M_h^{\frac{2}{p}} Q_h^{\frac{2}{p}}. \tag{17.23}$$

By the definition of an inner function, we know that

$$M_h^{\frac{2}{p}}(z) = c^{\frac{2}{p}} \exp\Big\{ -\frac{2}{p} \int_T \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \, \mathrm{d}\mu_h(t) \Big\} = c^{\frac{2}{p}} \exp\Big\{ -\int_T \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \, \mathrm{d}\mu_f(t) \Big\}, \tag{17.24}$$

where μ_h is a finite positive Borel measure on T and $\mu_f = \frac{2}{p}\mu_h$. Clearly, μ_f is also a finite positive Borel measure on T, so it follows from Theorem 17.15 that the right-most function is in fact an inner function. Let it be M_f.

Next, according to [62, Eqn. (1), p. 344], we see that

$$\begin{aligned} Q_h^{\frac{2}{p}}(z) &= \exp\Big\{ \frac{1}{2\pi} \int_T \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \cdot \frac{2}{p} \cdot \log |h^*(\mathrm{e}^{it})| \, \mathrm{d}t \Big\} \\ &= \exp\Big\{ \frac{1}{2\pi} \int_T \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \log \big|(h^*)^{\frac{2}{p}}(\mathrm{e}^{it})\big| \, \mathrm{d}t \Big\} \end{aligned}$$

$$= \exp\left\{\frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z}\log|f^*(e^{it})|\,dt\right\}. \tag{17.25}$$

Since $\log|h^*| \in L^1(T)$, we immediately have $\log|f^*| \in L^1(T)$. By Definition 17.14, the function (17.25) is an outer function and we let it be Q_f. Since $Q_h \in H^2$, Theorem 17.16 implies that $|h^*| \in L^2(T)$ and thus $|f^*| \in L^p(T)$. Using Theorem 17.16 again, we conclude that $Q_f \in H^p$. By substituting the expressions (17.24) and (17.25) into the formula (17.23), we obtain

$$f = M_f Q_f.$$

Finally, we note that the inequality

$$\log|h(0)| \leq \frac{1}{2\pi}\int_T \log|h^*(e^{it})|\,dt \tag{17.26}$$

is equivalent to the inequality

$$\log|f(0)| \leq \frac{1}{2\pi}\int_T \log|f^*(e^{it})|\,dt. \tag{17.27}$$

Hence equality holds in (17.27) if and only if equality holds in (17.26) if and only if M_h is constant if and only if M_f is constant too. Consequently, we have completed the proof of the problem. ∎

Problem 17.17

Rudin Chapter 17 Exercise 17.

Proof.

(a) Assume that $\frac{1}{\varphi} \in H^p$ for some $p > 0$. Since $\frac{1}{\varphi}$ has no zero in U and nonconstant, Theorem 17.10 (The Riesz Factorization Theorem) implies that there is a zero-free function $f \in H^2$ such that

$$\frac{1}{\varphi} = f^{\frac{2}{p}}.$$

Therefore, $\frac{1}{\varphi^{\frac{p}{2}}} = f$ is in H^2. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By the Parseval Theorem [62, Eqn. (6), p. 91] and also the proof of Theorem 17.12, we get

$$\sum_{n=0}^{\infty}|a_n|^2 = \lim_{r\to 1}\int_T |f_r|^2\,d\sigma = \lim_{r\to 1}\int_T \frac{1}{|\varphi|^p}\,d\sigma = \int_T \frac{1}{|\varphi^*|^p}\,d\sigma.$$

Since φ is an inner function in U, Definition 17.14 implies that $|\varphi^*| = 1$ a.e. on T and then

$$\sum_{n=0}^{\infty}|a_n|^2 = 1.$$

In particular, we have $|f(0)| = |a_0| \leq 1$ or equivalently, $|\varphi(0)| \geq 1$. By Theorem 17.15, every inner function M satisfies $|M(z)| \leq 1$ in U. Combining this fact and Theorem 10.24 (The Maximum Modulus Theorem), we establish that φ is constant, a contradiction. Consequently, $\frac{1}{\varphi} \notin H^p$ for all $p > 0$.

(b) By the hypotheses, we have

$$\varphi(z) = c \exp\Big\{ -\int_T \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\Big\},$$

where $|c| = 1$, μ is a finite positive Borel measure on T, and $\mu \perp m$. By Theorem 17.15 and Theorem 10.24 (The Maximum Modulus Theorem), we know that $\log|\varphi|$ is always negative, i.e., $0 < |\varphi(z)| < 1$ for every $z \in U$. Since $\varphi \in H(U)$ and $\varphi(z) \neq 0$ for all $z \in U$, it follows from Problem 11.5 that $\log|\varphi|$ is harmonic in U. Recall from [62, Eqn. (2), §11.5, p. 233] that

$$u(z) = -\log|\varphi(z)| = \int_T \mathrm{Re}\Big(\frac{e^{it}+z}{e^{it}-z}\Big)\, d\mu(t) = \int_T P(z, e^{it})\, d\mu(t)$$

which means $u = P[\,d\mu]$. Since $\mu \perp m$, it follows from Problem 11.19 that $u(re^{i\theta}) \to \infty$ a.e. $[\mu]$. Consequently, there exists an $e^{i\theta} \in T$ such that

$$\lim_{r\to 1} \varphi(re^{i\theta}) = 0.$$

We have completed the analysis of the problem. ■

Problem 17.18

Rudin Chapter 17 Exercise 18.

Proof. Suppose that

$$\varphi_\alpha(z) = \frac{\varphi(z)-\alpha}{1-\overline{\alpha}\varphi(z)}$$

for every $z \in U$. It is clear that φ_α has no zero in U because $\alpha \notin \varphi(U)$. If we can show that φ_α is nonconstant and inner, then it follows directly from Problem 17.17(b) that there is at least one $e^{i\theta} \in T$ such that $\varphi_\alpha(re^{i\theta}) \to 0$ as $r \to 1$. Equivalently, it means that

$$\lim_{r\to 1} \varphi(re^{i\theta}) = \alpha. \tag{17.28}$$

To this end, we directly apply the following result from [68, p. 323]:

Lemma 17.2

If ψ and φ are inner functions in U, then $\psi \circ \varphi$ is also inner.

Since $\psi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$ is clearly inner and $\varphi_\alpha = \psi_\alpha \circ \varphi$, we follow from Lemma 17.1 that φ_α is also inner. Now φ_α is nonconstant because φ is also nonconstant. Therefore, we conclude that the result (17.28) holds and we have completed the proof of the problem. ■

Problem 17.19

Rudin Chapter 17 Exercise 19.

Proof. Let $g = \frac{1}{f}$. Since $f, g \in H^1$ and f, g are not identically 0, we deduce from Theorem 17.17 (The Canonical Factorization Theorem) that

$$f = M_f Q_f \quad \text{and} \quad g = M_g Q_g$$

which imply that $1 = (M_f M_g)(Q_f Q_g)$. By Definition 17.14 and Theorem 17.15, finite products of inner functions and outer functions remain inner and outer respectively. Thus we can write

$$1 = MQ \tag{17.29}$$

for some inner and outer functions M and Q, where M has no zero in U.

We claim that this factorization (17.29) is unique up to a constant of modulus 1. Suppose that we have $1 = M_1 Q_1 = M_2 Q_2$. By Theorem 17.15, M_1 and M_2 can be expressed in the form [58, Eqn. (1), p. 342] which gives $|M_1(z)| = |M_2(z)| = 1$ on T. Therefore, we also have $|Q_1(z)| = |Q_2(z)| = 1$ on T. Since

$$\frac{Q_1}{Q_2} = \frac{M_2}{M_1} \quad \text{and} \quad \frac{Q_2}{Q_1} = \frac{M_1}{M_2},$$

both $\frac{Q_1}{Q_2}$ and $\frac{Q_2}{Q_1}$ are inner functions without zero in U. In other words, we have $\frac{Q_1}{Q_2}, \frac{Q_2}{Q_1} \in H(U)$. Since $|\frac{Q_1(z)}{Q_2(z)}| = |\frac{Q_2(z)}{Q_1(z)}| = 1$ on T, it follows from Theorem 10.24 (The Maximum Modulus Theorem) that $|\frac{Q_1(z)}{Q_2(z)}| \leq 1$ and $|\frac{Q_2(z)}{Q_1(z)}| \leq 1$ in U which imply that

$$Q_1(z) = cQ_2(z)$$

in U for some constant c with $|c| = 1$. Since Q is unique up to a constant of modulus 1, M is also unique up to a constant of modulus 1 in the factorization (17.29), as required.

By the definition of g and Theorem 17.17 (The Canonical Factorization Theorem), we know that $Q_g = \frac{1}{Q_f}$. Consequently, this fact and the above claim show immediately that $Q_f Q_g = M_f M_g = 1$. By Theorem 17.15 again, $M_f M_g = 1$ implies that $M_f = 1$ and hence $f = Q_f$, completing the proof of the problem. ■

Problem 17.20

Rudin Chapter 17 Exercise 20.

Proof. Given $\epsilon > 0$. Define $f_\epsilon(z) = f(z) + \epsilon$ in U. Then $\frac{1}{f_\epsilon}$ is bounded in U so that $\frac{1}{f_\epsilon} \in H^1$. Clearly, Problem 17.18 implies that $f_\epsilon = Q_{f_\epsilon}$, i.e.,

$$f_\epsilon(z) = c \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \log\left|(f_\epsilon)^*(\mathrm{e}^{it})\right| \mathrm{d}t\right\} \tag{17.30}$$

for some constant c with $|c| = 1$. By the definition, the functions $\log|(f_\epsilon)^*|$ decrease to $\log|f^*|$ as $\epsilon \to 0$. Next, we know from Theorem 17.17 (The Canonical Factorization Theorem) that $\log|f^*| \in L^1(T)$. Finally, we apply Problem 1.7 to the expression (17.30) to conclude that

$$f(z) = c \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \log|f^*(\mathrm{e}^{it})|\,\mathrm{d}t\right\} = Q_f(z)$$

for all $z \in U$. This completes the proof of the problem. ■

Problem 17.21

Rudin Chapter 17 Exercise 21.

Proof. Suppose first that $f = \frac{g}{h}$, where $g, h \in H^\infty$. There is no loss of generality to assume that $|g(z)| \leq 1$ and $|h(z)| \leq 1$. By the definition of $\log^+$, it is easy to see that

$$\int_{-\pi}^{\pi} \log^+ |f_r| \, dt = \int_{-\pi}^{\pi} \log^+ |f(re^{it})| \, dt \leq \int_{-\pi}^{\pi} \log |h(re^{it})| \, dt \tag{17.31}$$

for $0 \leq r < 1$. Since $h(z) \neq 0$ in U, Theorem 15.18 (Jensen's Formula) gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |h(re^{it})| \, dt = \log |h(0)|. \tag{17.32}$$

Combining the inequality (17.31) and the result (17.32), we conclude that $f \in N$.

Conversely, let $f \in N$ and $f \not\equiv 0$. According to Theorem 17.9, we may assume that f has no zero in U. By Problem 11.5, $u = \log |f|$ is harmonic in U. Thus the mean value property gives

$$\begin{aligned} u(0) &= \frac{1}{2\pi} \int_T u(r) \, dt \\ &= \frac{1}{2\pi} \int_T \log |f_r(e^{it})| \, dt \\ &= \frac{1}{2\pi} \int_T \log^+ |f_r(e^{it})| \, dt - \frac{1}{2\pi} \int_T \log^- |f_r(e^{it})| \, dt \\ &\leq \log \|f\|_0 - \frac{1}{2\pi} \int_T \log^- |f_r(e^{it})| \, dt \end{aligned} \tag{17.33}$$

for all $0 \leq r < 1$. Since $f \in N$ and the left-hand side of the equation (17.33) is *independent of* r, we know immediately that

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_T \log^- |f_r(e^{it})| \, dt < \infty$$

so that

$$\begin{aligned} \sup_{0 \leq r < 1} \|u_r\|_1 &= \sup_{r<1} \big\| \log |f| \big\|_1 \\ &= \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_T \big| \log |f_r(e^{it})| \big| \, dt \\ &= \sup_{0 \leq r < 1} \Big\{ \frac{1}{2\pi} \int_T \log^+ |f_r(e^{it})| \, dt + \frac{1}{2\pi} \int_T \log^- |f_r(e^{it})| \, dt \Big\} \\ &< \infty \end{aligned}$$

By Theorem 11.30(a), there is a unique complex Borel measure μ on T such that

$$u(z) = P[\, d\mu] = \int_T P(z, e^{it}) \, d\mu(e^{it}). \tag{17.34}$$

By the definition, u is real, so μ is in fact real. Using Theorem 6.14 (The Hahn Decomposition Theorem), we can write $\mu = \mu^+ - \mu^-$. Put $u_\pm = P[\, d\mu^\pm]$. Then both u_+ and u_- are nonnegative harmonic functions in U.

Since U is simply connected and f has no zero in U, it follows from Theorem 13.11 that $f = e^g$ for some $g \in H(U)$. We may assume that $g(0) = \log |f(0)|$ so that g is unique. Recall that $u = \log |f|$, so we have $u = \operatorname{Re} g$ and then the expression (17.34) implies that

$$g(z) = \int_T \frac{e^{it} + z}{e^{it} - z} \, d\mu(e^{it}).$$

If we denote

$$g^{\pm}(z) = \int_T \frac{e^{it} + z}{e^{it} - z} \, d\mu^{\pm}(e^{it}),$$

then we have $g = g^+ - g^-$ and $\operatorname{Re} g^{\pm} = u_{\pm} \geq 0$. Suppose that

$$g_1 = e^{-g^-} \quad \text{and} \quad g_2 = e^{-g^+}.$$

Clearly, $g_1, g_2 \in H(U)$ and they have no zero in U. In addition, we have

$$|g_1| = e^{-\operatorname{Re} g^-} = e^{-u_-} \leq 1.$$

Similarly, we have $|g_2| = e^{-u_+} \leq 1$. These mean that $g_1, g_2 \in H^\infty$ and furthermore,

$$\frac{g_1}{g_2} = \frac{\exp(-g^-)}{\exp(-g^+)} = \exp(g^+ - g^-) = e^g = f.$$

This completes the proof of the problem. ■

Problem 17.22

Rudin Chapter 17 Exercise 22.

Proof. Since $\log^+ t \leq |\log t|$ for every $0 < t < \infty$, the hypothesis gives

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta = 0. \tag{17.35}$$

Since $f \in H(U)$, Theorem 17.3 says that $\log^+ |f|$ is subharmonic in U. Since $\log^+ |f|$ is continuous and nonnegative in U, it follows from Theorem 17.5 that

$$0 \leq \int_{-\pi}^{\pi} \log^+ |f(r_1 e^{i\theta})| \, d\theta \leq \int_{-\pi}^{\pi} \log^+ |f(r_2 e^{i\theta})| \, d\theta \tag{17.36}$$

if $0 \leq r_1 < r_2 < 1$. Combining the limit (17.35) and the inequality (17.36), we deduce that

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta = 0$$

for every $0 \leq r < 1$. In other words, we have $\log^+ |f(z)| = 0$ and so $|f(z)| \leq 1$ for every $z \in U$, i.e., $f \in H^\infty$.

Recall from Definition 17.7 that $H^\infty \subseteq N$. The hypothesis yields that $f \equiv 0$, so Theorem 17.9 asserts that

$$f = Bg, \tag{17.37}$$

where B is the Blaschke product formed with the zeros of f and $g \in N$. Clearly, we always have $|g(z)| \leq 1$ in U. Furthermore, we see from the representation (17.37) that

$$0 = \lim_{r\to 1} \int_{-\pi}^{\pi} \big| \log |f(re^{i\theta})| \big| \, d\theta = \lim_{r\to 1} \int_{-\pi}^{\pi} \big| \log |g(re^{i\theta})| \big| \, d\theta = \lim_{r\to 1} \int_{-\pi}^{\pi} \left| \log \frac{1}{|g(re^{i\theta})|} \right| d\theta.$$

Therefore, the previous paragraph implies that $|\frac{1}{g(z)}| \leq 1$ in U so that $|g(z)| = 1$ in U. This means that g is a constant of modulus 1 and then f is a Blaschke product. We have completed the analysis of the problem. ■

Problem 17.23

Rudin Chapter 17 Exercise 23.

Proof. Let $M_1 \neq 0$ and $M_2 \neq 0$ be two closed S-invariant subspaces of H^2. By Theorem 17.21 (Beurling's Theorem), there exist inner functions φ_1 and φ_2 such that

$$M_1 = \varphi_1 H^2 \quad \text{and} \quad M_2 = \varphi_2 H^2.$$

By Theorem 17.15, we have

$$\varphi_1(z) = c_1 B_1(z) \exp\Big\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_1(t)\Big\} \tag{17.38}$$

and

$$\varphi_2(z) = c_2 B_2(z) \exp\Big\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_2(t)\Big\} \tag{17.39}$$

where c_1 and c_2 are constants such that $|c_1| = |c_2| = 1$, B_1 and B_2 are Blaschke products, and μ_1 and μ_2 are finite positive Borel measures on T such that $\mu_1, \mu_2 \perp m$.

We claim that $M_1 \subseteq M_2$ if and only if the quotient $\varphi = \frac{\varphi_1}{\varphi_2}$ is an inner function. This is equivalent to saying that every zero of B_2 is also a zero of B_1 with at least the same multiplicity and $\mu_2(E) \leq \mu_1(E)$ for every Borel subset E of T.

To prove the claim, we first suppose that every zero of B_2 is also a zero of B_1 with at least the same multiplicity and $\mu_2(E) \leq \mu_2(E)$ for every Borel subset E of T. Clearly, $\frac{B_1}{B_2}$ is another Blaschke product B. Next, it follows from the representations (17.38) and (17.39) that

$$\varphi(z) = \frac{\varphi_1(z)}{\varphi_2(z)} = \frac{c_1}{c_2} B(z) \exp\Big\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\big(\mu_1(t) - \mu_2(t)\big)\Big\}.$$

Define the measure μ on T by $\mu(E) = \mu_1(E) - \mu_2(E)$ for every Borel subset E of T. Since $\mu_2(E) \leq \mu_1(E)$, μ is a finite positive measure. Recall that $\mu_1, \mu_2 \perp m$, so Proposition 6.8 gives $\mu \perp m$. By Theorem 17.15, the function φ is inner.

Conversely, let $\varphi_1 H^2 \subseteq \varphi_2 H^2$. Since $1 \in H^2$, it follows that $\varphi_1 \in \varphi_2 H^2$ and then $\varphi_1 = \varphi_2 h$ for some $h \in H^2$. Since $|\varphi_1^*| = |\varphi_2^*| = 1$ a.e. on T, we have $|h^*| = 1$ a.e. on T. Consequently, $h^* \in L^\infty(T)$. By Problem 17.6, we know that $H^2 \subset H^1$. Then we deduce from Problem 17.8 that $h \in H^\infty$. By Definition 17.14, h is an inner function, so we may write

$$h(z) = cB(z) \exp\Big\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\Big\},$$

where c, B and μ satisfy the conditions of Theorem 17.15. Then we have

$$B_1(z) = B_2(z)B(z) \tag{17.40}$$

and

$$\exp\Big\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_1(t)\Big\} = \exp\Big\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\big(\mu(t) + \mu_2(t)\big)\Big\}. \tag{17.41}$$

Now the relation (17.40) implies that every zero of B_2 is also a zero of B_1 with at least the same multiplicity. Next, the expression (17.41) implies that there exists a real number C such that

$$\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_1(t) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\big(\mu(t) + \mu_2(t)\big) + iC$$

for all $z \in U$. Put $z = 0$ and recall that the measures are finite positive, we get $C = 0$. Recall from the formula following [62, Eqn. (8), p. 111] that

$$\frac{e^{it} + z}{e^{it} - z} = 1 + 2\sum_{n=1}^{\infty}(ze^{-it})^n, \tag{17.42}$$

where $z \in U$. Hence we obtain

$$\int_{-\pi}^{\pi}\Big(1 + 2\sum_{n=1}^{\infty} z^n e^{-int}\Big)\, d\mu_1(t) = \int_{-\pi}^{\pi}\Big(1 + 2\sum_{n=1}^{\infty} z^n e^{-int}\Big)\, d\big(\mu(t) + \mu_2(t)\big)$$

$$\int_{-\pi}^{\pi} d\mu_1(t) + 2\sum_{n=1}^{\infty}\Big[\int_{-\pi}^{\pi} e^{-int}\, d\mu_1(t)\Big] z^n = \int_{-\pi}^{\pi} d\big(\mu(t) + \mu_2(t)\big)$$
$$+ 2\sum_{n=1}^{\infty}\Big[\int_{-\pi}^{\pi} e^{-int}\, d\big(\mu(t) + \mu_2(t)\big)\Big] z^n.$$

It is easy to see that the power series converge in U, so they are holomorphic in U by Theorem 10.6. Since the power series representation of any $f \in H(U)$ is unique, we have

$$\int_{-\pi}^{\pi} d\mu_1(t) = \int_{-\pi}^{\pi} d\big(\mu(t) + \mu_2(t)\big) \tag{17.43}$$

and

$$\int_{-\pi}^{\pi} e^{-int}\, d\mu_1(t) = \int_{-\pi}^{\pi} e^{-int}\, d\big(\mu(t) + \mu_2(t)\big) \tag{17.44}$$

for every $n \in \mathbb{N}$. Since all the measures are positive, it yields from taking complex conjugates to the expression (17.44) that it also holds for all nonzero integers n. Combining this with the expression (17.43), we may conclude that

$$\int_{-\pi}^{\pi} e^{int}\, d[\mu(t) + \mu_2(t) - \mu_1(t)] = 0$$

for every $n \in \mathbb{Z}$. By Theorem 17.13 (The F. and M. Riesz Theorem), we have $\mu + \mu_2 - \mu_1 \ll m$. We know that $\mu + \mu_2 - \mu_1 \perp m$, so Proposition 6.8 implies that $\mu + \mu_2 - \mu_1 = 0$ or $\mu_1 = \mu + \mu_2$. This means that

$$\mu_2(E) \leq \mu_1(E)$$

for every Borel subset E of T, as desired. This completes the analysis of the problem. ■

17.4 A Projection of L^p onto H^p

Problem 17.24

Rudin Chapter 17 Exercise 24.

Proof. By the explanation in §17.24, Theorem 17.26 (M. Riesz's Theorem) is equivalent to saying that[h]

$$\sup_{0\leq r<1} \|v_r\|_p < \infty.$$

[h]See also [21, Theorem 4.1, p. 54]. More precisely, it says that if u is harmonic in U and it satisfies $\|u\|_p = \sup_{0\leq r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p\, d\theta < \infty$, then there exists a constant A_p, depending only on p, such that

$$\|v\|_p \leq A_p \|u\|_p,$$

where v is the harmonic conjugate of u.

Now we consider the function

$$f(z) = i\log\frac{1+z}{1-z} = u(z) + iv(z)$$

which maps U conformally onto the vertical strip $\{z \in \mathbb{C} \mid -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}\}$. In other words, u is bounded, but v is not so that M. Riesz's Theorem fails in the case $p = \infty$.

For $p = 1$, recall the facts [62, Eqn. (2) & (3), §11.5, p. 233] that

$$P_r(\theta) = \operatorname{Re}\Big(\frac{1+z}{1-z}\Big) \quad \text{and} \quad \sup_{0\leq r<1} \|P_r(\theta)\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,\mathrm{d}\theta < \infty.$$

Denote $f(z) = \frac{1+z}{1-z}$. If $f \in H^1$, then Theorem 17.11 gives

$$\|f\|_1 = \|f^*\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\frac{1+e^{it}}{1-e^{it}}\Big|\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}t}{|\tan\frac{t}{2}|} = \frac{1}{\pi}\int_0^{\pi}\frac{\mathrm{d}t}{\tan\frac{t}{2}} = \frac{2}{\pi}\Big[\ln\sin\frac{t}{2}\Big]_0^{\pi} = \infty$$

which is impossible. Hence we have $f \notin H^1$ and Theorem 17.26 (M. Riesz's Theorem) also fails in the case. This completes the proof of the problem. ■

Remark 17.1

Although Theorem 17.26 (M. Riesz's Theorem) fails to say that $\sup\limits_{0\leq r<1} \|v_r\|_1 < \infty$, it is true that $\sup\limits_{0\leq r<1} \|v_r\|_p < \infty$ for all $p < 1$. In fact, this is the content of the so-called Kolmogorov's Theorem, see [21, Theorem 4.2, p. 57].

Problem 17.25

Rudin Chapter 17 Exercise 25.

Proof.

(a) Let $z \in U$. Note that

$$\frac{1}{e^{it}-z} = \frac{1}{e^{it}}\sum_{n=0}^{\infty}\Big(\frac{z}{e^{it}}\Big)^n,$$

so we obtain

$$\begin{aligned}
(\psi f)(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}}\sum_{n=0}^{\infty}\Big(\frac{z}{e^{it}}\Big)^n f(e^{it})\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\,\mathrm{d}t\Big]\cdot z^n + \sum_{n=1}^{\infty}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\,\mathrm{d}t\Big]\cdot z^n \\
&= \sum_{n=0}^{\infty}\widehat{f}(n)z^n + \sum_{n=1}^{\infty}\widehat{f}(n)z^n \\
&= \sum_{n=0}^{\infty}2\widehat{f}(n)z^n - \widehat{f}(0). \qquad (17.45)
\end{aligned}$$

Define

$$F(z) = \frac{(\psi f)(z) + \widehat{f}(0)}{2} = \sum_{n=0}^{\infty}\widehat{f}(n)z^n.$$

Now Theorem 17.26 (M. Riesz's Theorem) implies that $F \in H^p$. By Theorem 17.11, $g = F^* \in L^p(T)$. Direct computation gives

$$\begin{aligned}
\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\mathrm{e}^{it})\mathrm{e}^{-int}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F^*(\mathrm{e}^{it})\mathrm{e}^{-int}\,\mathrm{d}t \\
&= \sum_{m=0}^{\infty} \widehat{f}(m)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(m-n)t}\,\mathrm{d}t \\
&= \begin{cases} \widehat{f}(n), & \text{for all } n \geq 0; \\ 0, & \text{for all } n < 0. \end{cases}
\end{aligned}$$

Next, we notice that Theorem 3.5 (Hölder's Inequality) gives

$$|\widehat{f}(n)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{it})\mathrm{e}^{-int}\,\mathrm{d}t\right| \leq \|f\|_1 \leq \|f\|_p \tag{17.46}$$

for every $n \in \mathbb{Z}$. Finally, we obtain from the inequality (17.46), Theorem 17.11 and Theorem 17.26 (M. Riesz's Theorem) that

$$\|g\|_p = \|F^*\|_p = \|F\|_p \leq \frac{|\widehat{f}(0)|}{2} + \frac{\|\psi f\|_p}{2} \leq \frac{\|f\|_p}{2} + A_p\|f\|_p = C_p\|f\|_p,$$

where $C_p = \frac{1}{2} + A_p$ which is a constant depending only on p. In addition, this means that the mapping $\Phi : L^p(T) \to L^p(T)$ defined by

$$\Phi(f) = g$$

is a bounded linear projection.

(b) Let $k > 0$. It follows from the representation (17.45) that we may define

$$\begin{aligned}
F(z) &= \frac{1}{2}\big[(\psi f)(z) + \widehat{f}(0) - 2\widehat{f}(0) - 2\widehat{f}(1)z - \cdots - 2\widehat{f}(k-1)z^{k-1}\big] \\
&= \sum_{n=k}^{\infty} \widehat{f}(n)z^n.
\end{aligned}$$

Similarly, we have $F \in H^p$, $g = F^* \in L^p(T)$ and then

$$\widehat{g}(n) = \sum_{m=k}^{\infty} \widehat{f}(m)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(m-n)t}\,\mathrm{d}t = \begin{cases} \widehat{f}(n), & \text{for all } n \geq k; \\ 0, & \text{for all } n < k. \end{cases} \tag{17.47}$$

Finally, if we combine the inequality (17.46), Theorem 17.11 and Theorem 17.26 (M. Riesz's Theorem), then we get

$$\begin{aligned}
\|g\|_p &= \|F\|_p \\
&\leq A_p\|f\|_p + \frac{|\widehat{f}(0)|}{2} + \sum_{n=0}^{k-1}|\widehat{f}(n)| \\
&\leq A_p\|f\|_p + \frac{\|f\|_p}{2} + \sum_{n=0}^{k-1}\|f\|_p
\end{aligned}$$

$$= C_{p,k} \cdot \|f\|_p,$$

where $C_{p,k} = A_p + k + \frac{1}{2}$, a constant depending only on p and k.

Similarly, if $k < 0$, then we define

$$\begin{aligned} F(z) &= \frac{1}{2}\big[(\psi f)(z) + \widehat{f}(0) + 2\widehat{f}(-1)z^{-1} + 2\widehat{f}(-2)z^{-2} + \cdots + 2\widehat{f}(k)z^k\big] \\ &= \sum_{n=k}^{\infty} \widehat{f}(n)z^n. \end{aligned}$$

Now if we define $g = F^*$, then it is easy to check that g satisfies the condition (17.47) so that

$$\|g\|_p \leq A_p\|f\|_p + \frac{|\widehat{f}(0)|}{2} + \sum_{n=0}^{k} |\widehat{f}(n)| = C_{p,k} \cdot \|f\|_p,$$

where $C_{p,k} = A_p + k + \frac{3}{2}$ in this case.

(c) We have to show that there exists a constant $M > 0$ such that $\|s_n\|_p \leq M$ for $n = 1, 2, \ldots$. Recall that

$$s_n(t) = \sum_{k=-n}^{n} \widehat{f}(k)e^{ikt} = g_1(t) - g_2(t),$$

where

$$g_1(t) = \sum_{k=-n}^{\infty} \widehat{f}(k)e^{int} \quad \text{and} \quad g_2(t) = \sum_{k=n+1}^{\infty} \widehat{f}(k)e^{int}.$$

Using part (b), there exists a constant C_p, depending only on p, such that

$$\|g_1\|_p \leq C_p\|f\|_p \quad \text{and} \quad \|g_2\|_p \leq C_p\|f\|_p$$

so that

$$\|s_n\|_p \leq \|g_1\|_p + \|g_2\|_p \leq 2C_p\|f\|_p \tag{17.48}$$

for all $n \in \mathbb{N}$. Since $f \in L^p(T)$, we may take $M = 2C_p\|f\|_p$ and we are done.

Given $\epsilon > 0$. Recall from Theorems 3.14 and 4.25 (Weierstrass Theorem) that there exists a trigonometric polynomial g such that

$$\|f - g\|_p < \frac{\epsilon}{1 + 2C_p}. \tag{17.49}$$

Clearly, if $n \geq \deg g$, then we have

$$s_n(g) = \sum_{-n}^{n} \widehat{g}(n)e^{int} = g(t) \tag{17.50}$$

for every $t \in T$. By Theorem 3.5 (Minkowski's Inequality), we see that

$$\|s_n(f) - f\|_p \leq \|f - g\|_p + \|g - s_n(g)\|_p + \|s_n(g) - s_n(f)\|_p.$$

For sufficiently large enough n, we get from the inequalities (17.48), (17.49) and the expression (17.50) that

$$\|s_n(f) - f\|_p < \frac{\epsilon}{1 + 2C_p} + \|s_n(f - g)\|_p \leq \frac{\epsilon}{1 + 2C_p} + 2C_p\|f - g\|_p < \epsilon$$

which implies the desired result

$$\lim_{n\to\infty} \|f - s_n(f)\|_p = 0.$$

(d) Suppose that $f \in L^p(T)$ and set $\widehat{f}(n) = 0$ for all $n < 0$. Using the estimate (17.46), we know that

$$\limsup_{n\to\infty} |\widehat{f}(n)|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|f\|_p^{\frac{1}{n}} = 1$$

which implies that $F \in H\big(D(0;R)\big)$ for some $R \geq 1$, where F is the function defined in part (a). In fact, F is the Poisson integral of f because

$$\begin{aligned}
P[f](z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(\mathrm{e}^{it})\,\mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} r^{|n|} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{it})\mathrm{e}^{in(\theta-t)}\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} \widehat{f}(n) r^n \mathrm{e}^{in\theta} \\
&= \sum_{n=0}^{\infty} \widehat{f}(n)(r\mathrm{e}^{i\theta})^n \\
&= F(z).
\end{aligned}$$

Thus it follows from Theorem 11.16 that $\|F_r\|_p \leq \|f\|_p$ for all $0 \leq r < 1$. By Definition 17.7, we conclude that $F \in H^p$.

Conversely, we suppose that $F \in H^p$. By Theorem 17.11, we have $f = F^* \in L^p(T)$. Since $1 < p < \infty$, we have $H^p \subset H^1$ so that F is the Cauchy integral of f, i.e.,

$$\begin{aligned}
F(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\mathrm{e}^{i\theta})}{1 - \mathrm{e}^{-i\theta}z}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\sum_{n=0}^{\infty} \mathrm{e}^{-in\theta} z^n\Big) f(\mathrm{e}^{i\theta})\,\mathrm{d}\theta \\
&= \sum_{n=0}^{\infty} \Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{i\theta})\mathrm{e}^{-in\theta}\,\mathrm{d}\theta\Big] z^n \\
&= \sum_{n=0}^{\infty} \widehat{f}(n) z^n.
\end{aligned}$$

We end the proof of the problem. ■

Remark 17.2

(a) Problem 17.25(a) can be treated as an equivalent form of Theorem 17.26 (M. Riesz Theorem), see the first paragraph of the proof of the theorem in [31, p. 152].

(b) We may regard H^p as a subspace of $L^p(T)$ by Theorem 17.11(d), where $1 \leq p \leq \infty$. In addition, H^p is closed in $L^p(T)$. To see this, let f be a limit point of H^p. Given $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\|f_n - f\|_p < \frac{\epsilon}{2}$, where $f_n \in H^p$. For $m, n \geq N$, the inequality

$$\|f_m - f_n\|_p \leq \|f_m - f\|_p + \|f - f_n\|_p < \epsilon$$

implies that $\{f_n\}$ is Cauchy. Note that H^p is Banach by Remark 17.8(c), so $f \in H^p$ and then H^p is closed in $L^p(T)$.

Problem 17.26

Rudin Chapter 17 Exercise 26.

Proof. Take $p = 2$. Denote $H = \psi h$ which is holomorphic in U. Therefore, we follow from the expression (17.42) that

$$\begin{aligned} H(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(1 + 2\sum_{n=1}^{\infty} z^n \mathrm{e}^{-int}\Big)h(\mathrm{e}^{it})\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\mathrm{e}^{it})\,\mathrm{d}t + 2\sum_{n=1}^{\infty} z^n\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\mathrm{e}^{it})\mathrm{e}^{-int}\,\mathrm{d}t\Big] \\ &= \widehat{h}(0) + \sum_{n=1}^{\infty} 2\widehat{h}(n) z^n \end{aligned}$$

and then

$$H(r\mathrm{e}^{i\theta}) = \widehat{h}(0) + \sum_{n=1}^{\infty} 2\widehat{h}(n)(r\mathrm{e}^{i\theta})^n = \sum_{n=-\infty}^{\infty} \widehat{H}(n)(r\mathrm{e}^{i\theta})^n, \tag{17.51}$$

where

$$\widehat{H}(n) = \begin{cases} 2\widehat{h}(n), & \text{if } n = 1, 2, \ldots; \\ \widehat{h}(0) & \text{if } n = 0; \\ 0, & \text{if } n = -1, -2, \ldots. \end{cases}$$

Using the Parseval Theorem [62, Eqn. (6), p. 91] and the Fourier series (17.51), we see that

$$\begin{aligned} \|H\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(r\mathrm{e}^{i\theta})|^2\,\mathrm{d}\theta \\ &= \sum_{n=-\infty}^{\infty} |\widehat{H}(n)|^2 \\ &= |\widehat{h}(0)|^2 + \sum_{n=1}^{\infty} |2\widehat{h}(n)|^2 \\ &\leq 4\sum_{n=-\infty}^{\infty} |\widehat{h}(n)|^2 + |\widehat{h}(0)|^2 \qquad (17.52) \\ &= 4\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(\mathrm{e}^{it})|^2\,\mathrm{d}t + |\widehat{h}(0)|^2 \\ &= 4\|h\|_2^2 + |\widehat{h}(0)|^2. \end{aligned}$$

When $\widehat{h}(n) = 0$ for $n = 0, -1, -2, \ldots,$[i] the equality (17.52) holds and we get $\|H\|_2 = 2\|h\|_2$. Hence the best possible value of A_2 is 2 and we complete the proof of the problem. ∎

[i]This is equivalent to

$$\int_{-\pi}^{\pi} h(\mathrm{e}^{i\theta})\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = 0$$

for $n = 0, -1, -2, \ldots$.

17.5 Miscellaneous Problems

Problem 17.27

Rudin Chapter 17 Exercise 27.

Proof. Define $g : U \to \mathbb{C}$ by

$$g(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

By Theorem 10.6, we have $f, g \in H(U)$ so that

$$|f'(z)| = \Big|\sum_{n=1}^{\infty} n a_n z^{n-1}\Big| \leq \sum_{n=1}^{\infty} n|a_n| \cdot |z|^{n-1} = g'(|z|)$$

and thus for every $x \in (0,1)$, we have

$$\int_0^x |f'(re^{i\theta})|\,dr \leq \int_0^x g'(r)\,dr = g(x) - g(0) \leq \sum_{n=0}^{\infty} |a_n| < \infty.$$

Since this is true for every $0 < x < 1$, we actually have

$$\int_0^1 |f'(re^{i\theta})|\,dr < \infty,$$

completing the proof of the problem. ■

Problem 17.28

Rudin Chapter 17 Exercise 28.

Proof. Suppose that $\{n_k\}$ is a sequence of positive integers.

- $|f'(z)| > \frac{n_k}{10k}$ **for all** $1 - \frac{1}{n_k} < |z| < 1 - \frac{1}{2n_k}$**.** Let n_1 be a sufficiently large positive integer and
$$n_k > 4k(k-1)\sqrt{e} \cdot \exp(n_{k-1}) \tag{17.53}$$
for $k \geq 2$. Then we have $n_k > k$ which implies $n_k > k > \frac{1}{k} > \frac{1}{n_k}$ and
$$\frac{1}{n_k^{\frac{1}{n_k}}} < \frac{1}{k^{\frac{1}{n_k}}} < n_k^{\frac{1}{n_k}}.$$
Thus the radius of convergence of the power series will be 1 and Theorem 10.6 implies that
$$f'(z) = \sum_{k=1}^{\infty} \frac{n_k z^{n_k - 1}}{k} = \sum_{p=1}^{k-1} \frac{n_p z^{n_p - 1}}{p} + \frac{n_k z^{n_k - 1}}{k} + \sum_{p=k+1}^{\infty} \frac{n_p z^{n_p - 1}}{p}$$
which implies
$$|f'(z)| \geq \frac{n_k |z|^{n_k - 1}}{k} - \Big|\sum_{p=1}^{k-1} \frac{n_p z^{n_p - 1}}{p}\Big| - \Big|\sum_{p=k+1}^{\infty} \frac{n_p z^{n_p - 1}}{p}\Big| \tag{17.54}$$

for all z satisfying $1-\frac{1}{n_k}<|z|<1-\frac{1}{2n_k}$. Since n_k is sufficiently large enough, we have

$$\frac{n_k|z|^{n_k-1}}{k}>\frac{n_k}{k}\Big(1-\frac{1}{n_k}\Big)^{n_k-1}\approx\frac{n_k}{\mathrm{e}k} \tag{17.55}$$

and the property (17.53) shows that

$$\begin{aligned}\Big|\sum_{p=1}^{k-1}\frac{n_p z^{n_p-1}}{p}\Big| &\leq \sum_{p=1}^{k-1} n_p|z|^{n_p-1}\\ &\leq \sum_{p=1}^{k-1}\frac{n_p}{p}\Big(1-\frac{1}{2n_p}\Big)^{n_p-1}\\ &\leq \sum_{p=1}^{k-1}\frac{n_p}{p}\Big[\Big(1-\frac{1}{2n_p}\Big)^{2n_p-2}\Big]^{\frac{1}{2}}\\ &\approx \sum_{p=1}^{k-1}\frac{n_p}{\sqrt{\mathrm{e}}p}\\ &\leq \frac{(k-1)n_{k-1}}{\sqrt{\mathrm{e}}}\\ &< \frac{n_k}{4\mathrm{e}k}. \end{aligned} \tag{17.56}$$

If $p\geq k+1$, then $n_p-1=\underbrace{\exp(\exp\cdots\exp}_{(p-k)\text{ iterations}} n_k)-1$. Therefore, we obtain

$$\begin{aligned}\Big|\sum_{p=k+1}^{\infty}\frac{n_p z^{n_p-1}}{p}\Big| &\leq \sum_{p=k+1}^{\infty}\frac{n_p}{p}\Big[\Big(1-\frac{1}{2n_k}\Big)^{2n_k}\Big]^{\frac{n_p-1}{2n_k}}\\ &\approx \sum_{p=k+1}^{\infty}\frac{n_p}{p\exp(\frac{n_p-1}{2n_k})}. \end{aligned} \tag{17.57}$$

Besides the property (17.53), we require that the sequence $\{n_k\}$ satisfies

$$\exp\Big(\frac{n_p-1}{2n_k}\Big)>2^{p-k}\times\frac{4\mathrm{e}kn_p}{pn_k}$$

for every $p\geq k+1$. Then the estimate (17.57) becomes

$$\Big|\sum_{p=k+1}^{\infty}\frac{n_p z^{n_p-1}}{p}\Big|<\frac{n_k}{4\mathrm{e}k}\sum_{p=k+1}^{\infty}\frac{1}{2^{p-k}}=\frac{n_k}{4\mathrm{e}k}. \tag{17.58}$$

Substituting the estimates (17.55), (17.56) and (17.58) into the inequality (17.54), we get

$$|f'(z)|>\frac{n_k}{\mathrm{e}k}-\frac{n_k}{4\mathrm{e}k}-\frac{n_k}{4\mathrm{e}k}=\frac{n_k}{2\mathrm{e}k}>\frac{n_k}{10k}$$

in $1-\frac{1}{n_k}<|z|<1-\frac{1}{2n_k}$.

- **The divergence of the integral.** It is clear from the first assertion that

$$\int_0^1|f'(r\mathrm{e}^{i\theta})|\,\mathrm{d}r\geq\sum_{k=1}^{\infty}\int_{1-\frac{1}{n_k}}^{1-\frac{1}{2n_k}}|f'(r\mathrm{e}^{i\theta})|\,\mathrm{d}r$$

$$\geq \sum_{k=1}^{\infty} \int_{1-\frac{1}{n_k}}^{1-\frac{1}{2n_k}} \frac{n_k}{10k}\,\mathrm{d}r$$
$$= \sum_{k=1}^{\infty} \frac{n_k}{10k}\Big(\frac{1}{n_k} - \frac{1}{2n_k}\Big)$$
$$= \sum_{k=1}^{\infty} \frac{1}{20k}$$

for every θ. Hence we have

$$\int_0^1 |f'(re^{i\theta})|\,\mathrm{d}r = \infty \tag{17.59}$$

for every θ.

- **The convergence of the limit.** Let $0 < R < 1$. By the definition of f, we have

$$\begin{aligned}\int_0^R f'(re^{i\theta})\,\mathrm{d}r &= \int_0^R \sum_{k=1}^{\infty} \frac{n_k r^{n_k-1}}{k} e^{i(n_k-1)\theta}\,\mathrm{d}r \\ &= \sum_{k=1}^{\infty} \frac{n_k}{k}\Big(\int_0^R r^{n_k-1}\,\mathrm{d}r\Big)e^{i(n_k-1)\theta} \\ &= \sum_{k=1}^{\infty} \frac{R^{n_k}}{k} e^{i(n_k-1)\theta} \\ &= e^{-i\theta} f(Re^{i\theta}). \end{aligned} \tag{17.60}$$

As $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, Theorem 17.12 implies that $f \in H^2$ and then we follow from Theorem 17.11 that $f^*(e^{i\theta})$ exists a.e. on T. Thus we deduce from the expression (17.60) that

$$\lim_{R\to 1} \int_0^R f'(re^{i\theta})\,\mathrm{d}r = e^{-i\theta} f^*(e^{i\theta})$$

exists for almost all θ.

- **The geometrical meaning of the integral (17.59).** If $f \in H(U)$, we denote

$$V(f;\theta) = \int_0^1 |f'(re^{i\theta})|\,\mathrm{d}r$$

which is the **total variation of** f on the radius of U terminating at the point $e^{i\theta}$. Hence the result (17.59) tells us that the length of the curve which is the image of the radius with angle θ under f is always infinite.

We complete the proof of the problem. ■

Problem 17.29

Rudin Chapter 17 Exercise 29.

Proof. Let $g \in L^p(T)$. Suppose that $g(e^{it}) = f^*(e^{it})$ a.e. for some $f \in H^p$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and $\widehat{f^*}(n)$ be the Fourier coefficients of its boundary function $f^*(\mathrm{e}^{it})$, i.e.,

$$\widehat{f^*}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-int} f^*(\mathrm{e}^{it})\,\mathrm{d}t \tag{17.61}$$

for all $n \in \mathbb{Z}$. The Taylor coefficients of f can be expressed in the form

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(r\mathrm{e}^{it})}{r^n \mathrm{e}^{int}}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{-n}\mathrm{e}^{-int} f(r\mathrm{e}^{it})\,\mathrm{d}t \tag{17.62}$$

for every $0 < r < 1$. Combining the coefficients (17.61) and (17.62), we get

$$\big|r^n a_n - \widehat{f^*}(n)\big| = \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-int}\big[f(r\mathrm{e}^{it}) - f^*(\mathrm{e}^{it})\big]\,\mathrm{d}t\Big| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(r\mathrm{e}^{it}) - f^*(\mathrm{e}^{it})\big|\,\mathrm{d}t$$

for every $n \in \mathbb{Z}$. Applying Theorem 17.11, we obtain

$$\big|r^n a_n - \widehat{f^*}(n)\big| \leq \|f_r - f^*\|_1 \to 0$$

as $r \to 1$. Therefore, we have

$$\widehat{f^*}(n) = \begin{cases} a_n, & \text{if } n \geq 0; \\ 0, & \text{if } n < 0. \end{cases}$$

By the hypothesis, we conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-int} g(\mathrm{e}^{it})\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-int} f^*(\mathrm{e}^{it})\,\mathrm{d}t = 0 \tag{17.63}$$

for all negative integers n.

Conversely, we suppose that the formula (17.63) holds for all negative integers n. In other words, $\widehat{g}(n) = 0$ for all negative integers n. Let

$$f(r\mathrm{e}^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) g(\mathrm{e}^{it})\,\mathrm{d}t, \tag{17.64}$$

where $0 < r < 1$. Since the Poisson kernel has the expansion

$$P_r(t) = 1 + \sum_{n=1}^{\infty} r^n(\mathrm{e}^{int} + \mathrm{e}^{-int}),$$

it follows from the formula (17.63) that

$$\begin{aligned} f(r\mathrm{e}^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) g(\mathrm{e}^{it})\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} r^n \mathrm{e}^{in(\theta - t)} g(\mathrm{e}^{it})\,\mathrm{d}t + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-in(\theta - t)} g(\mathrm{e}^{it})\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} (r\mathrm{e}^{i\theta})^n \Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-int} g(\mathrm{e}^{it})\,\mathrm{d}t\Big] \\ &= \sum_{n=0}^{\infty} \widehat{g}(n)(r\mathrm{e}^{i\theta})^n \end{aligned}$$

which means

$$f(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n$$

for every $z \in U$ and then $f \in H(U)$. By the integral (17.64) and [62, Eqn. (3), p. 233], it is easy to see that

$$\begin{aligned}\|f\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(r\mathrm{e}^{i\theta})|\,\mathrm{d}\theta\\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)|g(\mathrm{e}^{it})|\,\mathrm{d}t\Big]\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\,\mathrm{d}\theta\Big]\cdot|g(\mathrm{e}^{it})|\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\mathrm{e}^{it})|\,\mathrm{d}t\end{aligned}$$

so that $f \in H^1$.

By Theorem 17.11, we see that $f^* \in L^1(T)$. Now we consider

$$\Phi(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)f^*(\mathrm{e}^{it})\,\mathrm{d}t = P[f^*](z).$$

Let $z \in U$. For any fixed $0 < \rho < 1$, since $f \in H(U)$, it follows from Theorem 11.4 and the mean value property that

$$f(\rho z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)f(\rho\mathrm{e}^{it})\,\mathrm{d}t.$$

We observe from Theorem 17.11 that

$$\lim_{\rho\to 1}\int_{-\pi}^{\pi}|f(\rho\mathrm{e}^{it}) - f^*(\mathrm{e}^{it})|\,\mathrm{d}t \to 0,$$

so we obtain

$$f(z) = \lim_{\rho\to 1} f(\rho z) = \Phi(z)$$

for every $z \in U$, i.e.,

$$f(r\mathrm{e}^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)f^*(\mathrm{e}^{it})\,\mathrm{d}t. \tag{17.65}$$

Comparing the two integrals (17.64) and (17.65), we conclude immediately that $g(z) = f^*(z)$ a.e. on T. Since $g \in L^p(T)$, we know that $f^* \in L^p(T)$ and it yields from Problem 17.8 that $f \in H^p$, as required. Now we have completed the proof of the problem. ■

CHAPTER 18

Elementary Theory of Banach Algebras

18.1 Examples of Banach Spaces and Spectrums

Problem 18.1

Rudin Chapter 18 Exercise 1.

Proof. Let X be a Banach space and

$$B(X) = \{A : X \to X \,|\, A \text{ is linear and bounded with mentioned conditions}\}. \tag{18.1}$$

Denote $\|\cdot\|_X$ to be the norm of the Banach space X. The hypotheses ensure that an associative and distributive multiplication is well-defined in $B(X)$. For every $\alpha \in \mathbb{C}$, $A_1, A_2 \in B(X)$ and $x \in X$, we see that

$$\alpha(A_1A_2)(x) = \alpha A_1(A_2x) = A_1(\alpha A_2x) = (\alpha A_1)(A_2x),$$

i.e., $\alpha(A_1A_2) = A_1(\alpha A_2) = (\alpha A_1)A_2$. Thus $B(X)$ is a complex algebra.

We check Definition 5.2. By the definition, $\|A\|$ must be nonnegative. By the definition (18.1), there exists a positive constant M such that $\|Ax\|_X \leq M\|x\|_X$ for all $x \in X$. Therefore, we have

$$\|A\| = \sup \frac{\|Ax\|_X}{\|x\|_X} \leq M,$$

i.e., $\|A\|$ is a real number. Next, for all $A_1, A_2 \in B(X)$, we apply the fact that X is a Banach, so we have

$$\begin{aligned}
\|A_1 + A_2\| &= \sup \frac{\|(A_1 + A_2)(x)\|_X}{\|x\|_X} \\
&= \sup \frac{\|A_1x + A_2x\|_X}{\|x\|_X} \\
&\leq \sup \frac{\|A_1x\|_X + \|A_2x\|_X}{\|x\|_X} \\
&= \|A_1\| + \|A_2\|.
\end{aligned}$$

Furthermore, if $\alpha \in \mathbb{C}$ and $A \in B(X)$, then we obtain

$$\|\alpha A\| = \sup \frac{\|(\alpha A)(x)\|_X}{\|x\|_X} = |\alpha| \sup \frac{\|Ax\|_X}{\|x\|_X} = |\alpha| \cdot \|A\|.$$

Let $\|A\| = 0$. This means that $\|Ax\|_X = 0$ for all $x \in X$ so that $Ax = 0$ for all $x \in X$. Consequently, it must be the case $A = 0$ and then $B(X)$ is a normed linear space.

For $A_1, A_2 \in B(X)$, we see that

$$\begin{aligned}\|A_1A_2\| &= \sup \frac{\|(A_1A_2)(x)\|_X}{\|x\|_X} \\ &= \sup \frac{\|A_1(A_2x)\|_X}{\|A_2x\|_X} \times \frac{\|A_2x\|_X}{\|x\|_X} \\ &\leq \sup \frac{\|A_1(A_2x)\|_X}{\|A_2x\|_X} \times \sup \frac{\|A_2x\|_X}{\|x\|_X} \\ &\leq \|A_1\| \cdot \|A_2\|.\end{aligned}$$

Hence $B(X)$ is also a normed complex algebra.

We claim that $B(X)$ is a complete metric space. Fix $x \in X$. Given $\epsilon > 0$. Let $\{A_n\}$ be Cauchy with respect to the norm $\|\cdot\|$. Then it suffices to show that there exists an $A \in B(X)$ such that

$$\|A_n - A\| \to 0$$

as $n \to \infty$. Since there exists an $N \in \mathbb{N}$ such that $\|A_m - A_n\| < 1$ for all $n, m \geq N$. Using the triangle inequality of the norm $\|\cdot\|$, we see that

$$\|A_n\| < 1 + \|A_N\|$$

for all $n \geq N$. Denote $M = \max\{\|A_1\|, \|A_2\|, \ldots, \|A_{N-1}\|, 1 + \|A_N\|\}$. Thus we get

$$\|A_n\| < 1 + \|A_N\|$$

for all $n \in \mathbb{N}$. This means that

$$\|A_nx\|_X \leq M\|x\|_X \tag{18.2}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now, for all $x \in X$, we can establish

$$\|A_mx - A_nx\|_X = \|(A_m - A_n)x\|_X \leq \|A_m - A_n\| \cdot \|x\|_X \to 0$$

as $n, m \to \infty$. In other words, $\{A_nx\}$ is a Cauchy sequence in X. Since X is Banach, the sequence converges to an element in X, namely Ax. Then we can define the operator $A : X \to X$ by this and the linearity of taking limits implies the linearity of A. Besides, we follow from the inequality (18.2) that

$$\|Ax\|_X = \lim_{n\to\infty} \|A_nx\|_X \leq M\|x\|_X$$

for every $x \in X$. Consequently, it means that $A \in B(X)$.

Now it remains to verify that $\|A_m - A\| \to 0$ as $m \to \infty$. Recall that $\{A_n\}$ is Cauchy, so given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|A_m - A_n\| \leq \epsilon$$

whenever $n, m \geq N$. Therefore, for every $x \in X$, we have

$$\|A_mx - A_nx\|_X \leq \|A_m - A_n\| \cdot \|x\|_X < \epsilon \cdot \|x\|_X$$

whenever $n, m \geq N$. If $n \to \infty$, then we see that

$$\|A_mx - Ax\|_X \leq \epsilon \cdot \|x\|_X$$

for all $m \geq N$ and all $x \in X$. By the definition, we conclude that $\|A_m - A\| \leq \epsilon$ for all $m \geq N$. Hence this proves our claim that $B(X)$ is complete and then it is a Banach algebra, completing the proof of the problem. ∎

Problem 18.2

Rudin Chapter 18 Exercise 2.

Proof. Suppose that we have

$$X = \{\mathbf{v} = (z_1, z_2, \ldots, z_n) \,|\, z_1, z_2, \ldots, z_n \in \mathbb{C}\} = \mathbb{C}^n$$

is equipped with the norm $\|\cdot\|_{\mathbb{C}^n}$ and $B(\mathbb{C}^n)$ is the algebra of all bounded linear operators on $\mathbb{C}^n$. By Problem 18.1, $B(\mathbb{C}^n)$ is also a Banach algebra with the norm $\|\cdot\|$ given by

$$\|A\| = \sup \frac{\|A\mathbf{v}\|_{\mathbb{C}^n}}{\|\mathbf{v}\|_{\mathbb{C}^n}}.$$

Our target is to find

$$\sigma(A) = \{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ is not invertible}\}.$$

According to the explanation in [35, pp. 96, 97], every bounded linear operator A can be represented by a matrix with entries in $\mathbb{C}$. We also denote this matrix by A. Therefore, $A - \lambda I$ is not invertible if and only if

$$\det(A - \lambda I) = 0. \tag{18.3}$$

Since $\det(A-\lambda I) = 0$ is an equation in λ of order n, the Fundamental Theorem of Algebra ensures that it has at most n complex roots. Hence $\sigma(A)$ consists of at most n complex numbers and they are exactly the solutions of the equation (18.3), completing the proof of the problem. ∎

Problem 18.3

Rudin Chapter 18 Exercise 3.

Proof. Suppose that C is a positive constant such that $|\varphi(x)| \leq C$ a.e. on $\mathbb{R}$. Define the mapping $M_\varphi : L^2 \to L^2$ by

$$M_\varphi(f) = \varphi \times f.$$

Of course, it is true that $\varphi f \in L^2$, so M_φ is well-defined. Furthermore, the linearity of M_φ is clear. Recall that L^2 is a normed linear space with the norm $\|f\|_2$. Since we have

$$\|\varphi f\|_2 = \int_{\mathbb{R}} |\varphi(x) f(x)|^2 \,\mathrm{d}x \leq C\|f\|_2,$$

so we get

$$\begin{aligned}\|M_\varphi\| &= \sup\big\{\|M_\varphi(f)\| \,|\, f \in L^2 \text{ and } \|f\|_2 = 1\big\}\\ &= \sup\big\{\|\varphi f\|_2 \,|\, f \in L^2 \text{ and } \|f\|_2 = 1\big\}\\ &\leq C.\end{aligned}$$

By Definition 5.3, M_φ is bounded.

For the second assertion, recall from [62, Exercise 19, p. 74] that

$$R_\varphi = \big\{\lambda \in \mathbb{C} \,\big|\, m\big(\{x \,|\, |\varphi(x) - \lambda| < \epsilon\}\big) > 0 \text{ for every } \epsilon > 0\big\}. \tag{18.4}$$

Let I be the identity operator on L^2. We are required to prove that

$$\sigma(M_\varphi) = \{\lambda \in \mathbb{C} \,|\, M_\varphi - \lambda I \text{ is not invertible}\} = R_\varphi. \tag{18.5}$$

On the one hand, let $\lambda \notin R_\varphi$. Then $|\varphi(x)-\lambda| \geq \epsilon$ for some $\epsilon > 0$ a.e. on $\mathbb{R}$, so we have $\frac{1}{\varphi-\lambda} \in L^\infty(\mathbb{R})$ and this implies that the operator $M_{\frac{1}{\varphi-\lambda}}$ is bounded. Furthermore, it is easy to see that

$$M_{\frac{1}{\varphi-\lambda}}\big(M_\varphi(f)-\lambda f\big) = M_{\frac{1}{\varphi-\lambda}}(\varphi f - \lambda f) = f.$$

Thus we have $M_{\frac{1}{\varphi-\lambda}}$ is the inverse of $M_\varphi - \lambda I$ which means $\lambda \notin \sigma(M_\varphi)$.

On the other hand, let $\lambda \in R_\varphi$. For any $n \in \mathbb{N}$, we denote

$$S_n = \{x \,|\, |\varphi(x)-\lambda| < 2^{-n}\}.$$

The definition (18.4) reveals that $m(S_n) > 0$. Suppose that there exists an $N \geq n$ such that $0 < m(S_N) < \infty$. Otherwise, $m(S_n) = \infty$ for all $n \geq 1$ and this means that $\varphi(x) = \lambda$ for almost all $x \in \mathbb{R}$. In this case, we know that $R_\lambda = \{\lambda\}$. Clearly, $M_\lambda f - \lambda f = 0$, so $\lambda \in \sigma(M_\lambda)$. If $\mu \neq \lambda$, then $M_\lambda(f) - \mu f = (\lambda-\mu)f$ so that

$$M_{\frac{1}{\lambda-\mu}}\big(M_\lambda(f) - \mu f\big) = f.$$

Consequently, we obtain $\sigma(M_\lambda) = \{\lambda\}$ and then the equality (18.5) holds. Let $0 < m(S_N) < \infty$. Take $\phi_n = \chi_{S_N}$. Then we have

$$\|(M_\varphi - \lambda I)(\phi_n)\|^2 = \|\varphi\phi_n - \lambda\phi_n\|^2 = \int_{S_N} |\varphi(x)-\lambda|^2 \cdot |\phi_n(x)|^2 \,\mathrm{d}x \leq 2^{-2n}\|\phi_n\|^2$$

so that the operator $(M_\varphi - \lambda I)^{-1}$ is not bounded, i.e., $M_\varphi - \lambda I$ is not invertible. Hence we conclude that $\lambda \in \sigma(M_\varphi)$ and we have established the equality (18.5), completing the proof of the problem. ■

Problem 18.4

Rudin Chapter 18 Exercise 4.

Proof. Recall that $\ell^2 = \Big\{x = \{\xi_0,\xi_1,\xi_2,\ldots\} \,\Big|\, \|x\| = \Big\{\sum_{n=0}^\infty |\xi_n|^2\Big\}^{\frac{1}{2}} < \infty\Big\}$ and $S : \ell^2 \to \ell^2$ is given by

$$Sx = \{0,\xi_0,\xi_1,\ldots\}$$

which is a bounded linear operator on ℓ^2 and $\|S\| = 1$.[a] We want to determine

$$\sigma(S) = \{\lambda \in \mathbb{C} \,|\, S - \lambda I \text{ is not invertible}\}.$$

Since $\|S\| = 1$, Corollary 3 to Theorem 18.4 implies that

$$\sigma(S) \subseteq \overline{U}. \tag{18.6}$$

Take $0 < |\lambda| < 1$. Assume that $\lambda \notin \sigma(S)$. Then $S - \lambda I$ is invertible so that the equation

$$(S - \lambda I)x = y \tag{18.7}$$

has a unique solution $x \in \ell^2$ for every $y \in \ell^2$. If $y = (1,0,0,\ldots)$, then the equation (18.7) takes the system

$$-\lambda\xi_0 = 1 \quad \text{and} \quad \xi_n - \lambda\xi_{n+1} = 0$$

[a] In fact, S is called the **right-shift operator**.

for every $n = 0, 1, 2, \ldots$. Solving it, we obtain $\xi_n = -\lambda^{-(n+1)}$ for every $n \in \mathbb{N}$. Since $|\lambda| < 1$, we have $|\xi_n| = |\lambda|^{-(n+1)} >$ and thus $x \notin \ell^2$ which is a contradiction. Hence we must have $\lambda \in \sigma(S)$, i.e.,

$$\{\lambda \in \mathbb{C} \,|\, 0 < |\lambda| < 1\} \subseteq \sigma(S). \tag{18.8}$$

Finally, we observe from Theorem 18.6 that $\sigma(S)$ is a closed set, so we conclude from the set relations (18.6) and (18.8) that

$$\sigma(S) = \overline{U}.$$

We end the analysis of the problem. ■

18.2 Properties of Ideals and Homomorphisms

Problem 18.5

Rudin Chapter 18 Exercise 5.

Proof. Let M be an ideal of the commutative complex algebra A. Then M is a vector space and we note from §4.7 that $\overline{M}$ is also a vector space. Thus it remains to show that $a\overline{M} \subseteq \overline{M}$ and $\overline{M}a \subseteq \overline{M}$ for all $a \in A$. Fix $a \in A$ and consider $b \in \overline{M}$. Then there exists a sequence $\{b_n\} \subseteq M$ such that $b_n \to b$ as $n \to \infty$. By considering the sequences $\{ab_n\}$ and $\{b_n a\}$, since M is an ideal, it is true that

$$ab_n \in M \quad \text{and} \quad b_n a \in M$$

for all $n = 1, 2, \ldots$. Since a is fixed, the mapping $x \mapsto ax$ and $x \mapsto xa$ are both continuous on M. Therefore, $ab_n \to ab$ and $b_n a \to ba$ as $n \to \infty$. In other words, it is true that $ab, ba \in M$. This completes the proof of the problem. ■

Problem 18.6

Rudin Chapter 18 Exercise 6.

Proof. Let $C(X)$ be the algebra of all continuous complex functions on X with pointwise addition multiplication and the supremum norm. The constant function 1 is the unit element. Let I be an ideal in $C(X)$. We claim that either $I = C(X)$ or there exists a $p \in X$ such that

$$I = \{f \in C(X) \,|\, f(p) = 0\}. \tag{18.9}$$

Assume that, for every $p \in X$, there exists a continuous function $f \in I$ such that $f(p) \neq 0$. Since X is compact Hausdorff, the continuity of f implies that there exists an open set V_p containing p such that $f(x) \neq 0$ for all $x \in V_p$. Therefore, the collection $\{V_p\}$ forms an open covering of X. Since X is compact, there must exist a finite subcover. Call this subcover $V_1, V_2, \ldots, V_N$ and the corresponding functions $f_1, f_2, \ldots, f_N$ for some $N \in \mathbb{N}$. Define

$$F(x) = f_1^2(x) + f_2^2(x) + \cdots + f_N^2(x). \tag{18.10}$$

Since $f_k \in I$ and I is an ideal, it follows from Definition 18.12 that $F \in I$. For every $p \in X$, we have $f_k(p) \neq 0$ for some $k \in \{1, 2, \ldots, N\}$ so that $F(p) \neq 0$. Since F is continuous on the compact set X, it must attain a minimum. By the form (18.10), it is trivial that $F(x) > 0$ for all $x \in X$. This implies that $F^{-1}(x) = \frac{1}{F(x)}$ is the inverse of F in $C(X)$. However, we note from Definition 18.12 that *no* proper ideal contains an invertible element, so we have $I = C(X)$. Consequently, we have obtained our claim.

If I is maximal, then it has the form (18.9) for some $p \in X$. Assume that $I \subset J$ for some ideal J in $C(X)$, where $I \neq J$. Then there corresponds an $f \in J$ such that $f(p) \neq 0$. Since f is continuous, one can find a neighborhood V_p of p such that $f(x) \neq 0$ for all $x \in V_p$. The point set $\{p\}$ is compact by Theorem 2.4. According to Urysohn's Lemma, there exists an $g \in C(X)$ such that

$$\{p\} \prec g \prec V_p,$$

i.e., $g(p) = 1$ and $g(x) = 0$ for all $x \in X \setminus V_p$. Take $h = 1 - g$ which is also an element of $C(X)$ and it satisfies $h(p) = 0$ and $h(x) = 1$ for all $x \in X \setminus V_p$. As J has the unit, we have $h \in J$. Next, we define

$$H(x) = f^2(x) + h^2(x)$$

which is an element of J. Obviously, it is easy to check that $H(x) > 0$ on X. Since X is compact, H attains its minimum in X and thus H is bounded from below by a positive number. Therefore, its inverse $\frac{1}{H}$ belongs to $C(X)$ which asserts that $J = C(X)$ by the previous paragraph. Hence we have the expected conclusion that the ideal in the form (18.9) is maximal. This completes the analysis of the problem. ■

Problem 18.7

Rudin Chapter 18 Exercise 7.

Proof. Let e be the unit of A. Given $\lambda \notin \sigma(x)$. Then $x - \lambda e$ is invertible so that $(x - \lambda e)^{-1} \in A$. Since A is generated by a single element x, this means that there are polynomials P_n such that

$$P_n(x) \to (x - \lambda e)^{-1} \tag{18.11}$$

as $n \to \infty$ in A. If $z \in \sigma(x)$, then Theorem 18.17(b) ensures that $h(x) = z$ for some $h \in \Delta$. Since h is a complex homomorphism of A, we have $h(x^m) = z^m$ for every $m \in \mathbb{Z}$. By this and Theorem 18.17(e), we establish that

$$|P_n(z) - (\lambda - z)^{-1}| = \left|h\big(P_n(x) - (\lambda - x)^{-1}\big)\right| \leq \|P_n(x) - (\lambda - x)^{-1}\|. \tag{18.12}$$

Applying the result (18.11) to the inequality (18.12), we obtain the result that $P_n(z) \to (\lambda - z)^{-1}$ uniformly on $\sigma(x)$.

Assume that $\mathbb{C} \setminus \sigma(x)$ was disconnected. Let Ω be a (non-empty) bounded component of it. Fix $\lambda \in \Omega$, i.e., $\lambda \notin \sigma(x)$. Choose $\{P_n\}$ as above. For every $n \in \mathbb{N}$ and $z \in \sigma(x)$, we have

$$|(z - \lambda)P_n(z) - 1| = |z - \lambda| \cdot \left|P_n(z) - (z - \lambda)^{-1}\right| \leq \ell \cdot L_n, \tag{18.13}$$

where

$$\ell = \sup_{z \in \sigma(x)} |z - \lambda| \quad \text{and} \quad L_n = \sup_{z \in \sigma(x)} |P_n(z) - (z - \lambda)^{-1}|.$$

The compactness of $\sigma(x)$ by Theorem 18.6 asserts that both ℓ and L_n are finite. Furthermore, $\partial\Omega \subseteq \sigma(x)$. Next, we use Theorem 10.24 (The Maximum Modulus Theorem) to see that the inequality (18.13) also holds on Ω. In particular, we get

$$|P_n(z) - (z - \lambda)^{-1}| \leq \frac{\ell \cdot L_n}{|z - \lambda|} \tag{18.14}$$

for all $z \in \Omega \setminus \{\lambda\}$. Since Ω is a component, one can find a $\delta > 0$ small enough such that the circle $C(\lambda; \delta)$ lies in Ω. Therefore, we conclude from the estimate (18.14) that

$$2\pi i = \left|\int_{C(\lambda;\delta)} \left[P_n(z) - (z - \lambda)^{-1}\right] \mathrm{d}z\right| \leq \ell \cdot L_n \int_{C(\lambda;\delta)} \frac{\mathrm{d}z}{|z - \lambda|} = \frac{\ell L_n}{\delta} \cdot 2\pi\delta = 2\pi \ell L_n. \tag{18.15}$$

Notice that $L_n \to 0$ as $n \to \infty$, so the inequality (18.15) implies a contradiction. Hence the set $\mathbb{C} \setminus \sigma(x)$ must be connected, as required. This completes the proof of the problem. ■

Problem 18.8

Rudin Chapter 18 Exercise 8.

Proof. Since $\sum_{n=0}^{\infty} |c_n| < \infty$, there exists a positive constant M such that $|c_n| \leq M$. This implies that

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \leq 1,$$

i.e., the radius of convergence R of the power series satisfies $R \geq 1$. By Theorem 10.6, both f and then $\frac{1}{f}$ are holomorphic in a region containing $\overline{U}$.

Define $C_n = c_n$ for all $n \geq 0$ and $C_n = 0$ for all $n < \infty$. It is clear that

$$f(e^{it}) = \sum_{n=0}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} C_n e^{int} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |C_n| = \sum_{n=0}^{\infty} |c_n| < \infty.$$

Now the hypothesis $|f(z)| > 0$ for every $z \in \overline{U}$ implies that $f(e^{it}) \neq 0$ for every real t, so Theorem 18.21 (Wiener's Theorem) guarantees that f satisfies

$$\frac{1}{f(e^{it})} = \sum_{n=-\infty}^{\infty} \gamma_n e^{int} \quad \text{and} \quad \sum_{n=\infty}^{\infty} |\gamma_n| < \infty. \tag{18.16}$$

Since $\frac{1}{f}$ is holomorphic in a region containing $\overline{U}$, we have $\gamma_n = 0$ for all $n < 0$ and

$$\frac{1}{f(e^{it})} = \sum_{n=0}^{\infty} a_n e^{int} = \sum_{n=0}^{\infty} \gamma_n e^{int}$$

which implies immediately that $a_n = \gamma_n$ for all $n \geq 0$ by the Corollary following Theorem 10.18. Hence the second condition (18.16) gives the desired result that

$$\sum_{n=0}^{\infty} |a_n| < \infty.$$

This completes the proof of the problem. ■

Problem 18.9

Rudin Chapter 18 Exercise 9.

Proof. We note from Example 9.19(d) that we define the multiplication in $L^1(\mathbb{R})$ by convolution. Let $f, g \in L^1(\mathbb{R})$ and $\phi \in L^\infty(\mathbb{R})$. Here we employ the proof of [60, pp. 157, 158]: If I is a translation invariant subspace of $L^1(\mathbb{R})$, then we say that ϕ **annihilates** I if

$$(f * \phi)(x) = \int_{\mathbb{R}} f(x-y)\phi(y)\,\mathrm{d}m(y) = 0$$

for all $f \in I$ and $x \in \mathbb{R}$. On the one hand, we note that

$$(f * g * \phi)(0) = [(f * g) * \phi](0) = \int_{\mathbb{R}} (f * g)(0-y)\phi(y)\,\mathrm{d}m(y). \tag{18.17}$$

On the other hand, recall from [62, Example 9.19(d)] that $f * g = g * f$, so we have

$$(f * g * \phi)(0) = [g * (f * \phi)](0) = \int_{\mathbb{R}} g(0 - y)(f * \phi)(y)\,\mathrm{d}m(y). \tag{18.18}$$

In other words, the two integrals (18.17) and (18.18) are equal, i.e.,

$$\int_{\mathbb{R}} (f * g)(0 - y)\phi(y)\,\mathrm{d}m(y) = \int_{\mathbb{R}} g(0 - y)(f * \phi)(y)\,\mathrm{d}m(y). \tag{18.19}$$

Let I be a closed translation invariant subspace of the Banach space $L^1(\mathbb{R})$ and $\phi \in L^1(\mathbb{R})$ annihilate $f \in I$. Then $f * \phi = 0$ and we see from the right-hand side of the expression (18.19) that

$$(f * g) * \phi = 0 \tag{18.20}$$

for every $g \in L^1(\mathbb{R})$.[b] Recall the basic fact from Theorem 6.16 that $L^1(\mathbb{R})$ is *isometrically isomorphic* to the dual space of $L^\infty(\mathbb{R})$, i.e., $\big(L^1(\mathbb{R})\big)^* \cong L^\infty(\mathbb{R})$. By Remark 5.21, every $\phi \in L^\infty(\mathbb{R})$ is a bounded linear functional on $L^1(\mathbb{R})$. Assume that $f * g \notin I$. Since $\overline{I} = I$, there corresponds an $\varphi \in L^\infty(\mathbb{R})$ such that $f * \varphi = 0$ and $(f * g) * \varphi \neq 0$ by Theorem 5.19, but this contradicts the result (18.20). Hence we conclude that $f * g \in I$ so that I is an ideal.

Conversely, let I be a closed ideal and $f * \phi = 0$ for all $f \in I$. Assume that $F = f_{x_0} \notin I = \overline{I}$ for some $x_0 \in \mathbb{R}$.[c] By Theorem 5.19, there exists an $\varphi \in L^\infty(\mathbb{R})$ such that $f * \varphi = 0$ for all $f \in I$ but

$$F * \varphi \neq 0. \tag{18.21}$$

Now the hypotheses show that we have $f * g \in I$ for every $g \in L^1(\mathbb{R})$, so the left-hand side of the expression (18.19) gives $f * \phi$ annihilates every $g \in L^1(\mathbb{R})$. This implies that $f * \phi = 0$ or

$$0 = (f * \phi)(x) = \int_{\mathbb{R}} f(x - y)\phi(y)\,\mathrm{d}m(y)$$

for $x \in \mathbb{R}$. In other words, ϕ annihilates *every* translate of f. Particularly, this implies that $F * \varphi = 0$ which contradicts the result (18.21). Consequently, $f_x \in I$ for every $x \in \mathbb{R}$ which means it is translation invariant. Hence we have completed the analysis of the problem. ■

Remark 18.1

As [60, Theorem 7.1.2, p. 157] indicates, the result of Problem 18.9 is also valid if we replace $\mathbb{R}$ by any **locally compact abelian** group. In particular, Problem 18.9 remains true for the unit circle T.

Problem 18.10

Rudin Chapter 18 Exercise 10.

Proof. We prove the assertions one by one.

- $L^1(T)$ **is a commutative Banach algebra.** Suppose that $f, g, h \in L^1(T)$. It is clear that $f * (g + h) = f * g + f * h$, $(f + g) * h = f * h + g * h$ and $\alpha(f * g) = f * (\alpha g) = (\alpha f) * g$ for every $\alpha \in \mathbb{C}$. It also satisfies the associative law by an application of Theorem 8.8 (The Fubini Theorem)[d] Thus $L^1(T)$ is a complex algebra.

[b] By Theorem 8.14, we know that $f * g \in L^1(\mathbb{R})$.
[c] Recall that $f_{x_0}(y) = f(y - x_0)$, see Theorem 9.5.
[d] See also [62, Example 9.19(d), p. 190].

Recall from [62, p. 96] that $L^1(T)$ is a Banach space normed by $\|f\|_1$. Using Theorem 8.8 (The Fubini Theorem) again, we know that

$$\begin{aligned}
\|f*g\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|(f*g)(t)|\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\Big|\int_{-\pi}^{\pi}f(t-s)g(s)\,\mathrm{d}s\Big|\,\mathrm{d}t \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)|\cdot|g(s)|\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)|\,\mathrm{d}t\Big)\cdot|g(s)|\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\|f\|_1\cdot|g(s)|\,\mathrm{d}s \\
&= \|f\|_1\cdot\|g\|_1.
\end{aligned}$$

In other words, $L^1(T)$ is a Banach algebra by Definition 18.1. The definition implies immediately that $f*g=g*f$. Hence $L^1(T)$ is commutative.

- $L^1(T)$ **does not have a unit.** Assume that $e\in L^1(T)$ was a unit. Then $e*f=f$ for every $f\in L^1(T)$. Using similar argument as in the proof of Theorem 9.2(c) (The Convolution Theorem), we can show that
$$\widehat{h}(n)=\widehat{f}(n)\cdot\widehat{g}(n) \tag{18.22}$$
if $f,g\in L^1(T)$. Therefore, we have
$$\widehat{e}(n)=1$$
for every $n\in\mathbb{Z}$, but it contradicts the Riemann-Lebesgue Lemma [62, §5.14, p. 103].

- **Complex homomorphisms of $L^1(T)$.** Denote Δ_T to be the set of all complex homomorphisms of $L^1(T)$. Let $\varphi\in\Delta_T$ and $\varphi\neq 0$. By Theorem 18.17(e), φ is bounded by 1, so it follows from Theorem 6.16 that there is a *unique* $\beta\in L^\infty(T)$ such that
$$\varphi(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\beta(t)\,\mathrm{d}t. \tag{18.23}$$
On the one hand, we have
$$\varphi(f*g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(t)\beta(t)\,\mathrm{d}t=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(t-s)g(s)\beta(t)\,\mathrm{d}s\,\mathrm{d}t.$$
On the other hand, we obtain
$$\begin{aligned}
\varphi(f)\varphi(g) &= \Big[\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\beta(t)\,\mathrm{d}t\Big]\times\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi}g(s)\beta(s)\,\mathrm{d}s\Big] \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(t-s)g(s)\beta(t-s)\beta(s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$
The fact $\varphi(f*g)=\varphi(f)\varphi(g)$ asserts that $\beta(t)=\beta(t-s)\beta(s)$ a.e. on T or equivalently,
$$\beta(x+y)=\beta(x)\beta(y)$$
a.e. on T. Employing similar analysis as in the proof of Theorem 9.23, since β is periodic with period 1, it has the form
$$\beta(x)=\mathrm{e}^{-i\alpha x}$$

for a unique $\alpha \in \mathbb{R}$.[e] Since we must have $\beta(x + 2\pi) = \beta(x)$, α must be an integer. Put $\alpha = n$. Substituting this back into the integral (18.23), we get

$$\varphi(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt = \widehat{f}(n)$$

for a unique integer n.

- **I_E is a closed ideal in $L^1(T)$.** Let $E \subseteq \mathbb{Z}$ and

$$I_E = \big\{ f \in L^1(T) \,\big|\, \widehat{f}(n) = 0 \text{ for all } n \in E \big\}. \tag{18.24}$$

For any $f \in I_E$ and $g \in L^1(T)$, if we write $h = f * g$, then the formula (18.22) implies that

$$\widehat{h}(n) = \widehat{f}(n) \cdot \widehat{g}(n) = 0$$

for every $n \in E$. Thus we have $f * g \in I_E$ and I_E is an ideal in $L^1(T)$. For every $\alpha \in \mathbb{R}$, we follow from the definition that

$$\widehat{f}(n - \alpha) = \widehat{f}(n) e^{-i\alpha n} = 0$$

for every $n \in E$, so $f(x - \alpha) \in I_E$. This means that I_E contains every translate of f and Remark 18.1 ensures that I_E is closed.

- **Every closed ideal I in $L^1(T)$ has the form (18.24).** Let I be a closed ideal of $L^1(T)$. We have to prove that $I = I_E$ *for some* set $E \subseteq \mathbb{Z}$. Suppose that for each $n \in \mathbb{Z}$, there exists an $f_n \in I$ such that $\widehat{f_n}(n) \neq 0$. Put $g_n(t) = e^{int} \big[\widehat{f_n}(n)\big]^{-1} \in L^1(T)$. Then we have

$$(g_n * f_n)(t) = \frac{1}{\widehat{f_n}(n)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(s) e^{in(t-s)} \, ds = \frac{e^{int}}{\widehat{f_n}(n)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(s) e^{-ins} \, ds = e^{int}.$$

Since I is an ideal, we have $e^{int} \in I$ for every $n \in \mathbb{Z}$. Using Theorems 3.14, 4.25 and the fact that I is closed, we know that the set $\{e^{int} \,|\, n \in \mathbb{Z}\}$ is dense in $L^1(T)$. Thus we conclude that $I = L^1(T)$.

Without loss of generality, we may assume that $I \neq L^1(T)$. Then there exists an $n \in \mathbb{Z}$ such that $\widehat{f}(n) = 0$ for all $f \in I$. Let the collection of such integers be E, i.e.,

$$E = \{ n \in \mathbb{Z} \,|\, \widehat{f}(n) = 0 \text{ for all } f \in I \}. \tag{18.25}$$

Thus it is easy to see that $I \subseteq I_E$. If $n \notin E$, then there exists an $g \in I$ such that $\widehat{g}(n) \neq 0$. For simplicity, we may assume that this number is 1. Regarding e^{int} as an element of $L^1(T)$, we see that $(g * e^{in})(t) = -e^{int}$. Since I is an ideal, we have $g * e^{in} \in I$ and then $e^{int} \in I$. Thus I contains every trigonometric polynomial of the form

$$\sum a_n e^{int}$$

provided that $a_n = 0$ for all $n \in E$. Suppose that $f \in I_E$ and we consider the set

$$Z(f) = \big\{ n \in \mathbb{Z} \,\big|\, \widehat{f}(n) = 0 \big\}.$$

Now the definitions (18.24) and (18.25) imply that $E \subseteq Z(f)$, so it follows from Theorem 9.2(c) that if P is a trigonometric polynomial on T and $n \in E$, then we have

$$(\widehat{f * P})(n) = \widehat{f}(n) \times \widehat{P}(n) = 0,$$

[e] See also [23, Theorem 8.19, p. 247].

i.e., $E \subset Z(f * g)$. Using [60, Theorem 2.6.6, p. 51], we see that $\|f - f * P\|_1$ can be made as small as we want. Therefore, the closeness of I asserts that $f \in I$ which implies $I_E \subseteq I$. Hence we conclude that

$$I = I_E$$

as desired.

We end the proof of the problem. ■

Remark 18.2

For other classes of complex homomorphisms of specific Banach algebras, please refer to [81, §9, pp. 39 – 43].

Problem 18.11

Rudin Chapter 18 Exercise 11.

Proof. Notice that $\lambda, \mu \in \mathbb{C} \setminus \sigma(x)$. On the one hand, we have

$$\begin{aligned}(x - \lambda e)\big[R(\lambda, x) - R(\mu, x)\big](x - \mu e) &= \big[(x - \lambda e)(\lambda e - x)^{-1} - (x - \lambda e)(\mu e - x)^{-1}\big](x - \mu e) \\ &= -e(x - \mu e) - (x - \lambda e)(\mu e - x)^{-1}(x - \mu e) \\ &= -x + \mu e + x - \lambda e \\ &= (\mu - \lambda)e. \end{aligned} \tag{18.26}$$

On the other hand, we see that

$$\begin{aligned}(x - \lambda e)\big[(\mu - \lambda)R(\lambda, x)R(\mu, x)\big](x - \mu e) &= (\mu - \lambda)\big[(x - \lambda e)R(\lambda, x)R(\mu, x)(x - \mu e)\big] \\ &= (\mu - \lambda)e. \end{aligned} \tag{18.27}$$

It yields from the results (18.26) and (18.27) that

$$R(\lambda, x) - R(\mu, x) = (\mu - \lambda)R(\lambda, x)R(\mu, x) \tag{18.28}$$

holds for all $\lambda, \mu \in \mathbb{C} \setminus \sigma(x)$.

We follow from the identity (18.28) that

$$\frac{(x - \mu e)^{-1} - (x - \lambda e)^{-1}}{\mu - \lambda} = \frac{R(\lambda, x) - R(\mu, x)}{\mu - \lambda} = R(\lambda, x)R(\mu, x) \to R(\lambda, x)^2 = (x - \lambda e)^{-2}$$

as $\mu \to \lambda$. This is exactly [62, Eqn. (3), p. 359], so the argument in the proof of Theorem 18.5 can be applied directly. This completes the proof of the problem. ■

Remark 18.3

The result in Problem 18.11 is called **Hilbert's identity**.

Problem 18.12

Rudin Chapter 18 Exercise 12.

Proof. Denote $\mathcal{M}$ be the set of maximal ideals of A. Let M be a maximal ideal of A. By Theorem 18.17, we have $M = \ker h$ for some $h \in \Delta$. Conversely, if $h \in \Delta$, then it follows from the First Isomorphism Theorem [25, Theorem 16.2, p. 145] that

$$A/\ker h \cong h(A) = \mathbb{C}.$$

Since $\mathbb{C}$ is a field, $\ker h$ is a maximal ideal of A. In other words, there exists an one-to-one correspondence between $\mathcal{M}$ and Δ.

Let $\operatorname{rad} A$ be the radical of A. Suppose that $x \in \operatorname{rad} A$, i.e.,

$$x \in \bigcap_{M \in \mathcal{M}} M. \tag{18.29}$$

Now the previous paragraph yields that the set relation (18.29) is equivalent to the condition

$$x \in \bigcap_{h \in \Delta} h^{-1}(0)$$

which means that $h(x) = 0$ for every $h \in \Delta$. Consequently, statements (a) and (c) are equivalent.

Next, Theorem 18.17(b) means that the spectrum $\sigma(x)$ is exactly the set $\{h(x) \,|\, h \in \Delta\}$. Since $x \in \operatorname{rad} A$ if and only if $h(x) = 0$ for every $h \in \Delta$, this implies that $x \in \operatorname{rad} A$ if and only if $\sigma(x) = \{0\}$ if and only if $\|x^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ by Theorem 18.9 (The Spectral Radius Formula). Hence we have shown the three statements are equivalent, completing the proof of the problem. ■

Problem 18.13

Rudin Chapter 18 Exercise 13.

Proof. Let $X = C([0,1])$. Then X is a Hilbert space (and hence a Banach space). For each $f \in X$, we define

$$T(f)(t) = \int_0^t f(s)\,\mathrm{d}s,$$

where $t \in [0,1]$. Since $f \in X$, $T(f) \in X$ so that $T \in B(X)$, the algebra of all bounded linear operations on X. By Problem 18.1, $B(X)$ is a Banach space.

Particularly, we take $f(x) = 1$. Then we observe

$$T^2(f)(t) = \int_0^t \int_0^s \mathrm{d}u\,\mathrm{d}s = \int_0^t s\,\mathrm{d}s = \frac{t^2}{2}.$$

More generally, for $n = 1, 2, \ldots$, we obtain

$$T^n(f)(t) = \frac{t^n}{n!}$$

which implies that $T^n(f) \neq 0$ for all $n > 0$ and

$$\|T^n\| = \sup \frac{\|T^n(f)\|_\infty}{\|f\|_\infty} = \frac{1}{n!}. \tag{18.30}$$

Since $(n!)^{\frac{1}{n}} \to \infty$ as $n \to \infty$, we conclude immediately from the result (18.30) that

$$\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = 0.$$

This completes the analysis of the problem. ■

Problem 18.14

Rudin Chapter 18 Exercise 14.

Proof. By the definition, the function $\widehat{x} : \Delta \to \mathbb{C}$ is given by $\widehat{x}(h) = h(x)$ and we have the set $\widehat{A} = \{\widehat{x} \,|\, x \in A\}$. Denote the surjective mapping $G : A \to \widehat{A}$ by $G(x) = \widehat{x}$.

- **The mapping G is a homomorphism.** Suppose that $x, y \in A$, $\alpha \in \mathbb{C}$ and $h \in \Delta$. Then we see that

$$\widehat{(\alpha x)}(h) = h(\alpha x) = \alpha h(x) = (\alpha\widehat{x})(h)$$
$$\widehat{(x+y)}(h) = h(x+y) = h(x) + h(y) = \widehat{x}(h) + \widehat{y}(h) = (\widehat{x} + \widehat{y})(h)$$

  and

$$\widehat{xy}(h) = h(xy) = h(x)h(y) = \widehat{x}(h)\widehat{y}(h) = (\widehat{x}\widehat{y})(h).$$

  Therefore, the map G is a homomorphism. Its kernel consists of those $x \in A$ such that $h(x) = 0$ for every $h \in \Delta$. By Problem 18.12, it is exactly the radical of A, i.e., $\ker \widehat{x} = \operatorname{rad} A$.

  Combining the First Isomorphism Theorem and the previous result, we see that

$$A/\operatorname{rad} A = A/\ker \widehat{x} \cong \widehat{A}.$$

  Thus if $\operatorname{rad} A = \{0\}$,[f] then we get $A \cong \widehat{A}$ and G becomes an isomorphism.

- $\rho(x) = \|\widehat{x}\|_\infty = \sup\{|\widehat{x}(h)| \,|\, h \in \Delta\}$**.** By Theorem 18.17(e), we have $\rho(x) \geq |h(x)| = |\widehat{x}(h)|$ for every $h \in \Delta$ which means that

$$\rho(x) \geq \|\widehat{x}\|_\infty.$$

  For the other direction, λ belongs to the range of $\widehat{x}$ means that $\lambda = \widehat{x}(h) = h(x)$ for some $h \in \Delta$ and it follows from Theorem 18.17(b) that this happens if and only if $\lambda \in \sigma(x)$, so Definition 18.8 establishes

$$\rho(x) = \sup\{|\lambda| \,|\, \lambda \in \sigma(x)\} \leq \sup\{|\widehat{x}(h)| \,|\, h \in \Delta\} = \|x\|_\infty.$$

- **The range of $\widehat{x}$ is $\sigma(x)$.** The analysis in the previous part also implies that the range of the function $\widehat{x}$ is exactly the spectrum $\sigma(x)$.

We have completed the proof of the problem. ■

Problem 18.15

Rudin Chapter 18 Exercise 15.

Proof. Let $A_1 = \{(x, \lambda) \,|\, x \in A \text{ and } \lambda \in \mathbb{C}\}$ and $\|(x, \lambda)\| = \|x\| + |\lambda|$. For any $(x, \lambda), (y, \mu) \in A_1$, we define the multiplication in A_1 by

$$(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu). \tag{18.31}$$

[f] In this case, A is called **semisimple**.

- A_1 **is a commutative Banach algebra with unit.** It is easily checked that this is associative and distributive. Thus A_1 is a complex algebra. Furthermore, the element $(0, 1)$ is a unit for this multiplication because

$$(x, \lambda)(0, 1) = (x \cdot 0 + 1 \cdot x + \lambda \cdot 0, \lambda \cdot 1) = (x, \lambda) = (0, 1)(x, \lambda).$$

For any $(x, \lambda), (y, \mu) \in A_1$, we see that

$$\begin{aligned}\|(x, \lambda) + (y, \mu)\| &= \|(x + y, \lambda + \mu)\| \\ &= \|x + y\| + |\lambda + \mu| \\ &\leq \|x\| + \|y\| + |\lambda| + |\mu| \\ &= \|(x, \lambda)\| + \|(y, \mu)\|.\end{aligned}$$

If $\alpha \in \mathbb{C}$, then we have

$$\|\alpha(x, \lambda)\| = \|(\alpha x, \alpha\lambda)\| = \|\alpha x\| + |\alpha\lambda| = |\alpha| \cdot \|x\| + |\alpha| \cdot |\lambda| = |\alpha| \cdot \|(x, \lambda)\|.$$

As $\|(x, \lambda)\| = 0$ if and only if $\|x\| + |\lambda| = 0$ if and only if $\|x\| = 0$ if and only if $x = 0$ and $\lambda = 0$, A_1 is a normed linear space by Definition 5.2. Since $\|xy\| \leq \|x\| \cdot \|y\|$, we see that

$$\begin{aligned}\|(x, \lambda)(y, \mu)\| &= \|(xy + \mu x + \lambda y, \lambda\mu)\| \\ &= \|xy + \mu x + \lambda y\| + |\lambda\mu| \\ &\leq \|xy\| + \|\mu x\| + \|\lambda y\| + |\lambda| \cdot |\mu| \\ &\leq \|x\|\|y\| + |\mu| \cdot \|x\| + |\lambda| \cdot \|y\| + |\lambda| \cdot |\mu| \\ &= \big(\|x\| + |\lambda|\big) \cdot \big(\|y\| + |\mu|\big) \\ &= \|(x, \lambda)\| \cdot \|(y, \mu)\|.\end{aligned}$$

Since the spaces A and $\mathbb{C}$ are complete, A_1 is obviously complete and then it is a Banach algebra with unit by Definition 18.1. It is commutative because A and $\mathbb{C}$ are commutative so that $(y, \mu)(x, \lambda)$ equals to the right-hand side of the expression (18.31).

- **The mapping $x \mapsto (x, 0)$ is an isometric isomorphism of A onto a maximal ideal of A_1.** Let Φ be this mapping. It is trivial surjective. If $\Phi(x) = \Phi(y)$, then $(x, 0) = (y, 0)$ which means that $(x - y, 0) = 0$. Since $\|(x - y, 0)\| = 0$, we get $x = y$ and thus Φ is injective. Is is easily checked that Φ satisfies

$$\Phi(x + y) = \Phi(x) + \Phi(y),$$

so Φ is an isomorphism onto $\Phi(A)$. It is also isometric because

$$\|\Phi(x) - \Phi(y)\| = \|\Phi(x - y)\| = \|(x - y, 0)\| = \|x - y\|.$$

Now we may identify A with $\Phi(A) \subseteq A_1$. Since $(x, \lambda)(y, 0) = (xy + \lambda y, 0) \in \Phi(A)$ for every $(x, \lambda) \in A_1$ and $(y, 0) \in \Phi(A)$, $\Phi(A)$ is an ideal of A_1 by Definition 18.12. Since $A_1 = A \oplus \mathbb{C}$, we have $A_1/\Phi(A) \cong A_1/A \cong \mathbb{C}$ which implies that $\Phi(A)$ is a maximal ideal of A_1 as required.

This completes the proof of the problem. ■

18.3 The Commutative Banach algebra H^∞

Problem 18.16

Rudin Chapter 18 Exercise 16.

Proof. It is clear that H^∞ is a commutative complex algebra. Recall from §11.31 that its norm is defined by

$$\|f\|_\infty = \sup\{|f(z)| \mid z \in U\}.$$

This norm makes H^∞ satisfy Definition 5.2. Thus H^∞ is a normed linear space, so it is a normed complex algebra. The fact that H^∞ is complete has been shown in [62, Remark 17.8(c), p. 338], so H^∞ is a commutative Banach algebra by Definition 18.1. The element $1 \in H^\infty$ is easily seen to be its unit which gives the first assertion.

Suppose that $|\alpha| < 1$. Define $\Phi_\alpha : H^\infty \to \mathbb{C}$ by

$$\Phi_\alpha(f) = f(\alpha). \tag{18.32}$$

Then it satisfies $\Phi_\alpha(fg) = f(\alpha)g(\alpha) = \Phi_\alpha(f)\Phi_\alpha(g)$. For every constant a, the function $f(z) = a$ gives $\Phi_\alpha(f) = a$ so it is surjective. In other words, $\Phi_\alpha \in \Delta$. To see that there are complex homomorphisms of H^∞ other than the point homomorphisms (18.32), we let I be the set of functions $f \in H^\infty$ such that $f(\alpha) \to 0$ as $\alpha \to 1$ and $\alpha > 0$. Then it is easy to see that I is a proper ideal of H^∞. By Theorem 18.13, I is contained in a maximal ideal J of H^∞ which means that there exists a complex homomorphism of H^∞, say $\varphi \in \Delta$ such that $\varphi(f) = 0$ for all $f \in I$ by Theorem 18.17(a). However, $\varphi \neq \Phi_\alpha$ for all $\alpha \in U$ because there is no α such that $\Phi_\alpha(f) = f(\alpha) = 0$ for every $f \in I$. We have finished the proof of this problem. ∎

Problem 18.17

Rudin Chapter 18 Exercise 17.

Proof. By Problem 18.16, H^∞ is a commutative Banach algebra. Let $I = \{(z-1)^2 f \mid f \in H^\infty\}$. For any $f, g \in H^\infty$, we know that $fg \in H^\infty$. Thus if $(z-1)^2 f \in I$ and $g \in H^\infty$, then we have

$$g \cdot (z-1)^2 f = (z-1)^2 fg \in I.$$

By Definition 18.12, I is an ideal of H^∞.

Given $\epsilon > 0$. The function $f_\epsilon(z) = (1 + \epsilon - z)^{-1}$ belongs to H^∞ because

$$|1 + \epsilon - z| \geq |1 + \epsilon| - |z| > \epsilon$$

so that $|f_\epsilon(z)| < \epsilon^{-1}$ for all $z \in U$. We observe that

$$\left|(1-z)^2(1+\epsilon-z)^{-1} - (1-z)\right| = \left|\frac{\epsilon(1-z)}{1-z+\epsilon}\right| < \epsilon$$

for all $z \in U$. This means that $(z-1)^2 f_\epsilon(z)$ converges uniformly to $1 - z$ in U. However, we know that $1 - z \notin H^\infty$ which means that I is not closed. This ends the proof of the problem. ∎

Problem 18.18

Rudin Chapter 18 Exercise 18.

Proof. Denote $I = \{\varphi f \mid f \in H^\infty\}$. Of course, we have $\varphi f \in H^\infty$. Since $fg \in H^\infty$ for any $f, g \in H^\infty$, the space I is an ideal of H^∞ by Definition 18.12. Let $\{f_n\}$ be a sequence in H^∞ such that

$$\|\varphi f_n - g\|_\infty \to 0$$

as $n \to \infty$, where $g \in H^\infty$. Thus $\{\varphi f_n\}$ is a Cauchy sequence by [61, Theorem 3.11(a), p. 53] and hence so is $\{f_n\}$. Observing from Remark 17.8(c) that H^∞ is Banach, so it is complete. Then we have $f_n \to f \in H^\infty$ as $n \to \infty$ and this means that $\varphi f \in I$. Hence we complete the proof of the problem.

■

CHAPTER 19

Holomorphic Fourier Transforms

19.1 Problems on Entire Functions of Exponential Type

Problem 19.1

Rudin Chapter 19 Exercise 1.

Proof. By the hypothesis, we know that there exist some constants A and C such that

$$|f(z)| \leq Ce^{A|z|}$$

for all $z \in \mathbb{C}$. Suppose for simplicity that $\varphi(0) < \infty$, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty.$$

By Theorem 19.3 (The Paley and Wiener Theorem), there exists an $F \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(t)e^{itz} \, dt$$

for all $z \in \mathbb{C}$. Define F_1 and F_2 by

$$F_1(t) = \begin{cases} F(t), & \text{if } 0 \leq t < A; \\ 0, & \text{if } t \geq A \end{cases} \quad \text{and} \quad F_2(t) = \begin{cases} F(t), & \text{if } -A < t \leq 0; \\ 0, & \text{if } t \leq -A. \end{cases}$$

Then we may express f as

$$f(z) = f_1(z) + f_2(z) = \int_0^{\infty} F_1(t)e^{itz} \, dt + \int_{-\infty}^{0} F_2(t)e^{itz} \, dt. \tag{19.1}$$

By the definition, it is clear that $F_1 \in L^2(0, \infty)$, so we know from [62, Eqn. (3), p. 372] that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f_1(x + iy)|^2 \, dx \leq \int_0^{\infty} |F_1(t)|^2 \, dt < \infty \tag{19.2}$$

for every $y > 0$. For the second integral of the equation (19.1), we write

$$f_2(z) = \int_{-\infty}^{0} F_2(t)e^{itz} \, dt = \int_0^{\infty} \widetilde{F_2}(t)e^{-itz} \, dt,$$

where $\widetilde{F_2}(t) = F_2(-t)$. Since $F_2 \in L^2(-\infty, 0)$, we have $\widetilde{F_2} \in L^2(0, \infty)$. By similar argument as in [62, pp. 371, 372], we can show that f_2 is holomorphic in the lower half plane Π^- and if we write

$$f_2(x + iy) = \int_0^\infty \big[\widetilde{F_2}(t)\mathrm{e}^{ty}\big] \cdot \mathrm{e}^{-itx}\,\mathrm{d}t,$$

regard y as fixed, then Theorem 9.13 (The Plancherel Theorem) implies that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+iy)|^2\,\mathrm{d}x = \int_0^\infty \big|\widetilde{F_2}(t)\big|^2 \mathrm{e}^{2ty}\,\mathrm{d}t \leq \int_0^\infty \big|\widetilde{F_2}(t)\big|^2\,\mathrm{d}t < \infty \tag{19.3}$$

for every $y < 0$. Now we substitute the estimates (19.2) and (19.3) into the expression (19.1), we see from Theorem 3.8 that

$$\begin{aligned}
\frac{1}{2\pi}\varphi(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+iy)|^2\,\mathrm{d}x \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |f_1(x+iy)|^2\,\mathrm{d}x + \frac{1}{\pi}\int_{-\infty}^{\infty} |f_1(x+iy)|\cdot|f_2(x+iy)|\,\mathrm{d}x \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} |f_2(x+iy)|^2\,\mathrm{d}x \\
&\leq \int_0^\infty |F_1(t)|^2\,\mathrm{d}t + \frac{1}{\pi}\Big\{\int_{-\infty}^{\infty} |f_1(x+iy)|^2\,\mathrm{d}x\Big\}^{\frac{1}{2}}\cdot\Big\{\int_{-\infty}^{\infty} |f_2(x+iy)|^2\,\mathrm{d}x\Big\}^{\frac{1}{2}} \\
&\quad + \int_0^\infty \big|\widetilde{F_2}(t)\big|^2\,\mathrm{d}t \\
&= \|F_1\|_2^2 + 2\|F_1\|_2\cdot\big\|\widetilde{F_2}\big\|_2 + \big\|\widetilde{F_2}\big\|_2^2 \\
&= \big(\|F_1\|_2 + \|\widetilde{F_2}\|_2\big)^2 \\
&< \infty
\end{aligned}$$

for every real y. This proves the first assertion.

For the second assertion, we suppose that φ is a bounded function. By Theorem 19.3 (The Paley and Wiener Theorem), there exists an $F \in L^2(-A, A)$ such that

$$f(x+iy) = f(z) = \int_{-A}^{A} F(t)\mathrm{e}^{itz}\,\mathrm{d}t = \int_{-A}^{A} \big[F(t)\mathrm{e}^{-ty}\big]\cdot\mathrm{e}^{-itx}\,\mathrm{d}t \tag{19.4}$$

for all $z = x + iy \in \mathbb{C}$. If we extend $F(t) = 0$ for all $t \leq -A$ and $t \geq A$, then the integral (19.4) becomes

$$f(x+iy) = \int_{-\infty}^{\infty} [F(t)\mathrm{e}^{-ty}]\cdot\mathrm{e}^{-itx}\,\mathrm{d}t.$$

Observing from Theorem 9.13 (The Plancherel Theorem), we have

$$\varphi(y) = \int_{-\infty}^{\infty} |f(x+iy)|^2\,\mathrm{d}x = 2\pi\int_{-\infty}^{\infty} |F(t)\mathrm{e}^{-ty}|^2\,\mathrm{d}t = 2\pi\int_{-A}^{A} \mathrm{e}^{-2ty}|F(t)|^2\,\mathrm{d}t.$$

If we consider $y \to -\infty$, then the boundedness of φ implies that $F(t) = 0$ a.e. on $[0, \infty)$. Similarly, if $y \to \infty$, then we have $F(t) = 0$ a.e. on $(-\infty, 0]$. Thus we have $F(t) = 0$ a.e. on $\mathbb{R}$, so the integral representation (19.4) implies that $f = 0$ which ends the proof of the problem. ■

Problem 19.2

Rudin Chapter 19 Exercise 2.

Proof. Since f is of exponential type, there exist some constants A and C such that

$$|f(z)| \leq C\exp(A|z|) \tag{19.5}$$

for all $z \in \mathbb{C}$. Using rotation, we may assume that one of the nonparallel lines is the real line,[a] so we also have

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x < \infty.$$

Now Theorem 19.3 (The Paley and Wiener Theorem) asserts that there exists an $F \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(t)\mathrm{e}^{itz} \, \mathrm{d}t$$

for all z. Particularly, we take $z = x \in \mathbb{R}$ so that

$$|f(x)| \leq \int_{-A}^{A} |F(t)| \, \mathrm{d}t \leq (2A)^{\frac{1}{2}} \cdot \left\{ \int_{-A}^{A} |F(t)|^2 \, \mathrm{d}t \right\}^{\frac{1}{2}} < \infty$$

by Theorem 3.8. Thus f is bounded on the real line. Similarly, it can be shown that f is also bounded on the other line. Then we may apply Problem 19.3 to conclude that f is actually a constant. Since $f \in L^2$, this constant is in fact zero. Hence we obtain our desired result and this ends the proof of the problem. ∎

Problem 19.3

Rudin Chapter 19 Exercise 3.

Proof. Without loss of generality, we assume that the two nonparallel lines pass through the origin and f is bounded by 1 on these lines. Suppose that Δ is the open sector between the two lines with sectoral angle $\frac{\pi}{\beta} < \pi$ for some $\beta > 1$. Since f is entire, it is continuous on $\overline{\Delta}$ and holomorphic in Δ. Since f is of exponential type, the inequality (19.5) holds for all complex z. Take $1 < \alpha < \beta$. Then there exists a positive number R such that $A|z| \leq |z|^\alpha$ for all z with $|z| > R$. For $|z| \leq R$, we let $M = C\mathrm{e}^{AR}$ so that

$$|f(z)| \leq M\exp(|z|^\alpha)$$

for all $z \in \Delta$. Thus Problem 12.9 reveals that $|f(z)|$ is bounded in Δ which implies that f is a bounded entire function. Hence it follows from Theorem 10.23 (Liouville's Theorem) that f is constant, as desired. This ends the proof of the problem. ∎

Problem 19.4

Rudin Chapter 19 Exercise 4.

Proof. Now f is an entire function of exponential type.

- **The series converges if $|w| > A$.** By the hypothesis, it is true that $|f(z)| < \exp(|z|^\lambda)$ for all large enough $|z|$, where $\lambda > 1$. According to Problem 15.2, f is of order 1. We note from [11, Eqns. (2.1.6) & (2.2.12), pp. 8, 12] that

$$\limsup_{n\to\infty} |f^{(n)}(0)|^{\frac{1}{n}} = A.$$

[a] Notice that $f(\mathrm{e}^{i\theta}z)$ is also an entire function satisfying the inequality (19.5).

This means that

$$\limsup_{n\to\infty} |n!a_n|^{\frac{1}{n}} = A$$

and so the power series Φ converges if

$$\frac{1}{|w|} < \frac{1}{\limsup\limits_{n\to\infty} |n!a_n|^{\frac{1}{n}}} = \frac{1}{A},$$

i.e., $|w| > A$.[b]

- **The function f can be expressed as an integral.** By the power series of Φ, we see that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \Phi(w)\mathrm{e}^{wz}\,\mathrm{d}w &= \frac{1}{2\pi i}\int_\Gamma \Big(\sum_{n=0}^{\infty}\frac{n!a_n}{w^{n+1}}\Big)\times\Big(\sum_{k=0}^{\infty}\frac{z^k w^k}{k!}\Big)\,\mathrm{d}w\\
&= \frac{1}{2\pi i}\int_\Gamma \Big(\sum_{m=0}^{\infty}\frac{a_m z^m}{w}\Big)\,\mathrm{d}w\\
&= \frac{1}{2\pi i}\sum_{m=0}^{\infty} a_m z^m \int_\Gamma \frac{\mathrm{d}w}{w}\\
&= \sum_{m=0}^{\infty} a_m z^m\\
&= f(z),
\end{aligned}$$

where the term-by-term integration being justified by the uniform convergence of the series on Γ.[c]

- **Φ is the function which occurred in the proof of Theorem 19.3.** Recall from Remark 19.4 that the functions Φ_α are restrictions of a function holomorphic in the complement of the interval $[-iA, iA]$. Thus it suffices to prove that Φ is such function.

  Our first assertion and Theorem 10.6 ensure that the Borel transform Φ is holomorphic in the complement of $[-iA, iA]$. It remains to show that

$$\Phi|_{\Pi_\alpha} = \Phi_\alpha \tag{19.6}$$

  for every real α.[d] If $\operatorname{Re}(w\mathrm{e}^{i\alpha}) > 3A$, then we have

$$|\exp(-ws\mathrm{e}^{i\alpha})| = \big|\exp\big(-s\operatorname{Re}(w\mathrm{e}^{i\alpha})\big)\big| < \mathrm{e}^{-3As}. \tag{19.7}$$

  Furthermore, if we define $M_f(r) = \max\limits_{z\in \overline{D(0;r)}} |f(z)|$, then we follow from Theorem 10.26 (Cauchy's Estimates) that

$$|a_n| \leq \frac{M_f(r)}{r^n}$$

  and the remainder

$$R_n(s) = \sum_{k=n+1}^{\infty} a_k s^k$$

[b] The function $\Phi(w)$ in question is called the **Borel transform** of the function $f(z)$, see [11, §5.3, p. 73] or [36, §20, p. 84].

[c] The integral is sometimes called the **Pólya representation** of the function f.

[d] Indeed, the half plane Π_α is given by $\{w = x+iy \,|\, x\cos\alpha - y\sin\alpha > A\}$.

of the power series of f satisfies

$$|R_n(s)| \leq M_f(r) \sum_{k=n+1}^{\infty} \Big(\frac{s}{r}\Big)^k = \frac{M_f(r)}{1-\frac{s}{r}} \cdot \Big(\frac{s}{r}\Big)^{n+1}.$$

By putting $r = 2s$, we get

$$|R_n(s)| \leq \frac{\mathrm{e}^{2s}}{2^n} \tag{19.8}$$

Therefore, we deduce from the inequalities (19.7) and (19.8) that the series

$$\sum_{n=0}^{\infty} a_n \mathrm{e}^{-wz} z^n$$

converges uniformly on the ray $\Gamma_\alpha = \{se^{i\alpha} \,|\, s \geq 0\}$. Consequently, for every $w \in \Pi_\alpha$, we obtain

$$\Phi_\alpha(\omega) = \int_{\Gamma_\alpha} \mathrm{e}^{-wz} f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_{\Gamma_\alpha} z^n \mathrm{e}^{-wz}\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{n! a_n}{w^{n+1}} = \Phi(w)$$

which is exactly the expression (19.6).

We have completed the proof of the problem. ■

Problem 19.5

Rudin Chapter 19 Exercise 5.

Proof. Suppose that $f \in H(\Pi^+)$ and

$$\sup_{0<y<\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2\,\mathrm{d}x = C < \infty. \tag{19.9}$$

- **The Cauchy formula holds in Π^+.** Let $0 < \epsilon < y$ and $z = x + iy$. Let $r > 0$ be large and γ_r be the semicircle in Π^+ with radius r and centered at $i\epsilon$. Define Γ_r to be the union of γ_r and the line segment $[-r + i\epsilon, r + i\epsilon]$. Since r is large, we may assume that $z \notin \Gamma_r^*$. See Figure 19.1 below:

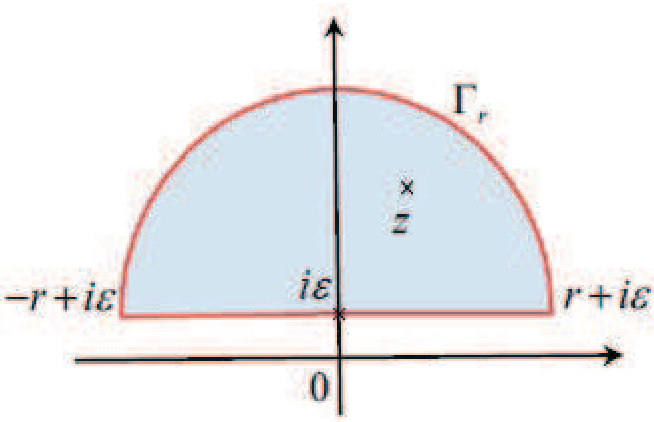


Figure 19.1: The closed contour Γ_r.

Using Theorem 10.15 (The Cauchy's Formula in a Convex Set), we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(\omega)}{\omega - z}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi i}\int_{-r}^{r}\frac{f(\xi+i\epsilon)}{\xi+i\epsilon-z}\,\mathrm{d}\xi + \frac{1}{2\pi i}\int_0^{\pi}\frac{ir\mathrm{e}^{i\theta}f(r\mathrm{e}^{i\theta}+i\epsilon)}{r\mathrm{e}^{i\theta}+i\epsilon-z}\,\mathrm{d}\theta. \end{aligned} \tag{19.10}$$

By Theorem 19.2 (The Paley and Wiener Theorem), there exists an $F \in L^2(0,\infty)$ such that

$$f(z) = \int_0^{\infty} F(t)\mathrm{e}^{itz}\,\mathrm{d}t,$$

where $z \in \Pi^+$. Take $z = r\mathrm{e}^{i\theta} + i\epsilon = r\cos\theta + i(r\sin\theta + \epsilon)$, where $0 < \theta < \pi$. Since $itz = -t(r\sin\theta+\epsilon) + itr\cos\theta$, we have

$$|\mathrm{e}^{itz}| = \mathrm{e}^{-t(r\sin\theta+\epsilon)} \leq \mathrm{e}^{-tr\sin\theta}, \tag{19.11}$$

where $0 < \theta < \pi$. According to Theorem 3.8 and the estimate (19.11), we obtain

$$|f(r\mathrm{e}^{i\theta}+i\epsilon)|^2 \leq \|F\|_2^2 \times \int_0^{\infty}\mathrm{e}^{-2tr\sin\theta}\,\mathrm{d}t = \|F\|_2^2 \times \frac{\mathrm{e}^{-2tr\sin\theta}}{-2r\sin\theta}\Big|_0^{\infty} = \frac{\|F\|_2^2}{2r\sin\theta}. \tag{19.12}$$

Substituting the bound (19.12) into the second integral in the formula (19.10), we obtain

$$\begin{aligned} \Big|\int_0^{\pi}\frac{ir\mathrm{e}^{i\theta}f(r\mathrm{e}^{i\theta}+i\epsilon)}{r\mathrm{e}^{i\theta}+i\epsilon-z}\,\mathrm{d}\theta\Big| &\leq \frac{r\|F\|_2}{\sqrt{2r}(r-|i\epsilon-z|)}\int_0^{\pi}\frac{\mathrm{d}\theta}{\sin^{\frac{1}{2}}\theta} \\ &= \frac{\sqrt{2r}\|F\|_2}{r-|i\epsilon-z|}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sin^{\frac{1}{2}}\theta}. \end{aligned} \tag{19.13}$$

By the inequality $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta}$ for $0 < \theta \leq \frac{\pi}{2}$, we can show that the integral in the estimate (19.13) is bounded so that

$$\lim_{r\to\infty}\int_0^{\pi}\frac{ir\mathrm{e}^{i\theta}f(r\mathrm{e}^{i\theta}+i\epsilon)}{r\mathrm{e}^{i\theta}+i\epsilon-z}\,\mathrm{d}\theta = 0$$

and consequently, the expression (19.10) gives

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi+i\epsilon)}{\xi+i\epsilon-z}\,\mathrm{d}\xi$$

for every $0 < \xi < y$.

- $f^*(x) = \lim\limits_{y\to 0} f(x+iy)$ **exists for almost all** x**.** Notice that the linear fractional transformation $\varphi(z) = i\frac{1+z}{1-z}$ maps U conformally onto Π^+ and $\varphi(T) = \mathbb{R}\cup\{\infty\}$. If we consider $g = f\circ\varphi \in H(U)$, then the condition (19.9) implies definitely that

$$\int_U |g(z)|^2\,\mathrm{d}\theta\,\mathrm{d}r < \infty.$$

Therefore, g is bounded for almost all $z \in U$ which means $\|g\|_\infty < \infty$, i.e., $g \in H^\infty$. By Theorem 11.32 (Fatou's Theorem), the limit

$$\lim_{r\to 1} g(r\mathrm{e}^{i\theta}) \tag{19.14}$$

exists for almost everywhere on $[-\pi,\pi]$. After transforming back to f, the result (19.14) means that

$$f^*(x) = \lim_{y\to 1} f(x+iy)$$

exists for almost all x.

- **The relation between f^* and F.** The second assertion implies

$$f^*(x) = \lim_{y\to 0} f(x+iy) = \lim_{y\to 0}\int_0^\infty F(t)\mathrm{e}^{-ty+itx}\,\mathrm{d}t. \tag{19.15}$$

Let $f_n(t) = F(t)\mathrm{e}^{-\frac{t}{n}+itx}$. Then we have $|f_1| \geq |f_2| \geq \cdots \geq 0$ and $f_n(t) \to F(t)\mathrm{e}^{itx}$ as $n\to\infty$ for almost every $t\in(0,\infty)$. It is clear from Theorem 3.8 that

$$\int_0^\infty |f_1(t)|\,\mathrm{d}t = \int_0^\infty |F(t)|\mathrm{e}^{-t}\,\mathrm{d}t \leq \|F\|_2 \times \Big\{\int_0^\infty \mathrm{e}^{-2t}\,\mathrm{d}t\Big\}^{\frac{1}{2}} = \frac{\|F\|_2}{\sqrt{2}} < \infty,$$

i.e., $|f_1| \in L^1(0,\infty)$. Hence we may apply Problem 1.7 to the limit (19.15) to get

$$f^*(x) = \int_0^\infty F(t)\mathrm{e}^{itx}\,\mathrm{d}t \tag{19.16}$$

for almost all real x. Since we may assume that F vanishes on $(-\infty,0)$ (see §19.1), we follow from the expression (19.16) that

$$f^*(x) = \sqrt{2\pi}\cdot\widehat{F}(-x)$$

for almost all $x\in\mathbb{R}$.

- **The case when $\epsilon = 0$.** Let $z = x+iy$ with $y>0$. Notice that

$$\Big|\frac{f(\xi+i\epsilon)}{\xi+i\epsilon-z}\Big| \leq \frac{|f(\xi+i\epsilon)|}{|\xi-z|}$$

for every $\xi\in(-\infty,\infty)$ and $0<\epsilon<y$. Thus it follows from Theorem 3.8 that

$$\begin{aligned}\int_{-\infty}^\infty \Big|\frac{f(\xi+i\epsilon)}{\xi+i\epsilon-z}\Big|\,\mathrm{d}\xi &\leq \int_{-\infty}^\infty \frac{|f(\xi+i\epsilon)|}{|\xi-x|}\,\mathrm{d}\xi\\ &\leq \Big\{\int_{-\infty}^\infty |f(\xi+i\epsilon)|^2\,\mathrm{d}\xi\Big\}^{\frac{1}{2}} \times \Big\{\int_{-\infty}^\infty \frac{\mathrm{d}\xi}{(\xi-x)^2}\Big\}^{\frac{1}{2}}\\ &\leq \sqrt{2C\pi}\cdot\Big\{\int_{-\infty}^\infty \frac{\mathrm{d}\xi}{(x-\xi)^2}\Big\}^{\frac{1}{2}}\\ &< \infty,\end{aligned}$$

so Theorem 1.34 (The Lebesgue's Dominated Convergence Theorem) ensures that

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f^*(\xi)}{\xi - z}\,\mathrm{d}\xi.$$

We complete the proof of the problem. ■

Problem 19.6

Rudin Chapter 19 Exercise 6.

Proof. Here we follow mainly [50, Theorem XII, pp. 16 – 20]. Since $0<\varphi<\mathrm{e}^\varphi$, we have $\log\varphi<\varphi$. Combining the hypothesis and Theorem 3.5 (Hölder's Inequality), we obtain

$$-\infty < \int_{-\infty}^\infty \frac{\log\varphi(x)}{1+x^2}\,\mathrm{d}x < \int_{-\infty}^\infty \frac{\varphi(x)}{1+x^2}\,\mathrm{d}x \leq \Big\{\int_{-\infty}^\infty |\varphi(x)|^2\,\mathrm{d}x\Big\}^{\frac{1}{2}} \times \Big\{\int_{-\infty}^\infty \frac{\mathrm{d}x}{(1+x^2)^2}\Big\}^{\frac{1}{2}} < \infty.$$

In other words, we have

$$\int_{-\infty}^{\infty} \frac{|\log \varphi(x)|}{1+x^2}\, \mathrm{d}x < \infty.$$

We write $z = x + iy$ with $y > 0$ and consider

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \log \varphi(t)\, \mathrm{d}t. \tag{19.17}$$

Using the half-plane version of Fatou's Theorem [55, Theorem 5.5, pp. 86, 87], we see that the function (19.17) is harmonic in Π^+ and

$$\lim_{y\to 0} u(x+iy) = \log \varphi(x) \quad \text{or} \quad \lim_{y\to 0} |f(x+iy)| = \varphi(x) \tag{19.18}$$

holds for almost all $x \in \mathbb{R}$.[e] Let $v(z)$ be its harmonic conjugate and write

$$f(z) = \exp(u(z) + iv(z)).$$

Because of [62, Eqn. (7), p. 63], we see that

$$|f(x+iy)| = \mathrm{e}^{u(z)} \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(t)y}{(x-t)^2+y^2}\, \mathrm{d}t,$$

it follows from Theorem 3.5 (Hölder's Inequality) that

$$\begin{aligned}
|f(x+iy)|^2 &\leq \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\varphi(t)y}{(x-t)^2+y^2}\, \mathrm{d}t \times \int_{-\infty}^{\infty} \frac{\varphi(s)y}{(x-s)^2+y^2}\, \mathrm{d}s \\
&\leq \frac{1}{\pi^2}\Big\{\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2 y}{(x-t)^2+y^2}\, \mathrm{d}t\Big\}^{\frac{1}{2}}\Big\{\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, \mathrm{d}t\Big\}^{\frac{1}{2}} \\
&\quad \times \Big\{\int_{-\infty}^{\infty} \frac{|\varphi(s)|^2 y}{(x-s)^2+y^2}\, \mathrm{d}s\Big\}^{\frac{1}{2}}\Big\{\int_{-\infty}^{\infty} \frac{y}{(x-s)^2+y^2}\, \mathrm{d}s\Big\}^{\frac{1}{2}} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2 y}{(x-t)^2+y^2}\, \mathrm{d}t
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x+iy)|^2\, \mathrm{d}x &\leq \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2 y}{(x-t)^2+y^2}\, \mathrm{d}t\, \mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} |\varphi(t)|^2\, \mathrm{d}t \cdot \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} |\varphi(t)|^2\, \mathrm{d}t.
\end{aligned}$$

In other words, we have established

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\, \mathrm{d}x < \infty.$$

According to Theorem 19.2 (The Paley-Wiener Theorem), there exists an $F \in L^2(-\infty, \infty)$ vanishing on $(-\infty, 0)$ such that

$$f(z) = \int_{-\infty}^{\infty} F(t)\mathrm{e}^{itz}\, \mathrm{d}t$$

---

[e] Here the function

$$\frac{y}{(x-t)^2+y^2}$$

is the **Poisson kernel** in the upper half plane Π^+. See also [7, pp. 145 – 147; Theorem 7.28, pp. 160, 161].

for all $z \in \Pi^+$. In particular, we have

$$\lim_{y\to 0} f(x+iy) = f(x) = \int_0^\infty F(t)\mathrm{e}^{itx}\,\mathrm{d}t = \int_{-\infty}^\infty F(t)\mathrm{e}^{itx}\,\mathrm{d}t = \widehat{F}(-x) \tag{19.19}$$

for $x \in \mathbb{R}$. If we denote $G(x) = \widehat{F}(-x)$, then we combine the results (19.18) and (19.19) to get

$$|G(x)| = \varphi(x).$$

Since $G \in L^2(-\infty,\infty)$, we derive from [78, Lemma 9.3, p. 286] that $\widehat{G}(x) = F(-x)$ which vanishes on $[0,\infty)$.

Conversely, we suppose that there exists an f with $|\widehat{f}| = \varphi$ such that $f(x) = 0$ for all $x \leq 0$. Let us write

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)\mathrm{e}^{-ixt}\,\mathrm{d}t \tag{19.20}$$

and

$$\psi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)\mathrm{e}^{-izt}\,\mathrm{d}t, \tag{19.21}$$

where $z \in \Pi^+$ and the integral in (19.21) is taken along a horizontal line in the z-plane. Certainly, we have $\psi \in H(\Pi^+)$ by §19.1. Suppose that we map Π^+ (conformally) onto U by $z = i\frac{\zeta+1}{\zeta-1}$. Write

$$k(\zeta) = \psi(z) = \psi\Big(i\frac{\zeta+1}{\zeta-1}\Big) \quad \text{and} \quad K(\mathrm{e}^{i\theta}) = \widehat{f}(x),$$

where $\zeta = r\mathrm{e}^{i\theta}$ and $0 \leq r < 1$ and $x = i\frac{\mathrm{e}^{i\theta}+1}{\mathrm{e}^{i\theta}-1}$. Then it is easily seen from Theorem 12.12 (The Hausdorff-Young Theorem) that

$$\int_{-\pi}^{\pi} |K(\mathrm{e}^{i\theta})|^2\,\mathrm{d}\theta = 2\int_{-\infty}^\infty \frac{|\widehat{f}(x)|^2}{1+x^2}\,\mathrm{d}x \leq 2\int_{-\infty}^\infty |\widehat{f}(x)|^2\,\mathrm{d}x = 2\|\widehat{f}\|_2^2 \leq 2\|f\|_2^2 = 2\|\varphi\|_2^2 < \infty.$$

Therefore, we have $K \in L^2(T)$. On the other hand, if $z = x+iy$, then the integral (19.20) implies that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} K(\mathrm{e}^{i\theta})P_r(\theta-\phi)\,\mathrm{d}\theta &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{y}{(x-t)^2+y^2}\widehat{f}(t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\infty}^\infty \frac{y}{(x-t)^2+y^2} \times \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)\mathrm{e}^{-it\xi}\,\mathrm{d}\xi\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi) \times \Big(\frac{1}{\pi}\int_{-\infty}^\infty \frac{\mathrm{e}^{-it\xi}y}{(x-t)^2+y^2}\,\mathrm{d}t\Big)\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)\mathrm{e}^{-ix\xi+y\xi}\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)\mathrm{e}^{-iz\xi}\,\mathrm{d}\xi \\
&= \psi(z) \\
&= k(r\mathrm{e}^{i\theta}).
\end{aligned}$$

By Definition 11.6, k is the Poisson integral of K, i.e., $k = P[K]$. Since $K \in L^2(T)$ and $k = P[K]$, it follows from Theorem 11.16 that

$$\int_{-\pi}^{\pi} \log^+|k(r\mathrm{e}^{i\theta})|\,\mathrm{d}\theta \leq \int_{-\pi}^{\pi} |k(r\mathrm{e}^{i\theta})|^2\,\mathrm{d}\theta \leq \int_{-\pi}^{\pi} |K(r\mathrm{e}^{i\theta})|^2\,\mathrm{d}\theta. \tag{19.22}$$

If $k(0) \neq 0$, then we apply Theorem 15.18 (Jensen's Formula) to obtain

$$\log |k(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |k(re^{i\theta})| \, d\theta. \tag{19.23}$$

Now the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |k(re^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |k(re^{i\theta})| \, d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |k(re^{i\theta})| \, d\theta$$

clearly implies

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \big| \log |k(re^{i\theta})| \big| \, d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |k(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |k(re^{i\theta})| \, d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \log^+ |k(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |k(re^{i\theta})| \, d\theta. \end{aligned} \tag{19.24}$$

Substituting the inequalities (19.22) and (19.23) into the formula (19.24), we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \big| \log |k(re^{i\theta})| \big| \, d\theta \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |K(re^{i\theta})|^2 \, d\theta - \log |k(0)| \tag{19.25}$$

for all $0 \leq r < 1$. If k has zero at 0 with multiplicity m, then the inequality (19.25) becomes

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \big| \log |k(re^{i\theta})| \big| \, d\theta \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |K(re^{i\theta})|^2 \, d\theta - \log \left| \frac{k(\zeta)}{\zeta^m} \right|_{\zeta=0} - m \log r. \tag{19.26}$$

for all $0 \leq r < 1$. By the definition, $\log |k(re^{i\theta})| \to \log |K(e^{i\theta})|$ as $r \to 1$ almost everywhere, so we conclude from the inequality (19.26) that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\log \varphi(x)|}{1+x^2} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\big| \log |\widehat{f}(x)| \big|}{1+x^2} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \big| \log |K(e^{i\theta})| \big| \, d\theta < \infty.$$

Consequently, this ensures that

$$\int_{-\infty}^{\infty} \log \varphi(x) \frac{dx}{1+x^2} > -\infty.$$

Hence we have completed the proof of the problem. ■

19.2 Quasi-analytic Classes and Borel's Theorem

Problem 19.7

Rudin Chapter 19 Exercise 7.

Proof. It is trivial that Condition (a) implies Condition (b). Conversely, suppose that Condition (b) holds. For each $\alpha \in E$, we *fix* a neighborhood V_α of α and put

$$\Omega = \bigcup_{\alpha \in E} V_\alpha.$$

It is clear that $E \subset \Omega$ and Ω is an open set. Now we define $F : \Omega \to \mathbb{C}$ as follows: Given $z \in \Omega$. Then we have $z \in V_\alpha$ for some V_α and we define

$$F(z) = F_\alpha(z). \tag{19.27}$$

We claim that $F \in H(\Omega)$ and $F(z) = f(z)$ for $z \in E$. If $z \in V_\beta$ for $\beta \neq \alpha$, then we have $V_\alpha \cap V_\beta \neq \varnothing$. Since $V_\alpha \cap V_\beta$ is an open set, there exists a $\delta > 0$ such that $z \in D(0;\delta) \subseteq V_\alpha \cap V_\beta$, so Theorem 10.18 says that $F_\alpha \equiv F_\beta$ in $V_\alpha \cap V_\beta$ and thus the formula (19.27) is well-defined. Furthermore, it is clear that F is holomorphic at every point of Ω. Finally, if $\zeta \in E$, then $\zeta \in V_\zeta$. Since $F_\zeta(z) = f(z)$ for all $z \in V_\zeta \cap E$, we must have

$$F(\zeta) = F_\zeta(\zeta) = f(\zeta).$$

This ends the proof of the problem. ■

Problem 19.8

Rudin Chapter 19 Exercise 8.

Proof. Since $n! \leq n^n$ for every $n \geq 1$, we have $\|D^n f\|_\infty \leq \beta_f B_f^n n! \leq \beta_f B_f^n n^n$. By Definition 19.6, we have $C\{n!\} \subseteq C\{n^n\}$. Recall the approximation to Stirling's formula [61, Exercise 20, p. 200] that

$$\frac{n^n}{\mathrm{e}^n n!} \sim \frac{1}{\sqrt{2\pi n}}$$

for large n. Thus there exists a constant $M > 0$ such that $n^n \leq M\mathrm{e}^n n!$ for all $n \geq 1$. If $f \in C\{n^n\}$, then we have

$$\|D^n f\|_\infty \leq \beta_f B_f^n n^n \leq (M\beta_f)(\mathrm{e}B_f)^n n!$$

which means $f \in C\{n!\}$. Consequently, we obtain the desired result that $C\{n!\} = C\{n^n\}$, completing the proof of the problem. ■

Problem 19.9

Rudin Chapter 19 Exercise 9.

Proof. Let $M_0 = 1$, $M_1 = 1$, $M_2 = 2$ and $M_n = n!(\log n)^n$ for every $n \geq 3$. Thus we always have $M_n^2 \leq M_{n-1}M_{n+1}$ for all $n = 1, 2, \ldots$. Using Theorem 19.11 (The Denjoy-Carleman Theorem), we know that $C\{M_n\}$ is quasi-analytic. If $f \in C\{n!\}$, then there exist positive constants β_f and B_f such that

$$\|D^n f\|_\infty \leq \beta_f B_f^n n!$$

for all $n \geq 0$. By the definition of $\{M_n\}$, we also have

$$\|D^n f\|_\infty \leq \beta_f B_f^n M_n$$

for every $n \geq 0$. Thus we have

$$C\{n!\} \subseteq C\{M_n\}.$$

The construction of an example f belonging to $C\{M_n\}$, but $f \notin C\{n!\}$ is basically motivated by [70, Theorem 1, p. 4]. Put $m_n = \frac{M_{n+1}}{M_n}$ for every $n = 0, 1, 2, \ldots$. Since

$$m_n - m_{n-1} = \frac{M_{n+1}}{M_n} - \frac{M_n}{M_{n-1}} = \frac{M_{n+1}M_{n-1} - M_n^2}{M_nM_{n-1}} \geq 0$$

for every $n \geq 0$. Thus $\{m_n\}$ is a positive increasing sequence. It is clear that $f \in C^\infty$. For every $n, k \in \mathbb{N}$, if $k \leq n$, then we see that

$$\frac{1}{m_n^{n-k}} = \frac{1}{m_n} \times \frac{1}{m_n} \times \cdots \times \frac{1}{m_n}$$

$$
\begin{aligned}
&\le \frac{1}{m_{n-1}} \times \frac{1}{m_{n-2}} \times \cdots \times \frac{1}{m_k}\\
&= \frac{M_{n-1}}{M_n} \times \frac{M_{n-2}}{M_{n-1}} \times \cdots \times \frac{M_k}{M_{k+1}}\\
&= \frac{M_k}{M_n}.
\end{aligned}
$$

If $k > n$, then we have

$$
\begin{aligned}
\frac{1}{m_n^{n-k}} &= m_n^{k-n}\\
&\le m_n \times m_{n+1} \times \cdots \times m_{k-1}\\
&= \frac{M_{n+1}}{M_n} \times \frac{M_{n+2}}{M_{n+1}} \times \cdots \times \frac{M_k}{M_{k-1}}\\
&= \frac{M_k}{M_n}.
\end{aligned}
$$

In other words, we obtain the estimate

$$
\frac{1}{m_n^{n-k}} \le \frac{M_k}{M_n}. \tag{19.28}
$$

Define

$$
f_n(x) = \frac{M_n}{(2m_n)^n} \mathrm{e}^{2m_n ix} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} f_n(x)
$$

for $x \in \mathbb{R}$. We first show that $f \in C\{M_n\}$. For every $k \ge 1$, we have

$$
f_n^{(k)}(x) = \frac{i^k M_n}{(2m_n)^{n-k}} \mathrm{e}^{2m_n ix}.
$$

Combining this and the estimate (19.28), we get

$$
|f^{(k)}(x)| \le \sum_{n=0}^{\infty} |f_n^{(k)}(x)| = \sum_{n=0}^{\infty} \frac{M_n}{(2m_n)^{n-k}} \le \sum_{n=0}^{\infty} \frac{M_n}{2^{n-k}} \cdot \frac{M_k}{M_n} = M_k \cdot \sum_{n=0}^{\infty} \frac{1}{2^{n-k}} \le 2 \cdot 2^k M_k \tag{19.29}
$$

for every $k \ge 1$ and $x \in \mathbb{R}$. Furthermore, we also have

$$
|f(x)| \le \sum_{n=0}^{\infty} |f_n(x)| = \sum_{n=0}^{\infty} \frac{M_n}{(2m_n)^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{M_n^{n+1}}{M_{n+1}^n}. \tag{19.30}
$$

By induction, we can show that $\frac{M_n^{n+1}}{M_{n+1}^n} \le 1$ for each $n \ge 0$. Therefore, the estimate (19.30) gives

$$
|f(x)| \le 2 = 2 \cdot 2^0 M_0
$$

for every $x \in \mathbb{R}$. Now we conclude from the estimates (19.29) and (19.30) that

$$
\|D^k f\|_\infty \le 2 \cdot 2^k M_k
$$

holds for every $k \ge 0$ so that $f \in C\{M_n\}$ as required.

Next, we want to show that $|f^{(k)}(0)| \ge M_k$ for every $k \ge 0$. It is obvious that

$$
f(0) = \sum_{n=0}^{\infty} \frac{M_n}{(2m_n)^n} = M_0 + \frac{M_1}{2m_1} + \cdots \ge 1 = M_0.
$$

In addition, if $k \geq 1$, then since every term $\frac{M_n}{(2m_n)^{n-k}}$ is positive for every $n = 0, 1, 2, \ldots$, we have

$$|f^{(k)}(0)| = |i^k| \sum_{n=0}^{\infty} \frac{M_n}{(2m_n)^{n-k}} \geq M_k. \tag{19.31}$$

Hence we have obtained what we want. If $f \in C\{n!\}$, then it must be true that

$$|f^{(k)}(0)| \leq \beta_f B_f^k k!$$

for some positive constants β_f and B_f. However, if k is sufficiently large so that $(\log k)^k \geq \beta_f B_f^k$, then this will certainly contradict the estimate (19.31). Hence we conclude that $f \notin C\{n!\}$ and then we complete the proof of the problem.

■

Problem 19.10

Rudin Chapter 19 Exercise 10.

Proof. Suppose that $\lambda = \sum_{n=1}^{\infty} \lambda_n$ is positive finite. Recall from Definition 2.9 that $C_c(\mathbb{R})$ is the collection of all continuous complex functions on $\mathbb{R}$ whose support is compact. Now we let g_0 to be the function modified from the (19.38) in such the way that $g_0(x) = 1$ for $-\lambda \leq x \leq \lambda$, $g_0(x) = 0$ for $|x| \geq 2\lambda$ and $0 \leq g_0(x) \leq 1$ if $x \in [-2\lambda, -\lambda] \cup [\lambda, 2\lambda]$. Then $g_0 \in C_c(\mathbb{R})$ and g_0 is integrable in $\mathbb{R}$. Write

$$\begin{aligned} g_n(x) &= g\big(\lambda_1, \lambda_2, \ldots, \lambda_n; g_0(x)\big) \\ &= \frac{1}{2^n \lambda_1 \lambda_2 \cdots \lambda_n} \int_{-\lambda_1}^{\lambda_1} \mathrm{d}t_1 \int_{-\lambda_2}^{\lambda_2} \mathrm{d}t_2 \cdots \int_{-\lambda_n}^{\lambda_n} g_0(x - t_1 - t_2 - \cdots - t_n)\, \mathrm{d}t_n. \end{aligned} \tag{19.32}$$

Since $|g_0(x)| \leq 1$ for every $x \in \mathbb{R}$, the definition (19.32) ensures that $|g_n(x)| \leq 1$ for every $n = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$. In other words, the family $\{g_n\}$ is (uniform) bounded in $\mathbb{R}$. Besides, if $|x| \geq 3\lambda$, then $|x - \lambda_1 - \lambda_2 - \cdots - \lambda_n| \geq 2\lambda$ so that $g_n(x) = 0$ there.

Obviously, we have

$$g\big(\lambda_1, \lambda_2, \ldots, \lambda_n; g_0(x)\big) = g\big(\lambda_1, \lambda_2, \ldots, \lambda_k; g\big(\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n; g_0(x)\big)\big). \tag{19.33}$$

By the definition (19.32) again, we see that

$$g_1(x) = \frac{1}{2\lambda_1} \int_{-\lambda_1}^{\lambda_1} g_0(x - t_1)\, \mathrm{d}t_1 = \frac{1}{2\lambda_1} \int_{x-\lambda_1}^{x+\lambda_1} g_0(t)\, \mathrm{d}t,$$

so the Fundamental Theorem of Calculus yields that $g_1(x)$ is differentiable in $\mathbb{R}$ and

$$g_1'(x) = \frac{1}{2\lambda_1}[g_0(x + \lambda_1) - g_0(x - \lambda_1)].$$

Next, we assume that the function $g_{n-1}(x)$ is continuous for any $n \geq 2$ in $\mathbb{R}$ and if $n \geq 2$, then the function $g_n(x) = g\big(\lambda_n; g(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}; g_0(x))\big) = g\big(\lambda_n; g_{n-1}(x)\big)$ is differentiable in $\mathbb{R}$ and it follows from the formula (19.33) that

$$g_n'(x) = \frac{\mathrm{d}}{\mathrm{d}x} g\big(\lambda_1, \lambda_2, \ldots, \lambda_n; g_0(x)\big)$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x} g\big(\lambda_1; g\big(\lambda_2, \lambda_3, \ldots, \lambda_n; g_0(x)\big)\big) \\
&= \frac{1}{2\lambda_1}\big[g\big(\lambda_2, \lambda_3, \ldots, \lambda_n; g_0(x+\lambda_1)\big) - g\big(\lambda_2, \lambda_3, \ldots, \lambda_n; g_0(x-\lambda_1)\big)\big] \qquad (19.34) \\
&= \frac{1}{2\lambda_1} g\big(\lambda_2, \lambda_3, \ldots, \lambda_n; g_0(x+\lambda_1) - g_0(x-\lambda_1)\big).
\end{aligned}
$$

This also implies that $g_n(x)$ has continuous derivatives of order $0, 1, \ldots, n-1$ in $\mathbb{R}$. Furthermore, if we combine the formula (19.34) and the Mean Value Theorem for Derivatives, we get, for every $x \in \mathbb{R}$, that

$$
|g_n'(x)| \le \max_{x\in\mathbb{R}} |g'\big(\lambda_2, \lambda_3, \ldots, \lambda_n; g_0(x)\big)| = \max_{x\in\mathbb{R}} |g_{n-1}'(x)| \le \cdots \le \max_{x\in\mathbb{R}} |g_2'(x)| < \infty. \qquad (19.35)
$$

For every $n \ge 2$ and any $x, y \in \mathbb{R}$, the bound (19.35) asserts that

$$
|g_n(x) - g_n(y)| = |x-y| \cdot |g_n'(\xi)| \le |x-y| \cdot \max_{x\in\mathbb{R}} |g_2'(x)|
$$

which shows that the family $\{g_n\}$ is equicontinuous on $\mathbb{R}$. Recall that $g_n(x) = 0$ outside $[-3\lambda, 3\lambda]$, so $\{g_n\}$ is actually equicontinuous on $[-3\lambda, 3\lambda]$. The fact $|g_n(x)| \le 1$ in $\mathbb{R}$ guarantees that $\{g_n\}$ converges pointwise on $\mathbb{R}$. Hence it asserts from [61, Exercise 16, p. 168] that $\{g_n\}$ converges uniformly to a continuous function g on $[-3\lambda, 3\lambda]$, i.e.,

$$
g(x) = \lim_{n\to\infty} g_n(x)
$$

for every $x \in [-3\lambda, 3\lambda]$. Since the argument also applies to any compact interval of $\mathbb{R}$, the function g is also continuous at the end points $\pm 3\lambda$. It is trivial that if $x \notin [-3\lambda, 3\lambda]$, then $g(x) = 0$. Hence we must have $g(\pm 3\lambda) = 0$.

Denote $g(x) = g\big(\lambda_1, \lambda_2, \ldots; g_0(x)\big)$. Then we know that

$$
\begin{aligned}
g(x) &= \lim_{n\to\infty} g\big(\lambda_1, \lambda_2, \ldots, \lambda_n; g_0(x)\big) \\
&= \lim_{n\to\infty} \frac{1}{2\lambda_1} \int_{-\lambda_1}^{\lambda_1} g\big(\lambda_2, \lambda_3, \ldots, \lambda_n; g_0(x-t)\big)\,\mathrm{d}t \\
&= \frac{1}{2\lambda_1} \int_{-\lambda_1}^{\lambda_1} g\big(\lambda_2, \lambda_3, \ldots; g_0(x-t)\big)\,\mathrm{d}t
\end{aligned}
$$

is true for all $x \in [-3\lambda, 3\lambda]$ which means that

$$
\begin{aligned}
g'(x) &= \frac{1}{2\lambda_1}\big[g\big(\lambda_2, \lambda_3, \ldots; g_0(x+\lambda_1)\big) - g\big(\lambda_2, \lambda_3, \ldots; g_0(x-\lambda_1)\big)\big] \\
&= g\Big(\lambda_2, \lambda_3, \ldots; \frac{g_0(x+\lambda_1) - g_0(x-\lambda_1)}{2\lambda_1}\Big) \qquad (19.36)
\end{aligned}
$$

holds in $[-3\lambda, 3\lambda]$. Using similar reasoning as the previous paragraph, it can be shown that g is also differentiable at the end points $\pm 3\lambda$ and the formula (19.36) holds in $\mathbb{R}$. In conclusion, we have $g \in C^\infty$.

Since $|\lambda_1 + \lambda_2 + \cdots + \lambda_n| < \lambda$, we obtain $g_0(-t_1 - t_2 - \cdots - t_n) = 1$ for all $-\lambda_k \le t_k \le \lambda_k$, where $1 \le k \le n$. Thus we note that

$$
g(0) = \lim_{n\to\infty} \frac{1}{2^n \lambda_1 \lambda_2 \cdots \lambda_n} \int_{-\lambda_1}^{\lambda_1} \mathrm{d}t_1 \int_{-\lambda_2}^{\lambda_2} \mathrm{d}t_2 \cdots \int_{-\lambda_n}^{\lambda_n} \mathrm{d}t_n = 1,
$$

so g is not identically zero in $\mathbb{R}$.

Put

$$\begin{aligned}
G_0(x) &= g_0(x) \\
G_1(x) &= \frac{G_0(x+\lambda_1) - G_0(x-\lambda_1)}{2\lambda_1}, \\
&\vdots \\
G_n(x) &= \frac{G_{n-1}(x+\lambda_n) - G_{n-1}(x-\lambda_n)}{2\lambda_n}.
\end{aligned}$$

Thus the formula (19.36) can be written as

$$g'(x) = g\big(\lambda_2, \lambda_3, \ldots; G_1(x)\big).$$

In fact, it is true that

$$g^{(n)}(x) = g\big(\lambda_{n+1}, \lambda_{n+2}, \ldots; G_n(x)\big) \tag{19.37}$$

for every $n = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$. Recall that $|g_0(x)| \leq 1$ on $\mathbb{R}$, so we have

$$|G_n(x)| \leq \frac{1}{\lambda_1\lambda_2\cdots\lambda_n} = M_n.$$

on $\mathbb{R}$. Thus it follows from the formula (19.37) that

$$|g^{(n)}(x)| \leq M_n$$

for all $n = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$. Consequently, $g \in C\{M_n\}$ which completes the proof of the problem.

■

Remark 19.1

The construction in Problem 19.10 follows basically the unpublished work of H. E. Bray which was quited in Mandelbrojt's article [38, pp. 79 – 84]. See also [32].

Problem 19.11

Rudin Chapter 19 Exercise 11.

Proof. An example of a function $\varphi \in C^\infty$ with the required properties can be found in [77, Problem 10.6, pp. 266, 267]. In fact, we start with

$$g(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

It is known that $g \in C^\infty$ and $g^{(m)}(0) = 0$ for all $m = 1, 2, \ldots$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \frac{g(2-|x|)}{g(2-|x|) + g(|x|-1)}. \tag{19.38}$$

Now it is easy to see that $\varphi \in C^\infty$. Furthermore, we have $\varphi(x) = 1$ for $-1 \leq x \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$ and $0 \leq \varphi(x) \leq 1$ if $x \in [-2,-1] \cup [1,2]$ so that

$$\operatorname{supp}\varphi \subseteq [-2, 2].$$

This completes the proof of the problem. ■

Problem 19.12

Rudin Chapter 19 Exercise 12.

Proof. Let φ be as in Problem 19.11. Set $\beta = \frac{\alpha_n}{n!}$ and $g_n(x) = \beta_n x^n \varphi(x)$. Take

$$f_n(x) = \frac{g_n(\lambda_n x)}{\lambda_n^n} = \beta_n x^n \varphi(\lambda_n x),$$

where λ_n is large enough. Fix the non-negative integer n, we notice that

$$(D^k f_n)(x) = \beta_n \sum_{m=0}^{k} \frac{n!}{(n-m)!} C_m^k \lambda_n^{k-m} x^{n-m} \varphi^{(m)}(\lambda_n x), \tag{19.39}$$

where $k = 0, 1, 2, \ldots, n-1$. Recall from the definition of φ in Problem 19.11 that

$$\operatorname{supp} \varphi^{(m)} \subseteq \operatorname{supp} \varphi \subseteq [-2, 2]$$

holds for every $m = 0, 1, \ldots, k$. If $\lambda_n x \notin [-2, 2]$, then $\varphi^{(m)}(\lambda_n x) = 0$ so that $(D^k f)(x) = 0$. If $\lambda_n x \in [-2, 2]$, then $|x| \leq \frac{2}{\lambda_n}$. Since $\varphi^{(m)}$ is continuous on $[-2, 2]$, there is a positive constant M such that $|\varphi^{(m)}(\lambda_n x)| \leq M$. Thus we obtain

$$\begin{aligned} |(D^k f_n)(x)| &\leq |\beta_n| M \sum_{m=0}^{k} \frac{n!}{(n-m)!} C_m^k \lambda_n^{k-m} \cdot \frac{2^{n-m}}{\lambda_n^{n-m}} \\ &\leq |\beta_n| M \sum_{m=0}^{k} \frac{2^n (n!)^2}{\lambda_n^{n-k}} \\ &\leq \frac{n 2^n (n!)^2 |\beta_n| M}{\lambda_n} \end{aligned} \tag{19.40}$$

for all $x \in \mathbb{R}$ and $k = 0, 1, \ldots, n-1$. Since λ_n can be chosen large enough, we observe from the estimate (19.40) that

$$\|D^k f_n\|_\infty < \frac{1}{2^n} \tag{19.41}$$

for all $k = 0, 1, \ldots, n-1$. Take $f = f_0 + f_1 + \cdots$.

It is clear that $f_0(0) + f_1(0) + \cdots = \alpha_0$. Besides, the result (19.41) ensures that the series $\{f_0' + f_1' + \cdots + f_n'\}$ converges uniformly on $\mathbb{R}$. Using [61, Theorem 7.17, p. 152], termwise differentiation is legitimate so that $f' = f_0' + f_1' + \cdots$. Now this argument can be applied repeatedly to show that $f \in C^\infty$. Next, it follows from the expression (19.39) that $(D^k f_n)(0) = 0$ for $k = 0, 1, \ldots, n-1$. Since $\varphi(x) = 1$ on $[-1, 1]$, $\varphi^{(n)}(0) = 0$ for all $n = 1, 2, \ldots$ and this implies that $(D^n f_m)(0) = 0$ for $m = 0, 1, \ldots, n-1$. Hence we have

$$\begin{aligned} (D^n f)(0) &= \big[(D^n f_0)(0) + (D^n f_1)(0) + \cdots + (D^n f_{n-1})(0)\big] + (D^n f_n)(0) \\ &\quad + \big[(D^n f_{n+1})(0) + (D^n f_{n+2})(0) + \cdots\big] \\ &= n! \beta_n \\ &= \alpha_n \end{aligned}$$

for every $n = 0, 1, 2, \ldots$, as required. This completes the proof of the problem.[f] ■

[f] Please also read [41] and [56].

Remark 19.2

Problem 19.12 is called **Borel's Theorem** which says that every power series is the Taylor series of some smooth function, see, for examples, [46, Theorem 1.5.4, p. 30] and [51].

Problem 19.13

Rudin Chapter 19 Exercise 13.

Proof. It suffices to prove that

$$\limsup_{n\to\infty}\Big(\frac{|(D^n f)(a)|}{n!}\Big)^{\frac{1}{n}}=\infty. \tag{19.42}$$

We follow the suggestion. Let $c_k=\lambda_k^{1-k}$, where the sequence $\{\lambda_k\}$ satisfies

$$c_k\lambda_k^k=\lambda_k>2\sum_{j=1}^{k-1}c_j\lambda_j^k=2(\lambda_1^k+\lambda_2^{k-1}+\cdots+\lambda_{k-1}^2)\quad\text{and}\quad\lambda_k>k^{2k}>1.$$

Fix the non-negative integer n, we have

$$\sum_{k=1}^{\infty}c_k\lambda_k^n=\sum_{k=1}^{n+1}\lambda_k^{n+1-k}+\sum_{k=n+2}^{\infty}\lambda_k^{n+1-k}<\sum_{k=1}^{n+1}\lambda_k^{n+1-k}+\sum_{k=n+2}^{\infty}\frac{1}{k^{2k}}<\infty.$$

We put

$$f(z)=\sum_{k=1}^{\infty}c_k e^{i\lambda_k x},$$

where $z=x+iy$. Let $f_k(x)=c_k e^{i\lambda_k x}$, where $k\in\mathbb{N}$. For each $n=0,1,2,\ldots$, we observe that

$$|(D^n f_k)(a)|=c_k\lambda_k^n=\lambda_k^{n+1-k}$$

for every $a\in\mathbb{R}$. Using similar argument as in the proof of Problem 19.12, it is easy to show that $f\in C^\infty$. The choices of our $\{c_k\}$ and $\{\lambda_k\}$ reveal that

$$|(D^n f)(a)|\geq|c_n\lambda_n^n|-\Big|\sum_{\substack{k=1\\k\neq n}}^{\infty}c_k\lambda_k^n\Big|=\lambda_n-\Big|\sum_{\substack{k=1\\k\neq n}}^{\infty}c_k\lambda_k^n\Big|>\frac{n^{2n}}{2}.$$

Combining this and Stirling's formula, for large enough n, we get

$$\Big(\frac{|(D^n f)(a)|}{n!}\Big)^{\frac{1}{n}}>\Big(\frac{n^{2n}}{2n!}\Big)^{\frac{1}{n}}\approx\Big(\frac{e^n n^{2n}}{2n^n\sqrt{2\pi n}}\Big)^{\frac{1}{n}}=\frac{en}{(8\pi)^{\frac{2}{n}}\cdot n^{\frac{2}{n}}}$$

which implies the result (19.42). Hence the power series

$$\sum_{n=0}^{\infty}\frac{(D^n f)(a)}{n!}(x-a)^n$$

has radius of convergence 0 for every $a\in\mathbb{R}$, completing the proof of the problem. ∎

Remark 19.3

Let $S = \{2^n \mid n \in \mathbb{N}\}$. Define

$$g(x) = \sum_{k\in S} \mathrm{e}^{-\sqrt{k}} \cos(kx).$$

Then it can be shown that g also satisfies the requirements of Problem 19.13.

Problem 19.14

Rudin Chapter 19 Exercise 14.

Proof. Suppose that $f \in C\{M_n\}$ has infinitely many zeros $\{x_n\}$ in $[0,1]$. Then $\{x_n\}$ has a convergent subsequence by the Bolzano-Weierstrass Theorem[g]. Without loss of generality, we may assume that $\{x_n\}$ is itself convergent, distinct, increasing and its limit is α. The continuity of f gives

$$f'(\alpha) = 0.$$

By the Mean Value Theorem for Derivatives, we see that $f'(\xi_n) = 0$ for some $\xi_n \in (x_n, x_{n+1})$ for all $n = 1, 2, \ldots$. The fact $x_n \to \alpha$ as $n \to \infty$ ensures that $\xi_n \to \alpha$ as $n \to \infty$. Since $f \in C^\infty$, the continuity of f' implies that

$$f'(\alpha) = \lim_{n\to\infty} f'(\xi_n) = 0.$$

This argument can be repeated to show that $f^{(n)}(\alpha) = 0$ for all $n = 0, 1, 2, \ldots$. Since $C\{M_n\}$ is quasi-analytic, Definition 19.8 gives $f(x) \equiv 0$ for all $x \in \mathbb{R}$, completing the proof of the problem. ∎

Problem 19.15

Rudin Chapter 19 Exercise 15.

Proof. Suppose that

$$X = \big\{f \in H(\mathbb{C}) \,\big|\, |f(z)| \leq C\mathrm{e}^{\pi|z|} \text{ for some } C > 0,\ f \in L^2\big([-\pi, \pi]\big)\big\}.$$

Recall from Definition 3.6 that the sequence space ℓ^2 is given by

$$\ell^2 = \Big\{\{f(n)\} \,\Big|\, \sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty\Big\}.$$

Define the map $\Phi : X \to \ell^2$ by

$$\Phi(f) = \{f(n)\}.$$

For any $\alpha, \beta \in \mathbb{C}$ and $f, g \in X$, it is clear that $|f(z)| \leq C_1\mathrm{e}^{\pi|z|}$ and $|g(z)| \leq C_2\mathrm{e}^{\pi|z|}$ for some positive constants C_1 and C_2. Therefore, we have $|\alpha f(z) + \beta g(z)| \leq \big(|\alpha|C_1 + |\beta|C_2\big)\mathrm{e}^{\pi|z|}$ and

$$\|\alpha f + \beta g\|_2 \leq |\alpha| \cdot \|f\|_2 + \beta \cdot \|g\|_2 < \infty$$

[g]See [79, Problem 5.25, pp. 68, 69]

by Theorem 3.9. This means that $\alpha f + \beta g \in X$. Besides, the fact

$$\sum_{n=-\infty}^{\infty} |\alpha f(n) + \beta g(n)|^2 \leq 2|\alpha|^2 \sum_{n=-\infty}^{\infty} |f(n)|^2 + 2|\beta|^2 \sum_{n=-\infty}^{\infty} |g(n)|^2 < \infty$$

implies that $\{\alpha f(n) + \beta g(n)\} \in \ell^2$. Thus the relation

$$\Phi(\alpha f + \beta g) = \{\alpha f(n) + \beta g(n)\} = \alpha\{f(n)\} + \beta\{g(n)\} = \alpha\Phi(f) + \beta\Phi(g)$$

holds, i.e, Φ is linear.

For every $f \in L^2(T)$, we define

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-izt}\,\mathrm{d}t.$$

Using the analysis in §19.2, one can show that F is entire, $|F(z)| \leq C\mathrm{e}^{\pi|z|}$ for all $z \in \mathbb{C}$ and $F \in L^2(-\infty,\infty)$. In other words, this means that $F \in X$. Furthermore, we note that $F = \widehat{f}$. Next, for each $n \in \mathbb{Z}$, recall from Definition 4.23 that $\{u_n(t) = \mathrm{e}^{int} \,|\, n \in \mathbb{Z}\}$ forms an orthonormal set in $L^2(T)$. Furthermore, we have

$$U_n(z) = \widehat{u}_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(n-z)t}\,\mathrm{d}t = \frac{1}{2\pi} \times \frac{\mathrm{e}^{i(n-z)\pi} - \mathrm{e}^{-i(n-z)\pi}}{i(n-z)} = \frac{\sin[(z-n)\pi]}{(z-n)\pi}$$

holds for every $n \in \mathbb{Z}$. Recall from [62, Example 4.5(b), p. 78] that

$$\langle f, g\rangle_T = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g}(t)\,\mathrm{d}t$$

defines an inner product in $L^2(T)$. Then it is easily checked that we can induce a norm $\|\cdot\|$ to X by defining

$$\|F\|_X = \sqrt{\langle F, F\rangle_X} = \sqrt{\langle f, f\rangle_T} = \|f\|_T, \tag{19.43}$$

so it makes X Hilbert. Of course, it follows from the expression (19.43) that

$$\langle U_n, U_m\rangle_X = \langle u_n, u_m\rangle_T = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\{U_n \,|\, n \in \mathbb{Z}\}$ forms an orthonormal set in X. Since $\{u_n \,|\, n \in \mathbb{Z}\}$ is maximal in $L^2(T)$, $\{U_n \,|\, n \in \mathbb{Z}\}$ is also maximal in X. By Theorem 4.18 (The Riesz-Fischer Theorem), it is true that

$$F(z) = \sum_{k\in\mathbb{Z}} c_k U_k(z),$$

where $c_1, c_2, \ldots$ are some scalars. Since F is entire, we have

$$F(n) = \lim_{z\to n}\sum_{k\in\mathbb{Z}} c_k U_k(z) = c_n U_n(n) = c_n$$

for every $n \in \mathbb{Z}$. Finally, as a Hilbert space X with a maximal orthonormal set $\{U_n \,|\, n \in \mathbb{Z}\}$, we conclude from §4.19 that the mapping

$$F \mapsto \langle F, U_n\rangle = c_n = F(n)$$

is a Hilbert space isomorphism of X onto $\ell^2(\mathbb{Z})$. This means that our map Φ is a bijection, completing the analysis of the problem.

■

Problem 19.16

Rudin Chapter 19 Exercise 16.

Proof. Since $|f(x)| \leq \mathrm{e}^{-|x|}$ on $\mathbb{R}$, $f \in L^2(-\infty, \infty)$. By the analysis in §19.1, its Fourier transform

$$\widehat{f}(z) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{itz}\,\mathrm{d}t$$

is holomorphic in Π^+. In particular, we have

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{itx}\,\mathrm{d}t$$

for every $x \in \mathbb{R}$. For every $n \geq 0$, the hypothesis implies that differentiation under the integral sign is legitimate[h] so that

$$|(\widehat{f})^{(n)}(x)| = \Big|\int_{-\infty}^{\infty} f(t)(it)^n\mathrm{e}^{itx}\,\mathrm{d}t\Big| \leq \int_{-\infty}^{\infty} \mathrm{e}^{-|t|}\cdot|t|^n\,\mathrm{d}t = 2n!.$$

Consequently, the power series

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

has at least 1 as its radius of convergence for every $a \in \mathbb{R}$ which means that $\widehat{f}$ is also holomorphic on $\mathbb{R}$. If $\widehat{f}$ has compact support, then $\widehat{f}$ vanishes on a set with a limit point. Hence Theorem 10.18 forces that $f \equiv 0$ a.e. on $\mathbb{R}$. This completes the proof of the problem. ■

[h] Of course, it follows from the **Leibniz's Rule** by Problem 10.16, where $\varphi(z,t) = f(t)\mathrm{e}^{itz}$. See also [3, Theorem 24.5, pp. 193, 194].

CHAPTER 20

Uniform Approximation by Polynomials

Problem 20.1

Rudin Chapter 20 Exercise 1.

Proof. We want to prove that if $\epsilon > 0$, $S^2 \setminus K$ has finitely many components, $f \in C(K)$ and $f \in H(K^\circ)$, then there exists a rational function R such that

$$|f(z) - R(z)| < \epsilon \tag{20.1}$$

for all $z \in K$.

Indeed, everything up to [62, p. 392] in the proof of Theorem 20.5 (Mergelyan's Theorem) remains the same. Let $S_1, S_2, \ldots, S_m$ be the (connected) components of $S^2 \setminus K$, i.e.,

$$S^2 \setminus K = S_1 \cup S_2 \cup \cdots \cup S_m.$$

Pick $\delta > 0$ very small. Recall also that $X = \{z \in \operatorname{supp}\Phi \mid \operatorname{dist}(z, S^2 \setminus K) \leq \delta\}$ is compact, so X contains no point which is "far within" K, see Figure 20.1 which shows that X is exactly the yellow part.

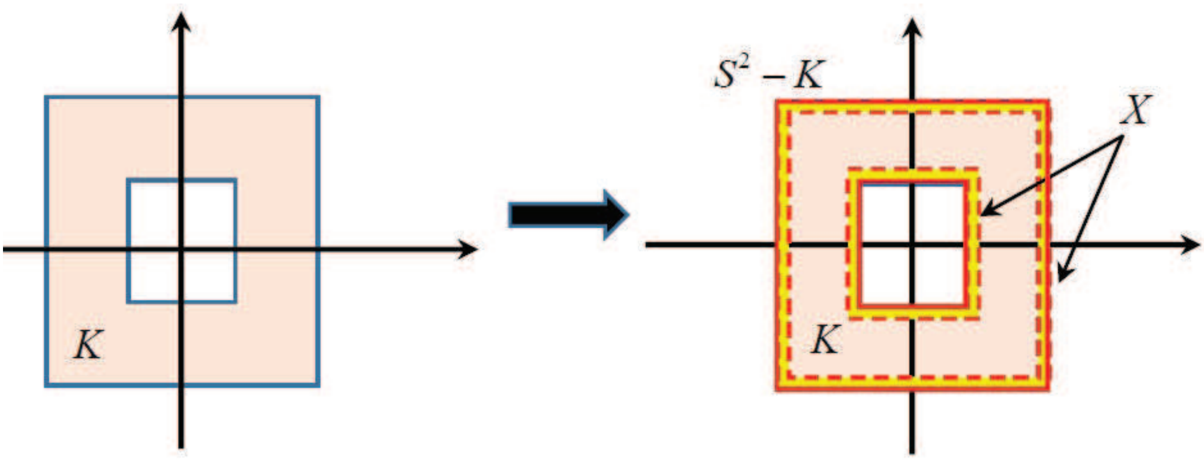


Figure 20.1: The compact set X.

Now we can cover X by finitely many open discs $D_1(p_1; 2\delta), D_2(p_2; 2\delta), \ldots, D_n(p_n; 2\delta)$, where $p_1, p_2, \ldots, p_n$ are points in $S^2 \setminus K$. We may assume that $n = m$ and $p_j \in S_j$ for $j = 1, 2, \ldots, n$.

Since each S_j is connected, there must be a curve from p_j to a point of $\partial D_j(p_j;2\delta)$ that is not of K. In other words, one can find a set $E_j \subset D_j(p_j;2\delta)$ such that E_j is a compact and connected subset of S_j, $\operatorname{diam} E_j \geq 2\delta$, $S^2 \setminus E_j$ is connected and $K \cap E_j = \varnothing$, where $j = 1, 2, \ldots, n$. We apply Lemma 20.2 with $r = 2\delta$ and follow the proof in [62, pp. 393, 394], we can obtain

$$|F(z) - \Phi(z)| < 6000\omega(z) \quad \text{and} \quad |f(z) - \Phi(z)| < \omega(\delta) \tag{20.2}$$

for all $z \in \Omega$, where $\Omega = S^2 \setminus (E_1 \cup E_2 \cup \cdots \cup E_n)$ which is an open set containing K. By the definition, we have

$$S^2 \setminus \Omega = E_1 \cup E_2 \cup \cdots \cup E_n.$$

Since $E_j \subset S_j$ for $j = 1, 2, \ldots, n$, one gets the set $A = \{p_1, p_2, \ldots, p_n\}$. Since $F \in H(\Omega)$ and $K \subseteq \Omega$, Theorem 13.9 (Rung's Theorem) implies that there exists a rational function $R(z)$ with poles only in A such that

$$|F(z) - R(z)| < \omega(\delta) \tag{20.3}$$

for all $z \in K$. Combining this and the inequalities (20.2) and (20.3), we have

$$|f(z) - R(z)| \leq |f(z) - \Phi(z)| + |\Phi(z) - F(z)| + |F(z) - R(z)| < 10000\omega(\delta)$$

for all $z \in K$. Since $\omega(\delta) \to 0$ as $\delta \to 0$, it yields the inequality (20.1) by choosing sufficiently small δ. Hence we have completed the proof of the problem. ■

Problem 20.2

Rudin Chapter 20 Exercise 2.

Proof. The set K is known as a **Swiss cheese set**.[a] The construction of such a sequence $\{D_n\}$ in U with the specific properties can be found in [27, pp. 344, 345]. In fact, we can also assume that

$$\sum_{n=1}^{\infty} r_n^2 < 1. \tag{20.4}$$

- **L is a bounded linear functional on $C(K)$.** It is easy to see that L is a linear functional on $C(K)$. According to [9, Theorem 4.10, p. 49], we have

  $$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq 2\pi \cdot \|f\|_\infty \quad \text{and} \quad \left|\int_{\gamma_n} f(z)\,\mathrm{d}z\right| \leq 2\pi r_n \cdot \|f\|_\infty$$

  for every $f \in C(K)$ and $n \in \mathbb{N}$. Therefore, we get

  $$|L(f)| \leq 2\pi\Big(1 + \sum_{n=1}^{\infty} r_n\Big) \cdot \|f\|_\infty < \infty$$

  which shows that L is bounded.

- **$L(R) = 0$ for every rational function R whose poles are outside K.** Let z_0 be a pole of R. Since $K = \overline{U} \setminus V$, we have either z_0 lies outside $\overline{U}$ or $z_0 \in D_n$ for exactly one n. If z_0 lies outside $\overline{U}$, then since $\Gamma([0,2\pi]), \gamma_n([0,2\pi]) \subseteq \overline{U}$, the integrals in $L(R)$ are both zero so that $L(R) = 0$ in this case. If $z_0 \in D_m$, then we have $z_0 \notin D_n$ for every $n \neq m$. In this case, we know that $\operatorname{Ind}_\Gamma(z_0) = \operatorname{Ind}_{\gamma_m}(z_0) = 1$ and $\operatorname{Ind}_{\gamma_n}(z_0) = 0$ for every $n \neq m$. Consequently, the integrals cancel for the principal part of R at the pole z_0 which gives $L(R) = 0$ when we express R in its partial fraction decomposition.

[a]This shows that Mergelyan's Theorem does not hold anymore if the finiteness of the components of $S^2 \setminus K$ is dropped.

- **There exists an $f \in C(K)$ for which $L(f) \neq 0$.** We take $f(z) = \overline{z}$ which belongs to $C(K)$. Obviously, we have

$$\int_\Gamma \overline{z}\, \mathrm{d}z = 1 \quad \text{and} \quad \int_{\gamma_n} \overline{z}\, \mathrm{d}z = 2\pi i r_n^2$$

for every $n \in \mathbb{N}$. Therefore, we obtain

$$L(f) = 2\pi i\Big(1 - \sum_{n=1}^{\infty} r_n^2\Big) \neq 0$$

by the hypothesis (20.4).

This completes the proof of the problem. ■

Problem 20.3

Rudin Chapter 20 Exercise 3.

Proof. Suppose that $E \subseteq D(0; r)$ is compact and connected, where $r > 0$. Let $\operatorname{diam} E \geq r$ and $\Omega = S^2 \setminus E$ be connected. Denote $X = \{f \in H(\Omega) \,|\, zf(z) \to 1 \text{ as } z \to \infty\}$. Now we recall the definitions of the conformal mappings $F : U \to \Omega$ and $g : U \to D(0; |a|^{-1})$ that

$$F(\omega) = \frac{a}{\omega} + \sum_{n=0}^{\infty} c_n \omega^n \quad \text{and} \quad g(z) = \frac{1}{a} F^{-1}(z), \tag{20.5}$$

where $\omega \in U$ and $z \in \Omega$. Without loss of generality, we may assume that $a > 0$. Assume that there was an $f \in X$ such that

$$\|g\|_\infty > \|f\|_\infty. \tag{20.6}$$

Since F^{-1} is a conformal mapping of Ω onto U, we have $\|F^{-1}\|_\infty = 1$ and thus the definition (20.5) gives

$$\|g\|_\infty = a^{-1}. \tag{20.7}$$

These two facts (20.6) and (20.7) combine to give $f(\Omega) \subseteq D(0; a^{-1})$. Next, we define the mapping $\varphi : U \to U$ by

$$\varphi(\omega) = af\big(F(\omega)\big).$$

Then it is easily checked that $\varphi \in H^\infty$ and $\varphi(0) = af\big(F(0)\big) = af(\infty) = 0$. Besides, we observe that

$$\varphi'(\omega) = af'\big(F(\omega)\big) \cdot F'(\omega).$$

By the definition (20.5), we have

$$F'(\omega) = -\frac{a}{\omega^2} + \sum_{n=1}^{\infty} n c_n \omega^{n-1}.$$

Since $zf(z) \to 1$ as $z \to \infty$, f has the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{a_{-n}}{z^n}$$

so that

$$f'(z) = \frac{-1}{z^2} + \sum_{n=0}^{\infty} \frac{-na_{-n}}{z^{n+1}}.$$

By the definition (20.5), we get

$$\varphi'(0) = \lim_{\omega\to 0} af'\big(F(\omega)\big) \cdot F'(\omega) = 1.$$

Hence it follows from Theorem 12.2 (The Schwarz Lemma) that $\varphi(\omega) = \lambda\omega$ for some constant λ with $|\lambda| = 1$ so that $af\big(F(\omega)\big) = \lambda\omega$. Substituting $\omega = F^{-1}(z)$ into this equation, we obtain

$$f(z) = \lambda\frac{F^{-1}(z)}{a} = \lambda g(z)$$

which implies that $\|f\|_\infty = \|g\|_\infty$, a contradiction to the inequality (20.6).

Put $\omega = F^{-1}(z)$. Then the definitions (20.5) imply

$$z = F\big(F^{-1}(z)\big) = \frac{a}{F^{-1}(z)} + c_0 + \sum_{n=1}^{\infty} c_n\big[F^{-1}(z)\big]^n = \frac{1}{g(z)} + c_0 + \sum_{n=1}^{\infty} c_n a^n g^n(z).$$

Rewrite it as

$$zg(z) = 1 + c_0 g(z) + \sum_{n=1}^{\infty} c_n a^n g^{n+1}(z). \tag{20.8}$$

Since

$$b = \frac{1}{2\pi i}\int_\Gamma zg(z)\,\mathrm{d}z,$$

where Γ is the positively oriented circle with center 0 and radius r, we may substitute the formula (20.8) into the integral to get

$$b = \frac{1}{2\pi i}\int_\Gamma \Big[1 + c_0 g(z) + \sum_{n=1}^{\infty} c_n a^n g^{n+1}(z)\Big]\,\mathrm{d}z.$$

Since $g \in X$, g has a simple zero at ∞ which shows that $\mathrm{Res}\,(g;\infty) = 1$ and $\mathrm{Res}\,\big(g^{n+1}(z);\infty\big) = 0$ for all $n \geq 1$. Hence we conclude from Theorem 10.42 (The Residue Theorem) that

$$b = \frac{c_0}{2\pi i}\int_\Gamma g(z)\,\mathrm{d}z + \sum_{n=1}^{\infty} c_n a^n \cdot \Big[\frac{1}{2\pi i}\int_\Gamma g^{n+1}(z)\,\mathrm{d}z\Big] = c_0$$

as desired. This proves the second assertion.

To prove the third assertion, we notice that since $F(0) = \infty$, we observe that F maps $C_R = \{\omega \mid |\omega| = R\}$ into the disk $D(0;r)$ for some $R < 1$ and sufficiently close to 1. Therefore, we obtain

$$|b| = \Big|\frac{1}{2\pi i}\int_{C_R} \frac{F(\omega)}{\omega}\,\mathrm{d}\omega\Big| < \frac{1}{2\pi} \times \frac{2\pi Rr}{R} = r$$

as desired, completing the proof of the problem. ■

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