Modelling with differential equations; higher order linear equations and boundary-value problems

MA221, Lecture 7

Reminder: Exam 1 is W, 9/25

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Proportionality

The equation

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = kA}$$

or, equivalently, $\frac{\mathrm{d}A}{\mathrm{d}t} \propto A$, is used to model scenarios in which the rate of change of a quantity A is *proportional* to the quantity A itself.

If the **constant of proportionality** k is positive, this equation models *growth*; if k is negative, it models *decay*.

This equation, and variations of it, can be used to model

- population change,
- radioactive decay,
- cooling/warming,
- spread of disease,
- and lots of other things.

Proportionality examples

Example 1: Let $P(t)$ denote the population (e.g., of a country) at time t. If $k > 0$, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ describes a scenario in which the population grows more quickly if the population is larger.

On the other hand, if $k < 0$, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ describes a scenario in which the population decays more quickly if the population is larger.

While this model is very simple and won't fully capture all information in real-world situation, it's a good starting point.

Proportionality examples

Example 2: Assume that in the absence of immigration, the growth rate of a country's population $P(t)$ satisfies $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ for $k > 0$. Write down a differential equation governing the growing population $P(t)$ when individuals are allowed to immigrate at a constant rate $r > 0$.

$$P = P_{NAT} + P_{IM} \Rightarrow \frac{dP}{dt} = \underbrace{\frac{dP_{NAT}}{dt}}_{=kP} + \underbrace{\frac{dP_{IM}}{dt}}_{r}$$

$$\boxed{\frac{dP}{dt} = kP + r}$$

Proportionality examples

Example 3: The model $\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP$ for $k > 0$ for the population of a country does not take into account the deaths. Assuming that birth rates and death rates are both proportional to population (with constants k_b and k_d, respectively), write down a differential equation to model the population.

$\frac{dP}{dt}$ is a "net rate": rate in − rate out

$$\frac{dP}{dt} = k_b P - k_d P$$
$$= (k_b - k_d) P$$

Proportionality examples

Example 4: In a given community, let $x(t)$ denote the number of people who have contracted a contagious disease and $y(t)$ denote the number of people who have not yet been exposed. It seems reasonable to assume that the rate $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ at which the disease spreads is proportional to the number of encounters, or interactions, between these two groups of people.

$\dfrac{dx}{dt}$ is "jointly proportional" to $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = k\,\underline{xy} \quad \text{or} \quad \frac{dx}{dt} = k\,x(t)\,y(t)$$

why xy? Suppose population $= 8$, #infected is 3.

$$\underbrace{xy}_{\text{upper bound for \# interactions.}} = 3 \cdot 5 = [\cdot\ \cdot\ \cdot][\cdot\ \cdot\ \cdot\ \cdot\ \cdot]$$

Proportionality examples

Example 5: Suppose that a large mixing tank initially holds 200 gallons of water in which 70 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 2 gallons per minute. When the solution is well-stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $A(t)$ for any time $t > 0$. What is $A(0)$?

$$\frac{dA}{dt} = \underbrace{\text{rate in}}_{=0} - \text{rate out}$$

70

$$= 0 - \frac{2 \text{ gal}}{1 \text{ min}} \cdot \frac{A(t) \text{ lbs.}}{200 \text{ gal}}$$

$$= -\frac{A(t)}{100}\left(\frac{\text{lbs}}{\text{min}}\right) \Rightarrow \frac{dA}{dt} = -\frac{1}{100}A$$

Higher order linear equations

Consider the equation

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x).$$

Note that this is equivalent to a linear first order equation (divide by $a_1(x)$ on both sides). Thus, one may define a LFO equation to be an equation of this form.

A second order linear equation is an equation of the form

$$a_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x).$$

More generally, an n-th order linear equation is an equation of the form

$$a_n(x)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + \ldots + a_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x).$$

Initial and boundary value problems for linear equations

A typical first order initial value problem may look like:

$$\text{Solve} \quad \frac{\mathrm{d}y}{\mathrm{d}x} + x^2 y = 0 \quad \text{subject to} \quad y(0) = 0$$

A standard second-order initial value problem may look like:

$$\text{Solve} \quad y'' + y = 0 \quad \text{subject to} \quad y'\left(\frac{\pi}{2}\right) = 1 \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = 1$$

- each IV conditions specifies the same point (e.g., $\pi/2$)
- one IV condition for each derivative of order < 2 (e.g., y, y')

By contrast, a **boundary value problem** may look like:

$$\text{Solve} \quad y'' + y = 0 \quad \text{subject to} \quad y'(0) = 1 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = 1$$

- BV conditions specify different points (e.g., $0, \pi/2$)
- not necessarily one BV condition per derivative

Initial and boundary value problems for linear equations

Fact: (*Theorem 4.1.1 in Zill*, for second order equations) Suppose that in the second order linear equation

$$a_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x),$$

the functions a_2, a_1, a_0, and g are continuous over the interval I, and that a_n is non-zero on I. Then for any x_0 in I, the IVP with constraints $y(x_0) = y_0$ and $y'(x_0) = y_1$ has a unique solution. $\Rightarrow$ Under mild conditions, IVPs have unique solutions

On the other hand, a given BVP may have zero, one, or many solutions. For example, take the differential equation

$$y'' - 2y' + 2y = 0.$$

This equation has a general solution of the form

$$y = Ae^x \cos x + Be^x \sin x.$$

Initial and boundary value problems for linear equations

$$y'' - 2y' + 2y = 0; y = Ae^x \cos x + Be^x \sin x$$

Depending on the boundary conditions, the number of solutions may vary:

- $y(0) = 1$ and $y(\pi) = -1$

$1 = y(0) = Ae^0 \cos(0) + Be^0 \sin(0) = A$

$-1 = y(\pi) = Ae^\pi \cos(\pi) + Be^\pi \sin(\pi) = A \cdot e^\pi(-1) \Rightarrow A = e^{-\pi}$

no solutions: $1 \neq e^{-\pi}$

- $y(0) = 1$ and $y'(\pi) = 0$

See below

- $y(0) = 0$ and $y(\pi) = 0$

$0 = y(0) = A \Rightarrow A = 0$

$0 = y(\pi) = A \cdot e^\pi(-1) \Rightarrow A = 0$

$y = Be^x \sin x$

infinitely many solutions

$1 = y(0) = A \Rightarrow A = 1.$ On the other hand,

$$y'(x) = [Ae^x \cos x + Be^x \sin x]'$$

$$= A[e^x \cos x - e^x \sin x]$$

$$+ B[e^x \sin x + e^x \cos x]$$

$$= (A+B)e^x \cos x + (B-A)e^x \sin x,$$

so ...

$$0 = y'(\pi) = (A+B)e^{\pi} \cos(\pi) + (B-A)e^{\pi} \sin \pi$$

$$= (A+B)e^{\pi}(-1)$$

$$\Rightarrow A + B = 0 \Rightarrow B = -A = -1.$$

solution: $\boxed{y = e^x \cos x - e^x \sin x}$

ONE SOLUTION!