1.1 Definitions and Terminology

Introduction The derivative dy/dx of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. The exponential function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$ and by the Chain Rule its first derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ on the right-hand side of the last equation by the symbol y, the derivative becomes

$$\frac{dy}{dx} = 0.2xy. \qquad (1)$$

Now imagine that a friend of yours simply hands you equation (1)—you have no idea how it was constructed—and asks, *What is the function represented by the symbol y*? You are now face to face with one of the basic problems in this course:

How do you solve an equation such as (1) for the function $y = \phi(x)$?

A Definition

The equation that we made up in (1) is called a **differential equation**. Before proceeding any further, let us consider a more precise definition of this concept.

Definition 1.1.1

Differential Equation

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE)**.

To talk about them, we shall classify differential equations according to **type, order,** and **linearity**.

Classification By Type

If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a *single* independent variable, it is said to be an **ordinary differential equation (ODE)**. An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a **partial differential equation**

(PDE). Our first example illustrates several of each type of differential equation.

Example 1

Types of Differential Equations

(a) The equations

an ODE can contain more than one unknown function

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \qquad (2)$$

are examples of ordinary differential equations.

(b) The following equations are partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad (3)$$

Notice in the third equation that there are two unknown functions and two independent variables in the PDE. This means u and v must be functions of *two or more* independent variables.

Notation

Throughout this text ordinary derivatives will be written by using either the **Leibniz notation** $dy/dx, d^2y/dx^2, d^3y/dx^3, \ldots$ or the **prime notation** $y', y'', y''', \ldots$. By using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 5y = e^x$ and $y'' - y' + 6y = 0$. Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''. In general, the nth derivative of y is written $d^n y/dx^n$ or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the equation

unknown function or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

it is immediately seen that the symbol x now represents a dependent variable, whereas the independent variable is t. You should also be aware that in physical sciences and engineering, Newton's **dot notation** (derogatorily referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time t. Thus the differential equation $d^2 s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation in (3) becomes $u_{xx} = u_{tt} - 2u_t$.

Classification By Order

The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

$$\overset{\text{second order}}{\frac{d^2y}{dx^2}} + 5\overset{\text{first order}}{\left(\frac{dy}{dx}\right)^3} - 4y = e^x$$

is a second-order ordinary differential equation. In Example 1, the first and third equations in (2) are first-order ODEs, whereas in (3) the first two equations are second-order PDEs. A first-order ordinary differential equation is sometimes written in the **differential form**

$$M(x, y)\, dx + N(x, y)\, dy = 0.$$

Example 2

Differential Form of a First-Order ODE

If we assume that y is the dependent variable in a first-order ODE, then recall from calculus that the differential dy is defined to be $dy = y'\,dx$.

(a) By dividing by the differential dx an alternative form of the equation $(y - x)\, dx + 4x\, dy = 0$ is given by

$$y - x + 4x\frac{dy}{dx} = 0$$

or equivalently

$$4x\frac{dy}{dx} + y = x.$$

(b) By multiplying the differential equation

$$6xy\,\frac{dy}{dx} + x^2 + y^2 = 0$$

by dx we see that the equation has the alternative differential form

$$\left(x^2+y^2\right)\,dx+6xy\,dy=0.$$

In symbols we can express an $n\text{th}$-order ordinary differential equation in one dependent variable by the general form

$$F(x,y,y',\ldots,y^{(n)})=0, \qquad (4)$$

where F is a real-valued function of $n+2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n+1$ variables. The differential equation

$$\frac{d^n y}{dx^n}=f(x,y,y',\ldots,y^{(n-1)}), \qquad (5)$$

where f is a real-valued continuous function, is referred to as the **normal form** of (4). Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx}=f(x,y)$$

and

$$\frac{d^2y}{dx^2}=f(x,y,y')$$

to represent general first- and second-order ordinary differential equations.

Example 3

Normal Form of an ODE

(a) By solving for the derivative dy/dx the normal form of the first-order differential equation

$$4x\frac{dy}{dx}+y=x$$

is

$$\frac{dy}{dx}=\frac{x-y}{4x}.$$

(b) By solving for the derivative y'' the normal form of the second-order differential equation

$$y'' - y' + 6y = 0$$

is

$$y'' = y' - 6y.$$

Classification By Linearity

An $n\text{th}$-order ordinary differential equation (4) is said to be **linear** if F is linear in $y, y', \ldots, y^{(n)}$. This means that an $n\text{th}$-order ODE is linear when (4) is
$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad (6)$$

Two important special cases of (6) are linear first-order $(n = 1)$ and linear second-order $(n = 2)$ DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

In the additive combination on the left-hand side of equation (6) we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation.

Example 4

Linear and Nonlinear ODEs

(a) The equations

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y + y = 0, \quad x^3\frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated in part (a) of Example 2 that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

$$\underbrace{(1-y)}_{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}}y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underbrace{\sin y}_{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}} = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + \underbrace{y^2}_{\substack{\text{nonlinear term:}\\ \text{power not 1}}} = 0$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

Solutions

As was stated previously, one of the goals in this course is to solve, or find solutions of, differential equations. In the next definition we consider the concept of a solution of an ordinary differential equation.

Definition 1.1.2

Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an $n\text{th}$-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an $n\text{th}$-order ordinary differential equation (4) is a function ϕ that possesses at least n derivatives and for which

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

We say that ϕ *satisfies* the differential equation on I. For our purposes we shall also assume that a solution ϕ is a real-valued function. In our introductory discussion we saw that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally, it will be convenient to denote a solution by the alternative symbol $y(x)$.

Interval of Definition

You cannot think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval I in Definition 1.1.2 is variously called the **interval of definition,** the **interval of existence,** the **interval of validity,** or the **domain of the solution** and can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

Example 5

Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}$; $y = \dfrac{1}{16}x^4$

(b) $y'' - 2y' + y = 0$; $y = xe^x$

Solution

One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x in the interval.

(a) From

left-hand side:

$$\frac{dy}{dx} = \frac{1}{16}(4 \cdot x^3) = \frac{1}{4}x^3,$$

right-hand side:

$$xy^{1/2} = x \cdot \left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3,$$

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \dfrac{1}{4}x^2$ is, by definition, the nonnegative square root of $\dfrac{1}{16}x^4$.

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number x,

left-hand side:

$$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$$

right-hand side:

0.

Note, too, that each differential equation in Example 5 possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution**.

Solution Curve

The graph of a solution ϕ of an ODE is called a **solution curve**. Since ϕ is a differentiable function, it is continuous on its interval I of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. Put another way, the domain of the function ϕ need not be the same as the interval I of definition (or domain) of the solution ϕ. Example 6 illustrates the difference.

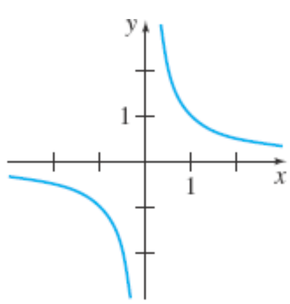
Example 6

Function versus Solution

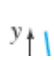
(a) The domain of $y = 1/x$, considered simply as a *function*, is the set of all real numbers x except 0. When we graph $y = 1/x$, we plot points in the xy-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in Figure 1.1.1(a). The function $y = 1/x$ is not differentiable at $x = 0$, since the y-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

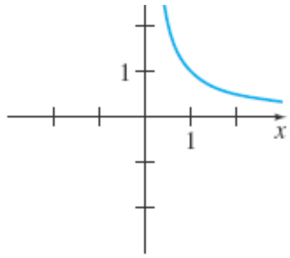
Figure 1.1.1

In Example 6 the function $y = 1/x$ is not the same as the solution $y = 1/x$



(a) function $y = 1/x$, $x \neq 0$





(b) solution $y = 1/x$, $(0, \infty)$

(b) Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$. (Verify.) But when we say that $y = 1/x$ is a *solution* of this DE, we mean that it is a function defined on an interval I on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval that does not contain 0, such as $(-3, -1)$, $\left(\frac{1}{2}, 10\right)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ for $-3 < x < -1$ and $\frac{1}{2} < x < 10$ are simply segments, or pieces, of the solution curves defined by $y = 1/x$ for $-\infty < x < 0$ and $0 < x < \infty$, respectively, it makes sense to take the interval I to be as large as possible. Thus we take I to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on $(0, \infty)$ is shown in Figure 1.1.1(b).

Explicit and Implicit Solutions

You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**. For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution ϕ

implicitly.

Definition 1.1.3

Implicit Solution of an ODE

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval I, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

It is beyond the scope of this course to investigate the conditions under which a relation $G(x, y) = 0$ defines a differentiable function ϕ. So we shall assume that if the formal implementation of a method of solution leads to a relation $G(x, y) = 0$, then there exists at least one function ϕ that satisfies both the relation (that is, $G(x, \phi(x)) = 0$) and the differential equation on an interval I. If the implicit solution $G(x, y) = 0$ is fairly simple, we may be able to solve for y in terms of x and obtain one or more explicit solutions. See (iv) in the Remarks.

Example 7

Verification of an Implicit Solution

The relation $x^2 + y^2 = 25$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \qquad (8)$$

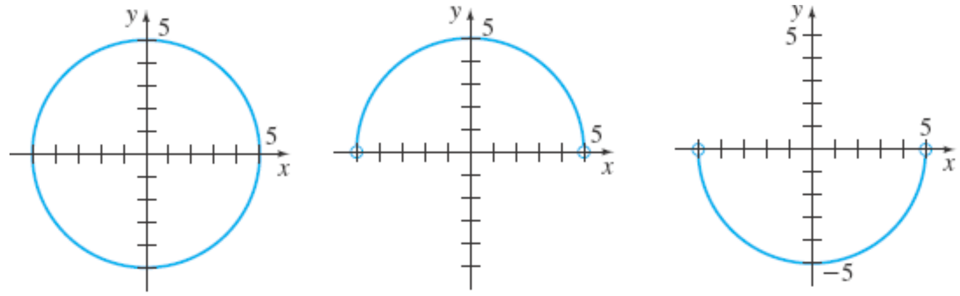
on the open interval $(-5, 5)$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \quad \text{or} \quad 2x + 2y\frac{dy}{dx} = 0. \qquad (9)$$

Solving the last equation in (9) for the symbol dy/dx gives (8). Moreover, solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $(-5, 5)$. The solution curves given in Figures 1.1.2(b) and 1.1.2(c) are segments of the graph of the implicit solution in Figure 1.1.2(a).

Figure 1.1.2

An implicit solution and two explicit solutions of (8) in Example 7



(a) implicit solution

$x^2 + y^2 = 25$

(b) explicit solution

$y_1 = \sqrt{25 - x^2}, -5 < x < 5$

(c) explicit solution

$y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

Because the distinction between an explicit solution and an implicit solution should be intuitively clear, we will not belabor the issue by always saying, "Here is an explicit (implicit) solution."

Families of Solutions

The study of differential equations is similar to that of integral calculus. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration. Analogously, we shall see in Chapter 2 that when solving a first-order differential equation $F(x, y, y') = 0$ we *usually* obtain a solution containing a single constant or parameter c. A solution of $F(x, y, y') = 0$ containing a constant c is a set of solutions $G(x, y, c) = 0$ called a **one-parameter family of solutions**. When solving an $n\mathrm{th}$-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameter(s). A solution of a differential equation that is free of parameters is called a **particular solution**.

The parameters in a family of solutions such as $G(x, y, c_1, c_2, \ldots, c_n) = 0$ are *arbitrary* up to a point. For example, proceeding as in (9) a relation $x^2 + y^2 = c$ formally satisfies (8) for any constant c. However, it is understood that the relation should always make sense in the real number system; thus, if $c = -25$ we cannot say that $x^2 + y^2 = -25$ is an implicit solution of the differential equation.

Example 8

Particular Solutions

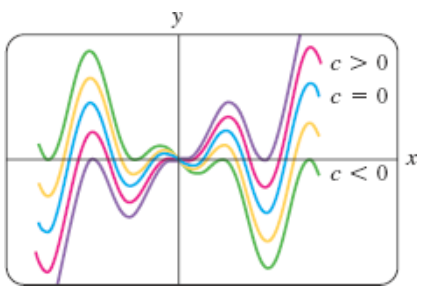
(a) For all real values of c, the one-parameter family $y = cx - x\cos x$ is an explicit solution of the linear first-order equation

$$xy' - y = x^2 \sin x$$

on the interval $(-\infty, \infty)$. (Verify.) Figure 1.1.3 shows the graphs of some particular solutions in this family for various choices of c. The solution $y = -x \cos x$, the blue graph in the figure, is a particular solution corresponding to $c = 0$.

Figure 1.1.3

Some solutions of DE in part (a) of Example 8



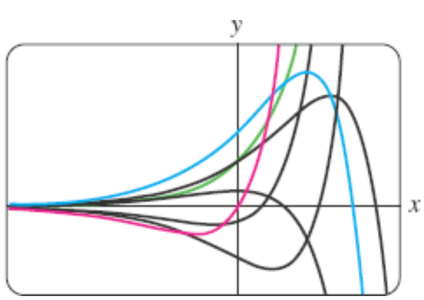
(b) The two-parameter family $y = c_1 e^x + c_2 x e^x$ is an explicit solution of the linear second-order equation

$$y'' - 2y' + y = 0$$

in part (b) of Example 5. (Verify.) In Figure 1.1.4 we have shown seven of the "double infinity" of solutions in the family. The solution curves in red, green, and blue are the graphs of the particular solutions $y = 5xe^x\,(c_1 = 0,\, c_2 = 5)$, $y = 3e^x\,(c_1 = 3,\, c_2 = 0)$, and $y = 5e^x - 2xe^x\,(c_1 = 5,\, c_2 = 2)$, respectively.

Figure 1.1.4

Some solutions of DE in part (b) of Example 8



Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the equation—that is, a solution that cannot be obtained by specializing *any* of the parameters in the family of solutions. Such an extra solution is called a **singular solution**. For example, we have seen that $y = \frac{1}{16}x^4$ and $y = 0$ are solutions of the differential equation $dy/dx = xy^{1/2}$ on $(-\infty, \infty)$. In Section 2.2 we shall demonstrate, by actually solving it, that the differential equation $dy/dx = xy^{1/2}$ possesses the one-parameter family of solutions $y = \left(\frac{1}{4}x^2 + c\right)^2$, $c \geq 0$. When $c = 0$, the resulting particular solution is $y = \frac{1}{16}x^4$. But notice that the trivial solution $y = 0$ is a singular solution since it is not a member of the family $y = \left(\frac{1}{4}x^2 + c\right)^2$; there is no way of assigning a value to the constant c to obtain $y = 0$.

In all the preceding examples we used x and y to denote the independent and dependent variables, respectively. But you should become accustomed to seeing and working with other symbols to denote these variables. For example, we could denote the independent variable by t and the dependent variable by x.

Example 9

Using Different Symbols

The functions $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where c_1 and c_2 are arbitrary constants or parameters, are both solutions of the linear differential equation

$$x'' + 16x = 0.$$

For $x = c_1 \cos 4t$ the first two derivatives with respect to t are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1 \cos 4t$. Substituting x'' and x then gives

$$x'' + 16x = -16c_1 \cos 4t + 16(c_1 \cos 4t) = 0.$$

In like manner, for $x = c_2 \sin 4t$ we have $x'' = -16c_2 \sin 4t$, and so

$$x'' + 16x = -16c_2 \sin 4t + 16(c_2 \sin 4t) = 0.$$

Finally, it is straightforward to verify that the linear combination of solutions, or the two-parameter family $x = c_1 \cos 4t + c_2 \sin 4t$, is also a solution of the differential equation.

The next example shows that a solution of a differential equation can be a piecewise-defined function.

Example 10

Piecewise-Defined Solution

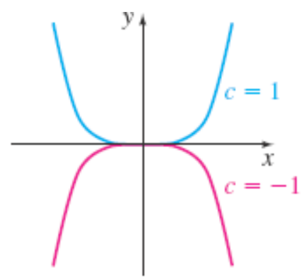
The one-parameter family of quartic monomial functions $y = cx^4$ is an explicit solution of the linear first-order equation

$$xy' - 4y = 0$$

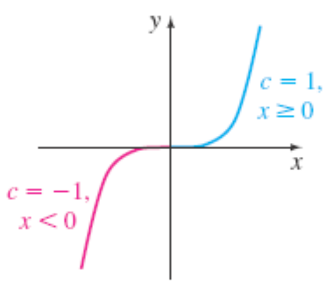
on the interval $(-\infty, \infty)$. (Verify.) The blue and red solution curves shown in Figure 1.1.5(a) are the graphs of $y = x^4$ and $y = -x^4$ and correspond to the choices $c = 1$ and $c = -1$, respectively.

Figure 1.1.5

Some solutions of DE in Example 10



(a) two explicit solutions



(b) piecewise-defined solution

The piecewise-defined differentiable function

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x > 0 \end{cases}$$

is also a solution of the differential equation but cannot be obtained from the family $y = cx^4$ by a single choice of c. As seen in Figure 1.1.5(b) the solution is constructed from the family by choosing $c = -1$ for $x < 0$ and $c = 1$ for $x \geq 0$.

Systems of Differential Equations

Up to this point we have been discussing single differential equations containing one unknown function. But often in theory, as well as in many applications, we must deal with systems of differential equations. A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if x and y denote dependent variables and t denotes the independent variable, then a system of two first-order differential equations is given by

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, y) \\ \frac{dy}{dt} &= g(t, x, y). \end{aligned} \qquad (10)$$

A **solution** of a system such as (10) is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, defined on a common interval I, that satisfy each equation of the system on this interval.

Remarks

i. It might not be apparent whether a first-order ODE written in differential form $M(x, y)\, dx + N(x, y)\, dy = 0$ is linear or nonlinear because there is nothing in this form that tells us which symbol denotes the dependent variable. See Problems 9 and 10 in Exercises 1.1.

ii. We will see in the chapters that follow that a solution of a differential equation may involve an **integral-defined function**. One way of defining a function F of a single variable x by means of a definite integral is:

$$F(x) = \int_a^x g(t)\, dt. \qquad (11)$$

If the integrand g in (11) is continuous on an interval $[a, b]$ and $a \le x \le b$, then the derivative form of the Fundamental Theorem of Calculus states that F is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x g(t)\; dt = g(x) \qquad (12)$$

The integral in (11) is often **nonelementary,** that is, an integral of a function g that does not have an elementary-function antiderivative. Elementary functions include the familiar functions studied in a typical precalculus course:

constant, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions,

as well as rational powers of these functions; finite combinations of these functions using addition, subtraction, multiplication, division; and function compositions. For example, even though e^{-t^2}, $\sqrt{1+t^3}$, and $\cos t^2$ are elementary functions, the integrals $\int e^{-t^2}\,dt$, $\int \sqrt{1+t^3}\,dt$, and $\int \cos t^2\,dt$ are nonelementary. See Problems 25, 26, 27, and 28 in Exercises 1.1. Also see Appendix A.

iii. Although the concept of a solution of a differential equation has been emphasized in this section, you should be aware that a DE does not necessarily have to possess a solution. See Problem 43 in Exercises 1.1. The question of whether a solution exists will be touched on in the next section.

iv. A few last words about implicit solutions of differential equations are in order. In Example 7 we were able to solve the relation $x^2+y^2=25$ for y in terms of x to get two explicit solutions, $\phi_1(x)=\sqrt{25-x^2}$ and $\phi_2(x)=-\sqrt{25-x^2}$, of the differential equation (8). But don't read too much into this one example. Unless it is easy or important or you are instructed to, there is usually no need to try to solve an implicit solution $G(x,y)=0$ for y explicitly in terms of x. Also do not misinterpret the second sentence following Definition 1.1.3. An implicit solution $G(x,y)=0$ can define a perfectly good differentiable function ϕ that is a solution of a DE, yet we might not be able to solve $G(x,y)=0$ using analytical methods such as algebra. The solution curve of ϕ may be a segment or piece of the graph of $G(x,y)=0$. See Problems 49 and 50 in Exercises 1.1. Also, read the discussion following Example 4 in Section 2.2.

v. It might not seem like a big deal to assume that $F(x,y,y',\ldots,y^{(n)})=0$ can be solved for $y^{(n)}$, but one should be a little bit careful here. There are exceptions, and there certainly are some problems connected with this assumption. See Problems 56 and 57 in Exercises 1.1.

vi. If *every* solution of an $n\mathrm{th}$-order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, \ldots, c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, \ldots, n$, we then say that the family is the **general solution** of the DE. In solving linear ODEs, we shall impose relatively simple restrictions on the coefficients of the equation; with these restrictions one can be assured that not only does a solution exist on an interval but also that a family of solutions yields all possible solutions. Nonlinear ODEs, with the exception of some first-order equations, are usually difficult or impossible to solve in terms of elementary functions. Furthermore, if we happen to obtain a family of solutions for a nonlinear equation, it is not obvious whether this family contains all solutions. On a practical level, then, the designation “general solution” is applied only to linear ODEs. Don’t be concerned about this concept at this point, but store the words “general solution” in the back of your mind—we will come back to this notion in Section 2.3 and again in Chapter 4.