Homework 3

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Question 1

Consider the power series

$$\sum_{n \geqslant 1} \frac{1}{n} z^n = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$$

a)

Does the series converge for $z = -i$? (Note: changed from $z = i$ in first version)

Sol:

To determine if the series $\sum_{n \geqslant 1} \frac{1}{n} z^n$ converges for $z = -i$, we can use the ratio test. The ratio test states that if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series converges absolutely.

Let's apply the ratio test to our series:

$$\lim_{n \to \infty} \left| \frac{\frac{1}{n+1}(-i)^{n+1}}{\frac{1}{n}(-i)^n} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1}(-i) \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} \right| = 1$$

Since the limit is 1, the ratio test is inconclusive. Therefore, we cannot determine if the series converges absolutely for $z = -i$ using the ratio test.

To determine if the series converges conditionally, we can use the alternating series test. The alternating series test states that if a series $\sum_{n \geqslant 1} (-1)^n b_n$ satisfies the conditions: (1) $b_n > 0$ for all n, (2) $b_{n+1} \leqslant b_n$ for all n, and (3) $\lim_{n \to \infty} b_n = 0$, then the series converges.

In our case, $b_n = \frac{1}{n}(-i)^n$. We can see that $b_n > 0$ for all n, and $b_{n+1} \leqslant b_n$ since $\frac{1}{n+1} \leqslant \frac{1}{n}$. Additionally, $\lim_{n \to \infty} \frac{1}{n} = 0$. Therefore, the series converges conditionally for $z = -i$.

b)

Does the series converge for $z = \frac{1}{2}(-1 + i\sqrt{3})$?

To determine whether the series converges for $z = \frac{1}{2}(-1 + i\sqrt{3})$, we can again use the ratio test.

The ratio test states that if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series converges absolutely.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n}\left(\frac{1}{2}(-1+i\sqrt{3})\right)^{n+1}}{\frac{1}{n}\left(\frac{1}{2}(-1+i\sqrt{3})\right)^{n}}\right|$$
$$= \left|\frac{\frac{1}{n+1}\left(\frac{-1+i\sqrt{3}}{2}\right)^{n+1}}{\frac{1}{n}\left(\frac{-1+i\sqrt{3}}{2}\right)^{n}}\right|$$
$$= \left|\frac{\frac{1}{n+1}\left(-1+i\sqrt{3}\right)}{\frac{1}{n}\cdot 2}\right|$$
$$= \left|\frac{\left(-1+\sqrt{3}i\right)n}{(n+1)\cdot 2}\right|$$

Given the sequence:

$$\frac{(-1+i\sqrt{3})n}{2(n+1)}$$

we need to find:

$$\lim_{n\to\infty} a_n$$

Separate the sequence into its real and imaginary parts:

Real part:

$$\mathrm{Re}\left(\frac{(-1+i\sqrt{3})n}{2(n+1)}\right) = \frac{-n}{2(n+1)}$$

Imaginary part:

$$\mathrm{Im}\left(\frac{(-1+i\sqrt{3})n}{2(n+1)}\right) = \frac{\sqrt{3}n}{2(n+1)}$$

Now, let's find the limit of each part as $(n \to \infty)$:

For the real part:

$$\lim_{n\to\infty} \frac{-n}{2(n+1)} = \lim_{n\to\infty} \frac{-1}{2+\frac{2}{n}}$$

As $(n \to \infty)$, the term $(\frac{2}{n})$ approaches 0, hence:

$$\lim_{n\to\infty} \frac{-n}{2(n+1)} = \frac{-1}{2}$$

For the imaginary part:

$$\lim_{n\to\infty} \frac{\sqrt{3}n}{2(n+1)} = \lim_{n\to\infty} \frac{\sqrt{3}}{2+\frac{2}{n}}$$

Similarly, as $(n \to \infty)$, the term $(\frac{2}{n})$ approaches 0, so:

$$\lim_{n\to\infty} \frac{\sqrt{3}n}{2(n+1)} = \frac{\sqrt{3}}{2}$$

Thus, the final result is a value less than 1, the series converges for $z = \frac{1}{2}(-1+i\sqrt{3})$

Question 2

We are given a power series $\sum_{n \geqslant 0} a_n(z-2)^n$ that is convergent for $z=-1$. What can we conclude about the series $$ \sum*{n \geqslant 2023} a*{n}(-2-2 i)^{n} ?

$Isitabsolutelyconvergent, isitdivergent, orcouldbeeither, dependingontheseries? * * * * * Sol : * * Todeterminetheconvergenceofthese$

Since the series $\sum_{n \geqslant 0} a_n(z-2)^n$ is convergent for $z=-1$, we know that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ for $z=-1$. Therefore, we have:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}(-2-2i)^{n+1}}{a_n(-2-2i)^n}\right| &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \cdot 2\sqrt{2} \\
&< 1 \cdot 2\sqrt{2} \\
&= 2\sqrt{2} \\
&< 1
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}(-2-2i)^{n+1}}{a_n(-2-2i)^n}\right| < 1$, we can conclude that the series $\sum_{n \geqslant 2023} a_n(-2-2i)^n$ is absolutely convergent.

Additional method: Given that the power series $\sum_{n \geqslant 0} a_n(z-2)^n$ is convergent for $z=-1$, we know that the series $\sum_{n \geqslant 0} a_n(-3)^n$ converges.

To analyze the convergence of the series $\sum_{n \geqslant 2023} a_n(-2-2i)^n$, we'll use the comparison test.

First, let's find the magnitude of $(-2-2i)^n$:

$$\begin{aligned}
(-2-2i)^n &= ((-2)(1+i))^n \\
&= (-2)^n(1+i)^n
\end{aligned}$$

The magnitude of $1+i$ is $\sqrt{2}$. So, the magnitude of $(1+i)^n$ is $(\sqrt{2})^n = 2^{n/2}$. Hence, the magnitude of $(-2-2i)^n$ is:

$$\begin{aligned}
2^n \cdot 2^{n/2} &= 2^{3n/2} \\
&= 2^{3n/2}
\end{aligned}$$

Thus, the magnitude of the general term for $\sum_{n \geqslant 2023} a_n(-2-2i)^n$ is: $|a_n| \cdot 2^{3n/2}$

Now, we know that $\sum_{n \geqslant 0} a_n(-3)^n$ converges. Therefore, $\sum_{n \geqslant 0} |a_n| \cdot 3^n$ is absolutely convergent.

Using the comparison test, we can compare the two series: $\sum_{n \geqslant 0} |a_n| \cdot 3^n$ and $\sum_{n \geqslant 2023} |a_n| \cdot 2^{3n/2}$

Since $2^{3n/2} \leq 3^n$ for all $n \geqslant 0$, it means that: $\sum_{n \geqslant 2023} |a_n| \cdot 2^{3n/2} \leq \sum_{n \geqslant 2023} |a_n| \cdot 3^n$

And since the series on the right-hand side is absolutely convergent, the series on the left-hand side is also absolutely convergent.

Therefore, the series $\sum_{n \geqslant 2023} a_n(-2-2i)^n$ is absolutely convergent.

Question 3

a)

Let $z=-1-2i$. Compute $\operatorname{Exp}(z), \log(z), z^i$, and i^z.

$\operatorname{Exp}(z)$

Let $z=-1-2i$. We want to compute $\operatorname{Exp}(z), \log(z), z^i$, and i^z.

To compute $\operatorname{Exp}(z)$, we can use the definition of the exponential function for complex numbers:

$$\operatorname{Exp}(z) = e^z = e^{-1-2i}$$

Using Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we can write

$$e^{-1-2i} = e^{-1}e^{-2i} = e^{-1}(\cos(-2) + i\sin(-2))$$

Simplifying further, we have

$$e^{-1-2i} = e^{-1}(\cos(-2) - i\sin(2))$$

Therefore, $\text{Exp}(z) = e^{-1}(\cos(-2) - i\sin(2))$.

$\log(z)$

To compute $\log(z)$, we can use the principal branch of the logarithm function. The principal branch is defined as $\log(z) = \ln|z| + i\arg(z)$, where $\arg(z)$ is the principal argument of z. In this case, $z = -1 - 2i$, so we have

$$\log(-1-2i) = \ln|-1-2i| + i\arg(-1-2i)$$

The magnitude of $-1-2i$ is $\sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$, and the principal argument can be found using the formula $\arg(z) = \arctan\left(\frac{\Im(z)}{\Re(z)}\right)$. In this case, $\Re(-1-2i) = -1$ and $\Im(-1-2i) = -2$, so we have

$$\arg(-1-2i) = \arctan\left(\frac{-2}{-1}\right) = \arctan(2) - \pi$$

Therefore,

$$\log(-1-2i) = \ln(\sqrt{5}) + i\arctan(2) - \pi$$

z^i

$z^i = \text{Exp}(i\log(z))$ or $z^i = e^{i\log(z)}$ Substituting the value of log(z)log(z) which we computed in the previous step, we get: $z^i = \text{Exp}(\ln(\sqrt{5}) + i\arctan(2))$

In this case, $z = -1 - 2i$ and $\log(z) = \ln(\sqrt{5}) + i\arctan(2)$, so we have

$$(-1-2i)^i = e^{i(\ln(\sqrt{5}) + i\arctan(2))}$$

Using Euler's formula, we can write

$$e^{i(\ln(\sqrt{5}) + i\arctan(2))} = e^{i\ln(\sqrt{5})}e^{-\arctan(2)}$$

Simplifying further, we have

$$e^{i\ln(\sqrt{5})}e^{-\arctan(2)} = (\cos(\ln(\sqrt{5})) + i\sin(\ln(\sqrt{5})))e^{-\arctan(2)}$$

Therefore, $(-1-2i)^i = (\cos(\ln(\sqrt{5})) + i\sin(\ln(\sqrt{5})))e^{-\arctan(2)}$.

i^z

b)

Let $z = -1 + ti$, with $t \in \mathbb{R}$. Compute $\text{Exp}(z), \log(z), z^i$, and i^z as functions of t.

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e^z

To compute $\text{Exp}(z)$, we can use the definition of the exponential function for complex numbers:

$$\text{Exp}(z) = e^z = e^{-1+ti}$$

Using Euler's formula, we can write

$$e^{-1+ti} = e^{-1}e^{ti} = e^{-1}(\cos(t) + i\sin(t))$$

Therefore, $\text{Exp}(z) = e^{-1}(\cos(t) + i\sin(t))$.

$\log(z)$

The complex logarithm is defined as $\log(z) = \ln|z| + i\arg(z)$

Here, $z = -1 + ti$, so $|z| = \sqrt{(-1)^2 + t^2} = \sqrt{1+t^2}$

The argument of z is $\arg(z) = \arctan\left(\frac{t}{-1}\right) = -\arctan(t)$

Therefore, $\log(z) = \ln(\sqrt{1+t^2}) - i\arctan(t)$

To compute $\log(z)$, we can use the principal branch of the logarithm function. The principal branch is defined as $\log(z) = \ln|z| + i\arg(z)$, where $\arg(z)$ is the principal argument of z. In this case, $z = -1 + ti$, so we have

$$\log(-1 + ti) = \ln|-1 + ti| + i\arg(-1 + ti)$$

z^i

We have $z = -1 + ti$, and we want to compute z^i.

First, we express z in polar form. The modulus of z is $|z| = \sqrt{(-1)^2 + t^2} = \sqrt{1+t^2}$, and the argument of z is $\arg(z) = \arctan\left(\frac{t}{-1}\right) = -\arctan(t)$.

So, $z = |z|(\cos(\arg(z)) + i\sin(\arg(z))) = \sqrt{1+t^2}(\cos(-\arctan(t)) + i\sin(-\arctan(t)))$.

Now, we can compute z^i using the formula $z^i = |z|^i(\cos(i\arg(z)) + i\sin(i\arg(z)))$.

Substituting the expressions for $|z|$ and $\arg(z)$, we get $z^i = (\sqrt{1+t^2})^i(\cos(-i\arctan(t)) + i\sin(-i\arctan(t)))$.

Simplifying, we get $z^i = \sqrt{1+t^2}^i \cdot e^{-\arctan(t)}$.

i^z

$i^z = e^{z\log(i)}$

Again, we're using the property of exponents that $a^{bc} = (a^b)^c = (a^c)^b$. We're applying this property with $a = e$, $b = z$, and $c = \log(i)$.

5. $i^z = e^{(-1+ti)\cdot i\frac{\pi}{2}} = e^{-\frac{\pi}{2} - \frac{\pi t}{2}i}$

This is just substituting the expression for z and $\log(i)$ that we computed earlier.

We have $z = -1 + ti$, and we want to compute i^z.

First, we express i in polar form. The modulus of i is $|i| = 1$, and the argument of i is $\arg(i) = \frac{\pi}{2}$.

So, $i = |i|(\cos(\arg(i)) + i\sin(\arg(i))) = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$.

Now, we can compute i^z using the formula $i^z = |i|^z(\cos(z\arg(i)) + i\sin(z\arg(i)))$.

Substituting the expressions for $|i|$ and $\arg(i)$, we get $i^z = 1^z(\cos((-1+ti)\frac{\pi}{2}) + i\sin((-1+ti)\frac{\pi}{2}))$.

Simplifying, we get $i^z = e^{-\frac{\pi}{2} - \frac{\pi t}{2}i}$.

So, the final results are:

$z^i = \sqrt{1+t^2}^i \cdot e^{-\arctan(t)}$

$i^z = e^{-\frac{\pi}{2} - \frac{\pi t}{2}i}$

Question 4

a)

Let $n > 0$ be a positive real number. Compute n^{-2+i}.

Using the exponential function, we can rewrite this as $e^{(-2+i)\ln(n)}$.

From the notes, we know that the exponential of a complex number $x+iy$ is given by $e^x(\cos y + i\sin y)$. Applying this to our problem, we get:

$$n^{-2+i} = e^{(-2+i)\ln(n)} = e^{-2\ln(n)}(\cos(\ln(n)) + i\sin(\ln(n))) = \frac{1}{n^2}(\cos(\ln(n)) + i\sin(\ln(n)))$$

b)

Show that the series

$$\sum_{n\geqslant 1} n^{-2+i}$$

is absolutely convergent.

A series $\sum_{n\geqslant 1} a_n$ is absolutely convergent if the series $\sum_{n\geqslant 1} |a_n|$ is convergent.

In this case, $a_n = n^{-2+i} = \frac{1}{n^2} \cdot (cos(\ln(n)) + isin(\ln(n)))$.

The absolute value of a_n is $|a_n| = |\frac{1}{n^2} \cdot (cos(\ln(n)) + isin(\ln(n)))| = \frac{1}{n^2} * |cos(\ln(n)) + isin(\ln(n))|$.

Since $|cos(\ln(n)) + isin(\ln(n))| \leq 2$ for all n, we have $|a_n| \leq \frac{2}{n^2}$.

The series $\sum_{n\geqslant 1} \frac{2}{n^2}$ is a convergent p-series with $p = 2 > 1$, so by the comparison test, the series $\sum_{n\geqslant 1} |a_n|$ is also convergent.

Therefore, the series $\sum_{n\geqslant 1} n^{-2+i}$ is absolutely convergent.

c)

Let $z = x + iy \in \mathbb{C}$ such that $x > 1$. Determine whether the series

$$\sum_{n\geqslant 1} \frac{1}{n^z}$$

is convergent or divergent.

Here, $z = x + iy$ with $x > 1$.

The series $\sum_{n\geqslant 1} \frac{1}{n^z} = \sum_{n\geqslant 1} \frac{1}{n^{x+iy}} = \sum_{n\geqslant 1} \frac{1}{n^x \cdot n^{iy}}$.

We know that $\sum_{n\geqslant 1} \frac{1}{n^x}$ is a convergent p-series since $x > 1$.

For $\sum_{n\geqslant 1} n^{iy}$, we can use Euler's formula again to get $\sum_{n\geqslant 1}(cos(y \cdot ln(n)) + isin(y \cdot \ln(n)))$.

This is a sum of a cosine and sine series, both of which are bounded.

Therefore, the product of a convergent series and a bounded series is also convergent.

So, the series $\sum_{n\geqslant 1} \frac{1}{n^z}$ is convergent.

Question 5

The Fibonacci sequence is given by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$, and the general term is

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n})$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Consider the power series

$$\sum_{n\geqslant 0} F_n z^n = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots = 0 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + \cdots$$

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a)

Determine the radius of convergence of the power series.

To determine the radius of convergence of the power series, we can use the ratio test. The ratio test states that if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

then the series converges absolutely. In this case, $a_n = F_n$, so we have

$$\lim_{n\to\infty}\left|\frac{F_{n+1}}{F_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{\sqrt{5}}\left(\phi^{n+1}-(-\phi)^{-(n+1)}\right)}{\frac{1}{\sqrt{5}}\left(\phi^{n}-(-\phi)^{-n}\right)}\right| = \lim_{n\to\infty}\left|\frac{\phi^{n+1}-(-\phi)^{-(n+1)}}{\phi^{n}-(-\phi)^{-n}}\right|$$

Simplifying further, we have

$$\lim_{n\to\infty}\left|\frac{\phi^{n+1}-(-\phi)^{-(n+1)}}{\phi^{n}-(-\phi)^{-n}}\right| = \lim_{n\to\infty}\left|\frac{\phi^{n+1}-(-\phi)^{-(n+1)}}{\phi^{n}-(-\phi)^{-n}}\right| = \lim_{n\to\infty}\left|\frac{\phi^{n+1}-(-\phi)^{-(n+1)}}{\phi^{n}-(-\phi)^{-n}}\right| = \phi$$

Therefore, the radius of convergence of the power series is ϕ.

b)

For z in the disk of convergence, determine the sum of the series. $\frac{\left|-1+\sqrt{3}i\right|\,|n|}{2\,|n+1|}$

For z in the disk of convergence, we can determine the sum of the series by evaluating the power series at z. In this case, the power series is

$$\sum_{n\geqslant 0} F_n z^n = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots = 0 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + \cdots$$

To find the sum of the series, we can substitute z into the power series and simplify. For example, if $z = 1$, we have

$$\sum_{n\geqslant 0} F_n (1)^n = F_0 + F_1(1) + F_2(1)^2 + F_3(1)^3 + \cdots = 0 + 1 + 1 + 2 + 3 + 5 + 8 + \cdots$$

which is the Fibonacci sequence. Therefore, for z in the disk of convergence, the sum of the series is the Fibonacci sequence.