QR Decomposition with Gram-Schmidt

The QR decomposition
or (QR factorization)
Of a matrix is a decomposition of the matrix
into an orthogonal matrix and
a triangular matrix.

A QR decomposition of a real square matrix A is a decomposition of A as
$A = QR$

- Where Q is an orthogonal matrix
  - $(Q^*Q = I)$
- R is an upper triangular matrix.

Note : If A is non-singular, the factorization is unique.

Gram-Schmidt Process

Consider the vectors to be the columns of A, so

$$A = [\, a_1 \quad | \quad a_2 \quad | \quad \cdots \quad | \quad a_n \,]$$

then

$$\begin{aligned}
u_1 &= a_1, & e_1 &= \frac{u_1}{\|u_1\|} \\
u_2 &= a_2 - (a_2 \cdot e_1)e_1, & e_2 &= \frac{u_2}{\|u_2\|} \\
&\vdots \\
u_{k+1} &= a_{k+1} - (a_{k+1} \cdot e_1)e_1 - \cdots - (a_{k+1} \cdot e_k)e_k, & e_{k+1} &= \frac{u_{k+1}}{\|u_{k+1}\|}
\end{aligned}$$

Note: $\|\cdot\|$ is the L_2 norm

the resulting QR factorization is

$$A = [\, a_1 \quad | \quad a_2 \quad | \quad \cdots \quad | \quad a_n \,] = [\, e_1 \quad | \quad e_2 \quad | \quad \cdots \quad | \quad e_n \,] \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & \cdots & a_n \cdot \\ 0 & a_2 \cdot e_2 & \cdots & a_n \cdot \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \cdot \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 4 \end{bmatrix}$$

with the vectors

$$a_1 = (1, 0, -1)^* \\ a_2 = (2, 3, 4)^*$$

All vectors consideerd are column vectors

$$\begin{aligned} u_1 &= a_1 && = (1, 0, -1) \\ e_1 &= \frac{u_1}{\|u_1\|} && = \frac{1}{\sqrt{2}}(1, 0, -1) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \\ u_2 &= a_2 - (a_2 \cdot e_1)e_1 && = (2, 3, 4) - \frac{6}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) = (1, 3, 1) \\ e_2 &= \frac{u_2}{\|u_2\|} && = \frac{1}{\sqrt{11}}(1, 3, 1) = \left(\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right) \end{aligned}$$

thus,

$$Q = [\, e_1 \quad | \quad e_2 \quad | \quad \cdots \quad | \quad e_n \,] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

$$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 \\ 0 & a_2 \cdot e_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & 3\sqrt{3} \end{bmatrix}$$

Orthonormalize v_1

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1. Orthonormalize v_2

$$\begin{aligned} u_2 &= v_2 - (v_2 \cdot e_1)e_1 \\ &= v_2 - \mathbf{proj}_{u_1}(v_2) \end{aligned}$$

- Calcuate $(v_2 \cdot e_1)e_1 \text{ or } \mathbf{proj}_{u_1}(v_2)$

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*Notes :* $\mathbf{proj}_{u_{1}}(v_{2}) = \cfrac{v_{2}\cdot u_{1}}{\|u_{1}\|^{2}}\cdot u_{1}$
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so,

$$v_2 = (2, 3, 4) \textbf{ and } u_1 = (1, 0, -1)$$

$\|u_1\| \ : \ \|(1, 0, -1)\| = \sqrt{(1)^2 + (0)^2 + 1^2} = \sqrt{2}$

$v_2 \cdot u_1 = (2, 3, 4)(1, 0, -1) = -2$

$\cfrac{v_2 \cdot u_1}{\|u_1\|^2} \cdot u_1 = \cfrac{-2}{\sqrt{2}^2} \cdot (1, 0, -1) = (-1, 0, 1)$

- calcuate $v_2 - \mathbf{proj}_{u_1}(v_2)$

$$\begin{aligned} u_2 &= v_2 - (v_2 \cdot e_1)e_1 \\ &= v_2 - \mathbf{proj}_{u_1}(v_2) \\ &= (2, 3, 4) - (-1, 0, 1) \\ &= (2 - (-1), 3 - (0), 4 - (1)) \\ &= (3, 3, 3) \end{aligned}$$

calcaute e_1

$\|u_2\| : \|(3, 3, 3)\| = \sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3}$

then, $e_2 = \cfrac{u_2}{\|u_2\|} = \cfrac{(3, 3, 3)}{3\sqrt{3}} = \left(\cfrac{1}{\sqrt{3}}, \cfrac{1}{\sqrt{3}}, \cfrac{1}{\sqrt{3}}\right)$

thus,

$$e_1 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$
$$e_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$