Solutions

Vector Bootcamp (Part 1)

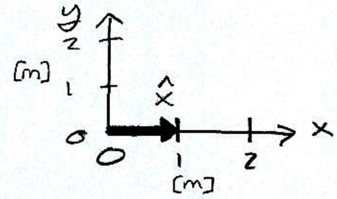
Vector:

- Cartesian form: $\boxed{\vec{v} \equiv v_x \hat{x} + v_y \hat{y}}$ (1)
- v_x and v_y are scalar components of $\vec{v}$ in each direction:

(scalars are numbers!) $\begin{pmatrix} v_x \equiv x \text{ component of } \vec{v} \equiv \text{signed amount of } \vec{v} \text{ in } x \text{ direction} \\ v_y \equiv y \text{ component of } \vec{v} \equiv \text{signed amount of } \vec{v} \text{ in } y \text{ direction} \end{pmatrix}$ (2)

- v_x and v_y can have any sign or be 0
- $\hat{x}$ and $\hat{y}$ are unit vectors (vectors of length 1)

($\hat{x}$ points in the $+x$ direction; $\hat{y}$ points in the $+y$ direction)



(3)

{① Make a vector with scalar component −3 in the x direction, and scalar component +4 in the y direction.}

- A scalar component is the entire number that multiplies a given unit vector
- So here, if −3 is the x-direction scalar component, then:

$$v_x = -3$$ {1}

- Similarly:

$$v_y = +4$$ {2}

- So by (1),

① $$\boxed{\vec{v} = -3\hat{x} + 4\hat{y}}$$ {3}

{② Make a vector with x component 0 and y component −4.1.}

$$v_x = 0$$ {1}

$$v_y = -4.1$$ {2}

② $$\boxed{\vec{v} = 0\hat{x} - 4.1\hat{y} = -4.1\hat{y}}$$ (if a component is 0, we don't have to write it.) {3}

{③ Make a vector with x component a and y component b, where a and b are both symbolic variables.}

$$v_x = a$$ {1}

$$v_y = b$$ {2}

③ $$\boxed{\vec{v} = a\hat{x} + b\hat{y}}$$ {3}

{④ Make a vector with x component 1 and y component 0.}

$$v_x = 1$$ {1}

$$v_y = 0$$ {2}

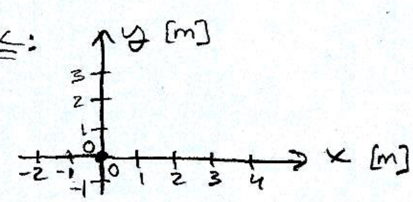
④ $$\boxed{\vec{v} = 1\hat{x} + 0\hat{y} = 1\hat{x} = \hat{x}}$$ (we can always omit 1 as $1\hat{u} = \hat{u}$) {3}

Coordinate System:

a set of perpendicular lines with arrows showing the positive direction for each coordinate, with labels!

- coordinates are things like x and y
- you should indicate the origin (0,0) as well

ex:



Length of a Vector:

- length of a vector in Cartesian form is its magnitude:

$$\boxed{|\vec{v}| \equiv \sqrt{v_x^2 + v_y^2}} \tag{4}$$

- note: unit vectors do not appear in $|\vec{v}|$; it only depends on scalars
- so $|\vec{v}|$ is also a scalar, but is nonnegative: $|\vec{v}| \geq 0$

{(5) Find the magnitude of each vector in problems (1)–(4). Are any of them unit vectors?}

- (5-1) $\vec{v} = -3\hat{x} + 4\hat{y}$
  $|\vec{v}| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$
  (5-1) $\boxed{|\vec{v}| = 5}$ ← (not a unit vector) {1}
- (5-2) $\vec{v} = -4.1\hat{y}$
  $|\vec{v}| = \sqrt{0^2 + (-4.1)^2} = \sqrt{(4.1)^2} = 4.1$
  (5-2) $\boxed{|\vec{v}| = 4.1}$ ← (not a unit vector) {2}
- (5-3) $\vec{v} = a\hat{x} + b\hat{y} \Rightarrow$ (5-3) $\boxed{|\vec{v}| = \sqrt{a^2+b^2}}$ ← (might be a unit vector, but only if $a^2+b^2=1$) {3}
- (5-4) $\vec{v} = \hat{x}$
  $|\vec{v}| = \sqrt{1^2 + 0^2} = \sqrt{1^2} = 1$ (5-4) $\boxed{|\vec{v}| = 1}$ ← (is a unit vector!) {4}

Unit Vectors:

- vectors of magnitude 1
- most general form (in 2D):

$$\boxed{\hat{u} \equiv \cos(\theta)\hat{x} + \sin(\theta)\hat{y}} \tag{5}$$

(here we stick to real-valued vectors)

- A unit vector in the direction of $\vec{v}$ can be found as:
  - note: $\vec{0} \equiv 0\hat{x} + 0\hat{y}$

$$\boxed{\hat{v} \equiv \frac{\vec{v}}{|\vec{v}|} = \frac{v_x}{\sqrt{v_x^2+v_y^2}}\hat{x} + \frac{v_y}{\sqrt{v_x^2+v_y^2}}\hat{y}} \tag{6}$$

(valid only for $\vec{v} \neq \vec{0}$)

{(6) What is a unit vector in the direction of nonunit vector $\vec{c} \equiv c_x\hat{x} + c_y\hat{y}$?}

(6) $\boxed{\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{c_x}{\sqrt{c_x^2+c_y^2}}\hat{x} + \frac{c_y}{\sqrt{c_x^2+c_y^2}}\hat{y}}$ {1}

{(7) What is the unit vector in (5) when $\theta = \frac{\pi}{2}$?}

$\hat{u} = \cos(\frac{\pi}{2})\hat{x} + \sin(\frac{\pi}{2})\hat{y} = 0\hat{x} + 1\hat{y} = \hat{y} \Rightarrow$ (7) $\boxed{\hat{u} = \hat{y}}$ {1}

{(8) What is the unit vector in (5) when $\theta = 0$?}

$\hat{u} = \cos(0)\hat{x} + \sin(0)\hat{y} = 1\hat{x} + 0\hat{y} = \hat{x} \Rightarrow$ (8) $\boxed{\hat{u} = \hat{x}}$ {1}

{(9) What is the unit vector in (5) when $\theta = \frac{\pi}{4}$?}

$\hat{u} = \cos(\frac{\pi}{4})\hat{x} + \sin(\frac{\pi}{4})\hat{y} = \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y}) \Rightarrow$ (9) $\boxed{\hat{u} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y})}$ {1}

{(10) Make a vector of magnitude 3 that points in the direction of $\vec{w} \equiv c\hat{x} + d\hat{y}$.}

$\vec{v} \equiv 3\hat{w}$; $\hat{w} = \frac{\vec{w}}{|\vec{w}|} = \frac{c}{\sqrt{c^2+d^2}}\hat{x} + \frac{d}{\sqrt{c^2+d^2}}\hat{y} \Rightarrow$ (10) $\boxed{\vec{v} = 3\left(\frac{c}{\sqrt{c^2+d^2}}\hat{x} + \frac{d}{\sqrt{c^2+d^2}}\hat{y}\right)}$ {1}

{(11) Make a vector of magnitude π that points in the direction opposite to $\vec{b} \equiv 2\hat{x} - \hat{y}$}

$\vec{v} \equiv -\pi\hat{b}$ {1}

$\hat{b} \equiv \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{2^2+(-1)^2}}\vec{b} = \frac{1}{\sqrt{5}}(2\hat{x}-\hat{y})$ {2}

$\Rightarrow$ (11) $\boxed{\vec{v} = -\frac{\pi}{\sqrt{5}}(2\hat{x}-\hat{y})}$ {3}

Projections of a Vector:

- Scalar projections: $\vec{v} \equiv v_x \hat{x} + v_y \hat{y}$ (7)

$$\boxed{\begin{array}{l} v_x \equiv \text{scalar projection of } \vec{v} \text{ onto } \hat{x} \\ v_y \equiv \text{scalar projection of } \vec{v} \text{ onto } \hat{y} \end{array}}$$ (are scalars) (8)

- vector projections:

$$\vec{v} \equiv \underbrace{v_x \hat{x}} + \underbrace{v_y \hat{y}}$$ (9)

$$\vec{v} \equiv \vec{v}_x + \vec{v}_y$$ (can also call $\vec{v}_x$ and $\vec{v}_y$ the vector components of $\vec{v}$) (10)

(Note: could also have scalar and vector projections onto some arbitrary vector $\vec{a}$, so then replace $\hat{x}$ with $\hat{a}$ to get v_a and $\vec{v}_a = v_a \hat{a}$. we'll learn how to find v_a later...)

$$\boxed{\begin{array}{l} \vec{v}_x \equiv v_x \hat{x} \equiv \text{vector projection of } \vec{v} \text{ onto } \hat{x} \\ \vec{v}_y \equiv v_y \hat{y} \equiv \text{vector projection of } \vec{v} \text{ onto } \hat{y} \end{array}}$$ (are vectors) (11)

{(12) what's the y-direction vector component of $\vec{c} \equiv c_x \hat{x} + c_y \hat{y}$?}

(12) $\boxed{\vec{c}_y = c_y \hat{y}}$ {1}

{(13) what's the x-direction vector component of $\vec{w} \equiv w_x \hat{x} + w_y \hat{y}$?}

(13) $\boxed{\vec{w}_x = w_x \hat{x}}$ {1}

{(14) Find the magnitudes of the vector components in (12) and (13).}

(14)
$$\boxed{\begin{array}{l} |\vec{c}_y| = \sqrt{0^2 + c_y^2} = \sqrt{c_y^2} = |c_y| \\ |\vec{w}_x| = \sqrt{w_x^2 + 0^2} = \sqrt{w_x^2} = |w_x| \end{array}}$$ {1} {2}

{(15) what's the y-direction vector component of $\vec{h} \equiv 7\hat{x} - \hat{y}$?}

(15) $\boxed{\vec{h}_y = -\hat{y}}$ {1}

{(16) what's the x-direction vector component of $\vec{g} \equiv 3\hat{y}$?}

(16) $\boxed{\vec{g}_x = 0\hat{x} = \vec{0}}$ {1}

{(17) Find the magnitude of the vecter projections in (15) and (16).}

(17)
$$\boxed{\begin{array}{l} |\vec{h}_y| = \sqrt{0^2 + (-1)^2} = \sqrt{1^2} = 1 \\ |\vec{g}_x| = \sqrt{0^2 + 0^2} = \sqrt{0} = 0 \end{array}}$$ {1} {2}

{(18) what's the scalar projection in the x direction of $\vec{w} \equiv w_x \hat{x} + w_y \hat{y}$?}

(18) $\boxed{w_x}$ {1}

{(19) what's the y-direction scalar projection of $\vec{h} \equiv 7\hat{x} - \hat{y}$?}

(19) $\boxed{h_y = -1}$ {1}

Vector Addition:

- vectors add component-wise. If $\vec{a} \equiv a_x\hat{x} + a_y\hat{y}$ and $\vec{b} \equiv b_x\hat{x} + b_y\hat{y}$, then

$$\boxed{\begin{aligned}\vec{c} &\equiv \vec{a} + \vec{b} \\ &= \underbrace{(a_x+b_x)}\hat{x} + \underbrace{(a_y+b_y)}\hat{y} \\ &\equiv c_x\hat{x} + c_y\hat{y}\end{aligned}} \quad (12)$$

$$\boxed{\begin{aligned}c_x &\equiv a_x+b_x \\ c_y &\equiv a_y+b_y\end{aligned}} \quad (13)$$

{20 what is the scalar y component of $\vec{c}$ in (12) in terms of the scalar components of $\vec{a}$ and $\vec{b}$?}

20 $$\boxed{c_y \equiv a_y + b_y}$$ {1}

{21 what's the vector projection in the x direction of $\vec{c}$ in (12) in terms of the components of $\vec{a}$ and $\vec{b}$?}

21 $$\boxed{\vec{c}_x \equiv c_x\hat{x} = (a_x+b_x)\hat{x}}$$ {1}

{22 what is the magnitude of $\vec{c}$ in (12) in terms of the scalor components of $\vec{a}$ and $\vec{b}$?}

22 $$\boxed{\begin{aligned}|\vec{c}| &\equiv \sqrt{c_x^2 + c_y^2} \\ &= \sqrt{(a_x+b_x)^2 + (a_y+b_y)^2}\end{aligned}}$$ {1} {2}

{23 what is the x-direction vector projection of the most general 2D unit vector $\hat{u} \equiv \cos(\theta)\hat{x} + \sin(\theta)\hat{y}$?}

$$\boxed{\tilde{\hat{u}}_x \equiv \cos(\theta)\hat{x}}$$

note: this is a vector, but not a unit vector, even though it's part of a unit vector, so I put a tilde over it to remind us of that! {1}

{24 Build the most general 2D vector possible by multiplying a general nonnegative number r times $\hat{u} \equiv \cos(\theta)\hat{x} + \sin(\theta)\hat{y}$. What is the magnitude of this vector? In what direction does it point?}

24
$$\vec{v} \equiv r\hat{u} = r(\cos(\theta)\hat{x} + \sin(\theta)\hat{y}) \quad \{1\}$$

$$|\vec{v}| = |r\hat{u}| = |r||\hat{u}| = r1 = r \quad \{2\}$$

$$\text{or} = \sqrt{[r\cos(\theta)]^2 + [r\sin(\theta)]^2} = \sqrt{r^2[\cos^2(\theta) + \sin^2(\theta)]}$$

$$= \sqrt{r^2 1} = \sqrt{r^2} = |r| = r$$

and $\vec{v}$ points in the direction of $\hat{u}$ by definition in {1} {3}

{25 Build the most general 2D vector in a different way by defining two vectors: $\vec{A} \equiv r\cos(\theta)\hat{x}$ and $\vec{B} \equiv r\sin(\theta)\hat{y}$, and then defining the most general vector as $\vec{C} \equiv \vec{A} + \vec{B}$. What is $\vec{C}$ in terms of r and θ? How does $\vec{C}$ compare to $\vec{v}$ in 24?}

25
$$\vec{A} \equiv r\cos(\theta)\hat{x}, \quad \vec{B} \equiv r\sin(\theta)\hat{y} \quad \{1\}$$

$$\vec{C} \equiv \vec{A} + \vec{B} = r\cos(\theta)\hat{x} + r\sin(\theta)\hat{y} = r(\cos(\theta)\hat{x} + \sin(\theta)\hat{y}) \quad \{2\}$$

$\vec{C} = \vec{v}$ from 24; the most general 2D vector {3}