A Brief Calculus Review

Overview:

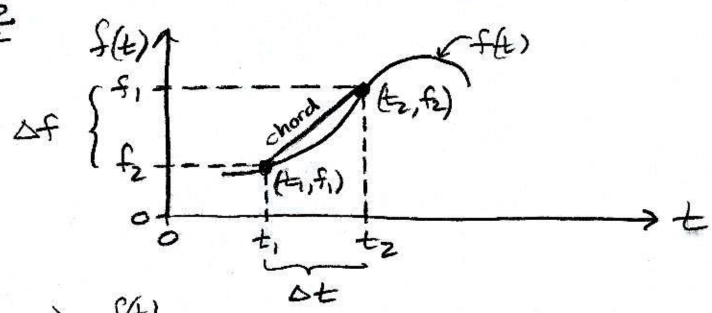
All of calculus basically is about:
- slopes ← (derivatives)
- areas ← (integrals)

(more generally, integrals describe lengths (1D), areas (2D), volumes (3D) etc.) (1)

Overview of Derivatives:

derivative $\equiv$ $\boxed{\text{slope of a curve at a point}}$ (2)

- average slope is the slope of a chord:

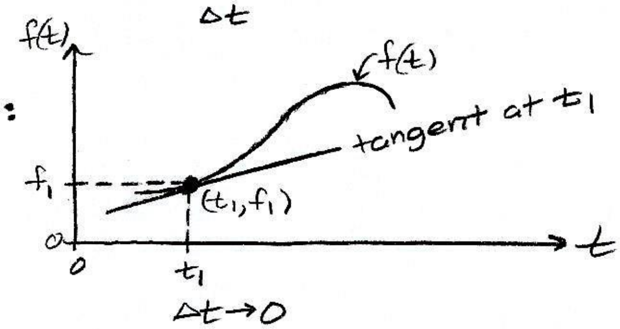


$$\boxed{\bar{m} \equiv \frac{\Delta f}{\Delta t} = \frac{f_2 - f_1}{t_2 - t_1}} \quad (3)$$

where

$$f_1 \equiv f(t_1)$$
$$f_2 \equiv f(t_2)$$

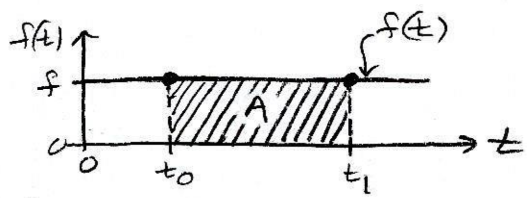
- the derivative (instantaneous slope) is the slope of the tangent line at a point:
  - take $\Delta t \to 0$ in (3)



$$\boxed{m \equiv \lim_{\Delta t \to 0} \left(\frac{\Delta f}{\Delta t} \right)} \quad (4)$$

Overview of Integrals:

- "Area" under a constant curve between t_0 and t_1 is height times width:

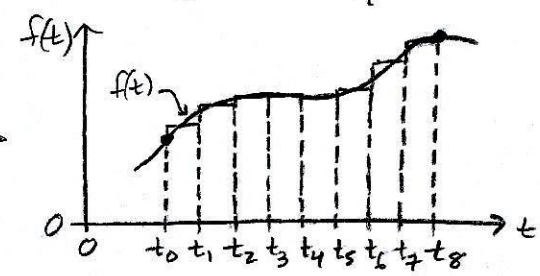


$$\boxed{A = f \Delta t} \quad (5)$$

$$\Delta t \equiv t_1 - t_0$$

- Approximate area under nonconstant curve betw. t_1 and t_2 is sum of n rectangle areas:

  ($n = 8$ here) →

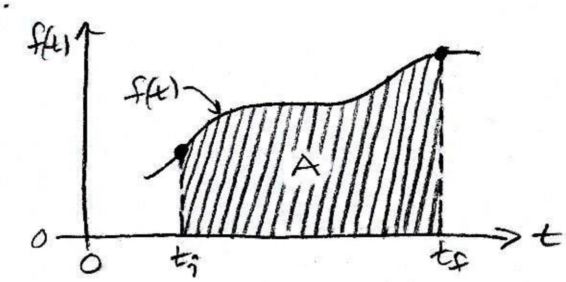


$$\boxed{A \approx \sum_{k=1}^{n} f(\bar{t}_k) \Delta t_k} \quad (6)$$

$$\bar{t}_k \equiv \frac{t_k + t_{k-1}}{2} \quad (7)$$

$$\Delta t_k \equiv t_k - t_{k-1} \quad (8)$$
$$= \frac{t_n - t_0}{n} = \frac{\Delta t}{n}$$

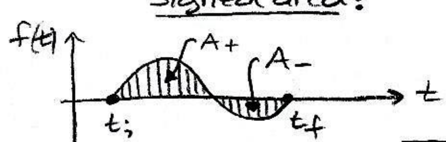
- Exact area under nonconstant curve betw. t_i and t_f is limit of sum in (6) as $\Delta t_k \to 0$ (implying $n \to \infty$)



$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{t}_k) \frac{\Delta t}{n} \quad (9)$$

$$\boxed{A = \int_{t_i}^{t_f} f(t)\, dt} \quad (10)$$

- Note: A is really a signed area:



$$A = A_+ + A_-$$
$$A_+ > 0, \; A_- < 0$$

$\boxed{\text{the area under } f(t) \text{ between } t_i \text{ and } t_f \text{ is the integral of } f(t) \text{ from } t_i \text{ to } t_f}$ (11)

Derivatives:

$$\boxed{\text{the derivative of } f(t) \text{ is the slope of the tangent line to } f(t) \text{ at } t.} \qquad (1)$$

- definition:

$$\boxed{\frac{df(t)}{dt} \equiv \lim_{\Delta t \to 0}\left(\frac{\Delta f(t)}{\Delta t}\right)} = \lim_{\Delta t \to 0}\left(\frac{f(t_2) - f(t_1)}{t_2 - t_1}\right) \qquad (2)$$

- to get a more useful form, let

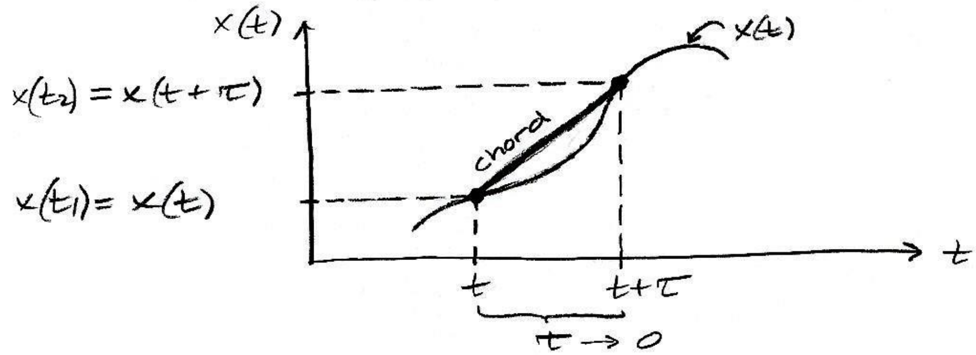
$$t_1 \equiv t \qquad (3)$$

$$t_2 \equiv t + \tau \qquad (4)$$

- so (2) becomes

$$\frac{df(t)}{dt} = \lim_{([t+\tau]-t)\to 0} \frac{f(t+\tau) - f(t)}{(t+\tau) - t}$$

$$\boxed{\frac{df(t)}{dt} = \lim_{\tau \to 0} \frac{f(t+\tau) - f(t)}{\tau}} \qquad (5)$$



(6)

- key points:
  - as $\tau \to 0$, it becomes $\tau = dt$ an "infinitesimal" (7)
  - since limit means only to approach zero, we say $\lim_{\tau\to 0} \tau = dt \equiv$ a "nonzero infinitesimal" (8)
  - so $\frac{1}{dt} \neq \infty$, and we can use dt algebraically (9)

- How to Compute Derivatives:
  - let $\lim_{\tau \to 0} \tau = dt \neq 0$ in $\frac{df(t)}{dt}$ (10)
  - simplify as much as possible (11)
  - discard any terms left with dt (take the "standard part") (12)

- ex:
  - Suppose $x(t) \equiv ct^2$
  - then

$$\frac{dx(t)}{dt} = \lim_{\tau\to 0}\left(\frac{x(t+\tau) - x(t)}{\tau}\right) = sp\left(\frac{x(t+dt) - x(t)}{dt}\right)$$

$$= sp\left(\frac{c(t+dt)^2 - c(t)^2}{dt}\right) = sp\left(\frac{c\left(t^2 + 2tdt + (dt)^2 - t^2\right)}{dt}\right)$$

(taking the standard part means finding the nearest real number that has no infinitesimal part)

$$= sp\left(\frac{2ctdt}{dt} + \frac{c(dt)^2}{dt}\right) = sp\left(2ct + cdt\right)$$

$$\boxed{\frac{dx(t)}{dt} = 2ct} \qquad (13)$$

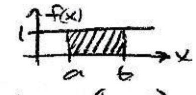
- In practice: (We try to break-down a problem to standard quantities whose derivatives we've already computed.) (14)

- ex: $\frac{d}{dt} c f(t) = c\frac{df(t)}{dt}$; $\frac{d}{dt} t^n = nt^{n-1}$; $\frac{d}{dt} e^{ct} = ce^{ct}$; $\frac{d}{dt} f(t)g(t) = \left(\frac{df}{dt}\right) g + f\left(\frac{dg}{dt}\right)$ (15)

How To Compute Definite Integrals:

- Simplest case:

$$\boxed{\int_a^b dx = x\Big|_a^b = b-a} \leftarrow \left(\text{Because } f(x)=1: \int_a^b 1\cdot dx : \quad A = 1\cdot(b-a) = b-a\right) \tag{1}$$



- Integrand x:

$$\int_a^b x\,dx = ? \tag{2}$$

- need to rewrite $x\,dx$ as something we can integrate already.
- recall that $x\,dx$ appears in the derivative of x^2:

  - if $\quad y = x^2 \quad (3)$
  - then $\quad \dfrac{dy}{dx} = 2x \quad (4)$

$$dy = 2x\,dx$$

(now we have rewritten the integrand in terms of a simpler variable) $\longrightarrow$

$$x\,dx = \frac{1}{2}dy \tag{5}$$

- so plug (5) into (2):
  - but always change the bounds too
  - then use the definition of the new variable to compute the bounds

$$\int_a^b x\,dx = \int_{y(a)}^{y(b)} \frac{1}{2}dy \qquad \begin{aligned} y(x) &\equiv x^2 \\ y(b) &= b^2 \\ y(a) &= a^2 \end{aligned}$$

$$= \frac{1}{2}\int_{a^2}^{b^2} dy = \frac{1}{2}\left[y\Big|_{a^2}^{b^2}\right] = \frac{1}{2}\left[b^2 - a^2\right]$$

$$\boxed{\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)} \tag{6}$$

- Indefinite Integrals:
  - variables of integration get primes (dummy variables)
  - upper limit(s) treated as unprimed variables
  - lower limits lumped into constant C
  - then bounds ignored and primes dropped

- ex: $\displaystyle\int dx \equiv \int_a^x dx' = x - a \equiv x + C \quad (C \equiv -a) \quad (7)$
  - notice: $\dfrac{d}{dx}(x+C) = 1 \quad (8)$

    or $\quad d(x+C) = dx \quad (9)$

- ex: $\displaystyle\int x\,dx \equiv \int_a^x x'\,dx' = \frac{1}{2}x^2 - \frac{1}{2}a^2 \equiv \frac{1}{2}x^2 + C \quad (10)$
  - notice: $\dfrac{d}{dx}\left(\dfrac{1}{2}x^2 + C\right) = x$

    or $\quad d\left(\frac{1}{2}x^2 + C\right) = x\,dx \quad (11)$

- since integrals give back the function whose derivative appears in the integrand, they are called antiderivatives:

$$\boxed{\begin{aligned} F(x) &\equiv \int f(x)\,dx \quad &&(\text{antiderivative}) \quad (12) \\ \frac{dF(x)}{dx} &= f(x) \quad &&(\text{derivative}) \quad (13) \end{aligned}}$$

- Definite Integrals from Indefinite Integrals:
  - ignore bounds
  - do indefinite version but drop C
  - evaluate difference of bounds

- ex: $\displaystyle\int_a^b x\,dx = \left[\int x\,dx\right]\Big|_a^b = \left[\frac{x^2}{2}\right]\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2} = \frac{1}{2}(b^2 - a^2) \quad (14)$