1. Let $||A||_{m,n} = \sup_{x \neq 0} \dfrac{||Ax||_m}{||x||_n}$ be the operator (induced) norm and let $||A||_F = \left(\sum |a_{ij}|^2\right)^{1/2}$ be the Frobenius norm of the matrix A. We denote the transpose of the matrix by A^T and the adjoint by A^*. Show that:
   (a) $||A||_\infty = \sup_{x \neq 0} \dfrac{||Ax||_\infty}{||x||_\infty} = \max_i \sum_j |a_{ij}|$ = maximum absolute row sum.
   (b) $||A||_1 = \sup_{x \neq 0} \dfrac{||Ax||_1}{||x||_1} = ||A^T||_\infty = \max_j \sum_i |a_{ij}|$ = maximum absolute column sum.
   (c) $||A||_2 = \sup_{x \neq 0} \dfrac{||Ax||_2}{||x||_2} = \sqrt{\lambda_{\max}(A^*A)}$, where $\lambda_{\max}$ denotes the maximum eigenvalue.
   (d) $||A||_2 = \sigma_1$, where σ_1 is the largest singular value of A
   (e) $||A||_2 = ||A^T||_2$.
   (f) If A is normal ($AA^* = A^*A = I$), then $||A||_2 = \max_i \lambda_i(A)$.
   (g) $||QAZ|| = ||A||$ if Q and Z are orthogonal (or unitary) for the Frobenius and the induced 2-norm.
   (h) $||A||_F = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$.
   (i) If A is diagonal, then $||A||_2 = \max_i |a_{ii}|$.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
   (a) Calculate the 2-norm of A using the definition.
   (b) Calculate the Singular Value Decomposition of A, i.e., $A = U\Sigma V^T$, with U and V orthogonal and Σ diagonal. Determine the 2-norm from singular values.

3. Let $A \in \mathbb{C}^{m\times m}$ be a square matrix with SVD $A = U\Sigma V^*$ and let $H = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$. Show that eigenvalues and eigenvectors of H are $\pm\sigma_i$ and $\dfrac{1}{\sqrt{2}}\begin{bmatrix} u_i \\ \pm u_i \end{bmatrix}$, where σ_i are diagonal elements of Σ and u_i are columns of U.

4. Let A be an $m \times m$ matrix with distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ and their corresponding eigenspaces $E_{\lambda_1}, \cdots, E_{\lambda_k}$. Let $\mathbb{B}_i$ be a basis for E_{λ_i}. Show that the set $\mathbb{B} = \mathbb{B}_1 \cup \cdots \cup \mathbb{B}_k$ is linearly independent.

5. Let A be an $m \times m$ Hermitian (self-adjoint) matrix ($A = A^*$). Show that:

(a) Eigenvalues of A are real.

(b) Eigenspaces corresponding to distinct eigenvalues are orthogonal.

(c) A is non-defective, i.e., A has a m linearly independent eigenvectors. Conclude that A is orthogonally diagonalizable.

6. Show that if $A = A^*$, then singular values of A are absolute value of eignvalues of A.

7. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Is A diagonalizable?

8. Let $P = P^2$ be a projection. If $S_1 = Range(P)$ and $S_2 = Null(P)$, we say that P is the projector onto S_1 along S_2. If S_1 and S_2 are orthogonal ($S_1 \perp S_2$), then P is called an orthogonal projection. Note that orthogonal projections are different from orthogonal matrices. Show that:

   (a) $S_1 \cap S_2 = 0$.

   (b) $I - P$ is a projection ($I - P$ is called the complementary projection).

   (c) $Range(I - P) = Null(P)$.

   (d) P is orthogonal if and only if $P = P^*$.

   (e) Let q be a unit vector. Show that $P_q = qq^*$ is a rank 1 orthogonal projector. What can you say about $P_{\perp q} = I - qq^*$? Generalize this to arbitrary non unit (non-zero) vectors.

   (f) Let Q be a unitary (orthonormal in real case) matrix ($Q^*Q = QQ^* = I$) with columns $\{q_1, \cdots, q_n\}$. Show that $P = QQ^*$ is the orthogonal projection onto the range of Q.

   (g) For a given full rank matrix A, the orthogonal projection onto the $Range(A)$ is given by $P = A(A^*A)^{-1}A^*$.

   (h) If P is orthogonal, then $I - 2P$ is unitary.

   (i) $||P||_2 \geq 1$ with equality if and only if P is orthogonal.

9. Let A be an $m \times n$ full rank matrix with $m \geq n$ with columns $\{a_j\}$. Orthogonal vectors produced by Gram-Schmidt can be written in terms of projections:

$$q_1 = \frac{P_1 a_1}{||P_1 a_1||}, \quad q_2 = \frac{P_2 a_2}{||P_2 a_2||}, \quad \cdots \quad , q_n = \frac{P_n a_n}{||P_n a_n||},$$

   where $P_1 = I$ and P_j for $j > 1$ is the orthogonal projection onto the space orthogonal to the range of $Q_{j-1} = [q_1 | \cdots | q_{j-1}]$.

   (a) Show that $P_j = I - Q_{j-1}Q_{j-1}^* = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1}$.

(b) In Classical Gram-Schmidt (CGS), we compute $v_j = P_j a_j$. In modified GM (MGS), we calculate $v_j = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1} a_j$:

for $j = 1, \cdots, n$ **do**
  $v_j = a_j$;
  for $i = 1, \cdots, j-1$ **do**
    $r_{ij} = q_i^* a_j$;
    $v_j = v_j - r_{ij} q_i$;
  end
  $r_{jj} = ||v_j||_2$;
  $q_j = v_j / r_{jj}$;
end

Algorithm 1: CGS

for $i = 1, \cdots, n$ **do**
  $v_i = a_i$;
end
for $i = 1, \cdots, n$ **do**
  $r_{ii} = ||v_i||_2$;
  $q_i = v_i / r_{ii}$;
  for $j = i+1, \cdots, n$ **do**
    $r_{ij} = q_i^* v_j$;
    $v_j = v_j - r_{ij} q_i$;
  end
end

Algorithm 2: MGS

Explain the reason why MGS is more stable than CGS.

(c) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix}$$

where ϵ is small so that $1 + \epsilon^2 = 1$ within roundoff errors. Calculate $q_2^T q_3$ using both algorithms. Comment on your observations.

10. Show that GS can be viewed as an orthogonal triangularization approach $A \underbrace{R_1 \cdots R_n}_{R^{-1}} = Q$, where R_i's are triangular matrices.

$$[v_1 | \cdots | v_n] = \begin{bmatrix} 1/r_{11} & -r_{12}/r_{11} & \cdots & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} [q_1 | v_2^{(2)} | \cdots | v_n^{(2)}].$$

Derive the recursions and write pseudo-code for QR factorization using the GS. Calculate the complexity of the algorithm.

11. Householder algorithm constructs a sequence of unitary matrices Q_k so that $Q_n \cdots Q_2 Q_1 A = R$ is upper triangular. Here

$$Q_k = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix}$$

where I is $(k-1) \times (k-1)$ identity matrix and F is an $(m-k+1) \times (m-k+1)$ unitary matrix that reflects the space $\mathcal{C}^{m-k+1}$ across the hyperplane H orthogonal to $v = ||x|| e_1 - x$.

(a) Describe the geometry of the Householder algorithm

(b) Derive a formula for the Householder reflector F ($F = I - 2\frac{vv^*}{v^*v}$).

(c) Write the pseudo-code of Householder QR factorization. What is the algorithmic complexity?

12. Let $u_1, \cdots, u_n$ be a sequence of unit vectors and let the first $i-1$ components of u_i be zero. Let $P = P_n P_{n-1} \cdots P_1$, where $P_i = I - 2u_i u_i^T$ is a Householder transformation. Show that there is an $n \times n$ lower triangular matrix T such that $P = I - UTU^T$, where $U = [u_1| \cdots |u_n]$. Provide an algorithm for computing the entries of T.

13. Let H be a *hyperplane* in $\mathbb{R}^n$ with orthonormal basis $\{u_1, u_2, \cdots, u_{n-1}\}$. A hyperplane can be characterized as a set of points orthogonal to a fixed non-zero vector v. Let P be the orthogonal projection onto H and let $P^\perp$ be its complementary projection.

(a) Describe P in terms of the basis of H.

(b) Describe P in terms of the basis of the complementary subspace $H^\perp$.

(c) Let $F = I - 2P^\perp$. Show that F is unitary and give a geometric interpretation of F.

14. (Overdetermined systems) Consider a system of linear equations $Ax = b$, where A is $m \times n$, $m \geq n$ with full (column) rank. This kind of system is called overdetermined. The least-squares solution is to find x that minimizes $||Ax - b||$. Show that this minimization is achieved by

$$Ax = b \iff A^T Ax = A^T b \iff x = (A^T A)^{-1} A^T b$$

The matrix $A^\dagger = (A^T A)^{-1} A^T$ is called Moore-Penrose pseudo-inverse.

15. (Underdetermined systems) For the system $Ax = b$, where A is $m \times n$, $m < n$ or if $rank(A) < n$, the $n \times n$ matrix $A^T A$ is non invertible. This is because $rank(A^T A) = rank(A) < n$ (prove this). In such cases, we can still define a pseudo-inverse using singular value decomposition. Let $A = U\Sigma V^T$ with U, $m \times m$ orthogonal, Σ, $m \times n$ diagonal and V, $n \times n$ orthogonal.

$$Ax = b \iff U\Sigma V^T x = b \iff \Sigma V^T x = U^T b$$

Define $y = V^T x$ and let $b' = U^T b$. Then the system $Ax = b$ is equivalent to the diagonal system:

$$\Sigma y = b'.$$

If we can solve this diagonal system for y, we can then compute $x = Vy$. The diagonal matrix Σ is of the form

$$\Sigma = \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$$

with $\tilde{\Sigma}$ is an $r \times r$ diagonal matrix of the form

$$\tilde{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}$$

where $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are the non-zero singular values of A, with $r \leq \min(m, n)$. Since Σ may not be invertible due to singularities, we resort to defining a pseudo inverse of Σ as

$$\Sigma^{-1} = \begin{bmatrix} \tilde{\Sigma}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\tilde{\Sigma}^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_r \end{bmatrix}$$

Now, let $\hat{y} = \Sigma^{-1}b'$. Show that this solution minimizes $||\Sigma y - b'||$ with the additional constraint that $||\hat{y}||$ is minimum among all possible solutions. The pseudo-inverse of A can be obtained by $A^\dagger = V\Sigma^{-1}U^T$.

16. Show that the Moore-Penrose pseudo-inverse of A satisfies the following identities

    (a) $AA^\dagger A = A$.

    (b) $A^\dagger AA^\dagger = A^\dagger$.

    (c) $A^\dagger A = (A^\dagger A)^T$.

    (d) $AA^\dagger = (AA^\dagger)^T$.

17. Cholesky Factorization: A $n \times n$ positive definite matrix A can be factorized into $A = LL^T$, where L is lower triangular. Derive the recursions for Cholesky Factorization as presented in the following pseudo code:

$\ell_{11} = \sqrt{a_{11}}$;
for $j = 2, \cdots, n$ **do**
  $\ell_{j1} = a_{j1}/\ell_{11}$;
end
for $i = 2, \cdots, n-1$ **do**
  $\ell_{ii} = (a_{ii} - \sum_{k=1}^{i-1} \ell_{ik}^2)^{1/2}$;
  for $j = i+1, \cdots, n$ **do**
    $\ell_{ji} = \left(a_{ji} - \sum_{k=1}^{i-1} \ell_{jk}\ell_{ik}\right)/\ell_{ii}$;
  end
  $\ell_{nn} = (a_{nn} - \sum_{k=1}^{n-1} \ell_{nk}^2)^{1/2}$;
end

18. Let $A = \begin{bmatrix} 4 & 8 \\ 8 & 32 \end{bmatrix}$
   (a) Use the LU decomposition to write $PA = LU$ using Gaussian elimination with partial pivoting.
   (b) Find the Cholesky decomposition of A, i.e., $A = LL^T$:
   $$A = \begin{bmatrix} a_{11} & w^* \\ w & K \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ w/\sqrt{a_{11}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K - ww^*/a_{11} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & w^*/\sqrt{a_{11}} \\ 0 & I \end{bmatrix}$$
   (c) Find LDL^T decomposition of A.

19. (Need for pivoting in Gaussian Eliminations). In the following use three-decimal digit floating point arithmetic. Let
   $$A = \begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix}$$
   (a) Show that $\kappa(A) = ||A||_\infty ||A^{-1}||_\infty \approx 4$. (Hence A is well conditioned).
   (b) Calculate the LU decomposition of A without partial pivoting and compute the product LU and compare with A. Comment on your results.
   (c) Solve the system $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using this LU decomposition. Compare with exact solution and comment on your results.
   (d) Repeat the above steps this time with partial pivoting and show that the problem is eliminated.

20. Let A be an $m \times m$ real matrix.
   (a) Describe the sequence of similarity transformations that can transform A to a Hessenberg matrix H. Describe the relation between eigenvalues and eigenvectors of A and H.

(b) Show that if A is symmetric, then the Hessenberg matrix H is tridiagonal.

(c) Show that the QR decomposition of a Hessenberg matrix $H = QR$, yields a Hessenberg matrix Q. Deduce that RQ remains a Hessenberg matrix.

21. Let A be a symmetric $m \times m$ matrix with distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ and let $W(A) = \left\{ \frac{x^T A x}{x^T x} \,\middle|\, x \in \mathbb{R}^m \right\}$ denote the set of all Rayleigh Quotients of A. Show that $W(A)$ is the convex hull of the eigenvalues of A, i.e.,

$$W(A) = \left\{ \sum_{i=1}^{k} \alpha_i \lambda_i \,\middle|\, \alpha_i \geq 0 \text{ and } \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$