Name: ***Solutions***

---

1. (a) Is the series $1 - y^2 + y^4 - y^6 + \cdots$ a geometric series for a fixed value of y? Identify the constant term and the ratio.

(b) Write the series in sigma notation.

(c) For which values of y is the series convergent? Find the sum of the series for those values.

Solution:

(a) Yes! with $a = 1$ and $r = -y^2$

(b) $\displaystyle\sum_{n=0}^{\infty} 1(-y^2)^n$

(c) The series is convergent: if $|r| = |-y^2| < 1 \Leftrightarrow |y| < 1 \Leftrightarrow -1 < y < 1$. For all $-1 < y < 1$:

$$\sum_{n=0}^{\infty} 1(-y^2)^n = \frac{a}{1-r} = \frac{1}{1+y^2}$$

2. A ball is dropped from the height of 10 feet and bounces. Each bounce is $\frac{3}{4}$ of the height of the bounce before. Thus, after the ball hits the floor for the first time, the ball rises to the height of $10(\frac{3}{4}) = 7.5$ feet, and after it hits the floor the second time it rises to the height of $7.5(\frac{3}{4}) = 10(\frac{3}{4})^2 = 5.625$ feet. (Assume there is no air resistance.)

(a) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the n^{th} time. Express your answer in closed form.

(b) Use your answer to part (a) to find the approximate total vertical distance the ball travels in the long run.

Solution: Let us denote the total distance the ball traveled after it hits the floor the n^{th} time by D_n.

(a)

$$\begin{aligned}
D_n &= 10 + 2(10)(3/4) + 2(10)(3/4)^2 + \cdots + 2(10)(3/4)^{n-1} \\
&= 10 + 20(3/4) + 20(3/4)^2 + \cdots + 20(3/4)^{n-1} \\
&= -10 + 10 + \left(10 + 20(3/4) + 20(3/4)^2 + \cdots + 20(3/4)^{n-1}\right) \\
&= -10 + \left(20 + 20(3/4) + 20(3/4)^2 + \cdots + 20(3/4)^{n-1}\right) \\
&= -10 + \left(\frac{20(1-(3/4)^n)}{1-(3/4)}\right) \\
&= -10 + \frac{20(1-(3/4)^n)}{1/4} \\
&= -10 + 80(1-(3/4)^n) \\
&= 70 - 80(3/4)^n
\end{aligned}$$

(b) So the approximate total vertical distance the ball travels in the long run is given by

$$\begin{aligned}\lim_{n\to\infty} D_n &= \lim_{n\to\infty}(70 - 80(3/4)^n)\\ &= 70 - 80\lim_{n\to\infty}(3/4)^n\\ &= 70 - 0 = 70 \text{ feet.}\end{aligned}$$

We can also simply note that the total distance is the sum of the geometric series with $a = 20$ and $r = 3/4$ minus 10 since the before it hits the floor for the first time there is only fall not a rise. Therefore the distance traveled in the long run is

$$-10 + \sum_{n=0}^{\infty} 20(3/4)^n = -10 + \frac{20}{1 - 3/4} = -10 + 80 = 70$$

3. Test each series for convergence/divergence.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\sqrt{\ln n})}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^4 n}{n^3}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n+2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3n+2}}$

Solution:

(a) Integral test applies, since $f(x) = \dfrac{1}{x(\sqrt{\ln x})}$ is positive, decreasing and continuous on $[2, \infty)$. Now $\displaystyle\int_2^{\infty} \frac{1}{x(\sqrt{\ln x})}dx = \int_{\ln 2}^{\infty} u^{-1/2}du = \left[2u^{1/2}\right]_2^{\infty}$ which is a divergent integral. So the series is divergent.

(b) Convergent in comparison with the convergent p-series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$, as $0 \le \dfrac{\sin^4 n}{n^3} \le \dfrac{1}{n^3}$.

(c) Divergent by the Divergent Test as $\displaystyle\lim_{n\to\infty}\frac{2n}{3n+2} = \frac{2}{3} \neq 0$.

(d) Convergent by the Alternating Series Test as $b_n = (-1)^n a_n$ where $a_n = \dfrac{1}{\sqrt{3n+2}}$. The sequence of a_n's is decreasing, positive and $\displaystyle\lim_{n\to\infty} a_n = 0$.

4. Sum of a telescoping series. Here, we find the sum of the telescoping series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(a) Recall from Calculus 1 that one can use partial fractions to rewrite:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Now, write the first few partial sums, s_1, s_2, and s_3 using the above identity. Notice the cancellations.

(b) Find a formula for s_n.

(c) Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n$

Solution:

(a)

- $s_1 = 1 - \frac{1}{2}$
- $s_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$
- $s_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$

(b) $s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \cdots\right) + \cdots + \left(\cdots - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$