Chapter 3
Interpolation

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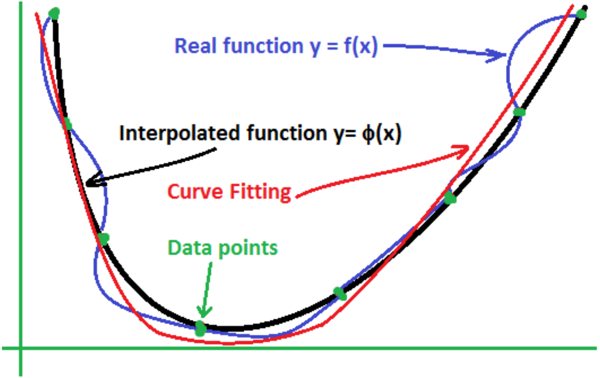
Numerical Methods (MCSC 202)

Curve fitting and Interpolation

Given the set of tabular values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ is not known, it is required to find a simpler function, say $\phi(x)$.
Curve fitting is the process of approximating $\phi(x)$ that approximately fits the data points, while, Interpolation is the process of approximating $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points.
The curve is not necessarily passing through data points in curve fitting, however, the curve must pass though every data points in interpolation.



Least square curve fitting procedure

Let the fitted curve is given by $Y = f(x)$. At $x = x_i$, the given ordinate is y_i and the corresponding functional value of the fitted curve is $f(x_i)$. If e_i is the error of approximation at $x = x_i$, then we have, $e_i = y_i - f(x_i)$.
If we write
$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_m - f(x_m)]^2 = e_1^2 + e_2^2 + e_3^2 + \cdots + e_m^2$
then the method of least squares consists in minimizing S.

Fitting a straight line

If $Y = a + bx$ be straight line fitted to the points (x_i, y_i), then
$S = [y_1 - (a + bx_1)]^2 + [y_2 - (a + bx_2)]^2 + \cdots + [y_m - (a + bx_m)]^2$
For S to be minimum, we have
$\frac{\partial S}{\partial a} = 0 = -2[y_1 - (a + bx_1)] - 2[y_2 - (a + bx_2)] - \cdots - 2[y_m - (a + bx_m)]$
$\implies ma + b(x_1 + x_2 + \cdots + x_m) = y_1 + y_2 + \cdots + y_m$
$\implies ma + b\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i \quad \cdots \quad (1)$
Similarly,
$\frac{\partial S}{\partial b} = 0 = -2x_1[y_1 - (a + bx_1)] - 2x_2[y_2 - (a + bx_2)] - \cdots - 2x_m[y_m - (a + bx_m)]$
$\implies a(x_1 + x_2 + \cdots + x_m) + b(x_1^2 + x_2^2 + \cdots + x_m^2) = x_1y_1 + x_2y_2 + \cdots + x_my_m$
$\implies a\sum_{i=1}^{m} x_i + b\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_iy_i \quad \cdots \quad (2)$
Equations (1) and (2) are called normal equations. Solving these for a and b and substituting in the equation $Y = a + bx$, we get the required fitted straight line.

Example

Fit a straight line from the data given $(0,-1),(2,5),(5,12),(7,20)$

Solution: Let $Y = a + bx$ **be required fitted straight line. The table of values are**

x	y	x^2	xy
0	-1	0	0
2	5	4	10
5	12	25	60
7	20	49	140
14	36	78	210

The normal equations are:

$4a + 14b = 36$

$14a + 78b = 210$

Solving, we get, $a = -1.1381$ **and** $b = 2.8966$

$\therefore Y = -1.1381 + 2.8966x$

Fitting exponential functions: $y = ab^x, y = ax^b$ and $y = ae^{bx}$

Given: $y = ax^b$

Taking log on both sides

$logy = log(ax^b)$

$logy = loga + logx^b$

$logy = loga + blogx$

$Y = A + bX$**, Which is a straight line. Where,** $Y = logy$**,** $A = loga$ **and** $X = logx$**.**

After calculation of A, we have $a = e^A$**.**

Fit a function of the form $y = ax^b$ **for the following data:**

x	61	26	7	2.6
y	350	400	500	600

Fitting qudratic polynomial

Let the quadratic polynomial is given by:

$y = a + bx + cx^2$

For normal equations, we take $\sum, \sum x$ **and** $\sum x^2$ **on both sides**

Taking $\sum$ **on both sides, we get**

$\sum y = \sum a + \sum bx + \sum cx^2 \implies \sum y = na + b\sum x + c\sum x^2$

Taking $\sum x$ **on both sides, we get**

$\sum xy = \sum xa + \sum bx^2 + \sum cx^3 \implies \sum xy = a\sum x + b\sum x^2 + c\sum x^3$

Taking $\sum x^2$ **on both sides, we get**

$\sum x^2y = \sum x^2a + \sum bx^3 + \sum cx^4 \implies \sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$

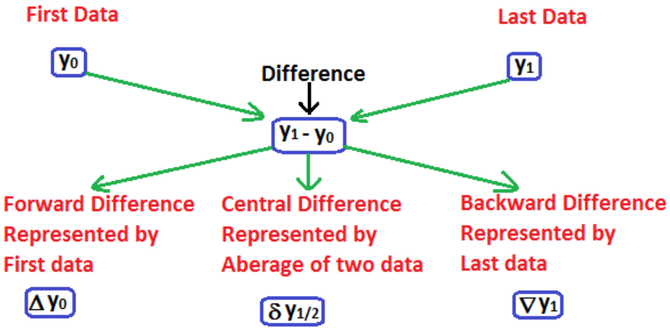
Fit a second degree parabola $y = a + bx + cx^2$ **to the data:**
$(1, 0.63), (3, 2.05), (4, 4.08), (6, 10.78)$

Finite Differences

Let y_0 and y_1 be two data points and $y_1 - y_0$ be difference such that:

- **First forward difference:** $\Delta y_0 = y_1 - y_0$
- **First backward difference:** $\nabla y_1 = y_1 - y_0$
- **First central difference:** $\delta y_{1/2} = y_1 - y_0$

The difference $y_1 - y_0$ is same but representation is different.



Forward differences

$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \cdots, \Delta y_{n-1} = y_n - y_{n-1}$ **are called first forward differences.**
$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \Delta^2 y_1 = \Delta y_2 - \Delta y_1, \cdots, \Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$ **are called second order differences and so on.**

$$\begin{aligned}\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)\\ &= y_2 - 2y_1 + y_0,\\ \Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)\\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward differences

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \cdots, \nabla y_n = y_n - y_{n-1}$ **are called first forward differences.**
$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \cdots, \nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$ **are called second order differences and so on.**

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0, \text{ etc.}$$

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Newton's backward difference interpolation formula

Given the set of $(n+1)$ **tabular values** $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ **of** x **and** y**, it is required to find a polynomial** $y_n(x)$**, such that** y **and** $y_n(x)$ **agree at the tabulated points. Let the domain values are in equidistant, i.e,** $x_i = x_0 + ih, \qquad i = 0, 1, 2, \cdots, n$.

Let the polynomial $y_n(x)$ **be written as (Starting from backward points)**

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots + a_n(x - x_n)(x - x_{n-1})(x - x_{n-2}) \cdots (x - x_1)$$

Substituting the data values (Starting from last data) successively, we get

$$(x_n, y_n) : y_n = a_0 \rightarrow a_0 = y_n$$

$$(x_{n-1}, y_{n-1}) : y_{n-1} = a_0 + a_1(x_{n-1} - x_n) \implies y_{n-1} = y_n + a_1(-h)$$

$$\implies a_1 = \frac{y_{n-1} - y_n}{-h} = \frac{\nabla y_n}{h}$$

$$(x_{n-2}, y_{n-2}) : y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\implies y_{n-2} = a_0 + a_1(-2h) + a_2(-2h)(-h)$$

$$\implies y_{n-2} = y_n + \frac{\nabla y_n}{h}(-2h) + a_2 \; 2h^2$$

$$\implies y_{n-2} = y_n + (y_n - y_{n-1})(-2) + a_2 \; 2h^2$$

$$\implies y_{n-2} - y_n = -2y_n + 2y_{n-1} + a_2 \; 2h^2$$

$$\implies y_{n-2} - 2y_{n-1} + y_n = a_2 \; 2h^2$$

$$\implies \nabla^2 y_n = a_2 \; 2h^2$$

$$\implies a_2 = \frac{\nabla^2 y_n}{2h^2} = \frac{\nabla^2 y_n}{2! \; h^2}, \cdots, a_n = \frac{\nabla^n y_n}{n! \; h^n}$$

Substituting these values in polynomial $y_n(x)$, we get

$$y_n(x) = y_n + (x-x_n)\frac{\nabla y_n}{h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 y_n}{2!h^2} + (x-x_n)(x-x_{n-1})(x-x_{n-2})\frac{\nabla^3 y_n}{3!h^3} + \cdots + (x-x_n)(x-x_{n-1})(x-x_{n-2})\cdots(x-x_1)\frac{\nabla^n y_n}{n!h^n}$$

Setting $x = x_n + ph$ we have,

$$x - x_n = (x_n + ph) - x_n = ph$$
$$x - x_{n-1} = (x_n + ph) - x_{n-1} = ph + x_n - x_{n-1} = ph + h = h(p+1)$$
$$x - x_{n-2} = (x_n + ph) - x_{n-2} = ph + x_n - x_{n-2} = ph + 2h = h(p+2)$$
$$\vdots$$
$$x - x_1 = (x_n + ph) - x_1 = ph + x_n - x_1 = ph + x_n - x_0 + x_0 - x_1 = ph + nh - h = h(p+n-1)$$

Substituting these values, we get

$$y_n(x) = y_n + (ph)\frac{\nabla y_n}{h} + (ph)[h(p+1)]\frac{\nabla^2 y_n}{2!h^2} + (ph)[h(p+1)][h(p+2)]\frac{\nabla^3 y_n}{3!h^3} + \cdots + (ph)[h(p+1)][h(p+2)]\cdots[h(p+n-1)]\frac{\nabla^n y_n}{n!h^n}$$

Therefore the final formula is

$$y_n(x) = y_n + p\frac{\nabla y_n}{1!} + p(p+1)\frac{\nabla^2 y_n}{2!} + p(p+1)(p+2)\frac{\nabla^3 y_n}{3!} + \cdots + p(p+1)(p+2)\cdots(p+n-1)\frac{\nabla^n y_n}{n!}$$

3.6 NEWTON'S FORMULAE FOR INTERPOLATION

Given the set of $(n+1)$ values, viz., $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. let

$$x_i = x_0 + ih, \quad i = 0, 1, 2, \ldots, n.$$

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$\left.\begin{aligned} y_n(x) = a_0 &+ a_1(x-x_0) + a_2(x-x_0)(x-x_1) \\ &+ a_3(x-x_0)(x-x_1)(x-x_2) + \cdots \\ &+ a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1}). \end{aligned}\right\} \tag{3.9}$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

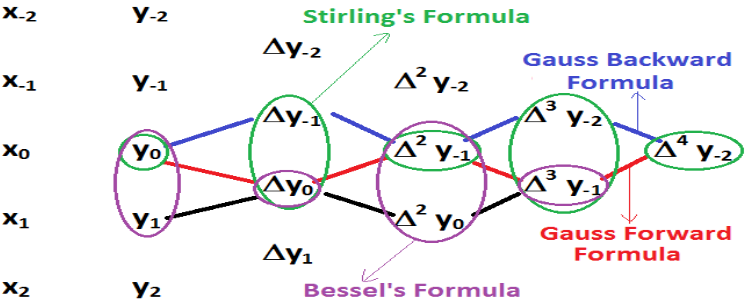
$$a_0 = y_0;\ a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h};\ a_2 = \frac{\Delta^2 y_0}{h^2 2!};\ a_3 = \frac{\Delta^3 y_0}{h^3 3!};\cdots;\ a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots, a_n$, Eq. (3.9) gives

$$\begin{aligned} y_n(x) = y_0 + p\Delta y_0 &+ \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \\ &+ \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0, \end{aligned} \tag{3.10}$$

which is *Newton's forward difference interpolation formula* and is useful for interpolation *near the beginning* of a set of tabular values.

Central difference formulas



Gauss Forward Formula:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Gauss Backward Formula:

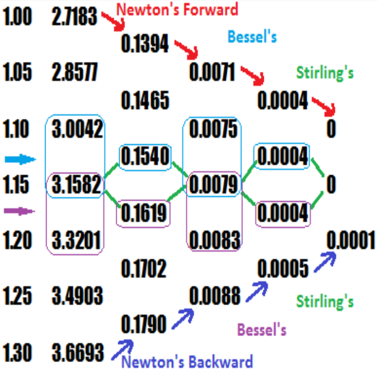
$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Stirling's Formula:

$$y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Bessel's Formula: $y_p = \frac{y_0+y_1}{2} + (p-1/2)\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots$

Estimate the values of $f(x)$ at $x = 0.98, 1.02, 1.12, 1.16, 1.27, 1.34$ from the given data values: $(1.00, 2.7183)$, $(1.05, 2.8577)$, $(1.10, 3.0042)$, $(1.15, 3.1582)$, $(1.20, 3.3201)$, $(1.25, 3.4903)$, $(1.30, 3.6693)$.



Given: $h = 0.05, x_0 = 1.00, x = 0.98$
$\implies p = \frac{0.98-1.00}{0.05} = -0.4$

Newton's forward formula: $y_n(x) = y_0 + p\frac{\Delta y_0}{1!} + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!}$.

$= 2.7183 + (-0.4)(0.1394) + \frac{(-0.4)(-1.4)}{2}(0.0071) + \frac{(-0.4)(-1.4)(-2.4)}{6}(0.0004)$

$= 2.6644$

Now, for $x = 1.02, p = \frac{1.02-1}{0.05} = 0.4$.

Then, $y_n = 2.77$.

For $x = 1.27$, **we have,** $h = 0.05, x_n = 1.30 \implies p = \frac{1.27-1.30}{0.05} = -0.6$

Newton's backward formula:

$y_n(x) = y_n + p\frac{\nabla y_n}{1!} + p(p+1)\frac{\nabla^2 y_n}{2!} + p(p+1)(p+2)\frac{\nabla^3 y_n}{3!}$.

$= 3.6693 + (-0.6)(0.1790) + \frac{(-0.6)(0.4)}{2}(0.0088) + \frac{(-0.6)(0.4)(1.4)}{6}(0.0005)$

$= 3.561$

Now, for $x = 1.34, p = \frac{1.34-1}{0.05} = 0.8$. **Then,** $y_n = 3.6726$.

Newton's Forward

Bessel's

Stirling's

1.00	2.7183				
		0.1394			
1.05	2.8577		0.0071		
		0.1465		0.0004	
1.10	3.0042		0.0075		0
		0.1540		0.0004	
1.15	3.1582		0.0079		0
		0.1619		0.0004	
1.20	3.3201		0.0083		0.0001
		0.1702		0.0005	
1.25	3.4903		0.0088		
		0.1790			
1.30	3.6693				

Stirling's

Bessel's

Newton's Backward

Given: $h = 0.05, x_0 = 1.15, x = 1.12$
$\implies p = \frac{1.12-1.15}{0.05} = -0.6$

Stirling's formula: $y_p = y_0 + p\frac{\Delta y_{-1}+\Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} = 3.1582 + (-0.6)(\frac{90.1540+0.1619)}{2} + \frac{(-0.6)^2}{2}(0.0079) + \frac{(-0.6)((-0.6)^2-1)}{6}(\frac{0.0004+0.0004}{2}) = 3.065$

Now, for $x = 1.16, p = \frac{1.16-1.15}{0.05} = 0.2$. **Then,** $y_n = 3.19$.

For $x = 1.12$, **we have,** $h = 0.05, x_0 = 1.10 \implies p = \frac{1.12-1.10}{0.05} = 0.4$

Bessel's formula:

$y_p = \frac{y_0+y_1}{2} + (p-1/2)\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2} + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1} = \frac{(3.0042+3.1582)}{2} + (-0.1)(0.1540) + \frac{(0.4)(-0.6)}{2}\frac{(0.0075+0.0079)}{2} + \frac{(0.4)(-0.6)(-0.1)}{6}(0.0004) = 3.065$

Now, for $x = 1.16, p = \frac{1.16-1.15}{0.05} = 0.2$. **Then,** $y_n = 3.19$.

Estimate the values of $f(x)$ **at** $x = 0.10, 0.15$ **from the given data values:** $(1.00, 2.7183)$, $(1.05, 2.8577)$, $(1.10, 3.0042)$, $(1.15, 3.1582)$, $(1.20, 3.3201)$, $(1.25, 3.4903)$, $(1.30, 3.6693)$.

Example 3.4 Find the cubic polynomial which takes the following values: $y(1) = 24$, $y(3) = 120$, $y(5) = 336$, and $y(7) = 720$. Hence, or otherwise, obtain the value of $y(8)$.

We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1 + 2p$ or $p = (x-1)/2$. Substituting this value of p in Eq. (3.10), we obtain

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$= x^3 + 6x^2 + 11x + 6.$$

Exercise

Find a cubic polynomial which takes the following values:
$(-3, -81), (-2, -37), (-1, -11), (0, 3), (1, 11), (2, 19), (3, 33)$

Interpolation for unequally spaced data points

Divided differences

Given the set of $(n+1)$ **tabular values** $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$**.**
Then the divided differences of order $1, 2, 3, \cdots$ **are defined by:**

First order: $[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$ **Second order:** $[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

Third order: $[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$

$\vdots$

n^{th} **order:** $[x_0, x_1, x_2, \cdots, x_n] = \frac{[x_1, x_2, \cdots, x_n] - [x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$

<u>Divided difference Table:</u>

$$
\begin{array}{llll}
x_0 \quad y_0 & & & \\
 & [x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0} & & \\
x_1 \quad y_1 & & [x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} & \\
 & [x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1} & & [x_0, x_1, x_2, x_3] = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \\
x_2 \quad y_2 & & [x_1, x_2, x_3] = \dfrac{[x_2, x_3] - [x_1, x_2]}{x_3 - x_1} & \\
 & [x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2} & & \\
x_3 \quad y_3 & & &
\end{array}
$$

3.10.1 Newton's General Interpolation Formula

We have, from the definition of divided differences,

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that

$$y = y_0 + (x - x_0)[x, x_0]. \tag{3.65}$$

Again,

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1},$$

which gives

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1].$$

Substituting this value of $[x, x_0]$ in (3.65), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]. \tag{3.66}$$

But

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

and so

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]. \tag{3.67}$$

Equation (3.66) now gives

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]. \end{aligned} \tag{3.68}$$

Proceeding in this way, we obtain

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots \\ &+ (x - x_0)(x - x_1)\ldots(x - x_n)[x, x_0, x_1, \ldots, x_n]. \end{aligned} \tag{3.69}$$

Example

Estimate the value of $f(301)$ from the following data values: $(300, 2.4771), (304, 2.4829), (305, 2.4843), (307, 2.4871)$.

The divided difference table is

300	2.4771		
		0.00145	
304	2.4829		−0.00001
		0.00140	
305	2.4843		0
		0.00140	
307	2.4871		

From Newton's divided difference formula, we have

$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots$

$f(301) = 2.4771 + (301 - 300)(0.00145) + (301 - 300)(301 - 304)(-0.00001)$

$f(301) = 2.4786$

Exercise

1. Estimate the value of $f(0.7)$ from the following data values: $(0, 1), (0.4, 1.8556), (0.9, 2.5868), (1.2, 2.1786), (1.5, 0.4167)$

2. Find $f(x)$ as a polynomial in x from the following data: $(-1, 3), (0, -6), (3, 39), (6, 822), (7, 1611)$

Lagrange's interpolation formula

Given the set of $(n+1)$ tabular values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. We derive Lagrange's interpolation formula using induction principle: When $n = 1$, we have two data points $(x_0, y_0), (x_1, y_1)$. The Lagrange's interpolation polynomial passing through these two points is

$L_1(x) = a_0(x - x_1) + a_1(x - x_0)$

Substituting data points successively, we get

$(x_0, y_0) : y_0 = a_0(x_0 - x_1) \implies a_0 = \frac{y_0}{(x_0 - x_1)}$

$(x_1, y_1) : y_1 = a_1(x_1 - x_0) \implies a_1 = \frac{y_1}{(x_1 - x_0)}$

Substituting in $L_1(x)$, we get Lagrange's linear interpolation formula:

$$L_1(x) = \frac{y_0}{x_0 - x_1}(x - x_1) + \frac{y_1}{x_1 - x_0}(x - x_0)$$

$$L_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$L_1(x) = l_0(x) y_0 + l_1(x) y_1$$

$$L_1(x) = \sum_{i=0}^{1} l_i(x) y_i,$$

where, $l_0(x) = \frac{x - x_1}{(x_0 - x_1)}$ and $l_1(x) = \frac{x - x_0}{(x_1 - x_0)}$ are called Lagrange interpolation coefficients.

Properties of Lagrange's interpolation coefficients

1. $l_0(x_0) = 1, l_0(x_1) = 0$

$l_1(x_0) = 0, l_1(x_1) = 1$

$$l_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

2. $\sum_{i=0}^{1} l_i(x) = l_0(x) + l_1(x)$

$= \frac{x - x_1}{(x_0 - x_1)} + \frac{x - x_0}{(x_1 - x_0)}$

$= 1$

When $n = 2$, we have three data points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. The Lagrange's interpolation polynomial passing through these three points is

$$L_2(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2) + a_2(x - x_0)(x - x_1)$$

Substituting data points successively, we get

$$(x_0, y_0) : y_0 = a_0(x_0 - x_1)(x_0 - x_2) \implies a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}$$
$$(x_1, y_1) : y_1 = a_1(x_1 - x_0)(x_1 - x_2) \implies a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)}$$
$$(x_2, y_2) : y_2 = a_2(x_2 - x_0)(x_2 - x_1) \implies a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

Substituting in $L_2(x)$, we get Lagrange's quadratic interpolation formula:

$$L_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$L_2(x) = l_0(x) y_0 + l_1(x) y_1 + l_2(x) y_2$$

$$L_2(x) = \sum_{i=0}^{2} l_i(x) y_i,$$

Proceeding in such a way, Lagrange's n^{th} degree polynomial is given by:

$$L_n(x) = \sum_{i=0}^{n} l_i(x) y_i,$$

Where,

$$l_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

Example

Estimate the value of $f(301)$ from the following data values: $(300, 2.4771), (304, 2.4829), (305, 2.4843), (307, 2.4871)$.

Since, we have 4 data points, Using Lagrange's third degree polynomial,

$$L_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$L_3(301) = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}2.4771 + \frac{(1)(-4)(-6)}{(4)(-1)(-3)}2.4829$$

$$+\frac{(1)(-3)(-6)}{(5)(1)(-2)}2.4843 + \frac{(1)(-3)(-4)}{(7)(3)(2)}2.4871$$

$$= 1.2739 + 4.9658 - 4.4717 + 0.7106$$

$$= 2.4786$$

Exercise

1. Estimate the value of $f(0.7)$ from the following data values:
$(0, 1), (0.4, 1.8556), (0.9, 2.5868), (1.2, 2.1786), (1.5, 0.4167)$

2. Find $f(x)$ as a polynomial in x from the following data:
$(0, -12), (1, 0), (3, 12), (4, 24)$

Practical Interpolation

1. **Interpolation for equally spaced data points**
   - For interpolation at the beginning of table values, Newton's forward formula is used.
   - For interpolation at the end of table values, Newton's backward formula is used.
   - For interpolation near the middle of set of values, the following are the choices
     - Stirling's formula if $-\frac{1}{4} \leq p \leq \frac{1}{4}$
     - Bessel's formula if $\frac{1}{4} \leq p \leq \frac{3}{4}$
2. **Interpolation for unequally spaced data points**
   - Newton's divided differences is best for manual calculation for more than 5 data values.
   - Lagrange's interpolation formula is best and fast in coding languages.

Consider Newton's forward interpolation formula:

$$y = y_0 + p\frac{\Delta y_0}{1!} + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + p(p-1)(p-2)(p-3)\frac{\Delta^4 y_0}{4!}\cdots$$

Then,

$$\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \frac{dp}{dx}\frac{d(y_0 + p\frac{\Delta y_0}{1!} + (p^2-p)\frac{\Delta^2 y_0}{2} + (p^3-3p^2+2p)\frac{\Delta^3 y_0}{6} + (p^4-6p^3+11p^2-6p)\frac{\Delta^4 y}{24} + \cdots)}{dx}$$

$$= \frac{dp}{dx}[\Delta y_0 + (2p-1)\frac{\Delta^2 y_0}{2} + (3p^2-6p+2)\frac{\Delta^3 y_0}{6} + (4p^3-18p^2+22p-6)\frac{\Delta^4 y}{24} + \cdots]$$

Since $x = x_0 + ph \implies \frac{dp}{dx} = \frac{1}{h}$,

Now, at $x = x_0 \implies x_0 = x_0 + ph \implies p = 0$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}(\Delta y_0 + -\frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5}\cdots)$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dp}\frac{dp}{dx} = \frac{1}{h^2}\left[(2)\frac{\Delta^2 y_0}{2} + (6p-6)\frac{\Delta^3 y_0}{6} + (12p^2-36p+22)\frac{\Delta^4 y_0}{24} + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots)$$

Exercise: Find dy/dx and d^2y/dx^2 at $x = x_n$ using Newton's backward formula and at $x = x_0$ using Stirling's formula.

(a) Newton's backward difference formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right) \tag{5.7}$$

and

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right). \tag{5.8}$$

(b) Stirling's formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left(\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + \cdots\right) \tag{5.9}$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right). \tag{5.10}$$

Find dy/dx and d^2y/dx^2 at $x = 1.2, 1.6$ and 2.2 from the data values:
$(1, 2.7183)$,
$(1.2, 3.3201)$,
$(1.4, 4.0522)$,
$(1.6, 4.9530)$,
$(1.8, 6.0496)$,
$(2.0, 7.3891)$,
$(2.2, 9.0250)$

1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Given: $h = 0.2, x_0 = 1.2$, **Using Newton's forward formula:**

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left(\Delta y_0 + -\frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} \cdots\right)$$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.008) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right)$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.2^2}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318$$

Numerical Integration

General problem: Given the set of $(n+1)$ tabular values $(x_0, y_0), (x_1, y_1)$, $(x_2, y_2), \cdots, (x_n, y_n)$ of a function $y = f(x)$, Integrate: $I = \int_a^b ydx$.
Let the interval $[a, b]$ be divided into n equal sub-intervals such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Clearly $x_n = x_0 + nh$. The integral becomes $I = \int_{x_0}^{x_n} ydx$.
Approximating y by Newton's forward interpolation formula, we obtain
$I = \int_{x_0}^{x_n} \left[y_0 + p\frac{\Delta y_0}{1!} + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + \cdots \right] dx$
Since $x = x_0 + ph, dx = hdp$,
When $x = x_0 \implies p = 0$ and when $x = x_n \implies p = n$
$I = \int_0^n \left[y_0 + p\frac{\Delta y_0}{1!} + (p^2 - p)\frac{\Delta^2 y_0}{2} + (p^3 - 3p^2 + 2p)\frac{\Delta^3 y_0}{6} + \cdots \right] hdp$
Integrating, we get
$I = \left[py_0 + \frac{p^2}{2}\frac{\Delta y_0}{1!} + (\frac{p^3}{3} - \frac{p^2}{2})\frac{\Delta^2 y_0}{2} + (\frac{p^4}{4} - p^3 + p^2)\frac{\Delta^3 y_0}{6} + \cdots \right]_0^n hdp$
$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{(2n^3 - 3n^2)}{12}\Delta^2 y_0 + \frac{(n^4 - 4n^3 + 4n^2)}{24}\Delta^3 y_0 + \cdots \right]$

$$I = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{(2n^2 - 3n)}{12}\Delta^2 y_0 + \frac{(n^3 - 4n^2 + 4n)}{24}\Delta^3 y_0 + \cdots \right] \qquad (1)$$

This equation (1) is called general integration formula.

Simpson's 1/3 Rule

If we set $n = 2$ in general formula, we have initial 3 data points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), then the differences higher than 2^{nd} order will become zero,

$\therefore$ **On** $[x_0, x_2] : \int_{x_0}^{x_2} ydx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{(2\times 2^2 - 3\times 2)}{12}\Delta^2 y_0\right]$

$$\begin{aligned} &= 2h\left[y_0 + \Delta y_0 + \tfrac{1}{6}\Delta^2 y_0\right] \\ &= 2h\left[y_0 + (y_1 - y_0) + \tfrac{1}{6}(y_2 - 2y_1 + y_0)\right] \\ &= 2h\left[y_0 + (y_1 - y_0) + \tfrac{1}{6}(y_2 - 2y_1 + y_0)\right] \\ &= \tfrac{h}{3}(y_0 + 4y_1 + y_2) \end{aligned}$$

Similarly, on $[x_2, x_4] : \int_{x_2}^{x_4} ydx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

$\vdots$

Finally, on $[x_{n-2}, x_n] : \int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$

Now, $\int_{x_0}^{x_n} ydx = \int_{x_0}^{x_2} ydx + \int_{x_2}^{x_4} ydx + \cdots + \int_{x_{n-2}}^{x_n} ydx$

$$\begin{aligned} \int_{x_0}^{x_n} ydx &= \tfrac{h}{3}(y_0 + 4y_1 + y_2) + \tfrac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \tfrac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \tfrac{h}{3}(y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n) \end{aligned}$$

Note: Simpson's 1/3 rule requires even number of subintervals.

Exercise: Derive integration formula for trapezoidal rule and Simpson's 3/8 rule by setting $n = 1$ and $n = 3$ respectively.

Example

Find, from the data, the area bounded by the curve and the x-axis:
$(7.47, 1.93), (7.48, 1.95), (7.49, 1.98), (7.50, 2.01), (7.51, 2.03), (7.52, 2.06)$

Solution: We know that: Area=$\int_{7.47}^{7.52} f(x)dx$

Here, $h = 0.01$**, total data points:** 6**, Total subintervals:** $n = 5$**.** $\therefore$ **we can use trapezoidal rule only.**

Trapezoidal rule: $\int_{x_0}^{x_n} ydx = \frac{h}{2}(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n)$

Area=$\int_{7.47}^{7.52} f(x)dx = \frac{0.01}{2}[1.93 + 2(1.95 + 1.98 + 2.01 + 2.03) + 2.06] = 0.0996$

Exercise

Find the volume of solid of revolution formed by rotating about the x-axis the area between the $x - axis$**, the lines** $x = 0$ **and** $x = 1$**, and a curve through the points:**
$(0, 1), (0.25, 0.9896), (0.5, 0.9589), (0.75, 0.9089), (1, 0.8415)$**.**

Hint: Volume=$\pi \int_a^b y^2 dx$

Example

Integrate: $I = \int_0^{1.5} \frac{e^x + x}{\sin x + 1} dx$

Given, $a = 0, b = 1.5, f(x) = \frac{e^x + x}{\sin x + 1}$**. Let us choose** $n = 6$**, so that** $h = \frac{b-a}{n} = \frac{1.5-0}{6} = 0.25$**. The functional values are:**

x	0	0.25	0.5	0.75	1	1.25	1.5
f(x)	1	1.2298	1.4524	1.7049	2.0192	2.4322	2.9946

Simpson's 1/3 rule:

$$I = \frac{h}{3}(y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n)$$

$$= \frac{0.25}{3}(1 + 4(1.2298 + 1.7049 + 2.4322) + 2(1.4524 + 2.0192) + 2.9946)$$

$$= 2.7004$$

Simpson's 3/8 rule:

$$I = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

$$I = \frac{3h}{8}(1 + 3(1.2298 + 1.4524 + 2.0192 + 2.4322) + 2(1.7049) + 2.9946)$$
$$= 2.7005$$

Exercise

Evaluate: $I = \int_0^1 \frac{1}{1+x} dx$ **using** $h = 0.125$**.**

Numerical double integration

Evaluate:
$I = \int_a^b \int_c^d f(x,y)dxdy = \int_0^2 \int_0^1 e^{x+y} dxdy$

Given: $a = 0, b = 2, c = 0, d = 1$, $f(x,y) = e^{x+y}$. **Let us choose** $n = 4$ **such that** $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$ **and** $k = \frac{d-c}{n} = \frac{1-0}{4} = 0.25$ **The corresponding functional value table is:**

f(x,y)	0	0.5	1	1.5	2
0	1.0000	1.6487	2.7183	4.4817	7.3891
0.25	1.2840	2.1170	3.4903	5.7546	9.4877
0.5	1.6487	2.7183	4.4817	7.3891	12.1825
0.75	2.1170	3.4903	5.7546	9.4877	15.6426
1	2.7183	4.4817	7.3891	12.1825	20.0855

Now, the solution is
$I = \int_0^2 \int_0^1 e^{x+y} dxdy = \frac{h}{2}\frac{k}{2}$**[sum of all multiples of corresponding elements of functional value table and multiplication table]**

Multiplication table (Trapezoidal rule):

×	1	2	2	2	1
1	1	2	2	2	1
2	2	4	4	4	2
2	2	4	4	4	2
2	2	4	4	4	2
1	1	2	2	2	1

$\therefore\ I = \left(\frac{0.5}{2}\right)\left(\frac{0.25}{2}\right)[1 \times 1 + 1.6487 \times 2 + 2.7183 \times 2 + \cdots + 20.0855 \times 1] = 11.2643$

Multiplication table (Simpson's rule):

×	1	4	2	4	1
1	1	4	2	4	1
4	4	16	8	16	4
2	2	8	4	8	2
4	4	16	8	16	4
1	1	4	2	4	1

$\therefore\ I = \left(\frac{0.5}{3}\right)\left(\frac{0.25}{3}\right)[1 \times 1 + 1.6487 \times 4 + 2.7183 \times 2 + \cdots + 20.0855 \times 1] = 10.9821$

Exercise: Evaluate
$\int_{-2}^2 \int_0^4 (x^2 - xy + y^2)dxdy$