DETERMINANT OF A MATRIX

Machine Learning course, Chapter 2, Mathematics, 4th Session

Exercise number 10

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Content

What is determinant?

- ***Determinant*** of the matrix is a special number that is calculated for square matrices.
- ***A matrix*** is an orderly arrangement of numbers arranged in certain rows and columns.
- For example this is a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

What is the use of this number?

The determinant provides information about the matrix that is useful in systems of linear equations. This number is useful for finding the ***inverse*** of the matrix, as well as in differential and integral equations and in many other situations.

If the determinant of the opposite matrix is ***zero***, then we know that the matrix is ***invertible***. Therefore, through the determinant, it is possible to determine the eigenvalues of a matrix or, in better words, a linear mapping of it.

What is determinant? (cont.)

- It should be noted that matrix analysis and the concept of determinants can even be used ***to solve the 3rd degree*** equation.
- The determinant symbol is two vertical lines on the sides of the English letter of the desired matrix. For example **|A|** means determinant of matrix A. In fact, this symbol is exactly the same as the "absolute value" symbol.
- The first condition for calculating the determinant is that the ***matrix must be square***, that is, the number of rows and columns must be equal. If this condition is met, the determinant of the matrix can be calculated with some simple arithmetical relations.

Determinant of a 2×2 matrix

- For a 2×2 matrix, i.e. a matrix that has 2 rows and 2 columns:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- The determinant is equal to:

$$|A| = ad - bc$$

- Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$|B| = (4 \times 8) - (6 \times 3)$$

$$|B| = 32 - 18$$

$$|B| = 14$$

Determinant of a 3 × 3 matrix

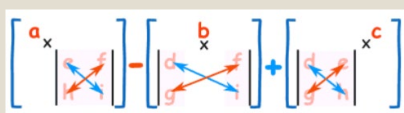
- For a 3 × 3 matrix, i.e. a matrix that has 3 rows and 3 columns:

$$A = \begin{bmatrix} a & b & c \\ d & \mathrm{e} & f \\ g & h & \mathrm{i} \end{bmatrix}$$

- The determinant is equal to:

$$|A| = a(ei - fh) - b(di - \mathrm{fg}) + c(dh - eg)$$

- It sounds complicated, but the working pattern is as follows:



To find the determinant of a 3x3 matrix, we perform the following steps:

- First, we calculate the determinant of the 2x2 matrix, which does not have any element in the rows and columns of a, and then we multiply a by this value.
- We repeat the same process for b and c.
- We add the obtained values together, but remember that we use the negative value of b in the sum.

- We have formulated the above in the form of the image below.

$$|A| = a \cdot \begin{vmatrix} \mathrm{e} & f \\ h & \mathrm{i} \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & \mathrm{i} \end{vmatrix} + c \cdot \begin{vmatrix} d & \mathrm{e} \\ g & h \end{vmatrix}$$

Determinant of a 3×3 matrix (cont.)

- Example:

$$B = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$|B| =$$

$$6 \times ((-2 \times 7) - (5 \times 8)) - 1 \times ((4 \times 7) - (5 \times 2)) + 1 \times ((4 \times 8) - (-2 \times 2))$$

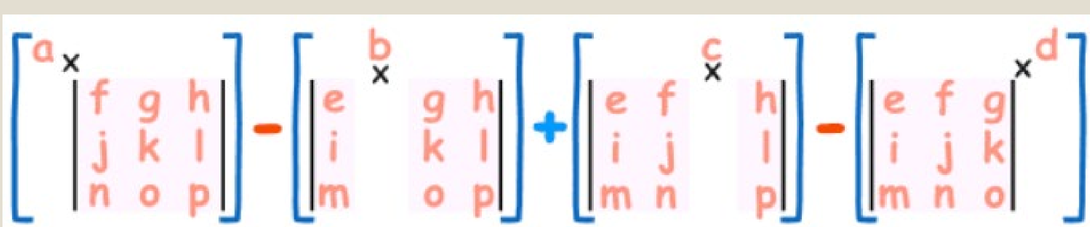
$$= 6 \times (-54) - 1 \times (18) + 1 \times (36)$$

$$= -306$$

Determinant of a 4×4 matrix

In the case of 4×4 matrices, the procedure will be as follows:

- ***Positive a*** is multiplied by the determinant of the matrix that is not in the rows and columns of ***a***.
- ***Negative b*** multiplied by the determinant of the matrix that is not in the row and column of ***b***.
- ***Positive c*** is multiplied by the determinant of the matrix that is not in the row and column of ***c***.
- ***Negative d*** multiplied by the determinant of the matrix that is not in row and column ***d***.



Determinant of a 4×4 matrix (cont.)

- Its formula can be shown in the following image:

$$|A| = a.\begin{vmatrix} f & g & h \\ j & k & \mathrm{l} \\ n & \mathrm{o} & p \end{vmatrix} - b.\begin{vmatrix} e & g & h \\ i & k & \mathrm{l} \\ m & \mathrm{o} & p \end{vmatrix} + c.\begin{vmatrix} e & f & h \\ i & j & \mathrm{l} \\ m & \mathrm{n} & p \end{vmatrix} - d.\begin{vmatrix} e & f & g \\ i & j & \mathrm{k} \\ m & \mathrm{n} & o \end{vmatrix}$$

- Pay attention to the " **+ - + -** " pattern (**+ *a*** … **- *b*** … **+ *c*** … **- *d*** …).
- You must remember this pattern.
- This pattern also applies to 5×5 matrices and above. But to calculate the determinants of these types of matrices, it is better to use software designed for this purpose.

There are other methods to calculate determinants;

This calculation method that we mentioned in the above section is called "***Laplace expansion***" method.

Determinant of a $n \times n$ matrix

The said pattern also applies to larger matrices:

- Multiply ***a*** by a matrix that is not in the row or column of ***a***.
- Then do the same for ***b*** and ***so on***.
- But always remember the **+-+-** pattern and make the obtained values negative one by one.

- Example:

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

- The solution is on the next page.

Determinant of a $n \times n$ matrix

$$= \underbrace{1\begin{bmatrix}0&2&0\\1&2&3\\3&0&0\end{bmatrix}}_{\mathrm{I}} \underbrace{- 2\begin{bmatrix}1&2&0\\0&2&3\\2&0&0\end{bmatrix}}_{\mathrm{II}} \underbrace{+ 3\begin{bmatrix}1&0&0\\0&1&3\\2&3&0\end{bmatrix}}_{\mathrm{III}} \underbrace{- 4\begin{bmatrix}1&0&2\\0&1&2\\2&3&0\end{bmatrix}}_{\mathrm{IV}}$$

$$\mathrm{I} = 1\left(0\begin{vmatrix}2&3\\0&0\end{vmatrix} - 2\begin{vmatrix}1&3\\3&0\end{vmatrix} + 0\begin{vmatrix}1&2\\3&0\end{vmatrix}\right) = 1\left(\overset{0}{\cancel{0(0-0)}} - \overset{18}{2(0-9)} + \overset{0}{\cancel{0(0-6)}}\right) = 18$$

$$\mathrm{II} = -2\left(1\begin{vmatrix}2&3\\0&0\end{vmatrix} - 2\begin{vmatrix}0&3\\2&0\end{vmatrix} + 0\begin{vmatrix}0&2\\2&0\end{vmatrix}\right) = -2\left(\overset{0}{1(\cancel{0}-0)} - \overset{12}{2(\cancel{0}-6)} + \overset{0}{0(\cancel{0}-4)}\right) = -24$$

$$\mathrm{III} = 3\left(1\begin{vmatrix}1&3\\3&0\end{vmatrix} - 0\begin{vmatrix}0&3\\2&0\end{vmatrix} + 0\begin{vmatrix}0&1\\2&3\end{vmatrix}\right) = 3\left(\overset{-9}{1(\cancel{0}-9)} - \overset{0}{0(\cancel{0}-6)} + \overset{0}{0(\cancel{0}-2)}\right) = -27$$

$$\mathrm{IV} = -4\left(1\begin{vmatrix}1&2\\3&0\end{vmatrix} - 0\begin{vmatrix}0&2\\2&0\end{vmatrix} + 2\begin{vmatrix}0&1\\2&3\end{vmatrix}\right) = -4\left(\overset{-6}{1(\cancel{0}-6)} - \overset{0}{0(\cancel{0}-4)} + \overset{-4}{2(\cancel{0}-2)}\right) = 40$$

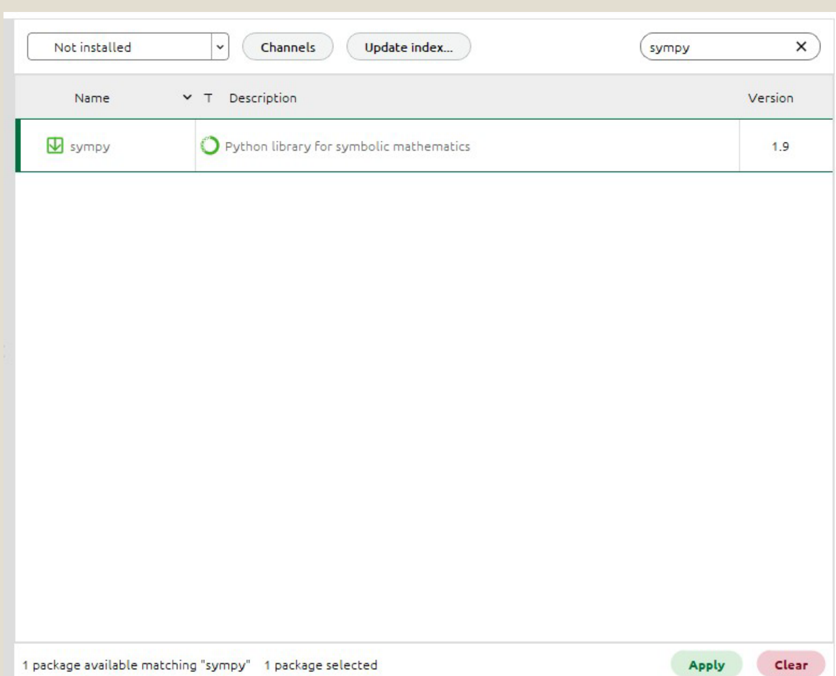
$$|B| = 18 - 24 - 27 + 40 = 7$$

Determinant computing by “SymPy”

What is ***SymPy***?

- ***SymPy*** is an open-source Python programming ***library*** for symbolic computation. It provides computer algebra capabilities either as a standalone application, as a library to other applications, or live on the web as SymPy Live or SymPy Gamma.

First of all install this library.



Determinant computing by “SymPy” (cont.)

Then do this coding parts: (I considered the previous matrix)

In [2]:
```python
from sympy import * # importing the library
init_printing(use_unicode=True) # this said to be necessary
```

To make a matrix in SymPy, use the 'Matrix' object. A matrix is constructed by providing a list of row vectors that make up the matrix.

In [10]:
```python
# this is the matrix which I earlier computed the determinant by hand
Matrix([[1, 2, 3, 4], [1, 0, 2, 0], [0, 1, 2, 3], [2, 3, 0, 0]])
```

Out[10]:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

In [11]:
```python
from sympy import shape # let's check out the shape of the matrix
```

In [5]:
```python
Q = Matrix([[1, 2, 3, 4], [1, 0, 2, 0], [0, 1, 2, 3], [2, 3, 0, 0]])
```

In [6]:
```python
shape(Q)
```

Out[6]: (4, 4)

Determinant

To compute the determinant of a matrix, use 'det'.

In [12]:
```python
Q.det() # as you can see, I also computed this answer by hand!
```

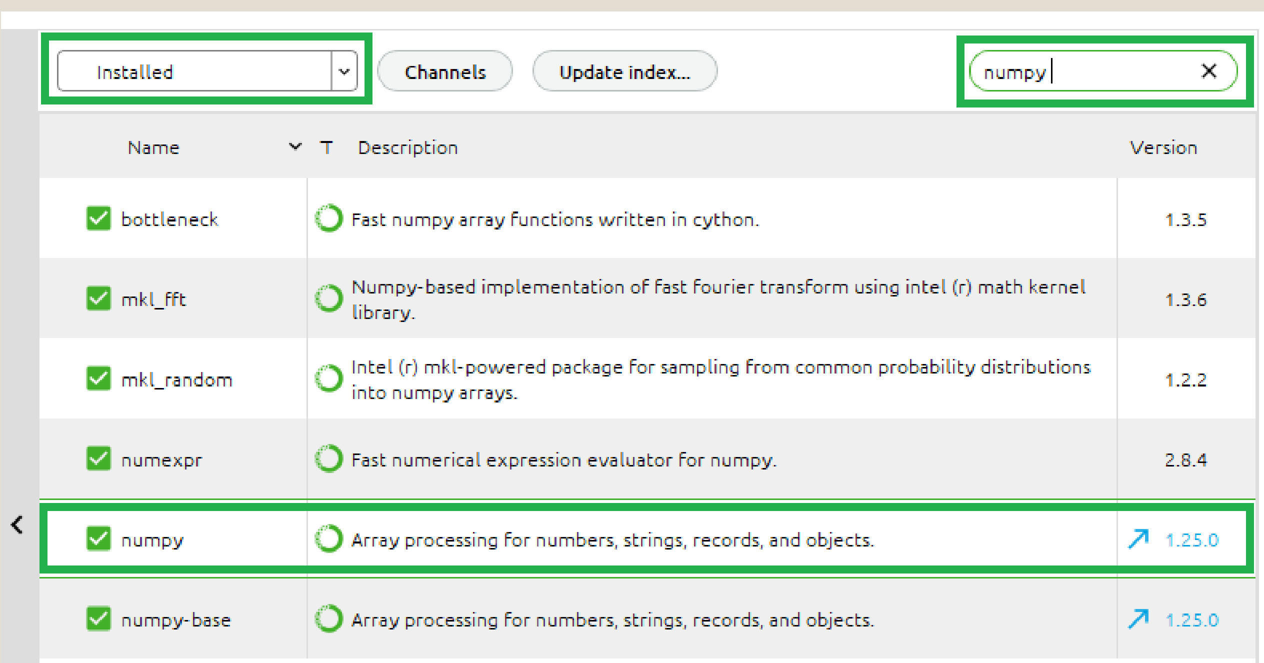
Out[12]: 7

As you can see the answer is **“7”**, as I computed earlier!

Determinant computing by “Numpy”

What is ***Numpy***?

- NumPy is a library for the Python programming language, adding support for large, multi-dimensional arrays and matrices, along with a large collection of high-level mathematical functions to operate on these arrays.
- I have this library already installed.



Determinant computing by “Numpy” (cont.)

- Then do this coding parts: (I considered the previous matrix)

Numpy

https://numpy.org/doc/stable/reference/generated/numpy.linalg.det.html

In [1]:
```
import numpy as np  # importing the library
```

Computing determinants for a stack of matrices:

In [2]:
```
# this is the matrix which I earlier computed the determinant by hand and also by SymPy
Q = np.array([[1, 2, 3, 4], [1, 0, 2, 0], [0, 1, 2, 3], [2, 3, 0, 0]])
Q
```

Out[2]:
```
array([[1, 2, 3, 4],
       [1, 0, 2, 0],
       [0, 1, 2, 3],
       [2, 3, 0, 0]])
```

In [3]:
```
# let's check out the shape of the matrix
Q.shape
```

Out[3]:
```
(4, 4)
```

Determinant

To compute the determinant of a matrix, use 'linalg.det'.

In [7]:
```
np.linalg.det(Q) # as you can see, I got almost the same answer by hand and by SymPy earlier!
```

Out[7]:
```
6.999999999999999
```

- As you can see the answer is almost **“7”**, as I computed earlier!

References

- https://docs.sympy.org/latest/tutorials/introtutorial/matrices.html#determinant
- https://numpy.org/doc/stable/reference/generated/numpy.linalg.det.html