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Unitaries Representing Inner Toral Polynomials and Behaviour of Pure Algebraic Isopairs at Non-regular Points

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Declaration

I do hereby declare that the work reported in this research report was exactly carried out by me under the supervision of Dr. U. D. Wijesooriya. It describes the result of my own independent work due reference has been made in the text. No part of this research report has been submitted earlier or concurrently for the same or any other degree.

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Abstract

Given a minimal inner toral polynomial $\mathfrak{p}$, there exists a unitary with a determinantal representation. We prove that if the minimal inner toral polynomial $\mathfrak{p}$ is a finite product of polynomials, then a unitary can be obtained using unitaries representing its factors. Also the converse can be done. Regular points are points from the zero set of an inner toral polynomial where the gradient of the polynomial is non zero. A pair of pure commuting isometries is called a pure $\mathfrak{p}$-isopair. Given a pure $\mathfrak{p}$- isopair with finite bimultiplicity, dimension of the intersection of kernels of the adjoints of given operator pair at a regular point is 1. We show that the converse is not true. We found general forms of unitaries representing all linear inner toral polynomials. Formed a conjecture about the dimension of such isopairs when the considered polynomial annihilates at the origin.

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Dedication

To all my loved ones.

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Chapter 1

INTRODUCTION

A polynomial $\mathfrak{p} = \mathfrak{p}(z,w) \in \mathbb{C}[z,w]$ is called an inner toral polynomial if the zero set of $\mathfrak{p}$, $Z(\mathfrak{p}) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$ where $\mathbb{D}$ is the open unit disk, $\mathbb{T}$ is the unit circle and $\mathbb{E}$ is the exterior of the closed unit disk. An inner toral polynomial $\mathfrak{p} = \mathfrak{p}(z,w) \in \mathbb{C}[z,w]$, a minimal inner toral polynomial if $\mathfrak{p}$ divides any $\mathfrak{q} \in \mathbb{C}[z,w]$ with the same zero set (i.e. when $Z(\mathfrak{p}) = Z(\mathfrak{q})$).

In 2005, Jim Agler and John McCarthy have proved that we can represent a minimal inner toral polynomial with a unitary block matrix with some conditions. We compute unitaries for all linear inner toral polynomials in the next chapters. We prove that if a higher degree inner toral polynomial can be written as a product of lower degree inner toral polynomials, then unitaries representing the first polynomial can be obtained by a special direct sum which we call the diagonal-wise direct sum. Converse of the above seems to be true with all the examples we worked with. Currently we are halfway through the proof.

Inner product is a function that gives out a scalar when two vectors from the considered vector space are multiplied together. This function has to satisfy a certain set of conditions to be called an inner product. The concept of a vector space with an inner product was first taken into the discussion in 1898 by an Italian mathematician Giuseppe Peano. Inner product space is a vector space with an additional structure, which is called the inner product. Inner product space can also be identified as the space with minimum requirements to define both the notion of length of a vector and the angle between two vectors.

Hilbert space is a complete inner product space. An inner product space is said to be complete if every Cauchy sequence of the considered inner product space converges to an element in the same space. These spaces are named after the German mathematician, David Hilbert who lived in 19^{th} and early 20^{th} century. David Hilbert, Erhard Schmidt and Frigyes Riesz initiated the study of these spaces in the first decade of 20^{th} century. Later, John Von Neumann who is a Hungarian mathematician coined the name "Hilbert space" to honour the service rendered towards this area by David Hilbert.

An isometry is a distance and angle preserving map between two Hilbert spaces. Shift operator and unitary operator are examples of isometries. Shift operator, which is also called the translation operator translates elements of the Hilbert space. A bounded linear operator which is bijective and between two Hilbert spaces is called a unitary if the adjoint of the operator is same as the inverse operator. Wold-Von Neumann decomposition which is usually referred to as the Wold decomposition, named after the Swedish mathematician Herman Wold and John Von Neumann, is a classification theorem for isometries on a given Hilbert space. The theorem states that every isometry can be represented as a direct sum of a unitary operator and copies of unilateral shift. An isometry is called pure if there is no unitary part in this decomposition.

The multiplicity of a pure isometry is defined as $\text{mult}(T) = \dim(\ker(T^*))$ where T^* is the adjoint operator of T. If $V = (S, T)$ is a pair of pure isometries, then the bimultiplicity of V is defined as $\text{bimult}(V) = (\text{mult}(S), \text{mult}(T))$. A pair of commuting pure isometries (S, T) is a pure algebraic isopair if there is a non zero polynomial $q \in \mathbb{C}[z, w]$ such that $q(S, T) = 0$. In 2002, Jim Agler, Greg Knese and John McCarthy has proved that for any given pure algebraic isopair V, we can find another polynomial in $\mathbb{C}[z, w]$ which is minimal in the sense that it satisfies V and also divides any other polynomial which satisfies V. This minimal polynomial is a minimal inner toral polynomial.

Regular point for a minimal inner toral polynomial $\mathfrak{p}$ is a point in $\mathbb{C}^2$ which is taken from the zero set of $\mathfrak{p}$ such that the gradient of $\mathfrak{p}$ at that point is non-zero. In 2018, U. D. Wijesooriya has proved that for a minimal inner toral polynomial $\mathfrak{p}$ with bidegree (n, m), if (S, T) is a pure $\mathfrak{p}$-isopair with bimultiplicity (m, n) and (λ, μ) is regular then, $\dim[\, \ker(S - \lambda I)^* \cap \ker(T - \mu I)^*] = 1$.

We fix the minimal inner toral polynomial be $\mathfrak{p} = p(z, w) = z^2 - w^2$ and compute the given dimension at $(0, 0)$ which is not a regular point. By this, we provide plenty of counterexamples to show that the converse of this theorem does not hold. Also during the process of computing the dimension we compute all the unitary matrices with $-1, 0, 1$ as entries and representing $\mathfrak{p}$.

Further, we compute the dimension for higher degree inner toral polynomials at $(0, 0)$. With the help of examples, we have observed a pattern and it is presented as a conjecture which could be helpful in computing the dimension.

Chapter 2

PRELIMINARIES

2.1 Hilbert Space

Definition 2.1. *A vector space (or a linear space) over a field $\mathbb{F}$ (mostly $\mathbb{R}$ or $\mathbb{C}$) is a set X of elements (Vectors) together with two operations,(addition and scalar multiplication), satisfying following eight axioms.*

(i) (Associativity of addition) $x + (y + z) = (x + y) + z$ for every $x, y, z \in X$.

(ii) (Commutativity of addition) $x + y = y + x$ for every $x, y \in X$.

(iii) (Identity element of addition) There exists an element $x \in X$, called the zero vector, such that $x + 0 = x$ for all $x \in X$.

(iv) (Inverse element of addition) For every $x \in X$, there exists an element $-x \in X$, called the additive inverse of x, such that $x + (-x) = 0$.

(v) (Compatibility of scalar multiplication with field multiplication) $\alpha(\beta x) = (\alpha\beta)x$ for every $\alpha, \beta \in \mathbb{F}$.

(vi) (Identity element of scalar multiplication) $1x = x$, where 1 denotes the multiplicative identity in $\mathbb{F}$.

(vii) *(Distributivity of scalar multiplication with respect to vector addition)* $\alpha(x+y) = \alpha x + \alpha y$ *for every* $\alpha \in \mathbb{F}$ *and* $x, y \in X$.

(viii) *(Distributivity of scalar multiplication with respect to field addition)* $(\alpha + \beta)x = \alpha x + \beta x$ *for every* $\alpha, \beta \in \mathbb{F}$ *and* $x \in X$.

Definition 2.2. *Let* X *be a vector space over* $\mathbb{F}$ *and* M *be a non empty subset of* X. *If* $\alpha x_1 + \beta x_2 \in X$ *whenever* $x_1, x_2 \in X$ *and* $\alpha, \beta \in \mathbb{F}$, *then* M *is a linear subspace of* X.

Equivalently, we could say M is a linear subspace if M is a vector space under the operations of X. It is denoted as $M \leqslant X$.

Definition 2.3. *Let* X *be a vector space over* $\mathbb{F}$. *A function* $|| \;\; || : X \longrightarrow \mathbb{R}$ *is said to be a norm on* X, *if*

(i) $||x|| \geq 0$ *for all* $x \in X$,

(ii) $||x|| = 0$ *if and only if* $x = 0$,

(iii) $||\alpha x|| = \alpha||x||$ *for all* $x \in X$ *and for all* $\alpha \in \mathbb{F}$,

(iv) $||x + y|| \leq ||x|| + ||y||$ *for all* $x, y \in X$.

The pair $(X, || \;\; ||)$ *is called a normed vector space (or a normed space or a normed linear space).*

Theorem 2.1. *Every normed vector space is a metric space.*

Proof. Let $(X, || \;\; ||)$ be a normed vector space.Define

$$d : X \times X \longrightarrow \mathbb{R}$$

by $d(x, y) = ||x - y||$ for all $x, y \in X$. It is easily followed that d is a metric on X. Therefore X is a metric space. $\square$

Definition 2.4. *Let X be a vector space over a field $\mathbb{F}$. A map $\langle \cdot \rangle : X \times X \longrightarrow \mathbb{F}$ is called an inner product if*

(i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ *for all* $x, y \in X$,

(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *for all* $x, y, z \in X$,

(iii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ *for all* $x, y \in X$ *and* $\alpha \in \mathbb{F}$,

(iv) $\langle x, x\rangle \geq 0$ *for all* $x \in X$,

(v) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

The pair $(X, \langle \cdot \rangle)$ is called the inner product space.

Example 2.1. *The map $\langle \ , \ \rangle : \mathbb{C}^N \times \mathbb{C}^N \longrightarrow \mathbb{C}$ given by*

$$\langle z, w\rangle = \sum_{k=1}^{N} z_k \overline{w}_k$$

is an inner product on $\mathbb{C}^N$.

Example 2.2. *Let $C[a, b] = \{f | f : [a, b] \longrightarrow \mathbb{F}, f \text{ is continuous}\}$. The map $\langle \ , \ \rangle : C[a, b] \times C[a, b] \longrightarrow \mathbb{F}$ given by*

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt$$

is an inner product on $C[a, b]$.

By an inner product defined on a vector space, a norm can be induced. Also we proved that a norm can induce a metric. So a metric can be induced by an inner product.

Definition 2.5. *Let $(X, \langle \cdot \rangle)$ be an inner product space. Define $|| \, || : X \longrightarrow \mathbb{R}$ by $||x|| = \sqrt{\langle \, x, x\rangle}$ for all $x \in X$. $|| \quad ||$ is a norm on X and we call it the norm induced by the inner product. Define $d : X \times X \longrightarrow \mathbb{R}$ by $d(x, y) = ||x - y||$ where $|| \, ||$ is the norm induced by the inner product.*

Recall that d is a metric on X and we call it the metric induced by the inner product.

Definition 2.6. *Let $(X, \langle \cdot \rangle)$ be an inner product space and let d be the metric induced by the inner product. If (X, d) is complete, then we call $(X, \langle \cdot \rangle)$ a Hilbert space.*

Example 2.3. *$\mathbb{C}^N$ is a Hilbert space with respect to the inner product defined in example 2.1 above.*

Example 2.4. *$C[a, b]$ is not a Hilbert space with respect to the inner product defined in example 2.2.*

2.2 Isometry

An operator is a mapping between two spaces. The term operator is usually used when the considered spaces are vector spaces. In general operators act on elements of a space to produce elements on the other space. In this section we pay attention isometries which are linear.

A map between two vector spaces which is compatible with their linear structures is called a linear operator. It can be defined formally as follows.

Definition 2.7. *Let X, Y be vector spaces over the field $\mathbb{F}$. Let T be an operator from X to Y. Define $T : X \longrightarrow Y$. If*

(i) $T(x + y) = Tx + Ty$ for all $x, y \in X$ and

(ii) $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and $\alpha \in \mathbb{F}$

then, T is linear.

Throughout this report since we look at isometries between Hilbert spaces, let us define an isometry between two Hilbert spaces.

Definition 2.8. *Let $(H, \langle \cdot, \cdot \rangle_H)$ and $(K, \langle \cdot, \cdot \rangle_K)$ be Hilbert spaces. A linear map $T : H \to K$ is an isometry if and only if $\langle g, h \rangle_H = \langle Tg, Th \rangle_K$ for all $g, h \in H$.*

Let us consider an image on the plane. This can be altered by using different operations. All these can be taken as a combination of translation (which moves the image in any direction), reflection (which flips the image over a line), rotation (which rotates the image) and dilation (which enlarge or reduce the image). Therefore it is clear that the distance between any two points of the first image is same as the distance between the same two points in the final image in translation, reflection and rotation. So translation, reflection and rotation are isometries. But in dilation since the size of the image changes, it is not an isometry.

Definition 2.9. *Let $(H, \langle \cdot, \cdot \rangle_H)$ and $(K, \langle \cdot, \cdot \rangle_K)$ be Hilbert spaces. A linear map $T : H \to K$ is bounded if there exists $C > 0$ such that $\langle Tg, Th \rangle_K \leq C \langle g, h \rangle_H$ for all $g, h \in H$.*

Definition 2.10. *If $T : H \to K$ is a bounded linear operator, where H and K are Hilbert spaces, then the unique operator $W : K \to H$ such that $\langle Tx, y \rangle_K = \langle x, Wy \rangle_H$ for all $x \in H$ and $y \in K$ is called the adjoint of T and denoted by T^* [i.e. $W = T^*$].*

We will be mainly focusing on shift operators and unitary operators on Hilbert spaces in coming sections. Shift operators and unitary operators are examples for isometries, meaning that they preserve distance. Following are few examples of shift operators on different spaces.

- On sequence spaces $S_R : (a_1, a_2, a_3, ...) \to (0, a_1, a_2, a_3, ...)$
- On function spaces $S_t : f(x) \to f(x + t)$
- On polynomials $M_x : P(x) \to xP(x)$

Definition 2.11. *A bounded linear operator $T : H \to K$ where H and K are Hilbert spaces is said to be unitary if T is bijective and $T^* = T^{-1}$.*

In the above definition $T^* = T^{-1}$ is equivalent to saying that, $T^*T = TT^* = I$ where I is the identity map. Unitary matrices, which will be discussed in later sections are precisely the unitary operators on finite-dimensional Hilbert spaces. So one could say that the notion of a unitary operator is a generalization of the notion of a unitary matrix. Further, Fourier operator is a unitary operator which performs the Fourier transform.

Definition 2.12. *Let $T : H \to H$ is a bounded linear operator. A subspace $M \subseteq H$ is called a reducing subspace for T, if $TM \subseteq M$ and $TM^{\perp} \subseteq M^{\perp}$.*

In other words, for $T : H \to H$ is a bounded linear operator and $M \leqslant H$, M is said to be an invariant subspace for T if $Th \in M$ whenever $h \in M$. This leads to a result which states that, $M \oplus M^{\perp} = H$ when M is a linear subspace of H. A stronger version of this, which is called the "Wold Decomposition" is stated below.

Theorem 2.2 (**Wold Decomposition**)**.** *If T is an isometry on a Hilbert space H, then there exist two reducing subspaces for T, say K and L, such that*

- $H = K \oplus L$,
- $S = T\mid_K$ *is a shift operator and*
- $U = T\mid_L$ *is a unitary operator.*

An isometry is said to be pure if does not contain a unitary part. We define multiplicity for such isometries.

Definition 2.13. *The multiplicity of a pure isometry T is defined as*

$$\mathit{mult}(T) = \dim(\ker(T^*))$$

where T^* is the adjoint of T.

Definition 2.14. *Let S, T be two pure isometries and $V = (S, T)$. Define the bimultiplicity of V by $\mathit{bimult}(V) = (\mathit{mult}(S), \mathit{mult}(T))$.*

Theorem 2.3 (Identity Theorem)**.** *If two holomorphic functions f and g on a domain D agree on a set S which has an accumulation point c in D, then $f = g$ on a disk in D centered at c.*

Chapter 3

UNITARIES REPRESENTING INNER TORAL POLYNOMIALS

3.1 Inner Toral Polynomial

Definition 3.1. *A polynomial* $\mathfrak{p} = \mathfrak{p}(z, w) \in \mathbb{C}[z, w]$ *is called an inner toral polynomial if the zero set of* $\mathfrak{p}$*,*

$$Z(\mathfrak{p}) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$$

where $\mathbb{D}$ *is the open unit disk,* $\mathbb{T}$ *is the unit circle and* $\mathbb{E}$ *is the exterior of the closed unit disk.*

In other words, if $(z, w) \in Z(\mathfrak{p})$ then,

$$|z| < 1 \;\Leftrightarrow |w| < 1 \text{ or}$$

$$|z| = 1 \;\Leftrightarrow |w| = 1 \text{ or}$$

$$|z| > 1 \;\Leftrightarrow |w| > 1.$$

Example 3.1. $z - w,\ z^2 - w^2,\ z + iw,\ z^3 - w^2$ *are examples for inner toral polynomials*

Example 3.2. *$z - w + 1$, $z^2 - w - z$ are not inner toral polynomials. Since $(0,1)$ satisfies $z - w + 1 = 0$, but it violates the required condition. And $(1,0)$ satisfies $z^2 - w - z = 0$, but it also violates the conditions required by an inner toral polynomial.*

Definition 3.2. *We say an inner toral polynomial $\mathfrak{p} = \mathfrak{p}(z,w) \in \mathbb{C}[z,w]$, a minimal inner toral polynomial if $\mathfrak{p}$ divides any $\mathfrak{q} \in \mathbb{C}[z,w]$ with $Z(\mathfrak{p}) = Z(\mathfrak{q})$.*

Example 3.3. *Let $\mathfrak{q} = 2z - 2w$ and its zero set be $Z(\mathfrak{q})$. Zero set of this polynomial is given by the set of points of the form (a,a) where $a \in \mathbb{C}$. Consider the polynomial $\mathfrak{p} = z - w$. It is clear that $Z(\mathfrak{p}) = Z(\mathfrak{q})$ and $\mathfrak{p}|\mathfrak{q}$. Therefore, $\mathfrak{p}$ is a minimal inner toral polynomial but $\mathfrak{q}$ is not.*

3.2 Unitary Matrices

Definition 3.3. *A square matrix U is called an unitary matrix if its conjugate transpose U^* is its inverse. That is, $UU^* = U^*U = I$.*

In the paper *Distinguished Varieties* by Jim Agler and John McCarthy they have proved that for a given minimal inner toral polynomial, there exist a unitary matrix which represents the polynomial. Further, it should satisfy few properties as stated below.

Theorem 3.1 (**Jim Agler, Jhon McCarthy**, Distinguished Varieties)**.** *For a minimal inner toral polynomial $\mathfrak{p}(z,w)$ of bidegree (n,m), there exist an $(m+n) \times (m+n)$ unitary matrix U such that*

$$U = \begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{array}{c} m \\ n \end{array} \end{array}$$

- $\det \begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} = k\ \mathfrak{p}(z,w)$ *where k is a constant and*
- *if* $\det(D - \lambda I_n) = 0$, *then* $|\lambda| \neq 1$.

Let $U(z,w) = \begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix}$.

Here, A, B, C and D are matrices of sizes $m \times m$, $n \times m$, $m \times n$ and $n \times n$ respectively. We call such U unitaries representing minimal inner toral polynomial $\mathfrak{p}(z,w)$.

Example 3.4. *For the polynomial* $p(z,w) = z^2 - w^2$.

Consider

$$\mathfrak{U}_1 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

First we should check if this is a unitary matrix. For that we should check if $\mathfrak{U}_1\mathfrak{U}_1^* = \mathfrak{U}_1^*\mathfrak{U}_1 = I_4$ *where* I_4 *is the idetity matrix of order* 4. $\mathfrak{U}_1^* = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$. *Therefore,* $\mathfrak{U}_1\mathfrak{U}_1^* = \mathfrak{U}_1^*\mathfrak{U}_1 = I_4$.

It is clear that the sizes of the blocks are according to the bi-degree of the polynomial $p(z,w)$. *Since the block* D *is the zero matrix, it has no uni modular eigenvalues.*

Let us consider the determinant of the shifted matrix.

i.e. $\det \left(\begin{array}{cc|cc} 0 - w & 0 & 0 & z \cdot 1 \\ 0 & 0 - w & z \cdot 1 & 0 \\ \hline 1 & 0 & 0 - 1 & 0 \\ 0 & 1 & 0 & 0 - 1 \end{array}\right) = (-1)(z^2 - w^2).$

Hence, $\mathfrak{U}_1$ *is a unitary representing the polynomial* $p(z,w)$.

Definition 3.4. *For an inner toral polynomial* $\mathfrak{p}$, *the set*

$$\mathfrak{V}(p) = Z(\mathfrak{p}) \cap \mathbb{D}^2$$

is called a distinguished variety.

Theorem 3.2 (**Greg Knese**, Polynomials Defining Distinguished Varities). *Let $\mathfrak{p} = \mathfrak{p}(z, w)$ be a minimal inner toral polynomial of bi-degree (n, m) and* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a unitary representing $\mathfrak{p}$. There exist matrix polynomials $Q(z, w)$ and $P(z, w)$ such that*

$$\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} \begin{pmatrix} Q(z,w) \\ P(z,w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } (z, w) \in \mathfrak{V}(\mathfrak{p}).$$

These matrix polynomials $Q(z, w)$ and $P(z, w)$ should satisfy the following properties too.

- $Q(z, w)$ is of bi-degree at most $(n, m - 1)$ and $P(z, w)$ is of bi-degree at most $(n - 1, m)$.
- $Q(z, w)$ has size $m \times 1$ and $P(z, w)$ has size $n \times 1$.
- Q and P have at most finitely many common zeros on $\mathfrak{V}(\mathfrak{p})$.

Such (Q, P) pairs are called admissible pairs.

Example 3.5. *Consider the unitaries representing the minimal inner toral polynomial $z^3 - w^2$.*

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_{(U_1)}(z, w) = \begin{pmatrix} 1 \\ w \end{pmatrix} \quad \text{and } P_{(U_1)}(z, w) = \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix}.$$

$$U_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Q_{(U_2)}(z,w) = \begin{pmatrix} w \\ z^2 \end{pmatrix} \text{ and } P_{(U_2)}(z,w) = \begin{pmatrix} w \\ z^2 \\ zw \end{pmatrix}.$$

(Q_{U_1}, P_{U_1}) and (Q_{U_2}, P_{U_2}) are admissible pairs.

3.3 Diagonal-wise Direct Sum

Definition 3.5. *Let U_i be a block matrix of the following form for $i = 1,\ 2,\ \ldots\ ,\ s$ with given sizes.*

$$U_i = \begin{array}{cc} \begin{array}{cc} m_i & n_i \end{array} & \\ \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} & \begin{array}{c} m_i \\ n_i \end{array} \end{array}$$

$$U_1 \oplus_D U_2 \oplus_D \ldots \oplus_D U_s = \begin{array}{cc} \begin{array}{cccccccc} m_1 & m_2 & \ldots & m_s & n_1 & n_2 & \ldots & n_s \end{array} & \\ \left(\begin{array}{cccc|cccc} A_1 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_s & 0 & 0 & 0 & B_s \\ \hline C_1 & 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & D_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C_s & 0 & 0 & 0 & D_s \end{array}\right) & \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_s \\ n_1 \\ n_2 \\ \vdots \\ n_s \end{array} \end{array}$$

This is called the diagonal-wise direct sum and denoted by $\oplus_D$.

3.4 Results

Let $\mathfrak{p} = \mathfrak{p}(z,w) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3...\mathfrak{p}_s$ be an minimal inner toral polynomial. Let bi-degree of the polynomial $\mathfrak{p}$ and $\mathfrak{p}_i$ be (n,m) and (n_i,m_i) for all $i = 1,2,...,s$ respectively. And let U and U_i be a square matrices of order $(m+n)$ and (m_i+n_i) for all $i = 1,2,...,s$ respectively.

Theorem 3.3 (Theorem 1)**.** *Let U_i be an unitary representing the polynomial $\mathfrak{p}_i$ for each $i = 1,2,...,s$. If the matrix U is obtained as above by the diagonal-wise direct sum of U_i's, then U is a unitary which represents the polynomial $\mathfrak{p}$.*

Proof. Let $U_k = \begin{array}{cc} \begin{array}{cc} m_k & n_k \end{array} & \\ \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} & \begin{array}{c} m_k \\ n_k \end{array} \end{array}$ for each $k = 1,2,...,s$. Let $\mathfrak{p}_k$ be a minimal inner toral polynomial and U_k be a unitary representing $\mathfrak{p}_k$. Therefore, from theorem 3.1, $\det(U_k(z,w)) = c_k\ \mathfrak{p}_k(z,w)$. Further, there exists Q_k and P_k matrix polynomials such that,

$$U_k(z,w)\begin{pmatrix} Q_k(z,w) \\ P_k(z,w) \end{pmatrix} = 0 \quad \text{for all} \ \ (z,w) \in \mathfrak{V}_k. \tag{3.1}$$

Let

$$U = \begin{array}{cc} \begin{array}{cccccccc} m_1 & m_2 & \cdots & m_s & n_1 & n_2 & \cdots & n_s \end{array} & \\ \left(\begin{array}{cccc|cccc} A_1 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_s & 0 & 0 & 0 & B_s \\ \hline C_1 & 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & D_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C_s & 0 & 0 & 0 & D_s \end{array}\right) & \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_s \\ n_1 \\ n_2 \\ \vdots \\ n_s \end{array} \end{array}$$

Now from equation (3.1),

$$\left(\begin{array}{cccc|cccc} A_1 - wI_{m1} & 0 & 0 & 0 & zB_1 & 0 & 0 & 0 \\ 0 & A_2 - wI_{m2} & 0 & 0 & 0 & zB_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_s - wI_{ms} & 0 & 0 & 0 & zB_s \\ \hline C_1 & 0 & 0 & 0 & zD_1 - I_{n1} & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & zD_2 - I_{n2} & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C_s & 0 & 0 & 0 & zD_s - I_{n3} \end{array}\right) \begin{pmatrix} Q_1(z,w) \\ Q_2(z,w) \\ \vdots \\ Q_s(z,w) \\ P_1(z,w) \\ P_2(z,w) \\ \vdots \\ P_s(z,w) \end{pmatrix} = 0 \tag{3.2}$$

But, now we cannot guarantee that (Q, P) has only finitely many common zeros. So we replace

$$\begin{aligned} Q_1 &\ \ by\ \ \mathfrak{p}_2 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_s \cdot Q_1 \\ Q_2 &\ \ by\ \ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_s \cdot Q_2 \\ &\vdots \\ Q_{s-1} &\ \ by\ \ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_s \cdot Q_{s-1} \\ Q_s &\ \ by\ \ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_{s-1} \cdot Q_s \end{aligned}$$

$$\begin{aligned} P_1 &\ \ by\ \ \mathfrak{p}_2 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_s \cdot P_1 \\ P_2 &\ \ by\ \ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_s \cdot P_2 \\ &\vdots \\ P_{s-1} &\ \ by\ \ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_s \cdot P_{s-1} \\ P_s &\ \ by\ \ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot \ldots \cdot \mathfrak{p}_{s-1} \cdot P_s \end{aligned}$$

Now new (Q, P) can be written as follows.

$$Q = \begin{pmatrix} \mathfrak{p}_2 \cdot \mathfrak{p}_3 \cdot ... \cdot \mathfrak{p}_s \cdot Q_1 \\ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot ... \cdot \mathfrak{p}_s \cdot Q_2 \\ \vdots \\ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot ... \cdot \mathfrak{p}_{s-1} \cdot Q_s \end{pmatrix}$$

$$P = \begin{pmatrix} \mathfrak{p}_2 \cdot \mathfrak{p}_3 \cdot ... \cdot \mathfrak{p}_s \cdot P_1 \\ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot ... \cdot \mathfrak{p}_s \cdot P_2 \\ \vdots \\ \mathfrak{p}_1 \cdot \mathfrak{p}_3 \cdot ... \cdot \mathfrak{p}_{s-1} \cdot P_s \end{pmatrix}$$

So,

$$U(z,w) \begin{pmatrix} Q(z,w) \\ P(z,w) \end{pmatrix} = 0 \quad \text{for all} \quad (z,w) \in \mathfrak{V}. \tag{3.3}$$

Since Q and P have only finitely many common zeroes, Q and P vanishes almost everywhere in $\mathfrak{V}$. So, $\det(U(z,w)) = 0$ almost everywhere in $\mathfrak{V}$. By the identity theorem, $\det(U(z,w)) = 0$ everywhere in $\mathfrak{V}$. Since $\mathfrak{p}$ is a minimal inner toral polynomial and $\det(U(z,w)) = c \cdot \mathfrak{p}$ has bidegree atmost (n, m),

$$\det(U(z,w)) = c\ \mathfrak{p}(z,w).$$

$\square$

Converse of the theorem also seems to be true. That is, if U is a unitary of the above form representing the polynomial $\mathfrak{p}$, then U_i is a unitary representing $\mathfrak{p}_i$ for each $i = 1, 2, ..., s$. But, for now we have one sub case to be proven in order to call it a theorem. Proposed proof is as follows.

Let $\mathfrak{V}(\mathfrak{p})$ and $\mathfrak{V}(\mathfrak{p}_i)$ be distinguished varieties define on the zero set of $\mathfrak{p}$ and $\mathfrak{p}_i$ respectively for $i = 1, 2, ..., s$.

Let $U = \begin{array}{cc} \begin{array}{cccccccc} m_1 & m_2 & \dots & m_s & n_1 & n_2 & \dots & n_s \end{array} & \\ \left(\begin{array}{cccc|cccc} A_1 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_s & 0 & 0 & 0 & B_s \\ \hline C_1 & 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & D_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C_s & 0 & 0 & 0 & D_s \end{array}\right) & \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_s \\ n_1 \\ n_2 \\ \vdots \\ n_s \end{array} \end{array}$ be a unitary representing $\mathfrak{p}$. There exists matrix polynomials (Q, P) such that

$$\left(\begin{array}{cccc|cccc} A_1 - wI_{m1} & 0 & 0 & 0 & zB_1 & 0 & 0 & 0 \\ 0 & A_2 - wI_{m2} & 0 & 0 & 0 & zB_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_s - wI_{ms} & 0 & 0 & 0 & zB_s \\ \hline C_1 & 0 & 0 & 0 & zD_1 - I_{n1} & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & zD_2 - I_{n2} & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C_s & 0 & 0 & 0 & zD_s - I_{n3} \end{array}\right) \begin{pmatrix} Q_1(z,w) \\ Q_2(z,w) \\ \vdots \\ Q_s(z,w) \\ P_1(z,w) \\ P_2(z,w) \\ \vdots \\ P_s(z,w) \end{pmatrix} = 0 \tag{3.4}$$

for all $(z, w) \in \mathfrak{V}(\mathfrak{p})$.

$$Q = \begin{pmatrix} Q_1(z,w) \\ Q_2(z,w) \\ \vdots \\ Q_s(z,w) \end{pmatrix} \quad \text{with degree atmost} \quad (n, m-1).$$

Now,

$$P = \begin{pmatrix} P_1(z,w) \\ P_2(z,w) \\ \vdots \\ P_s(z,w) \end{pmatrix} \quad \text{with degree atmost} \quad (n-1, m).$$

Q and P have only finitely many common zeros.

From equation 3.4,

$$\begin{pmatrix} A_k - wI_m & zB_k \\ C_k & zD_k - I_n \end{pmatrix} \begin{pmatrix} Q_k(z,w) \\ P_k(z,w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } (z,w) \in \mathfrak{V}(\mathfrak{p}).$$

Therefore, equation 3.4 hold for all $(z,w) \in \mathfrak{V}(\mathfrak{p}_k)$.

<u>Case 1 : $Q_k(z,w) \neq 0$ or $P_k(z,w) \neq 0$</u>

Since

$$\begin{pmatrix} A_k - wI_m & zB_k \\ C_k & zD_k - I_n \end{pmatrix} \begin{pmatrix} Q_k(z,w) \\ P_k(z,w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } (z,w) \in \mathfrak{V}(\mathfrak{p}_k).$$

Assume $\det \begin{pmatrix} A_k - wI_m & zB_k \\ C_k & zD_k - I_n \end{pmatrix} \neq 0$, then $\begin{pmatrix} Q_k(z,w) \\ P_k(z,w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is a contradiction.

Therefore, $\det \begin{pmatrix} A_k - wI_m & zB_k \\ C_k & zD_k - I_n \end{pmatrix} = 0.$

Hence, $\det \begin{pmatrix} A_k - wI_m & zB_k \\ C_k & zD_k - I_n \end{pmatrix} = c_k.\mathfrak{p}_k.$

<u>Case 2 : $Q_k(z,w) = 0$ and $P_k(z,w) = 0$</u>

Yet to be proven.

Chapter 4

PURE ALGEBRAIC ISOPAIRS

4.1 Pure $\mathfrak{p}$-isopair

Definition 4.1. *A pair of commuting pure isometries* (S, T) *is a pure algebraic isopair if there is a non zero polynomial* $q \in \mathbb{C}[z, w]$ *such that* $q(S, T) = 0$.

In the paper "Algebraic Pairs of Isometries" by Jim Agler, Greg Knese and John McCarthy, they have proved for a given pure algebraic isopair which annihilates a polynomial, there exist a minimal inner toral polynomial which divides the first polynomial and annihilates the isopair.

Theorem 4.1 (**Jim Agler, Greg Knese, John McCarthy**,Algebraic Pairs of Isometries)**.** *If* (S, T) *is a pure algebraic isopair, then there exists* $\mathfrak{p} = \mathfrak{p}(z, w) \in \mathbb{C}[z, w]$ *such that* $\mathfrak{p}(S, T) = 0$ *and* $\mathfrak{p}$ *divides* q *for any* q *with* $q(S, T) = 0$.

Since $\mathfrak{p}$ divides any polynomial with the zero set $Z(\mathfrak{p})$, $\mathfrak{p}$ is the minimal inner toral polynomial which is annihilated by the isopair (S, T). Therefore, the pair (S, T) is referred to as pure $\mathfrak{p}$-isopair.

4.2 Regular point

For the ease of study a selected set of points from the zero of the polynomial is considered and they are called regular points. The points which are in the zero set and gives zero gradient for the polynomial at the considered point are referred to as irregular points in the discussion.

Definition 4.2. *We say a point $(\lambda, \mu) \in \mathbb{C}^2$ is a regular point for $p(z, w) \in \mathbb{C}[z, w]$ if $(\lambda, \mu) \in Z(p)$ and*

$$\nabla p(\lambda, \mu) = \left(\frac{\partial p}{\partial z}, \frac{\partial p}{\partial w} \right) \Big|_{(\lambda,\mu)} \neq (0, 0)$$

where $Z(p)$ is the zero set of $p(z, w)$.

Example 4.1. *Consider $p(z, w) = z^2 - w^2$. Here $\nabla p(\lambda, \mu) = (2z, 2w) \neq (0, 0)$ for any $(\lambda, \mu) \in Z(p) \setminus \{(0, 0)\}$. Therefore, any $(\lambda, \mu) \in Z(p) \setminus \{(0, 0)\}$ is a regular point of $p(z, w)$.*

Example 4.2. *Zero set of polynomials such as $z - w$, $z + w$, $z + iw$ contains only regular points, since the gradient never becomes zero in those.*

4.3 Results

In 2008 Udeni Wijesooriya, has proved the following theorem in the paper, "Algebraic Pairs of Pure Commuting Isometries with Finite Multiplicity". In this chapter we will check if the converse of this theorem is true.

Theorem 4.2 (**U. D. Wijesooriya**, Algebraic Pairs of Pure Commuting Isometries with Finite Bimultiplicity)**.** *Let $\mathfrak{p}(z, w)$ be a minimal inner toral polynomial of bi-degree (n, m) and (S, T) be a pure $\mathfrak{p}$-isopair, with bimultiplicity (m, n). If $(\lambda, \mu) \in \mathbb{D}^2$ is a regular point for $\mathfrak{p}$, then*

$$\dim[\,\ker(S - \lambda I)^* \cap \ker(T - \mu I)^*] = 1.$$

Considered converse of the theorem can be stated as follows.

Let $\mathfrak{p}(z,w)$ be a minimal inner toral polynomial of bi-degree (n,m) and (S,T) be a pure $\mathfrak{p}$-isopair, with bimultiplicity (m,n). If $\dim[\ \ker(S-\lambda I)^* \cap \ker(T-\mu I)^*] = 1$, then $(\lambda,\mu) \in \mathbb{D}^2$ is a regular point for $\mathfrak{p}$.

After several computations we found counter examples where the converse of this theorem does not hold. In order to discuss the counter example we need the following definitions and theorems.

In the paper "Nevanlinna-Pick Interpolation on Distinguished Varieties in the Bidisk" by Michael Jury, Greg Knese and Scott McCullough, they have defined a function named admissible kernel and proved that the pair M_z and M_w that are multiplication operators by z and w respectively on Hilbert space generated by the admissible kernel is a pure algebraic isopair.

Definition 4.3. *A pair P and Q as above determines the kernel $K : \mathfrak{V} \times \mathfrak{V} \to \mathbb{C}$*

$$K((z,w),(\zeta,\eta)) = \frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1-w\overline{\eta}}$$

on $\mathfrak{V} \times \mathfrak{V}$. K is called the admissible kernel.

Theorem 4.3 (**M. T. Jury, G. Knese, S. McCullough**, Nevanlinna-Pick Interpolation on Distinguished Varietiesin the Bidisk)**.** *Let K be an admissible kernel and write*

$$S = M_z, \qquad T = M_w$$

for the multiplication operators on $H^2(K)$.

- *S and T are pure commuting isometries and*
- *$\mathfrak{p}(S,T) = 0$.*

4.3.1 Counter Example

Let $\mathfrak{p} = z^2 - w^2$ be the minimal inner toral polynomial. Zero set of this polynomial, $Z(\mathfrak{p})$ contains the point $(0,0)$, but it is not a irregular point. Consider $U = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right)$.

Conjugate transpose of U can be written as $U^* = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$. Now, it is clear that $UU^* = U^*U = I_4$ where I_4 is the identity matrix of order 4.

The determinant of $\left(\begin{array}{cc|cc} 0-w & 1 & 0 & 0 \\ 0 & 0-w & z\cdot 1 & 0 \\ \hline 0 & 0 & 0-1 & z\cdot 1 \\ 1 & 0 & 0 & 0-1 \end{array}\right)$ is $(-1)(z^2-w^2)$. Therefore, U is a unitary matrix which represents $\mathfrak{p}$.

Using the theorem 3.2, we can say that an admissible pair (Q, P) exists.

Take $Q = \begin{pmatrix} 1 \\ w \end{pmatrix}$ and $P = \begin{pmatrix} z \\ 1 \end{pmatrix}$.

Consider

$$\left(\begin{array}{cc|cc} 0-w & 1 & 0 & 0 \\ 0 & 0-w & z\cdot 1 & 0 \\ \hline 0 & 0 & 0-1 & z\cdot 1 \\ 1 & 0 & 0 & 0-1 \end{array}\right)\begin{pmatrix} 1 \\ w \\ z \\ 1 \end{pmatrix}.$$

It is clear that the product becomes the zero matrix when $(z, w) \in \mathfrak{V}(\mathfrak{p})$. Hence the pair (Q, P) is an admissible pair.

From the definition 4.3 the admissible kernel can be computed as follows.

$$
\begin{aligned}
K((z,w),(\zeta,\eta)) &= \frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} \\
&= \frac{\begin{pmatrix} 1 & w \end{pmatrix}\begin{pmatrix} 1 \\ \eta \end{pmatrix}}{1-z\overline{\zeta}} \\
&= (1+w\eta)(1+z\overline{\zeta}+(z\overline{\zeta})^2+(z\overline{\zeta})^3+...)
\end{aligned}
$$

where z and ζ are restricted to the open bi-disk (which allows us to use the Taylor series expansion).

Hence, basis elements of the Hilbert space $H^2(K)$ can be written as follows.
$\{1, z, z^2, z^3, ..., w, wz, wz^2, wz^3, ...\}$.
Here $H^2(K)$ is the Hilbert space generated by the admissible kernel K.

In order to compute the dimension we need to see what happens to elements when two operators M_z and M_w act on them. Practically it is not easy to observe the behaviour of all the elements. It is enough to observe the behaviour pattern of basis elements to compute the dimension.

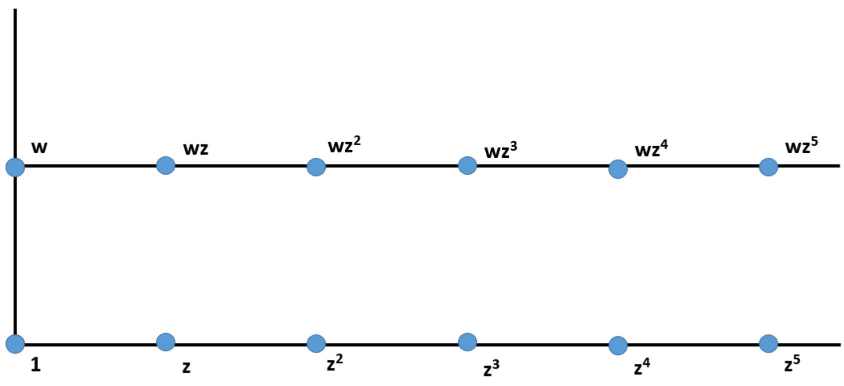

Figure 4.1: Basis elements of the Generated Hilbert space $H^2(K)$.

This figure represents the basis elements of the generated Hilbert space.

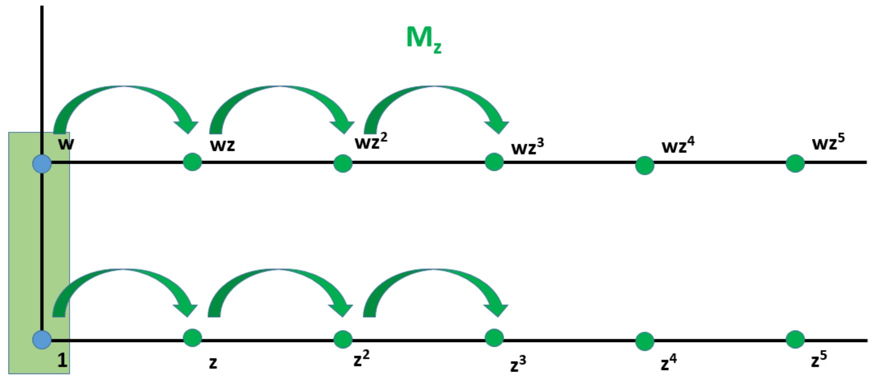

Figure 4.2: Operation of Multiplication by z.

This figure shows us how each basis element change when the operator M_z acts on basis elements. From this it is clear that when $(M_z)^*$ which is the adjoint of the operator M_z acts on basis elements, each element except $\{1,\ w\}$ has an element which produces the considered element when multiplied by z. Therefore $\ker(M_z)^* = \overline{\text{span}\{1, w\}}$.

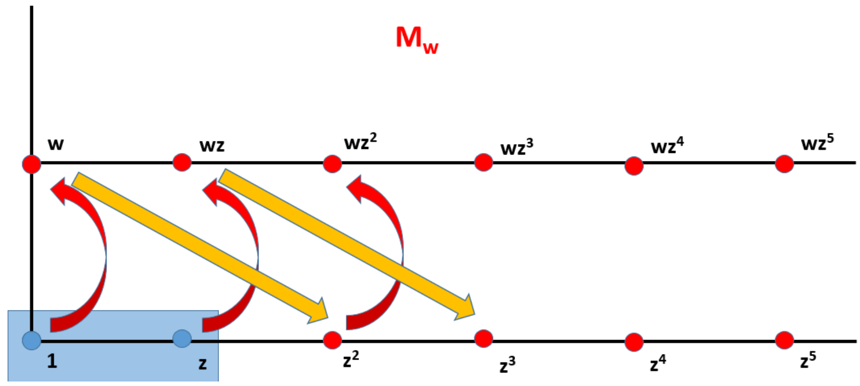

Figure 4.3: Operation of Multiplication by w.

This figure shows us how each basis element change when the operator M_w acts on basis elements. From this it is clear that when $(M_w)^*$ which is the adjoint of the operator M_w acts on basis elements, each element except $\{1,\ z\}$ has an element which produces the considered element when multiplied by w. Therefore $\ker(M_w)^* = \overline{\text{span}\{1, z\}}$.

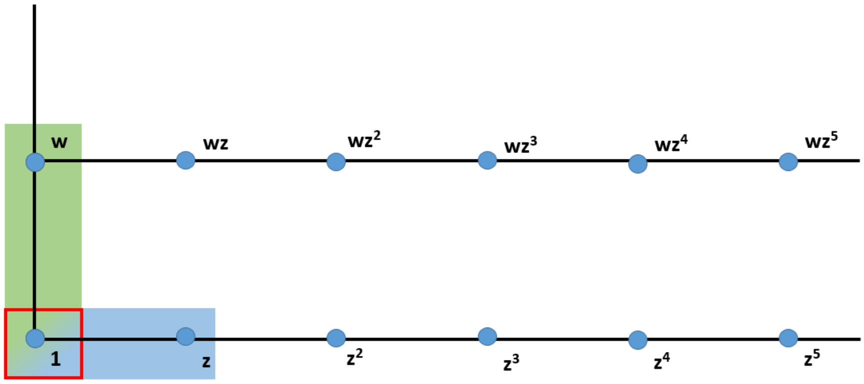

Figure 4.4: Dimension Computation.

From figure 4.2, $\ker(M_z)^* = \overline{\text{span}\{1, w\}}$

From figure 4.3, $\ker(M_w)^* = \overline{\text{span}\{1, z\}}$

$\dim[\ \ker(M_z)^* \cap \ker(M_w)^*] = 1$

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 1$$

Hence, the converse of the Theorem is not true.

4.3.2 More Examples

In order to check if the converse of the Theorem is true we computed all the unitaries with entries $-1, 0, 1$, which are representing the polynomial $\mathfrak{p} = z^2 - w^2$. 32 out of all 64 unitaries lead to counter examples of the above form. All 32 unitaries can be written as follows.

$$\left(\begin{array}{cc|cc} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ \hline 0 & 0 & 0 & d \\ c & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & a & 0 & 0 \\ 0 & 0 & 0 & b \\ \hline c & 0 & 0 & 0 \\ 0 & 0 & d & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ \hline 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & d \\ 0 & c & 0 & 0 \end{array}\right)$$

where $a \cdot b \cdot c \cdot d = 1$ with $a, b, c, d = \pm 1$.

All the computation related to a representative from the first set is already done. Next, consider a representative from the second set.

Let $U = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ is a unitary representing $\mathfrak{p}$ and $Q = \begin{pmatrix} 1 \\ w \end{pmatrix}$ and $P = \begin{pmatrix} 1 \\ z \end{pmatrix}$ be an admissible pair for U.

Since Q is the same matrix, everything from this point onwards behaves in the same manner as the previous example. Now we may consider one representative from the third type.

Let $U = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ is a unitary representing $\mathfrak{p}$. For this unitary, $Q = \begin{pmatrix} w \\ 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 \\ z \end{pmatrix}$ give an admissible pair.

Admissible kernel for obtain (Q, P),

$$
\begin{aligned}
K((z,w),(\zeta,\eta)) &= \frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}} \\
&= \frac{\begin{pmatrix} w & 1 \end{pmatrix}\begin{pmatrix} \eta \\ 1 \end{pmatrix}}{1 - z\overline{\zeta}} \\
&= (w\eta + 1)(1 + z\overline{\zeta} + (z\overline{\zeta})^2 + (z\overline{\zeta})^3 + ...)
\end{aligned}
$$

where z and ζ are restricted to the open bi-disk (which allows us to use the Taylor series expansion).

Hence, basis elements of the Hilbert space $H^2(K)$ can be written as follows.
$\{1, z, z^2, z^3, ..., w, wz, wz^2, wz^3, ...\}$.
Here $H^2(K)$ is the Hilbert space generated by the admissible kernel K.

Once again the basis elements are the same. Therefore, the dimension computation is same from this point onwards. Similarly when a representative from the last set is taken, we have to go through the same steps in the computation which means that it is the same computation as shown above.

Next, consider the following 32 matrices which are represented in the compact form. Those 32 are unitaries representing the polynomial $\mathfrak{p}$ such that the computed dimension is 2.

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ \hline 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ \hline c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{array}\right)$$

where $a \cdot b \cdot c \cdot d = 1$ with $a, b, c, d = \pm 1$.

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 2.$$

Let us consider the unitary with $a = b = c = d = 1$ in the first type.

Let $U = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$.

Now, $Q = \begin{pmatrix} w \\ z \end{pmatrix}$ and $P = \begin{pmatrix} w \\ z \end{pmatrix}$ gives an admissible pair for U.

For the given adissible pair,

$$\begin{aligned} K((z,w),(\zeta,\eta)) &= \frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} \\ &= \frac{\begin{pmatrix} w & z \end{pmatrix}\begin{pmatrix} \eta \\ \zeta \end{pmatrix}}{1-z\overline{\zeta}} \\ &= (w\eta + z\zeta)(1 + z\overline{\zeta} + (z\overline{\zeta})^2 + (z\overline{\zeta})^3 + ...) \end{aligned}$$

where z and ζ are restricted to the open bi-disk (which allows us to use the Taylor series expansion).

Hence, basis elements of the Hilbert space $H^2(K)$ can be written as follows.
$\{z, z^2, z^3, ..., w, wz, wz^2, wz^3, ...\}$.
Here $H^2(K)$ is the Hilbert space generated by the admissible kernel K. From this onwards computations are same as the above example. It can be briefly explained as follows.

Nodes in this figure represents basis elements in the genereted Hilbert space.

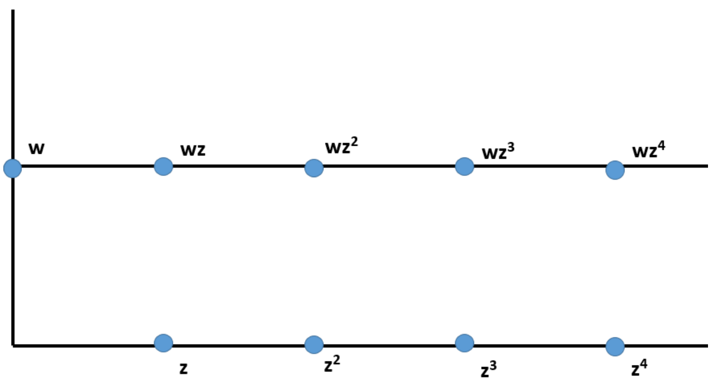

Figure 4.5: Basis elements of the Generated Hilbert space $H^2(K)$.

Next figure shows how basis elements behave under multiplication by z. So it is clear that the kernel of M_z^* is the closed span of $\{z, w\}$. Therefore, $\ker(M_z)^* = \overline{\text{span}\{z, w\}}$.

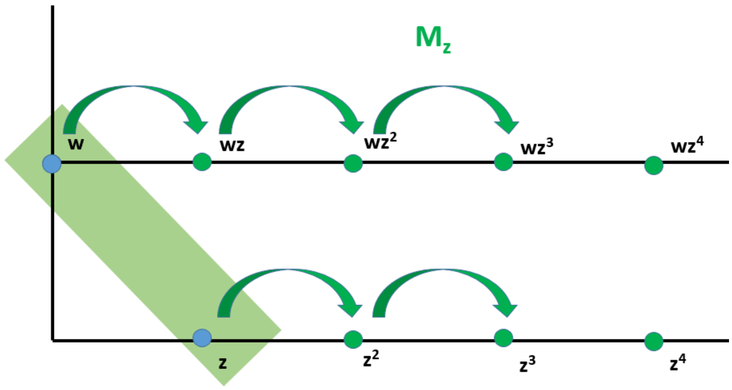

Figure 4.6: Operation of Multiplication by z.

Next figure shows how basis elements behave under multiplication by w. So it is clear that the kernel of M_w^* is the closed span of $\{z, w\}$. Therefore, $\ker(M_w)^* = \overline{\text{span}\{z, w\}}$.

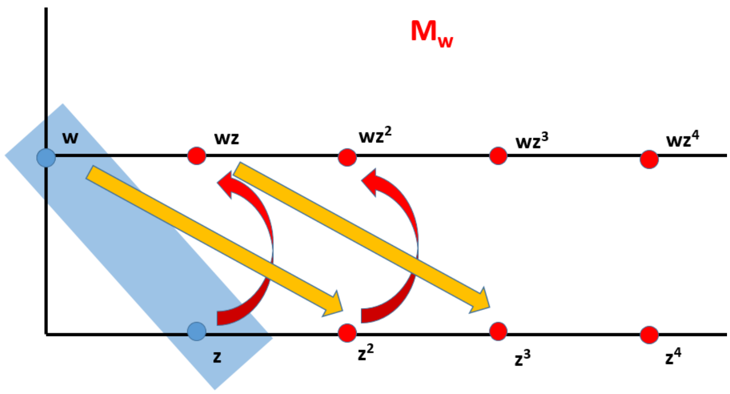

Figure 4.7: Operation of Multiplication by w.

Finally, to compute the dimension of the intersection of kernels of adjoint operators of M_z and M_w, following graph can be used.

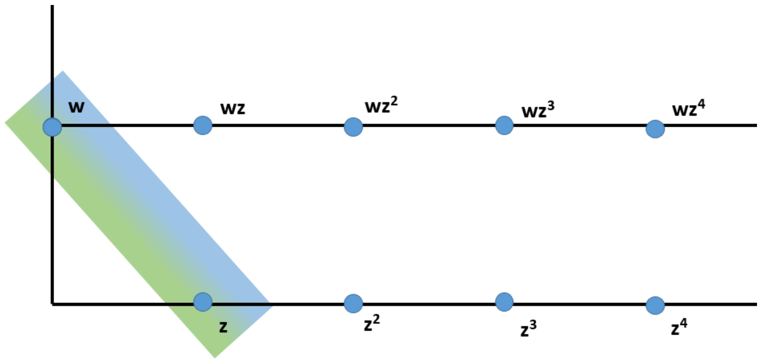

Figure 4.8: Dimension Computation.

From figure 4.6, $\ker(M_z)^* = \overline{\text{span}\{z, w\}}$

From figure 4.5, $\ker(M_w)^* = \overline{\text{span}\{z, w\}}$

$\dim[\ \ker(M_z)^* \cap \ker(M_w)^*] = 2.$

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 2.$$

By a similar computation we can show that the dimension becomes 2 when we start with the given unitaries.

Chapter 5

FURTHER OBSERVATIONS

5.1 Linear inner toral polynomials

We can say that there are only 4 linear inner toral polynomials since any linear inner toral polynomial can be obtained by a constant multiple of the following.

(i) $z + w$,

(ii) $z - w$,

(iii) $z + iw$ and

(iv) $z - iw$.

In this section we give the general form of all the unitaries representing each of the above polynomials.

Let us consider the polynomial $\mathfrak{p}_1 = z + w$.

Take any unitary $\mathcal{U}_1$,

$\mathcal{U}_1 = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$ where $a, b, c, d \in \mathbb{C}$ representing $\mathfrak{p}_1$.

Further, $\det\left(\begin{array}{c|c} a-w & zb \\ \hline c & zd-1 \end{array}\right) = k(z+w)$ By the comparison of coefficients we obtain, $k=1,\ a=0,\ d=0,\ b\cdot c=-1$. Since $d=0$, it has no uni modular eigenvalues. Since U is a unitary, $|b|=|c|=1$. Therefore, unitaries representing $\mathfrak{p}_1$ can be written as follows.

Theorem 5.1. *$\mathcal{U}$ is a unitary of the polynomial $\mathfrak{p}(z,w)=z+w$ if and only if*

$$\mathcal{U} = \begin{pmatrix} 0 & t \\ -\bar{t} & 0 \end{pmatrix}$$

where $t \in \mathbb{C}$ and $|t|=1$.

Similarly by computations of the same fashion we can prove the following theorems.

Theorem 5.2. *$\mathcal{U}$ is a unitary of the polynomial $\mathfrak{p}(z,w)=z-w$ if and only if*

$$\mathcal{U} = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}$$

where $t \in \mathbb{C}$ and $|t|=1$.

Theorem 5.3. *$\mathcal{U}$ is a unitary of the polynomial $\mathfrak{p}(z,w)=z+iw$ if and only if*

$$\mathcal{U} = \begin{pmatrix} 0 & t \\ \bar{t}i & 0 \end{pmatrix}$$

where $t \in \mathbb{C}$ and $|t|=1$.

Theorem 5.4. *$\mathcal{U}$ is a unitary of the polynomial $\mathfrak{p}(z,w)=z-iw$ if and only if*

$$\mathcal{U} = \begin{pmatrix} 0 & t \\ -\bar{t}i & 0 \end{pmatrix}$$

where $t \in \mathbb{C}$ and $|t|=1$.

These 4 theorems cover all linear inner toral polynomials since any other linear inner toral polynomial can be obtained as a constant multiple of one of the above 4.

5.2 Further examples

In this section we look at few examples and later in this section we form a conjecture based on the observations.

Let (S,T) be a pure $\mathfrak{p}$-isopair with bimultiplicity (m,n) and let $\mathfrak{p}$ be a minimal inner toral polynomial with $(0,0) \in Z(\mathfrak{p})$.

For a give pure $\mathfrak{p}$-isopair, (S,T) with bimultiplicity (m,n), we can find a unitary representing $\mathfrak{p}$. Following theorem which was proved by Jim Agler, Greg Knese and John McCarthy guarantees the existence of a unitary.

Theorem 5.5. *If (S,T) is a pure $\mathfrak{p}$-isopair of finite bimultiplicity (m,n), then there exists an $m \times m$ matrix-valued rational inner function $\Phi(z)$, such that*

$$(S,T) \cong (M_z, M_{\Phi(z)}).$$

These $\Phi(z)$ has the form

$$\Phi(z) = A + zB(I_n - zD)^{-1}C$$

where

$$U = \begin{array}{cc} \begin{array}{cc} m & n \end{array} & \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{array}{c} m \\ n \end{array} \end{array}$$

is a unitary such that

$$\det \begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} = k\ \mathfrak{p}(z,w)$$

where k is a constant.

5.2.1 Let $\mathfrak{p}_1 = z^2 - w^3$

Clearly, $\mathcal{U}_{1(\mathfrak{p}1)} = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$ is a unitary representing $\mathfrak{p}_1$.

Next, $Q = \begin{pmatrix} 1 \\ w \\ w^2 \end{pmatrix}$ and $P = \begin{pmatrix} z \\ 1 \end{pmatrix}$ give an admissible pair for this unitary.

$$\begin{aligned} K((z,w),(\zeta,\eta)) &= \frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} \\ &= \frac{\begin{pmatrix} 1 & w & w^2 \end{pmatrix}\begin{pmatrix} 1 \\ \eta \\ \eta^2 \end{pmatrix}}{1-z\overline{\zeta}} \\ &= (1+w\eta+w^2\eta^2)(1+z\overline{\zeta}+(z\overline{\zeta})^2+(z\overline{\zeta})^3+...) \end{aligned}$$

where z and ζ are restricted to the open bi-disk (which allows us to use the Taylor series expansion).

Hence, basis elements of the Hilbert space $H^2(K)$ can be written as follows.
$\{1, z, z^2, z^3, ..., w, wz, wz^2, wz^3, ..., w^2, w^2z, w^2z^2, w^2z^3, ...\}$.
Here $H^2(K)$ is the Hilbert space generated by the admissible kernel K.

Next, the required dimension is computed using pictures.

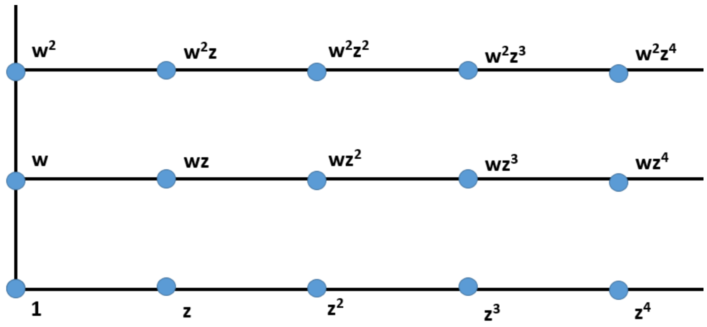

Figure 5.1: Basis elements of the Generated Hilbert space $H^2(K)$.

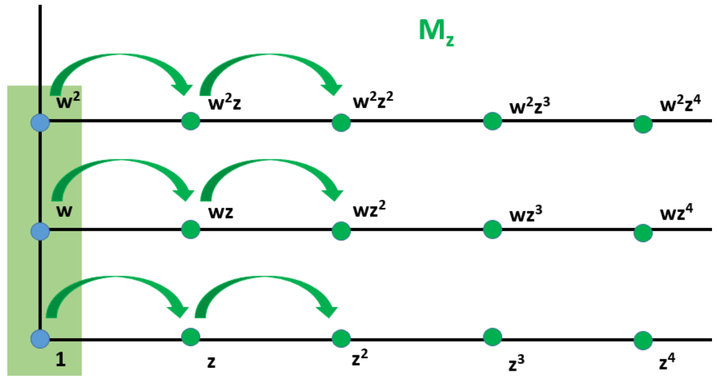

Figure 5.2: Operation of Multiplication by z.

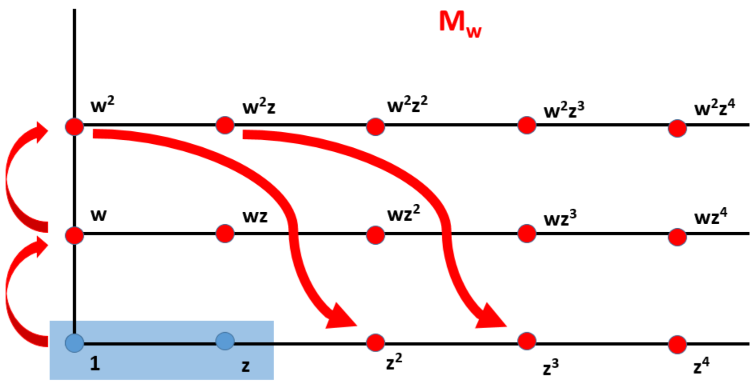

Figure 5.3: Operation of Multiplication by w.

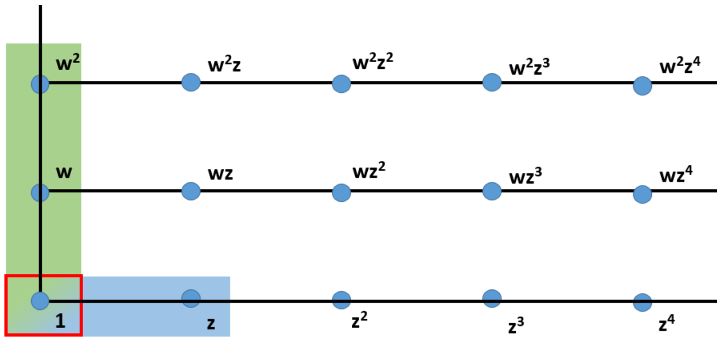

Figure 5.4: Dimension Computation.

Information on previous images can be summarized as follows.

From figure 5.2, $\ker(M_z)^* = \overline{\text{span}\{1, w, w^2\}}$

From figure 5.3, $\ker(M_w)^* = \overline{\text{span}\{1, z\}}$

$\dim[\ \ker(M_z)^* \cap \ker(M_w)^*] = 1$

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 1.$$

Now, let us consider the following unitary which represents $\mathfrak{p}_1$.

$$\mathcal{U}_{2(\mathfrak{p}1)} = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right) \text{ is a unitary representing } \mathfrak{p}_1.$$

When the same computation is done as above we get,

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 2.$$

5.2.2 Let $\mathfrak{p}_2 = z^3 - w^2$

Similarly, following dimensions can be obtained using the same procedure.

When, $\mathcal{U}_{1(\mathfrak{p}2)} = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$ is a unitary representing $\mathfrak{p}_2$.

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 1.$$

And when, $\mathcal{U}_{2(\mathfrak{p}2)} = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right)$ is a unitary representing $\mathfrak{p}_2$.

$$\dim[\ \ker(M_z - 0I)^* \cap \ker(M_w - 0I)^*] = 2.$$

Further, other than these examples we can consider the examples already discussed in above chapters. Then from those observations we can form the following conjecture.

5.3 Conjecture

Let $\mathfrak{p}$ be a minimal inner toral polynomial with bidegree (n, m) and $(0, 0) \in Z(\mathfrak{p})$. If (S, T) is a pure $\mathfrak{p}$-isopair, with bimultiplicilty (m, n), then

$$\dim[\ \ker(S - 0I)^* \cap \ker(T - 0I)^*] = m - m_0$$

where m_0 is the number of non zero entries in the block matrix A of the considered unitary representing $\mathfrak{p}$.

Bibliography

[1] J. Agler, G. Knese, J. E. McCarthy (2012), **Algebraic Pairs of Isometries**, *J. Operator Theory*, 67(2012), 215 – 236..

[2] J. Agler, J. E. McCarthy (2005), **Distinguished Varieties**, *Acta Math* 194(2005), 133 – 153.

[3] J. B. Conway (1990). **A Course in Functional Analysis (2^{nd} Edition)**, *ISBN 0-387-97245-5.*

[4] M. T. Jury, G. Knese, S. McCullough (2012). **Nevanlinna-Pick Interpolation on Distinguished Varieties in the Bidisk**, *SciVerse ScienceDirect- Journal of Functional Analysis* 262(2012), 3812–3838.

[5] G. Knese (2010) **Polynomials Defining Distinguished Varities**, *American Mathematical Society* Volume 362, 5635 – 5655.

[6] E. Kreyszig (1978) **Introductory Functional Analysis with Applications**, *ISBN 0-471-50731-8.*

[7] M. Rosenblum, J. Rovnyak (1985) **Hardy Classes and Operator Theory**, *ISBN 0-486-69536-0.*

[8] U. D. Wijesooriya (2018) **Algebraic Pairs of Pure Commuting Isometries with Finite Multiplicity**, *J.Operator Theory* 79:2(2018), 507 – 527.