W.D. Henshaw **Math 6840: Problem Set 1** **Due:** Thurs. Jan 23, 2020.

General note: For computer exercises please provide listings of all computer programs that were used to generate the results. Please present your results in a clear and neat manner (bonus marks may be given for presentation).

1. (20 points) Consider the initial-boundary-value problem (IBVP) for the heat equation,

$$\begin{aligned}
&u_t = \kappa u_{xx}, && a < x < b, && 0 \le t \le t_{\text{final}}, \\
&u(a,t) = g_a(t), && u(b,t) = g_b(t), && t > 0 \\
&u(x,0) = \sin(k\pi x), && && a < x < b,
\end{aligned}$$

with $a = 0$, $b = 1$, and where k is an integer.

(a) Find the exact solution when $g_a(t) = 0$ and $g_b(t) = 0$ which is of the form $u(x,t) = \hat{u}(t)\,\sin(k\pi x)$.

(b) Write down the FE-CD2 scheme (forward-Euler in time and second-order central difference in space) given in class for the discrete solution U_i^n. Be sure to specify the initial conditions and boundary conditions and indicate where each equation is applied in index space.

(c) Write a Matlab code to solve heat equation IBVP using the FE-CD2 scheme. Use values $k = 2$, $\kappa = .1$, and $t_{\text{final}} = 1$. Solve the problem for $N_x = 10 \times 2^m$, $m = 1, 2, 3$ using a time-step that satisfies $\kappa\Delta t/\Delta x^2 = 1/4$ (adjust this time-step, as shown in class, to exactly reach $t = t_{\text{final}}$). Print m, N_x, Δt and the max-norm errors E_m at $t = 1$, for each m, along with the ratios E_{m-1}/E_m, and estimated convergence rates. Use the print statements:

```
fprintf(' t=%10.4e: Nx=%3d Nt=%4d dt=%9.3e maxErr=%8.2e',t,Nx,Nt,dt,errMax(m));
if( m==1 ) fprintf('\n'); else fprintf(' ratio=%8.2e, rate=%4.2f\n',errMax(m-1)/errMax(m),...
                                       log2(errMax(m-1)/errMax(m))); end;
```

Confirm that the ratio approaches 4 and rates approach 2.

(d) For $m = 1$, plot the computed and exact solution at $t = 1$ on the same graph. On a second graph, plot the errors (signed error, not absolute value) as a grid-function for m=1,2,3 (i.e. all errors on the same graph). In each case, label axes and provide a title and legend.

2. (20 points) Manufactured solutions. Consider the initial-boundary-value problem (IBVP) for the forced heat equation,

$$\begin{aligned}
&u_t = \kappa u_{xx} + f(x,t), && a < x < b, && 0 \le t \le t_{\text{final}}, \\
&u(a,t) = g_a(t), && u(b,t) = g_b(t), && t > 0 \\
&u(x,0) = u_0(x), && && a < x < b,
\end{aligned}$$

with $a = 0$, and $b = 1$.

(a) Polynomial manufactured solution. How should $f(x,t)$, $u_0(x)$, $g_a(t)$ and $g_b(t)$ be chosen so that the following polynomial is an exact solution:

$$u^e(x,t) = (b_0 + b_1 t + b_2 t^2)\,(c_0 + c_1 x + c_2 x^2). \tag{1}$$

(b) Starting from your Matlab code for question 1, implement the FE-CD2 scheme for the forced heat equation. For forward-Euler in time, evaluate the forcing $f(x,t)$ at the old time when solving the PDE. Choose $f(x,t)$, $u_0(x)$, $g_a(t)$ and $g_b(t)$ so that (1) is the exact solution.

Perform a grid refinement study as in 1(c) for the two cases

(i) Choose $b_0 = .5$, $b_1 = .7$, $b_2 = 0$ and $c_0 = 1$, $c_1 = .8$, $c_2 = .6$.

(ii) Choose $b_0 = .5$, $b_1 = .7$, $b_2 = .9$ and $c_0 = 1$, $c_1 = .8$, $c_2 = .6$.

In the first case confirm the the scheme is exact (up to round-off errors). In the second case confirm the ratio of the errors approaches 4.

For case (ii) provide plots as for problem (1) : for $m = 1$ plot the computed and exact solution at $t = 1$ on the same graph. On a second graph, plot the errors for m=1,2,3 (i.e. all errors on the same graph). In each case, label axes and provide a title and legend.

(b) Trigonometric manufactured solution. Repeat 2(b) but for the trigonometric manufactured solution

$$u^e(x,t) = \cos(k_t t)\,\cos(k_x x) \tag{2}$$

with $k_t = \pi$ and $k_x = 7\pi$. Perform a grid refinement study and confirm that the ratio of the errors approaches 4.

As for problem (1), for $m = 1$ plot the computed and exact solution at $t = 1$ on the same graph. On a second graph, plot the errors for m=1,2,3 (i.e. all errors on the same graph). In each case, label axes and provide a title and legend.

3. (10 points)

(a) Find the most accurate approximation to $\partial_x u$ of the form

$$\frac{\partial u}{\partial x}(x) \approx c_{-1}u(x-h) + c_0 u(x) + c_1 u(x+h) + c_2 u(x+2h).$$

Provide the explicit form of the leading term in the truncation error. For what degrees of polynomial is this approximation exact?

(b) Find the most accurate approximation to $\partial_x^2 u$ of the form

$$\frac{\partial^2 u}{\partial x^2}(x) \approx c_{-2}u(x-2h) + c_{-1}u(x-h) + c_0 u(x) + c_1 u(x+h) + c_2 u(x+2h).$$

Provide the explicit form of the leading term in the truncation error. For what degrees of polynomial is this approximation exact?