CONVERSE, INVERSE AND CONTRAPOSITIVE

1. An implication is logically equivalent to it's contrapositive.
2. The converse and inverse of an implication are logically equivalent
3. An implication is not equivalent to it's converse.

BICONDITIONAL

If p and q are statement variables,
the biconditional of p and q
is "**p if, and only if, q**" and is denoted **$p \leftrightarrow q$**.

The words **if and only if** are sometimes abbreviated **iff**.
The double headed arrow "$\leftrightarrow$" is the **biconditional operator**.

TRUTH TABLE FOR

$p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLES

1. "$1+1 = 3$ if and only if earth is flat"
   TRUE

2. "Sky is blue iff $1 = 0$"
   FALSE

3. "Milk is white iff birds lay eggs"
   TRUE

EXAMPLES

4. "33 is divisible by 4 if and only if horse has four legs"

   FALSE

5. "$x > 5$ iff $x^2 > 25$"

   FALSE

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow r)$
T	T	T			
T	F	F			
F	T	F			
F	F	T			

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow r)$
T	T		T	T	
T	F		F	T	
F	T		T	F	
F	F		T	T	

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to r)$
			T	T	T
			F	T	F
			T	F	F
			T	T	T

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow r)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

REPHRASING BICONDITIONAL

$p \leftrightarrow q$ is also expressed as:

"p is necessary and sufficient for q"

"if p then q, and conversely"

"p is equivalent to q"

EXAMPLES

1. If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.

Solution:
You buy an ice cream cone if and only if it is hot outside.

EXAMPLES

2. For you to win the contest it is necessary and sufficient that you have the only winning ticket.

Solution:

You win the contest if and only if you hold the only winning ticket.

EXAMPLES

3. If you read the news paper every day, you will be informed and conversely.

Solution:

You will be informed if and only if you read the news paper every day.

TRUTH TABLE FOR

$(p \to q) \leftrightarrow (\sim q \to \sim p)$

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T				
T	F	F				
F	T	T				
F	F	T				

TRUTH TABLE FOR

$(p \to q) \leftrightarrow (\sim q \to \sim p)$

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
			F	F	T	
			T	F	F	
			F	T	T	
			T	T	T	

TRUTH TABLE FOR

$$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$$

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
		T			T	T
		F			F	T
		T			T	T
		T			T	T

TRUTH TABLE FOR

$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T		T		
T	T		T		
T	F		F		
T	F		F		
F	T		F		
F	T		F		
F	F		T		
F	F		T		

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
	T	T		T	
	T	F		F	
	F	T		F	
	F	F		T	
	T	T		T	
	T	F		F	
	F	T		F	
	F	F		T	

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
			T	T	T
			T	F	F
			F	F	T
			F	T	F
			F	T	F
			F	F	T
			T	F	F
			T	T	T

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	T	T

$p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

$p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
		T	F			
		F	T			
		T	F			
		F	T			
		T	F			
		F	T			
		T	F			
		F	T			

$p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T			F	F		
T			T	T		
T			F	F		
T			T	T		
F			F	F		
F			T	F		
F			F	F		
F			T	F		

$p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
	T	T			T	
	T	F			T	
	F	T			T	
	F	F			F	
	T	T			T	
	T	F			T	
	F	T			T	
	F	F			F	

$p \wedge \sim r \leftrightarrow q \vee r$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
				F	T	F
				T	T	T
				F	T	F
				T	F	F
				F	T	F
				F	T	F
				F	T	F
				F	F	T

SHOW THAT

$$\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q)$
T	T	F	F		
T	F	F	T		
F	T	T	F		
F	F	T	T		

SHOW THAT

$$\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T			F		F
T			T		T
F			F		T
F			T		F

SHOW THAT

$$\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
	T	F		F	
	F	F		T	
	T	T		T	
	F	T		F	

SHOW THAT

$$\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

SHOW THAT

$$\sim(p \oplus q) \equiv p \leftrightarrow q$$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T	F		
T	F	T		
F	T	T		
F	F	F		

SHOW THAT

$$\sim(p \oplus q) \equiv p \leftrightarrow q$$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
		F	T	
		T	F	
		T	F	
		F	T	

SHOW THAT

$$\sim(p \oplus q) \equiv p \leftrightarrow q$$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T			T
T	F			F
F	T			F
F	F			T

SHOW THAT

$$\sim(p \oplus q) \equiv p \leftrightarrow q$$

p	q	$p \oplus q$	$\sim(p \oplus q)$	$p \leftrightarrow q$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

LAWS OF LOGIC

1. Commutative Law:

$$p \leftrightarrow q \equiv q \leftrightarrow p$$

2. Implication Laws:

$$p \to q \equiv \sim p \vee q$$
$$\equiv \sim (p \wedge \sim q)$$

LAWS OF LOGIC

3. Exportation Law:

$$(p \wedge q) \to r \equiv p \to (q \to r)$$

4. Equivalence:

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

5. Reductio ad absurdum:

$$p \to q \equiv (p \wedge \sim q) \to c$$

APPLICATION

1. $p \wedge \sim q \rightarrow r$

SOLUTION

$p \wedge \sim q \rightarrow r$

$\equiv (p \wedge \sim q) \rightarrow r$ order of operations

$\equiv \sim (p \wedge \sim q) \vee r$ implication law

APPLICATION

2. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$

SOLUTION

$(p \rightarrow r) \leftrightarrow (q \rightarrow r)$

$$\equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$$

implication law

$$\equiv [(\sim p \vee r) \rightarrow (\sim q \vee r)] \wedge [(\sim q \vee r) \rightarrow (\sim p \vee r)]$$

equivalence of biconditional

$$\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$$

implication law

EXERCISE

$\sim(p \to q) \to p$

SOLUTION

Statement	Reason
$\sim(p \to q) \to p$	Given statement form
$\equiv\sim[\sim(p \wedge \sim q)] \to p$	Implication law
$\equiv(p \wedge \sim q)] \to p$	Double negation law
$\equiv\sim(p \wedge \sim q)] \vee p$	Implication law

EXERCISE

$\sim(p \to q) \to p$

SOLUTION

Statement	Reason
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law of $\vee$
$\equiv q \vee (\sim p \vee p)$	Associative law of $\vee$
$\equiv q \vee t$	Negation law
$\equiv t$	Universal bound law