REFLEXIVE RELATION

Let R be a relation on a set A. R is **reflexive if, and only if,** for all $a \in A$, $(a, a) \in R$. Or **equivalently** aRa. That is, **each element** of A is related to itself.

REFLEXIVE RELATION

REMARK:

R is not reflexive iff there is an element "a" in A such that $(a, a) \notin R$. That is, some element "a" of A is not related to itself.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

We define relations R_1, R_2, R_3, R_4 on A as follows:

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

$R_3 =$
$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

EXAMPLE

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

R_2 is not reflexive, because $(4, 4) \notin R_2$.

EXAMPLE

$R_3 =$
$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.

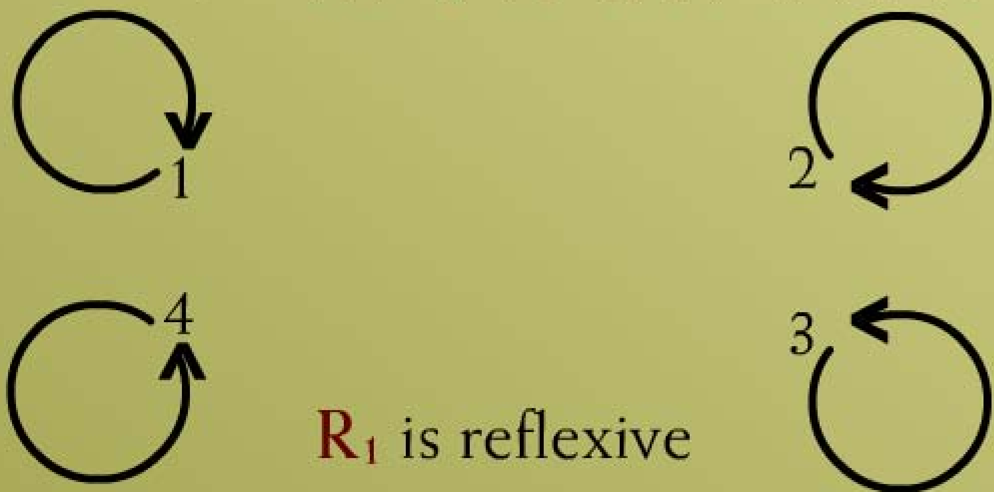
$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

R_4 is not reflexive, because $(1, 1) \notin R_4$,
$(3, 3) \notin R4$

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$



R_1 is reflexive

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

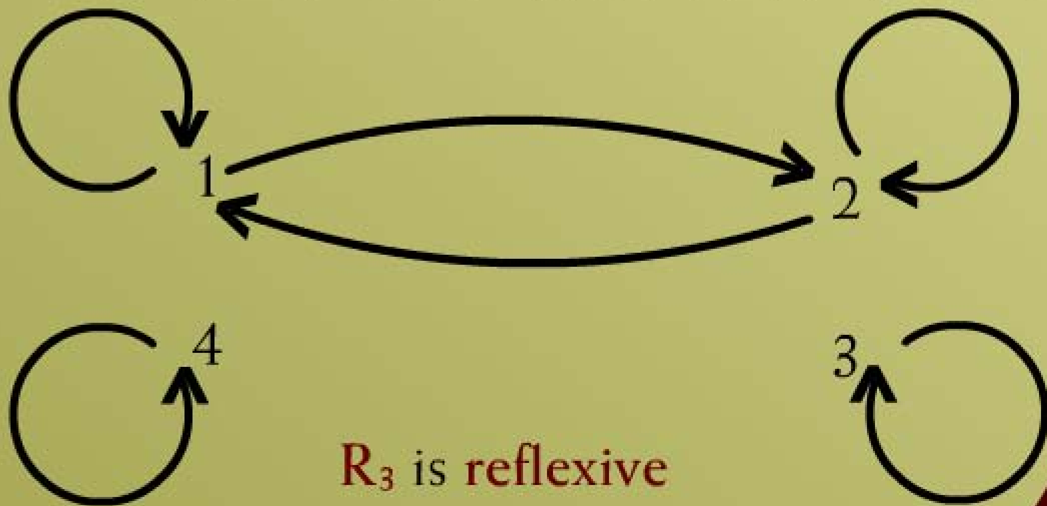
R_2 is not **reflexive**, as there is no loop at 4.

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let

$A = \{1, 2, 3, 4\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$



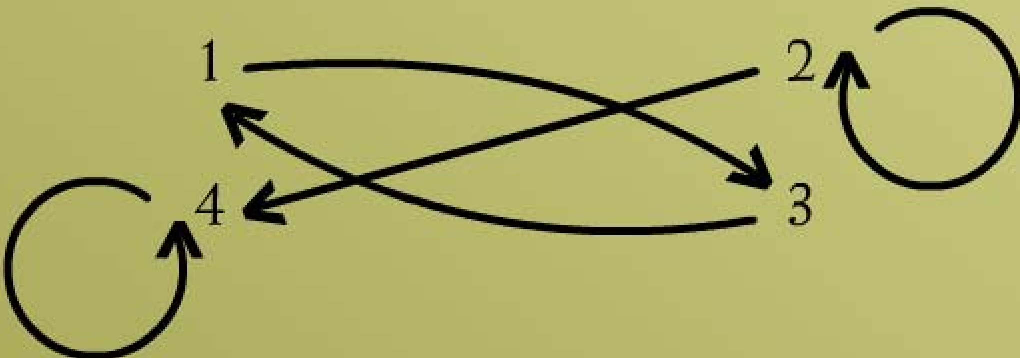
R_3 is reflexive

DIRECTED GRAPH OF A REFLEXIVE RELATION

Let

$A = \{1, 2, 3, 4\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$



R_4 is not **reflexive**, as there are **no loops** at 1and 3.

MATRIX REPRESENTATION OF A REFLEXIVE RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$. A Relation R on A is reflexive if and only if $(a_i, a_i) \in R \ \forall\ i = 1, 2, \ldots, n$.

Accordingly, R is **reflexive** if all the elements on the **main diagonal** of the matrix M representing R are equal to 1.

EXAMPLE

Let $A = \{a_1, a_2, \ldots, a_n\}$.

$R = \{(1,1), (1,3), (2,2), (3,2), (3,3)\}$

R is reflexive

$$M = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

SYMMETRIC RELATION

Let R be a relation on a set A. R is symmetric if, and only if, for all a, $b \in A$, if $(a, b) \in R$ then $(b, a) \in R$. That is, if aRb then bRa.

REMARK:

R is not symmetric iff there are elements a and b in A such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

R_1, R_2, R_3, R_4

$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$

R_1 is symmetric.

$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

R_2 is symmetric.

EXAMPLE

$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

R_3 is not symmetric, because $(2,3) \in R_3$ but $(3,2) \notin R_3$.

$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

R_4 is not symmetric, because $(4,3) \in R_4$ but $(3,4) \notin R_3$.

DIRECTED GRAPH OF A SYMMETRIC RELATION

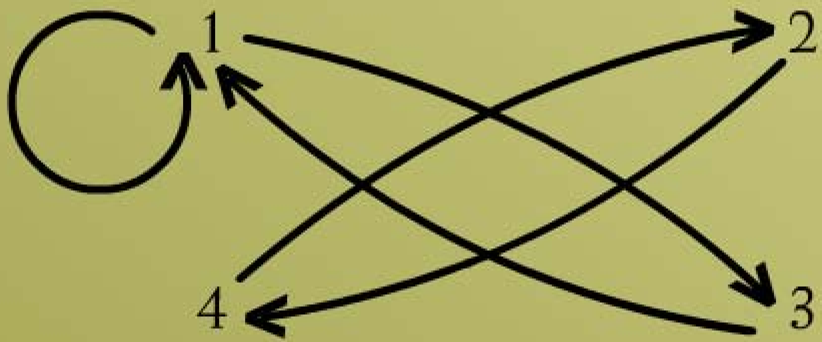
For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

EXAMPLE

Let

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$



R_1 is symmetric

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

R_2 is symmetricis.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

R_3 is not symmetric since there are arrows from 2 to 3 and from 3 to 4 but not conversely.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

R_4 is not symmetric since there is an arrow from 4 to 3 but no arrow from 3 to 4 symmetricis.

MATRIX REPRESENTATION OF A SYMMETRIC RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$. A relation R on A is symmetric if and only if for all $a_i, a_j \in A$, if $(a_i, a_j) \in R$ then $(a_j, a_i) \in R$.

Accordingly, R is symmetric if the elements in the ith row are the same as the elements in the ith column of the matrix M representing R.

EXAMPLE

Let $A = \{1,2,3\}$

$R = \{(1,3), (2,2), (3,1), (3,3)\}$

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

M is symmetric i.e $M = M^t$

EXAMPLE

Let R be a relation on a set A.
R is **transitive if and only if**
for all a, b, c $\in$ A, if (a, b) $\in$ R
and (b, c) $\in$ R then (a, c) $\in$ R.

That is,if aRb and bRc then aRc.

EXAMPLE

In words, if any one element is related to a second and that second element is related to a third, then the first is related to the third.

REMARK:

R is not transitive iff there are elements a, b, c in A such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$ define relations.

$$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$

$$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$$

EXAMPLE

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

R_1 is transitive. Because $(1,2) \in R_1$ and $(2,3) \in R_1 \Rightarrow (1,3) \in R_1$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

R_2 is not transitive. Because $(1,2) \in R_2$ and $(2,3) \in R_2$ but $(1,3) \notin R_2$

$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

R_3 is transitive. Because $(2,3) \in R_3$ and $(3,4) \in R_3 \Rightarrow (2,4) \in R_3$

DIRECTED GRAPH OF A TRANSITIVE RELATION

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

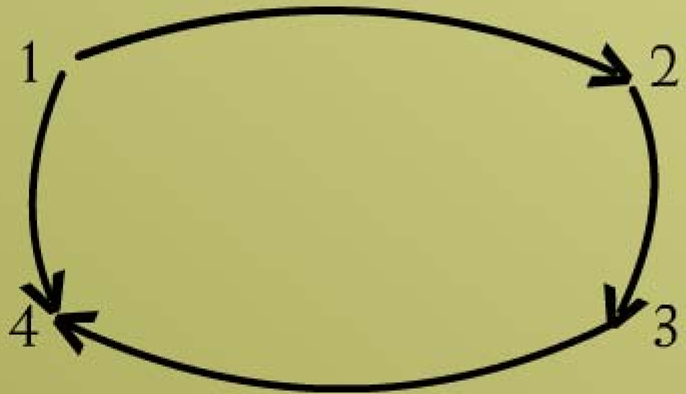
$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

R_1 is transitive.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$



R_2 is not **transitive** since there is **an arrow** from 1 to 2 and from 2 to 3 but **no arrow** from 1 to 3 directly.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

R_3 is transitive

EXERCISE

Let $A = \{0, 1, 2\}$

$$R = \{(0,2), (1,1), (2,0)\}$$

1. Is R reflexive? Symmetric? Transitive?
2. Which ordered pairs are needed in R to make it a reflexive and transitive relation.

SOLUTION

$R = \{(0,2), (1,1), (2,0)\}$

1. R is not reflexive, since $0 \in A$ but $(0, 0) \notin R$.

R is symmetric.

Because $(0,2) \in R \Rightarrow (2,0) \in R$

R is not transitive, since $(0, 2)$ & $(2, 0) \in R$ but $(0, 0) \notin R$.

SOLUTION

$$R = \{(0,2), (1,1), (2,0)\}$$

1. R is not reflexive.
   R to be reflexive it must contains $(0,0)$ and $(2,2)$.

2. R is not transitive.
   For R to be transitive it must contain $(0,0)$ and $(2,2)$.

EXERCISE

Define a relation L on the set of real numbers R be defined as follows:

$$\text{for all } x, y \in R, \qquad x\,L\,y \Leftrightarrow x < y.$$

a. Is L reflexive?
b. Is L symmetric?
c. Is L transitive?

SOLUTION

$$x\ L\ y \Leftrightarrow x < y$$

a. L is not **reflexive**, because $x < x$ for any **real number** x.

For example $1 \nless 1$

b. L is not **symmetric**, because for all $x, y \in R$, if $x < y$ then $y < x$

For example $1 < 2$ and $2 \nless 1$

SOLUTION

$$x \, L \, y \Leftrightarrow x < y$$

c. L is **transitive**, because for all, $x, y, z \in R$, if $x < y$ and $y < z$, then $x < z$.
(by **transitive law** of order of **real numbers**).

Note: These properties are independent of each other.

EQUIVALENCE RELATION

Let A be a non-empty set and R a binary relation on A. R is an equivalence relation if, and only if, R is reflexive, symmetric, and transitive.

EXAMPLE

Let

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (2,4), (3,3), (4,2), (4,4)\}$

R is reflexive, symmetric and transitive.

CONGRUENCES RELATION

Let m and n be integers and d be a positive integer.

The notation $m \equiv n \pmod{d}$

means that

$d \mid (m - n)$ {d divides m minus n}

$\Leftrightarrow$ There exists an integer k such that

$(m - n) = d \cdot k$

EXAMPLE

a. Is $22 \equiv 1 (\bmod 3)$? b. Is $-5 \equiv +10 \ (\bmod 3)$?

c. Is $7 \equiv 7 \ (\bmod 4)$? d. Is $14 \equiv 4 \ (\bmod 4)$?

Solution:

a. Since $22\text{-}1 = 21$

21 is divisible by 3

Hence $3 \mid (22\text{-}1)$, and so $22 \equiv 1 \ (\bmod 3)$

b. Since $-5 - 10 = -15$

-15 is divisible by 3

Hence $3 \mid ((-5)\text{-}10)$, and so $-5 \equiv 10 \ (\bmod 3)$

Solution contd...

c. Since $7 - 7 = 0$

Hence $3 \mid (7\text{-}7)$, and so $7 \equiv 7 \pmod 4$

d. Since $14 - 4 = 10$ but 4 does not divide 10

Hence $14 \equiv 4 \pmod 4$.