

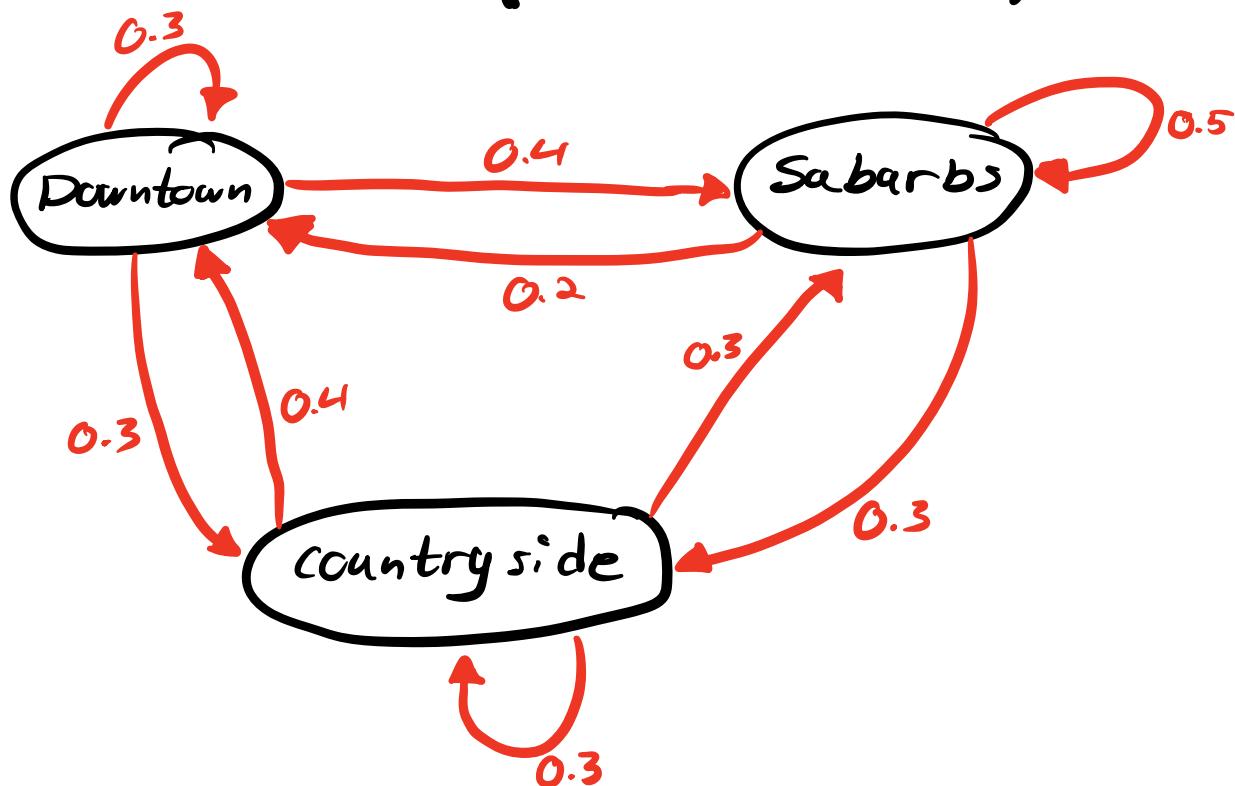
Problem ①

①

1. If a truck is currently in the suburbs, what is the probability that it will be in the downtown region after two time steps? [1.5p]

Transition Matrix:

$$P = \begin{matrix} & \text{downtown} & \text{suburbs} & \text{countryside} \\ \text{downtown} & 0.3 & 0.4 & 0.3 \\ \text{suburbs} & 0.2 & 0.5 & 0.3 \\ \text{countryside} & 0.4 & 0.3 & 0.3 \end{matrix}$$



So: The transition matrix is P above.

We want the probability of starting in suburbs and being in Downtown after two time steps

In markov chains, two-step transitions are

compacted with the matrix:

$$P \cdot P = P^2$$

So: Use matmul to multiply, then look at the element at row Suburbs, column Downtown.

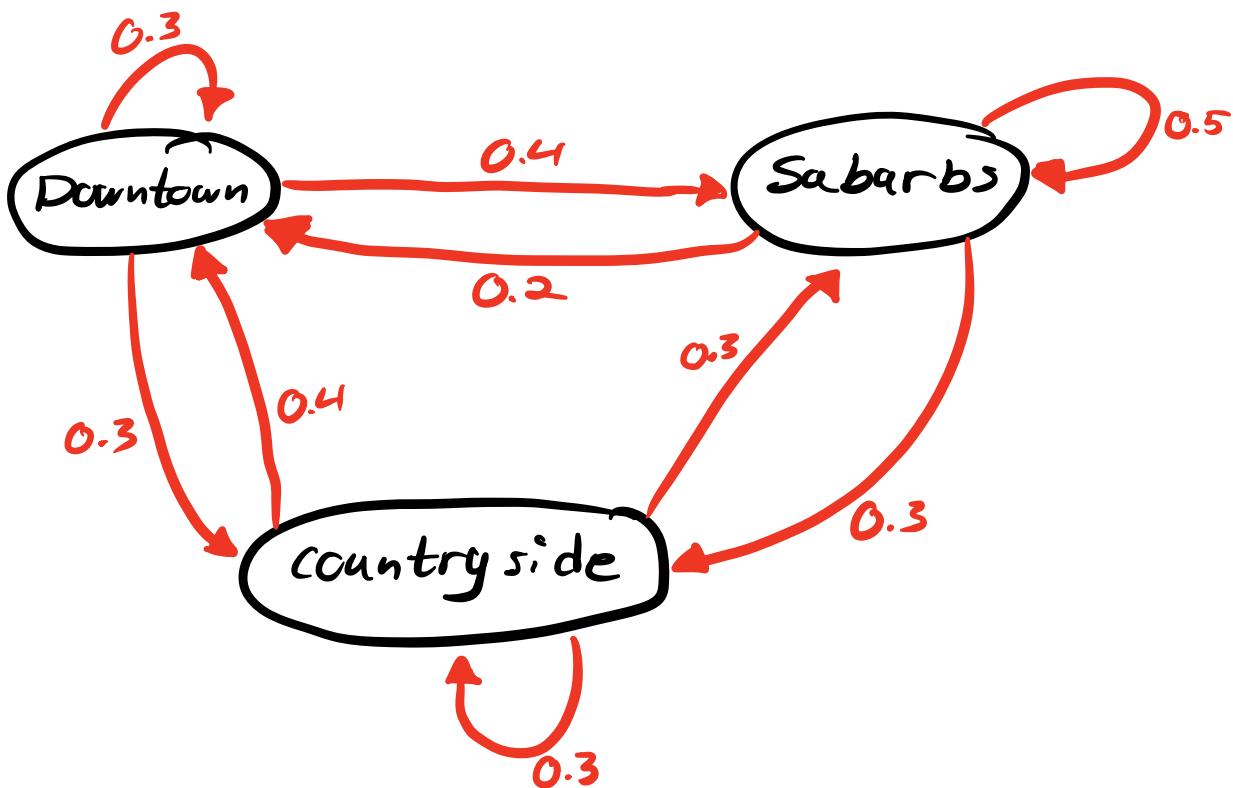
This gives 0.28:

Problem ①

②

2. If a truck is currently in the suburbs, what is the probability that it will be in the downtown region **the first time** after two time steps? [1.5p]

This is done by looking at all possible paths, and then sum them up. We look at transition matrix.



Path 1: Suburbs \rightarrow Suburbs \rightarrow Downtown

$$\Rightarrow 0.5 \cdot 0.2 = 0.1$$

Path 2: Suburbs \rightarrow country side \rightarrow Down town

$$\Rightarrow 0.3 \cdot 0.4 = 0.12$$

Total: $0.1 + 0.12 = \underline{\underline{0.22}}$

Problem ①:

③

- A Markov chain is irreducible if:

- You can get from every state to every other state
- As long as there is some path with positive probability.

* If matrix has no zeros at all, it is trivially irreducible, because all transitions can occur in 1 step

* This does not need to necessarily happen in one step

So: No zeros in matrix \rightarrow irreducible?

Answer:

- In this case it's true since you can go from each state to each other state.

There's no zeros in markov chain.

Problem ①

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4. What is the stationary distribution? [1.5p]

We label the stationary distribution as

$$\pi = (\pi_1, \pi_2, \pi_3) = (\text{downtown}, \text{suburbs}, \text{countryside})$$

Transition matrix is still the same:

$$\begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$$

Stationary distribution satisfies: $\pi \cdot P = \pi$

$$\Rightarrow \begin{cases} \pi_1 = 0.3\pi_1 + 0.2\pi_2 + 0.4\pi_3 \\ \pi_2 = 0.4\pi_1 + 0.5\pi_2 + 0.3\pi_3 \\ \pi_3 = 0.3\pi_1 + 0.3\pi_2 + 0.3\pi_3 \end{cases}$$

and we also know that: $\boxed{\pi_1 + \pi_2 + \pi_3 = 1}$

Put these 4 equations into matrices and solve:

Problem ①

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5. Advanced question: What is the expected number of steps until the first time one enters the downtown region having started in the suburbs region. Hint: to get within 1 decimal point, it is enough to compute the probabilities for hitting times below 30. [2p]

Goal:

Compute the expected number of steps until the chain first enters **Downtown**, starting in **Suburbs**.

We label the states as:

- 0 = Downtown (D)
- 1 = Suburbs (S)
- 2 = Countryside (C)

Transition matrix:

$$P = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$$

1. Define the hitting time

Let

$$T = \min\{n \geq 0 : X_n \text{ is in Downtown}\}$$

and define for each state i :

$$h(i) = \mathbb{E}[T \mid X_0 = i].$$

We want

$$h(S).$$

Since downtown is the target:

$$h(D) = 0.$$

Let:

- $h_S = h(S)$
- $h_C = h(C)$.

2. Use first-step decomposition

For any non-target state i :

$$h(i) = 1 + \sum_j P_{ij}h(j).$$

Suburbs row of P

$$h_S = 1 + 0.2h_D + 0.5h_S + 0.3h_C.$$

Since $h_D = 0$:

$$h_S = 1 + 0.5h_S + 0.3h_C.$$

Countryside row

$$h_C = 1 + 0.4h_D + 0.3h_S + 0.3h_C.$$

Again $h_D = 0$:

$$h_C = 1 + 0.3h_S + 0.3h_C.$$

3. Rearranging the system

Equation for h_S :

$$0.5h_S - 0.3h_C = 1.$$

Equation for h_C :

$$-0.3h_S + 0.7h_C = 1.$$

System:

$$\begin{cases} 0.5h_S - 0.3h_C = 1 \\ -0.3h_S + 0.7h_C = 1 \end{cases}$$

4. Solve the system

Multiply the first equation by 3:

$$1.5h_S - 0.9h_C = 3$$

Multiply the second by 5:

$$-1.5h_S + 3.5h_C = 5$$

Add them:

$$2.6h_C = 8$$

Thus:

$$h_C = \frac{40}{13}.$$

Plug into the first equation:

$$0.5h_S = 1 + 0.3 \cdot \frac{40}{13} = \frac{25}{13}.$$

Therefore:

$$h_S = \frac{50}{13} \approx 3.8461538.$$

✓ FINAL ANSWER

$$\mathbb{E}[T \mid X_0 = \text{Suburbs}] = \frac{50}{13} \approx 3.8 \text{ steps}$$

Problem ③ :

Assignment 2, PROBLEM 3

Maximum Points = 4

Derive the maximum likelihood estimate for n IID samples from a random variable with the following probability density function:

$$f(x; \lambda) = \frac{1}{24} \lambda^5 x^4 \exp(-\lambda x), \quad \text{where, } \lambda > 0, x > 0$$

You can solve the MLE by hand (using pencil paper or using key-strokes). Present your solution as the return value of a function called `def MLEForAssignment2Problem3(x)`, where `x` is a list of n input data points.

$$f(x; \lambda) = \frac{1}{24} \cdot \lambda^5 \cdot x^4 \cdot e^{-\lambda x}$$

$$\log(f(x; \lambda))$$

$$\Rightarrow \log\left(\frac{1}{24}\right) + 5\log(\lambda) + 4\log(x) - \lambda x$$

$$\log\left(\frac{1}{24}\right) = \log(24^{-1}) = -\log(24)$$

$$\Rightarrow -n\log(24) + 5n\log(\lambda) + 4 \sum_{i=1}^n \log(x_i) - \lambda \sum_{i=1}^n x_i$$

The maximum likelihood estimate (MLE) is the value of the parameter (here λ) that maximizes the likelihood of observing our data.

$$\text{MLE: } \hat{\lambda} = \underset{\lambda}{\operatorname{argmax}} \underbrace{l(\lambda)}_{\text{loss}}$$

We maximize a function by looking for critical points:

$$\frac{d}{d\lambda} l(\lambda) = 0$$

- If slope positive ($l'(\lambda) > 0$), function is increasing
- If slope negative ($l'(\lambda) < 0$), function is decreasing

• At a peak or valley, the slope is flat: $\ell'(\lambda) = 0$.

So, solving $\ell'(\lambda) = 0$ tells where maxima or minima can occur.

Once we have derivative = 0, we check second derivative:

- $\ell''(\lambda) < 0$: it's concave down \rightarrow maximum
- $\ell''(\lambda) > 0$: it's concave up \rightarrow minimum.

So: $\ell'(\lambda) = 0$ gives: $\frac{d}{dx} \lambda = \frac{1}{x}$.

$$5n(\log \lambda) - \lambda \sum_{i=1}^n x_i$$

$$\Rightarrow 5n \frac{1}{\lambda} - \sum_{i=1}^n x_i = 5n \lambda^{-1} - \sum_{i=1}^n x_i$$

$$\ell''(\lambda) = \frac{-2}{\lambda^2} = 5n \frac{1}{\lambda^2} = -\frac{5n}{\lambda^2} < 0$$

So this is indeed a maximum.

So: $\ell'(\lambda) = 0 \Rightarrow \frac{5n}{\lambda} - \sum_{i=1}^n x_i = 0$

$$\Rightarrow \frac{5n}{\lambda} = \sum_{i=1}^n x_i$$

$$\Rightarrow \hat{\lambda} = \frac{5n}{\sum_{i=1}^n x_i}, \quad \sum_{i=1}^n x_i$$

$\sum_{i=1}^n x_i = n \bar{x}$, since we need mean of x .

$$\Rightarrow \hat{\lambda} = \frac{5n}{n \bar{x}} \Rightarrow \frac{5}{\bar{x}}$$

Problem ④