Exact equation :-

The equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

How to get solution?

$$\int M\,dx + \int N\,dy = c$$

(Treating y constant) (Only those term which are free from x)

Q! Check whether the equation is exact or not then find the solution

$$(3x^2 + 2e^y)\,dx + (2xe^y + 3y^2)\,dy = 0$$

Soln

$$M(x,y) = 3x^2 + 2e^y$$

$$N(x,y) = 2xe^y + 3y^2$$

$$\frac{\partial M}{\partial y} = 2e^y$$

$$\frac{\partial N}{\partial x} = 2e^y$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ eqn is exact

$$\int M\,dx = \int (3x^2 + 2e^y)\,dx$$

$$= x^3 + 2xe^y$$

$$\cancel{\int N\,dy} \int 3y^2\,dy = y^3$$

$$\int (3x^2+2e^y)\,dx + \int 3y^2\,dy = c$$

$$\Rightarrow x^3 + 2xe^y + y^3 = c$$

Q:- What about $e^x(\cos y\,dx \mp \sin y\,dy) = 0$

$y(0) = 0$

Sol:

$$e^x \cos y\,dx - e^x \sin y\,dy = 0$$

$$\frac{\partial M}{\partial y} = -\sin y\, e^x$$

$$\frac{\partial N}{\partial x} = -e^x \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ eqn is exact

Now

$$\int e^x \cos y\,dx + \int 0\,dy = c$$

$$\Rightarrow e^x \cos y = c \quad \text{——— (1)}$$

Given, $y(0) = 0$

From (1)

$$e^0 \cos 0 = c$$

$$\Rightarrow c = 1$$

Now soln is

$$\boxed{e^x \cos y = 1}$$

$$\int (3x^2+2e^y)\,dx + \int 3y^2\,dy = c$$

$$\Rightarrow x^3 + 2xe^y + y^3 = c$$

Q:- What about $e^x(\cos y\,dx \mp \sin y\,dy) = 0$

$y(0) = 0$

Sol:

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$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ eqn is exact

Now

$$\int e^x \cos y\,dx + \int 0\,dy = c$$

$$\Rightarrow e^x \cos y = c \quad \text{--- (1)}$$

Given, $y(0) = 0$

From (1)

$$e^0 \cos 0 = c$$

$$\Rightarrow c = 1$$

Now soln is

$$\boxed{e^x \cos y = 1}$$

Integrating Factor: (I.F) / μ

Integrating factor is a function $f(x,y)$ which is multiplied with a non-exact equation, it will be exact

Ex: $y\,dx - x\,dy = 0$ ——— ①

the above eqn is not exact

Let $f(x,y) = \frac{1}{y^2}$

Now multiply $f(x,y)$ in eqn ① we get

$$\frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow \underbrace{\frac{1}{y}}_{M}\,dx - \underbrace{\frac{x}{y^2}}_{N}\,dy = 0 \quad \text{——— ②}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

$$\frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ eqn ② is exact

$$\int \frac{1}{y}\,dx + \int 0\,dy = c$$

$$\Rightarrow \frac{x}{y} = c \quad \text{(solution)}$$

Let $g(x,y) = \dfrac{1}{\frac{1}{2}(y^2-x^2)}$

Now multiply $g(x,y)$ in eqn (1) we get

$$\underbrace{\frac{2y}{y^2-x^2}}_{M} dy - \underbrace{\frac{2x}{y^2-x^2}}_{N} dy = 0 \quad \text{———} (3)$$

$$\frac{\partial M}{\partial y} = \frac{2(y^2-x^2) - 2y \times 2y}{(y^2-x^2)^2}$$

$$= \frac{2y^2-2x^2-4y^2}{(y^2-x^2)^2}$$

$$= \frac{-2x^2-2y^2}{(y^2-x^2)^2}$$

$$=$$

$$\frac{\partial N}{\partial x} = \frac{-2(y^2-x^2) + 2x \times (-2x)}{(y^2-x^2)^2}$$

$$= \frac{-2y^2+2x^2-4x^2}{(y^2-x^2)^2}$$

$$= \frac{-2x^2+2y^2}{(y^2-x^2)^2}$$

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ eqn (3) is exact

How to find integrating factor?

1. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ then $I.F = e^{\int f(x)dx}$

2. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = g(y)$ then $I.F = e^{\int -g(y)\, dy}$

3. If M & N are homogeneous function of same degree and $M.x + N.y \neq 0$ then $I.F = \dfrac{1}{Mx + Ny}$. But if $Mx + N.y = 0$ then $\dfrac{1}{xy}, \dfrac{1}{x^2}, \dfrac{1}{y^2}$ may be taken as integrating factor.

x

X —— X

1 Q! Find the solution of the following diff. eqn

$$(5x^3+12x^2+6y^2)dx + 6xy\,dy = 0$$

Soln Given,

$$\underbrace{(5x^3+12x^2+6y^2)}_{M}dx + \underbrace{6xy}_{N}dy = 0 \quad \text{—— (1)}$$

$$\frac{\partial M}{\partial y} = 12y \quad , \quad \frac{\partial N}{\partial x} = 6y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{(Not exact equation)}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 12y - 6y = 6y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6y}{6xy} = \frac{1}{x} = f(x)$$

Now,

$$I.F = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$\Rightarrow \mu = x$$

Multiply I.F in eqn (1)

$$\underbrace{(5x^4 + 12x^3 + 6xy^2)}_{M'}dx + \overbrace{6x^2y}^{N'}dy = 0$$

(Exact equation)

Solution :

$$\int M'dx + \int N'dy = c$$

(Treating 'y' as constant) (Taking the term which are rest from x)

$$\Rightarrow \int (5x^4 + 12x^3 + 6xy^2)\,dx + \int 0\, dy = c$$

$$\Rightarrow \boxed{x^5 + 3x^4 + 3x^2y^2 = c}$$

↳ solution

Q:- Find solution for the following differential equation

$$(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$$

Sol

Given, $\underbrace{(3x^2y^3e^y + y^3 + y^2)}_{M}\,dx + \underbrace{(x^3y^3e^y - xy)}_{N}\,dy = 0$ —— (1)

$$\frac{\partial M}{\partial y} = 9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y$$

$$\frac{\partial N}{\partial x} = 3x^2y^3e^y - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{(not exact eq}^n\text{)}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 9x^2y^2e^y + 3y^2 + 3y$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} \neq \cancel{\frac{9x^2y^2e^y + 3y^2 + 3y}{9x^2y^2e^y + 3x^2y^3e^y}}$$

$$= \frac{9x^2y^2e^y + 3y^2 + 3y}{3x^2y^3e^y + y^3 + y^2} = \frac{3(3x^2y^2e^y + y^2 + y)}{y(3x^2y^2e^y + y^2 + y)}$$

$$= \frac{3}{y} = g(y)$$

Now,

$$I.F = u = e^{\int g(y)\,dy} = e^{\int -\frac{3}{y}\,dy}$$

$$= e^{-3\ln y}$$

$$= \frac{1}{y^3}$$

Now multiplying I.F in eqn (1)

$$\underbrace{\left(3x^2e^y + 1 + \frac{1}{y}\right)}_{M'} dx + \underbrace{\left(x^3e^y - \frac{x}{y^2}\right)}_{N'} dy = 0$$

(exact eqn.)

To get solution:

$$\int M'\,dx + \int N'\,dy = c$$

(Taking 'y' as constant) (Taking the term rest from x)

$$\Rightarrow \int \left(3x^2e^y + 1 + \frac{1}{y}\right) dx + \int 0\,dy = c$$

$$\Rightarrow \boxed{x^3e^y + x + \frac{x}{y} = c}$$

solution

Linear equation :-

↳ In the form of : $\frac{dy}{dx} + P(x)y = Q(x)$

$I.F = \mu = e^{\int P dx}$

To write : $y\mu = \int Q\mu \, dx$ (To find solⁿ of diff. eqⁿ)

eg : $y' - y = e^{2x}$

solⁿ $I.F = \mu = e^{\int P dx} = e^{\int (-1) dx} = e^{-x}$

$y\mu = \int Q\mu \, dx$

$\Rightarrow \; ye^{-x} = \int e^{2x} \cdot e^{-x} dx$

$\Rightarrow \; \boxed{ye^{-x} = e^{x} + c}$

↳ solution of diff. eqⁿ

Q1. Find the solution of the following differential equation

$$\frac{dy}{dx} + y\tan x = \sin 2x, \quad y(0) = 1$$

Sol^n Given, $$\frac{dy}{dx} + \underbrace{y\tan x}_{P(x)} = \overbrace{\sin 2x}^{Q(x)} \quad \text{--- (i)}$$

→ Linear diff. eq^n

$$I.F = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x = u$$

~~Multiplying 'u' in eq (i) we get~~

~~$\sec x \frac{dy}{dx} +$~~

Solution :-

$$y u = \int Q u\,dx$$

$$\Rightarrow y\sec x = \int \sin 2x \sec x\,dx$$

$$\Rightarrow y\sec x = 2\int \sin x\,dx$$

$$\Rightarrow y\sec x = -2\cos x + c$$

$$\Rightarrow y\sec x + 2\cos x = c \quad \text{--- (2)}$$

Given $y(0) = 1$ (when $x = 0 \Rightarrow y = 1$)

From (2)

$$1 \times \sec 0 + 2 \times \cos 0 = c$$

$$\Rightarrow 1 \times 1 + 2 \times 1 = c$$

$$\Rightarrow c = 3$$

Now solution will be

$$\boxed{y \sec x + 2\cos x = 3}$$

Bernoulli's equation :-

$$\frac{dy}{dx} + P(x)y = q(x)y^a \quad a \in R \quad \text{——— (1)}$$

$$u = y^{1-a}$$

$$\frac{du}{dx} = (1-a)y^{-a}\frac{dy}{dx}$$

Dividing y^a in both sides in eqn (1)

$$y^{-a}\frac{dy}{dx} + P(x)y^{1-a} = q(x)$$

$$\Rightarrow \frac{1}{(1-a)}\left(\frac{du}{dx}\right) + uP(x) = q(x)$$

$$\Rightarrow \frac{du}{dx} + (1-a)uP(x) = (1-a)q(x)$$

Apply procedure of linear equation

Q: solve $y' + 4xy + xy^3 = 0$

Soln

$$y' + \underbrace{4x}_{P(x)}y = \underbrace{-x}_{q(x)}y^3 \quad \text{——— (1)}$$

Let $u = y^{1-3} = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}y'$$

Dividing y^3 in eqn (1) we get

$$y^{-3}y' + 4xy^{-2} = -x$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dx} + 4xu = -x$$

$$\Rightarrow \frac{du}{dx} - 8xu = 2x \quad \text{——— (2)}$$

$$\text{I.F} = u = e^{\int -8x\,dx}$$

$$= e^{-4x^2}$$

sol^n for (2)

$$u \times (\text{I.F}) = \int 2x e^{-4x^2} dx$$

$$\Rightarrow u \cdot e^{-4x^2} = \int 2x e^{-4x^2} dx$$

$$\Rightarrow u \cdot e^{-4x^2} = \frac{-1}{4}\int e^{z} dz$$

$$\Rightarrow u \cdot e^{-4x^2} = \frac{-1}{4} e^{z} \cancel{dx} + c$$

$$\Rightarrow u \cdot e^{-4x^2} = -\frac{1}{4} e^{-4x^2} + c$$

$$\Rightarrow u = -\frac{1}{4} + ce^{4x^2}$$

$$\Rightarrow \boxed{y^{-2} = ce^{4x^2} + -\frac{1}{4}}$$ → solution

put $-4x^2 = z$

$\Rightarrow -8x\,dx = dz$

$\Rightarrow 2x\,dx = -\frac{dz}{4}$

Application 1st order differential equation :-

(1) The type rate of change of temperature $\frac{dT}{dt}$ ($T(t)$ is the temperature) is directly proportinal to the difference of temperatures (T) and temperature of surrounding (T_A).

That means, $\frac{dT}{dt} \propto (T - T_A)$

Let the temperature of the room at $t=0$ is $66°F$ at $t=2$, $63°F$. If temperature of surrounding is $32°F$. Then find the temperature of the room after 10 mins

Sol^n

$$\frac{dT}{dt} \propto (T - T_A)$$

$$\Rightarrow \frac{dT}{dt} = K(T - T_A)$$

$$\Rightarrow \frac{dT}{T - T_A} = K dt$$

$$\Rightarrow \ln(T - T_A) = Kt + C$$

$$\Rightarrow T - T_A = e^{Kt+c}$$

$$\Rightarrow T = e^{Kt+c} + T_A$$

or $T(t) = e^{Kt+c} + T_A$

at $t=0$, $T = 66$

$$66 = e^{K\times 0 + c} + 32$$

$$\Rightarrow e^{c} = 34$$

$$\Rightarrow c = \ln 34$$

Now,

$$T(t) = e^{Kt+c} + T_A$$

at $t = 2$, $T = 63$

$$63 = e^{K \times 2 + \ln 34} + 32$$

$$\Rightarrow \quad 31 = 34\, e^{2K}$$

$$\Rightarrow \quad e^{2K} = \frac{31}{34}.$$

$$\Rightarrow \quad K = \frac{1}{2} \ln \frac{31}{34}$$

Now,

$$T(t) = e^{\left(\frac{1}{2} \ln \frac{31}{34}\right) t + \ln 34} + T_A$$

put $t = 10$

$$\Rightarrow \quad T(10) = e^{\left(\frac{1}{2} \ln \frac{31}{34}\right) \times 10 + \ln 34} + 32$$

$$= e^{\ln \left(\frac{31}{34}\right)^5 + \ln 34} + 32$$

$$= 34 \cdot \left(\frac{31}{34}\right)^5 + 32$$

$$= \frac{(31)^5}{(34)^4} + 32$$

Malthus law of population growth :-

Let $P(t)$ be the population of particular species at time t.
The rate of change of population at time 't' is proportional to the population present at that time.

$$\frac{dP(t)}{dt} \propto P(t)$$

At $t=0$ population of particular species is 5.3 million. At $t=1$, $P(t) = 4.8$ million. Find population of the species at $t=10$.

Sol^n

$$\frac{d\,P(t)}{dt} \propto P(t)$$

$$\Rightarrow \frac{d\,P(t)}{dt} = K\,P(t)$$

$$\Rightarrow \frac{d\,P(t)}{P(t)} = K\,dt$$

$$\Rightarrow \ln(P(t)) = Kt + C$$

$$\Rightarrow P(t) = e^{Kt+C} \quad \text{--- (1)}$$

at $t=0$, $P(0) = 5.3$

$$5.3 = e^{K\times 0 + C}$$

$$\Rightarrow 5.3 = e^{C}$$

$$\Rightarrow C = \ln 5.3$$

at $t=1$, $P(1) = 4.8$

$$4.8 = e^{K\times 1 + \ln 5.3}$$

$$\Rightarrow 4.8 = 5.3e^{K}$$

$$\Rightarrow K = \ln\left(\frac{4.8}{5.3}\right)$$

Now,

$$P(t) = e^{\left(\ln\left(\frac{4.8}{5.3}\right)\right)t + \ln(5.3)}$$

at $t=10$

$$P(10) = e^{10\ln\left(\frac{4.8}{5.3}\right) + \ln(5.3)}$$

$$\Rightarrow P(10) = \left(\frac{4.8}{5.3}\right)^{10} \cdot (5.3)$$

Voltage drop (E_L) across an inductor $= L\frac{dI}{dt}$

" " (E_C) " a capacitor $= \frac{1}{C}q$

$E_R = RI$

$$L\frac{dI}{dt} + RI = E(t)$$

$$\Rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right)I = \frac{1}{L}E(t)$$

Q: If $L = 0.1H$, $R = 5\Omega$, $E(t) = 12$ then :- Find the ~~~~ diff. eqn & current

Soln Diff. eqn will be :-

$$\frac{dI}{dt} + \left(\frac{R}{L}\right)I = \frac{1}{L}E(t)$$

$$\Rightarrow \frac{dI}{dt} + \left(\frac{5}{0.1}\right)I = \frac{1}{0.1}\times 12$$

$$\Rightarrow \boxed{\frac{dI}{dt} + 50I = 120} \rightarrow \text{Linear differential eqn}$$

$$\text{I.F} = \mu = e^{\int 50\,dt} = e^{50t}$$

solution:- $I\cdot\mu = \int Q\mu\,dt$

$$\Rightarrow I\cdot e^{50t} = \int 120e^{50t}\,dt$$

$$\Rightarrow \boxed{I\cdot e^{50t} = \frac{12}{5}e^{50t} + c}$$

→ solution.

Current, $I = \frac{12}{5} + c\cdot e^{-50t}$

Q: If $E(t) = E_0 \sin\omega t$ then find the current :- ?

Sol^n

D.E will be

$$L\frac{dI}{dt} + IR = E_0 \sin\omega t$$

$$\Rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right) I = \left(\frac{E_0}{L}\right)\sin\omega t$$

$$I.F = \mu = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

$$I \cdot \mu = \int \frac{E_0}{L}\sin\omega t \cdot \mu \, dt$$

$$\Rightarrow I e^{\frac{R}{L}t} = \frac{E_0}{L}\int \sin\omega t \cdot e^{\frac{R}{L}t} dt$$

$$\Rightarrow I e^{\frac{R}{L}t} = \frac{E_0}{L} \frac{e^{\frac{R}{L}t}}{\left(\frac{R}{L}\right)^2 + \omega^2} \times \left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + c$$

$$\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\Rightarrow I = \frac{E_0 L}{R^2+\omega^2 L^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + c e^{-\frac{R}{L}t}$$

Dt- 02/02/2024

Second order homogenous equation with constant coefficient

General form:-

$$y'' + ay' + by = 0$$

Solution

Step-1: Put $y'' = \lambda^2$, $y' = \lambda$, $y = 1$

Step-2: Characteristic equation became,

$$\lambda^2 + a\lambda + b = 0$$

Step-3: Solve the characteristic equation

(A) If $\lambda = \lambda_1, \lambda_2$

Then gen. soln,

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

(B) If $\lambda = \lambda_1 = \lambda_2$

Then general soln

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

(C) Let $\lambda = \alpha \pm i\beta$

The general soln,

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Q:- $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 1$

Sol^n put $y'' = \lambda^2$, $y' = \lambda$, $y = 1$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 2, 1$$

general sol^n,

$$y = c_1 e^{2x} + c_2 e^{x} \quad \text{——— (1)}$$

$y(0) = 1$

From (1) $\quad 1 = c_1 e^0 + c_2 e^0$

$$\Rightarrow c_1 + c_2 = 1 \quad \text{——— (2)}$$

From (1) $\quad y' = 2c_1 e^{2x} + c_2 e^{x}$

$y'(0) = 1$

$$1 = 2c_1 e^0 + c_2 e^0$$

$$\Rightarrow 2c_1 + c_2 = 1 \quad \text{——— (3)}$$

From (2) & (3)

$$c_1 = 0$$

$$c_2 = 1$$

Now,

$$y = 0 \times e^{2x} + 1 \times e^{x}$$

$$\Rightarrow \boxed{y = e^{x}}$$ solution

Q1. let ~~2nd~~ $y = e^{2x}(C_1 \cos x + C_2 \sin x)$ be the solution of second order ~~diff.~~ ~~eqn~~ linear homogenous eq. Then Find eqn?

Soln

$$y = e^{2x}(C_1 \cos x + C_2 \sin x) \quad // \quad y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

when solution of quadratic eqn is

$$\alpha = 2, \quad \beta = 1$$

$$\alpha \pm i\beta = 2 \pm i \qquad \underset{2+i}{\text{root1}}, \ \underset{2-i}{\text{root2}}$$

$$\cancel{\lambda^2 - (\alpha+\beta)\lambda + \alpha\beta = 0}$$

$$\Rightarrow \cancel{\lambda^2 - 3\lambda + 2 = 0}$$

$$\Rightarrow \cancel{\boxed{y'' - 3y' + 2y = 0}}$$

~~diff. eqn~~

$$\lambda^2 - (\text{root1} + \text{root2})\lambda + (\text{root1} \times \text{root2}) = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

$$\Rightarrow \boxed{y'' - 4y' + 5y = 0}$$

Diff. eqn

Cauchy Euler equation:

1 $\hookrightarrow x^2y'' + axy' + by = 0$

Characteristic equation: $m^2 + (a-1)m + b = 0$

(m is variable)

① If m_1 & m_2 are ~~dis~~ different real roots then

general solution,

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

② If $m = m_1 = m_2$

general solution,

$$y = (c_1 + c_2 \ln x) x^m$$

③ If $m = u \pm iv$

general solution,

$$y = x^u \left(A\cos(v\ln x) + B\sin(v\ln x)\right)$$

Q: $x^2y'' - 2xy' + 2y = 0$, $y(1) = 1.5$, $y'(1) = 1$

Find general solution

Sol^n

~~$m^2 + 0$~~

$x^2y'' - 2xy' + 2y = 0$

$(x^2y'' + axy' + by = 0)$

$a = -2$, $b = 2$

Characteristic equation:

$m^2 + (\cancel{a}-1)m + b = 0$

$\Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow \quad m = 2, 1$

general solution

$$y = C_1 x^2 + C_2 x \quad \text{—— (1)}$$

$y(1) = 1.5$

so, $\quad 1.5 = C_1 \times 1^2 + C_2 \times 1$

$\Rightarrow \quad C_1 + C_2 = 1.5 \quad \text{—— (2)}$

from (1) $\quad y' = 2C_1 x + C_2 \quad \text{—— (3)}$

$y'(1) = 1$

so, $\quad 1 = 2 \times C_1 \times 1 + C_2$

$\Rightarrow \quad 2C_1 + C_2 = 1 \quad \text{—— (4)}$

From (2) & (4)

$C_1 = -0.5$

$C_2 = 2$

Now general solution will be

$$\boxed{y = -0.5x^2 + 2x}$$

Q! $(x^2D^2 - 3xD + 4)y = 0$ Find general solution

$\underline{\text{sol}^n}$ $(x^2D^2 - 3xD + 4)y = 0$

$\Rightarrow x^2D^2y - 3xDy + 4y = 0$

$\Rightarrow x^2y'' - 3xy' + 4y = 0$

Remember
$D^2y = y''$
$Dy = y'$

$a = -3,\ b = 4$ $\quad (x^2y'' + axy' + by = 0)$

Characteristic equation!

$m^2 + (a-1)m + b = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2$

general sol^n, $y = (c_1 + c_2 \ln x)x^m$

$\Rightarrow \boxed{y = (c_1 + c_2 \ln x)x^2}$

Q!- $x^2y'' + 3xy' + 4 = 0$, find solution.

$\underline{\text{sol}^n}$ $x^2y'' + 3xy' + 4 = 0$ $\quad (x^2y'' + axy' + by = 0)$

$a = 3,\ b = 4$

Charactetic eq^n!

$m^2 + (a-1)m + b = 0$

$\Rightarrow m^2 + 2m + 4 = 0$

$\Rightarrow m = \dfrac{-2 \pm \sqrt{4 - 4\times1\times4}}{2}$

$\Rightarrow m = \dfrac{-2 \pm \sqrt{-12}}{2}$

$m = -1 \pm \sqrt{3}\,i$

$\alpha = -1, \ \beta = \sqrt{3}$

general sol^n

$y = x^{\alpha}\left[A\cos(\beta \ln x) + B\sin(\beta \ln x)\right]$

$$\boxed{y = x^{-1}\left[A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)\right]}$$

Q. $y'' + 4y' + 3y = 0$

$\frac{8}{8}$

sol:

$y'' + 4y' + 3y = 0$

$\Rightarrow$ now

$\lambda^2 + 4\lambda + 3 = 0$

$\Rightarrow (\lambda+3)(\lambda+1) = 0$

$\Rightarrow \lambda = -3, -1$

general sol^n

$$\boxed{y = c_1 e^{-3x} + c_2 e^{-x}}$$

Sum rule :-

If $r(x)$ is the sum of two or more functions in column 1 then take y_p as the sum of corresponding function of column 2

Q:- $y'' + 2y' - 35y = 12e^{5x} + 37\sin 2x$

solution = ??

Sol^n

$$y'' + 2y' - 35y = 12e^{5x} + 37\sin 2x \quad \text{—— (1)}$$

To find y_c

$$y'' + 2y' - 35y = 0$$

$$\lambda^2 + 2\lambda - 35 = 0$$

$$\Rightarrow \lambda = -7, 5$$

$$y_c = c_1 e^{-7x} + c_2 e^{5x}$$

To find y_p

$r(x) = 12e^{5x} + 37\sin 5x$

$$y_p = ce^{5x} + K\sin 5x + M\cos 5x$$

$\cancel{\text{put } y = y_p \text{ in eqn ①}}$

$$\cancel{y_p'' + 2y_p' - 35y_p = 12e^{5x} + 37\sin 5x}$$

$$\cancel{\Rightarrow 25ce^{5x} - 25K\sin 5x - 25M\cos 5x}$$

$$\cancel{+ 2(5ce^{5x} + 5K\cos 5x}$$

After modification,

$$y_p = cxe^{5x} + K\sin 5x + M\cos 5x$$

$$y_p' = 5cxe^{5x} + ce^{5x} + 5K\cos 5x - 5M\sin 5x$$

$$y_p'' = 25cxe^{5x} + 5ce^{5x} + 5ce^{5x} - 25K\sin 5x - 25M\cos 5x$$

putting $y = y_p$ in eqn ①

$$25cxe^{5x} + 5ce^{5x} + 5ce^{5x} - 25K\sin 5x - 25M\cos 5x$$
$$+ 2(5cxe^{5x} + ce^{5x} + 5K\cos 5x - 5M\sin 5x)$$
$$- 35(cxe^{5x} + K\sin 5x + M\cos 5x) = 12e^{5x} + 37\sin 5x$$

$\Rightarrow 12ce^{5x} + (-60K - 10M)\sin 5x + (10K - 60M)\cos 5x$

$= 12e^{5x} + 37\sin 5x$

comparing both the sides we will get

$12c = 12$ & $-60K - 10M = 37$

$\Rightarrow \boxed{c = 1}$ & $10K - 60M = 0$

so, $\boxed{K = \frac{-3}{5}}$, $\boxed{M = \frac{-1}{10}}$

Nxo) Solution: $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{-7x} + c_2 e^{5x} + cx e^{5x} + K \sin 5x + M \cos 5x$$

$$\Rightarrow \boxed{y = c_1 e^{-7x} + c_2 e^{5x} + x e^{5x} - \frac{3}{5} \sin 5x - \frac{1}{10} \cos 5x}$$

Solution of non homogenous eqn by variation of parameters

$$y'' + ay' + by = r(x)$$

<u>Step 1</u> Make it homogenous and find

$$y_c = c_1 y_1 + c_2 y_2$$

<u>Step 2</u> Find Wronskian by the formula

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

<u>Step 3</u> $$y_p = -y_1 \int \frac{y_2 r}{W} + y_2 \int \frac{y_1 r}{W}$$

Q! $y'' + y = \sec x$, solution = ??

<u>sol^n</u>

$$y'' + y = \sec x \quad \text{—} (1)$$

<u>S-1</u> Making homogenous

$$y'' + y = 0$$

$y \to \lambda$ $\quad \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i \quad (\equiv \overset{0}{\overset{\shortparallel}{\alpha}} \pm i \overset{1}{\overset{\shortparallel}{\beta}})$

$$y_c = e^{\alpha x}(A \sin \beta x + B \cos \beta x)$$

$$y_c = A \sin x + B \cos x$$

$$(\equiv c_2 y_2 + c_1 y_1)$$

Step 2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \cancel{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

S-3

$$y_p = -y_1 \int \frac{y_2 r}{W} + y_2 \int \frac{y_1 r}{W}$$

$$= -\cos x \int \frac{\sin x \cdot \sec x \, dx}{1} + \sin x \int \frac{\cos x \sec x \, dx}{1}$$

$$= -\cos x \ln|\sec x| + \sin x \cdot x$$

$$= \cos x \ln|\cos x| + x \sin x$$

Solution:

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = A\sin x + B\cos x + \cos \ln|\cos x| + x\sin x}$$

Q! $x^2y'' - 4xy' + 6y = 21x^{-4}$, solution = ?

Sol^n

S-1 $\quad x^2y'' - 4xy' + 6y = 0$ (Making as homogenous)

$m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 3, 3$

$y_c = c_1x^2 + c_2x^3$

S-2

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$= 3x^4 - 2x^{4} = x^4$

$$y_p = -x^2\int \frac{x^3 \cdot 21x^{-4}}{x^4}\,dx + x^3\int \frac{x^2 \cdot 21x^{-4}}{x^4}\,dx$$

$= -x^2\int 21x^{-5}\,dx + x^3\int 21x^{-6}\,dx$

$= -21x^2 \times \frac{x^{-4}}{-4} + 21x^3 \times \frac{x^{-5}}{-5}$

$= \frac{21}{4}x^{-2} - \frac{21}{5}x^{-2}$

$= x^{-2}\left(\frac{105-84}{20}\right)$

$= \frac{21}{20}x^{-2}$

solution! $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 x^2 + c_2 x^3 + \frac{21}{20} x^{-2}}$$

Power series solution of 2nd differential equation :-

power series $\rightarrow$ $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)$

↳ Radius of convergence of power series will be determined by

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

or $$R = \frac{1}{\lim_{n \to \infty} |a_n|^{1/n}}$$

↳ Circle of of convergence: $|x - x_0| = R$

Q! Find radius of convergence

1. $\sum_{m=0}^{\infty} x^m m(m+1)$

2. $\sum_{m=0}^{\infty} \frac{x^{2m}}{m!}$

3. $\sum_{m=0}^{\infty} \frac{1}{3^m} (x-3)$

4. $\sum_{n=0}^{\infty} (n-2)(n+1) x^n$

Sol^n

(1) $\sum_{m=0}^{\infty} x^{m} m(m+1)$ $\longrightarrow$ multiple of $m=1$

so, $R = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right|$

$a_m = m(m+1)$

$a_{m+1} = (m+1)(m+2)$

$$R = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right|$$

$$= \lim_{m\to\infty} \frac{m(m+1)}{(m+1)(m+2)}$$

$$= \lim_{m\to\infty} \frac{m}{m+2}$$

$$= \lim_{m\to\infty} \frac{1}{1+\frac{2}{m}} = 1$$

(2) $\sum_{m=0}^{\infty} \frac{x^{2m}}{m!}$ $\longrightarrow$ (multiple of $m=2$)

$\Rightarrow R^2 = \lim_{m\to\infty} \frac{a_m}{a_{m+1}}$

$$a_m = \frac{1}{m!}$$

$$a_{m+1} = \frac{1}{(m+1)!}$$

$$R^2 = \lim_{m\to\infty} \frac{a_m}{a_{m+1}}$$

$$= \lim_{m\to\infty} \frac{\frac{1}{m!}}{\frac{1}{(m+1)!}}$$

$$= \lim_{m\to\infty} (m+1) = \infty \quad \Rightarrow R = \infty$$

(3) $\sum_{m=0}^{\infty} \frac{1}{3^m}(x-3)^{2m-1000} \rightarrow$ (multiple of $m = 2$)

so $\rightarrow R^2 = \lim_{m\to\infty} \frac{a_m}{a_{m+1}}$

$a_m = \frac{1}{3^m}$

$a_{m+1} = \frac{1}{3^{m+1}}$

$R^2 = \lim_{m\to\infty} \frac{a_m}{a_{m+1}}$

$= \lim_{m\to\infty} \frac{\frac{1}{3^m}}{\frac{1}{3^{m+1}}}$

$= 3$

$\Rightarrow R = \sqrt{3}$

circle of convergence: $|x-3| = \sqrt{3}$

(4) $\sum_{n=0}^{\infty} (n-2)(n+1) x^n \rightarrow$ (multiple of $n = 1$)

so $\rightarrow R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$

$a_n = (n-2)(n+1)$

$a_{n+1} = (n-1)(n+2)$

$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(n-2)(n+1)}{(n-1)(n+2)}$

$= \lim_{n\to\infty} \frac{n^2-n-2}{n^2+n-2}$

$= \lim_{n\to\infty} \frac{1-\frac{1}{n}-\frac{2}{n^2}}{1+\frac{1}{n}-\frac{2}{n^2}}$

$= 1$

Conversion of power series (changing only m/n li value)

e.g.: $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{--- (1)}$$

If we want → The series will start from $n=1$

then $(n=0 \to n=1)$ that mean $n \to n-1$

Now in (1) Replacing n by $n-1$

$$\sum_{n-1=0}^{\infty} (-1)^{n-1} \frac{x^{2(n-1)-1}}{(2(n-1)+1)!}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-3}}{(2n-3)!}$$

Q! Find power series solution of

$$y' - y = 0$$

soln

S-1 Make co-efficient of highest order derivative part '1'

$$y' - y = 0 \quad \text{---} (1)$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=0}^{\infty} a_m m x^{m-1}$$

$$= \sum_{m=1}^{\infty} a_m m x^{m-1}$$

From (1)

$$\sum_{m=1}^{\infty} a_m m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} a_{m+1}(m+1) x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[\{a_{m+1}(m+1) - a_m\} x^m\right] = 0 \quad \text{---} (2)$$

Comparing co-effocient of x (or x^n)

$$(m+1)\,a_{m+1} - a_m = 0$$

$$\Rightarrow \quad a_{m+1} = \frac{a_m}{m+1}$$

For $m=0 \rightarrow \quad a_1 = \frac{a_0}{1}$

For $m=1 \rightarrow \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$

For $m=2 \rightarrow \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3\cdot 2!} = \frac{a_0}{3!}$

From (2)

$$\cancel{\sum_{m=0}^{\infty} \frac{a_0}{(m+1)!}\cdot(m+1) = \frac{a_0}{m!}}$$

Now solution:-

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + \frac{a_0}{1!}x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots$$

Q! Find power series solution

1. $y'' = y'$ 2. $y' = 3x^3 y$

Sol^n

(1) $y'' = y'$ ——— (1)

$\Rightarrow y'' - y' = 0$

Now,

$$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{——— (2)}$$ (Will be the solution)

$$\Rightarrow y' = \sum_{m=0}^{\infty} a_m \cdot m x^{m-1}$$

$$= \sum_{m=1}^{\infty} a_m \cdot m x^{m-1}$$

$$y'' = \sum_{m=1}^{\infty} a_m \cdot m \cdot (m-1) x^{m-2}$$

$$\Rightarrow y'' = \sum_{m=2}^{\infty} a_m \cdot m \cdot (m-1) x^{m-2}$$

From eq^n (1)

$$\sum_{m=2}^{\infty} a_m \cdot m(m-1) x^{m-2} = \sum_{m=1}^{\infty} a_m \cdot m x^{m-1}$$

$$\Rightarrow \sum_{m=0}^{\infty} a_{m+2} \cdot (m+2)(m+1) x^m = \sum_{m=0}^{\infty} a_{m+1} \cdot (m+1) x^m$$

$$\Rightarrow \sum_{m=0}^{\infty} \{a_{m+2}(m+2)(m+1) - a_{m+1}(m+1)\} x^m = 0$$

Comparing both the sides

$$a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) = 0$$

$$\Rightarrow (m+2)\, a_{m+2} = a_{m+1}$$

$$\Rightarrow a_{m+2} = \frac{a_{m+1}}{m+2}$$

For $m=0 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_1}{2!}$

For $m=1 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{3 \cdot 2!} = \frac{a_1}{3!}$

For $m=2 \rightarrow a_4 = \frac{a_3}{4} = \frac{a_1}{4 \cdot 3!} = \frac{a_1}{4!}$

Now solution:

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 x + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 x + a_1 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \cdots$$

$$\Rightarrow \sum_{m=0}^{\infty} \{a_{m+2}(m+2)(m+1) - a_{m+1}(m+1)\} x^m = 0$$

Comparing both the sides

$$a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) = 0$$

$$\Rightarrow (m+2)\, a_{m+2} = a_{m+1}$$

$$\Rightarrow a_{m+2} = \frac{a_{m+1}}{m+2}$$

For $m=0 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_1}{2!}$

For $m=1 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{3 \cdot 2!} = \frac{a_1}{3!}$

For $m=2 \rightarrow a_4 = \frac{a_3}{4} = \frac{a_1}{4 \cdot 3!} = \frac{a_1}{4!}$

Now solution:

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 x + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 x + a_1 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \cdots$$

Legendary equation :-

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

Solution : $y = c_1 y_1(x) + c_2 y_2(x)$

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)}{6!}x^6 + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots$$

For any integer 'n' (positive integer) one of the solution ($y_1(x)$) other will be series.

Polynomial solution of legendary eqn when multiplied with a constant is called lengendo polynomial.
The formula for lendendo polynomial exliplitly can be written as

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

$$M = \begin{cases} n/2 \\ \frac{n-1}{2} \end{cases}$$ which will be integer that will be taken

when $n=0$, $M=0$

$$P_0(x) = \sum_{m=0}^{0} \frac{(-1)^m (2\times 0-2m)!\ x^{0-2m}}{2^0\, m!\, (0-m)!\, (0-2m)!}$$

$$= 1$$

when $n=1$, $M=0$

$$P_1(x) = \sum_{m=0}^{0} \frac{(-1)^m (2\times 1-2m)!\ x^{1-2m}}{2^1\, m!\, (1-m)!\, (1-2m)!}$$

$$= x$$

Rodrigue's formula:-

$\hookrightarrow$ $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Polynomial solution of legendary equation when multiplied with a constant is called legendary polynomial

Proof:-

Let $v = (x^2-1)^n$

$$\frac{dv}{dx} = 2nx(x^2-1)^{n-1}$$

$$\Rightarrow \frac{dv}{dx} = 2nx\frac{(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nxv$$

$$\Rightarrow (1-x^2)\frac{dv}{dx} + 2nxv = 0$$

Differentiating above eqn we get

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nx\frac{dv}{dx} + 2nv = 0$$

$$\Rightarrow (1-x^2)\frac{d^2v}{dx^2} + (2n-2)x\frac{dv}{dx} + 2nv = 0$$

Differenting the above eqn 'n' times using Lebling rule we get

$$(1-x^2)\frac{d^{n+2}}{dx^{n+2}} - 2nx\frac{d^{n+1}v}{dx^{n+1}} - 2n\frac{(n-1)}{2!}\frac{d^n v}{dx^n}$$
$$+ (2n-2)\left\{x\frac{d^{n+1}v}{dx^{n+1}} + n\frac{d^n v}{dx^n}\right\} + 2n\frac{d^n v}{dx^n} = 0$$

$$(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} + \{-2nx + (2n-2)x\}\frac{d^{n+1}v}{dx^{n+1}}$$
$$+\{-n(n-1) + (2n-2)n + 2n\}\frac{d^n v}{dx^n} = 0$$

$$(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} - 2x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n} = 0$$

Let $W = \dfrac{d^n v}{dx^n}$

$$\Rightarrow (1-x^2)\frac{d^2 W}{dx^2} - 2x\frac{dW}{dx} + n(n+1)W = 0$$

W is a Polynomial solution of legendary equation

Legendary polynomial $= P_n(x) = cW$

<u>Note</u>: $P_n(1) = 1 \ \forall n \in N$

Now,

$$P_n(x) = c\,\frac{d^n}{dx^n}(v)$$

$$= c\,\frac{d^n}{dx^n}(x^2-1)^n$$

$$= c\left[\frac{d^n}{dx^n}\left\{(x+1)^n (x-1)^n\right\}\right]$$

$$= c\left\{(x-1)^n \frac{d^n}{dx^n}(x+1)^n + n(x-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n \right.$$

$$\left. \text{-----} + \frac{d^n}{dx^n}\left\{(x-1)^n\right\}\cdot(x+1)\right\}$$

$$P_n(1) = c\{0+0+0\ldots + n!\cdot 2^n\}$$

$$\Rightarrow 1 = c\cdot n!\,2^n$$

$$\Rightarrow \boxed{c = \frac{1}{2^n n!}}$$

Now,

$$P_n(x) = c.\,w$$

$$P_n(x) = \frac{1}{2^n n!}\,w$$

$$\Rightarrow P_n(x) = \frac{1}{2^n\, n!}\,\frac{d^n}{dx^n}(x^2-1)^n$$

Q: show that $\int_{-1} P_m(x) P_n(x)\,dx = 0, \; m \neq n$

sol^n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{——— (1)}$$

Here $P_m(x)$ & P_n both are the solution of the above equation.

Now,

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \quad \text{——— (2)}$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad \text{——— (3)}$$

Multiply P_n in eq^n (2) and P_m in eq^n (3) and subtract

$$(1-x^2)P_m''P_n - 2xP_m'P_n + m(m+1)P_mP_n = 0$$

$$(1-x^2)P_n''P_m - 2xP_n'P_m + n(n+1)P_nP_m = 0$$

$$(-) \qquad (+) \qquad (-)$$

$$(1-x^2)\left[P_m''P_n - P_n''P_m\right] - 2x\left[P_m'P_n - P_n'P_m\right]$$

$$+\left[m(m+1) - n(n+1)\right]P_mP_n = 0 \quad \text{——— (4)}$$

Let

$$z = (1-x^2)(P_m'P_n - P_n'P_m)$$

$$z' = (1-x^2)(P_m''P_n + P_m'P_n' - P_n'P_m' - P_n''P_m)$$

$$- 2x(P_m'P_n - P_n'P_m)$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)(P_m'P_n - P_n'P_m)\right] = (1-x^2)(P_m''P_n - P_n''P_m)$$

$$- 2x(P_m'P_n - P_n'P_m)$$

From eq^n (4) we get

$$\frac{d}{dx}\left[(1-x^2)\,(P_m' P_n - P_m P_n')\right] + (m-n)(m+n+1)\,P_m P_n = 0$$

Integrating the above eq from -1 to $+1$ we get

$$\left[(1-x^2)\cancel{(P_m' P_n - P_m P_n')}^{\,0}\right]_{-1}^{1} + (m-n)(m+n+1)\int_{-1}^{1} P_m P_n = 0$$

$$\Rightarrow \int_{-1}^{1} P_m P_n = 0 \quad (\text{for } m \neq n)$$

$$\Rightarrow \text{ or } \boxed{\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0}$$

Module-3

Scalar point function :-

$f(x,y,z) = x^2+y^2+z^2$

$f(1,2,-1) = 1+4+1 = 6$

Vector point function :-

$f(x,y,z) = xy\hat{i} + y^2z\hat{j} + z^2\hat{k}$

$f(1,-2,-1) = -2\hat{i} - 4\hat{j} + \hat{k}$

Scalar field :-

→ A scalar point function when made equal to a constant is called a scalar field

eg: $f(x,y,z) = x^2+y^2+z^2 = 4$

Gradient of a scalar point function :-

→ $\nabla f = \frac{\partial}{\partial x} f \cdot \hat{i} + \frac{\partial}{\partial y} f \hat{j} + \frac{\partial}{\partial z} f \hat{k}$

where $\nabla \rightarrow$ nabla

$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

eg: $f(x) = x^2+y^2+z^2$

grad $f = \nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Directional derivative of f at the point P in the direction of a vector $\vec{b}$ is defined as:

$$\boxed{D_b f = \text{grad} f \Big]_P \cdot \frac{\vec{b}}{|\vec{b}|}}$$

Q: Find directional derivative of $f = 2x^2 + 3y^2 + z^2$ at $P(2,1,3)$ in the direction of $\hat{i} - 2\hat{k}$.

Sol^n: $f = 2x^2 + 3y^2 + z^2$, $\vec{b} = \hat{i} - 2\hat{k}$

$$\text{grad} f = 4x\,\hat{i} + 6y\,\hat{j} + 2z\,\hat{k}$$

$$\text{grad} f \Big]_{P(2,1,3)} = 8\hat{i} + 6\hat{j} + 6\hat{k}$$

$$D_b f = \text{grad} f \Big]_{(2,1,3)} \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$= (8\hat{i} + 6\hat{j} + 6\hat{k}) \cdot \frac{(\hat{i} - 2\hat{k})}{\sqrt{1+4}}$$

$$= \frac{8 - 12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

Note

→ If f is a scalar point function having continuous 1st partial derivatives, at point 'P' then gradient of f is ~~[illegible]~~ is not a zero vector then f increase maximum in the direction of 'P'.

Let $r(t)$ lies on the surface:

$$f(x, y, z) = c \quad \text{——— (1)}$$

$$f(x(t), y(t), z(t)) = c \quad \text{——— (2)}$$

Let $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$

Differentiating eqn (2) we get

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial x}{\partial t}\hat{i} + \frac{\partial y}{\partial t}\hat{j} + \frac{\partial z}{\partial t}\hat{k}\right) = 0$$

$$\Rightarrow \nabla f \cdot \frac{dr}{dt} = 0$$

or $\quad \nabla f \cdot r' = 0$

∇f is a vector perpendicular to the surface

Laplace equation :-

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Let V be a vector point function then

Divergence of a vector point function

↳ If V be a vector point function

$$\operatorname{div} V = \nabla \cdot V$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note

If $\operatorname{div} V = 0 \Rightarrow$ condition of incompressibility

Q: (1) $\operatorname{div}(\operatorname{grad} f) = \nabla^2 f$

(2) $\nabla f g = f \nabla g + g \nabla f$

(3) Directional derivative = ??

If $f = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3,0,4)$ & in the direction of $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

(4) $\nabla^2 f = ??$

where $f = \cosh^2 x - \sinh^2 y$

solution:

(1) LHS = $\operatorname{div}(\operatorname{grad} f)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \operatorname{grad} f$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

Divergence of a vector point function

$\hookrightarrow$ If V is a vector point function

$$\text{div}\, V = \nabla \cdot V$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note

If $\text{div}\, V = 0 \Rightarrow$ condition of incompressibility.

Q: (1) $\text{div}\,(\text{grad}\, f) = \nabla^2 f$

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solution:

(1) LHS $= \text{div}(\text{grad}\, f)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \text{grad}\, f$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} + \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} + \frac{\partial}{\partial z} \cdot \frac{\partial f}{\partial z}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

$$= \nabla^2 f = RHS$$

~~(2) Let $f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$~~

~~$g = g_x \hat{i} + g_y \hat{j} + g_z \hat{k}$~~

~~LHS = $\nabla f \cdot g$~~

~~$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (f_x g_x + f_y g_y + f_z g_z)$~~

$=$

(2) ~~Let~~ $\nabla f g = f \nabla g + g \nabla f$

LHS = $\nabla f g$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) f g$$

$$= \frac{\partial (fg)}{\partial x} \hat{i} + \frac{\partial (fg)}{\partial y} \hat{j} + \frac{\partial (fg)}{\partial z} \hat{k}$$

$$= \left(f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right) \hat{i} + \left(f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y} \right) \hat{j}$$

$$+ \left(f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z} \right) \hat{k}$$

$$= f \left(\frac{\partial g}{\partial x} \hat{i} + \frac{\partial g}{\partial y} \hat{j} + \frac{\partial g}{\partial z} \hat{k} \right) + g \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right.$$

$$= f \nabla g + g \nabla f = RHS$$

(3) Given, $f = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

$$\text{grad} f = \frac{-x}{(x^2+y^2+z^2)^{3/2}}\hat{i} - \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{j}$$

$$- \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

$$\text{grad} f \Big]_{(3,0,4)} = \frac{-3}{125}\hat{i} - \frac{4}{125}\hat{k}$$

$$\text{directional derivative} = \text{grad} f \Big]_{(3,0,4)} \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= \left(\frac{-3}{125}\hat{i} - \frac{4}{125}\hat{j}\right) \cdot \left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} \times \frac{-7}{125} = \frac{-7}{125\sqrt{3}}$$

1) Given $f_0 = \cosh^2 x - \sinh^2 y$

$$\frac{\partial f}{\partial x} = 2\cosh x \cdot \sinh x$$

$$\frac{\partial^2 f}{\partial x^2} = 2\sinh x \cdot \sinh x + 2\cosh x \cdot \cosh x$$

$$= 2\sinh^2 x + 2\cosh^2 x$$

$$\frac{\partial f}{\partial y} = -2\sinh y \cdot \cosh y$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cosh^2 y - 2\sinh^2 y$$

$$\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial z^2} = 0$$

Now,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 2\sinh^2 x + 2\cosh^2 x - 2\cosh^2 y - 2\sinh^2 y + 0$$

$$= 2\times\left(\frac{e^x + e^{-x}}{2}\right)^2 + 2\times\left(\frac{e^x - e^{-x}}{2}\right)^2 - 2\times\frac{(e^y - e^{-y})^2}{2^2} - 2\times\left(\frac{e^y + e^{-y}}{2}\right)^2$$

$$= \frac{2(e^{2x} + e^{-2x})}{2} - 2\,\frac{(e^{2y} + e^{-2y})}{2}$$

$$= e^{2x} + e^{-2x} - e^{2y} - e^{-2y}$$

Curl of a vector point function :-

→ Let $V = yz\hat{i} + xz\hat{j} + yx\hat{k}$

Now,

$$\text{curl } V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & yx \end{vmatrix} = \nabla \times V$$

$$= \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - z)$$

$$= \vec{0}$$

Q: If $V = x^2y\hat{i} + y^3\hat{j} + z^2xy\hat{k}$

then curl $V = ?$

Sol^n $\text{curl } V = \vec{\nabla} \times V$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^3 & z^2xy \end{vmatrix}$$

$$= \hat{i}(z^2x - 0) - \hat{j}(z^2y - 0) + \hat{k}(0 - x^2)$$

$$= z^2x\hat{i} - z^2y\hat{j} - x^2\hat{k}$$

Q1:- Find scalar point function f whose gradient is $yz\hat{i} + xz\hat{j} + xy\hat{k}$

Sol^n Given $\text{grad} f = yz\hat{i} + xz\hat{j} + xy\hat{k}$ ——— (1)

$\Rightarrow \quad \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ ——— (2)

Comparing (1) & (2)

(i) $\frac{\partial f}{\partial x} = yz$; (ii) $\frac{\partial f}{\partial y} = xz$, (iii) $\frac{\partial f}{\partial z} = xy$

$\Rightarrow \quad \partial f = yz\,\partial x$

$\Rightarrow \quad f = yzx + C_1(y, z)$ ——— (3)

again $\cancel{\frac{\partial f}{\partial y} = xz}$

$\Rightarrow \cancel{\partial f = xz\,\partial y}$

Derivating eq^n (3) w.r.t y we get

$\frac{\partial f}{\partial y} = zx + \frac{\partial}{\partial y} C_1(y, z)$ ——— (4)

Comparing eq^n (ii) & (4) we get

$\frac{\partial}{\partial y} C_1(y, z) = 0$

$\Rightarrow \partial C_1(y, z) = 0\,\partial y$

$\Rightarrow C_2(y, z) = C_2(z)$

From (2)

$$f = yzx + G_2(z) \quad \text{--- (5)}$$

$$\Rightarrow \frac{\partial f}{\partial z} = yx + \frac{d}{dz} G_2(z) \quad \text{--- (6)}$$

Comparing eqn (6) with eqn (iii) we get

$$\frac{d}{dz} G_2(z) = 0$$

$$\Rightarrow G_2(z) = C$$

From (5) $\boxed{f = yzx + C}$ (Ans.)

$\Rightarrow \cancel{\frac{\partial f}{\partial x} = yz}$

Q:- Find the scalar point function whose gradient is $x^2y\hat{i} + y^3\hat{j} + z^2xy\hat{k}$

Soln Given, grad $f = x^2y\hat{i} + y^3\hat{j} + z^2xy\hat{k}$

$$\Rightarrow \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = x^2y\hat{i} + y^3\hat{j} + z^2xy\hat{k}$$

Comparing both the sides,

$$\frac{\partial f}{\partial x} = x^2y \text{ --- (i)}, \quad \frac{\partial f}{\partial y} = y^3 \text{ --- (ii)}, \quad \frac{\partial f}{\partial z} = z^2xy \text{ --- (iii)}$$

$$\Rightarrow \partial f = x^2y\,\partial x$$

$$\Rightarrow f = \frac{x^3}{3}y + G(y,z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x^3}{3} + \frac{\partial}{\partial y} G(y,z) = y^3 \quad \text{(From (ii))}$$

we can't compare bothsides

Here we can't find the scalar point function

Q:- Prove that curl (grad f) $= \vec{0}$ where f has 2nd order continuous partial derivative

Sol^n:

LHS = curl (grad f)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= \vec{0} = \text{RHS}$$

Q. ① If $f = xy^2$, $g = x^2y$
verify $\nabla(fg) = f\,\nabla g + g\,\nabla f$

② If $f = e^x \cos y$, $P(3, \pi, 0)$
$a = 2\hat{i} + 3\hat{j}$, find $D_a f$

③ If $f = x^2 y z^2$, verify div(grad f) $= \nabla^2 f$

④ Find f if $\nabla f = [y e^x, e^x, 1]$

Soln

① Given $f = xy^2$, $g = x^2y$

LHS $= \nabla(fg) = \nabla(xy^2 \cdot x^2y)$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^3y^3)$

$= 3x^2y^3\,\hat{i} + 3x^3y^2\,\hat{j}$

4

RHS $= f\,\nabla g + g\,\nabla f$

$= xy^2\left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)x^2y\right\}$

$+ x^2y\left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)xy^2\right\}$

$= xy^2 \times (2xy\,\hat{i} + x^2\,\hat{j})$

$+ x^2y\,(y^2\,\hat{i} + 2xy\,\hat{j})$

$= 2x^2y^3\,\hat{i} + x^3y^2\,\hat{j} + x^2y^3\,\hat{i} + 2x^3y^2\,\hat{j}$

$= 3x^2y^3\,\hat{i} + 3x^3y^2\,\hat{j}$

$\therefore$ LHS = RHS

(2) Given $f = e^x \cos y$ $\quad P(2,\pi,0)$, $a = 2\hat{i} + 3\hat{j}$

$$\text{grad} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) e^x \cos y$$

$$= e^x \cos y\,\hat{i} - e^x \sin y\,\hat{j}$$

$$\text{grad} f\Big]_{(2,\pi,0)} = e^2 \cos\pi\,\hat{i} - e^2 \sin\pi\,\hat{j}$$

$$= -e^2\hat{i}$$

$$D_a f = \text{grad} f\Big]_{(2,\pi,0)} \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (-e^2\hat{i}) \cdot \frac{(2\hat{i} + 3\hat{j})}{\sqrt{2^2 + 3^2}}$$

$$= \frac{-2e^2}{\sqrt{13}}$$

4

(3) Given, $f = x^2 y z^2$

LHS $= \text{div}(\text{grad} f)$

$$= \text{div}\left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) x^2 y z^2\right\}$$

$$= \text{div}\left(2xyz^2\hat{i} + x^2 z^2\hat{j} + 2x^2 y z\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(2xyz^2\hat{i} + x^2z^2\hat{j} + 2x^2yz\hat{k}\right)$$

$= 2yz^2 + 2x^2y$

RHS $= \nabla^2 f$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2xyz^2) + \frac{\partial}{\partial y}(x^2z^2) + \frac{\partial}{\partial z}(2x^2yz)$$

4 $= 2yz^2 + 2x^2y$

$\therefore$ LHS = RHS

(4) Given $\nabla f = [ye^x, e^x, 1]$

$$\Rightarrow \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = ye^x\hat{i} + e^x\hat{j} + \hat{k}$$

$\frac{\partial f}{\partial x} = ye^x$ —— (1)

$\frac{\partial f}{\partial y} = e^x$ —— (2)

$\frac{\partial f}{\partial z} = 1$ —— (3)

$\Rightarrow f = ye^x + c_1(y,z)$

$\Rightarrow \frac{\partial f}{\partial y} = e^x + \frac{\partial}{\partial y} c_1(y,z)$

$\Rightarrow e^x = e^x + \frac{\partial}{\partial y} c_1(y,z)$

$\Rightarrow \cancel{c_1(y,z) = 0}$

$\Rightarrow c_1(y,z) = c_2(z)$

$\Rightarrow f = ye^{x} + C_2(z)$

$\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(C_2(z)) \Rightarrow 1 = \frac{\partial}{\partial z}(C_2(z))$

$\Rightarrow \cancel{1} \; \cancel{z+c} \; \cancel{C_2(z)} \Rightarrow z + c = C_2(z)$

Now,

$f = ye^{x} + z + c$

Q1:- If $f = xyz$, $g = x^2y$
verify $\nabla(fg) = f\nabla g + g\nabla f$

Sol^n Given, $f = xyz$, $g = x^2y$

LHS $= \nabla(fg)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (xyz \cdot x^2y)$$

$$= \frac{\partial}{\partial x}(x^3y^2z)\,\hat{i} + \frac{\partial}{\partial y}(x^3y^2z)\,\hat{j} + \frac{\partial}{\partial z}(x^3y^2z)\,\hat{k}$$

$$= 3x^2y^2z\,\hat{i} + 2x^3yz\,\hat{j} + x^3y^2\,\hat{k}$$

RHS $= f\nabla g + g\nabla f$

$$= (xyz)\left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) x^2y\right\}$$
$$+ x^2y\left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) xyz\right\}$$

$$= (xyz)(2xy\,\hat{i} + x^2\,\hat{j})$$
$$+ x^2y\,(yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k})$$

$$= 2x^2y^2z\,\hat{i} + x^3yz\,\hat{j} + x^2y^2z\,\hat{i} + x^3yz\,\hat{j} + x^3y^2\,\hat{k}$$

$$= 3x^2y^2z\,\hat{i} + 2x^3yz\,\hat{j} + x^3y^2\,\hat{k}$$

Now, LHS = RHS