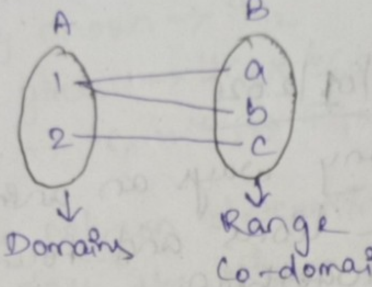
Relations on set:-

Let A and B be two sets. A relation R from A to B

$R: A \to B$ is a subset of $A \times B$.



$R: \{(1,a), (1,b), (2,c)\}$.

Ex: $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3)\}$

: Define a Relation $R: A \to B$ by a divides b.

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3)\}$.

Domain = $\{1, 2, 3\}$

Range = $\{1, 2, 3\}$.

Inverse of R:

Let $R: A \to B$ be a relation then the inverse of R denoted as

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

$$R^{-1} = \{(1,1), (2,1), (3,1), (2,2), (3,3)\}$$

Identity Relation:

$I = \{(a, a) \mid a \in A\}$

Properties of Binary Relation:

A binary relation $R \subseteq A \times A$ is called

(i) Reflexive if $\forall x\ (x, x) \in R$.

(ii) symmetric if $\forall (x, y).\ ((x, y) \in R \Rightarrow (y, x) \in R)$

(iii) Anti-symmet...
if $\forall (x, y$ ...

(iv) Transitive ...
if $\forall (x, y$ ...

Ex:

Q) $\leq$ is ...
Reflexive ...
$\leq$, ...

Q) $=$ is it ...
Yes

Q) $<$ is ...
No.

Q) If $x \leq$ ...

Q) $\{(x, y) \mid$ ...
Yes.
Is it ...

→ If A ...
set ...
relatio...

Ex: If ...

Q) If ...
and ...

Composit...
Let R...

(iii) Anti - symmetric
if $\forall (x,y) ((x,y) \in R \wedge (y,x) \in R \Rightarrow x = y)$.

(iv) Transitive property
if $\forall (x,y,z) ((x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R)$

Ex:

Q) $\leq$ is it a reflexive relation?
Reflexive as $=$ is present.
$\leq$, $=$ reflexive.

Q) $=$ is it a symmetric relation?
Yes

Q) $<$ is it a symmetric relation?
No.

Q) If $x \leq y$ then it is an anti-symmetric relation.

Q) $\{(x,y) \mid x+y=3\}$. Is it a symmetric relation?
Yes.
Is it anti-symmetric? No.

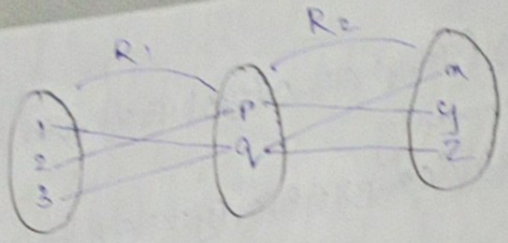
→ If A is a set with m elements and B is a set with n elements then the number of possible relations from $A \to B$ is $2^{m \times n}$.

Ex: If $m=4, n=2$ then the no. of relations
$$2^{4 \times 2} = 2^8.$$

Q) If there are 1024 relations from set A to B and $|B| = 2$, $|A| = ?$
$$|B| = 2 \quad \Rightarrow 1024 = 2^{10}$$
$$\Rightarrow |A| = 5.$$

Composition of Relations:

Let $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$ then
$$R_1 \circ R_2 = \{(x,z) \mid (x,y) \in R_1 \wedge (y,z) \in R_2\}$$



$(1, x) \in R_1 \circ R_2$

$\Rightarrow ((1,q) \wedge (q,x)) \; (1,x) \in R_1 R_2$

$(2,y) = ((2,p) \wedge (p,y)) \in R_1 \circ R_2 \; R_1 R_2)$.

<u>Ex:</u> $A = \{1, 3, 5, 7\}$, $B = \{x, y, z\}$, $C = \{2, 4, 6\}$

$R_1 : A \to B$, $R_1 = \{(1,x), (3,y), (5,x), (7,z)\}$

$R_2 : B \to C$, $R_2 = $ ~~{(1,2), (3,6), (5,2)}~~ $\{(x,2), (y,6)\}$

then $R_1 \circ R_2 = \{ \}$

$R_2 \circ R_1 = \{ \}$

$R_1 \circ R_2 = \{(1,2) \wedge (3,6)\}$

$R_2^{-1} \circ R_1^{-1} = \{$

<u>Equivalence Relations:</u>-

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

<u>Ex:</u> $R = \{(a,b) \in Z^+ \times Z^+ : a \text{ divides } b\}$

Is it an equivalence relation.

It is not an equivalence relation since symmetric property does not satisfy the above relation.

<u>Ex:</u> Show that the relation $R = \{(a,b) \mid a \equiv b \bmod m\}$.

Is it a equivalence relation

$m \mid (b-a)$,

$b - a = km$

→ Reflexive: aRa since m divides $a - a = 0$

→ Symmetric: $(a,b) \in R$

$\Rightarrow a - b = km$ for some integer k

$\Rightarrow -(a-b) = (-k) m$

$\Rightarrow b-a = (-k)m$

$\Rightarrow (b,a) \in R$

R is symmetric.

→ Transitive:

suppose aRb and bRc

$\Rightarrow a-b = km$ and $b-c = lm$ (for some integer k, l)

Adding the above relation

$a-b+b-c = km+lm$

$\Rightarrow a-c = \underbrace{(k+l)}_{\text{integer}}m$

$\Rightarrow m$ divides $a-c$

or $(a,c) \in R$ or R is transitive.

since R is reflexive, symmetric & transitive, then

R is an equivalence relation.

Reflexive and symmetric closure :-

Let R be a relation on a set A

Then,

$R_A = ((a,a) \mid a \in A)$

$R \cup R_A$ is Reflexive closure of R.

$R \cup R^{-1}$ is symmetric closure of R.

In other words if $(a,b) \in R$, then

add (b,a).

Transitive Closure :-

If R is a relation on a set A with n elements

Then

$R^* = R \cup R^2 \cup R^3 \cup \ldots\ldots \cup R^n$

Ex: $A = \{1, 2, 3\}$

$R = \{(1,2), (2,3), (3,3)\}$

$R^* = R \cup R^2 \cup R^3$

$R^2 = R \circ R = \{(1,3), (2,3), (3,3)\}$

$R^3 = R^2 \cup R = \{(1,3), (2,3), (3,3)\}$

Transitive closure of $R = R \cup R^2 \cup R^3$

$= \{(1,2), (1,3), (2,3), (3,3)\}$

→ $R = \{(1,2), (2,3), (3,1)\}$ and transitive closure of R reflexive, symmetric

$R = \{(1,2), (2,3), (3,1)\}$

Reflexive closure:

$R_A = \{(1,1), (2,2), (3,3)\}$

$R \cup R_A = \{(1,2), (2,3), (3,1), (1,1), (2,2), (3,3)\}$.

Symmetric closure:

$R^{-1} = \{(2,1), (3,2), (3,3)\}$.

$R \cup R^{-1} = \{(1,2), (2,3), (3,3), (2,1), (3,2), (3,3)\}$.

Transitive closure of R:

$R^2 = R \circ R = \{(1,3), (2,1), (3,2)\}$

$R^3 = R^2 \circ R = \{(1,1), (2,2), (3,2)\}$.

Transitive closure of R

$R \cup R^2 \cup R^3$

$= \{(1,1), (3,2), (2,2), (3,3), (2,1), (1,3)\}$

Posets (Partially Ordered set):-

A partially ordered set or a poset is a pair

A non-empty set.

$P = (X; \leq)$ where X is a non-empty set.

$\leq$ is a partial order.

i.e. for $x, y \in X$

1. $x \leq x$ (reflexive)
2. $x \leq y$, $y \leq x$ implies $x = y$ (Antisymmetric)
3. $x \leq y$, $y \leq z$ implies $x \leq z$ (transitive).

↳ partial ordered set.

→ A relation R is said to be partially ordered relation if it is reflexive, anti-symmetric and transitive.

Q $(Z, \geq)$, partially order relation. Check

$Z \, Z \, Z \rightarrow$

(i) Reflexive -

$a \geq a \cdot \forall a \in Z$

$\therefore \geq$ - reflexive

(ii) Anti-symmetric -

$a \geq b, \; b \geq a, \; \forall a, b \in Z$

$\Rightarrow a = b.$

$\therefore \geq$ - anti-symmetric.

(iii) Transitive:-

$a \geq b, \; b \geq c \; \forall a, b \in Z$

$\Rightarrow a \geq c$

$\therefore \geq$ is transitive

$\therefore$ This relation is partially ordered relation.

Q) Show that the following divides relation are partially ordered on the set a

(i) $m|n$, 'm' divides 'n' where

$A = N$ (set of all natural number)

(ii) $m|n$, 'm' divides 'n' where

A = the set of +ve integers.

(a) divisors of 36.

(i) Reflexive

$\frac{n}{n} \; \forall n \in A$

$\frac{n}{n}$ - Reflexive

(ii) Anti-symmetric

$\frac{m}{n}, \frac{n}{m} \; \forall m, n \in Z$

$\Rightarrow m = n$

$\therefore m/n$ is anti-symmetric.

(iii) Transitive

$m|n \,;\, n|p, \forall n, m \in Z$

$\therefore m|p$ (transitive).

$\therefore$ The above relation is a partially ordered relation.

all three are

$n|n \rightarrow$ here the relation is reflexive

$\rightarrow$ If $m|n$ and $n|m$

$\Rightarrow m = n \rightarrow$ anti symmetric

$\rightarrow$ If $m|n$ and $n|p$

$\Rightarrow m|p$ transitive

(b)
(ii) $m|n$, 'm' divides n where

$A = 1, 2, 3, 4, 6, 9, 12, 18, 36.$

Reflexive

Representation of posets or Hasse diagram:-

A graphical representation of a partial order relation in which all arrow pointing upward is known as Hasse diagram.

Procedure:-

step-1: Draw diagram of relation.

step-2: Remove self loops.

step-3: Remove all transitive edges.

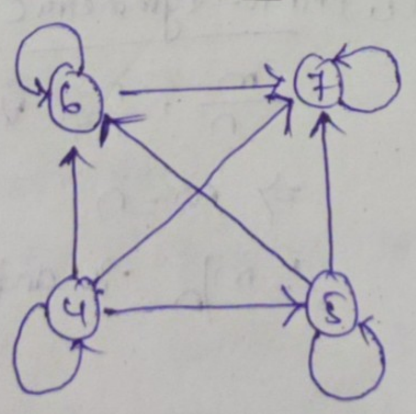
step-4: Arrange all edges pointing upward

step-5: Replace circles by dots as vertex.

Ex: Considering the set $A = \{4, 5, 6, 7\}$. Let R be the relation $\leq$ on A. Draw the Hasse diagram of R.

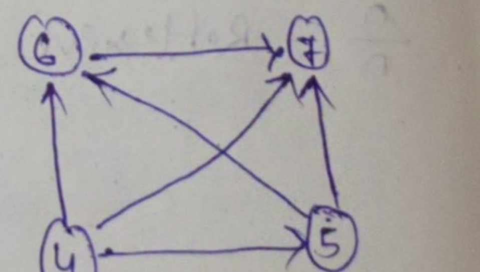
$R = \{(4,4), (4,5), (4,6), (4,7), (5,5), (5,6), (5,7), (6,6), (6,7), (7,7)\}$

Step 1:



Step 2:

Removing self loop $\rightarrow$ (4,4) (5,5), (6,6) (7,7)

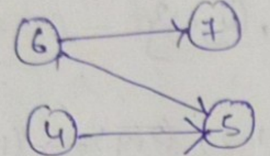


Step 3: (Remove transitive edges)

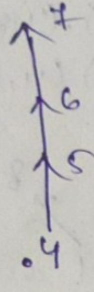
a=b b=c

(4,5) (5,6) , (5,6)(6,7)

(4,6) (5,7)



Step-4:

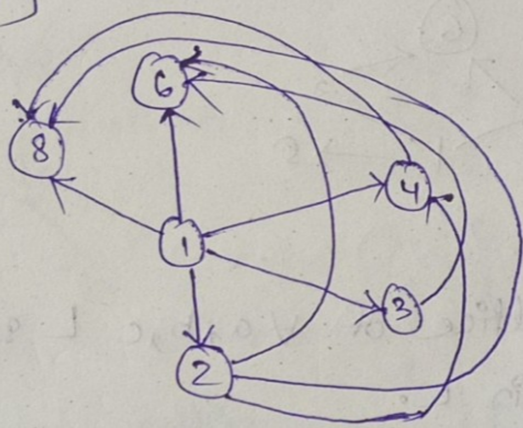


This is our required Hasse diagram.

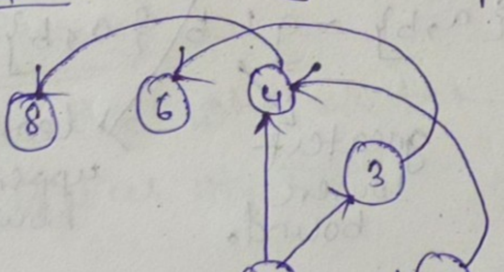
Ex:- Let A = {1, 2, 3, 4, 6, 8} be ordered by the relation 'a' divides 'b'.

R = [(1,2), (1,3), (1,4), (1,6), (1,8), (2,4), (2,6), (2,8), (3,6), (4,8)]

Step 1:



Step 2: Removing self loop



Step 3:

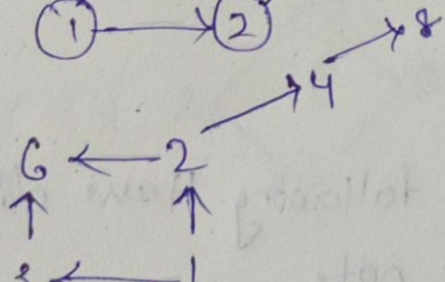
Remove transitive edge

(1,2), (2,4) → (1,4) ×

(1,3), (3,6) → (1,6) ×

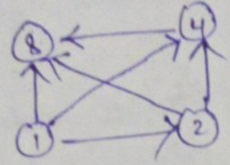
(1,4), (4,8) → (1,8) ×

(2,4), (4,8) → (2,8) ×

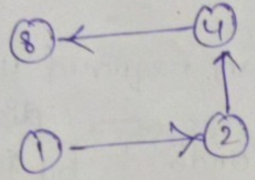


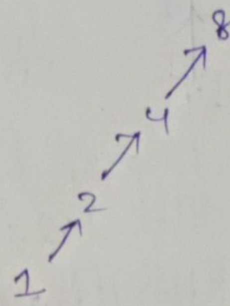
Ex:- $D_8 = \{1, 2, 4, 8\}$, The relation is divisibility.

$R = \{(1,2), \underset{\times}{(1,4)}, \underset{\times}{(1,8)}, (2,4), \underset{\times}{(2,8)}, (4,8)\}$

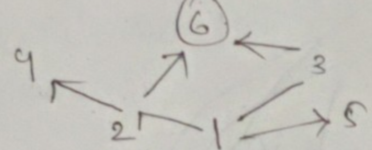


$(1,2)\ (2,4) \rightarrow (1,4)$
$(1,4) . (4,8) \rightarrow (1,8)$
$(2,4)\ (4,8) \rightarrow (2,8)$.





Ex: $R = \{1, 2, 3, 4, 5, 6\}$ divisibility.



Lattice:-

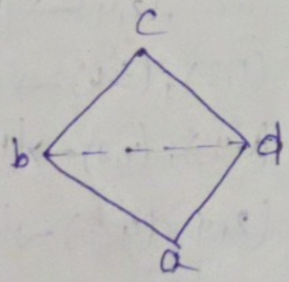
A poset is a lattice of $\forall a, b, c\ L$ sup$\{a,b\}$ and inf $\{a,b\}$ exist in L.

1. $a \vee b = a$ join b = sup $\{a, b\}$ = sup $\{a, b\}$
2. $a \wedge b = a$ meet b = inf $\{a, b\}$ = glb $\{a, b\}$

greatest lower bound.

least upper bound

Ex:



Determine whether the following Hasse diagram represents a lattice or not.

Ex.

lub

∨	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

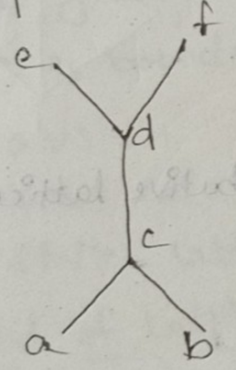
Since each subset of two elements has least upper bound. So, L contain lub

glb

∨	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Since each subset of two elements has least upper bound. So L become a Lattice and glb

Ex

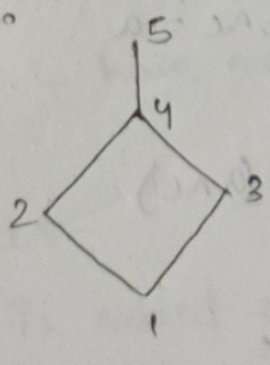


∨	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	x
f	f	f	f	f	x	f

It is not a lattice.

Assignment

Determine whether the following diagram represent a lattices or not.



Properties of lattice and distributive lattices

Let $(L, \leq)$ be a lattice $\forall a, b \in L$

1) Idempotent law, $a \vee a = a$
$a \wedge a = a$

2) Commutative law, $\forall a, b \in L$
$a \vee b = b \vee a$
$a \wedge b = b \wedge a$

3) Associative law = $\forall a, b, c \in L$
$(a \cup b) \vee c = a \vee (b \cup c)$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

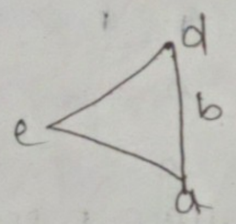
4) Absorption law for $a, b \in L$
(a) $a \vee (a \wedge b) = a$
(b) $a \wedge (a \vee b) = a$

Distributive Lattice

A lattice $(L, \leq)$ is called a distributive lattice if for any $a, b, c \in L$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Q)



Check whether it is a distributive lattice or not.

$a \vee (b \wedge c) = a \vee a = a$
$(a \vee b) \wedge (a \vee c) = b \wedge c = a$

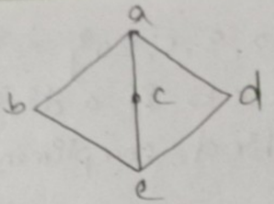
$\therefore$ LHS = RHS

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

LHS = $a \wedge (b \vee c) =$

RHS = $(a \wedge b) \vee (a \wedge c) =$

Ex:-



$b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$ whether it is distributive lattices or not.

LHS $= b \wedge (c \vee d) = b \wedge a = b$
RHS $= (b \wedge c) \vee (b \wedge d) = e \vee e = e$.

LHS $\neq$ RHS
Hence, it is a distributive lattice.

Bounded Lattice: A Lattice $(L, \leq)$ is called bounded lattice if it has greatest element 1 and least element 0. or L has greatest and least element in L.

→ $(L, \leq)$ is bounded then for any element $a \in L$

$a \vee 1 = 1$ $\quad$ $a \vee 0 = a$
$a \wedge 1 = a$ $\quad$ $a \wedge 0 = 0$.

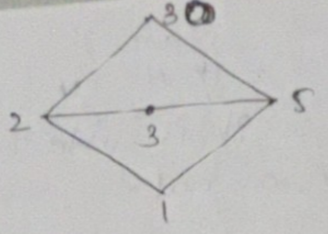
Every finite lattice is bounded.

Complement of Lattice:-
Complement of an element in a lattice $(L, \leq)$ in bounded lattice with least element '0' and greatest element 1.
$a \in L$, $a' \in L$ is called the complement of a.
$a \vee a' = 1$, $a \wedge a' = 0$.

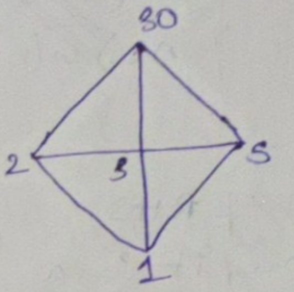
→ It is not necessary that every element has a complement
→ An element $a \in L$ have more than one complement also.

Complimented Lattice:-
A lattice $(L, \leq)$ is called complimented
(i) L is bound
(ii) Every element $a \in L$ has a compliment.

Ex:



A = {1, 2, 3, 5, 30} and $a \le b$ if a divides b Then find compliment of 2.

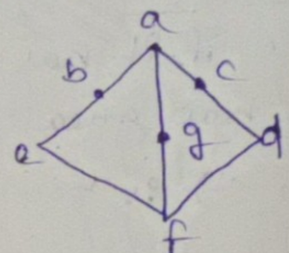


glb = 1
lub = 30

$2 \vee 2' = 30$
$2 \wedge 2' = 1$

$2 \vee 3 = 30$
$2 \wedge 3 = 1$
$2' = 3$

Complement of 2 is 3.
Complement of 5 is 3.



If 'a' and 'b' are the elements in a distributive lattice L and if a' complement of a then

(i) $(a')' = a$
(ii) $(a \vee b)' = a' \wedge b'$
(iii) $(a \wedge b)' = a' \vee b'$
(iv) $a \vee (a' \wedge b) = a \vee b$
(v) $a \wedge (a' \vee b) = a \wedge b$

<u>Boolean Algebra</u>:- A non-empty set B with two binary operation (+) and (·), a unary operation (') and two distinct element '0' and '1' is called a Boolean algebra denoted by (B, +, ·, ', 0, 1) ~~iff the~~ following properties are satisfied.

<u>Axioms of Boolean Algebra</u>

If a, b, c ∈ B, then

1) <u>Commutative Law</u>:-

$a + b = b + a$
$a \cdot b = b \cdot a$

2) Distributive

$a+(b\cdot c) = (a+b)\cdot(a+c)$

$a\cdot(b+c) = (a\cdot b)+(a\cdot c)$

3) Identity law -

$a+0=a$

$a\cdot 1=a$

4) Complement law -

$a+a'=1$

$a\cdot a'=0$

Some basic result of Boolean algebra

Let $a, b, c \in B$ then

Idempotent laws

$a+a=a$

$a\cdot a=a.$

$a = a+0$ (identity ~~law~~ rules)

$= a+(aa')$ (complement rules)

$= (a+a)\cdot(a+a')$ (distributive rules)

$= (a+a)\cdot(1)$ (complement rule)

$= a+a$

$\therefore a+a=a$ [Hence proved]

$a\cdot a=a$

$a\cdot a = a\cdot a+0$ (identity law)

$= a\cdot a+a\cdot a'$ (complement rule)

$= a(a+a')$ (distributive law)

$= a\cdot 1$ (complement rule)

$= a$ (idempotent laws).

$\therefore a\cdot a = a$

2) Boundness laws :-

(a) $a+1=1$

(b) $a\cdot 0=0.$

(a) $1 = a + a'$ (complement rule)
$= a + (a'.1)$ (identity law)
$= (a+a').(a+1)$ (distributive law)
$= 1.(a+1)$ (complement rule)
$= (a+1).1$ (commutative rules)
$= a+1$ (identity rules)

$\therefore \boxed{a+1 = 1}$

(b) $a.0 = 0$
$0 = a.a'$ (complement rule)
$= a.(a'+0)$ (identity law)
$= (a.a') + (a.0)$ (distributive law)
$= 0 + (a.0)$ (complement rule)
$= (a.0)+0$ (commutative rules)
$= a.0$ (identity rules)

$\therefore \boxed{a.0 = 0}$

Absorption law :-

Let $a, b \in B$
(a) $a + (a.b) = a$
(b) $a.(a+b) = a$

(a) $a + (a.b) = (a.1) + (a.b)$ (identity law)
$= a(1+b)$ (distributive)
$= a(b+1)$ (commutative law)
$= a.1$ ($b+1 = 1$ bounded law)
$= a$ (identity law).

(b) $a.(a+b) = (a+0).(a+b)$
Assignment
$=$