Length of a Curve

→ If equation given in cartesian. i.e, $y = f(x)$, then length of the curve from a to b, $L = \int_a^b \{[f'(x)]^2 + 1\}^{1/2}$

→ Ex :- Find length of the curve, $y = \log \sec x$
$x = 0$ to $\pi/3$.

$$f'(x) = \frac{d}{dx}(\log \sec x).$$

$$= \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x.$$

$$= \tan x.$$

$$L = \int \{[\tan^2 x]_a^b + 1\}^{1/2}$$

$$= \int^{\pi/3} \{\tan^2 x + 1\}^{1/2}$$

$$= \int_0^{\pi/3} (\sec^2 x)^{1/2}$$

$$\Rightarrow \int_0^{\pi/3} \sec x$$

$$\Rightarrow [\log(\sec x + \tan x)]_0^{\pi/3}$$

$$\Rightarrow \log\left(\sec \frac{\pi}{3} + \tan \frac{\pi}{3}\right)$$

$$\Rightarrow \log(2 + \sqrt{3}.)$$

→ If eq. given in parametric form $x = f(t)$ & $y = g(e)$, then length of the chord $= \int_{t_1}^{t_2} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt.$

$$y^2 = 4ax$$
$$x = t$$
$$y = 4at$$

Ex :- Find length of the curve $x = a\cos^3 t$
$y = a\sin^3 t$.

Sol :- $\frac{dx}{dt} = -3a\cos^2 t \sin t$.

$\boxed{\text{If } y = -y \text{, the symmetric around both axis, same for } x \text{ \& } -x.}$

$$\frac{dy}{dt} = 3a\sin^2 t \cos t.$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^4 \sin^2 t + 9a\sin^4 t\cos^2 t.$$
$$= 9a^2\sin^2 t\cos^2 t$$

$$\Rightarrow \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} = 3a\sin t\cos t.$$

Length of the chord from 0 to $\frac{\pi}{2}$.

$$\Rightarrow 3a \int_0^{\pi/2} \sin t \cos t \, dt.$$

$$\Rightarrow \frac{3a}{2} \int_0^{\pi/2} 2\sin t \cos t \, dt.$$

$$\Rightarrow \frac{3a}{2} \int_0^{\pi/2} \sin 2t.$$

$$\Rightarrow \frac{3a}{2}\left[\sin^2 t\right]_0^{\pi/2} \Rightarrow \frac{3a}{2}[1-0]$$
$$\Rightarrow \frac{3a}{2}.$$

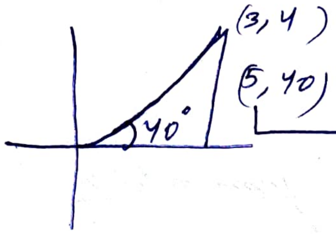
$\rightarrow$ $r = f(\theta)$, if the given eq. in the plane for $\rightarrow$

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} \, d\theta$$

Ex :-

Q) Find total length $r = a[1 - \cos\theta]$.

Polar cordinate

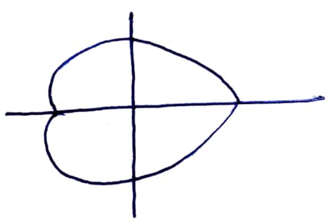


$\rightarrow$ Pythagoras theorem

$\cos\theta$ = even function
$\sin\theta$ = odd "

Cardio Curve

Sol :-



$$r' = \frac{dr}{dt} = a\sin\theta$$

$$\Rightarrow \sqrt{r^2 + (r')^2}$$

$$\Rightarrow \left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{1/2}.$$

$$\Rightarrow a\left[1 + \cos^2\theta - 2\cos\theta + \sin^2\theta\right]^{1/2}.$$

$$\Rightarrow a\left[2 - 2\cos\theta\right]^{1/2}.$$

$$\Rightarrow a\sqrt{2}\sqrt{[1-\cos\theta]}$$

$$\rightarrow a\sqrt{2} \cdot \sqrt{2}\sin\frac{\theta}{2}.$$

$$\Rightarrow 2a\sin\frac{\theta}{2}.$$

$$A_1 = a\sqrt{2}\int_0^{\pi} \sqrt{2}\sin\frac{\theta}{2}\, d\theta$$

d

$$\Rightarrow -2a\left[2\cos\frac{\theta}{2}\right]_0^{\pi} \Rightarrow -4a[-1] \Rightarrow 4a.$$

Total length = 8a.

Area bet the curves

→ If $f(x)$ & $g(x)$ are a continuous function & $f(x) > g(x)$ $\forall$

($\forall$ - for all x)

→ Area bet the curves $f(x)$ & $g(x)$ from $x = a$ to b is

$$\int_a^b (f(x) - g(x))\,dx.$$

Ex:-

Find area bet $y = \sec^2 x$, $y = \sin x$ from 0 to $\frac{\pi}{4}$.

$\cancel{f(x) = \int \sec^2 x\,dx}$. $\sec^2 x > \sin x$ $\forall$ $x \in [0$ to $\frac{\pi}{4}]$.

$$\Rightarrow \int_0^{\pi/4} (\sec^2 x - \sin x)\,dx.$$

$$\Rightarrow \left(\tan x + \cos x\right)_0^{\pi/4}.$$

$$\Rightarrow 1 + \frac{1}{\sqrt{2}} - 0 - 1 = \frac{1}{\sqrt{2}}.$$

Ex:

$y = 2 - x^2$

$y = -x$.

Find area bet the curve.

$2 - x^2 = -x$

$x^2 - x - 2 = 0.$

$x^2 - 2x + x - 2 = 0$

$x = (2, -1)$

$\Rightarrow 2-x^2 > -x \;\; \forall x.$

$\Rightarrow x \in [-1, 2].$

$\cancel{2-x^2>0}$ $\quad 2-x^2 > -x$

$2-x^2+x > 0.$

$-x^2+x+2 > 0$

$-x^2+2x-x+2 > 0$

$-(x+1)(x-2) > 0.$

$(x+1)(x-2) < 0.$

$$\Rightarrow \int_{-1}^{2} (2-x^2+x)\,dx$$

$$\Rightarrow \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$\Rightarrow 4 - \frac{8}{3} + \frac{4}{2} - \left[-2 + \frac{1}{3} - \frac{1}{2}\right]$$

$$\Rightarrow \frac{24 - 16 + \cancel{12} - \cancel{12} + 20 - 3}{6}$$

$\begin{array}{r} 26 \\ -19 \\ \hline 5 \end{array}$

$$\Rightarrow \frac{5}{6}.$$

~~Ex:~~ $a^2y^2 = x^3(2a-x).$

Find the area of the curve.

Ans). $a^2y^2 = x^3(2a-x).$

when $y=0$

$0 = \cancel{2}\,2ax^3 - x \Rightarrow \cancel{2ax^3 = x}$

$\cancel{2a = x^2}$

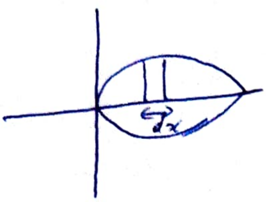
$y^2 = \frac{x^3(2a-x)}{a^2}$

$y = \frac{x^{3/2}\sqrt{2a-x}}{a^{\cancel{2}\,1/2}}$

$x^{3/2}\sqrt{2a-x} = 0$

$x = 2a$

$0 = 2a - x \Rightarrow x = 2a.$



$$A = 2\int_0^{2a} y\, dx$$

$$= \frac{2}{a}\int_0^{2a} x^{3/2}(2a - x)^{1/2}\, dx.$$

$$\cancel{\Rightarrow \frac{2}{a}\int x^{3/2}\left(2a^{1/2} - x^{1/2}\right) dx.}$$

$$\cancel{\Rightarrow \frac{2}{a}\int_0^{2a} 2a^{1/2}}$$

Let. $x = 2a\sin^2\theta.$

$$\cancel{\Rightarrow \frac{2}{a}\int_0^{\pi/2}}$$

$$\Rightarrow \frac{2}{a}\int_0^{\pi/2} (2a)^{3/2}\sin^3\theta\,(2a)^{1/2}\cos\theta \cdot 4a\sin\theta\cos\theta\, d\theta.$$

$$\Rightarrow 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2 d\theta.$$

$$\Rightarrow 32a^2\int_0^{\pi/2} \sin^4\theta\, d\theta - 32a^2\int_0^{\pi/2} \sin^6\theta\, d\theta.$$

Formula

$$\Rightarrow \int_0^{\pi/2} \sin^n\theta\, d\theta.$$

$$\Rightarrow \int_0^{\pi/2} \cos^n\theta - \frac{(n-1)(n-3)\ldots\ldots}{n(n-2)\ldots\ldots}$$

$$\Rightarrow y = \frac{\pi}{2}\left[32a^2\frac{(4-1)(4-3)}{(4-0)(4-2)} - \frac{32a^2(6-1)(6-3)(6-5)}{(6-0)(6-2)(6-4)}\right].$$

$$= \frac{\pi}{2}\, 32a^2\left\{\frac{3}{8} - \frac{15}{48}\right\}$$

Volume of Revolution

→ If a plane area bounded by the curve $y = f(x)$ within the coordinate $x = a$ & b. & revolves around x-axis then volume of solid so generated is $V = \int_a^b \pi y^2 dx$.

Ex:-

Find vol. of solid generated by revolving the semi circle $x^2+y^2=a^2$ around x-axis.

Sol:- $V = 2\pi \int_0^a (a^2 - x^2) dx$.

$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a$.

$\Rightarrow 2\pi \left[a^3 - \frac{a^3}{3}\right]$.

$\Rightarrow 2\pi \times \frac{2a^3}{3} = \frac{4a^3\pi}{3}$.

Ex:-

Find v. of paraboloid generated by revolving the parabola $y^2 = 4ax$ around x- axis within the limit $x = a$ to h.

Sol:- $V = \pi \int_a^h 4ax\, dx$. (If a full circle revolves forms a sphere same goes for a semi.)

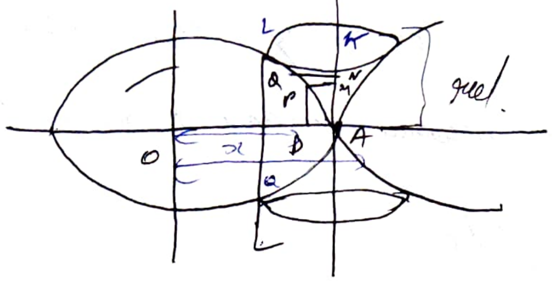
$= \pi \left[\frac{4ax^2}{2}\right]_a^h$

$\Rightarrow \pi \times \left[2ax^2\right]_a^h$

$\Rightarrow \pi \times \left[2ah^2 - 2a^3\right]$

$\Rightarrow 2a\pi \left[h^2 - a^2\right]$

Ex :-



The part of the elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut off by a lattus rectum revolves ~~around~~ about the tangent at the nearest tangent. Find v. of the reel that generated.

~~Ans.~~

Sol :-

~~Sol.~~ V of the elementary strip of revolution about line $x=0$,

$$\Rightarrow \pi (PM)^2 MN.$$

$$\Rightarrow \pi (a-x)^2 \cdot \delta y.$$

$$V = 2 \times \text{volume generated by } \cdot LAKL.$$

Coordinate of lactus rectum.

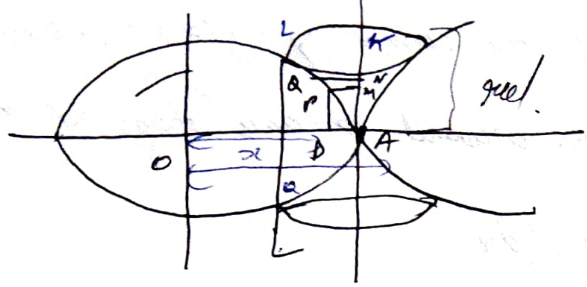
$$\boxed{= 2 \int_0^{b^2/a} \pi \{a^2 - 2ax + x^2\} dy.}$$

$$\Rightarrow 2\pi \int_0^{b^2/a} \{a^2 - 2a\frac{a}{b}\sqrt{b^2-y^2} + \frac{a^2}{b^2}(b^2-y^2\} \delta y.$$

$$\Rightarrow \frac{2\pi a^2}{b^2} \int_0^{b^2/a} \{2b^2 - 2b\sqrt{b^2-y^2} - y^2\} dy$$

$$\Rightarrow \frac{2\pi a^2}{b^2} \{2b^2 y - 2b\left[\frac{1}{2}y\sqrt{b^2-y^2} + \frac{1}{2}b^2 \sin^{-1}\frac{y}{b}\right] - \frac{y^3}{3}\}_0^{b^2/a}.$$

Ex :-



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Coordinate of lactus rectum.

$$\boxed{= 2 \int_0^{b^2/a} \pi \left\{a^2-2ax+x^2\right\} dy.}$$

$$\Rightarrow 2\pi \int_0^{b^2/a} \left\{a^2 - 2a\,\frac{a}{b}\sqrt{b^2-y^2} + \frac{a^2}{b^2}(b^2-y^2\right\} \delta y.$$

$$\Rightarrow \frac{2\pi a^2}{b^2} \int_0^{b^2/a} \left\{2b^2 - 2b\sqrt{b^2-y^2} - y^2\right\} dy.$$

$$\Rightarrow \frac{2\pi a^2}{b^2} \left\{2b^2 y - 2b\left[\frac{1}{2} y\sqrt{b^2-y^2} + \frac{1}{2} b^2 \sin^{-1}\frac{y}{b}\right] - \frac{y^3}{3}\right\}_0^{b^2/a}.$$

$\Rightarrow \frac{2\pi b}{3a} \{ 6a^2 b - 3ab\sqrt{a^2-b^2} - 3a^2 \sin^{-1}\frac{b}{a} - b^3 \}.$

———— × ————

★ **Improper Integral :-**

→ The Integral of the form $\int_a^b f(x)\,dx$. if a or (&) b is infinity then this is called I I of first ~~kind~~ kind.

→ If f(x) is unbounded for any $x \in [a, b]$ then the integral is formed I I of 2nd Kind. → Ex - $\int_0^{\pi/2} \sec x\,dx$
unbounded

Ex 1:-

$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{c\to\infty} \int_1^c \frac{dx}{x^{3/2}}$$

$$= \lim_{c\to\infty} \int_1^c x^{-3/2}\,dx$$

$$\Rightarrow \lim_{c\to\infty} \left[\frac{x^{-1/2}}{-1/2} \right]_1^c$$

$$\Rightarrow \lim_{c\to\infty} \left[\frac{-2}{\sqrt{x}} \right]_1^c$$

$$\Rightarrow -2 \lim_{c\to\infty} \left(\frac{1}{\sqrt{c}} - 1 \right)$$

$$\Rightarrow -2(-1) = +2.$$

Integral is convergent.

Ex 2 :-

$$\int_1^{\infty} \frac{dx}{\sqrt{x}}$$

$$\Rightarrow 2 \lim_{c \to \infty} \left[x^{1/2} \right]_1^c$$

$$\Rightarrow 2 \lim_{c \to \infty} \left[\sqrt{c} - 1 \right].$$

$$\Rightarrow \infty$$ Integral is divergent.

Ex 3 :- $\int_{-2}^{2} \frac{1}{x} dx$. (II of 2nd kind).

$$\Rightarrow \int_{-2}^{0} \frac{1}{x} dx + \int_0^2 \frac{1}{x} dx.$$

$$\Rightarrow \lim_{c \to 0} \int_{-2}^{c} \frac{1}{x} dx + \lim_{d \to 0} \int_d^2 \frac{1}{x} dx.$$

$\Rightarrow$

* Comparison :-

Let $f(x)$, $g(x)$ be 2 functions which are bounded & integrable in the interval (a, ∞). & $g(x) > 0$, $|f(x)| < g(x) \ \forall x (a, \infty)$

If $\int_a^{\infty} g(x) dx$ converges then $\int_a^{\infty} f(x) dx$ also converges.

Alternative,

If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$, $l \neq 0$, ~~l~~ then both integral

$\int_a^\infty f\,dx$, $\int_a^\infty g\,dx$. converge & diverge together.

~~Ex~~ test

Show that $\int_a^\infty \frac{dx}{x^n}$ $a > 0$.

1) converge for $n > 1$

2) Diverge " $n \leq 1$

Sol :- $\lim_{\epsilon \to \infty} \int_a^\epsilon \frac{dx}{x^n}$.

$$= \lim_{\epsilon \to \infty} \left[\frac{x^{1-n}}{1-n} \right]_a^\epsilon$$

$$\Rightarrow \left[\frac{\epsilon^{1-n}}{1-n} - \frac{a^{1-n}}{1-n} \right] = I.$$

case I). Let $n > 1$, $1 - n < 0$.

$\frac{1}{\epsilon^{n-1}} \to 0$, $\epsilon \to \infty$.

So I converges.

case II) Let $n < 1$.

$1 - n > 0$.

$\epsilon^{1-n} > 0$

$\epsilon \to \infty$, $\epsilon^{1-n} \to \infty$.

So, I diverges.

Case - III) $n=1$

$$\lim_{\epsilon \to \infty} \int_a^{\epsilon} \frac{dx}{x}$$

$$\Rightarrow \lim_{\epsilon \to \infty} [\ln x]_a^{\epsilon}$$

$$\Rightarrow (\ln \epsilon - \ln a)$$

$$\Rightarrow \infty.$$

This is divergent.

* Test for Convergents :-

1) $\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx.$

$= f(x) = \frac{\cos mx}{x^2+a^2}.$

$$|f(x)| = \left|\frac{\cos mx}{x^2+a^2}\right| \leq \frac{1}{x^2+a^2} < \frac{1}{x^2}.$$

Since $\int_a^{\infty} \frac{dx}{x^2}$ converge how.

$$\Rightarrow \int_a^{\infty} \frac{\cos mx}{x^2+a^2} dx \text{ converges}$$

$\int_0^a \frac{\cos mx}{x^2+a^2}$ is a propal integral also converges.

* The μ - Test :-

Let $f(x)$ be bounded as integrable ~~list~~ in the (a, ∞), $a > 0$.

a) If $\mu > 1$, $\lim_{x \to \infty} x^{\mu} f(x)$ except, then $\int_0^{\infty}$ is golden light.

$x \to \infty$

b) If $\mu < 1$, $\lim$ $\lim$, x^{μ} yield exist but $\lim_{x \to \infty} x^{\mu} = 0$ or ~~to~~ to ∞ then $x \to \infty$ $\int_a^{\infty} f(x)\, dx$ divergent.

Ex :-

$$\int_1^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$$

Sol :- $\Rightarrow f(x) = \dfrac{1}{x^{5/6}(x^{-1/2}+1)}$

~~Let $\mu = \frac{5}{6}$ then $\lim x^{3/6}$~~

Let $\mu = \dfrac{5}{6}$ $\lim_{x \to \infty} x^{5/6} \dfrac{1}{(x^{5/6} 1 + x^{a-1/2})}$

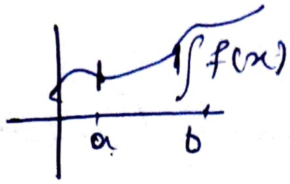
By μ test the given I is divergent.

μ Test

$$f(x)$$

$$\lim_{x\to\infty} x^{\mu} f(x)\,dx.$$

$\downarrow$



$\mu > 1$

finite $\rightarrow$ converge.

$\mu \leq 1$ $\lim_{x\to\infty} x^{\mu} f(x)\,dx$ $\rightarrow$ ~~converge~~ diverge.

Q

$$\int_1^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}.$$

$$\cancel{\lim_{x\to\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}} \rightarrow 0$$ $\rightarrow$ Test fail

Hence take out $x^{1/2}$ from parantheses.

$$\lim_{x\to\infty} \frac{dx}{x^{1/3+\frac{1}{2}}(x^{-1/2}+1)}$$ $\boxed{\mu = 5/6.}$ $\Rightarrow \lim_{x\to\infty} \frac{dx}{\left(\frac{1}{x^{1/2}}\to 0 + 1\right)}$

$$\lim_{x\to\infty} \frac{dx}{x^{5/6}(x^{-1/2}+1)}$$ x^{μ} $= 1$ Ans

$$\lim_{x\to\infty} \frac{\cancel{x^{5/6}}\,dx}{\cancel{x^{5/6}}(x^{-1/2}+1)}$$

$\therefore$ $f(x)$ is divergent.

* The μ - Test :-

Let $f(x)$ be bounded as intigerable in the (a, ∞), $a > 0$.

a) If $\mu > 1$, $\lim_{x\to\infty} x^{\mu} f(x)$ excist, then $\int_a^\infty$ is ... light.

b) If $\mu < 1$, $\lim_{x\to\infty} x^{\mu}$ ... exist but ... to ... $\neq 0$ or ... to ∞ then $\int_a^\infty f(x)\,dx$ divergent.

Ex :-

$$\int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$$

Sol :- $\Rightarrow f(x) = \dfrac{1}{x^{5/6}(x^{-1/2}+1)}$

~~Let $\mu = \frac{5}{6}$ then $\lim x^{33/6}$~~

Let $\mu = \frac{5}{6}$ $\lim_{x\to\infty} x^{5/6} \dfrac{1}{(x^{5/6} 1 + x^{-1/2})}$

By μ test the given I is divergent.

Ex:-

$$\int_0^\infty \frac{x\,dx}{(1+x)^3} = \int_0^a \frac{x\,dx}{(1+x)^3} + \int_a^\infty \frac{x\,dx}{(1+x)^3}$$

$$f(x) = \frac{x}{x^3\left(\frac{1}{x}+1\right)^3}$$

convergent

$$= \frac{1}{x^2\left[\frac{1}{x}+1\right]^3}$$

Let $u = \frac{1}{x^2}$

$$\lim_{x \to \infty} x^2 \frac{1}{\left(1+\frac{1}{x}\right)^3}$$

* Abel test :-

If $\int_a^\infty f(x)\,dx$ converges $\phi(x)$ bounded & manotic for $x >$

then $\int_a^\infty f(x)\,\phi(x)\,dx$ convergent.

Ex:-

Ex:-

$$\int_0^\infty \frac{x\,dx}{(1+x)^3} = \int_0^a \frac{x\,dx}{(1+x)^3} + \int_a^\infty \frac{x\,dx}{(1+x)^3}$$

$$f(x) = \frac{x}{x^3\left[\frac{1}{x}+1\right]^3}$$

convergent

$$= \frac{1}{x^2\left[\frac{1}{x}+1\right]^3}$$

Let $u = x^2 \cdot q$

$$\lim_{x \to \infty} x^2 \frac{1}{\left[1+\frac{1}{x}\right]^3}$$

★ Abel test :-

If $\int_a^\infty f(x)dx$ converges $\phi(x)$ bounded & monotonic for $x \geq a$,

then $\int_a^\infty f(x)\,\phi(x)\,dx$ convergent.

Ex:-

$$\int_a^\infty \frac{(1-e^{-x})\cos x}{x^2}dx. \qquad a>0$$

Sol:-

$$\int_a^\infty \frac{\cos x}{x^2}dx = \int_a^\infty f(x)dx.$$

$\left|\frac{\cos x}{x^2}\right| < \frac{1}{x^2}$, $\int_a^\infty \frac{1}{x^2}dx$ is convergent.

$\Rightarrow$ By comparison test $\int_0^\infty f(x)\,dx$ is convergent.

★ **Direchlet's Test** :-

If $f(x)$ is bounded (limit na thika) & monotonic (either entirely ↑ or dec ↓) in the interval $0<x<\infty$,

$\lim_{x\to\infty} f(x) = 0$, $\left|\int_a^x \phi(x)\,dx\right|$ is bounded for all finite x, then

$\int_a^\infty f(x)\phi(x)\,dx$. convergent.

$\int_a^\infty e^{-x} \cdot \frac{\cos x}{x^3}$.

Integration by parts

$\int uv\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$

★ **Gamma Function** :-

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \ , \ n>0}$$ (first kind of Improper I).

show that .

i) $\Gamma(1) = 0$.

$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -\left[e^{-x}\right]_0^\infty$

$\Rightarrow -[0-1]$

$= 1$.

$\int uv\,dx = u\int v\,dx - \int\left(u' \cdot \int v\,dx\right)dx$

$e^{-x} = v$

$x^n = u$

ii) $\Gamma(n+1) = n\Gamma(n)$.

$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$.

$-\int_0^\infty n x^{n-1} \frac{e^{-x}}{-1}$

$\Rightarrow \left[x^n \cdot \frac{e^{-x}}{-1}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\,dx$.

$\Rightarrow n\Gamma(n)$.

iii) $\Gamma(n+1) = n!$

$\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

continuing this process.

$\Gamma(n+1) = n(n-1)(n-2)\ldots\ldots\ldots 2\cdot 1\,\Gamma(1)$.

★ **Beta function :-**

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

$m, n > 0$.

show that,

$$\beta(m,n) = \beta(n,m).$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

$$\beta(m,n = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\,dx.$$

$$\Rightarrow \beta(n,m)$$

2) Find $\beta(m,1) = \int_0^1 x^{m-1}\,dx = \frac{1}{m}$.

3) $\cancel{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.}$

Sol:- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$

$= \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 - \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}\,dx.$

$= \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}\,dx.$

$= \frac{n-1}{m}\int_0^1 x^{(m+1)-1}(1-x)^{(n-1)-1}\,dx.$

so that

$= \frac{n-1}{m}\,\beta(m+1, n-1).$

4). Express in β- function.

$\int_0^1 x^m(1-x^2)^n\,dx, \quad m>-1, \; n>-1.$

Sol:- Let $x^2 = y$.

$= \int_0^1 x^{m-1}(1-x^2)^n x\,dx =$

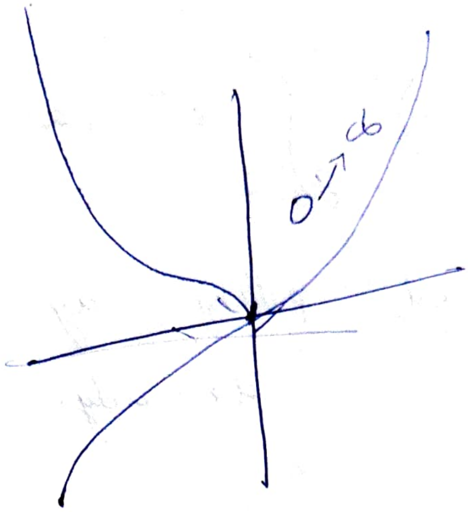
$= \int_0^1 y^{\frac{(m-1)}{2}}(1-y)^n \frac{dy}{2} \quad = \frac{1}{2}\int_0^1 y^{\frac{m+1}{2}-1}(1-y)^{(n+1)-1}\,dy$

$= \frac{1}{2}\,B\left(\frac{m+1}{2}, n+1\right).$

Ex-

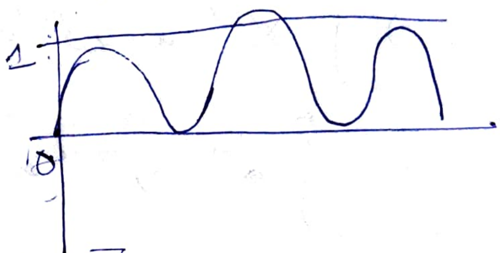
$$\int_0^{\infty} e^{-x} \frac{\cos x}{x^3}$$

$f(x) = $ — $\psi(x) = $ —



$e^{-x} = \lim_{x \to 0} \frac{1}{e^x} = 0.$

$f(x) = e^{-x}.$ $\psi(x) = \frac{\cos x}{x^3}$



$$\int_0^{\infty} e^{-x} \frac{\cos x}{x^3} \rightarrow \text{converges}$$

[∵ Directtlets Test].

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2). Express in β-function.

$$\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx.$$

Ans). $\int_0^1 x^2 (1-x^5)^{-\frac{1}{2}} dx = \int_0^1 \frac{x^2}{x^4} (1-x^5)^{-\frac{1}{2}} x^4 dx.$

$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$

$\boxed{m = n = \text{is single variable.}}$

$\cancel{\Rightarrow \int_0^1 \frac{x^2}{x^4} (1-x}$

Let $y = x^5$

$dy = 5x^4 dx$

$\frac{1}{5} dy = x^4 dx.$

$\Rightarrow \frac{1}{5} \int_0^1 y^{-\frac{2}{5}} (1-y)^{-\frac{1}{2}} dy.$

$\Rightarrow \frac{1}{5}\int_0^1 \overbrace{y^{\frac{3}{5}}}^{m-1=-\frac{2}{5}} * \overbrace{(1-y)^{\frac{1}{2}-1}}^{n-1} dy = \frac{1}{5}\beta\left(\frac{3}{5}, \frac{1}{2}\right).$

3). $\int_0^a x^{n-1}(a-x)^{m-1}dx = a^{m+n-1}\beta(m,n).$

sol :- putting $x = ay.$

$dx = a\,dy.$

$\Rightarrow \int_0^a x^{n-1}(a-x)^{m-1}dx.$

$\Rightarrow \int_0^1 a^{n-1}y^{n-1}a^{m-1}(1-y)^{m-1}a\,dy.$

when $x = 0$

$ay = 0.$

$y = 0.$

$x = a$

$ay = a.$

$y = 1.$

$\Rightarrow a^{m+n-1}\int_0^1 y^{n-1}(1-y)^{m-1}dy.$

$\Rightarrow a^{m+n-1}\beta(m,n).$

* $\boxed{B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$