Mathematics

D1 - 03/02/2025

Proposition :-

A Proposition is a statement which is either true or false ~~mot~~ but not both

1) $2+3=6$ (proposition)
2) Delhi is capital of India ([redacted] proposition)
3) $x+1=2$ (not a proposition)

Negation:-

P : Today is friday.

~P: Today is not friday.

Truth table

P	~P
T	F
F	T

Conjuction :-

If P and q are two propositions then the proposition "p and q" denoted as "$p \wedge q$" is a proposition that is true when both p and q are true else false then the proposition $p \wedge q$ is called as conjuction of p&q.

eg: P : Today is friday.

q: It is raining today.

$p \wedge q$: Today is friday and it is raining

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:-

Let p and q are two proposition. If the proposition "p or q" denoted as '$p \vee q$' is a proposition that is false when both p and q are false otherwise true then the proposition $p \vee q$ is called as disjunction.

Truth table:-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusion:-

Let p and q be proposition. If the exclusive or of p and q denoted as $p \oplus q$ is a proposition that is true when exactly one of p and q is true and is false otherwise.

p: You can have tea.

q: You can have coffee.

$p \oplus q$: You can either have coffee or tea but not both.

Truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication :

Let p and q be two propositions. If implication $p \to q$ is a proposition that is false when p is true and q is false and is true otherwise.

p: It is sunny today.
q: We will go to the beach
$p \to q$: If it is sunny today then we will go to the beach

Truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional statement: $(\leftrightarrow)$

Let p and q be propositions. The biconditional proposition $p \leftrightarrow q$ is "p if and only if q" or "p iff q" is a proposition, it is true when both p and q have same true value and is false.

p: You can take flight.
q: You buy a ticket.
$p \leftrightarrow q$: You can take flight if and only if you buy a ticket

Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

~~$q \leftrightarrow p$ is the converse of $p \leftrightarrow q$~~
~~$\sim p \leftrightarrow \sim q$ is the converse of $p \leftrightarrow q$~~
~~$\sim q \leftrightarrow \sim p$ is the contrapositive $p \leftrightarrow q$~~

$q \rightarrow p$ is the converse of $p \rightarrow q$

$\sim p \rightarrow \sim q$ is the converse of $p \rightarrow q$

$\sim q \rightarrow \sim p$ is the contrapositive $p \rightarrow q$

p	q	p → q	q → p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

p ∨ ~q

p	q	~q	p ∨ ~q
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

p ∧ ~q

p	q	~q	p ∧ ~q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

p	~p	p ∧ ~p	p ∨ ~p
T	F	F	T
F	T	F	T

p ∨ ~p → Tatology

p ∧ ~p → contradiction

If $x = 2$, ⇒ $x^2 = 4$

If $x^2 = 4$ then $x = 2$

T

p → q   q → p T

T   T

F   T

$(p \vee \sim q) \rightarrow (p \wedge q)$

p	q	$\sim q$	$p \vee \sim q$	$p \wedge q$	$(p \vee \sim q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

contingency

Logical equivalent:-

If P, Q both are composition and $P \leftrightarrow Q$ is a totology then $P \equiv Q$

e.g: $\sim(p \vee q)$ and $\sim p \wedge \sim q$ both are logical equivalent

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$	$\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

so, $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow \sim p \vee q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

H.W

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logical equivalent

Logical equivalent	Laws
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity law
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination law
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent law
$\sim(\sim p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	~~Double negation~~ commutative law
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative law
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distribution law
$\sim(p \wedge q) \equiv \sim p \vee \sim q$ $\sim(p \vee q) \equiv \sim p \wedge \sim q$	Demorgan law
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption law
$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv f$	Negation law
$p \to q \equiv \sim p \vee q$	Implication law

$p \to q \equiv \sim q \to \sim p$ law of contrapositive

Logical equivalent involving implication:

1) $p \to q \equiv \sim p \vee q$

2) $p \to q \equiv \sim q \to \sim p$

3) $p \vee q \equiv \sim p \to q$

4) $p \wedge q \equiv \sim(p \to \sim q)$

5) $\sim(p \to q) \equiv p \wedge \sim q$

6) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

7) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

8) $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

9) $(p \to q) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical equivalent involving biconditional statement:

1) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

2) $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

3) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

4) $\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

eg: show that $\sim(p \to q) \equiv (p \wedge \sim q)$ without using truth table.

$\sim(p \to q)$

$\equiv \sim(\sim p \vee q)$ (Implication law)

$\equiv \sim(\sim p) \wedge \sim q$ Demorgan's law

$\equiv p \wedge \sim q$ (Double negation law)

e.g: show that $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$

$\sim(p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$ (Demorgan's law)

$\equiv \sim p \wedge (\sim(\sim p) \vee \sim q)$ (Demorgan's law)

$\equiv \sim p \wedge (p \vee \sim q)$ Double negation law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ (Distributive law)

$\equiv f \vee (\sim p \wedge \sim q)$ (Negation law)

$\equiv (\sim p \wedge \sim q) \vee f$ (Commutative law)

$\equiv \sim p \wedge \sim q$ (Identity law)

Q: Show that $(p \wedge q) \rightarrow (p \vee q)$ is tautology

Sol^n

$(p \wedge q) \rightarrow (p \vee q) \equiv \sim(p \wedge q) \vee (p \vee q)$ (Implication law)

$\equiv (\sim p \vee \sim q) \vee (p \vee q)$ (De morgan law)

~~$\equiv \sim p \vee (p \vee q)$~~

$\equiv (\sim q \vee \sim p) \vee (p \vee q)$ (commutative law)

$\equiv (\sim q \vee q) \vee (\sim p \vee p)$ (Associative law)

$\equiv T \vee T$ (Identity law)

$\equiv T$ (Idempotent law)

Predicate :-

↳ statement involving variable.

eg: $P(x)$: $x > 3$

$P(4)$: $4 > 3$ → True

$P(1)$: $1 > 3$ → False

Quantifier :-

↳ A predicate which is true for a given range.

eg: $P(x)$: $x^3 > 0$

~~P(x)~~ $P(x)$ True $\forall x \in (0, \infty)$

$\therefore$ $P(x)$ is a quantifier

(1) Universal quantifier :

Quantifier

D1 - 10/02/2025
| 3/3 | P |

Existential quantifier: ($\exists$)

$P(x)$ is proposition
$\exists x \ P(x)$

e.g: $P(x)$: $x > 5$ Domain $= R$
$\exists x \ P(x)$

Note: $\sim \forall x \ P(x) = \exists x \sim P(x)$
$\sim \exists x \ P(x) = \forall x \sim P(x)$

Logical implication:
→ Two proposition P & Q are said to be logical implication if $P \to Q$ tautology

$P_1, P_2 - P_3 - - - , P_n \quad Q$

$P_1 \wedge P_2 \wedge - - - \wedge P_n \longrightarrow Q$

Rule of inference :-

Rule of inference	Tautology	Name
$\begin{array}{c} p \\ p \to q \\ \hline \therefore q \end{array}$	$[p \land (p \to q)] \to q$	Modus ponens
$\begin{array}{c} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$	$[\sim q \land (p \to q)] \to \sim p$	Modus tollens
$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	$[(p \to q) \land (q \to r)] \to (p \to r)$	hypothetical syllogism
$\begin{array}{c} p \lor q \\ \sim p \\ \hline \therefore q \end{array}$	$[(p \lor q) \land (\sim p)] \to q$	Disjunctive syllogism
$\begin{array}{c} p \\ \hline \therefore p \lor q \end{array}$	$p \to (p \lor q)$	Addition
$\begin{array}{c} p \land q \\ \hline \therefore p \end{array}$ or $\begin{array}{c} p \land q \\ \hline \therefore q \end{array}$	$p \land q \to p$ $p \land q \to q$	simplification
$\begin{array}{c} p \\ q \\ \hline \therefore p \land q \end{array}$	$p \land q \to p \land q$	Conjunction
$\begin{array}{c} p \lor q \\ \sim p \lor r \\ \hline \therefore q \lor r \end{array}$	$[(p \lor q) \land (\sim p \lor r)] \to q \lor r$	Resolution

Rule of constructive dilemma:

$p \to q$

$r \to s$

$p \lor r$

$\therefore q \lor s$

Sol^n

$p \to q$

$r \to s$

$\sim p \to r$

$\therefore p \to q$ (syllogism)

$\sim p \to s$

$\therefore p \to q$ (contrapositive rule)

$\sim s \to p$

$\therefore \sim s \to q$

$\therefore s \lor q$

$\therefore q \lor s$ (commutative)

Rule of distructive dilemma:-

$p \to q$

$r \to s$

$\sim q \lor \sim s$

$\therefore \sim p \lor \sim r$

Proof:

$p \to q$

$r \to s$

$\sim q \lor \sim s$

$p \to q$

$r \to s$

$q \to \sim s$

$r \to s$

$p \to \sim s$

Q.! Verify the validity of the argument:

$$\begin{array}{l} (p \vee r) \rightarrow q \\ s \rightarrow p \\ s \\ \hline \therefore q \end{array}$$

Proof:

$$\begin{array}{l} (p \vee r) \rightarrow q \\ s \rightarrow p \\ s \\ \hline \sim(p \vee r) \vee q \\ p \quad \text{(Modus pones)} \\ \hline (\sim p \wedge \sim r) \vee q \quad \text{(Demorgan's law)} \\ p \\ \hline (\sim p \vee q) \wedge (\sim r \vee q) \quad \text{(Distributive law)} \\ p \\ \hline \end{array}$$

~~[illegible]~~

$$\begin{array}{l} \sim p \vee q \quad \text{(simplification)} \\ p \\ \hline p \rightarrow q \\ p \\ \hline q \quad \text{(Modus pones)} \end{array}$$

Sequence:

$f : N \rightarrow R$

Denoted by $\langle a_n \rangle$ or $\{a_n\}$

e.g.: $a_n = \frac{1}{n}$

01 - 14/02/2025
$\boxed{4/4\,P}$

Q1. Prove that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$

Sol^n

Let $P(n) = 2^0+2^1+2^2+\cdots+2^n = 2^{n+1}-1$

We have to prove that $P(n)$ is true

Now,

~~$P(1) = 2^0+2^1 = 3$~~

~~or LHS $= P(1) = 3$~~

~~RHS $= 2^{1+1}-1 = 3$~~

~~Here LHS = RHS~~

Now,

$P(1)$: LHS $= 2^0+2^1 = 3$

RHS $= 2^{1+1}-1 = 3$

as LHS = RHS so, $P(1)$ is true

Assume $P(K)$ is true

That means

$$1+2+2^2+\cdots+2^K = 2^{K+1}-1 \quad \text{—— (i)}$$

Now, we'll check for $P(K+1)$

$P(K+1)$: $1+2+2^2+\cdots+2^K+2^{K+1} = 2^{(K+1)+1}-1$

LHS $= 1+2+2^2+\cdots+2^K+2^{K+1}$

$= 2^{K+1}-1+2^{K+1}$

$= 2\times 2^{K+1}-1$

$= 2^{(K+1)+1}-1$ = RHS

Here, $P(K+1)$ is true

$\therefore$ $P(n)$ is true

Q: Prove that $n < 2^n$

Sol^n Let $P(n): n < 2^n$

Now, $\underline{P(1)}$: $LHS = 1$

$RHS = 2^1 = 2$.

$LHS < RHS$ or $1 < 2$ —— (1)

so, $P(1)$ is true

Assume that $P(K)$ is true (Inductive hypothesis)

or $P(K): K < 2^K$ is true. —— (2)

For $P(K+1)$:

$P(K+1): (K+1) < 2^{K+1}$

We know, $K < 2^K$ (From (1))

$\Rightarrow K+1 < 2^K + 1$ (Adding 1 in both side)

$\Rightarrow K+1 < 2^K + 1 + 1$ (Adding 1 in RHS)

$\Rightarrow K+1 < 2^K + 2$

$\Rightarrow K+1 < 2^{K+1}$

Hence $P(K+1)$ is true

We know,

$$K < 2^K \quad (\text{From } (2))$$

~~Similarly~~

$$1 < K \quad (\text{Assumption})$$

$2 \cdot 2^K$

$2^K + 2^K$

Combing both inequality

$$K+1 < 2^K + K \text{ --- (i)}$$

$$K < 2^K$$

$$\Rightarrow 2^K + K < 2^K + 2^K$$

$$\Rightarrow 2^K + K < 2^{K+1} \text{ --- (ii)}$$

from (i) & (ii) $K+1 < 2^{K+1}$

Hence $P(K+1)$ is true

$\therefore$ $P(n)$ is true.

Q! Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Sol:

Let $P(n)$: $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

For $P(1)$! $LHS = 1^2 = 1$

$RHS = \frac{1(1+1)(2+1)}{6} = 1$

$LHS = RHS$

$\Rightarrow$ $P(1)$ is true

Let us assume $P(K)$ $(K > 1)$ is true

(Inductive hypothesis)

OR,

$P(K)$: $1^2 + 2^2 + \cdots + K^2 = \frac{K(K+1)(2K+1)}{6}$ is true

For P(K+1)

$(K+1)(K+2)(2K+3)$

$$P(K+1):\ 1^2+2^2+\cdots+K^2+(K+1)^2 = \frac{(K+1)(K+1+1)(2(K+1)+1)}{6}$$

$$LHS = 1^2+2^2+\cdots+K^2+(K+1)^2$$

$$= \frac{K(K+1)(2K+1)}{6}+(K+1)^2$$

$$= \frac{K(K+1)(2K+1)+6(K+1)^2}{6}$$

$$= \frac{(K+1)\{K(2K+1)+6(K+1)\}}{6}$$

$$= \frac{(K+1)(2K^2+7K+6)}{6}$$

$$= \frac{(K+1)(2K^2+3K+4K+6)}{6}$$

$$= \frac{(K+1)\{K(2K+3)+2(2K+3)\}}{6}$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}$$

$$= \frac{(K+1)(K+1+1)(2(K+1)+1)}{6}$$

$$= RHS$$

Hence $P(K+1)$ is true

$\therefore$ $P(n)$ is true

Q: Prove that $6^{n+2} + 7^{2n+1}$ is divisible by 43.

Sol^n:

Let $P(n)$: $6^{n+2} + 7^{2n+1}$ is divisible by 43

For P(1)

$LHS = 6^3 + 7^3 = 216 + 343$
$= 559$
$= 13 \times 43$
$= 43 \times 13$
(Divisible by 43)

So, P(1) is ~~true~~

So, P(1) is true

Let assume that P(K) is true

or $6^{K+2} + 7^{2K+1}$ is divisible by 43

Let $6^{K+2} + 7^{2K+1} = 43m$ ($m \in N$)

Now, $P(K+1)$: $6^{K+3} + 7^{2K+3}$ is divisible by 43

we'll prove P(K+1) is true.

$$6^{K+3} + 7^{2K+3} = 6^{K+2} \cdot 6 + 7^{2K+1} \cdot 49$$

$$= 6^{K+2} \cdot 6 + 6 \cdot 7^{2K+1} + 43 \cdot 7^{2K+1}$$

$$= 6(6^{K+2} + 7^{2K+1}) + 43 \times 7^{2K+1}$$

$$= 6 \times 43m + 43 \times 7^{2K+1}$$

$$= 43(6m + 7^{2K+1})$$

(Is divisible by 43)

Hence $P(K+1)$ is true.

$\therefore$ $P(n)$ is true.

Type 2: Strong induction:-

1) Basic step: $P(1)$ is true

2) Inductive hypothesis: $P(1) \cap P(2) \cap \cdots \cap P(K)$ is true

3) Inductive step: we prove $P(1) \cap P(2) \cap \cdots P(K) \rightarrow P(K+1)$

e.g:

Counting principle :-

Permutation: (order arrangement)

$$^nP_r = \frac{n!}{(n-r)!}$$

e.g: $n = 100$ people

1st prize

2nd "

3rd "

$$^{100}P_3 = \frac{100!}{3!} = 970200$$

e.g: How many permutation of the letter ABCDEFGH contain the string ABC

sol: $6!$

Combination: (selection)

$$^nC_r = C(n,r) = \frac{n!}{(n-r)!\, r!} = \frac{^nP_r}{r!}$$

e.g: calculate the number of diagonal in hexagon

$$= {^6C_2} - 6 = \frac{6!}{4!\,2!} - 6$$

$$= \frac{6\times 5}{2} - 6$$

$$= 15 - 6 = 9$$

For octagon
no. of diagonal $= {}^8C_2 - 8$

$$= \frac{8!}{2!\,6!} - 8$$

$$= \frac{8 \times 7}{2} - 8$$

$$= 28 - 8 = 20$$

Principle of inclusion and exclusion :-

Let A be a set
$n(A)$, $|A|$ → cardinality of A

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$$

$$= n(A - B) + n(B - A)$$

$$n(A) = n(A - B) + n(A \cap B)$$

$$n(A \cup B) = n(A) + n(B) \quad (\text{if } A \cap B = \phi)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Q1: How many positive integer not exceeding 1000 are divisible by 7 or 11.

Soln

$|A|$ = no. of +ve integer divisible by 7 $= \left[\frac{1000}{7}\right] = 142$

$|B|$ = " " " 11 $= \left[\frac{1000}{11}\right] = 90$

$|A \cap B|$ = no. of " " by both 7 and 11

= " " by lcm of 7 and 11

= " " by $77 = \left[\frac{1000}{77}\right] = 12$

$|A \cup B| = 142 + 90 - 12$

$= 220$

Pigeonhole principle :-

hole　　　　pigeon

If there are k ~~pigeon~~ and k+1 or more ~~hole~~

then there exist at least one whole which

contain more than one pigeon

MODULE-2

Dt - 17/01/2025

| 5/5 | P |

Relation :-

A relation R from a set A to a set B is a subset of A×B.

Denoted by:

$R : A \to B$ or aRb

$R = \{(a,b) \mid a \in A, b \in B\}$

eg: $A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3\}$

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3)\}$

$R = \{(a,b) \mid a \in A, b \in B$ and a divides $b\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3)\}$

Note:
$|A| = m$, $|B| = n$ then no. of relation from A to B is $2^{m \times n}$

Types of relation:

1) Identity relation:-

$I = \{(a,a) : \forall a \in A\}$

2) Inverse relation:-

$R^{-1} = \{(b,a) ; \forall a \in A, b \in B\}$

Properties of relation:-

(1) Reflexive relation:- (If $a \in A$ then $(a,a) \in R$ or aRa)

A relation R is said to be reflexive if $(a,a) \in R$ $\forall a \in A$

2) symmetric relation:- (If $(a,b) \in R$ then $(b,a) \in R$

A relation R is said to be symmetric if for every $(a,b) \in R$ then $(b,a) \in R$

3) Transitive relation: (If aRb & bRc then aRc)

A relation R on a set A is said to be transitive if for every $(a,b) \in R$ & $(b,c) \in R$, $(a,c) \in R$

4) Irreflexive relation ($(a,a) \notin R$ $\forall a \in A$)

A relation R on a set A is said to be irreflexive if $\forall a \in A$, $(a,a) \notin R$

5) Nonreflexive relation:-

A relation R on a set A is said to be nonreflexive if it is neither reflexive nor irreflexive.

e.g. $R : A \to A$ & $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (1,3)\}$

Here R is not reflexive as $(3,3) \notin R$

R is not irreflexive as $(1,1) \in R$

So, R is nonreflexive

6) Antisymmetric relation:-

A relation R on a set A is said to be antisymmetric if aRb & $bRa \Rightarrow a = b$.

OR

if $a \neq b$ — either aRb or bRa

— neither aRb nor bRa

e.g.: ① $R : A \to A$ $A = \{1, 2, 3\}$

$R = \{(a, b) \mid a \in A, b \in R \text{ \& } a \leq b\}$

$= \{(1,1), (1,2), (1,3),$
$(2,2), (2,3), (3,2)\}$

②

Q: Give an example of a relation which is
(i) Reflexive, transitive but not symmetric
(ii) symmetric, transitive but not reflexive
(iii) Reflexive, symmetric but not transitive

Sol: Let $A =$

7) Equivalence relation :-

A relation R on a set A is said to an equivl. relation if it is reflexive, symmetric, transitive

Q: show that congruence module relation is an equivalence relation

Dt - 28/02/2025

7/8 | P

8) Partial order relation (Reflexive, transitive and antisymmetric)

A relation R on a set A said to be partial. relation if R is reflexive, anti-symmetric & transitive.

e.g: $R = \{ a, b \in \mathbb{Z} \mid a \text{ divides } b \}$

Reflexive: clearly $a \mid a$

Transitive: Let $a \mid b$ & $b \mid c$

$\Rightarrow a = K_1 b$ & $b = K_2 c$

$\Rightarrow a = K_1 K_2 c \Rightarrow a \mid c$

01 -

Dt 28/02/2025

$\boxed{7/8 \mid R}$

$\Rightarrow (a,a) \in R$

Antisymmetric: Let $a|b$ & $b|a$ $(a,b) \in R$ or $(b,a) \in R$

so $a = b$, For $a \neq b$ either but not both

$\Rightarrow$ Relation is antisymmetric

Transitive:

Let aRb & bRc

$\Rightarrow a|b$ & $b|c$

$\Rightarrow b = k_1 a$ & $c = k_2 b$

$\Rightarrow c = k_2 k_1 a$

$\Rightarrow a|c$

$\Rightarrow (a,c) \in R$ or aRc

so R is transitive

As R is reflexive, antisymmetric & transitive so R is partial order relation.

It will be better to give an example

~~Q: Prove that R = {(a,b) | a~~

Q: Prove that $R = \{a, b \in P(A) \mid a \subseteq b\}$

Soln

Given $R = \{a, b \in P(A) \mid a \subseteq b\}$

Let consider an example,

$A = \{p, q, r\}$

$P(A) = \{\{\}, \{p\}, \{q\}, \{r\}, \{p,q\}, \{q,r\}, \{r,p\}, \{p,q,r\}\}$

Checking for reflexive:

we can write $a \subseteq a \quad \forall a \in P(A)$

(Improper subset)

For example: Let $a = \{p\}$

we can write $\{p\} \subseteq \{p\}$

So, R is reflexive —— (1)

Checking for antisymmetric:

Let $x, y \in P(A)$, xRy & yRx

That means $x \subseteq y$ —(i) & $y \subseteq x$ —(ii)

From (i) & (ii) $x = y$ —(iii)

Let $a, b \in P(A)$ & $a \neq b$

then either aRb or bRa (either a is a subset of b or b is a subset of a)

otherwise neither $a R b$ nor $b R a$
(neither a is a subset of b nor b is a subset of a)

So, R is antisymmetric —— (2)

<u>Checking for transitive:</u>

Let $x, y, z \in P(A)$, xRy, yRz

That means

$$x \subseteq y \text{ \& } y \subseteq z$$
$$\Rightarrow x \subseteq z$$
$$\Rightarrow xRz$$

For example: $x = \{p\}$, $y = \{p, q\}$, $z = \{p, q, r\}$

clearly $x \subseteq y$ & $y \subseteq z$

& $x \subseteq z$

So, R is transitive — (3)

From the statement (1), (2) & (3) we can conclude that R is a partial order relation

Q: Prove that $R = \{a, b \in \mathbb{Z} \mid a \leq b\}$ is a partial order relation

[Do it yourself]

* D_n = {set of all divisors of n

Q: show that '|' (divides) D_n is a partial order relation}.

Partal order set (POSET)

A partal order relation R on a set A is known as partial order set

e.g: $(\mathbb{Z}, |)$, $(P(A), \subseteq)$, $(D_n, |)$, $(\mathbb{Z}, \leq)$

POSET

~~Clouser of reflex~~

Clouser of relation:

1. Reflexive clouser: (R^r)

↳ The reflexive clouser of a relation R is the smallest reflexive relation that contains R as a subset.

e.g: Let $A = \{1, 2, 3, 4\}$

$R: A \to A$, $R = \{(1,2), (2,1), (3,3), (4,4)\}$

R is not a reflexive relation as $(2,2) \notin R$

$$\boxed{R^r = R \cup I_A}$$

where I_A = identity relation

$\Rightarrow R^r = R \cup \{(1,1), (2,2), (3,3), (4,4)\}$

$\Rightarrow R^r = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$

2.5 Symmetric closure : (R^s)

→ The symmetric closure of a relation R is a smallest symmetric relation that contains R as a subset.

e.g: $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (4,3), (2,2), (2,1), (3,1)\}$

Here R is not symmetric as $(4,3) \in R$ but $(3,4) \notin R$

& $(3,1) \in R$ but $(1,3) \notin R$

Now, adding elements of $(3,4)$ & $(1,3)$

$R^s = R \cup \{(3,4), (1,3)\}$

or $R^s = \{(1,2), (4,3), (2,2), (2,1), (3,4), (1,3), (3,1)\}$

OR in another way we can also write $\boxed{R^s = R \cup R^{-1}}$

3. Transitive closure (R^*)

→ The relation obtained by adding least number of order pairs to ensure transitivity is called transitive closure of relation R.

→ If A is a set and R is a relation on A the transitive closure of R on A satisfy the property (i) R^* is transitive

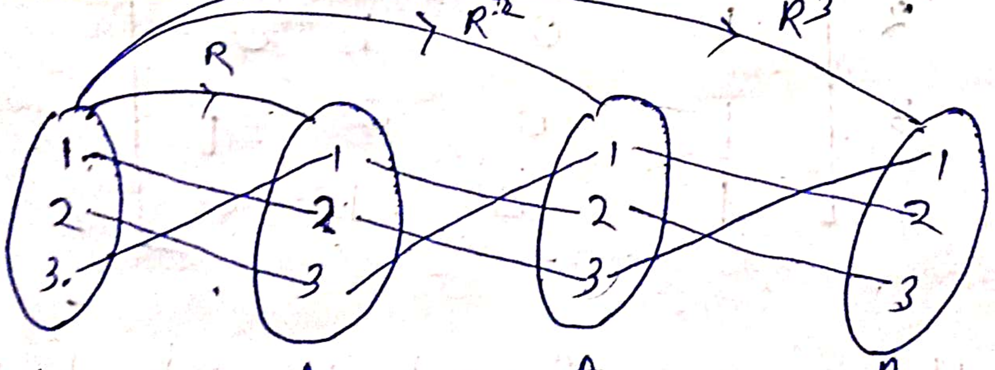
(ii) $R \subseteq R^*$

(iii) If S is any other transitive relation that R then $R^* \subseteq S$

e.g: Let $A = \{1, 2, 3\}$

& $R = \{(1,2), (2,3), (3,1)\}$

> If $n(A) = K$ & $R: A \to A$
> then $R^{(*)} = R \cup R^2 \cup R^3 \cup \dots R^K$
> $R^K = RoRoR \dots K$ times



$R^2 = \{(1,3), (2,1), (3,2)\}$

$R^3 = R^2 o R$

$= \{(1,1), (2,2), (3,3)\}$

Now,

$R^{(+)} = R \cup R^2 \cup R^3$

$= \{(1,2), (2,3), (3,1)\} \cup \{(1,3) (2,1), (3,2)\}$

$\cup \{(1,1), (2,2), (3,3)\}$

$= \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (3,1), (1,3)\}$

Warshall's algorithm :- (To find transitive closure)

By using warshall's algorithm find the transitive closer of the relation $R = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$ on a set $A = \{1,2,3,4\}$

Sol^n

S-1 Representation of relation in matrix format

$$R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Since there a 4 element in set A we need 4 steps to find transitive closer of relation R

In step-1 we'll consider 1st coloumn & 1st row of the above matrix (C_1 & R_1).
~~Row~~ write all position where 1 is present in column 1 ($C1$)

$C1 = \{2, 3, 4\}$

also write all position when 1 is present in Row 1 ($R1$)

$R1 = \phi$

Now take the cross product of $C1$ & $R1$

$C1 \times R1 = \phi$

$\therefore$ No new addition

In step-2 we'll consider 2nd coloumn & 2nd row of the above matrix (C_2 & R_2).
Write all position when 1 is present in $C2$ & R_2

$C_2 = \phi, \quad R_2 = \{1, 3\}$

$C_2 \times R_2 = \phi$

$\therefore$ No new addition

In step-3 we'll consider 3rd coloumn & 3rd row of the above matrix (C_3 & R_3)
write all position where 1 is present in C_3 & R_3

$C_3 = \{2, 4\}, \quad R_3 = \{1, 4\}$

$C_3 \times R_3 = \{(2,1), (2,4), (4,1), (4,4)\}$

After adding the element of $C_3 \times R_3$ on the matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}$$

In step-4 we'll consider 4th column & 4th row of the updated matrix

write all position where 1 is present in C_4 & R_4

$C_4 = \{2, 3, 4\}$ $R_4 = \{1, 3, 4\}$

$C_4 \times R_4 = \{(2,1), (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4)\}$

After adding the element of $C_4 \times R_4$ in the matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}$$

$R^* = \{(2,1), (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4)\}$