Existence of Laplace Transform

If $f(t)$ is defined & piecewise continuous on every finite interval on the semiaxis $t \geq 0$ & satisfies growth restriction $\forall\ t \geq 0$ i.e. $|f(t)| \leq Me^{kt}$. then Laplace transform of $f(t)$ exists. $s > k$.

Proof

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \underbrace{\int_0^{t_0} e^{-st} f(t)\,dt}_{I_1} + \underbrace{\int_{t_0}^\infty e^{-st} f(t)\,dt}_{I_2}.$$

$= I_1$ exist

$$|I_2| = \left|\int_{t_0}^\infty e^{-st} f(t)\,dt\right| \leq \int_{t_0}^\infty \left|e^{-st} f(t)\right| dt$$

$$= \int_{t_0}^\infty |e^{-st}|\,|f(t)|\,dt$$

$$\leq \int_{t_0}^\infty e^{-st} M e^{kt}\,dt$$

$s > k$

$$\leq M \int_{t_0}^\infty e^{-st} e^{kt}\,dt \quad \frac{M}{(s-k)}.$$

$$\le M \int_{t_0}^{\infty} e^{(-s+k)t}\, dt$$

$$\le \frac{M}{k+s}\left[e^{(-s+k)t}\right]_{t_0}^{\infty}$$

$$\le \frac{M\, e^{(-s+k)t_0}}{s-k} = \frac{M\, e^{(k-s)t_0}}{k-s}$$

$$= -M\left[\frac{e^{-(s-k)t_0}}{k-s}\right]_{t_0}^{\infty}$$

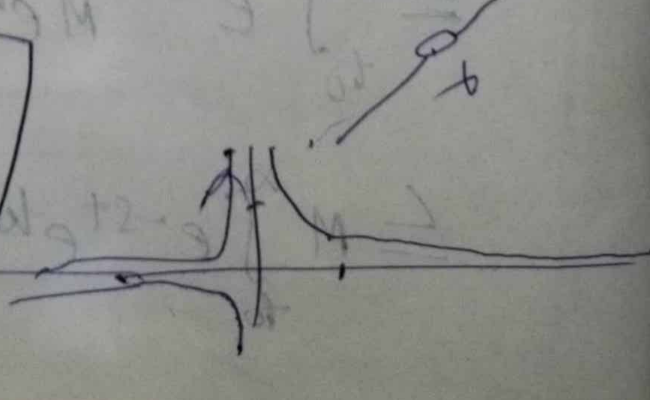
$$= \frac{M\, e^{(k-s)t_0}}{k-s}$$

Ex Find $L(f(t))$

$$f(t) = \begin{cases} t/k & 0 < t < k \\ 2 & t > k \end{cases}$$

$$I = \int_0^k e^{-st}\, t/k\, dt + \int_k^{\infty} e^{-st}\, 2\, dt$$

$$\boxed{\cancel{= \left[\frac{1}{ks^2}\right]_0^k + \left[\frac{2}{s}\right]_k^{\infty}} \qquad \cancel{= \frac{1}{ks^2} + \frac{2}{s}}}$$



$$= \frac{1}{k}\int_0^k e^{-st} t \, dt + 2\int_k^\infty e^{-st} dt$$

$$= \frac{1}{k}\left[\frac{-t}{s} e^{-st} - \int_0^k e^{-st} dt\right]_0^k + \frac{-2}{s}\left[e^{-st}\right]_k^\infty$$

$$= \frac{1}{k}\left\{\left[\frac{-t}{s} e^{-st}\right]_0^k - \frac{1}{s^2}\left[e^{-st}\right]_0^k\right\} + \frac{-2}{s}\left[e^{-st}\right]_k^\infty$$

$$= \frac{1}{k}\left[\frac{-k}{s} e^{-sk} - \frac{1}{s^2} e^{-sk}\right] - \frac{2}{s}\left(-e^{-sk}\right)$$

$$= \frac{-e^{-sk}}{s} - \frac{e^{-sk}}{ks^2} + \frac{2e^{-sk}}{s} - \frac{1}{ks^2}$$

$$= \frac{e^{-sk}}{s} - \frac{e^{-sk}}{ks^2} - \frac{1}{ks^2}$$

$$= \frac{e^{-sk}}{s} - \frac{e^{-sk}}{ks^2} - \frac{1}{ks^2}$$

Q

Assignment-1

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$$

Exist/not

Find

$L(f(t)) = ?$

$\frac{1}{t-1}$

Inverse Laplace (L^{-1})

$$L(f(t)) = F(s)$$

$$F(t) = L^{-1}(f(s))$$

$L^{-1}\left(\frac{1}{s}\right) = 1$, $L^{-1}\left(\frac{1}{s^2}\right) = t$, $L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$

Shifting Property in inverse Laplace

If $L^{-1}(f(s)) = f(t)$

$$L^{-1}(f(s-a)) = e^{at} L^{-1}(f(s))$$

Ex

$$L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t} t$$

$$L^{-1}\left(\frac{s}{(s+1)^3}\right) \cancel{= s L^{-1}\left(\frac{1}{(s+1)^3}\right) = \frac{s e^{-t} t^2}{2}}$$

$$= L^{-1} \frac{s}{\quad}$$

Ex

$$L^{-1}\left(\frac{s}{(s+1)^3}\right) = L^{-1}\left(\frac{s+1-1}{(s+1)^3}\right) = L^{-1}\left(\frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}\right)$$

$$= te^{-t} - \left(\frac{1}{2}t^2e^{-t}\right)$$

$$= te^{-t} - \frac{t^2}{2}e^{-t}.$$

~~Ex~~ **Assignment**

Ans

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t \quad = L^{-1}\left(\frac{2}{2(s^2+2^2)}\right)$$

$$= \frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= \frac{1}{2}\sin 2t.$$

$$L^{-1}\left(\frac{s}{s^2+2s+2}\right) = L^{-1}\left(\frac{s+1-1}{(s+1)^2+1}\right)$$

$$= L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$=$$

Multiply by t property

If $L(f(t)) = F(s)$ $f(t)$

Then $\boxed{L(t \cdot f(t)) = \frac{-d}{ds} F(s)}$

Proof

$F(s) = L(f(t))$

$\Rightarrow F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\Rightarrow \frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt$

$\Rightarrow \frac{d}{ds} F(s) = \int_0^\infty -t \cdot e^{-st} f(t)\,dt$

$\Rightarrow -\frac{d}{ds} F(s) = \int_0^\infty t\, e^{-st} f(t)\,dt$

$= \int_0^\infty e^{-st} (t \cdot f(t))\,dt$

$= L(t \cdot f(t))$

Proved

Ex

$$L(t\sin\omega t) = \frac{-d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= -\omega \frac{d}{ds}\left(s^2+\omega^2\right)^{-1}$$

$$= \frac{2s\omega}{(s^2+\omega^2)^2}$$

Ex

$$L(te^{2t}) = \frac{-d}{ds}\left(\frac{1}{s-2}\right)$$

$$= \frac{1}{(s-2)^2}$$

Ex

$$L(t^2 f(t)) = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$\boxed{L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} f(s)}$$

Laplace transform of derivatives :-

The transforms of the 1st & 2nd derivative of $f(t)$ satisfy

(i) $L(f') = s\,L(f) - f(0)$

(ii) $L(f'') = s^2 L(f) - s\,f(0) - f'(0)$

(i) hold if $f(t)$ is continuous $\forall\, t \geq 0$ & satisfies the growth restriction & $f'(t)$ is piecewise continuous on every finite interval. on $t \geq 0$.

Proof

Suppose f' is continuous

$$L(f') = \int_0^\infty \underbrace{e^{-st}}_{1st} \underbrace{f'(t)}_{2nd}\, dt$$

$$= e^{-st}\left[f(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f(t)\, dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= sL(f) + \left[0 - f(0)\right]$$

$$= sL(f) - f(0)$$ Proved

(ii) hold if $f(t)$ & $f'(t)$ both are cont. $\forall\, t \geq 0$ & satisfies growth restric^n & $f''(t)$ is piecewise cond^n on every finite interval on $t \geq 0$.

$$L(f'') = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \left[e^{-st} f'(t)\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= 0 - f'(0) + s\left(sL(f) - f(0)\right)$$

$$= -f'(0) + s^2 L(f) - sf(0)$$

$$= s^2 L(f) - sf(0) - f'(0)$$

Proved

$s^n L(f(t)) - s^{n-1} f(0) - \cdots$

$$\boxed{L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots f^{n-1}(0)}$$

Ex

$f(t) = t \sin\omega t$ $\quad f'(t) = \dfrac{t \sin\omega t \; \omega \sin\omega t}{\sin\omega t}$

$L(f') = s\,L(f(t)) - f(0)$

$= s \times L(t \sin\omega t) - 0$

$= s \times \dfrac{-d}{ds}\left(\dfrac{\omega}{\omega^2+s^2}\right) - 0$

$= s \times \dfrac{+2s\omega}{(s^2+\omega^2)^2} = \dfrac{2s^2\omega}{(s^2+\omega^2)^2}$

$L(f'') = s^2 L(f(t)) - sf(0) - f'(0)$

$= \dfrac{2s^3\omega}{(s^2+\omega^2)^2} - (s \times 0) - 0$

$= \dfrac{2s^3\omega}{(s^2+\omega^2)^2}$

Laplace transform of the integral of a funcn

Let $L(f(t)) = F(s)$; which is piecewise cont. for $t \geq 0$ & satisfies a growth restricn. Then for $s > 0$, $s > k$ & $t > 0$.

(i) $L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s)$

(ii) $\int_0^t f(\tau)\,d\tau = L^{-1}\left(\frac{1}{s}F(s)\right)$

Proof

Suppose $\int_0^t f(\tau)d\tau = g(t)$

$f(t)$ cont.

$g(t)$ is piecewise cont.

$$|g(t)| = \left|\int_0^t f(\tau)\,d\tau\right| \leq \int_0^t |f(\tau)|d\tau$$

$$\leq \int_0^t Me^{k\tau}d\tau$$

$$\leq \frac{M}{k}\left[e^{k\tau}\right]_0^t$$

$$= \frac{M}{k}\left(e^{kt}-1\right)$$

$f(t)=g'(t)$ except at points at which $f(t)$ is descontinuous.

$$L(f(t)) = L(g'(t)) = s\,L(g(t)) - \cancelto{0}{g(0)} =$$

$$\Rightarrow L(f(t)) = s \times L\, g(t)$$

$$\Rightarrow L f(t) = s\,L(g(t))$$

$$\Rightarrow F(s) = sL(g(t))$$

$$\Rightarrow \frac{1}{s}F(s) = L(g(t)) = L\left[\int_0^t f(\tau)d\tau\right]$$

$\Rightarrow L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)\,d\tau$

$3.14 = \pi$

Q Find the Laplace inverse of $\frac{1}{s(s^2+w^2)}$

$L^{-1}\left(\frac{1}{s(s^2+w^2)}\right) = \frac{1}{w}\int_0^t \sin w\tau\, d\tau$

$= \frac{-1}{w^2}\left[(\cos w\tau)\right]_0^t$

$= \frac{-1}{w^2}(\cos wt - 1)$

Sol^n of ODE$_s$ by L.T.

Step-1 We consider a differential eq^n & initial cond

Step-2 Take L.T. on both side.

Step-3 Use the result of L.T. of derivative then put the initial cond^n.

Step-4 Separate $Y = Y(s)$.

Step-5 Find $y(t)$ by using inverse Laplace transform

$y(t) = L^{-1}(Y)$

Ex

$y(0)=1$ & $y'(0)=7$

$y'' - 2y' - 3y = 0$

$L(y'' - 2y' - 3y) = L(0)$

$\Rightarrow L(y'') - 2L(y') - 3L(y) = 0$

$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) - 2sL(y) + 2y(0)$
$- 3L(y) = 0$

$\Rightarrow s^2 L(y) - s - 7 - 2sL(y) + 2 - 3L(y) = 0$

$\Rightarrow s^2 L(y) - 2sL(y) - 3L(y) - s - 5 = 0$

$\Rightarrow s^2 L(y) - 2sL(y) - 3L(y) = s+5$

$\Rightarrow (s^2 - 2s - 3)\, L(y) = s+5$

$\Rightarrow L(y) = \dfrac{s+5}{s^2 - 2s - 3}$ $\qquad (s-3)(s+1)$

$\Rightarrow y = L^{-1}\left(\dfrac{s+5}{s^2-2s-3}\right)$

$= L^{-1}\left(\dfrac{s-3+8}{(s-3)(s+1)}\right)$

$= L^{-1}\left(\dfrac{1}{s+1}\right) + L^{-1}\left(\dfrac{8}{(s-3)(s+1)}\right)$

$= e^{-1}$

$1/8$

1

e^{a}

$$= L^{-1}\left(\frac{s+5}{s^2-2s+1-4}\right)$$

$$= L^{-1}\left(\frac{s+5}{(s-1)^2-4}\right)$$

$$= L^{-1}\left(\frac{s-1}{(s-1)^2-2^2}\right) + \frac{6}{(s-1)^2-2^2}$$

$$= e^{t}\cosh 2t + \frac{6e^{t}}{2}\sinh 2t$$

$$= e^{t}\cosh 2t + 3e^{t}\sinh 2t.$$

Q. **Assignment**

$y'' + 4y = 0$ $\qquad y(0) = 1,\ y'(0) = 1$

$L(y'') + 4L(y) = L(0)$

Integration of transform

$$L\left(\frac{f(t)}{t}\right) = \int_{s}^{\infty} f(\bar{s})\, d\bar{s}$$

$$\frac{f(t)}{t} = L^{-1}\left(\int_{s}^{\infty} f(\bar{s})\, d\bar{s}\right)$$

$\frac{s}{1+s^2} = s$

Q/

$$L\left(\frac{\sin \omega t}{t}\right) = \int_s^\infty \frac{\omega}{\bar{s}^2+\omega^2}\, d\bar{s}$$

$$= \frac{1}{\omega}\left[\tan^{-1}\frac{\bar{s}}{\omega}\right]_s^\infty$$

$$= \frac{1}{\omega}\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{\omega}\right)$$

$$= \frac{1}{\omega}\cot^{-1}\frac{s}{\omega}$$

$$= \frac{1}{\omega}\tan^{-1}\frac{\omega}{s}$$ Possible

$\frac{1}{1+x^2}\,dt$ $\tan^{-1}\frac{t}{a}$

Q/

$$L\left(\frac{e^{at}}{t}\right) = \int_s^\infty \frac{1}{\bar{s}-a}\, d\bar{s}$$

$$= \left[\log(\bar{s}-a)\right]_s^\infty$$

$$= \ln \infty - \log(s-a)$$

→ Not define & unbounded.

So, growth restric^n ✗

Q/

$$L\left(\frac{\cos t}{t}\right) = \int_s^\infty \frac{\bar{s}}{\bar{s}^2+1}\, d\bar{s}$$

$$= \frac{1}{2}\int_{s^2+1}^\infty \frac{dt}{t}$$

$\bar{s}^2+1 = t$

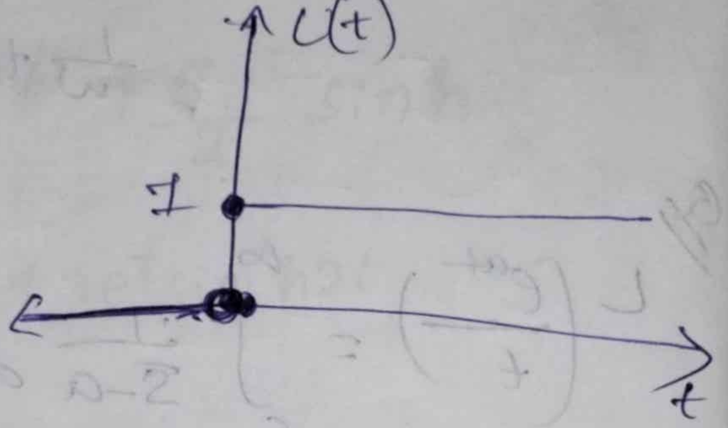
$\Rightarrow \bar{s}\, d\bar{s} = \frac{dt}{2}$

Unit step func^n / Heaviside step func^n

The func^n $u(t-a)$ is 0 for $t<a$ & has a jump of size 1 at $t \geq a$

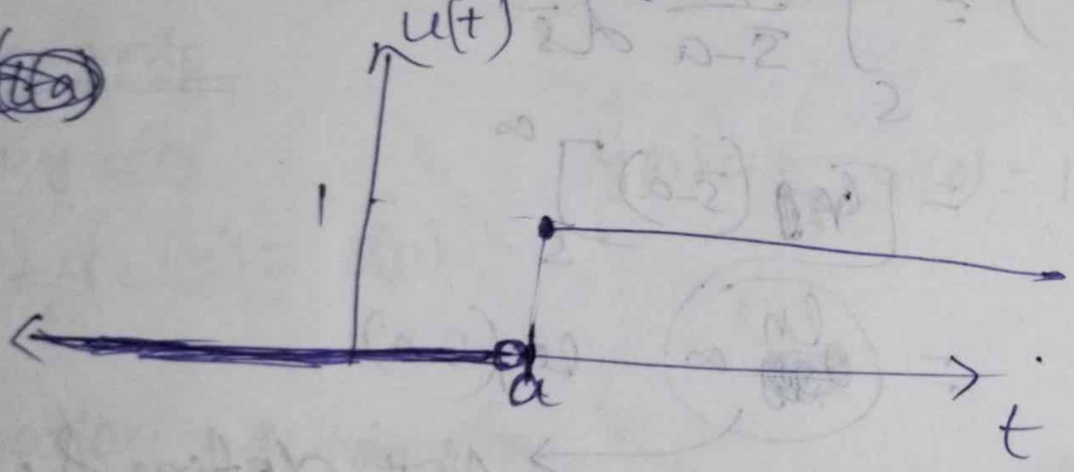
$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$$



0

$1/(s-a)$

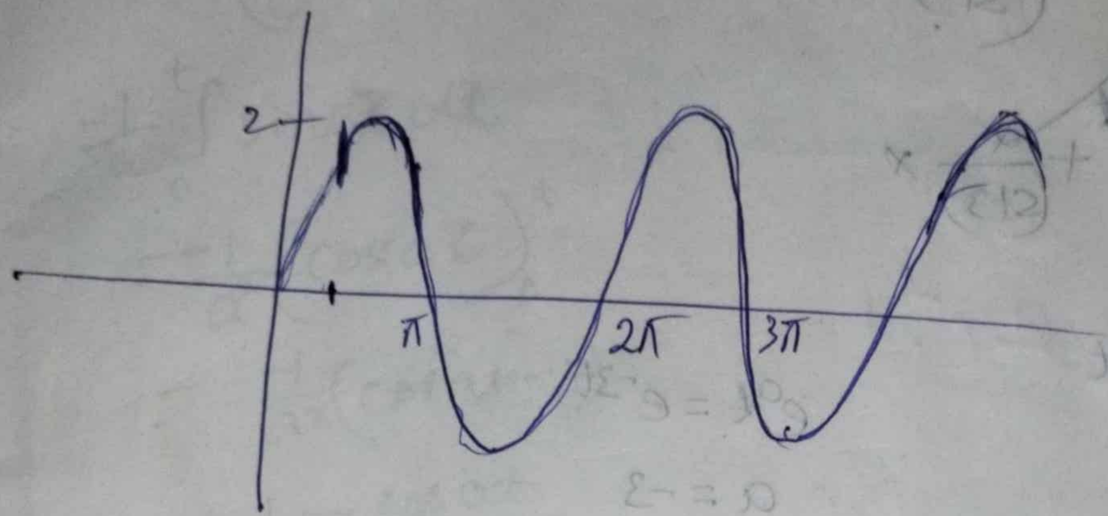


$$L(u(t)) = \int_0^\infty e^{-st}\, u(t)\, dt = 1/s$$

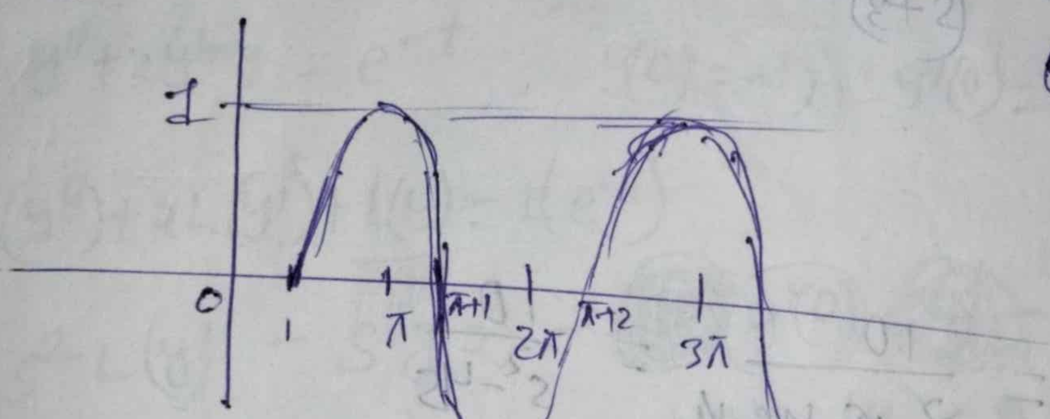
$$L(u(t-a)) = \int_0^a e^{-st} \times 0 + \int_a^\infty e^{-st} \times 1\, dt$$

$$\boxed{\frac{e^{-as}}{s}} \text{ Ans} = \frac{-1}{s}\left[e^{-st}\right]_a^\infty = \frac{-1}{s}(0-e^{-as}$$

$f(t) = 2\sin t \quad f(t)\, u(t-1)$



$f(t-1) = 2\sin(t-1)$



e^{at}

Q//

$f(t) = t^2 e^{-3t}$

$L\,f(t) = \dfrac{2}{(s+3)^3}$

$L\left(f(t) = t^2 e^{-3t}\right) = \int_0^\infty e^{-st} \cdot e^{-3t} \cdot t^2\, dt$

$= \int_0^\infty e^{-(3+s)t} \cdot t^2\, dt$

$= \dfrac{t^2}{-(3+s)} e^{-(3+s)t} - \int_0 \dfrac{2t}{-(3+s)} e^{-(3+s)t}\, dt$

$$= \frac{-t^2}{3+s} e^{-(3+s)t} + \frac{2}{(s+3)} \int t \cdot e^{-(3+s)t}\, dt$$

$$= \frac{-t^2}{3+s} e^{-(3+s)t} + \frac{2}{(s+3)} \times$$

$$f(t) = t^2 \cdot e^{-3t} \qquad e^{at} = e^{-3t}$$

$$a = -3$$

$$n = 2$$

$$L f(t) = f(s-a)$$

$$= \frac{2!}{(s-a)^3} = \frac{2}{(s+3)^3}$$

Q/ $5e^{2t} \sinh 2t$

$$= \frac{5 \times 2}{(s-2)^2 - 4} = \frac{10}{s^2 + 4 - 4s - 4} = \frac{10}{s^2 - 4s}$$

Q/ $L(f(t)) = \frac{1}{s(s^2+a^2)}$ find $f(t)$

$$f(t) = L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$$

$$= \int_0^t \frac{1}{(s^2+a^2)}\, ds$$

$$= \left[\frac{-2s}{(s^2+a^2)^2}\right]_0^t = \frac{-2t}{(t^2+a^2)^2}$$

$= L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$

$= \frac{1}{a}\int_0^t \sin a\tau \, d\tau$

$= \frac{-1}{a^2}\left(\cos a\tau\right)_0^t$

$= \frac{-1}{a^2}(\cos at - 1)$

$= \frac{1 - \cos at}{a^2}$

Q/

$y'' + 2y' + y = e^{-t} \qquad y(0) = -1, \quad y'(0) = 1$

$L(y'') + 2L(y') + L(y) = L(e^{-t})$

$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) + 2sL(y) - 2y(0) + L(y) = \frac{1}{s+1}$

$\Rightarrow s^2 L(y) + s - 1 = \frac{1}{s+1}$

$\Rightarrow s^2 L(y) = \frac{1}{s+1} - s + 1$

$\Rightarrow L(y) = \frac{1-s}{s^2(s+1)}$

$\Rightarrow y = L^{-1}\left(\frac{1-s}{s^2(s+1)}\right)$

$\Rightarrow L(y)(s^2+2s+1) + s - 1 + 2 = \frac{1}{s+1}$

$\Rightarrow L(y)(s^2+2s+1) = \frac{-1-s}{s+1}$

$\Rightarrow L(y) = \frac{-1-s}{(1+s)(s^2+2s+1)}$

$\Rightarrow y = L^{-1} \frac{-(1+s)}{(1+s)(s^2+2s+1)}$

$$\Rightarrow y = L^{-1} \frac{-(1+s)}{(1+s)\,(s^2+2s+1)}$$

$$e = L^{-1} \frac{-1}{(s+1)^2}$$

$$= -e^{-t} \cdot t$$

$$\Rightarrow L(y) = \frac{1}{s+1} - (s+1)$$

$$\Rightarrow L(y) = \frac{1-(s+1)^2}{s+1\,(s^2+2s+1)}$$

$$\Rightarrow y = L^{-1}\left(\frac{1}{(s+1)\,(s+1)^2}\right) - L^{-1}\frac{1}{s+1}$$

$$= L^{-1}\frac{1}{(s+1)^3} - L^{-1}\frac{1}{s+1}$$

$$= \frac{t^2}{2}e^{-t} - e^{-t}$$

$$= e^{-t}\left(\frac{t^2}{2} - 1\right)$$

6/

$y' + 3y = 10\sin t \qquad y(0) = 0$

$L(y') + 3L(y) = 10L(\sin t)$

$\Rightarrow sL(y) - y(0) + 3L(y) = \frac{10}{s^2+1}$

$\Rightarrow L(y)(s+3) = \frac{10}{1+s^2}$

$\Rightarrow y = L^{-1}\left(\frac{10}{(1+s^2)(s+3)}\right)$

$= L^{-1}\left(\frac{10}{s+3+s^3+3s^2}\right)$

$= 10 \times L^{-1} \frac{1}{s^3+3s^2+s+}$

$= 10 \times L^{-1} \frac{1}{(s+3)[(s+3)^2-(2\sqrt{2})^2]}$

$= 10 \times \int_0^t \frac{1}{2\sqrt{2}} e^{-3t} \sinh 2\sqrt{2}t \, dt$

$= 10 \times L^{-1}\left(\frac{1}{(1+s^2)(s+3)}\right)$

$= 10 \times L^{-1} \frac{s}{1+s^2} \frac{1}{(s+3)}$

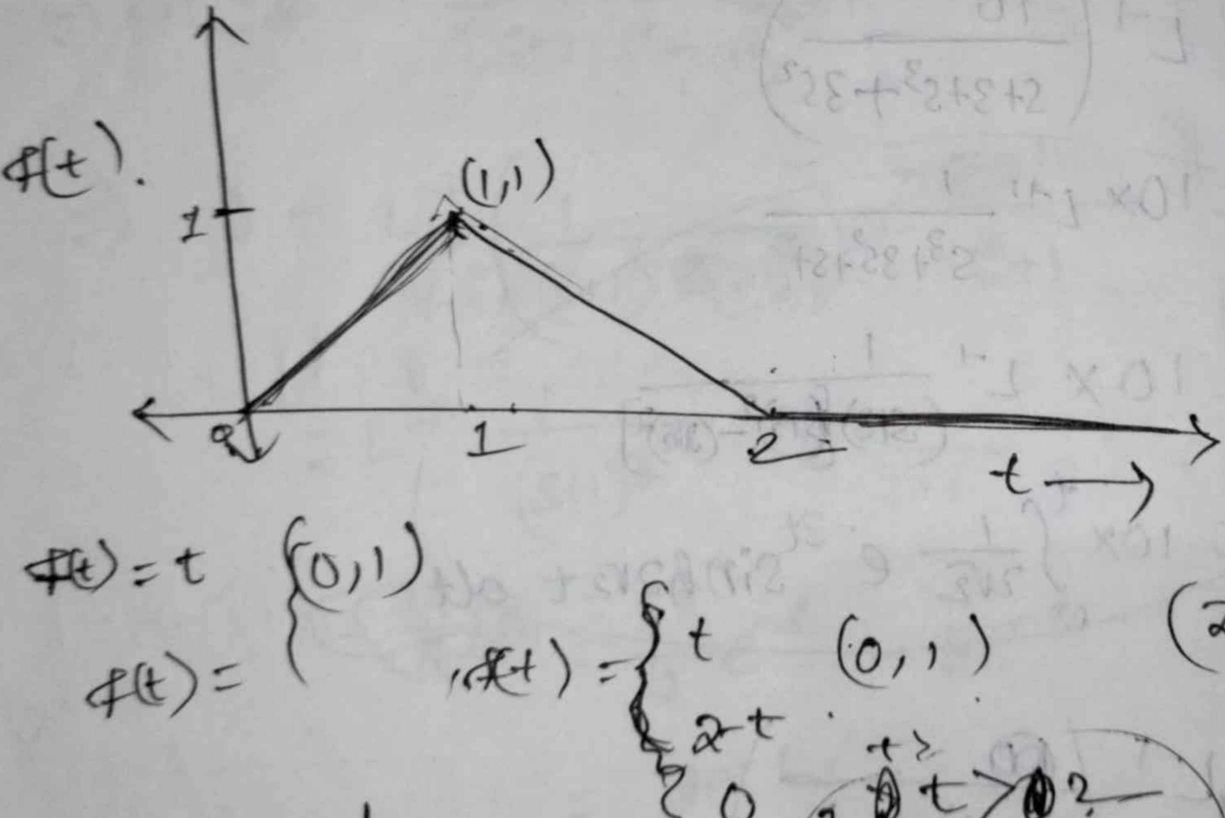
HW

① $y'' - y' - 2y = 0$, $y(0) = 8$, $y'(0) = 7$

② $L(f) = \frac{1}{s^2+4s}$ find $f(t)$

③ $y'' + y = 2\cos t$, $y(0) = 3$, $y'(0) = 4$.

Q. Find the L.T. of the following fig.



$f(t) = t \quad (0,1)$

$f(t) = $, $f(t) = \begin{cases} t & (0,1) \\ 2-t & \\ 0 & t > 2 \end{cases}$ $(2-t)$.

$L f(t) = \int_0^1 t\, e^{-st}\, dt + \int_1^2 (2-t) e^{-st}\, dt + \int_2^\infty 0\, e^{-st}\, dt$

$= \frac{t}{-s}\left[e^{-st}\right]_0^1 - \int_0^1 \frac{e^{-st}}{-s}\, dt$

$= \frac{-t}{s}\left(e^{-s} - 1\right) \left[\frac{e^{-st}}{s^2}\right]_0^1 = \frac{-(e^{-s}-1)t}{s} - \left(\frac{e^{-s}-1}{s^2}\right)$

$$= \int_1^2 (2-t)\, e^{-st}\, dt$$

$$= \frac{t-2}{s}\left(e^{-2s} - e^{-s}\right) - \left(\frac{e^{-2s}-1}{s^2}\right)$$

$$= \frac{-t(e^{s}-1)}{s} - \left(\frac{e^{-s}-1}{s^2}\right) + \frac{t-2}{s}\left(e^{-2s}-e^{-s}\right) + \left(\frac{e^{-2s}-1}{s^2}\right)$$

$$= \frac{e^{-s}-1}{s}\left((t-2)e^{-s} - t\right) + \cdot$$

Imp

Inverse.

Integral

ODE.

Q) Find the Laplace transform of $t^2 \sin 3t$.

$$L\left(t^2 \sin(3t)\right) = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) \qquad (s^2+9)^{-2}$$

$$= \frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right) =$$

$= -6 \frac{d}{ds} \frac{s}{(s^2+9)^2}$

$= -6 \frac{(s^2+9)^2 \times 1 - 2s \times (s^2+9) \times 2s}{(s^2+9)^4}$

$= \frac{-6 \left((s^2+9)^2 - 4s^2(s^2+9)\right)}{(s^2+9)^4}$

$= -6 \times \frac{(s^4 + 81 + 18s^2 - 4s^4 - 36s^2)}{(s^2+9)^4}$

$= -6 \left(\frac{-3s^4 - 18s^2 + 81}{(s^2+9)^4}\right)$

$= \frac{18s^4 + 108s^2 - 486}{(s^2+9)^4}$

Q// Find I.L.T. of $\frac{1}{s(s+1)}$.

$\frac{\frac{1}{s^2(s+1)}}{\frac{1}{s(s^2+a^2)}}$

(i) $$L^{-1}\left(\frac{1}{s(s+1)}\right) = \int_0^t e^{-t}\,dt = -\left[e^{-t}\right]_0^t$$

$$= -\left(e^{-t}-1\right)$$

$$= 1-e^{-t}.$$

(ii) $$L^{-1}\left(\frac{1}{s^2(s+1)}\right) = L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s(s+1)}\right) = \int_0^t \left(1-e^{-t}\right)dt$$

$$= t-\left(1-e^{-t}\right)$$

$$= t-1+e^{-t}$$

(iii) $$L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t \frac{1}{a}\sin at\,dt$$

$$= \frac{-1}{a^2}\left[\cos at\right]_0^t$$

$$= \frac{-1}{a^2}\left(\cos at-1\right) = \frac{(1-\cos at)}{a^2}$$

Convolution

$$F(s) = L\{f(t)\}$$

$$f(\tau) \sim f(t)$$

$$G(s) = L\{g(t)\}$$

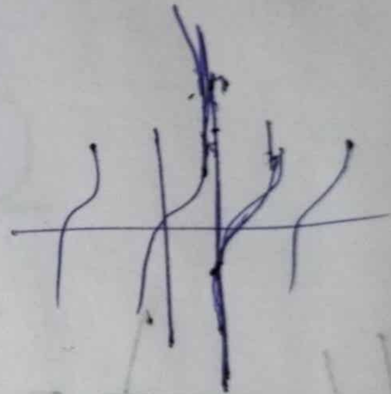
$$H(s) = F(s)\cdot G(s)$$

$$L^{-1}\{H(s)\} = h(t) = f(t) * g(t) = f*g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

Q//

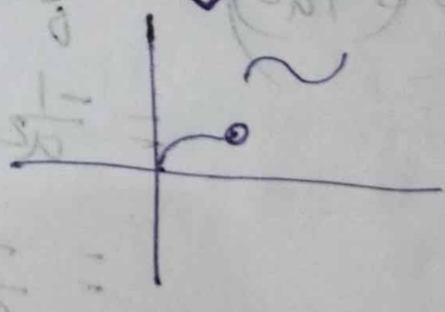
$$y(t) - \int_0^t y(\tau)\cdot\sin(t-\tau)\,d\tau = t$$

$$\Rightarrow Y - Y\cdot\frac{1}{s^2+1} = \frac{1}{s^2}$$



$$\Rightarrow Y\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2}$$

$$\Rightarrow Y\left(\frac{\cancel{1}+s^2\cancel{-1}}{s^2+1}\right) = \frac{1}{s^2}$$

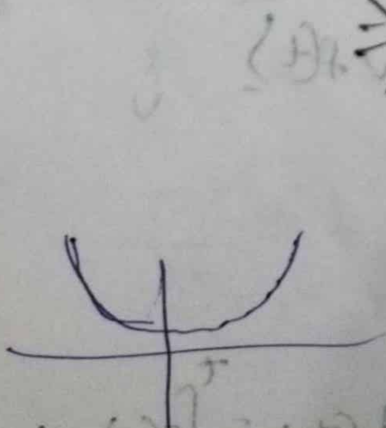


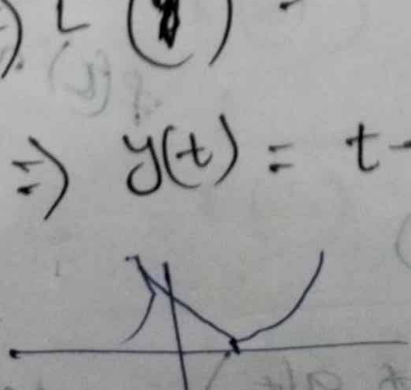
$$\Rightarrow Y = \frac{s^2+1}{s^4}$$

$$\Rightarrow Y = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\Rightarrow L^{-1}(Y) = L^{-1}\left(\frac{1}{s^2} + \frac{1}{s^4}\right)$$

$$\Rightarrow y(t) = t - \frac{1}{6t^3}$$





Q-1

Does the existence condition of Laplace transform is necessary or sufficient justify your answer.

⇒ Necessary not sufficient :->

For example $f(t) = \frac{1}{\sqrt{t}}$, clearly this func^n is not continuous at $t=0$. However we can find the L.T. of $f(t)$.

$f(t) = t^{-1/2}$.

$\frac{n!}{s^{n+1}}$

$L f(t) = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$

$\frac{\Gamma(n+1)}{s^{n+1}}$

Q-2

Does the L.T. of tan t and cot t exist? Justify your answer.

$\frac{1}{t^2-4}$ Dirichlet func^n

$$I_Q(t) = \begin{cases} 1, & t \in Q \\ 0, & t \notin Q. \end{cases}$$

Second Shifting Theorem (t-shifting theorem)

$$\boxed{\begin{array}{l} \text{If } [f(t)] = F(s) \text{, then} \\ L[[u(t-a)]\cdot[f(t-a)]] = e^{-as}F(s) \end{array}}$$

LHS

$$L[[u(t-a)]\cdot[f(t-a)]]$$

$$= \int_0^\infty e^{-st}\, u(t-a)\cdot f(t-a)\, dt$$

$$= \int_0^a e^{-st}\, \cancel{0\cdot f(t-a)\, dt} + \int_a^\infty e^{-st}\cdot 1\cdot f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Let $t-a = z$, $t = a+z$

$\Rightarrow dt = dz$

$t = a,\ z = 0$

$t = \infty,\ z = \infty$

$$= \int_0^\infty e^{-s(a+z)} f(z)\, dz$$

$$= \int_0^\infty e^{-sz}\cdot e^{-as} f(z)\, dz$$

$$= e^{-as}\int_0^\infty e^{-sz} f(z)\, dz$$

$$= e^{-as}F(s) = \text{RHS}$$

Proved

Ex PYQ

Find the L.T. of $f(t) = \begin{cases} 2 & 0 < t < 1 \\ t^2/2 & 1 < t < \pi/2 \\ \cos t & t > \pi/2 \end{cases}$

Sol^n

First, we can write the func^n $f(t)$ in terms of unit step func^n

$$f(t) = 2\left(u(t-0) - u(t-1)\right) + \frac{t^2}{2}\left(u(t-1) - u\left(t-\frac{\pi}{2}\right)\right) + \cos t \left(u\left(t-\frac{\pi}{2}\right)\right)$$

$$= \underbrace{2u(t) - 2u(t-1)}_{f_1(t)} + \underbrace{\frac{t^2}{2}u(t-1) - \frac{t^2}{2}u\left(t-\frac{\pi}{2}\right)}_{f_2(t)} + \underbrace{\cos t\, u\left(t-\frac{\pi}{2}\right)}_{f_3(t)}$$

$$f_1(t) = 2u(t) - 2u(t-1)$$

$$\Rightarrow L f_1(t) = L\left[2u(t) - 2u(t-1)\right]$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s}$$

$$f_2(t) = \frac{t^2}{2}u(t-1) - \frac{t^2}{2}u\left(t-\frac{\pi}{2}\right)$$

$$= \frac{(t-1+1)^2}{2}u(t-1) - \frac{\left(t-\frac{\pi}{2}+\frac{\pi}{2}\right)^2}{2}u\left(t-\frac{\pi}{2}\right)$$

$$= \frac{(t-1)^2+1+2(t-1)}{2}u(t-1) - \frac{\left(t-\frac{\pi}{2}\right)^2 + \frac{\pi^2}{4} + \pi\left(t-\frac{\pi}{2}\right)}{2}u\left(t-\frac{\pi}{2}\right)$$

$$L f_2(t) = \frac{1}{2}\Big[e^{-s}\frac{2}{s^3} + e^{-s}\frac{2}{s^2} + \frac{e^{-s}}{s},$$

$$- e^{-\pi/2 s}\frac{2}{s^3} - \frac{\pi e^{-\pi/2 s}}{s^2} - \frac{\pi^2}{4}\frac{e^{-\pi/2 s}}{s}\Big]$$

$$= \frac{1}{2}$$

$$2\ f_3(t) = \cos t . u(t-\pi/2)$$

$$= \cos(t-\pi/2+\pi/2) . u(t-\pi/2)$$

$$= \big[\cos(t-\pi/2) . \cancel{\cos\pi/2}^{\,0} - \sin(t-\pi/2)\sin\pi/2\big]\, u(t-\pi/2)$$

$$= -\sin\left(t-\frac{\pi}{2}\right) u(t-\pi/2)$$

$$= -e^{-\pi/2 s}\frac{1}{s^2+1} = \frac{-e^{-\pi/2 s}}{s^2+1}$$

Note

$$\boxed{L[f(t)u(t-a)] = e^{-as} L[f(t+a)]}$$

Find ILT of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s^2+2)^2}$$

find $L\, f(t)$

$$f(t) = \begin{cases} e^{t-a} & t > a \\ 0 & t < a \end{cases} = e^{t-a}\begin{cases} 1 & t > a \\ 0 & t < a. \end{cases}$$

$$\Rightarrow f(t) = e^{t-a}\, u(t-a)$$

$$\Rightarrow L(f(t)) = e^{-as} f(s)$$

$$= e^{-as} L(e^{t}).$$

$$= \frac{e^{-as}}{s-1}$$

$$f(t) = \begin{cases} \sin(t-\pi/6) & t > \pi/6 \\ 0 & t \geq \pi/6 \end{cases}$$

$$\Rightarrow f(t) = \sin(t-\pi/6)\begin{cases} 1 & t > \pi/6 \\ 0 & t \geq \pi/6 \end{cases}$$

$$\Rightarrow L(f(t)) = e^{-\pi/6\, s}\, L(\sin t)$$

$$= \frac{e^{-\pi s/6}}{s^2+1}$$

$$f(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$$

$\sin \pi t$

$$\cancel{= e^{-s} \frac{\pi}{\pi(s^2+\pi^2)}}$$

$$\cancel{= e^{-s} \sin}$$

1st

$$f(s)_1 = e^{-1s} \times \frac{\pi}{\pi(s^2+\pi^2)}$$

$$= \frac{\sin \pi(t-1)}{\pi} u(t-1)$$

2nd

$$f(s)_2 = \frac{\sin \pi(t-2)}{\pi} u(t-2)$$

3rd

$$f(s)_3 = \frac{e^{-3s}}{(s+2)^2} = e^{-3s}\frac{1}{(s+2)^2}$$

$$\cancel{= e^{-3s}\frac{1}{(s+3-1)^2}}$$

$$\cancel{= e^{-3s}\frac{1}{(s+3)^2+1-2(s+3)}}$$

$$= e^{-3s}\frac{1}{(s+2)^2}$$

$$= u(t-3)(t-3)e^{-2(t-3)}$$

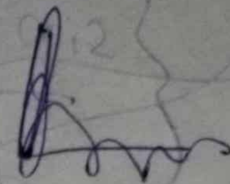
$$= u(t-3)(t-3)e^{6-2t}$$

$$f(t) = \begin{cases} \dfrac{\sin \pi(t-1)\, u(t-1)}{\pi} & 0<t<1 \\ \sin \pi(t-2)\, u(t-2) & 1<t<2 \end{cases}$$

$$f(t) = \begin{cases} 0 & 0<t<1 \\ \dfrac{\sin \pi(t-1)}{\pi} & 1<t<2 \\ \dfrac{\sin \pi(t-2)}{\pi} + \dfrac{\sin \pi(t-1)}{\pi} & 2<t<3 \\ e^{-2(t-3)}(t-3) & t>3 \end{cases}$$

$$= \begin{cases} 0 & 0<t<1 \\ \dfrac{\sin \pi(t-1)}{\pi} & 1<t<2 \\ \cancel{\sin \pi t} \xrightarrow{} 0 - \sin \pi t = 0 & 2<t<3 \\ (t-3)\, e^{-2(t-3)} & \end{cases}$$

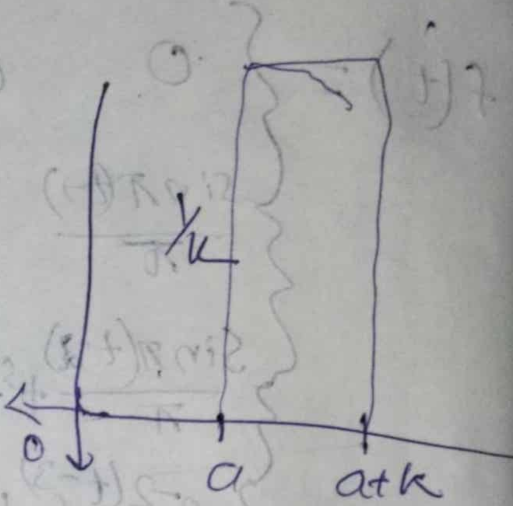
Dirac's delta function



Consider

$$f_k(t-a) = \begin{cases} 1/k & a \le t \le a+k \quad t \in [a, a+k]. \\ 0 & \text{otherwise } (-\infty, a) \cup (a+k, \infty) \end{cases}$$

k is +ve & small



$$S_\infty = \int_0^\infty f_k(t-a)\, dt$$

$$= 0 + 1 + 0$$

$$= 1$$

Area $= 1/k * k = 1$

$$\lim_{k \to 0} f_k(t-a) = \delta(t-a) \quad \leftarrow \text{Dirac's delta funct}^n.$$

$$\delta(t-a) = \begin{cases} \infty & a \le t \le a+k \quad \left(\frac{1}{k} = \frac{1}{0} = \infty\right) \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\int_0^\infty \delta(t-a)\, dt = \int_0^\infty \lim_{k \to 0} f_k(t-a)\, dt = \underline{1}}$$

Shifting property

For a continuous function $g(t)$

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$$

LHS

$$\int_0^\infty g(t) \lim_{k\to 0} f_k(t-a)\,dt$$

$$= \int_0^a \cdot \_\_\_ \,dt + \lim_{k\to 0} \int_a^{a+k} f_k(t-a)\,g(t)\,dt + \int \cdot \_\_\_ \,dt$$

$$= \lim_{k\to 0} \int_a^{a+k} \frac{1}{k}\,g(t)\,dt \,.$$

$$= \lim_{k\to 0} \frac{1}{k}\, G(a+k) - G(a) \,.$$

$$= \lim_{k\to 0} \frac{G(a+k) - G(a)}{k}$$

$$= G'(a) = g(a)$$

Assignment - Next class

$L(\delta(t-a)) = ?$

$$= \int_0^\infty e^{-st}\,\delta(t-a)\,dt$$

$$= \int_0^\infty e^{-st} \lim_{k\to 0} f_k(t-a)\,dt$$

$$= \lim_{k\to 0} \int_0^\infty e^{-st} f_k(t-a)\,dt$$

$$= \lim_{k\to 0} \int_a^{a+k} \frac{e^{-st}}{k}\,dt$$

$$= \lim_{k\to 0} \frac{1}{k}\,\frac{e^{-s(a+k)} - e^{-sa}}{-s}$$

$$= \lim_{k\to 0} \frac{-s e^{-s(a+k)}}{-s}$$

$$= e^{-as}$$

EX

$y'' + 3y' + 2y = u(t-1) - u(t-2)$ $y(0) = 0,\ y'(0) = 0.$

$L(y'' + 3y' + 2y) = L(u(t-1) - u(t-2))$

$\Rightarrow s^2 L(y) - s\cdot\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 3sL(y) - 3\cancel{y(0)}^{0} + 2L(y)$

$= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

$\Rightarrow L(y)(s^2 + 3s + 2) = \frac{e^{-s} - e^{-2s}}{s}$

$$\Rightarrow y = L^{-1}\left(\frac{e^{-s}-e^{-2s}}{s(s^2+3s+2)}\right)$$

$$= L^{-1}\left(\frac{e^{-s}-e^{-2s}}{s(s+2)(s+1)}\right)$$

$$= L^{-1}\left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right) e^{-s} - e^{-2s}$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$= \frac{A(s^2+3s+2) + B(s^2+2) + C(s^2+s)}{}$$

$$= L^{-1}\left(\frac{\frac{1}{2}e^{-s}}{s} - \frac{e^{-s}}{s+1} + \frac{\frac{1}{2}e^{-s}}{s+2} - \frac{\frac{1}{2}e^{-2s}}{s} + \frac{e^{-2s}}{s+1} - \frac{\frac{1}{2}e^{-2s}}{s+2}\right)$$

$A+B+C = 0$

$3A+C = 0$

$2A+2B = 1$

$\Rightarrow A+B = \frac{1}{2}$

$$= \frac{1}{2}u(t-1) - e^{-(t-1)} \cdot u(t-1) + \frac{1}{2} e^{-2(t-1)} u(t-1) - \frac{1}{2}u(t-2) + e^{-(t-2)}u(t-2)$$

$$L(f(t-a)\,u(t-a)) = e^{-as} f(s)$$

$$-\frac{1}{2}u(t-2)\,e^{-2(t-2)}$$

Q

$$L^{-1}\left(\frac{1}{S(s-a)}\right) \quad f(s)=\frac{1}{s},\ g(s)=\frac{1}{s-a}$$

$f(t)=1,\ g(t)=e^{at}$

$$h(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau \qquad g(t-\tau)=e^{a(t-\tau)}$$

$$=\int_0^t e^{a(t-\tau)}\,d\tau$$

$$=\frac{-1}{a}\left[e^{a(t-\tau)}\right]_0^t$$

$$=\frac{-1}{a}\left[e^{0}-e^{at}\right]$$

$$=\frac{-1}{a}\left(1-e^{at}\right)=\frac{1}{a}\left(e^{at}-1\right)$$

Q//

$$L^{-1}\frac{1}{(S^2+w^2)^2}=L^{-1}\frac{1}{(S^2+w^2)(S^2+w^2)}$$

$f(s)=\frac{1}{s^2+w^2}$

$f(t)=\frac{1}{w}\sin wt$

$g(t)=\frac{\sin w(t-\tau)}{w}$

$$=\frac{1}{w^2}\int_0^t \sin w\tau \sin w(t-\tau)\,d\tau$$

$$=\frac{1}{w^2}\,\frac{1}{2}\int_0^t \cos(w\tau-wt+w\tau)-\cos(w\tau+wt-w\tau)\,d\tau$$

$2\sin A\cdot\sin B=$

$$=\frac{1}{2w^2}\int_0^t \cancel{\cos w\tau\cos(2w\tau-w\tau)\,d\tau}$$

$\cos A\cos B+\sin A\sin B=\cos(A-B)-$

$-(\cos A\cos B+\sin A\sin B)\ \cos(A+B)$

$$= \frac{1}{2w^2}\int_0^t \cos(2w\tau - wt) - \cos wt \, d\tau$$

$$= \frac{1}{2w^2}\left[\frac{1}{2w}\Big[\sin(2w\tau - wt)\Big]_0^t - \Big[\tau\cos wt\Big]_0^t\right]$$

$$= \frac{1}{2w^2}\left[\frac{1}{2w}\sin wt - \sin wt -\right.$$

Q//

MODULE-2 Fourier Series

Properties of periodic function

① If $f(x)$ has period P it also has the period $2P$.... $3P$... nP.

$$f(x+nP) = f(x)$$

② If $f(x)$ & $g(x)$ have period 2π then $af(x)+bg(x)$ has period 2π. a, b = any constant.

③ . $\cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$. 2π

$a_0 \cos 0x + b_0 \sin 0x + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx$.

Convergent. Period $= 2\pi$

As the above series is convergent & period is 2π so, we can name it as $f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$

$$\Rightarrow \boxed{f(x) = a_0 + \sum_{1}^{\infty} a_n \cos nx + b_n \sin nx.}$$

(Period is 2π) Fourier series.

a_0, a_n, b_n are called fourier co-efficient

The convergency dependes upon a_0, a_n and b_n.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

Theorem

The trigonometry system is orthogonal on the interval $-\pi \le x \le \pi$.

i.e. integral of the product of any two trigonometric funcn over the interval is 0.

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0$$

$\left[\begin{array}{l} m, n \in \mathbb{Z} \\ \& \ m \neq n. \end{array}\right]$

$$\int_{-\pi}^{\pi} \sin mx . \cos nx\, dx = 0 \quad [m \neq n \text{ or } m = n]$$

Proof

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right) dx$$

$= 2\pi a_0 + 0 + 0$

$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$

$\Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx.$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$

$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} \underbrace{a_0\cos mx}_{\to 0} + \sum_{n=1}^{\infty} a_n\cos nx\cos mx + b_n \sin nx \cos mx$

$= \frac{a_0}{m}\underbrace{\left[\sin mx\right]_{-\pi}^{\pi}}_{\to 0} + a_n\int_{-\pi}^{\pi}\cos^2 nx\,dx + 0$

$= \frac{a_0}{m}\underbrace{\left(\sin m\pi + \sin m\pi\right)}_{\to 0} + a_n\int_{-\pi}^{\pi}\frac{1+\cos 2nx}{2}\,dx + 0$

$= a_n\left(\pi + \frac{1}{2}\times\frac{1}{2}\underbrace{\left(\sin 2n\pi + \sin 2n\pi\right)}_{0}\right) + 0$ $\qquad \frac{1}{2}(\pi+\pi)$

$= a_n\pi + 0$

$= a_n\pi$

$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} \cancel{a_0 \sin mx} + \sum_{n=1}^{\infty} \left(\cancel{a_n \cos nx \sin mx} + b_n \sin nx \sin mx \right) dx$$

$$= b_n \int_{-\pi}^{\pi} \frac{1-\cos 2mx}{2}\,dx$$

$$= b_n \pi$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Ex

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

$-k(0+\pi)$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= \frac{1}{2\pi}(-k\pi + k\pi)$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -k\cos nx\,dx \quad \frac{1}{\pi}\int_{0}^{\pi} k\sin nx\,dx$$

$$= \frac{-k}{\pi}\frac{1}{n}\cancel{\sin nx}^{0}$$

$$= \frac{k}{n\pi}\left[\cos nx\right]_0^{\pi} = \frac{-k}{n\pi}\cos n\pi - \cos 0$$

$$b_n = \frac{2k}{n\pi}(1-\cos n\pi)$$

$$\cos n\pi = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

$$1-\cos n\pi = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$f(x) = \cancelto{0}{a_0} + \sum_{n=1}^{\infty} \cancelto{0}{a_n \cos nx} + b_n \cos nx$$

$$= 0 + \sum_{n=1}^{\infty} \frac{2k}{n\pi}(1-\cos n\pi)\sin nx$$

$$= \sin nx \left[\frac{2k}{\pi}\times 2 + \frac{2k}{2\pi}\times 0 + \frac{2k}{3\pi}\times 2 + \frac{2k}{4\pi}\times 0 \dots\right]$$

$$= \sin nx \left[\frac{4k}{\pi} + \frac{4k}{3\pi} + \frac{4k}{5\pi} \dots\right]$$

$$= \frac{4k}{\pi}\sin nx \left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right)$$

$$= \frac{4k}{\pi}\sin nx \sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\right]$$

$= \frac{4k}{\pi} \sin x + \frac{4k}{3\pi} \sin 3x + \cdots$

$= \frac{4k}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots \right)$

$= \frac{4\pi}{\pi} \left[\frac{\sin(2n-1)x}{2n-1} \right]_{n=1}^{\infty}$

Fourier series of f(x) of any period p = 2L

Let f(x) be func^n of x defined in a interval (-L, L)

Let us choose a new variable

$v = \frac{\pi}{l} x$

x varies in (-l to l)

v varies in (-π to π)

$f(x) = f\left(\frac{l v}{\pi}\right) = F(v)$ Period is 2π

$F(v) = a_0 + \sum_{n=1}^{\infty} a_n \cos nv + b_n \sin nv$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(v)\, dv$ $\quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(v) \cos nv\, dv$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(v) \sin nv \, dv$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(v) \, dv$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{vL}{\pi}\right) dv$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, d\frac{\pi x}{L}$$

$\frac{l v}{\pi} = x$

$v \to -\pi, \; \frac{lx - \pi}{\pi}$

$$= \frac{1}{2\pi} \times \frac{\pi}{l} \int_{-l}^{l} f(x) \, dx$$

$$= \frac{1}{2l} \int_{-l}^{l} f(x) \, dx$$

Q. Assignment

Find the fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

Even funcⁿ — $x^2, \cos x$

Fourier series of even function

If a funcⁿ $f(x)$ is even in $(-\pi, \pi)$ then $b_n = 0$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

For period $2l$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$b_n = 0$.

Fourier cosine series.

If $f(x)$ is an odd func^n in interval $(-\pi, \pi)$.

Then $a_n = 0$ & $a_0 = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$a_0 = 0, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\text{odd}}{f(x)}\, \overset{\text{even}}{\cos nx}\, dx$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\text{odd}}{f(x)}\, \overset{\text{odd}}{\sin nx}\, dx \quad (\text{so even}).$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx\, dx$$

In case of period $2l$.

$$b_n = \frac{2}{l}\int_{0}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Ex

If $f(x) = x^2$. $\quad l = 2$

Find fourier series of this func^n.

$$a_0 = \frac{1}{2l}\int_0^l x^2\, dx$$

$$= \frac{1}{4} \times \frac{1}{3} l^{3}$$

$$= \frac{l^2}{3} = \frac{4}{3}$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos \frac{n\pi x}{l}\, dx$$

$$\cancel{= \frac{2}{l}\left[\frac{1}{3}[x^3]_0^l \cos \frac{n\pi x}{l} - \int_0^l\right.}$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\frac{x^2 l}{n\pi}\sin\frac{n\pi x}{l}\right]_0^l - \frac{2l}{n\pi}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left[\frac{l^3}{n\pi}\sin n\pi\right] - \frac{2l}{n\pi}\int_0^l x\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\frac{l^3}{n\pi}\sin n\pi - \frac{2l}{n\pi}\left[\left[x\sin\frac{n\pi x}{l}\right]_0^l + \cos\frac{n\pi x}{l}\right]_0^l\right.$$

$$= \frac{2}{2}\left[\frac{8}{n\pi}\sin n\pi - \frac{2\times 2}{n\pi}\left[l\sin\frac{n\pi}{l} + \cos\frac{n\pi}{l}\right]\right.$$

$$= \frac{8}{n\pi}\sin n\pi - \frac{4}{n\pi}(\sin n\pi + \cos n\pi)$$

$$= \frac{4}{n\pi}(2\sin n\pi - \cancel{2\sin n\pi} - \cos n\pi)$$

$$= -\frac{4}{n\pi}(\sin n\pi + \cos n\pi)$$

$$b_n = 0 \qquad = \frac{16}{n^2\pi^2}\cos n\pi$$

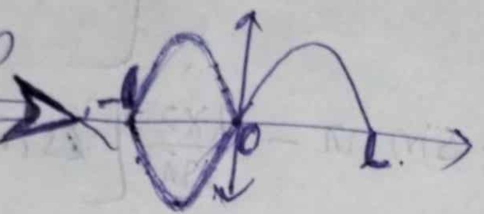
$$\frac{16}{n^2\pi^2}\cos n\pi = \begin{cases} \frac{-16}{n^2\pi^2} & \text{if } n \text{ is odd} \\ \frac{16}{n^2\pi^2} & \text{if } n \text{ is even.} \end{cases}$$

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2} \cos n\pi \cos\frac{n\pi x}{2}$$

$$\Rightarrow x^2 = \frac{4}{3} + \left(\frac{-16}{\pi^2} \cdot \cos\frac{\pi x}{2} + \frac{16}{2^2\pi^2} \cdot \cos \pi x - \frac{16}{3^2\pi^2} \cdot \cos\frac{3\pi x}{2}\right)$$

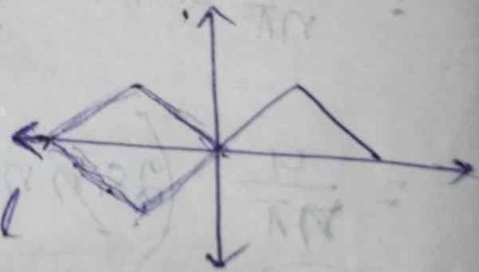
$$\Rightarrow x^2 = \frac{4}{3} - \frac{16}{\pi^2} \cos\frac{\pi x}{2} + \frac{16}{2^2\pi^2} \cos \pi x - \frac{16}{3^2\pi^2} \cos\frac{3\pi x}{2}$$

Half range expansion



Ex- Find two half range expansions of

$$f(x) = \begin{cases} \frac{2kx}{l} & \text{if } 0 < x < l/2 \\ \frac{2k(l-x)}{l} & \text{if } l/2 < x < l \end{cases}$$



- Even

$$a_0 = \frac{1}{l} \int_0^l f(x)\,dx$$

$$= \frac{1}{l}\left[\int_0^{l/2} \frac{2kx}{l}\,dx + \int_{l/2}^{l} \frac{2k(l-x)}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\frac{1}{2k} \times \frac{1}{2} \times \frac{l^2}{4} + \frac{1}{2k} \times \frac{1}{2} \times \frac{l^2}{4}\right]$$

$$= \frac{1}{l} \frac{l^3}{8k} = \frac{l^2}{8k}$$

Solution of PDE by separation of variable method

$U_{xx} - u = 0$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} - u = 0$

$\Rightarrow (D^2 - 1) u = 0$

$m = \pm 1$

$u = \overset{A(y)\ \text{const funcn}}{c_1} e^{x} + \overset{B(y)\ \text{const funcn}}{c_2} e^{-x}$

$\Rightarrow \varphi(x,y) = A(y) e^{x} + B(y) e^{-x}$.

Ex $U_{xy} = -U_x$

$\Rightarrow U_{xy} + U_x = 0$

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{-\partial u}{\partial x}$ $\qquad \frac{\partial u}{\partial x} = P$

$\Rightarrow \frac{\partial P}{\partial y} = -P$

$\Rightarrow \int \frac{\partial P}{-P} = \int \partial y$

$\Rightarrow \ln P = -y + c_1(x)$

$\Rightarrow P = e^{-y} c_2(x)$

$\Rightarrow \frac{\partial u}{\partial x} = e^{-y} c_2(x)$

$\Rightarrow \int \partial u = \int e^{-y} c_2(x)\, \partial x$

$\Rightarrow u = e^{-y}\int c_2(x)\,dx + c_3(y)$

$\Rightarrow \boxed{u = e^{-y} c_4(x) + c_3(y)}$

Ex

Let $u(x,t) = F(x)\cdot g(t)$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

Let $u(x,t) = F(x)\cdot G(t)$

So $\frac{\partial u}{\partial x} = G(t)F'(x)$, $\frac{\partial u}{\partial t} = F(x)\dot{G}(t)$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

$\Rightarrow G(t)F'(x) = 2F(x)\dot{G}(t) + F(x)\cdot G(t)$

$\Rightarrow G(t)F'(x) = F(x)\left(2\dot{G}(t) + G(t)\right)$

$$\Rightarrow \frac{F'(x)}{F(x)} = \frac{2\dot{G}(t) + G(t)}{G(t)} = k$$

So

$\frac{F'(x)}{F(x)} = k$

$\Rightarrow F'(x) = kF(x)$

$\Rightarrow \frac{df(x)}{dx} = kf(x)$

$\Rightarrow \frac{d\,f(x)}{f(x)} = k\,dx \Rightarrow \ln f(x) = kx$

$\Rightarrow \boxed{f(x) = C_1 e^{kx}}$

$$\frac{2\dot{G}(t) + G(t)}{G(t)} = k$$

$$2\frac{\dot{G}(t)+G(t)}{G(t)}=k$$

$$\Rightarrow \int \frac{\dot{G}(t)}{G(t)} = \int \frac{k-1}{2}$$

$$\Rightarrow G(t) = c_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$\boxed{u(x,t) = c_1 e^{kx} \cdot c_2 e^{\left(\frac{k-1}{2}\right)t}}$$

Let given $u(x,0) = 6e^{-3x}$

$t=0$,

$$u(x,0) = c_1 e^{kx} \cdot c_2 e^{\left(\frac{k-1}{2}\right)\times 0}$$

$$\Rightarrow u(x,0) = c_1 c_2 e^{kx}$$

$$\Rightarrow 6e^{-3x} = c_1 c_2 e^{kx}$$

$$\Rightarrow c_1 c_2 = 6 \qquad k = -3$$

$$\boxed{u(x,t) = 6e^{-3x} e^{-2t}}$$

Ans

Sol^n of one dimensional wave eq^n by separation of variable method

$x=0$ $\qquad\qquad$ $x=l$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{M}$$

$$u(0,t) = 0 = u(l,t) \quad \text{[Boundary cond}^n\text{]}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t} = 0, \text{ at } t=0$$

Let

$$u(x,t) = F(x)\, G(t).$$

$$\frac{\partial u}{\partial x} = F'(x)\, G(t), \quad \frac{\partial^2 u}{\partial x^2} = G(t)\, F''(x)$$

$$\frac{\partial^2 u}{\partial t^2} = F(x)\, \ddot{G}(t)$$

$y'' - k$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow F(x)\, \ddot{G}(t) = \frac{T}{M}\, G(t)\, F''(x)$$

$$\Rightarrow \frac{\ddot{G}(t)}{G(t)} = c^2 \frac{F''(x)}{F(x)} = K.$$

$\Rightarrow c^2 F''(x) - kF(x) = 0$

$\Rightarrow (c^2 D^2 - k) F(x) = 0$

$\Rightarrow c^2 D^2 - k = 0$

$\Rightarrow D = \frac{\sqrt{k}}{c}$

$F(x) = c_1 e^{\frac{\sqrt{k}}{c}x} + c_2 e^{-\frac{\sqrt{k}}{c}x}$

$\ddot{G}(t) - k\, G(t) = 0$

$\Rightarrow (D^2 - k)\, G(t) = 0$

$\Rightarrow D = \sqrt{k}$

$G(t) = c_3 e^{\sqrt{k}\,t} + c_4 e^{-\sqrt{k}\,t}$

Case - I

~~If $k = 0$~~

$F''(x) = 0$ $\qquad$ $\ddot{G}(t) = 0$

$F(x) = c_1 + c_2$

$= Cx$

If

$k = 0$

$G(t) = c_3 + c_4$

$= Ct$

$u = \text{const} = Cx + Ct$

If $k \to +ve$

Case - I

If $k = 0$

$F''(x) = 0$

$F(x) = c_1 x + c_2$

$\ddot{G}(t) = 0$

$G(t) = c_3 t + c_4$

$u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

Case-2

If $K = -ve = -p^2$ (Always negative)

$$\frac{c^2 F''(x)}{F(x)} = k = -p^2 \qquad\Big|\qquad \ddot{G}(t) + p^2 G(t) = 0$$

D^2

$D^2F(x)$

$$\Rightarrow F''(x) + \frac{p^2}{c^2} F(x) = 0 \qquad\Big|\qquad \Rightarrow (D^2 + p^2)\, G(t) = 0$$

$$\Rightarrow \left(D^2 + \frac{p^2}{c^2}\right) F(x) = 0 \qquad\Big|\qquad \Rightarrow D = \pm ip$$

$$\cancel{\Rightarrow \ddot{G}(t) + p^2 G(t) = 0}$$

$C_1 e^x$

$$\Rightarrow D = \sqrt{\frac{-p^2}{c^2}} = \pm \frac{p}{c} i$$

$$F(x) = C_5 \cos\frac{p}{c}x + C_6 \sin\frac{p}{c}x$$

$$G(t) = C_7 \cos pt + C_8 \sin pt$$

$$U(x,t) = \left(C_5 \cos\frac{p}{c}x + C_6 \sin\frac{p}{c}x\right)\left(C_7 \cos pt + C_8 \sin pt\right)$$

Case-3

$K > 0$, $K = p^2$ (Always +ve)

$$\frac{c^2 F''(x)}{F(x)} = p^2 \qquad\Big|\qquad \ddot{G}(t) - p^2 G(t) = 0$$

$$\Rightarrow D^2 - \frac{p^2}{c^2} = 0 \qquad\Big|\qquad D = \pm p$$

$$\Rightarrow D = \pm \frac{p}{c}$$

$$F(x) = C_9 e^{p/c\, x} + C_{10} e^{-p/c\, x} \qquad\Big|\qquad G(t) = C_{11} e^{pt} + C_{12} e^{-pt}$$

$$U(x,t) = \left(C_9 e^{p/c\, x} + C_{10} e^{-p/c\, x}\right)\left(C_{11} e^{pt} + C_{12} e^{-pt}\right)$$

Ya parse 2-3 pages nahina

Q. A tightly stretched flexible string has it's end point fixed at $x=0$ & $x=l$ $0 \le x \le l$ at $t=0$, the string has given a shape defined by $f(x) = \mu x(l-x)$, where μ is a constant & then relesed. Find the displacement at the string.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary cond^n $u(0,t) = u(l,t) = 0$

Initial cond^n. $u(x,0) = f(x) = \mu x(l-x)$

$u(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$

$u(0,t) = 0$ $\quad$ $u(l,t) = 0$

$u(0,t) = A \cdot (C\cos cpt + D\sin cpt) = 0$

$\Rightarrow A = 0.$

$u(l,t) = 0$

$\Rightarrow u(l,t) = (A\cos pl + B\sin pl)\underbrace{G(t)}_{\neq 0} = 0$

So, $\overset{0}{A}\cos pl + B\sin pl = 0$

$\Rightarrow B\sin pl = 0$

$\Rightarrow \sin pl = 0$

$\Rightarrow \sin pl = \sin n\pi$

$\Rightarrow pl = n\pi$

$\Rightarrow p = \frac{n\pi}{l}$

$u(x,t) = B\sin px\,(\cos cpt + D\sin cpt)$

$= \sin px\,(A_1 \cos cpt + B_1 \sin cpt)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{cn\pi}{l}t + B_n \sin\frac{cn\pi}{l}t\right)$$

$\because \frac{\partial u}{\partial t}(x,0) = 0$

$\Rightarrow$ ~~$\frac{\sin n\pi x}{l}\left(A_n \sin\frac{cn\pi}{l}t \times \frac{l}{cn\pi} + B_n \cos\frac{cn\pi}{l}t \times \ldots\right)$~~

$\Rightarrow$ ~~$\sin$~~

$$\Rightarrow \left[\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(-\frac{cn\pi}{l}A_n \sin\frac{cn\pi}{l}t + B_n\frac{cn\pi}{l}\cos\frac{cn\pi}{l}t\right)\right]_{t=0} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\cdot\frac{cn\pi}{l}B_n = 0$$

$\Rightarrow B_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{cn\pi}{l}t\right)$$

$u(x,0) = f(x) = \mu x(l-x)$

$$\Rightarrow \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = f(x)$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cdot X \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\mu}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= -x(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \int (l-2x)\times\frac{l}{n\pi}\cos\frac{n\pi x}{l}$$

$$= x(x-l)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l}{n\pi}\left[(l-2x)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - \frac{l}{n\pi}2\sin\frac{n\pi x}{l}\right]$$

$$\frac{2\mu}{l}\left[\frac{(x^2-lx)\,l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\,(l-2x)\sin\frac{n\pi x}{l} + 2\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2\mu}{l}\left(0 - \right.$$

$$\Rightarrow A_n = \frac{4\cdot\mu l^2}{n^3\pi^3}(1-\cos n\pi)$$

$$A_n = \begin{cases} \dfrac{8\mu l^2}{n^3\pi^3} & n = 1, 3, \ldots \\ 0 & n = 2, 4 \ldots \end{cases}$$

$$U(x,t) = \sum_{n=1}^{\infty} \frac{4\mu l^2}{n^3\pi^3}\cdot\sin\frac{n\pi x}{l}\cos\frac{cn\pi}{l}t.$$

$$= \frac{8\mu l^2}{\pi^3}\sin\frac{\pi x}{l}\cdot\cos\frac{c\pi t}{l}.$$

Sol^n of 1-D heat flow eq^n

C = diffusing constant.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = 0 \quad . \quad U(l,t) = 0$

Initial cond^n. $u(x,0) = f(x)$

Let $u(x,t) = F(x)\, G(t)$

$$\frac{\partial u}{\partial t} = F(x)\,\dot{G}(t) \qquad \frac{\partial^2 u}{\partial x^2} = F''(x)\, G(t)$$

$$F(x)\,\dot{G}(t) = c^2 F''(x)\, G(t) = k$$

$$\Rightarrow \frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

$$\Rightarrow F''(x) = k\, F(x)$$

$$\Rightarrow F''(x) - k\,F(x) = 0$$

$\dot{G}(t) - c^2 k\, G(t) = $

G_a

$D^2 - k = 0$

$\Rightarrow D = \sqrt{k}$

When $k = 0$

$F(x) = C_1 x + C_2 \quad | \quad G(t) = C_3$

$U(x,t) = (C_1 x + C_2)\, C_3$

Case-2 $k > 0$ $k = p^2$

$U(x,t) = \left(C_4 e^{px} + C_5 e^{-px}\right) C_6\, e^{c^2 p^2 t}$

$t \uparrow \; U(x,t) \uparrow$ so,

Case-3

$k < 0 \quad k = -p^2$

$U(x,t) = (C_7 \cos px + C_8 \sin px)\, C_9 e^{-p^2 ct}$

$\dot{G}(t) =$

$\dot{G}(t) + p^2c^2 \cdot G(t) = 0.$

$\Rightarrow D - p^2c^2 = 0$

$\Rightarrow D = p^2c^2$

$\frac{dG}{dt} = -p^2 ct$

$\Rightarrow \frac{dG}{\cancel{G}} = e^{-p^2 ct}$

$\Rightarrow G(t) = e^{\cdot}$

$\Rightarrow$ Initial cond^n find

~~Assignment~~

Q/ Find the temperature in a laterally insulated bar of length L whole end are kept at temp $0^\circ C$ assuming that the initial cond^n

$$f(x) = \begin{cases} x & \text{if } 0 < x < l/2 \\ l - x & l/2 < x < l \end{cases}$$