MATHEMATICS - III

L-1 (10.08.2022) (03.00-04.00 p.m)

Laplace Transform:

A function f(t) defined in $t>0$, by

$$\bar{f}(s) = F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

is called the Laplace transform of $f(t)$

Existence of Laplace Transform:

(i) $|f(t)| \le Me^{\alpha t}$

where M & α are constants

i.e. the function is to be bounded.

(ii) The function is piecewise continuous (in every sub-interval, to which the function may be divided, it is continuous)

{Qn. Write the existence of Laplace Transform?
or. which type of function has Laplace transform?}

e.g.: e^{t^2} is not bounded
hence, Laplace Transform doesn't exist

Inverse Laplace Transform:

$L\{f(t)\} = F(s)$

$\Rightarrow f(t) = L^{-1}\{F(s)\}$

Qn. Find the Laplace Transform for/of $\sin t$ & $\cos t$ from exponential function

$$L\{e^{it}\} = \frac{1}{s-i} = \frac{s+i}{(s-i)(s+i)} = \frac{s+i}{s^2+1} = \frac{s}{s^2+1} + i\frac{1}{s^2+1}$$

see rem(2)

$$\Rightarrow L\{\cos t\} = \frac{s}{s^2+1} \quad \& \quad L\{\sin t\} = \frac{1}{s^2+1}$$

Remember:

	$f(t)$	$L\{f(t)\} = F(s)$	$L^{-1}\{F(s)\}$
(1)	1	$1/s$	$L^{-1}\{1/s\} = 1$
(2)	e^{at}	$\frac{1}{s-a}$	$L^{-1}\{\frac{1}{s-a}\} = e^{at}$
(3)	$\cos at$	$\frac{s}{s^2+a^2}$	$L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
(4)	$\sin at$	$\frac{a}{s^2+a^2}$	$L^{-1}\{\frac{a}{s^2+a^2}\} = \sin at$
(5)	$\cosh at$	$\frac{s}{s^2-a^2}$	$L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$
(6)	$\sinh at$	$\frac{a}{s^2-a^2}$	$L^{-1}\{\frac{a}{s^2-a^2}\} = \sinh at$

* $L\{t^n\} = \frac{\Gamma_{n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}$ $\Rightarrow L\{t^{-1/2}\} = \frac{\Gamma_{1/2}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$ $(\because \Gamma_{1/2} = \pi)$

First Shifting Theorem:

$$L\{e^{at} f(t)\} = F(s-a)$$

e.g.: $L\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$, $L\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$

$$L\{e^{at}\sinh\omega t\} = \frac{\omega}{(s-a)^2-\omega^2}$$

Qn. Find the inverse Laplace Transform of $\frac{3s-137}{s^2+2s+401}$

Soln:
$$L^{-1}\left\{\frac{3s-137}{s^2+2s+401}\right\} = L^{-1}\left\{\frac{3s+3-140}{(s+1)^2+20^2}\right\}$$

$$= L^{-1}\left\{\frac{3(s+1)}{(s+1)^2+20^2} - 7\frac{20}{(s+1)^2+20^2}\right\}$$

$$= 3e^{-t}\cos 20t - 7e^{-t}\sin 20t$$ -- (Ans)

L-2 (17.08.2022) (02:00-03:00 pm)

Linearity of Laplace Transform:

Let $f(t)$ & $g(t)$ be any two functions whose Laplace transform exist, then for any two constants 'α' & 'β', then

$$L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)]$$

Qn. Prove that, $L\{\sinh\omega t\} = \frac{\omega}{s^2-\omega^2}$ using definition.

Soln:
$$\sinh\omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$$

$$\Rightarrow L\{\sinh\omega t\} = \int_0^\infty e^{-st}\left(\frac{e^{\omega t}-e^{-\omega t}}{2}\right)dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-\omega)t}dt - \int_0^\infty e^{-(s+\omega)}dt\right]$$

$$= \frac{1}{2}\cdot\frac{1}{s-\omega} + \frac{1}{2}\frac{1}{s+\omega}$$

$$= \frac{1}{2}\left[\frac{s+\omega+s-\omega}{s^2-\omega^2}\right] = \frac{1}{2}\frac{2s}{s^2-\omega^2} = \frac{s}{s^2-\omega^2}.$$ -- (proved)

Qn. Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 0, & 0 \le t \le 2 \\ K, & t \ge 2 \end{cases}$$

Solⁿ:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 0 \cdot dt + \int_2^\infty e^{-st} K\,dt$$

$$= K \int_2^\infty e^{-st}\,dt$$

$$= -\frac{K}{s}\left[e^{-st}\right]_2^\infty$$

$$= -\frac{K}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{K e^{-2s}}{s} \quad \text{.. (Ans)}$$

Qn. Find the Laplace transform of $\cos^2 at$.

Solⁿ:

$$L\{\cos^2 at\} = \int_0^\infty e^{-st} \frac{1+\cos 2at}{2}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}(1)\,dt + \frac{1}{2}\int_0^\infty e^{-st}\cos 2at\,dt$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+4a^2)} \quad \text{.. (Ans)}$$

Qn. Write the inverse Laplace transform of $\frac{2s+5}{s^2+25}$

Solⁿ:

$$L^{-1}\left\{\frac{2s+5}{s^2+25}\right\} = L^{-1}\left\{\frac{2s}{s^2+5^2}\right\} + L^{-1}\left\{\frac{5}{s^2+5^2}\right\}$$

$$= 2L^{-1}\left\{\frac{s}{s^2+5^2}\right\} + L^{-1}\left\{\frac{5}{s^2+5^2}\right\}$$

$$= 2\cos 5t + \sin 5t \quad \text{-- (Ans)}$$

Qn. Find $L^{-1}\left\{\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right\}$

Solⁿ: Let

$$\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)} = \frac{a}{(s-1)} + \frac{b}{(s-2)} + \frac{c}{(s+3)}$$

$$\Rightarrow 5s^2+3s-16 = a(s-2)(s+3) + b(s-1)(s+3) + c(s-1)(s-2)$$

For $s=2$, $5b = 20+6-16 = 10 \Rightarrow b = 2$

For $s=1$, $-4a = 5+3-16 \Rightarrow a = 2$

For $s=-3$, $20c = 45-9-16 \Rightarrow c = 1$

$$\Rightarrow \frac{5s^2+3s-16}{(s-1)(s-2)(s+3)} = \frac{2}{s-1} + \frac{2}{s-2} + \frac{1}{s+3}$$

$$\therefore L^{-1}\left[\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right] = L^{-1}\left[\frac{2}{s-1} + \frac{2}{s-2} + \frac{1}{s+3}\right]$$

$$= 2e^{t} + 2e^{2t} + e^{-3t} \quad \text{.. (Ans)}$$

Transforms of derivatives:

The transforms of the first and second derivatives of $f(t)$ satisfy

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$
$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$
$$\vdots$$
$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - \dots - f^{n-1}(0)$$

Example: Find the Laplace transform of $f(t) = t\sin\omega t$

Sol^n:

$f(t) = t\sin\omega t \quad , f(0) = 0$

$\Rightarrow f'(t) = \omega t\cos\omega t + \sin\omega t \quad , f'(0) = 0$

$\Rightarrow f''(t) = -t\omega^2\sin\omega t + 2\omega\cos\omega t$

$= 2\omega\cos\omega t - t\omega^2\sin\omega t$

$$L\{f''(t)\} = 2\omega\frac{s}{s^2+\omega^2} - \omega^2 L\{f(t)\}$$

$$\Rightarrow s^2L\{f(t)\} + \omega^2L\{f(t)\} = \frac{2\omega s}{s^2+\omega^2}$$

$$\Rightarrow L\{f(t)\} = \frac{2\omega s}{(s^2+\omega^2)^2} \quad \text{.. (Ans)}$$

Homework: Find Laplace transform using differentiation.

(a) $f(t) = t\cos 5t$

Sol^n:

$f(t) = t\cos 5t \quad , f(0) = 0$

$\Rightarrow f'(t) = -5t\sin 5t + \cos 5t \quad , f'(0) = 1$

$\Rightarrow f''(t) = (-25t\cos 5t) + 2(-5\sin 5t) = -25f(t) - 10\sin 5t$

$$L\{f''(t)\} = -25L\{f(t)\} - 10\frac{5}{s^2+5^2}$$

$$\Rightarrow s^2L\{f(t)\} + 25L\{f(t)\} - 1 = -\frac{50}{s^2+5^2} \quad \Rightarrow (s^2+5^2)L\{f(t)\} = 1 - \frac{50}{s^2+5^2}$$

$$\Rightarrow (s^2+5^2)L\{f(t)\} = \frac{-50}{(s^2+5^2)} + 1 = \frac{(s^2+5^2) - 2(5)^2}{(s^2+5^2)} = \frac{s^2-5^2}{s^2+5^2}$$

$$\Rightarrow L\{f(t)\} = \frac{s^2-5^2}{(s^2+5^2)^2} \quad \text{.. (Ans)}$$

(b) $f(t) = t e^{Kt}$

Sol^n: $f(t) = te^{Kt}$, $f(0) = 0$

$\Rightarrow f'(t) = Kte^{Kt} + e^{Kt}$, $f'(0) = 1$

$\Rightarrow f''(t) = K^2 t e^{Kt} + Ke^{Kt} + Ke^{Kt}$

$L\{f''(t)\} = K^2 L\{f(t)\} + 2K L\{e^{Kt}\}$

$\Rightarrow s^2 L\{f(t)\} - 1 = K^2 L\{f(t)\} + \frac{2K}{s-K}$

$\Rightarrow (s^2 - K^2) L\{f(t)\} = \frac{2K}{s-K} + 1$.

$\Rightarrow L\{f(t)\} = \frac{2K}{(s-K)(s^2-K^2)} + \frac{1}{s^2-K^2}$

$= \frac{2K + s - K}{(s-K)(s^2-K^2)} = \frac{s+K}{(s-K)(s^2-K^2)}$

$= \frac{(s+K)}{(s-K)(s+K)(s-K)}$

$\Rightarrow L\{f(t)\} = \frac{1}{(s-K)^2}$ . ..(Ans)

Imp

$L\{f(t)\} = F(s)$

$\Rightarrow L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

Similarly,

$L\{t.e^{Kt}\} = (-1) \frac{d}{ds} L\{e^{Kt}\}$

$= (-1) \frac{d}{ds} \left(\frac{1}{s-K}\right)$

$= (-1) \cdot \frac{-1}{(s-K)^2}$

$= \frac{1}{(s-K)^2}$ ..(Ans)

• L-3 (22.08.2022) (09-10 am)

Qn. Solve the initial value problem by Laplace Transform.

$y'' + 4y = 0$, $y(0) = 2.8$

Sol^n: Given that $y' + 4y = 0$

$sL(y) - y(0) + 4L(y) = 0$

$\Rightarrow (s+4)L(y) - 2.8 = 0$

$\Rightarrow L(y) = \frac{2.8}{s+4} = \frac{2.8}{s+4}$

$\Rightarrow y = L^{-1}\{2.8 \left(\frac{1}{s+4}\right)\}$

$= 2.8\, e^{-4t}$ .. (Ans)

Homework:

$y'' + 7y' + 12y = 21 e^{3t}$, $y(0) = 3.5$, $y'(0) = 0$

Qn $F(s) = \frac{1}{s^4 + s^2} = \frac{1}{s^2(s^2+1)} = \frac{as+b}{s^2} + \frac{cs+d}{s^2+1}$ (say)

where a, b, c & d are determinable constant

$\Rightarrow (s^2+1)(as+b) + s^2(cs+d) = 1$ . .

$$F(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t. \quad \text{.. (Ans)}$$

Homework

(i) $F(s) = \dfrac{2}{s^3+9s}$ (iii) $F(s) = \dfrac{1}{s^4+\pi^2 s^2}$

(ii) $F(s) = \dfrac{1}{s^4-4s^2}$

Unit step function: (Heaviside function) (u or H)

The unit step function is defined as.

$$u(t-a) = \begin{cases} 1 & , t \geq a \\ 0 & , t < a \end{cases}$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$
$$= \int_a^\infty e^{-st} dt$$
$$= \frac{e^{-as}}{s}.$$

Second shifting theorem: (we use unit step function)

If $f(t)$ has the transform $F(s)$, then the shifted function

$$g(t) = f(t-a)\, u(t-a) = \begin{cases} 0 & , t < a \\ f(t-a) & , t > a \end{cases}$$

has the Laplace transform $e^{-as} F(s)$.

i.e. the Laplace transform of $f(t-a)u(t-a)$

$$L\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$$

Ex: $f(t) = \begin{cases} \sin(t - \pi/3) & , t > \pi/3 \\ 0 & , t < \pi/3 \end{cases}$

$$= \sin(t - \pi/3) \begin{cases} 1 & , t > \pi/3 \\ 0 & , t < \pi/3 \end{cases} = \sin(t - \pi/3)\, \underbrace{u(t - \pi/3)}_{\text{unit step function}}$$

$L(\sin t) = \frac{1}{s^2+1}$

So, $L\{f(t)\} = L\{\sin(t-\pi/3)\, u(t-\pi/3)\}$

$$= e^{-\pi/3 s}\left(\frac{1}{s^2+1}\right) \quad \text{[Using second shifting theorem]}$$

.. (Ans.)

Qu $f(t) = \begin{cases} e^{t-a} & , t>a \\ 0 & , t<a \end{cases}$

Soln: $f(t) = e^{t-a} \begin{cases} 1 & , t>a \\ 0 & , t<a \end{cases}$

$$= e^{t-a}\, u(t-a)$$

$$\Rightarrow L\{f(t)\} = L\{e^{t-a}\, u(t-a)\}$$

$$= e^{-as}\, L\{e^{t}\}$$

$$= e^{-as}/(s-1) \quad \text{.. (Ans.)} \quad \left(\because L\{e^{t}\} = \frac{1}{s-1}\right)$$

Qn. Given $F(s) = e^{-\pi/3 s}\left(\frac{1}{s^2+1}\right)$

Find $L^{-1}\{F(s)\}$.

Soln: $f(t) = L^{-1}\left\{e^{-\pi/3 s}\left(\frac{1}{s^2+1}\right)\right\}$, $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

$$= \sin(t-\pi/3) \cdot u(t-\pi/3)$$

$$= \sin(t-\pi/3) \begin{cases} 1 & , t>\pi/3 \\ 0 & , t<\pi/3 \end{cases}$$

$$= \begin{cases} \sin(t-\pi/3) & , t>\pi/3 \\ 0 & , t<\pi/3 \end{cases}$$

.. (Ans.)

Dirac Delta Function:

Laplace transform of dirac delta function

$$L\{\delta(t-a)\} = e^{-as}$$

where $\delta(t-a) = \begin{cases} \infty & , t=a \\ 0 & , \text{otherwise} \end{cases}$

and $\int_0^\infty \delta(t-a)\,dt = 1$

$$\left[L^{-1}\left\{ \frac{F(s)}{s} \right\} = \int_0^t f(\ldots \right.$$

Example:

$y'' + y = \delta(t - 2\pi)$

where $y(0) = 10$ & $y'(0) = 0$

Sol^n:

$$s^2 L(y) - s y(0) - y'(0) + L(y) = e^{-2\pi s}$$

$$\Rightarrow (s^2+1) L(y) - 10s = e^{-2\pi s}$$

$$\Rightarrow (s^2+1) L(y) = e^{-2\pi s} + 10s$$

$$\Rightarrow L(y) = \frac{e^{-2\pi s} + 10s}{s^2+1}$$

$$= 10 \frac{s}{s^2+1} + e^{-2\pi s} \frac{1}{s^2+1}$$

$$\Rightarrow y = 10 L^{-1}\left\{ \frac{s}{s^2+1} \right\} + L^{-1}\left\{ e^{-2\pi s} \frac{1}{s^2+1} \right\}, \quad L^{-1}\left\{ \frac{1}{s^2+1} \right\} = \sin t$$

$$= 10 \cos t + \sin(t - 2\pi)\, u(t - 2\pi)$$ [Using second shifting t...

$$= 10 \cos t + \begin{cases} \sin(t-2\pi), & t > 2\pi \\ 0, & t < 2\pi \end{cases}$$

$$= \begin{cases} 10\cos t + \sin(t-2\pi), & t > 2\pi \\ 10 \cos t, & 0 < t < 2\pi \end{cases}$$ .. (Ans)

Imp

↳ Convulation Integral Equation:

Suppose $F(s)$ & $G(s)$ be two Laplace Transforms are given.

Let $f(t)$ & $g(t)$ be their corresponding Inverse Laplace Transform

then $h(t) = f * g$, called the convulation of f & g

$$= \int_0^t f(\tau)\, g(t - \tau)\, d\tau$$

Ex: Using convulation, find the inverse Laplace transform of

$$H(s) = \frac{1}{s(s-a)}$$

Sol^n:

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\{F(s) G(s)\}$$

$$= L^{-1}\left\{ \frac{1}{s}\left(\frac{1}{s-a} \right) \right\}$$

$$= 1 * e^{at}$$

$$= \int_0^t e^{a\tau}\, d\tau, \quad f(t) = e^{a\tau}, \ g(t-\tau) = 1$$

$$= \left[a^{-1} e^{a\tau} \right]_0^t$$

$$\Rightarrow h(t) = a^{-1} e^{at} - a^{-1} \cdot 1$$

$$= \frac{e^{at} - 1}{a}$$ .. (Ans)

$$L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+1)}\right\} \qquad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$= \sin t * \sin t$$

$$= \int_0^t \sin\tau . \sin(t-\tau)d\tau$$

$$= \int_0^t \sin\tau\ (\sin t\cos\tau - \cos t\sin\tau)\,d\tau$$

$$= \frac{\sin t}{2}\int_0^t \sin 2\tau - \cos t\int_0^t \sin^2\tau\, d\tau$$

$$= -\frac{\sin t}{2}\cdot\frac{1}{2}\cos 2\tau\Big]_0^t - \cos t\int_0^t \frac{1-\cos 2\tau}{2}d\tau$$

$$= \frac{\sin t}{4}\cos 2\tau\Big]_t^0 - \frac{\cos t}{2}\left[t - \frac{\sin 2\tau}{2}\right]_0^t$$

$$= \frac{\sin t}{4}(1-\cos 2t) - \frac{\cos t}{2}\left(t - \frac{\sin 2t}{2}\right)$$

$$= \frac{\sin t}{4} - \frac{\cos 2t\sin t}{4} - \frac{t\cos t}{2} + \frac{\cos t\sin 2t}{4}$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{\sin t}{4}$$

$$= \frac{1}{2}\left(\frac{\sin t}{2} - t\cos t\right). \qquad \text{..(Ans)}$$

Homework:

$$y'' + 7y' + 12y = 21e^{3t},\ y(0) = 3.5,\ y'(0) = 0$$

Sol:

$$s^2L(y) - s\,y(0) - \cancel{y'(0)}^{0} + 7sL(y) - 7y(0) + 12L(y) = 21\cdot\frac{1}{s-3}$$

$$\Rightarrow (s^2+7s)L(y) - s(3.5) - 7(3.5) + 12L(y) = \frac{21}{s-3}$$

$$\Rightarrow (s^2+7s+12)L(y) - (s+7)3.5 = \frac{21}{s-3}$$

$$\Rightarrow (s^2+7s+12)L(y) = \frac{21}{s-3} + 3.5(s+7) = \frac{21 + 3.5s(s+7) - 10.5(s+7)}{s-3}$$

$$\Rightarrow L(y) = \frac{3.5s^2 + 14.5s - 52.5}{(s-3)(s^2+7s+12)}$$

$$= \frac{3.5s^2 + 14.5s - 52.5}{(s-3)(s+3)(s+4)}$$

$\rightarrow$

$$F(s) = \frac{1}{s^4 - 4s^2} = \frac{4}{4s^2(s^2-4)} = \frac{1}{4}\left[\frac{1}{s^2-4} - \frac{1}{s^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s^2-4} - \frac{1}{s^2}\right\}$$

$$= \frac{1}{4}\left(\frac{\sinh 2t}{2} - t\right)$$

$$= \frac{\sinh 2t}{8} - \frac{t}{4}. \quad \text{.. (Ans)}$$

(iii) $$F(s) = \frac{1}{s^4 + \pi^2 s^2} = \frac{1}{s^2(s^2+\pi^2)} = \frac{s^2+\pi^2 - s^2}{\pi^2 s^2(s^2+\pi^2)}$$

$$= \frac{1}{\pi^2}\left[\frac{1}{s^2} - \frac{1}{s^2+\pi^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{\pi^2} L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+\pi^2}\right\}$$

$$= \frac{1}{\pi^2}\left[t - \frac{\sin \pi t}{\pi}\right]$$

$$= \frac{t}{\pi^2} - \frac{\sin \pi t}{\pi^3} \quad \text{.. (Ans)}$$

(c) $F(s) = \frac{2}{s^3+9s} = \frac{2}{s(s^2+9)} = 2\,\frac{1}{s}\cdot\frac{1}{s^2+9}$

$\therefore L^{-1}\{F(s)\} = 2\,L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+9}\right\}$

$= 2\left(1 * \frac{\sin 3t}{3}\right)$ [Using convolution theorem]

$= \frac{2}{3}\int_0^t \sin 3\tau\, d\tau$

$= -\frac{2}{9}\cos 3\tau\Big]_0^t$

$= -\frac{2}{9}(\cos 3t - 1)$

$= \frac{2}{9} - \frac{2}{9}\cos 3t.$ ... (Ans)

Homework ctd...

$$L(y) = \frac{3.5s^2+14.5s-52.5}{(s-3)(s+3)(s+4)} = \frac{a}{s-3} + \frac{b}{s+3} + \frac{c}{s+4}$$

$\Rightarrow 3.5s^2+14.5s-52.5 = a(s+3)(s+4) + b(s-3)(s+4) + c(s-3)(s+3)$

For $s=3$, $3.5(9)+14.5(3)-52.5 = a(6)(7)$

$\Rightarrow 42a = 22.5$

$\Rightarrow a = 0.54$

For $s=-3$, $3.5(9)+14.5(-3)-52.5 = b(-6)$

$\Rightarrow -6b = -64.5$

$\Rightarrow b = 10.75$

For $s=-4$, $3.5(16)+14.5(-4)-52.5 = c(-7)(-1)$

$\Rightarrow 7c = -54.5$

$\Rightarrow c = -7.79$

$L(y) = 0.54\frac{1}{s-3} + 10.75\frac{1}{s+3} - 7.79\frac{1}{s+4}$

$\Rightarrow y = 0.54\,L^{-1}\left\{\frac{1}{s-3}\right\} + 10.75\,L^{-1}\left\{\frac{1}{s+3}\right\} + (-7.79)\,L^{-1}\left\{\frac{1}{s+4}\right\}$

$= 0.54e^{3t} + 10.75e^{-3t} - 7.79e^{-4t}.$ ...(Ans)

$0.54 \simeq 0.5357142857$, $7.79 \simeq 7.7857142857$

L-4
(28.09
(07.09

Transform of integrals:

Let $F(s)$ denote the transform of a function $f(t)$, which is piecewise continuous for $t>0$, then we can write

$$L \int_0^t f(\tau)d\tau = \frac{1}{s} F(s)$$

$$\Rightarrow \int_0^t f(\tau)d\tau = L^{-1}\left\{\frac{1}{s}.F(s)\right\}$$

(i) Multiplication of 't'

$$L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

(ii) Division by 't'

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$$

Example: (a) $L\left\{te^{-t}\right\} = (-1)^1 \frac{d}{ds} F(s)$

where $f(t) = e^{-t}$, $F(s) = L\left\{e^{-t}\right\} = \frac{1}{s+1}$

$$\therefore \quad L\left\{te^{-t}\right\} = -\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

$$= -\left(-\frac{1}{(s+1)^2}\right) = \frac{1}{(s+1)^2} \quad \text{.. (Ans.)}$$

(b) $L\left\{t^2 \sin 3t\right\} = (-1)^2 \frac{d^2}{ds^2} L(\sin 3t)$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= 3 \frac{d^2}{ds^2}(s^2+9)^{-1}$$

$$= 3 \frac{d}{ds}\left\{-1(s^2+9)^{-2}\right\}.2s$$

$$= +6.2(s^3+9s)^{-3}(3s^2+9)$$

$$= 12(3s^2+9)/(s^3+9s)^3 \quad \text{.. (Ans.)}$$

(c) $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\left\{\sin t\right\}ds = \int_s^\infty \frac{1}{s^2+1}ds$

$$= \left.\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s. \quad \text{.. (Ans.)}$$

(d) $L\left\{\frac{1-e^{t}}{t}\right\} = \int_s^{\infty} L\{1-e^{t}\}ds$

$= \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s-1}\right)ds$

$= \int_s^{\infty}\frac{1}{s}ds - \int_s^{\infty}\frac{1}{s-1}ds$

$= \ln|s|\Big]_s^{\infty} - \ln|s-1|\Big]_s^{\infty}$

$= \ln\infty - \ln s - \ln\infty + \ln|s-1| \quad (\infty-1\to\infty)$

$= \ln\left|\frac{s-1}{s}\right|$. ..(Ans)

(e) Given $F(s) = \frac{6}{(s+1)^2}$, $f(t) = ?$

Sol^n: $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty}\frac{6}{(s+1)^2}ds$

$= -6\int_s^{\infty}\frac{-1}{(s+1)^2}ds$

$= \frac{-6}{s+1}\Big]_s^{\infty}$

$= \frac{6}{s+1}\Big]_{\infty}^{s} = \frac{6}{s+1}$

$\Rightarrow \frac{f(t)}{t} = L^{-1}\left\{\frac{6}{s+1}\right\} = 6e^{-t}$

$\Rightarrow f(t) = 6te^{-t}$. ..(Ans)

(f) $F(s) = \cot^{-1}\frac{s}{\pi}$, Find $f(t)$.

Sol^n: $L\{t f(t)\} = -\frac{d}{ds}\cot^{-1}\frac{s}{\pi}$

$= -\frac{d}{ds}\tan^{-1}\frac{\pi}{s}$

$= -\frac{1}{\frac{\pi^2}{s^2}+1}\left(\frac{-\pi}{s^2}\right) = \frac{-s^2}{s^2+\pi^2}\cdot\left(\frac{-\pi}{s^2}\right) = \frac{\pi}{s^2+\pi^2}$

$\Rightarrow t f(t) = +L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$= +L^{-1}\left\{\frac{s}{s}\cdot\frac{\pi}{s^2+\pi^2}\right\}$

$= +\sin\pi t \quad \Rightarrow f(t) = \frac{\sin\pi t}{t}$. ..(Ans)

Q. Solve the integral equation

$$y(t) - \int_0^t y(\tau)\cdot \sin(t-\tau)\,d\tau = t$$

Sol^n:

$$\Rightarrow y(t) - y(t) * \sin(t) = t$$

$$\Rightarrow L\{y(t)\} - L\{y(t)\}\cdot \frac{1}{s^2+1} = \frac{1}{s^2}$$

$$\Rightarrow \left(\frac{s^2+1-1}{s^2+1}\right) L\{y(t)\} = \frac{1}{s^2}$$

$$\Rightarrow L\{y(t)\} = \frac{s^2+1}{s^4}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{s^2+1}{s^4}\right\} = L^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^4}\right\}$$

$$= t + \frac{1}{3!}t^3$$

$$\Rightarrow y(t) = t + \frac{t^3}{6}. \quad \text{.. (Ans)}$$

Q. Given $H(s) = \dfrac{1}{(s-3)(s+5)}$, $h(t) = ?$

Sol^n:

$$h(t) = L^{-1}\left\{\frac{1}{(s-3)(s+5)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-3}\cdot\frac{1}{s+5}\right\}$$

$$= e^{3t} * e^{-5t}$$

$$= \int_0^t e^{3\tau}\cdot e^{-5(t-\tau)}\,d\tau$$

$$= \int_0^t e^{3\tau+5\tau-5t}\,d\tau$$

$$= \int_0^t e^{8\tau-5t}\,d\tau = \int_0^t e^{(8\tau-5t)}\,d\tau$$

$$= \frac{1}{8}\left. e^{(8\tau-5t)}\right]_0^t$$

$$= \frac{1}{8}\left[e^{8t-5t} - e^{0-5t}\right]$$

$$= \frac{1}{8}\left\{e^{3t} - e^{-5t}\right\}. \quad \text{.. (Ans)}$$

L-5 (05.09.2022) (09:00 - 10:00 a.m.)

F-Period
sin x → 2π
sin nx → $\frac{2\pi}{n}$

* Convergence
  - The function is piecewise continuous in every finite interval.
  - L.H.L. & R.H.L. exist within the boundary/interval.
  - The value of Fourier integral is the average of L.H.L. & R.H.L. at a point of discontinuity.

* Fourier integral:

f(x) can be represented as: $f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

Example: Find the Fourier integral of $f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$

Sol^n:

$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$

$= \frac{1}{\pi}\int_{-1}^{1} \cos\omega v\, dv$

$= \frac{1}{\pi\omega} \sin\omega v \Big]_{-1}^{1}$

$= \frac{1}{\pi\omega}[\sin\omega - \sin(-\omega)]$

$= \frac{2\sin\omega}{\pi\omega}$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

$= \frac{1}{\pi}\int_{-1}^{1} \sin\omega v\, dv$

$= -\frac{1}{\omega\pi}[\cos\omega]_{-1}^{1}$

$= 0$

The Fourier integral will be

$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$

$= \int_0^{\infty} \frac{2\sin\omega}{\pi\omega}\cos\omega x\, d\omega$

$= \frac{2}{\pi}\int_0^{\infty} \frac{\sin\omega}{\omega}\cos\omega x\, d\omega = \begin{cases} 1, & |x|<1 \\ 1/2, & |x|=1 \\ 0, & |x|>1 \end{cases}$

$\int_0^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}$

x = '1' is the point of discontinuity.

L.H.L. ← $\lim_{|x|\to 1^-} f(x) = 1$, $\lim_{|x|\to 1^+} f(x) = 0$ → R.H.L.

At $|x| = 1$, the Fourier integral $= \frac{1+0}{2} = \frac{1}{2}$

Ex:2 $\displaystyle\int \frac{\cos \omega x \; \omega \sin \omega x}{1+\omega^2} d\omega = \begin{cases} 0 & , x<0 \\ \pi/2 & , x=0 \\ \pi e^{-x} & , x>0 \end{cases}$

Sol^n:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos \omega v \, dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} 0 \cos \omega v \, dv + \frac{1}{\pi}\int_{0}^{\infty} \pi e^{-v} \cos \omega v \, dv$$

$$= \int_{0}^{\infty} e^{-v} \cos \omega v \, dv = I_1 \text{ (say)}$$

$$= \left(\int_0^{\infty} e^{-v} dv\right) \cos \omega v - \int_0^{\infty} \frac{d}{dv}\cos \omega v \int e^{-v} dv . \, dv$$

$$= -e^{-v}\cos \omega v\Big]_0^{\infty} + \int_0^{\infty} \omega \sin \omega v(-e^{-v}) dv$$

$$= -e^{-v}\cos \omega v\Big]_0^{\infty} - \omega \sin \omega v(-e^{-v})\Big]_0^{\infty} + \int_0^{\infty} \omega^2 \cos \omega v(-e^{-v}) dv$$

$$= +1 + (-\omega^2) I_1$$

$$\Rightarrow (1+\omega^2) I_1 = +1$$

$$\Rightarrow I_1 = \frac{+1}{1+\omega^2} \Rightarrow A(\omega) = \frac{1}{1+\omega^2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin \omega v \, dv$$

$$= \frac{1}{\pi}\int_0^{\infty} \pi e^{-v} \sin \omega v \, dv$$

$$= \int_0^{\infty} e^{-v} \sin \omega v \, dv = I_2 \text{ (say)}$$

$$= -\sin \omega v \, e^{-v}\Big]_0^{\infty} + \int_0^{\infty} \omega \cos \omega v \, e^{-v} dv$$

$$= -(0-0) + \omega \cos \omega v(-e^{-v})\Big]_0^{\infty} - (+\omega^2)\int_0^{\infty} \sin \omega v \, e^{-v} dv$$

$$\Rightarrow I_2 = \omega(0+1) + (-\omega^2) I_2$$

$$\Rightarrow (1+\omega^2) I_2 = \omega$$

$$\Rightarrow I_2 = \frac{\omega}{1+\omega^2} \Rightarrow B(\omega) = \frac{\omega}{1+\omega^2}$$

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw$$

$$= \int_0^\infty \left[\frac{1}{1+w^2}\cos wx + \frac{w}{1+w^2}\sin wx\right] dw$$

$$= \int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw$$

$x=0$ is the point of discontinuity.

$$\lim_{x\to 0^-} f(x) = 0 \quad , \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \pi e^{-x} = \pi$$

$\therefore$ At $x=0$, Fourier integral $= \frac{\pi+0}{2} = \frac{\pi}{2}$. .. (proved)

* Fourier sine integral:

$$f(x) = \int_0^\infty B(w)\sin wx\,dw$$

where $B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv$

* Fourier cosine integral:

$$f(x) = \int_0^\infty A(w)\cos wx\,dw$$

Where $A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$

Remember

$$\int e^{av}\sin bv\,dv = \frac{e^{av}}{a^2+b^2}(a\sin bv - b\cos bv)$$

$$\int e^{av}\cos bv\,dv = \frac{e^{av}}{a^2+b^2}(a\cos bv + b\sin bv)$$

Ex:3 $f(x) = e^{-Kx}$. find Fourier integral.

Sol^n: $A(w) = \frac{2}{\pi}\int_0^\infty e^{-Kv}\cos wv\,dv$

$$= \frac{2}{\pi}\left[\left[-K e^{-Kv}\cos wv\right]_0^\infty + \left(-K\int_0^\infty w\sin wv\, e^{-Kv}\,dv\right)\right]$$

$$= \frac{2}{\pi}\left[+Ke^0\cos 0 - K\left\{ w\sin wv\,(-Ke^{-Kv})\Big]_0^\infty + Kw^2\int_0^\infty e^{-Kv}\cos wv\,dv\right\}\right.$$

$$\Rightarrow \int_0^\infty e^{-Kv}\cos wv\,dv = K - K^2w^2\int_0^\infty e^{-Kv}\cos wv\,dv = I \text{ (say)}$$

$$\Rightarrow (1+K^2w^2)I = K \Rightarrow I = \frac{K}{1+K^2w^2} \Rightarrow \frac{2}{\pi}\int_0^\infty e^{-Kv}\cos wv\,dv = \frac{2K}{\pi(1+K^2w^2)}$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty e^{-Kv}\cos\omega v\,dv = \frac{2}{\pi}\frac{e^{-Kv}}{(-K)^2+\omega^2}(-K\cos\omega v + \omega\sin\omega v)\Big]_0^\infty$$

$$= \frac{2}{\pi}\left(0 - \frac{1(-K\cos 0 + \omega\sin 0)}{K^2+\omega^2}\right)$$

$$= \frac{2}{\pi}\left(\frac{K}{K^2+\omega^2}\right) = \frac{2K}{\pi(K^2+\omega^2)}$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-Kv}\sin\omega v\,dv = \frac{2}{\pi}\frac{e^{-Kv}}{(-K)^2+\omega^2}(-K\sin\omega v - \omega\cos\omega v)\Big]_0^\infty$$

$$= \frac{2}{\pi}\left(0 - \frac{(-K\sin 0 - \omega\cos 0)}{K^2+\omega^2}\right)$$

$$= \frac{2\omega}{\pi(K^2+\omega^2)}$$

$$\therefore f(x) = \int_0^\infty (A(\omega)\cos\omega v + B(\omega)\sin\omega v)\,d\omega$$

$$= \int_0^\infty \left[\frac{2K}{\pi(K^2+\omega^2)}\cos\omega v + \frac{2\omega}{\pi(K^2+\omega^2)}\sin\omega v\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{K\cos\omega v + \omega\sin\omega v}{K^2+\omega^2}\,d\omega$$

- L-6 (06.09.2022) (04:00-05:00 p.m.)

Qu) Express $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ as a Fourier sine integral & evaluate $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda\,d\lambda$.

Sol^n:

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv$$

$$= \frac{2}{\pi}\int_0^\pi \sin\omega v\,dv$$

$$= -\frac{2}{\pi\omega}\cos\omega v\Big]_0^\pi$$

$$= -\frac{2}{\pi\omega}(\cos\pi\omega - 1)$$

$$= \frac{2}{\pi\omega}(1-\cos\pi\omega)$$

$$\therefore f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega$$

$$= \int_0^\infty \frac{2}{\pi\omega}(1-\cos\pi\omega)\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(1-\cos\pi\lambda)}{\lambda}\sin x\lambda\,d\lambda$$

$$\Rightarrow \int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \pi/2, & 0 < x < \pi \\ \pi/4, & x = \pi \\ 0, & x > \pi \end{cases}$$

At $x = \pi$, Fourier integral gives $\frac{\text{R.H.L}+\text{L.H.L}}{2} = \frac{0+1}{2} = \frac{1}{2}$. $\therefore f(\pi) = \frac{1}{2}$

Ans.

Ex) Show that $\int_0^\infty \frac{\sin w \cos wx \, dw}{w} = \begin{cases} \pi/2 & , x<1 \\ \pi/4 & , x=1 \\ 0 & , x>1 \end{cases}$

Sol^n:

For Fourier cosine integral,

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv$$

$$= \frac{2}{\pi}\int_0^1 \frac{\pi}{2} \cos wv \, dv$$

$$= \frac{1}{w}\sin wv \Big]_0^1$$

$$= \frac{\sin w}{w}$$

$$f(x) = \int_0^\infty A(w)\cos wx \, dw$$

$$= \int_0^\infty \frac{\sin w \cos wx}{w} dw$$

$x=1$ is the point of discontinuity.

$$\lim_{x\to 1^-} f(x) = \frac{\pi}{2}, \quad \lim_{x\to 1^+} f(x) = 0$$

$\therefore$ Fourier integral at $x=1$ is

$$\frac{0+\frac{\pi}{2}}{2} = \frac{\pi}{4}.$$

..(proved)

Ex-2 Find the Fourier cosine integral of $f(x) = \begin{cases} x^2 , & 0<x<1 \\ 0 , & x>1 \end{cases}$

Sol^n:

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv$$

$$= \frac{2}{\pi}\int_0^1 v^2 \cos wv \, dv$$

Let $I = \int v^2 \cos wv \, dv.$

$$= \frac{v^2}{w}\sin wv - \int 2v \cdot \frac{\sin wv}{w} dv$$

$$= \frac{v^2}{w}\sin wv + \frac{2v}{w^2}\cos wv + \int \frac{2}{w^2}(-\cos wv)\, dv$$

$$= \frac{v^2}{w}\sin wv + \frac{2v}{w^2}\cos wv - \frac{2}{w^3}\sin wv + C$$

$$\therefore \int_0^1 v^2 \cos wv \, dv = \frac{\sin w}{w} + \left(-\frac{2\sin w}{w^3}\right) + \frac{2}{w^2}\cos w$$

$$\therefore A(w) = \frac{2}{\pi}\left(\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2\sin w}{w^3}\right)$$

* **Fourier Transform:**

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

Inverse Fourier Transform: $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

Q.1 Find the Fourier Transform of $f(x) = \begin{cases} e^{-x}, & |x| < a \\ 0, & \text{otherwise} \end{cases}$

sol^n:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-x} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-(1+i\omega)x}\, dx$$

$$= \frac{-1}{\sqrt{2\pi}\,(1+i\omega)} \left. e^{-(1+i\omega)x} \right]_{-a}^{a}$$

$$= \frac{-e^{-(1+i\omega)a} + e^{(1+i\omega)a}}{\sqrt{2\pi}\,(1+i\omega)} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{2a(1+i\omega)} - 1}{(1+i\omega)\, e^{a(1+i\omega)}} \right]$$

Q.2 Find $F(\omega)$ when $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

sol^n:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} x^2 e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-(i\omega)^{-1} x^2 e^{-i\omega x} + \int 2x (i\omega)^{-1} e^{-i\omega x}\, dx \right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[-(i\omega)^{-1} x^2 e^{-i\omega x} + 2(i\omega)^{-1} x\, (-i\omega)^{-1} e^{-i\omega x} + 2(i^2\omega^2)^{-1} e^{-i\omega x}\, dx \right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[-(i\omega)^{-1} x^2 e^{-i\omega x} + (-2(i^2\omega^2)^{-1} x\, e^{-i\omega x}) + 2(i^2\omega^2)^{-1} (-i\omega)^{-1} e^{-i\omega x} \right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[-(i\omega)^{-1} x^2 e^{-i\omega x} + 2(\omega^2)^{-1} x\, e^{-i\omega x} + 2(i\omega^3)^{-1} e^{-i\omega x} \right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}}\left[-(i\omega)^{-1}e^{-i\omega} + 2(\omega^3)^{-1}e^{-i\omega} + 2(i\omega^3)^{-1}e^{-i\omega} - 2(i\omega^3)^{-1}\right]$$

$$= \frac{(2(i\omega^3)^{-1} + 2(\omega^2)^{-1} - (i\omega)^{-1})\, e^{-i\omega} - 2(i\omega^3)^{-1}}{\sqrt{2\pi}}$$

$$= \left\{\frac{2}{i\omega^3}e^{-i\omega} + \frac{2}{\omega^2}e^{-i\omega} - \frac{e^{-i\omega}}{i\omega} - \frac{2}{i\omega^3}\right\}\frac{1}{\sqrt{2\pi}}$$

$$= \frac{2}{\omega\sqrt{2\pi}}\left\{\left(\frac{1}{i\omega^2} + \frac{2}{\omega} - \frac{1}{i}\right)e^{-i\omega} - \frac{1}{i\omega^2}\right\} \quad \text{.. (Ans)}$$

& Linearity Property:

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

L-7 (12.09.2022) (09:00 -10:00 a.m.)

& Fourier cosine transform:

(Page No: 23)

Q. Find the fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Sol^n:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-i\omega x}\, dx$$

$$= \frac{-1}{\sqrt{2\pi}\, i\omega}\left. e^{-i\omega x}\right]_{-1}^{1}$$

$$= \frac{-1}{\sqrt{2\pi}\, i\omega}\left[e^{-i\omega x} - e^{i\omega x}\right]$$

$$= \frac{e^{i\omega x} - e^{-i\omega x}}{i\omega\sqrt{2\pi}} \quad \text{.. (Ans.)}$$

$$= \frac{2(\cos\omega x + i\sin\omega x) - 1}{i\omega\sqrt{2\pi}\,(\cos\omega x + i\sin\omega x)}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2\sin\omega}{\omega} = \sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega}\, e^{i\omega x}\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, e^{i\omega x}\, d\omega = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, e^{i\omega x}\, d\omega = \begin{cases} \pi & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

For $x = 0$,

$$\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \pi/2 .$$

.. (Ans)

* Fourier Cosine Transform:

$F_c[f(x)] = F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\omega x\, dx$ → Fourier cosine Transform

$F_c^{-1}(\omega) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos\omega x\, d\omega$ → Inverse of Fourier cosine Transform.

* Fourier sine Transform:

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\omega x\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega) \sin\omega x\, d\omega$$

Qn! Find Fourier Cosine and Sine transform of $f(x) = \begin{cases} K & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$

Soln:

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\omega x\, dx = \sqrt{\frac{2}{\pi}} \int_0^{a} K\cos\omega x\, dx = \sqrt{\frac{2}{\pi}}\, \frac{K}{\omega} \Big[\sin\omega x\Big]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{K \sin\omega a}{\omega} .$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} K \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{K}{\omega}\right) \cos\omega x \Big]_0^a$$

$$= -\sqrt{\frac{2}{\pi}} \frac{K(\cos\omega a - 1)}{\omega}$$

$$= \sqrt{\frac{2}{\pi}} \frac{K(1-\cos\omega a)}{\omega} \quad \text{.. (Ans)}$$

Qn 2 $f(x) = e^{-x}$. Find the fourier cosine Transform.

Sol^n:

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+\omega^2} (-\cos\omega x + \omega \sin(-x))\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-e^{-x}}{1+\omega^2} (\omega \sin x + \cos\omega x)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{+1}{1+\omega^2} \cdot 1\right) = \frac{\sqrt{2/\pi}}{1+\omega^2} \quad \text{.. (Ans.)}$$

Qn. 3 $f(x) = e^{-ax}$. Find the Fourier cosine Transform.

Sol^n:

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+\omega^2} [-a\cos\omega x + \omega \sin(-ax)]\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{a^2+\omega^2} (+1)\right)$$

$$= \frac{\sqrt{2/\pi}}{a^2+\omega^2} \quad \text{.. (Ans.)}$$

Homework

Find Fourier cosine Transform of $f(x) = \begin{cases} x^2, & \text{for } 0 < x < 1 \\ 0, & \text{for } x > 1 \end{cases}$

* The Linearity property of Fourier Transform:

$$F_c[\alpha f+\beta g] = \alpha F(f) + \beta F(g)$$

Proof:

$$F_c[\alpha f+\beta g] = \sqrt{\frac{2}{\pi}}\int_0^\infty [\alpha f+\beta g]\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty \alpha f\cos wx\,dx + \int_0^\infty \beta g\cos wx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\alpha\int_0^\infty f\cos wx\,dx + \beta\int_0^\infty g\cos wx\,dx\right]$$

$$= \alpha\sqrt{\frac{2}{\pi}}\int_0^\infty f\cos wx\,dx + \beta\sqrt{\frac{2}{\pi}}\int_0^\infty g\cos wx\,dx$$

$$= \alpha F(f) + \beta F(g).$$ ∴ (proved.)

* Cosine and sine Transform of derivatives:

If (i) $f(x)$ is continuous on $[0,\infty)$

(ii) $f'(x)$ is piecewise continuous function in every finite interval

(iii) $\lim_{x\to\infty} f(x) = 0$

Then, fourier cosine of $f'(x)$, $F_c[f'(x)] = wF_s[f(x)] - \sqrt{\frac{2}{\pi}}f(0)$ —(a)

• Fourier sine of $f'(x)$, $F_s[f'(x)] = -wF_c[f(x)]$ —(b)

proof (a) $F_c[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\cos wx.f(x)\right]_0^\infty - \int_0^\infty -w\sin wx\, f(x)\,dx\right]$$

$$= w\int_0^\infty \sqrt{\frac{2}{\pi}}f(x)\sin wx\,dx + \sqrt{\frac{2}{\pi}}\left.f(x)\cos wx\right]_0^\infty$$

$(\because \lim_{x\to\infty} f(x) = 0)$ $= wF_s[f(x)] - \frac{\sqrt{2}f(0)}{\sqrt{\pi}} = wF_s[f(x)] - \sqrt{\frac{2}{\pi}}f(0)$

∴ (proved)

proof (b) $F_s[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx$

$(\because \lim_{x\to\infty} f(x) = 0)$

$$= \sqrt{\frac{2}{\pi}}\left[\left.(f(x)\sin wx\right]_0^\infty - \int_0^\infty w\cos wx\, f(x)\,dx\right]$$

$$= -w\sqrt{\frac{2}{\pi}}\int f(x)\cos wx\,dx = -wF_c[f(x)].$$ ∴ (proved)

ex:) Find $F_c(e^{-ax}) = ?$ $F_c[f''(x)] = ?$

$F_c[f''(x)] = wF_s[f'(x)] - \sqrt{\frac{2}{\pi}}f'(0)$

$F_s[f''(x)] = -w F_c(f'(x))$

Sol^n:

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos wx \, dx$$

Convulation of Fourier Transforms:

The convulation of two functions $f(x)$ & $g(x)$ over the interval $(-\infty, \infty)$ is defined by

$$f * g = \int_{-\infty}^{\infty} f(u) g(x-u) \, du$$

The Fourier transform of the convulation of $f(x)$ & $g(x)$ is the product of their Fourier Transforms

i.e. $F[f(x) * g(x)] = F[f(x)] . F[g(x)] \sqrt{2\pi}$ (proof at p: 28)

Qu1 Find the Fourier Transform of $x e^{-x^2}$

Qu.2 Find the Fourier Transform of e^{-ax^2}

L-8 (13.09.2022) (04:00 - 05:00 p.m.)

$$F[f*g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp\; e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, e^{-i\omega x}\, dx\, dp$$ [changing the order of integration]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(q)\, e^{-i\omega(p+q)}\, dq\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{-i\omega p}\, dp \int_{-\infty}^{\infty} g(q)\, e^{-i\omega q}\, dq$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} F[f*g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{-i\omega p}\, dp \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(q)\, e^{-i\omega q}\, dq$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} F[f*g] = F[f] \cdot F[g]$$

$$\Rightarrow F[f*g] = \sqrt{2\pi}\, F[f]\, F[g]$$ .. (proved)

Properties of Fourier Transform:

(1) Linear property

(2) Change of scale property

If $F(s)$ is the fourier Transform of the function $f(x)$, then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad \text{if } F[f(x)] = F(s)$$

↳ This is also valid in Laplace Transform as:

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad \text{if } L\{f(t)\} = F(s)$$

(3) Shifting Property

If $F(s)$ is the Fourier Transform of the function $f(x)$, then

$$F\{f(x-a)\} = e^{isa} F(s)$$

(4) Parseval's identity of Fourier Transform

If $F(s)$ and $G(s)$ are the Fourier Transform of $f(x)$ & $g(x)$ then

complex conjugate

(i) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, \bar{G}(s)\, ds = \int_{-\infty}^{\infty} f(x)\, \bar{g}(x)\, dx$$

(ii) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2\, dx$$

* **Laplace Transform of Periodic Function:**

- $f(t+T) = f(t)$
- $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-st}}$

Probability:

- **Random variable:**

A real valued function X defined on a sample space S of a random experiment $X: S \to \mathbb{R}$ is called a random variable.

- **Event:**

The event consists of all outcomes for which $X = x$.

- **Probability Distribution:**

The probability distribution $f(x)$ of a random variable X is a description of the set of possible values of X along with the probability associated with each of possible values x

- **Cumulative Distributive Function:**

A cumulative distributive function for a random variable 'X' is defined by $F(x) = P(X \leq x)$ defined in the interval $(-\infty < x < \infty)$, $x \in$

→ **Properties of $F(x)$:**

- If $a<b$, probability of $a \leq X \leq b$,

$$P(a \leq X \leq b) = F(b) - F(a)$$

- $P(a \leq X \leq b) = P(X=a) + F(b) - F(a)$

Random variable
- Discrete Random variable
- Continuous Random variable

- A discrete random variable is a random variable with a finite range.
- If the range of a random variable 'X' is an interval of real numbers then 'X' is a continuous random variable.

→ According to probability distribution, random variable is divided into two types:

(a) Discrete probability distribution
(b) Continuous probability distribution

- **Probability Mass Function:**

If (i) $f(x) > 0$
& (ii) $\sum f(x) = 1$

are satisfied by a discrete probability function $f(x)$, then it is called a probability mass function.

→ **Properties of Discrete Probability Function:**

→ Mean:

$$\mu = \sum x f(x)$$

→ Variance:

$$\sigma^2 = \sum (x-\mu)^2 f(x)$$

$$\begin{aligned}
R.H.S. &= \sum (x-\mu)^2 f(x) \\
&= \sum (x^2+\mu^2-2x\mu) f(x) \\
&= \sum x^2 f(x) + \sum \mu^2 f(x) - 2\mu \sum x f(x) \\
&= \sum x^2 f(x) + \mu^2 . 1 - 2\mu . \mu \\
&= \sum x^2 f(x) - \mu^2 \\
&= \mu' - \mu^2, \quad \text{where } \mu' = \sum x^2 f(x)
\end{aligned}$$

Example

If 'X' is a random variable with $f(x) = Kx^2$, $x = 1, 2, 3$
Find mean & variance.

Solⁿ:

$$\begin{aligned}
\mu &= \sum x f(x) \\
&= \sum_{x=1}^{3} Kx^3 \\
&= K \sum^{3} x^3
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \sum (x-\mu)^2 f(x) \\
&= \sum x^2 f(x) - \mu^2 \\
&= \sum^{3} x^2 K x^2 - \mu^2
\end{aligned}$$

3)

- L-9 (14.09.2022) (02:00-03:00 p.m.)

Class Test

- L-10 (20.09.2022) (04:00-05:00 p.m.)

$E(x)$ is called the expectation of 'x', which is same as mean.

$$\mu = \sum x f(x) = E(x)$$

In discrete,

$$\sigma^2 = \sum x^2 f(x) - \mu^2$$

In continuous

$$\sigma^2 = \int_a^b x^2 f(x)\,dx - \mu^2 = E(x^2) - (E(x))^2$$

discrete	continuous
probability mass function	probability density function
$\sum$ (summation)	$\int$ (integration)
$\sum f(x) = 1$	$\int_a^b f(x)dx = 1$
$f(x) \geq 0$	$f(x) \geq 0$

Transformation of mean & variance:

→ If X is a random variable with mean μ_x and variance σ_x^2, then the random variable $Y = a_2 X + a_1$ has mean

$$\mu_y = a_1 + a_2 \mu_x$$
$$\sigma_y^2 = a_2^2 \sigma_x^2$$

→ The random variable $Z = \frac{X-\mu}{\sigma}$ has mean '0' & variance '1'.

standardized random variable corresponding 'X'.

Qu. If x is a random variable with density function $f(x) = 2x$, $0 \leq x \leq 1$. $Y = -4X + 5$. Find mean & variance of Y.

sol^n:

mean of X, $\mu_x = \int_0^1 x \cdot 2x\,dx$

$$= \frac{2}{3} x^3\Big]_0^1$$

$$= \frac{2}{3} = E(X)$$

variance of X, $\sigma_x^2 = \int_0^1 x^2 \cdot 2x\,dx - \left(\frac{2}{3}\right)^2$

$E(x^2) - (E(x))^2$

$$= \frac{1}{2} x^4\Big]_0^1 - \frac{4}{9}$$

$$= \frac{9-8}{18} = \frac{1}{18}$$

$$Y = a_2 X + a_1 = -4X + 5$$
$$\Rightarrow a_2 = -4 \quad \& \quad a_1 = 5$$

$$\therefore \mu_y = a_1 + a_2 \mu_x = 5 - 4 \cdot \frac{2}{3} = \frac{15-8}{3} = \frac{7}{3}.$$
$$\& \ \sigma_y^2 = a_2^2 \sigma_x^2 = 16 \cdot \frac{1}{18} = \frac{8}{9}.$$ -- (Ans.)

* Binomial Distribution:

If we perform a series of independent trials such that for each trial, 'p' is the probability of success and 'q' that of a failure, then the probability of 'r' success in 'n' trials (we can write as)

$$^nC_r \, p^r q^{n-r}$$

$f(x) = {}^nC_x p^x q^{n-x}$ (Binomial distribution).

Then, mean $\mu = \sum_{x=0}^{n} x f(x)$

$$= \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x({}^nC_x) p^x q^{n-x}$$

$$^nC_x = \frac{n!}{x!(n-x)!}$$

$$^nC_x = \frac{n}{x} \, {}^{n-1}C_{x-1}$$

$$= \sum_{x=1}^{n} x \, \frac{n}{x} \, {}^{n-1}C_{x-1} \, p \cdot p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \, (p+q)^{n-1}$$ — Binomial $\exp^n$ $[\because p+q=1]$

$$\Rightarrow \boxed{\mu = np}$$

$$\sigma^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \sum_{x=0}^{n} x^2 f(x) = \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} [x(x-1) + x] \, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x(x-1) \, {}^nC_x p^x q^{n-x} + \sum_{x=1}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1)\, {}^{n}C_{x}\, p^{x} q^{n-x} + \mu$$

$$= \sum_{x=2}^{n} \cancel{x(x-1)} \frac{n}{\cancel{x}} \cdot \frac{n-1}{\cancel{(x-1)}}\, {}^{n-2}C_{x-2}\, p^{2} p^{x-2} \cdot q^{(n-2)-(x-2)} + \mu$$

$$= n(n-1)p^{2} \underbrace{\sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{(n-2)-(x-2)}}_{B.E.} + \mu$$

$$= n(n-1)p^{2} \cdot [p+q]^{n-2} + \mu \qquad [\because p+q=1]$$

$$= n(n-1)p^{2} + \mu$$

$$\therefore \sigma^{2} = E(x^{2}) - (E(x))^{2}$$

$$= n(n-1)p^{2} + np - n^{2}p^{2}$$

$$= np - np^{2}$$

$$= np(1-p)$$

$$\Rightarrow \boxed{\sigma^{2} = npq}$$

$$\sigma = \sqrt{npq}$$

L-11 (26.09.2022) (09:00-10:00 a.m.)

Ex: Sum of faces in single throw of two dice then find its probability distribution.

X_i :	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$:	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

1,1   1,2   1,3   1,4   1,5   1,6

2,1   2,2   2,3   2,4   2,5   2,6

3,1   3,2   3,3   3,4   3,5   3,6

4,1   4,2   4,3   4,4   4,5   4,6

5,1   5,2   5,3   5,4   5,5   5,6

6,1   6,2   6,3   6,4   6,5   6,6

$F(x)$:	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{15}{36}$	$\frac{7}{12}$	$\frac{13}{18}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{35}{36}$	1

$$P(7 < x \le 11) = F(11) - F(7) = \frac{35}{36} - \frac{7}{12} = \frac{14}{36} = \frac{7}{18}$$

$$P(4 < x \le 9) = F(9) - F(4)$$

Cumulative Distribution:

If $F(x)$ is the sum of all possibilities from the beginning to 'x' i.e.

$$F(x) = \sum f(x_i) \quad , x_i \leq x$$

then $F(x)$ is called cumulative distribution function.

Q.2 A bag contains four white and six red balls. 2 balls chosen randomly. X is a random variable, which says no white ball. Design a mass function and find mean & variance.

Qn.3 The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability

(i) exactly 2 will be defective

(ii) atleast 2 will be defective

(iii) none will be defective.

Q.4 Let X be a continuous random variable with density $f(x) = Kx^2 e^{-x}$, $x \geq$

Find mean and variance.

Sol^n(4) The given density function is $f(x) = Kx^2 e^{-x}$, $x \geq 0$

$\therefore$ Mean, $\mu = \int_0^\infty x K x^2 e^{-x} dx$

$= K \int_0^\infty x^3 e^{-x} dx$

$= \frac{1}{2} \left[\cancel{\left[-x^3 e^{-x}\right]_0^\infty} + \int_0^\infty 3x^2 e^{-x} dx \right]$

$= \frac{3}{2} \int_0^\infty x^2 e^{-x} dx$

$= \frac{3}{2} \times 2 = 3 \quad = E(x)$

$\int_0^\infty f(x)\, dx = 1$ (prop.)

$\Rightarrow K \int_0^\infty x^2 e^{-x} dx = 1$

$\Rightarrow K\left[\cancel{\left[-x^2 e^{-x}\right]_0^\infty} + 2\int_0^\infty x e^{-x} dx \right] = 1$

$\Rightarrow 2K\left[\cancel{\left(-x e^{-x}\right)_0^\infty} + \int_0^\infty e^{-x} dx \right] = 1$

$\Rightarrow 2K = 1$

$\Rightarrow K = \frac{1}{2}$

$$\boxed{\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!}$$

$\therefore$ variance, $\sigma^2 = E(x^2) - (E(x))^2$

$E(x^2) = \int_0^\infty x^2 f(x)\, dx$

$= K \int_0^\infty x^4 e^{-x} dx$

$= \frac{1}{2} \cdot \overbrace{4 \cdot 3 \cdot 2}^{\Gamma(4+1)} = 12$

$\therefore \sigma^2 = 12 - 3^2$

$= 12 - 9$

$= 3.$

$\underline{\text{sol}^n}$ (2) Total number of balls = $\overset{\curvearrowright 4W}{4} + \overset{\to 6R}{6} = 10$

Total no. of ways of choosing two balls = ${}^{10}C_2$

Total possible set of no white ball = ${}^{6}C_2$

$$\therefore P(2R) = \frac{{}^6C_2}{{}^{10}C_2} = \frac{6!}{4!\,2!} \Big/ \frac{10!}{8!\,2!}$$

$$= \frac{6\times 5}{10\times 9} = \frac{1}{3}.$$

$\underline{\text{sol}^n}$ (3) $n = 12$,

$$p = \frac{1}{10}, \quad q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}.$$

(i) $P(x=2) = {}^nC_x\, p^x\, q^{n-x}$

$$= {}^{12}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{10}$$

$$= \frac{12!}{10!\,2!}(0.01)(0.3487)$$

$$= \frac{\cancel{12}^{6}\times 11}{\cancel{2}} = 0.23$$

(iii) $P(0) = {}^{12}C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{12}$

$$= 0.282$$

(ii) $P(x \geq 2) = 1 - P(x<2)$

$$= 1 - [P(0) + P(1)]$$

$$= 1 - [\,{}^{12}C_0\, p^0 q^{12} + {}^{12}C_1\, p^1 q^{11}]$$

$$= 1 - \left[\left(\frac{9}{10}\right)^{12} + 12 \cdot \frac{1}{10}\cdot\left(\frac{9}{10}\right)^{11}\right]$$

$$= 1 - \left(\frac{9}{10}\right)^{11}\left[\frac{9+12}{10}\right] = 1 - (0.3138 \times 2.1) = 0.341$$

L-12 (27.09.2022) (04:00-05:00 p.m)

Distribution function:

Let 'X' be a random variable, the function 'F' defined for all real 'x' i.e.

$f(x) = P(X \le x) = \{\omega : X(\omega) \le x\}$, $-\infty < x < \infty$

→ Properties

(1) If 'F' is the distribution function of the random variable 'X', for a<b

- $P(a < X \le b) = F(b) - F(a)$
- $P(a \le X \le b) = P(X = a) + F(b) - F(a)$
- $P(a \le X < b) = F(b) - F(a) - P(X = b) + P(X = a)$
- $P(a < X < b) = F(b) - F(a) - P(X = b)$

(2) If F is one-dimensional random variable 'X', then

(i) $0 \le F(x) \le 1$

(ii) if $x < y$,

$F(x) < F(y)$

(3) $F(-\infty) = \lim_{x \to -\infty} f(x) = 0$

(4) $F(\infty) = \lim_{x \to \infty} f(x) = 1$

Probability Mass Function:

If 'X' is a discrete random variable with distinct values; $x_1, x_2, \dots, x_n$

then the function $P_X(x) = \begin{cases} P(X = x_i) = P_i, & X = x_i \\ 0, & \text{otherwise} \end{cases}$

Ex: A random variable 'X' given as

X	0	1	2	3	4	5	6	7
P(x)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$
F(x)	0	K	3K	5K	8K	K^2+8K	$3K^2+8K$	$10K^2+9K$

find probability of $x < 6$. $P(0 < X < 5) = ?$

Sol^n:

$\sum P(x) = 1$

$\Rightarrow 10K^2 + 9K = 1$

$\Rightarrow 10K^2 + 10K - K - 1 = 0$

$\Rightarrow 10K(K+1) - 1(K+1) = 0$

$\Rightarrow K = \frac{1}{10}$, $K = -1$ (imp.)

$\Rightarrow K = \frac{1}{10}$

$P(x < 6) = P(0 \le x < 6)$

$= P(0 \le x \le 5)$

$= P(0) + F(5) - F(0)$

$= K^2 + 8K - 0$

$= (\frac{1}{10})^2 + 8(\frac{1}{10})$

$= 0.01 + 0.8$

$= 0.81$

Q. A variable 'x' is distributed at random between the values 0 & 4. And its probability density function $F(x) = Kx^3(4-x)$. Find the mean and variance.

Sol^n:

$X: \quad 0 \ldots 4$

$$\int_0^4 Kx^3(4-x)^2 dx = 1$$

$$\Rightarrow K\left[x^4 - \frac{8}{5}x^5 + \frac{x^6}{6}\right]_0^4$$

$$\Rightarrow K[256 - 1638.4 +$$

$$\Rightarrow K =$$

* Continuous distribution function:

If 'X' is a continuous random variable with probability density function $f(x)$, then the function

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(t)\,dt$$ is the continuous distribution function

* Properties of distribution function:

(i) $P(a \leq X < b) = F(b) - F(a)$

(ii) $P(a < X < b) = P(a < X \leq b) = P(a \leq X \leq b) = \int_a^b f(t)dt$

Q. 10 coins are thrown simultaneously. Find the probability of getting at least 7 heads.

Sol^n:

* **Poisson Distribution:** (limiting case of binomial distribution)

Poisson's distribution is a limiting case of binomial distribution under the following conditions:

(i) $n \to \infty$

(ii) $p \to 0$

(iii) $np = \lambda$ is finite

- $p = \frac{\lambda}{n}$
- $p + q = 1$
- $q = 1 - \frac{\lambda}{n}$

$$f(x) = {}^nC_x \, p^x q^{n-x}$$

$$= {}^nC_x \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n\,(n-1)!}{((n-1)-(x-1))!\; x\,(x-1)!} \left(\frac{\lambda}{n}\right)^{x-1} \left(\frac{\lambda}{n}\right) \left(1 - \frac{\lambda}{n}\right)^{(n-1)-(x-1)}$$

$$= \frac{n}{x} \cdot {}^{n-1}C_{x-1} \frac{\lambda}{n} \left(\frac{\lambda}{n}\right)^{x-1} \left(1 - \frac{\lambda}{n}\right)^{(n-1)-(x-1)}$$

$$= \frac{\lambda}{x} \cdot {}^{n-1}C_{x-1} \left(\frac{\lambda}{n}\right)^{x-1} \left(1 - \frac{\lambda}{n}\right)^{(n-1)-(x-1)}$$

$$= \frac{\lambda}{x} \left(\frac{\lambda}{n} + 1 - \frac{\lambda}{n}\right)^{n-1}$$

$$= \frac{\lambda}{x}$$

L-13

Online

$\bar{x}$

(27.10) A discrete random variable X which has the following probability function:

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \ldots, \infty$$

Poission variate

Poission Distribution

Qn Prove that poission distribution is the limiting case of binomial distribution under the following three conditions:

(i) $n \to \infty$

(ii) $p \to 0$

(iii) $np = \lambda =$ finite

Proof: Probability mass function in binomial distribution is

$$P(x) = {}^nC_x \, p^x \, q^{n-x}$$

Applying limiting conditions,

$$\lim_{n\to\infty} P(x) = \lim_{n\to\infty} {}^nC_x\, p^x q^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$\left[\because p+q=1 \Rightarrow q = 1-p\right.$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x-1)(n-x)!}{x!\,(n-x)!} p^x(1-p)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x-1)}{x!} p^x (1-p)^{n-x}$$

$\left[\text{Again } np = \lambda \Rightarrow p = \frac{\lambda}{n}\right.$

$$= \lim_{n\to\infty} \frac{n^x (1-1/n)(1-2/n)\cdots(1-x-1/n)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{1}{x!}\lambda^x \lim_{n\to\infty} \frac{n^x}{n^x} (1-1/n)(1-2/n)\cdots(1-x-1/n)\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{1}{x!}\lambda^x \lim_{n\to\infty} (1-1/n)(1-2/n)\cdots(1-x-1/n) \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{1}{x!}\lambda^x . 1 . \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{1}{x!}\lambda^x . \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n . \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{1}{x!}\lambda^x e^{-\lambda} \qquad \left(\because \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a\right)$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

which is the poission's distribution mass function.

Hence proved.

Mean (μ)

Mean = Expectation of X

$= E(X)$

$$= \sum_{x=0}^{\infty} x p(x)$$

$$= \sum_{x=0}^{\infty} \frac{x\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x}{x}\frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} . \frac{\lambda . \lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda} \qquad \left(\because \sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a\right)$$

$$\Rightarrow \boxed{\mu = \lambda}$$

or Mean = λ for Poission's distribution.

Variance (σ^2)

$$\sigma^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} \frac{x^2 \lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x^{\cancel{2}}}{\cancel{x}(x-1)} \frac{\lambda^x e^{-\lambda}}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{x-1+1}{x-1} \cdot \frac{\lambda^2 \cdot \lambda^{x-2} e^{-\lambda}}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^2 \cdot \lambda^{x-2} e^{-\lambda}}{(x-2)!} + \sum_{x=0}^{\infty} \frac{1}{x-1} \cdot \frac{\lambda \lambda^{x-1} e^{-\lambda}}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} (\lambda^2 e^{-\lambda}) \frac{\lambda^{x-2}}{(x-2)!} + \sum_{x=0}^{\infty} (\lambda e^{-\lambda}) \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda^2 + \lambda$$

$$\therefore \sigma^2 = E(x^2) - (E(x))^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$\Rightarrow \boxed{\sigma^2 = \lambda}$ is the variance for poission's distribution.

Example: Envelope are sold 100 numbers in a pack and found that 3% are defective in a pack. If we draw at random, find the probability of no defective and two defective envelope.

sol^n: Given that, $p = 0.03$

$n = 100$

$\mu = np = 3 = \lambda$ for Poission's distribution.

$\therefore$ Probability mass function,

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-3} 3^x}{x!}$$

(i) For no defective product, $x = 0$

$$f(x=0) = \frac{e^{-3} 3^0}{0!} = e^{-3} = 0.0498$$

(ii) For two defective envelopes, $x = 2$

$$\therefore f(x=2) = \frac{e^{-3}3^2}{2!} = e^{-3}\frac{9}{2} = 4.5e^{-3}$$

$$\Rightarrow f(x=2) = 0.224 \quad \text{(Ans)}$$

Qu: 10 % of a product is defective, sample of 10 products are taken. Find probability of exactly 2 defective using binomial distribution and poission distribution.

Sol^n: Given $p = 0.1$
$q = 0.9$

$x = 2$
and $n = 10$

(i) Using binomial distribution,

Mass function is $p(x) = {}^nC_x\, p^x q^{n-x}$

$$\therefore P(x=2) = {}^{10}C_2\, p^2 q^{10-2}$$

$$= {}^{10}C_2\, (0.1)^2 (0.9)^8$$

$$= \frac{10!}{8!\,2!}(0.01)(0.43)$$

$$= \frac{10\times 9}{2!} \times \frac{8!}{8!}(0.0043)$$

$$= 45(0.0043)$$

$$\Rightarrow P(x=2) = 0.1935$$

(ii) Using poission distribution,

Mass function is $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$\lambda = np = 0.1\times 10 = 1$$

$$\therefore f(x) = \frac{e^{-1}(1)^x}{x!} = \frac{1}{e\,x!}$$

For $x = 2$,

$$f(x=2) = \frac{1}{e\,2!} = \frac{1}{2e} = 0.1839$$

.. (Ans)

Qu. Prove that $\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$

i.e. $\frac{e^{-\lambda}\lambda^x}{x!}$ is the probability mass function for discrete random variable.

Sol^n:
$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

For a probability mass function, $\sum_{x=1}^{n} f(x) = 1$

$$\text{L.H.S.} = \sum_{x=1}^{n} \frac{e^{-\lambda}\lambda^x}{x!} \qquad (\because n \to \infty \text{ in poission distribution})$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \cdots \right]$$

$= e^{-\lambda} g(\lambda)$

. which is the taylor series expansion of e^{λ}

$e g(\lambda)$

$$= e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \cdots \right] \qquad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1 = \text{R.H.S.}$$

$$\therefore \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$$

$\Rightarrow$ Probability mass function for poission distribution is $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$.

.. (proved)

L-14 (online) (28.10.2022) (07-08 pm)

Qn A hospital receives an average 5 emergency calls in 10 minutes interval. Find the probability that there are at most 2 emergency calls in that given interval of 10 minutes.

Sol^n: With the constant given time,

mean emergency calls, $\mu = 5 = \lambda$.

Using Poission distribution, mass function will be

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \frac{e^{-5} 5^x}{x!}$$

Again, probability of atmost 2 emergency calls will be

$$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= \frac{e^{-5} 5^0}{0!} + \frac{e^{-5} 5^1}{1!} + \frac{e^{-5} 5^2}{2!}$$

$$= e^{-5} + 5e^{-5} + \frac{25}{2} e^{-5}$$

$$= 18.5 e^{-5}$$

$$= 0.1247$$

... (Ans)

* Hypergeometric Distribution:

Let there be N things (say screws), M out of which are defective and n things are drawn at random.

(a) with replacement:-

Probability of 'x' number of defective things

$$= f(x) = {}^nC_x \left(\frac{M}{N}\right)^x \left(1 - \frac{M}{N}\right)^{n-x}$$

(b) without replacement:

Probability of 'x' number of defective things

$$f(x) = {}^MC_x \; {}^{N-M}C_{n-x} / {}^NC_n$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

is the selection of 'n' things taken 'r' at a time.

Ex: 1 In a box, there are 20 screws, 5 of which are defective. If three screws are drawn at random, find the probability of getting no defective one defective, two defective screws.

(a) with replacement

(b) without replacement.

Soln: (a) With Replacement:

$$N = 20, \; M = 5, \; n = 3$$

$$f(x) = {}^3C_x \left(\frac{5}{20}\right)^x \left(\frac{15}{20}\right)^{3-x} = {}^3C_x \left(\frac{1}{4^x}\right) \left(\frac{3}{4}\right)^{3-x}$$

$$\Rightarrow f(x) = {}^3C_x \cdot \frac{3^{3-x}}{4^x \cdot 4^{3-x}}$$

$$= {}^3C_x \frac{3^{3-x}}{64}$$

$$\Rightarrow f(x) = \frac{1}{64}\, {}^3C_x\, (3)^{3-x}$$

is the probability of 'x' number of defective screws.

(i) When $x=0$, probability of no defective screw,

$$f(0) = \frac{1}{64}\, {}^3C_0\, (3)^3$$

$$\Rightarrow f(0) = \frac{27}{64} \quad (\because {}^3C_0 = 1)$$

$$\simeq 0.422$$

(ii) When $x = 1$, probability of one defective screw,

$$f(1) = \frac{1}{64}\, {}^3C_1\, 3^2$$

$$\Rightarrow f(1) = \frac{27}{64} \quad (\because {}^3C_1 = 3)$$

$$\simeq 0.422$$

(iii) When $x = 2$, probability of two defective screws,

$$f(2) = \frac{1}{64}\, {}^3C_2 \cdot 3^1$$

$$\Rightarrow f(2) = \frac{9}{64} \quad (\because {}^3C_2 = 3)$$

$$\simeq 0.14$$

(Ans)

(b) without Replacement,

Given $N = 20$ = total number of screws

$M = 5$ = number of defective screws

$n = 3$ = number of screw drawn at random

$$f(x) = \frac{{}^MC_x\; {}^{N-M}C_{n-x}}{{}^NC_n} = \frac{{}^5C_x\; {}^{15}C_{3-x}}{{}^{20}C_3}$$ is the probability of 'x' number of defective screws.

$\therefore$ Probability of no defective screws

$$P(0) = \frac{{}^5C_0\; {}^{15}C_3}{{}^{20}C_3} = \frac{15!}{3!\,12!} \times \frac{3!\,17!}{20!} = \frac{15\times14\times13}{20\times19\times18} = 0.399$$

For one def. screw,

$$P(1) = \frac{{}^5C_1\; {}^{15}C_2}{{}^{20}C_3} = \frac{5\times15!}{2!\times13!} \times \frac{3!\times17!}{20!} = \frac{5\times3\times15\times14\times\cancel{13!}}{\cancel{2!}\times\cancel{13!}} \times \frac{\cancel{2!}\times\cancel{17!}}{20\times19\times18\times\cancel{17!}} = 0.46$$

For two def. screws,

$$P(2) = \frac{{}^5C_2\; {}^{15}C_1}{{}^{20}C_3} = \frac{5!\times15}{2!\,3!} \times \frac{3!\times17!}{20!} = \frac{5\times4\times3\times15}{20\times19\times18} = 0.1316$$

(Ans)

* Normal Distribution:

A continuous random variable has the following probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad \text{——— (1)}$$

where, $-\infty < x < \infty$

$-\infty < \mu < 0$

$\sigma > 0$

is called normal variate and its distribution is called normal distribution which is denoted by $X \sim N(\mu, \sigma)$

Mean ↲ ↳ standard Deviation

Mean and variance of Normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} \sigma\, dz$$

put $\frac{x-\mu}{\sigma} = z \Rightarrow x = \sigma z + \mu$

$\Rightarrow \frac{dx}{\sigma} = dz$

$\Rightarrow dx = \sigma dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu e^{-z^2/2} dz + \cancel{\int_{-\infty}^{\infty} \sigma z e^{-z^2/2} dz}^{\,0} \right]$$

(odd)

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \mu e^{-z^2/2} dz$$

put $z^2/2 = p \Rightarrow z = \sqrt{2p}$

$\Rightarrow \frac{2z\,dz}{2} = dp$

$\Rightarrow z\,dz = dp$

$\Rightarrow dz = \frac{dp}{z} = \frac{dp}{\sqrt{2p}}$

z	0	∞
p	0	∞

$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-p} \frac{dp}{\sqrt{2p}}$$

$$= \frac{2\mu}{\sqrt{2}\sqrt{2\pi}} \int_0^{\infty} p^{-1/2} e^{-p} dp$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} p^{1/2 - 1} e^{-p} dp \quad \text{(defn of gamma function)}$$

$$\Rightarrow E(x) = \frac{\mu}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} = \mu \quad (\because \Gamma\tfrac{1}{2} = \sqrt{\pi})$$

$$\Rightarrow \boxed{E(x) = \mu}$$

variance:

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \cdot \frac{1}{\sigma\sqrt{2\pi}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu+\sigma z)^2 e^{-z^2/2} \sigma\,dz$$

put $\frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z$

$\Rightarrow \frac{dx}{\sigma} = dz$

$\Rightarrow dx = \sigma\,dz$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu^2 e^{-z^2/2} dz + \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-z^2/2} dz + 2\int_{-\infty}^{\infty} \underbrace{\cancel{\mu\sigma z e^{-z^2/2}}}_{(odd)} dz \right] \quad \to 0$$

$$= \frac{2}{\sqrt{2\pi}} \mu^2 \int_0^{\infty} e^{-z^2/2} dz + \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2} dz$$

put $\frac{z^2}{2} = p \Rightarrow z = \sqrt{2p}$

$\Rightarrow \frac{2z\,dz}{2} = dp$

$\Rightarrow z\,dz = dp$

$\Rightarrow dz = \frac{dp}{z}$

$\Rightarrow dz = \frac{dp}{\sqrt{2p}}$

$$= \frac{2\mu^2}{\sqrt{2\pi}} \int_0^{\infty} e^{-p} \frac{dp}{\sqrt{2p}} + \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2p\, e^{-p} \frac{dp}{\sqrt{2p}}$$

$$= \frac{2\mu^2}{2\sqrt{\pi}} \int_0^{\infty} p^{-1/2} e^{-p} dp + \frac{4\sigma^2}{\sqrt{2\pi}\sqrt{2}} \int_0^{\infty} p^{1/2} e^{-p} dp$$

$$= \frac{\mu^2}{\sqrt{\pi}} \int_0^{\infty} p^{\frac{1}{2}-1} e^{-p} dp + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} p^{3/2-1} e^{-p} dp$$

$$= \frac{\mu^2}{\sqrt{\pi}} \Gamma\tfrac{1}{2} + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\tfrac{3}{2}$$

$$= \frac{\mu^2}{\sqrt{\pi}} \sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\tfrac{1}{2} \qquad \left(\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right)$$

$$= \mu^2 + \frac{2\sigma^2}{2\sqrt{\pi}} \cdot \sqrt{\pi} \qquad \& \left(\Gamma_{n+1} = n\Gamma_n \Rightarrow \Gamma\tfrac{3}{2} = \tfrac{1}{2}\Gamma\tfrac{1}{2} \right)$$

$$\Rightarrow E(x^2) = \mu^2 + \sigma^2$$

$\therefore$ Variance $= E(x^2) - (E(x))^2$

$= \mu^2 + \sigma^2 - \mu^2$

$\Rightarrow \boxed{Var(X) = \sigma^2}$

Ex: 1 The distribution of marks X is normal with mean 30 and standard deviation 6.25. How many students in a class of 200 students will get marks between 20 and 40?

$\underline{\text{Sol}^n}$: Given, $\mu = 30$, $\sigma = 6.25$

$$P(20 < x < 40) = \phi\left(\frac{40-\mu}{\sigma}\right) - \phi\left(\frac{20-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{40-30}{6.25}\right) - \phi\left(\frac{20-30}{6.25}\right)$$

$$= \phi(1.6) - \phi(-1.6)$$

$$= \phi(1.6) - 1 + \phi(1.6)$$

$$= 2\phi(1.6) - 1$$

$$= 2 \times 0.9452 - 1 = 0.8904$$

(From normal table, $\phi(1.6) = 0.9452$)

So, number of students with mark between 20 and 40 = 200×0.8904

$= 178.08 \simeq 178$.

... (Ans)

$\underline{\text{Ex:2}}$ The breaking strength of a material is normally distributed with mean 64.5 kg and standard deviation 33 kg. Find the probability that it will be at least 60 kg?

$\underline{\text{Sol}^n}$: Given mean, $\mu = 64.5$

standard deviation $\sigma = 3.3$

Probability of breaking strength to be at least 60 kg,

$$P(X \geq 60) = 1 - P(X < 60)$$

$$= 1 - \phi\left(\frac{60-\mu}{\sigma}\right)$$

$$= 1 - \phi\left(\frac{60-64.5}{3.3}\right)$$

$$= 1 - \phi(-1.36)$$

$$= 1 - 1 + \phi(1.36)$$

$$= 0.9131$$

... (Ans)

For corresponding 'standardized normal distribution' with mean 0 and standard deviation 1, $F(z)$ is denoted by $\phi(z)$

&

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, dz$$

$$\boxed{F(x) = \phi\left(\frac{x-\mu}{\sigma}\right)}$$

Normal probabilities for intervals:

The probability that a normal random variable X with mean μ and standard deviation σ assume any value in an interval $a < x \le b$ is

$$P(a < x \le b) = F(b) - F(a) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

* Imp: $\phi(x) = 1 - \phi(-x) \Rightarrow \boxed{\phi(x) + \phi(-x) = 1.}$

proof: For a standard normal distribution,

$P = 1$ is the total probability

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{-1} e^{-x^2/2}\,dx + \int_{-1}^{\infty} e^{-x^2/2}\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{-1} e^{-x^2/2}\,dx + \int_{-\infty}^{1} e^{-x^2/2}\,dx = 1 \quad \left[\because \int_{-a}^{-b} f(x^2)\,dx = \int_{b}^{a} f(x^2)\,dx\right]$$

$$\Rightarrow \phi(-1) + \phi(1) = 1$$

$$\Rightarrow \phi(1) = 1 - \phi(-1)$$

illustrated below

In a similar way,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{-z} e^{-x^2/2}\,dx + \int_{-z}^{\infty} e^{-x^2/2}\,dx = 1$$

In the second integral, put $x = -y$

$\Rightarrow x^2 = y^2$

Again, $dx = -dy$

x	$-z$	∞
y	z	$-\infty$

$$\Rightarrow \int_{-\infty}^{-z} e^{-x^2/2}\,dx + \int_{z}^{-\infty} e^{-y^2/2}(-dy) = 1$$

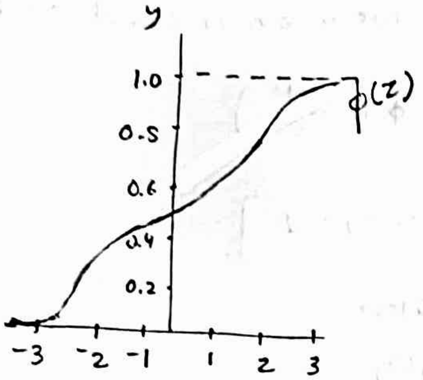
$$\Rightarrow \int_{-\infty}^{-z} e^{-x^2/2}\,dx + \int_{-\infty}^{z} e^{-y^2/2}\,dy = 1$$

$$\Rightarrow \phi(-z) + \phi(z) = 1$$

$$\Rightarrow \boxed{\phi(-z) = 1 - \phi(z)}$$

Hence proved.

Realization:



$\phi(z) + \phi(-z) = 1$

Important Numeric Values:

(a) $P(\mu - \sigma < X \leq \mu + \sigma) = 68\%$

$$= \phi\left(\frac{\mu+\sigma-\mu}{\sigma}\right) - \phi\left(\frac{\mu-\sigma-\mu}{\sigma}\right)$$

$$= \phi(1) - \phi(-1)$$

$$= 2\phi(1) - 1$$

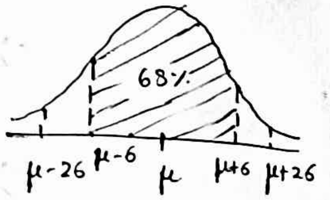
$$= 2 \times 0.8413 - 1$$

$$= 0.6826$$

$$= 68.26\%$$

Similarly, (b) $P(\mu - 2\sigma < X \leq \mu + 2\sigma) = 95.5\%$

& (c) $P(\mu - 3\sigma < X \leq \mu + 3\sigma) = 99.7\%$



(Normal distribution)

Add^{n}al

More precisely,

(a) $P(\mu - 1.96\sigma < X \leq \mu + 1.96\sigma) = 95\%$

(b) $P(\mu - 2.58\sigma < X \leq \mu + 2.58\sigma) = 99\%$

(c) $P(\mu - 3.29\sigma < X \leq \mu + 3.29\sigma) = 99.9\%$

Qn The daily income of an employee is normal with mean Rs. 10000 and standard deviation Rs. 1000. Find probability that the income will

(i) lay between Rs 9000 to Rs. 11000.

(ii) be atleast Rs. 11000.

Sol^n: Given, $\mu = 10000$ & $\sigma = 1000$

$$\therefore P(9000 < X \leq 11000) = \phi\left(\frac{11000-\mu}{\sigma}\right) - \phi\left(\frac{9000-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{11000-10000}{1000}\right) - \phi\left(\frac{9000-10000}{1000}\right)$$

$$= \phi(1) - \phi(-1)$$

$$= 0.6826$$

Again, $P(X \geq 11000) = 1 - P(X < 11000)$

$$= 1 - \phi\left(\frac{11000-10000}{1000}\right)$$

$$= 1 - \phi(1)$$

$$= 1 - 0.8413$$

$$= 0.1587.$$

.. (Ans)

L- 15 (online) (29.10.2022) (05:00 - 06:00 p.m.)

Qn Find the value of C if $P(X \leq c) = 95\%$ where 'X' is a normal random variable with mean $\mu = 5$ and variance 0.04.

Soln: Given $\mu = 5$, $\sigma^2 = 0.04 \Rightarrow \sigma = 0.2$

$P(X \leq c) = 0.95$ (given)

$$\Rightarrow \phi\left(\frac{c-\mu}{\sigma}\right) = 0.95$$

$$\Rightarrow \phi\left(\frac{c-5}{0.2}\right) = 0.95 = \phi(1.645)$$ $(\because z(\phi) = z(0.95) = 1.645)$

$$\Rightarrow \frac{c-5}{0.2} = 1.645$$

$$\Rightarrow c - 5 = 1.645 \times 0.2$$

$$\Rightarrow c = 0.329 + 5 = 5.329.$$ .. (Ans)

Soln to Qn 2 Page 34 dtd 26.09.2022 (L-11)

Qn A bag contains four white and six red balls. 2 balls chosen randomly. X is a random variable, which says no white ball. Design a mass function and find mean and variance.

Soln: Two balls are drawn at random.

Let 'r' be the number of white balls drawn.

Then '2-r' number of non-white balls will remain in that.

Total number of ways 2 balls are drawn is $^{10}C_2$.

Number of ways 'r' number of white balls and '2-r' number of non-white balls are drawn is $^4C_r \, ^6C_{2-r}$

Probability of 'r' white balls is

$$P(X = r) = f(r) = \frac{^4C_r \times {}^6C_{2-r}}{^{10}C_2}$$ ..(mass function)

x :	0	1	2
$f(x)$:	$1/3$	$24/45$	$6/45$

Mean : $E(X)$ $= 0 + \frac{24}{45} + 2 \times \frac{6}{45} = \frac{36}{45} = 0.8$

$E(X^2)$: $0 + 1 \times \frac{24}{45} + 2^2 \times \frac{6}{45} = \frac{48}{45} = 1.067$

$\therefore$ Variance $= E(X^2) - (E(X))^2 = 0.427$ .. (Ans)

Qn: Find the value of 'c' if $P(X \geq c) = 99\%$, $\sigma = 0.2, \mu = 5$

Sol^n:

$P(X \geq c) = 99\%$

$\Rightarrow 1 - P(X < c) = 99\%$

$\Rightarrow 1 - \phi\left(\frac{c-5}{0.2}\right) = 0.99$

$\Rightarrow \phi\left(\frac{c-5}{0.2}\right) = 1 - 0.99 = 0.01 = \phi(-2.326)$

$\Rightarrow \frac{c-5}{0.2} = -2.326$

$\Rightarrow c - 5 = -0.4652$

$\Rightarrow c = 4.5348$ .. (Ans)

* When you have to find $\phi(z)$ value, follow table A7.
* But, if you have to find inverse value or $z(\phi)$, follow table A8.

M-4

Random Sampling:

→ Samples are selected at random.
→ Samples are well-defined.
→ Each item of sample has same chance of being selected.

Ex: Find mean & variance of the sample 0 2 6 4 8.

Sol^n:

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{5}(0+2+6+4+8) = 4$

$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{4}\left[(0-4)^2 + (2-4)^2 + (4-4)^2 + (6-4)^2 + (8-4)^2\right]$

$= \frac{1}{4}\left[4^2 + 2^2 + 2^2 + 4^2\right]$

$= 4 + 1 + 1 + 4 = 10$ .. (Ans)

Maximum Likelihood Estimation:

→ Maximum likelihood estimation is a method of estimating the parameter (e.g: t, σ etc...) of a probability distribution by maximizing a likelihood function, so that under the assumed statistical model, the observed data is most probable.

Procedure to find the maximum likelihood estimation:

(i) Given a function $f(x)$ and parameter P

(ii) $l = f(x_1).f(x_2)....f(x_n)$

(iii) $\ln l = \ln[f(x_1).f(x_2)....f(x_n)]$

(iv) For maximum likelihood estimate, $\frac{\partial}{\partial p}(\ln l) = 0$ and solve it to find the parameter 'p'.

Ex: Find the maximeem likelihood estimate $f(x) = \theta e^{-\theta x}, x > 0$ ($\theta \to$ parameter)

Sol^n: Given $f(x) = \theta e^{-\theta x}$

$$f(x_1) = \theta e^{-\theta x_1}$$
$$f(x_2) = \theta e^{-\theta x_2}$$
$$\vdots$$
$$f(x_n) = \theta e^{-\theta x_n}$$

Then,

$$l = f(x_1) f(x_2) \cdots f(x_n)$$
$$= \theta e^{-\theta x_1} \cdot \theta e^{-\theta x_2} \cdots \theta e^{-\theta x_n}$$
$$\Rightarrow l = \theta^n e^{-\theta x_1 - \theta x_2 - \cdots - \theta x_n}$$
$$= \theta^n e^{-\theta(x_1 + x_2 + \cdots + x_n)}$$
$$= \theta^n e^{-\theta n \bar{x}}$$

$\because \bar{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

$\Rightarrow x_1 + x_2 + \cdots + x_n = n\bar{x}$

Taking logarithm on both side,

$$\ln l = \ln(\theta^n e^{-\theta n \bar{x}})$$
$$= \ln \theta^n + \ln e^{-\theta n \bar{x}} \qquad (\because \ln(ab) = \ln a + \ln b)$$
$$= n \ln \theta + (-\theta n \bar{x}) \qquad (\because \ln x^n = n \ln x \ \& \ \ln e^x = x)$$
$$\Rightarrow \ln l = n \ln \theta - \theta n \bar{x}$$

Differentiating w.r.t. 'θ',

$$\frac{\partial}{\partial \theta}(\ln l) = \frac{\partial}{\partial \theta}(n \ln \theta - \theta n \bar{x})$$
$$= n \frac{\partial}{\partial \theta} \ln \theta + (-n\bar{x}) \frac{\partial}{\partial \theta} \theta$$
$$= n \cdot \frac{1}{\theta} - n\bar{x}$$
$$= \frac{n}{\theta} - n\bar{x}.$$

Again, $\frac{\partial}{\partial \theta}(\ln l) = 0$

$$\Rightarrow \frac{n}{\theta} - n\bar{x} = 0 \Rightarrow \frac{n}{\theta} = n\bar{x}$$

$$\Rightarrow \theta = \frac{n}{n\bar{x}} = \boxed{\frac{n}{x_1 + x_2 + \cdots + x_n}} \quad \text{.. (Ans)}$$

$$\Rightarrow \boxed{\theta = \frac{1}{\bar{x}}}.$$

The maximum likelihood parameter was found to be

$$\theta = \frac{n}{x_1 + x_2 + \cdots + x_n}.$$

Task

Find the maximum likelihood estimate of binomial distribution using parameter 'P'.

Task

Find the maximum likelihood estimate of binomial distribution using parameter 'p'.

sol^n: Suppose $X_1, X_2, \ldots, X_n \sim$ Binomial $\underbrace{(m, p)}_{\text{binomial parameters}}$ where 'm' is known

$n \to$ no. of observations

The probability mass function of 'x_i' is given by

$$P(x_i \mid m, p) = {}^mC_{x_i}\, p^{x_i} (1-p)^{m-x_i}, \quad x_i \in \{0, 1, \ldots, m\}$$

The likelihood function is

$$L(P) = \prod_{i=1}^{n} P(x_i \mid m, p)$$

(product of all the respective marginal distributions)

$$= \prod_{i=1}^{n} {}^mC_{x_i}\, p^{x_i} (1-p)^{m-x_i}$$

$$= \prod_{i=1}^{n} {}^mC_{x_i} \prod_{i=1}^{n} p^{x_i} \prod_{i=1}^{n} (1-p)^{m-x_i}$$

$$= \prod_{i=1}^{n} {}^mC_{x_i}\, p^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n} (m-x_i)}$$

$$= \prod_{i=1}^{n} {}^mC_{x_i}\, p^{\sum_{i=1}^{n} x_i} (1-p)^{nm - \sum_{i=1}^{n} x_i}$$

Again,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \sum_{i=1}^{n} x_i = n\bar{x}$$

$$\Rightarrow L = \prod_{i=1}^{n} {}^mC_{x_i}\, p^{n\bar{x}} (1-p)^{nm - n\bar{x}}$$

The log likelihood function is given by

$$\ln L = \ln \left[\prod_{i=1}^{n} {}^mC_{x_i}\, p^{n\bar{x}} (1-p)^{nm-n\bar{x}} \right]$$

$$= \ln\left(\prod_{i=1}^{n} {}^mC_{x_i}\right) + \ln\left(p^{n\bar{x}}\right) + \ln\left[(1-p)^{n(m-\bar{x})}\right]$$

$$= \ln\left(\prod_{i=1}^{n} {}^mC_{x_i}\right) + n\bar{x} \ln p + n(m-\bar{x}) \ln(1-p)$$

Differentiating with respect to 'p' gives

$$\frac{\partial}{\partial p}(\ln L) = 0 + n\bar{x} \cdot \frac{1}{p} + n(m-\bar{x}) \cdot \frac{-1}{1-p}$$

$$\Rightarrow \frac{\partial}{\partial p}(\ln L) = \frac{1}{p} n\bar{x} - \frac{1}{1-p} n(m-\bar{x})$$

To find the MLE of p, we solve

$$\frac{\partial}{\partial p}(\ln L) = 0$$

$$\Rightarrow \frac{1}{p} n\bar{x} - \frac{1}{1-p} n(m-\bar{x}) = 0$$

$$\Rightarrow \frac{1}{p} n\bar{x} = \frac{1}{1-p} n(m-\bar{x})$$

$$\Rightarrow \frac{1-p}{p} = \frac{nm - n\bar{x}}{n\bar{x}}$$

$$\Rightarrow \frac{1}{p} - 1 = \frac{m}{\bar{x}} - 1$$

$$\Rightarrow \frac{1}{p} = \frac{m}{\bar{x}}$$

$$\Rightarrow p = \frac{\bar{x}}{m}$$

$$\Rightarrow \boxed{p = \frac{1}{nm}\sum_{i=1}^{n} x_i}$$

To prove that $\ln L$ is maximized by $p = \frac{1}{nm}\sum_{i=1}^{n} x_i$, we have to show that,

$$\frac{\partial^2}{\partial p^2}(\ln L) < 0$$

$$\frac{\partial^2}{\partial p^2}(\ln L) = -\frac{1}{p^2} n\bar{x} - \frac{1}{(1-p)^2} n(m-\bar{x}) = -\left[\frac{1}{p^2} n\bar{x} + \frac{1}{(1-p)^2} n(m-\bar{x})\right]$$

$$\Rightarrow \frac{\partial^2}{\partial p^2}(\ln L) = -\left[\underbrace{\frac{1}{p^2}}_{>0} \underbrace{n\bar{x}}_{\geq 0} + \underbrace{\frac{1}{(1-p)^2}}_{>0} \underbrace{n(m-\bar{x})}_{\geq 0}\right] < 0$$

Hence, the MLE of binomial distribution using parameter 'p' is

$$\boxed{p = \frac{1}{nm}\sum_{i=1}^{n} x_i} \quad \text{.. (Ans)}$$

L 16 (Jyoti Sir) (31.10.2022) (09:00 - 10:00 a.m.)

$$f(t) = \begin{cases} 2, & 0 < t < 1 \\ t^2/2, & 1 < t < \pi/2 \\ \cos t, & \frac{\pi}{2} < t \end{cases}$$

→ discontinuous (unit step function)

Formal def^n (integration) or → second shifting theorem

$\boxed{\text{exam}}$

$\boxed{\text{Kreyszig book example}}$

First we have to write $f(t)$ as a multiple of unit step function.

Second shifting property

if $L[f(t)] = F(s)$, then

$L[f(t-a)\,u(t-a)] = e^{-as} F(s)$

$$f(t) = 2(u(t-\underbrace{0}_{\text{lower limit}}) - u(t-\underbrace{1}_{\text{upper limit}}))$$
$$+ \frac{t^2}{2}(u(t-1) - u(t-\pi/2))$$
$$+ \cos t\, u(t - \pi/2)$$

Let $f_1(t) = 2(u(t-0) - u(t-1)) = 2(u(t) - u(t-1))$

$$\Rightarrow L[f_1(t)] = 2\left[\frac{e^0}{s} - \frac{e^{-s}}{s}\right] \qquad \left[\because L[u(t-a)] = \frac{e^{-as}}{s}\right]$$
$$= \frac{2}{s} - \frac{2e^{-s}}{s}$$

Let $f_2(t) = \frac{t^2}{2}[u(t-1) - u(t-\pi/2)]$

$$= \frac{1}{2}\left[(t-1+1)^2 u(t-1)\right] - \frac{1}{2}\left[(t-\pi/2+\pi/2)^2 u(t-\pi/2)\right]$$
$$= \frac{1}{2}\left[(t-1)^2 u(t-1) + 2(t-1)u(t-1) + u(t-1)\right]$$
$$- \frac{1}{2}\left[(t-\pi/2)^2 u(t-\pi/2) + 2(t-\pi/2)u(t-\pi/2)\frac{\pi}{2} + \frac{\pi^2 u}{2^2}(t-\pi/2)\right]$$

$$\Rightarrow L[f_2(t)] = \frac{1}{2}L\Big[(t-1)^2 u(t-1) + 2(t-1)u(t-1) + u(t-1) - (t-\pi/2)^2 u(t-\pi/2)$$
$$- 2(t-\pi/2)u(t-\pi/2)\frac{\pi}{2} - \frac{\pi^2}{4}u(t-\pi/2)\Big]$$

$$= \frac{1}{2}\left[e^{-s}\cdot\frac{2}{s^3} + \frac{2}{s^2}e^{-s} + \frac{e^{-s}}{s} - \frac{2}{s^3}e^{-\frac{\pi}{2}s} - \frac{2}{s^2}e^{-\pi/2 s}\cdot\frac{\pi}{2} - \frac{\pi^2}{4}\frac{e^{-\pi/2 s}}{s}\right]$$

$$= \left[\frac{e^{-s}}{s^3} + \frac{e^{-s}}{s^2} + \frac{e^{-s}}{2s} - \frac{e^{-\pi/2 s}}{s^3} - \frac{\pi e^{-\pi/2 s}}{2s^2} - \frac{\pi^2 e^{-\pi/2 s}}{8s}\right]$$

$f(t) = t^2$

$\Rightarrow L\{f(t)\} = L\{t^2\} = \frac{2!}{s^3}$

Let $f_3(t) = \cos t\; u(t-\pi/2) = \sin(\frac{\pi}{2} - t)\, u(t-\pi/2)$

$$\left(= \cos(t - \pi/2 + \pi/2)\, u(t-\pi/2)\right)$$

$$= -\sin(t-\pi/2)\, u(t-\pi/2)$$

$\cos t = -\sin(t - \frac{\pi}{2})$

$\uparrow$

$\sin(\frac{\pi}{2} - t)$

$$\Rightarrow L\{f_3(t)\} = L[-\sin(t-\pi/2)\,u(t-\pi/2)]$$
$$= -\frac{1}{s^2+1}e^{-\pi/2 s}$$

Hence $L[f(t)] = L[f_1(t)] + L[f_2(t)] + L[f_3(t)]$

$$= \frac{3}{s} - \frac{2e^{-s}}{s} + \frac{e^{-s}}{s^3} + \frac{e^{-s}}{s^2} + \frac{e^{-s}}{2s} - \frac{e^{-\frac{\pi}{2}s}}{s^3} - \frac{\pi e^{-\frac{\pi}{2}s}}{2s^2} - \frac{\pi^2 e^{-\frac{\pi}{2}s}}{8s} - \frac{e^{-\frac{\pi}{2}s}}{s^2+1}$$

.. (Ans)

e.g: $y'' + 3y' + 4y = \begin{cases} 4t+1, & 0 < t \le 1 \\ 8, & t > 1 \end{cases}$

Proof for second shifting theorem:

We know $u(t-a) = \begin{cases} 1, & t \ge a \\ 0, & t < a \end{cases}$

L.H.S. $L[f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}dt$ [def^n of Laplace Transform]

$$= \int_0^a \cancel{0 \cdot f(t-a)e^{-st}dt}^{\,0} + \int_a^\infty 1 \cdot f(t-a)e^{-st}dt$$

$$= \int_a^\infty f(t-a)e^{-st}dt$$

Let $t-a = z \Rightarrow t = z+a$

$\Rightarrow dt = dz$

$$= \int_0^\infty f(z)e^{-s(z+a)}dz$$

t	a	∞
z	0	∞

when $t = a$, $z = 0$

$t \to \infty$, $z \to \infty$

$$= e^{-as}\int_0^\infty f(z)e^{-sz}dz$$

$$= e^{-as}\int_0^\infty f(t)e^{-st}dt$$

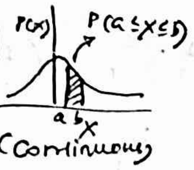
$$= e^{-as}F(s)$$, where $F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

= R.H.S.

.. (proved)

L-17 (Jyoti Sir) (02.11.2022) (02:00 -03:00 p.m)

But in case of continuous random variable, the graph gives a smooth curve. i.e. we can find probability at each and every point.



$(-\infty - \infty)$ (Continuous)

↳ Integration

↳ Probability density function.

Probability distribution function of discrete random variable — diff

diff — (continuous random variable) $f(x)$ or $p(x)$ (small p of x)

Discrete Random Variable

The graph of the outcomes is straight line. We cannot find probability at each and every point.

e.g: LUDO DICE Probability

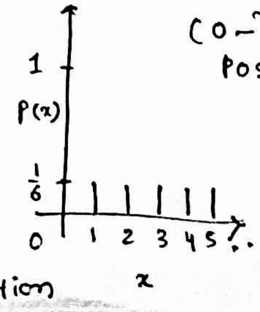
$P(1) = \frac{1}{6}$

$P(2) = \frac{1}{6}$

⋮

↳ Summation

↳ Probability mass function



$(0 - \infty)$ Positive

Distribution Function: $(F(x))$

↳ cumulative addition of probability mass function or density function. ($f(x)$ or $p(x)$)

for discrete random variable, $F(x) = \sum_{x_i \le x} f(x_i)$

for continuous random variable, $F(x) = \int_{-\infty}^{x} f(v)\,dv$

semi-open
semi-close
open
close

$P(a \le X \le b) = F(b) - F(a)$

same for $a < X < b$
$a \le X < b$
$a < X \le b$

Qn. Find the distribution function & then find probability.

- Poission distribution:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$\mu = E(X) = \lambda$, $\sigma^2 = \lambda$

(Imp) Find the mean & variance of (a) Binomial distribution (b) poisson distribution

Qn The mean & variance of a binomial distribution are 12 and 4. Then find probability of $x > 1$. $P(X>1) = ?$

Soln:

Mean $= np = 12$

Variance $= npq = 4$

$\therefore q = \frac{\text{variance}}{\text{mean}} = \frac{1}{3}$

Again, $np = 12$

$\Rightarrow n = \frac{12}{p} = \frac{12}{2} \times 3 = 18$

$p = 1 - q = \frac{2}{3}$.

$\therefore P(x>1) = 1 - P(x \le 1)$

$= 1 - [P(0) + P(1)]$

$= 1 - \left[{}^nC_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^n + {}^nC_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{n-1}\right]$

$= 1 - \left[\left(\frac{1}{3}\right)^n + n \cdot \frac{2}{3} \cdot \left(\frac{1}{3}\right)^{n-1}\right]$

$= 1 - \left[\left(\frac{1}{3}\right)^{18} + 12\left(\frac{1}{3}\right)^{17}\right]$

$= 1 - \left(\frac{1}{3}\right)^{17}\left(\frac{37}{3}\right)$

$= 0.99$

$\therefore$ (Ans.)

Qn. Fifteen coins are tossed simultaneously. Find the probability of getting atleast two heads.

Sol^n: Probability of getting one head on one coin toss, $p = \frac{1}{2}$

$$q = 1 - p = \frac{1}{2}$$

Number of trials $n = 15$

Probability of getting at least two head,

$$P(x \geq 2) = 1 - p(x < 2)$$
$$= 1 - [P(0) + P(1)]$$
$$= 1 - {}^{15}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{15} - {}^{15}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{14}$$
$$= 1 - \left(\frac{1}{2}\right)^{15} - 15\left(\frac{1}{2}\right)^{15}$$
$$= 1 - 16\left(\frac{1}{2}\right)^{15}$$
$$= 1 - \left(\frac{1}{2}\right)^{11} = 0.9995$$

First we have to choose which distribution should be applied.

- small & finite 'n' success & failure
  → Binomial distribution

Poisson
- Book 500 pages error
- Packet of defective material
- dates in a particular interval
- air accident in a time
- crowded, bike crossing level crossing

- Box red ball, white ball etc.. with two subcases:

with replacement : binomial
without replacement : hypergeometric

Qn. Find the mean & variance of $f(x) = e^{-2x}, x \geq 0$

Sol^n: Mean, $\mu = E(x)$

$$= \int_0^\infty x e^{-2x} dx$$
$$= \left[-\frac{1}{2} x e^{-2x}\right]_0^\infty - \int_0^\infty \frac{dx}{dx} \cdot -\frac{1}{2} e^{-2x} dx$$
$$= \frac{1}{2} \cdot 0 \cdot e^0 + \frac{1}{2} \cdot \left[-\frac{1}{2} e^{-2x}\right]_0^\infty$$
$$= \frac{1}{4}$$

$$E(x^2) = \int_0^\infty x^2 e^{-2x} dx$$
$$= 0 + \frac{1}{2}\int_0^\infty 2x e^{-2x} dx$$
$$= \frac{1}{4}$$

$$\therefore \sigma^2 = E(x^2) - (E(x))^2$$
$$= \frac{1}{4} - \left(\frac{1}{4}\right)^2$$
$$= \frac{1}{4} - \frac{1}{16}$$
$$= \frac{3}{16} \quad \text{— (Ans)}$$

Qn.
$$P(x) = \begin{cases} \dfrac{3!}{x!(3-x)!}\left(\dfrac{1}{2}\right)^3, & x = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

probability mass function.

(absent) L - 18 (07/11/2022) (09:00 - 10:00 a.m)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \left(z = \frac{x-\mu}{\sigma}\right)$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$ → probability distribution function.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

$$F(x) = \phi(z) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

* $P(a < x \leq b) = F(b) - F(a) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-x}{\sigma}\right)$

Statistics

Maximum Likelihood Estimation:

(i) for binomial distribution

$$f(x) = {}^nC_x p^x q^{n-x}$$

$$f(x_1) = {}^nC_{x_1} p^{x_1} q^{n-x_1}$$

$$f(x_2) = {}^nC_{x_2} p^{x_2} q^{n-x_2}$$

$$\vdots$$

$$f(x_n) = {}^nC_{x_n} p^{x_n} q^{n-x_n}$$

$$l = f(x_1).f(x_2)\cdots\cdots f(x_n)$$

$$= {}^nC_{x_1} p^{x_1}(1-p)^{n-x_1} . {}^nC_{x_2} p^{x_2}(1-p)^{n-x_2} \cdots {}^nC_{x_n} p^{x_n}(1-p)^{n-x_n}$$

$$l = {}^nC_{x_1} {}^nC_{x_2} \cdots {}^nC_{x_n} p^{x_1+x_2+\cdots+x_n} (1-p)^{n^2-(x_1+x_2+\cdots+x_n)}$$

$$= {}^nC_{x_1} {}^nC_{x_2} \cdots {}^nC_{x_n} p^{n\bar{x}} (1-p)^{n^2-n\bar{x}} \qquad \left(\because \bar{x} = \frac{x_1+x_2+\cdots+x_n}{n}\right)$$

Taking log on both sides

$$\ln l = \ln {}^nC_{x_1} + \ln {}^nC_{x_2} + \cdots + \ln {}^nC_{x_n} + \ln p^{n\bar{x}} + \ln(1-p)^{n^2-n\bar{x}}$$

$$\Rightarrow \ln l = \ln {}^nC_{x_1} + \ln {}^nC_{x_2} + \cdots + \ln {}^nC_{x_n} + (n\bar{x})\ln p + (n^2-n\bar{x})\ln(1-p)$$

Partial derivative of above eq^n,

$$\frac{\partial}{\partial p}(\ln l) = 0$$

$$\Rightarrow \frac{n\bar{x}}{p} + (n^2 - n\bar{x}) \frac{1}{1-p}(-1) = 0$$

$$\Rightarrow \frac{n\bar{x}}{p} - \frac{(n^2 - n\bar{x})}{1-p} = 0$$

$$\Rightarrow \frac{n\bar{x}(1-p) - (n^2 - n\bar{x})(p)}{p(1-p)} = 0$$

$$\Rightarrow \frac{n\bar{x} - n\bar{x}p - n^2p + n\bar{x}p}{p(1-p)} = 0$$

$$\Rightarrow n\bar{x} - n^2p = 0 \cdot p(1-p) = 0$$

$$\Rightarrow n\bar{x} = n^2p$$

$$\Rightarrow \bar{x} = np$$

$$\Rightarrow \boxed{p = \frac{\bar{x}}{n}}$$

Confidence interval:

Confidence interval for an unknown parameter 'θ' is defined as that interval $x_1 < \theta < x_2$ where estimating the parameter 'θ' for mean 'μ' and variance σ^2 has high probability denoted by 'Γ' (gamma) (95%, 99%, ...) called confidence interval.

→ Confidence interval for mean 'μ' of normal distribution (where variance σ^2 is known, mean μ is also known)

Step:1 Choose a confidence level (Γ is confidence level)
95% or 99%

Step:2 Find critical value 'c' from normal table.

Γ	0.9	0.95	0.99	0.999
c	1.645	1.96	2.576	3.291

Step-3 Compute mean $\bar{x}$ of sample $x_1, x_2, \ldots$

Step-4 Compute $K = \frac{c\sigma}{\sqrt{n}}$ ('σ' is known, 'n' is known)

Step-5 Confidence interval for mean 'μ' is

CONF $\Gamma(\bar{x} - K \leq x \leq \bar{x} + K)$

(P.T.O.)

Ex: Find 95 % confidence interval for mean 'μ' of normal distribution with standard deviation $\sigma = 1.2$. Using sample 10, 10, 8, 12, 10, 11, 10, 11

Ans: $\sigma^2 = (1.2)^2$

$x_1 = 10$, $x_2 = 10$, $x_3 = 8$, $x_4 = 12$, $x_5 = 10$, $x_6 = 11$, $x_7 = 10$, $x_8 = 11$

$$\bar{x} = \frac{10+10+8+12+10+11+10+11}{8}$$

$= 10.25$

$\gamma = 0.95$

Then $c = 1.96$

$$K = \frac{c\sigma}{\sqrt{n}}$$

$$= \frac{(1.96)(1.2)}{\sqrt{8}}$$

$= 0.83$

$CONF_\gamma\ (10.25 - 0.83 \le \mu \le 10.25 + 0.83)$

$CONF_\gamma\ (9.41 \le \mu \le 11.08)$.

.. (Ans)

L-19 (09.11.2022) (02:00-03:00 p.m.)

Q. Determine a 95 % confidence interval for the mean of a normal distribution with variance $\sigma^2 = 9$, using a sample of $n = 100$ values with mean $\bar{x} = 5$.

Sol^n: Given $\sigma^2 = 9 \Rightarrow \sigma = 3$

$\gamma = 0.95$

Then $c = 1.96$

$$K = \frac{c\sigma}{\sqrt{n}}$$

$$= \frac{1.96 \times 3}{\sqrt{100}}$$

$$= \frac{5.88}{10}$$

$= 0.588$

$CONF_\gamma\ (5 - 0.588 \le \mu \le 5 + 0.588)$

$CONF_\gamma\ (4.412 \le \mu \le 5.588)$

.. (Ans)

Confidence interval for mean μ with unknown variance σ^2

Step:1 Choose confidence level either 95% or 99%.

Step:2 Find solution 'c' of the equation $F(c) = \frac{1+\gamma}{2}$

previous (If σ^2 is known then find the value of 'c') from T-distribution table with $n-1$ degrees of freedom.

no. of samples

step:3 Compute mean $\bar{x}$ and variance $s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

step:4 Compute $K = \frac{cs}{\sqrt{n}}$

$$\text{CONF}_{\underset{r}{95\%}} \{ \bar{x} - K \le \mu \le \bar{x} + K \}$$

Ex: Find confidence interval $r = 99\%$

samples: 144 147 146 142 144 σ^2 is unknown.

sol^n: $r = 0.99$

$$F(c) = \frac{1+r}{2} = \frac{1+0.99}{2} = 0.995$$

$n = 5$, degrees of freedom $= 4$

value of c for $F(c) = 0.995$ with 4 degrees of freedom $= 4.60$

$$\text{Mean}, \bar{x} = \frac{144+147+146+142+144}{5} = 144.6$$

$$\therefore s^2 = \frac{1}{4} \sum_{i=1}^{5} (x_i - \bar{x})^2$$

$$= \frac{1}{4} \{ (144-144.6)^2 + (147-144.6)^2 + (146-144.6)^2 + (142-144.6)^2 + (144-144.6)^2 \}$$

$$= \frac{1}{4} \{ (0.6)^2 + (2.4)^2 + (1.4)^2 + (2.6)^2 + (0.6)^2 \}$$

$$= \frac{15.2}{4} = 3.8$$

$$\Rightarrow s = 1.949$$

$$\therefore K = \frac{4.60 \times 1.949}{\sqrt{5}} = 4.009$$

$$\text{CONF}_{99\%} (144.6 - 4.009 \le \mu \le 144.6 + 4.009)$$

$$\text{CONF}_{99\%} (140.591 \le \mu \le 148.609)$$

.. (Ans)

* **Confidence interval for variance σ^2 of normal distribution:**

step:1 Choose a confidence level r

step:2 Find solution 'c_1' & 'c_2' from $F(c_1) = \frac{1-r}{2}$ and $F(c_2) = \frac{1+r}{2}$ from chi-square distribution table with '$n-1$' degrees of freedom.

step:3 Compute s^2 using $s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

step:4 Compute $K_1 = \frac{(n-1)s^2}{c_1}$

$K_2 = \frac{(n-1)s^2}{c_2}$

step:5 $CONF_r \{ K_2 \le \sigma^2 \le K_1 \}$

Ex: find 95 % confidence interval for the variance of normal distribution using sample

89 84 87 81 89 86 91 90 78 89 87 99 83 89 $(n=14)$

Sol: $r = 95\% = 0.95$

$F(c_1) = \frac{1-r}{2} = \frac{1-0.95}{2} = 0.025$

$F(c_2) = \frac{1+r}{2} = \frac{1+0.95}{2} = 0.975$

$c_1 = 5.01$ for 13 degrees of freedom

$c_2 = 24.74$ for 13 degrees of freedom

$$\bar{x} = \frac{89+84+87+81+89+86+91+90+78+89+87+99+83+89}{14}$$

$= 87.2857$

$$s^2 = \frac{1}{13} \sum_{i=1}^{14} (x_i - \bar{x})^2$$

$$= \frac{1}{13} \left\{ (89-87.2857)^2 + (84-87.2857)^2 + \dots + (89-87.2857)^2 \right\}$$

$\Rightarrow s^2 = 25.14286$

$\Rightarrow s = 5.0143$

$CONF_{0.95} (13.21 \le \sigma^2 \le 65.25)$

L-20 (14.11.2022) (09:00 – 10:00 a.m)

Ex: Construct a 99% confidence interval for the true mean weight loss if 16 persons on diet control after one month had a mean weight loss of 3.4 kg with standard deviation of 0.68 kg.

Sol^n:

$\gamma = 99\% = 0.99$

$n = 16$

$\bar{x} = 3.4$ kg

$\sigma = 0.66$ kg $\qquad c = 2.576$

$$\therefore K = \frac{c\sigma}{\sqrt{n}} = \frac{2.576 \times 0.68}{\sqrt{16}}$$

$$(\Rightarrow n-1 = 15 \text{ degree}) \qquad = \frac{2.576 \times 0.68}{4}$$

$$= 0.43$$

$$CONF_{0.99}(\bar{x} - K \le \mu \le \bar{x} + K)$$

$$CONF_{0.99}(2.97 \le \mu \le 3.83). \qquad \text{.. (Ans)}$$

Ex: Determine a 99% confidence interval for a mean of a normal population with standard deviation 2.5 using the sample 30.8, 30, 29.9, 30.1, 31.7, 34

Sol^n:

$$\bar{x} = \frac{30.8 + 30 + 29.9 + 30.1 + 31.7 + 34}{6}$$

$$= 31.08$$

$\gamma = 0.99$, $c = 2.576$ $\qquad \sigma = 2.5$

$$K = \frac{c\sigma}{\sqrt{n}} = \frac{2.576 \times 2.5}{\sqrt{6}} = \frac{6.44}{2.449} = 2.629$$

$$CONF_{0.99}(\bar{x} - K \le \mu \le \bar{x} + K)$$

$$= CONF_{0.99}(31.08 - 2.62 \le \mu \le 31.08 + 2.62)$$

$$= CONF_{0.99}(28.46 \le \mu \le 33.7)$$

Ex: Determine a 95% confidence interval for the mean μ of a normal population with variance $\sigma^2 = 16$. Using a sample size 200 with mean = 74.81.

Sol^n:

$\gamma = 0.95$, $c = 1.96$

$$K = \frac{c\sigma}{\sqrt{n}} = \frac{1.96 \times 4}{\sqrt{200}} = \frac{7.84}{14.14} = 0.55$$

$$CONF_{0.95}(\bar{x} - K \le \mu \le \bar{x} + K)$$

$$= CONF_{0.95}(74.81 - 0.55 \le \mu \le 74.81 + 0.55)$$

$$= CONF_{0.95}(74.26 \le \mu \le 75.36)$$

Normal Distribution Questions

Ex: The breaking strength of a material is normally distributed with mean 64.5 kg and standard deviation 3.3 kg. Find the probability that it will be at least 60 kg.

Sol^n^:

$$\phi(z) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

$$P(a \le x \le b) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(x \geq 60) = 1 - P(x < 60)$$

$$= 1 - \phi\left(\frac{60-\mu}{\sigma}\right)$$

$$= 1 - \phi\left(\frac{60-64.5}{3.3}\right)$$

$$= 1 - \phi(-1.36)$$

$$= 1 - 1 + \phi(1.36)$$

$$= 0.9131. \quad \text{.. (Ans)}$$

Q^n^ Daily income of an employee is normal with $\mu = 10{,}000$, $\sigma = 1{,}000$. Find probability that $P(x \le 11000) = ?$

Sol^n^: We know that

$$P(a \le x \le b) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

here $a = 0$,

so $$P(x \le 11000) = \phi\left(\frac{11000-10000}{1000}\right)$$

$$= \phi(1)$$

$$= 0.8413. \quad \text{.. (Ans)}$$

(absent) L-21 (15.11.2022) (03:30-04:30 p.m.)

Testing of hypothesis:

The principal objective of statistical inference is to draw inference about the population on the basis of data collected by sampling from the populations.

Statistical inference consists two major areas:

(i) estimation

(ii) test of hypotheses

In test of hypotheses, a statement about a parameter of the population is tested for its validity.

* Statistical decisions:

They are conclusions about the population parameters on the basis of a random sample from the population.

* Statistical hypothesis:

It is an assumption about the population distribution parameters, when more than one population is considered. Statistical hypothesis consists of relationship between the parameter of population.

There are two types of hypothesis:

(i) Null hypothesis

& (ii) Alternative hypothesis

(1) Null hypothesis:

It is the statistical hypothesis which is to be actually tested for acceptance or rejection. It is denoted by H_0 (null hypothesis). In null hypothesis these statements are equal. (=)

$$\mu = \mu_0 \quad , \quad \sigma = \sigma .$$

(2) Alternative hypothesis:

It is a composite hypothesis involving statements expressed as inequalities. (<, >, ≠) denoted by

$$H_1 \quad \begin{cases} \mu > \mu_0 \\ \mu < \mu_0 \\ \mu \neq \mu_0 \end{cases}$$

Test of hypothesis decides whether a statement concerning a parameter is true or false instead of estimating the value of the parameter. Since, the test is based on sample observation, the decision of acceptance or rejection of the null hypothesis is always subjected to some errors.

(1) Type-I error (rejecting the true hypothesis)

If a null hypothesis is rejected instead of acceptance, Type-I error has occured.

(2) Type-II error (accepting the false hypothesis)

If a null hypothesis is accepted instead of rejection, then type-II error has occured.

→ It is denoted by level of significance.

* Level of significance:

It is a test of the probability of commiting type-I error, denoted by 'α'. can be 5%, 1%. Level of significance measure's the amount of error associated in taking decision.

can be 5% or 1%.

Probability of type-I error = probability of rejection of H_0 which is true = α.

Probability of type-II error = P(acceptance of H_0 which is false) = β'.

* Critical Region:

The critical region is the region of rejection of null hypothesis. The area of the critical region = level of significance.

Critical region always lies on the tail of the distribution. Depending on the nature of alternative hypothesis, critical region may lie on one side or both sides of the tail.

* Critical value:

It is the value of the test statistics S_α which divides the area under the probability curve into critical region & non-critical region.

* One-tailed Test and Two-tailed Test:

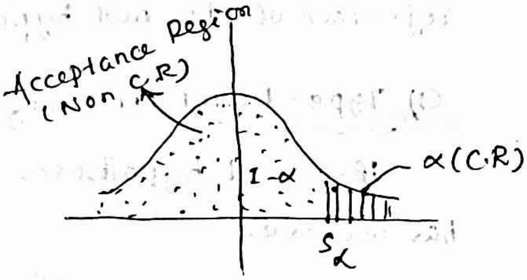
↳ depends on alternative hypothesis

(1) Right one tailed test:

When the alternative hypothesis H_1 is of the greater than type i.e.

$$H_1 : \mu > \mu_0 \quad \text{or} \quad H_1 : \sigma_1^2 > \sigma_2^2$$

then the entire critical region of area 'α' lies on the right side tail of the probability density curve.



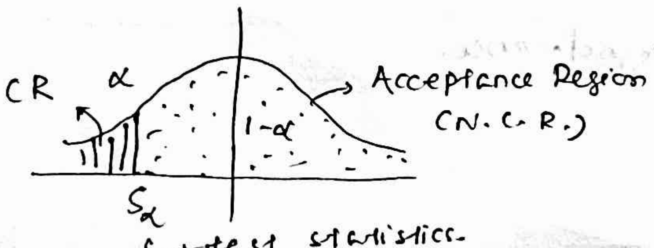
S_α is the test statistics

(2) Left one tailed Test:

When the alternative hypothesis H_1 is of the less than type, i.e.

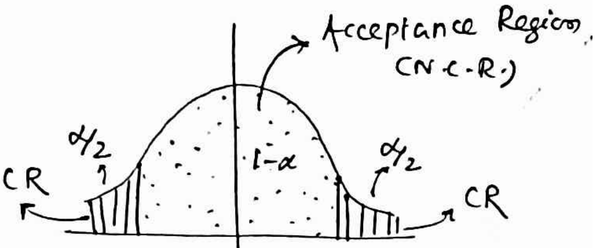
$$H_1 : \mu < \mu_0 \quad \text{or} \quad H_1 : \sigma_1^2 < \sigma_2^2$$

then the entire critical region of area 'α' lies on the left side tail of the probability density curve.



(3) Two - tailed test:

If associative hypothesis of the not equals type, i.e.



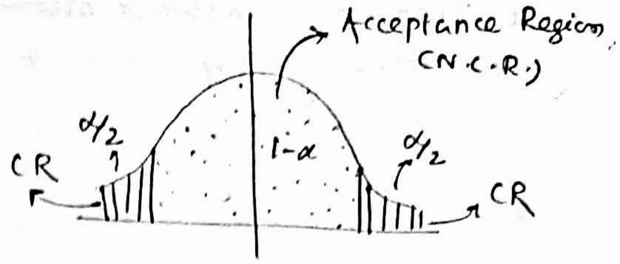
$$H_1 : \mu_1 \neq \mu_0 \text{ or } H_1 : \sigma_1^2 \neq \sigma_2^2$$

then the critical region lies on both sides of the right & left tails of the curve such that the critical region of area $\alpha/2$ lies on the right tail and critical region of area $\alpha/2$ lies on the left tail

(3) Two - tailed test:

If associative hypothesis of the not equals type, i.e.

$$H_1 : \mu_1 \neq \mu_0 \text{ or } H_1 : \sigma_1^2 \neq \sigma_2^2$$



then the critical region lies on both sides of the right & left tails of the curve such that the critical region of area $\alpha/2$ lies on the right tail and critical region of area $\alpha/2$ lies on the left tail.

L-22 (22.11.2022) (04:00-05:00 p.m)

Standard normal variant → z

There are three types of test of hypothesis:

Type I Test for μ of the normal distribution with known σ^2

Let $x_1, x_2, \ldots, x_n$ be random sample of size 'n' from a large population with mean μ' and variance 'σ^2'. Let 'c' be the critical point and $z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

$P(z \leq c) = 1-\alpha$(level of significance)

$P(z > c) = \alpha$

Estimation of parameter → $z(\phi)$

Test of hypothesis → $z(D)$

Ex: when $|z| < c$, then null hypothesis is accepted.

A random sample of 400 shoes has an average length of 10 cm. Can this be considered as a sample from a large population with mean as 10.2 cm and standard deviation 2.25 cm. Here (0.05 level of significance)

$n = 400$

mean, $\bar{x} = 10$ cm

$\mu = 10.2$ cm

and $\sigma = 2.25$ cm.

(standard normal variant)

$$\therefore z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{10-10.2}{2.25/20} = -\frac{4}{2.25} = -1.778$$

$|z| = 1.778 < c = 1.96$ (at 5% level of significance)

(for 95%)

→ Acceptance region

i.e. Null Hypothesis θ_0 is accepted.

Type-II Test for mean μ of the normal distribution with unknown σ^2. (and σ^2 is unknown)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size 'n' is drawn from a normal population with mean μ' and unknown 'σ^2', then

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \quad \text{and} \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Then the variant of the statistics is variant of student t-distribution and is defined as

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}} \quad \text{with } n-1 \text{ degrees of freedom.}$$

Let 'c' be the critical value at a significance value 'α'

$P(|t| > c) = \alpha$

$P(|t| \leq c) = 1-\alpha$

Ex: Test $\mu = 0$ against alternative $\mu > 0$, assuming normally & using the sample

* Observed value is always less than the critical value for acceptance of null hypothesis

1 -1 1 3 -8 6 0 . $n = 7$

$$\bar{x} = \frac{2}{7} = 0.2857$$

$$s^2 = \frac{1}{6}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{6}[(1-0.2857)^2 + (-1-0.2857)^2 + (1-0.2857)^2 + (3-0.2857)^2$$
$$(-8-0.2857)^2 + (6-0.2857)^2 + (0-0.2857)^2]$$

$$= \frac{1}{6}(111.42857143)$$

$$= 18.57$$

$\Rightarrow s = 4.31$

level of significance
$\alpha = 5\%$, with 6 degrees of freedom,
$c = 1.94$

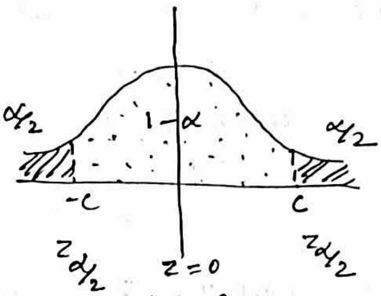
$$\therefore t = \frac{0.2857}{4.31/\sqrt{7}} = 0.1754$$

$|t| < c$ i.e. null hypotheses 'θ_0' is accepted.

L-23 (23.11.2022) (02:00-03:00 p.m)

* When no. of samples > 30, we use Z-distribution.
else, we use t-distribution ($n < 30$)

$$P(|z| \le c) = 1 - \alpha$$
$$P(|z| > c) = \alpha$$



$z < -c$ $|z| < c$ $z \ge c$
$-c < z < c$

* Accepting alternative hypothesis increases the error. (more than 5 %)

$$\eta = 1 - \beta$$

- large sample → z-distribution
- number of sample is less

Type-III

Test for variance of normal distribution.

→ Use sample size $n = 15$, sample variance $s^2 = 13$ from a normal population. Test the hypothesis $\sigma^2 = \sigma_0^2 = 10$ against alternative hypothesis $\sigma^2 = \sigma_1^2 = 20$.

Soln: We choose level of significance, $\alpha = 5\% = 0.05$
$n = 15$
$s^2 = 13$

$\sigma^2 = \sigma_0^2 = 10$

If the hypothesis is true,

$$y = (n-1)\frac{s^2}{\sigma_0^2}$$ has a chi-square distribution with

$(n-1)$ degrees of freedom.

Estimation of Chi-square distribution

Here, $y = 14 \times \frac{13}{10} = \frac{182}{10} = 18.2$

from chi-square distribution, for 14 d.o.f with 5% level of significance,

$c = 23.68$

So, $y < c$ and hence the hypothesis is accepted.

Alt. hyp: (62% error)
To find the error of alternative hypothesis
F(11.84)
if alternative hypotheses is accepted.

* Regression & correlation:

Equation of regression line

i.e. $(y-\bar{y}) = b(x-\bar{x})$

of sample $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

where $\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$ & $\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n}$

and 'b' is the regression coefficient $= \frac{S_{xy}}{S_x^2}$

where $S_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Correlation coefficient $r = \frac{S_{xy}}{S_x S_y} = \frac{S_{xy}}{\sqrt{S_x^2 \cdot S_y^2}}$

Ex: Find the equation of regression line from the following data:

x	0	2	4	6	8
y	5	15	10	30	40

Soln:

$\bar{x} = \frac{0+2+4+6+8}{5} = 4$

$\bar{y} = \frac{5+15+10+30+40}{5} = 20$

$S_x^2 = \frac{1}{4}[(0-4)^2 + (2-4)^2 + (4-4)^2 + (6-4)^2 + (8-4)^2]$

$= \frac{1}{4}(16+4+4+16)$

$$s_y^2 = \frac{1}{4}\left[(5-20)^2 + (15-20)^2 + (10-20)^2 + (30-20)^2 + (40-20^2)\right]$$

$$= \frac{1}{4}(850)$$

$$= 212.5$$

$$S_{xy} = \frac{1}{4}\left[(0-4)(5-20) + (2-4)(15-20) + (4-4)(10-20) + (6-4)(30-20) + (8-4)(40-20)\right]$$

$$= \frac{1}{4}\left[60 + 10 + 20 + 80\right]$$

$$= \frac{170}{4} = 42.5$$

$\therefore$ regression coefficient, $b = \frac{S_{xy}}{S_x^2} = 4.25$

Hence, the regression line will be

$$(y-20) = 4.25(x-4)$$

$$\Rightarrow y = 4.25x + 3$$

.. (Ans.)

correlation coefficient,

$$r = \frac{42.5}{46.097}$$

$$= 0.92197$$

.. (Ans.)

L-24 (28.11.2022) (09:00-10:00 a.m.)

M-5

Definition of Line Integral:

A line integral of a vector function $\vec{F}$ is defined over a curve 'c' as

$$\int_C F(r) = \int_a^b F(r(t))\, r'(t)\, dt$$

where 'c' is the path of integration

a → initial point

b → terminal point

$dr \to dx\hat{i} + dy\hat{j} + dz\hat{k}$

Ex: Find the value of the line integral $F = [-y, xy]$ and 'c' is the circular arc from 'A' to 'B'.

(Take radius = 1)

Sol:

$$F = [-y, xy]$$

$$= -y\hat{i} + xy\hat{j}$$

$$r(t) = [\cos t, \sin t]$$

$$= \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$F(r(t)) = -\sin t\,\hat{i} + \cos t \sin t\,\hat{j}$$

$$\int_C F(r) = \int_0^{\pi/2} (-\sin t\,\hat{i} + \cos t \sin t\,\hat{j})(-\sin t\,\hat{i} + \cos t\,\hat{j})\, dt$$

$$= \int_0^{\pi/2} (\sin^2 t + \cos^2 t \sin t)\, dt$$

$$= \int_0^{\pi/2} \frac{1-\cos 2t}{2}\, dt + \int_0^{\pi/2} -(\cos^2 t \, d\cos t)$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2} - \frac{1}{3}[\cos^3 t]_0^{\pi/2}$$

$$\left(\frac{\pi}{4} - \frac{1}{3}\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] - \frac{1}{3}[0 - 1]$$

$$= \frac{\pi}{4} + \frac{1}{3} \quad \text{.. (Ans)}$$

Line Integral in Space:

Ex Find the value of the line integral $F(r) = [z \ \ x \ \ y]$ and 'c' is the helix.

$r(t) = [\cos t \ \ \sin t \ \ 3t]$ $\qquad 0 - 2\pi$

Sol^n: Given, $F(r) = [z \ \ x \ \ y]$

$$= z\hat{i} + x\hat{j} + y\hat{k}$$

$r(t) = [\cos t \ \ \sin t \ \ 3t]$ $\qquad r'(t) = -\sin t\hat{i} + \cos t\hat{j} + 3\hat{k}$

$$= \cos t\hat{i} + \sin t\hat{j} + 3t\hat{k}$$

$$\therefore F(r(t)) = 3t\hat{i} + \cos t\hat{j} + \sin t\hat{k}$$

$$\therefore \int_C F(r) = \int_0^{2\pi} (3t\hat{i} + \cos t\hat{j} + \sin t\hat{k})(-\sin t\hat{i} + \cos t\hat{j} + 3\hat{k})\, dt$$

$$= \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)\, dt$$

$$= -3\int_0^{2\pi} t\sin t\, dt + \int_0^{2\pi} \cos^2 t\, dt + 3\int_0^{2\pi} \sin t\, dt \qquad 7\pi$$

$$= -3\left[-t\cos t + \sin t\right]_0^{2\pi} + \int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)dt + \left(-3\,[\cancel{\cos t\, dt}]_0^{2\pi}\right)$$

$$= \left[3t\cos t + (-3[\sin t])\right]_0^{2\pi} + \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} - 3.0$$

$$= (6\pi - 0) + \frac{1}{2}[2\pi]$$

$$= 6\pi + \pi$$

$$= 7\pi \quad \text{.. (Ans)}$$

Qn Evaluate the integral for $F(r) = [5x \quad xy \quad x^2z]$ along two different paths with same initial point $A(0,0,0)$ & same terminal point $B=(1,1,1)$

(a) C_1 is the straight line segment

Here the parametric representation $r_1(t) = [t, t, t]$

(b) C_2 is the parabolic arc $r_2(t) = [t, t, t^2]$

Sol^n:

$r_1(t) = [t, t, t]$ $\qquad F(r_1(t)) = [5t \quad t^2 \quad t^3]$

$r_1'(t) = [1, 1, 1] = \hat{i}+\hat{j}+\hat{k}$ $\qquad = 5t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$\therefore \int_{C_1} F(r) = \int_0^1 (5t + t^2 + t^3)\,dt$$

$$= \frac{5}{2}t^2\Big]_0^1 + \frac{1}{3}t^3\Big]_0^1 + \frac{1}{4}t^4\Big]_0^1$$

$$= \frac{5}{2} + \frac{1}{3} + \frac{1}{4}$$

$$= \frac{30+4+3}{12}$$

$$= \frac{37}{12} \qquad \text{(Ans.)}$$

$r_2(t) = [t, t, t^2]$ $\qquad F(r_2(t)) = [5t \quad t^2 \quad t^4]$

$r_2'(t) = [1, 1, 2t] = \hat{i}+\hat{j}+2t\hat{k}$ $\qquad = 5t\hat{i} + t^2\hat{j} + t^4\hat{k}$

$$\therefore \int_{C_2} F(r) = \int_0^1 (5t + t^2 + 2t^5)\,dt$$

$$= \frac{5}{2} + \frac{1}{3} + \frac{1}{3}$$

$$= \frac{15+2+2}{6}$$

$$= \frac{19}{6} \qquad \text{.. (Ans.)}$$

Properties of Line integral:

(i) $\displaystyle\int_C kF\,dr = k\int_C F\,dr$

(ii) $\displaystyle\int_C (F+G)\,dr = \int_C F\,dr + \int_C G\,dr$

(iii) $\displaystyle\int_C F\,dr = \int_{C_1} F\,dr + \int_{C_2} F\,dr$

(iv) $\displaystyle\int_a^b F\,dr = -\int_b^a F\,dr$

Line integrals independent of path:

$$\int_C F(r)\,dr = \int (F_1\,dx + F_2\,dy + F_3\,dz) \quad \text{—(1)}$$

A line integral (1) with continuous function F_1, F_2, F_3 in a domain 'D' in space is independent of path in 'D' if and only if $F = [F_1\ F_2\ F_3]$ (Gradient F) is the gradient of some function 'f' in 'D'.

i.e. $F = \text{grad} f$

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

→ e.g: Show that the integral $\int_C F(r)\,dr = \int (2x\,dx + 2y\,dy + 4z\,dz)$ is path independent in any domain in space and find its value in the integration from A (0, 0, 0) to B (2, 2, 2).

Sol^n

$F = [2x\ \ 2y\ \ 4z]$

$f = [x^2\ \ y^2\ \ 2z^2]$ is independent of path.

(convenient path)

Let $r(t) = [t\ \ t\ \ t]$, $r'(t) = [1\ \ 1\ \ 1] = \hat{i} + \hat{j} + \hat{k}$

$f(r(t)) = [t^2\ \ t^2\ \ 2t^2]$

$$\int_C f(r)\,dr = \int_0^2 (t^2 + t^2 + 2t^2)\,dt$$

$$= \frac{4}{3} t^3 \Big]_0^2 = \frac{32}{3}. \quad \text{.. (Ans)}$$

L-25 (29.11.2022) (04:00 - 05:00 p.m)

$F(r(t)) = 2t\hat{i} + 2t\hat{j} + 4t\hat{k}$

$$\int_0^2 F\, r'(t)\,dt = \int_0^2 (2t + 2t + 4t)\,dt = \frac{8}{2} t^2 \Big]_0^2 = 16 \quad \text{.. (Ans)}$$

Exactness:

If $\text{curl} f = 0$, then integral I is exact

Ex: $I = \int_C 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$

Show that 'I' is exact.

Sol^n

$$\text{curl} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + z\cos yz)\right] + \left[\frac{\partial}{\partial z}(2xyz^2) - \frac{\partial}{\partial x}(2x^2yz + y\cos yz)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(x^2z^2 + z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]\hat{k}$$

$$= \left(2x^2z + \cos yz - yz\sin yz\right)\hat{i}$$
$$- \left(2zx^2 + \cos yz - zy\sin yz\right)\hat{i}$$

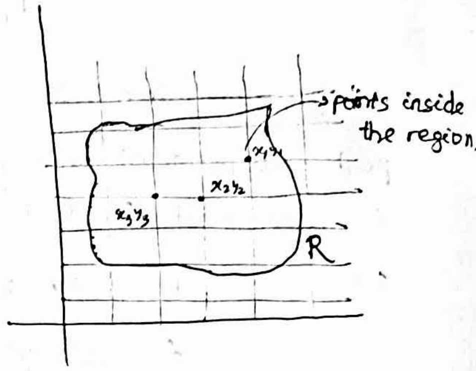
$+$

$+$

$\Rightarrow$ curl $f = 0$

$\Rightarrow$ '$\bar{f}$' is exact. .. (proved)

Double integral:



$$\iint_R f(x,y)\,dx\,dy$$

or $$\iint_R f(x,y)\,dA$$

is given by

$J_1 = f(x_0, y_0)\Delta A_0$

$J_2 = f(x_1, y_1)\Delta A_1$

$$J_n = \sum_{K=1}^{n} f(x_K, y_K)\Delta A_K$$

where 'A_K' is the area of the K^{th} rectangle.

$$\int_a^b \int_{g(x)}^{h(x)} [f(x,y)\,dy]\,dx \rightarrow$$ First integrate $f(x,y)$ w.r.t. 'y' then w.r.t. 'x'

$$\int_c^d \int_{p(y)}^{q(y)} [f(x,y)\,dx]\,dy \rightarrow$$ First integrate $f(x,y)$ w.r.t. 'x' then w.r.t. 'y'.

Applications of double integral:

(i) The area 'A' of a region 'R' in xy-plane, $A = \iint_R dx\,dy$

(ii) Volume $V = \iint_R f(x,y)\,dx\,dy$

↳ volume of the surface $z = f(x,y)$

(iii) Let $f(x,y)$ be the density of a distribution of mass in xy-plane.

Then, mass 'M' $= \iint_R f(x,y)\,dx\,dy$

(iv) Center of gravity $\bar{x} = \frac{1}{M}\iint_R x f(x,y)\,dx\,dy$

$$\bar{y} = \frac{1}{M}\iint_R y f(x,y)\,dx\,dy$$

(v) Moment of inertia

$$I_x = \iint_R y^2 f(x,y)\,dx\,dy, \quad I_y = \iint_R x^2 f(x,y)\,dx\,dy$$

(vi) Polar moment of inertia

$$I_0 = I_x + I_y$$

Ex: Let $f(x,y) = 1$ be the density of mass in the region R': $0 \le y \le \sqrt{1-x^2}$ & $0 \le x \le 1$

find the center of gravity and moment of inertia.

sol^n:

$$M = \iint_R f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \cancel{f(x,y)}^{1}\,dy\,dx$$

$$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} dy\right) dx$$

$$= \int_0^1 \sqrt{1-x^2}\,dx$$

put $x = \sin\theta$

$\Rightarrow \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{4}\cdot(\sin\pi - \sin 0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\bar{x} = \frac{4}{\pi}\int_0^1\int_0^{\sqrt{1-x^2}} x\,dx\,dy$$

$$= \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx$$

put $x = \sin\theta$

$\Rightarrow \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos^2\theta\,\sin\theta\,d\theta$$

$$= -\frac{4}{\pi}\int_0^{\pi/2} \cos^2\theta\,d\cos\theta$$

$$= -\frac{4}{3\pi}\left[\cos^3\theta\right]_0^{\pi/2}$$

$$= -\frac{4}{3\pi}(-1)$$

$$= \frac{4}{3\pi} \quad \text{.. (Ans)}$$

$$I_y = \int_0^1\int_0^{\sqrt{1-x^2}} x^2\,dx\,dy$$

$$= \int_0^1 x^2\sqrt{1-x^2}\,dx$$

$\bar{x} = \frac{4}{3\pi}$

$\bar{y} = \bar{x}$

$I_x = \frac{\pi}{16}$

$I_y = \frac{\pi}{8}$

$I_0 = \frac{\pi}{16} + \frac{\pi}{8}$

$= \frac{3\pi}{16}$

$$\bar{y} = \frac{4}{\pi}\int_0^1\int_0^{\sqrt{1-x^2}} y\,dx\,dy = \frac{4}{2\pi}\int_0^1 \left[y^2\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{4}{2\pi}\int_0^1 (1-x^2)\,dx = \frac{2}{\pi}\left(x - \frac{x^3}{3}\right)_0^1 = \frac{4}{3\pi} \quad \text{.. (Ans)}$$

Change of variables in double integral:

For a definite integral, we change 'x' to 'u' & 'y' to 'v' by using Jacobian.

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

R ↳ xy-plane R^* ↳ uv-plane

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

→ In polar co-ordinate, we use

$x = r\cos\theta,\ y = r\sin\theta$

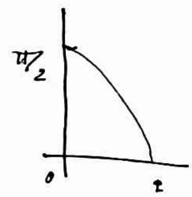
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

R ↳ xy plane R^* ↳ r-θ (polar)

Since this R^* is the region in r-θ plane (polar co-ordinates).

- How to find double integral?



Ex: $f(x,y) = 1$ Find I_x

$$\int_0^1\int_0^{\pi/2} r^2\sin^2\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\sin^2\theta\,dr\,d\theta$$

↓ .. limit?

$$= \frac{1}{4}\int_0^{\pi/2} \left.r^4\right]_0^1 \sin^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{8}\left[\left.\theta\right]_0^{\pi/2} - \left.\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}\right]$$

$$= \frac{1}{8}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{16}$$

... (Ans).

L-26 (30.11.2022) (02:00 - 03:00 p.m)

Green's Theorem:

(Transformation between double integral & line integral)

Let 'R' be a closed bounded region in xy plane whose boundary 'c' consists of finitely many smooth curves.

Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in same domain containing 'R'. Then

$$\iint_R \left(-\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)$$

closed bounded region

Verification of Green's theorem:

Ex: Let $F_1 = y^2 - 7y$ $\therefore \frac{\partial F_1}{\partial y} = 2y - 7$

and $F_2 = 2xy + 2x$ & $\frac{\partial F_2}{\partial x} = 2y + 2$

'c' is the circle i.e. $x^2 + y^2 = 1$

$\hookrightarrow y = 0 \Rightarrow x = \pm 1$ $\quad -1 \le x \le 1$, $y^2 = 1 - x^2 \Rightarrow y = \pm\sqrt{1-x^2}$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

L.H.S $= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R (2y + 2 - 2y + 7)\,dx\,dy$

$= \int\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 9\,dx\,dy$

$= 9 \times 2 \int_{-1}^{1} \sqrt{1-x^2}\,dx$

$= 18 \times 2 \int_0^1 \sqrt{1-x^2}\,dx$ put $x = \sin\theta$

$\Rightarrow \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\,d\theta$

$= 36 \int_0^{\pi/2} \cos^2\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$= \frac{36}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$

$= 18 \left\{ [\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} \right\}$

$= 18\left(\frac{\pi}{2} + 0\right)$

$= 9\pi$ .. (L.H.S)

Limit

- $x^2 + y^2 = 1$ put $y = 0$

$\Rightarrow x = \pm 1$

$-1 \le x \le 1$

- $y^2 = 1 - x^2$

$\Rightarrow y = \pm\sqrt{1-x^2}$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

R.H.S. $= \oint_C (F_1\,dx + F_2\,dy)$

$= \oint_C (CF_1\,dx) + F_2\,dy)$

For a circle, parametric function $r(t) = [\cos t, \sin t]$

$r'(t) = [-\sin t, \cos t]$

$\Rightarrow x'(t) = -\sin t$, $y'(t) = \cos t$.

$F_1(r(t)) = \sin^2 t - 7\sin t$

$F_2(r(t)) = 2\cos t \sin t + 2\cos t = \sin 2t + 2\cos t$

$$\text{R.H.S.} = \oint_C \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} \right) dl$$

$$= \int_0^{2\pi} (\sin^2 t - 7\sin t)(-\sin t) \, dt + \int_0^{2\pi} (\sin 2t + 2\cos t) \cos t \hat{j}) \, dt$$

$$= \int_0^{2\pi}$$

Applications of Green's theorem:

(1) Area of a plane region as a line integral over the boundary.

Here we choose

$$F_1 = 0, \; F_2 = x \quad \text{and} \quad F_1 = -y, \; F_2 = 0$$

$$\iint_R dx\,dy = \oint_C x\,dy, \quad \iint_R dx\,dy = -\oint_C y\,dx$$

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx) \quad \text{(Area)}$$

eg. To find the area of ellipse. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

let $x = a\cos t$, $x' = -a\sin t$

& $y = b\sin t$, $y' = b\cos t$

$$A = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{ab}{2}[t]_0^{2\pi} = \pi ab.$$

$$\cancel{\left[= \frac{ab}{2}\int_0^{2\pi}\left[\frac{1+\cos 2t}{2}\,dt\right] + \int_0^{2\pi}\left[\frac{1-\cos 2t}{2}\right]dt\right]}$$

(2) Area of a plane region in polar co-ordinates.

In polar co-ordinates, $x = r\cos\theta$, $x' = -r\sin\theta$

$y = r\sin\theta$, $y' = r\cos\theta$

$$\therefore \text{Area} = \frac{1}{2}\oint_C r^2\,d\theta$$

$$\left(\frac{1}{2}\oint_C (r^2\cos^2\theta + r^2\sin^2\theta)\,d\theta\right)$$

Suppose $r = a(1-\cos\theta)$ where $0 \le \theta \le 2\pi$

$$\therefore \text{Area} = \frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} (1 + \cos^2\theta - 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[[\theta - 2\sin\theta]_0^{2\pi} + \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta\right]$$

$$= \frac{a^2}{2}\left[(2\pi - 0) + \frac{1}{2}\left([\theta]_0^{2\pi} + \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right)\right] = \frac{a^2}{2}[2\pi + \pi] = \frac{3\pi a^2}{2}.$$

∴ (Ans)

Surface Integral:

$z = f(x, y)$

$x^2+y^2+z^2 = a^2$ (sphere)

$\Rightarrow x^2+y^2+z^2-a^2=0$.

[The parametric representation of a (circular) cylinder is] $\longrightarrow x$

circular cylinder $x^2+y^2 = a^2$, $-1 \le z \le 1$

has a radius 'a' and height '2'. $\hookrightarrow |z|$

- The parametric representation will be

$r(u, v) = [a \cos u, a \sin u, v]$ —— $\boxed{\text{cylinder}}$

- Parametric representation of sphere is

$r(u,v) = [a\cos v \cos u, a \cos v \sin u, a \sin v]$ $\quad 0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$

Another representation:

$r(u, v) = [a \cos u \sin v, a \sin u \sin v, a \cos v]$ $\quad 0 \le u \le 2\pi$, $0 \le v \le \pi$

$z = \sqrt{x^2+y^2}$ —— $\boxed{\text{sphere}}$

- The parametric representation of cone is

$r(u,v) = [u \cos v, u \sin v, u]$ —— $\boxed{\text{Cone}}$

Tangent plane and surface normal:

Normal vector, $N = r_u \times r_v$

r_u is $\dfrac{\partial r}{\partial u}$

r_v is $\dfrac{\partial r}{\partial v}$

Unit normal vector, $n = \dfrac{1}{|N|} N$

$g(x, y, z) = 0$ (e.g: $x^2+y^2+z^2-a^2=0$)

$\Rightarrow n = \dfrac{1}{|\operatorname{grad} g|} \cdot \operatorname{grad} g$

Ex: $g(x,y,z) = x^2+y^2+z^2-a^2 = 0$

Find the normal vector.

$\therefore$ Normal vector, $N = \operatorname{grad} g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore |N| = \sqrt{4x^2+4y^2+4z^2}$

(Unit normal vectors)

$$\therefore n = (|N|)^{-1} N$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{(x^2+y^2+z^2)}}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{x}{a}\hat{i} + \frac{y}{a}\hat{j} + \frac{z}{a}\hat{k}$$

$(\because x^2+y^2+z^2-a^2=0$
$\Rightarrow x^2+y^2+z^2=a^2$
$\Rightarrow \sqrt{x^2+y^2+z^2}=a)$

E.g: $g(x,y,z) = -z + \sqrt{x^2+y^2} = 0 \Rightarrow \sqrt{x^2+y^2} = z$

$$N = \text{grad } g = \frac{1}{2\sqrt{x^2+y^2}}\,2x\hat{i} + \frac{1}{2\sqrt{x^2+y^2}}\,2y\hat{j} - \hat{k}$$

$$= \frac{x}{z}\hat{i} + \frac{y}{z}\hat{j} - \hat{k}$$

$$|N| = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} = \sqrt{\frac{x^2+y^2+z^2}{z^2}}$$

$$= \frac{1}{z}\sqrt{z^2+z^2}$$

$$\therefore n = (|N|)^{-1} N =$$

L- (03.12.2022) (11:00 -

★ Surface Integral:

$r(u,v) = [x(u,v), y(u,v), z(u,v)]$

Normal vector, $N = r_u \times r_v$ and $n = \frac{1}{|N|} N$

$$\iint_S F \cdot n \, dA = \iint_R F(r(u,v)) \, N(u,v) \, du \, dv$$ → Flux integral (surface integral over S)

when $n = [\cos\alpha, \cos\beta, \cos\gamma]$

we can write

$$\iint_S F \cdot n \, dA = \iint_R (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) \, dA$$

★ Flux through a Surface:

Compute the flux of water through the parabolic cylinder $S: y = x^2$.

$0 \le x \le 2$, $0 \le z \le 3$

★ Probability
statistics
Kreyszig → all example & exercises

If the velocity vector $V = F = [3z^2, 6, 6xz]$

Sol^n: Let $x = u$ & $z = v$

then $y = u^2$ $\quad (\because y = x^2)$

$\therefore r(u,v) = [u, u^2, v]$

$r_u = [1, 2u, 0] \quad \left(\frac{\partial r}{\partial u}\right)$

& $r_v = [0, 0, 1] \quad \left(\frac{\partial r}{\partial v}\right)$

$\therefore N = r_u \times r_v$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2u\hat{i} + (-\hat{j}) = 2u\hat{i} - 1\hat{j}$$

$F = [3z^2, 6, 6xz]$

$= 3z^2\hat{i} + 6\hat{j} + 6xz\hat{k}$

$\therefore F(u,v) = 3v^2\hat{i} + 6\hat{j} + 6uv\hat{k}$

$F(r(u,v)) \cdot N(u,v) = 6uv^2 \hat{i} - 6\hat{j}\cdot\hat{j}$

$= 6uv^2 - 6$

$$\therefore \iint_R F(r(u,v)) \cdot N(u,v) = \int_0^2\int_0^3 (6uv^2 - 6)\,du\,dv$$

$$= \int_0^2 [3v^2u^2 - 6u]_0^3\,dv$$

$$= \int_0^2 [27v^2 - 18]\,dv$$

check limit

$$= [9v^3 - 18v]_0^2$$

$$= 72 - 36$$

$$= 36$$

X .. (Ans)

$$= \int_0^2 [3v^2u^2 - 6u]_0^2\,dv$$

$$= \int_0^3 [12v^2 - 12]\,dv$$

$$= [4v^3 - 12v]_0^3$$

$$= 108 - 36$$

$$= 72 \quad \text{.. (Ans)}$$

✓

★ Gauss Divergence Theorem:

It is the transformation between volume integral and surface integral.

Let 'T' be a closed and bounded region in space whose boundary is a piecewise smooth orientable surface 'S'. Let $F(x,y,z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing 'T', then

$$\iiint_T \text{div}\, F\,dv = \iint_S F\,n\,dA$$

If $n = [\cos\alpha, \cos\beta, \cos\gamma]$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S [F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma]\, dA$$

Evaluation of a surface integral by the divergence theorem

$$I = \iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$$

where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and $0 \le z \le b$ (circular disc)

Sol^n: $F_1 = x^3$

$F_2 = x^2y$

& $F_3 = x^2z$

$x^2+y^2=a^2$

for $y=0$, $x = \pm a$

$-a \le x \le a$

Then, $y^2 = a^2 - x^2$

$\Rightarrow y = \pm\sqrt{a^2-x^2}$

$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

$$\therefore \operatorname{div} F = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$\iiint_S \left(\frac{F_1}{dx} + \frac{F_2}{dy} + \frac{F_3}{dz}\right) dx\,dy\,dz$$

$$= 3x^2 + x^2 + x^2$$

$$= 5x^2$$

$$\therefore I = \iiint_T (\operatorname{div} F)\, dx\,dy\,dz$$

$$= \int_0^b \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 5x^2\, dx\,dy\,dz$$

$$= 2\int_0^b \int_{-\sqrt{a^2}}^{a} \sqrt{a^2-x^2}\cdot 5x^2\,dx\,dz$$

$$= 2\int_0^b 2\int_0^a \sqrt{a^2-x^2}\, 5x^2\,dx\,dz$$

put $x = a\sin\theta$

$\Rightarrow \sqrt{a^2-x^2} = a\cos\theta$

when $\begin{cases} x=0 \\ \theta=0 \end{cases}$

$x = a$

$\theta = \pi/2$

$dx = a\cos\theta$

$$= 4\int_0^b 5\int_0^{\pi/2} a^3\cos\theta\,\sin^2\theta\, a\cos\theta\, d\theta\, dz$$

$$= 20a^4\int_0^b \int_0^{\pi/2} \cos^2\theta\sin^2\theta\, d\theta\, dz$$

$$= 20a^4\int_0^b \int_0^{\pi/2} \frac{\sin^2 2\theta}{4}\, d\theta\, dz$$

$$= 5a^4\int_0^b \int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\, d\theta\, dz$$

$$= \frac{5}{2}a^4\int_0^b \left(\theta - \frac{1}{4}\sin 4\theta\right)_0^{\pi/2} dz$$

$$= \frac{5}{2}a^4\int_0^b \left(\frac{\pi}{2}\right) dz$$

$$= \frac{5\pi}{4}a^4 b \quad \text{.. (Ans)}$$

* Stoke's Theorem:

(Transformation between surface integral and line integral)

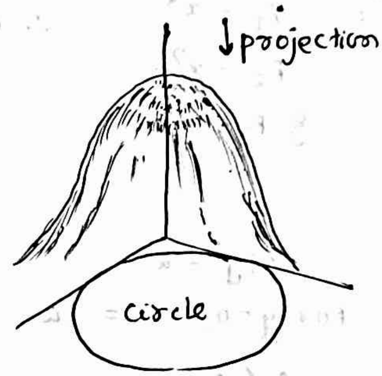
Let 'S' be a piecewise smooth oriented surface in space and let the boundary of 'S' be a piecewise smooth simple closed curve 'C'.

Let 'F' be a continuous vector function that has continuous first partial derivatives in a domain in space containing 'S'. Then

$$\iint_S (\text{curl } F \cdot n)\, dA = \oint_C F\, r'(s)\, ds$$

(Surface Integral & Line Integral)

Paraboloid:



Q. Verify Stoke's theorem:

Given $F = [y \quad z \quad x]$

$z = f(x,y) = 1 - (x^2+y^2)$

'S' = paraboloid

$r(s) = [\cos(s), \sin(s), 0]$

$\therefore r'(s) = [-\sin(s), \cos(s), 0]$

$F(r(s)) = [\sin(s), 0, \cos(s)]$

$\therefore \oint_C F(r(s)) \cdot r'(s) \cdot ds$

$= \int_0^{2\pi} -\sin^2 s\, ds$

$= -\int_0^{2\pi} \frac{1-\cos 2s}{2}\, ds$

$= -\frac{1}{2}\left(s - \frac{\sin 2s}{2}\right)_0^{2\pi}$

$= -\frac{1}{2}(2\pi)$

$= -\pi.$

(R.H.S.)

R.H.S.:

$\iint_S (\text{curl} F \cdot n)\, dA$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$= \left(\frac{\partial x}{\partial y} - 1\right)\hat{i} + \left(\frac{\partial y}{\partial z} - 1\right)\hat{j} + \left(\frac{\partial z}{\partial x} - 1\right)\hat{k}$

$= \frac{\partial x}{\partial y}\hat{i} + \frac{\partial y}{\partial z}\hat{j} + \frac{\partial z}{\partial x}\hat{k} - (\hat{i}+\hat{j}+\hat{k})$

$= -\hat{i} - \hat{j} - \hat{k}$

$g(x,y,z) = 1 - (x^2+y^2) - z$

$z - 1 + (x^2+y^2) \Rightarrow N = [2x \quad 2y \quad 1]$

$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).g$

$N = \text{grad}\, g = -2x\hat{i} - 2y\hat{j} + \hat{k}$

$= [-2x \quad -2y \quad +1]$

$z = f(x,y) = 1 - (x^2+y^2)$

$\Rightarrow f(x,y) = 1 - (x^2+y^2) - z = 0$

$|\text{grad}\, g| = \sqrt{4x^2+4y^2+1}$

$\times(-1)$

$n.dA = N\,dx\,dy$ —— (how?)

$N = |N|n$

$N\,dx\,dy = |N|n\,dx\,dy$

$= n\,|N|\,dx\,dy$

$= n\,dA$

$|N| = \sqrt{x^2+y^2}$

$\Rightarrow |N|\,dx\,dy = dA$

dx dy dA

$$\iint_S \text{curl}\, F\, n\, dA = \iint_S \text{curl}\, F.N\,dx\,dy$$

$$= \iint_S (-\hat{i} - \hat{j} - \hat{k})(-2x\hat{i} - 2y\hat{j} + \hat{k})\,dx\,dy$$

$$= -\iint_S (2x + 2y + 1)\,dx\,dy$$

put $x = r\cos\theta$

$y = r\sin\theta$

$$= -\int_0^1\int_0^{2\pi} (2r\cos\theta + 2r\sin\theta + 1)\,r\,dr\,d\theta$$

$\left(\frac{dx}{dr} = \cos\theta\right) \ldots J = r$

$dx\,dy = r\,dr\,d\theta$

$$= -\int_0^1\int_0^{2\pi} (2r\cos\theta + 2r\sin\theta + 1)\,d\theta\; r\,dr$$

$$= -\int_0^1 \left[2r\sin\theta + (-2r\cos\theta) + \theta\right]_0^{2\pi} r\,dr$$

$$= -\int_0^1 2\pi\, r\,dr$$

$$= -\frac{2\pi}{2}\; r^2\Big]_0^1$$

$= -\pi \;=$ R.H.S.

∴ (proved.)