

Exact equation:

The equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

How to get solution?

$$\int M dx + \int N dy = C$$

(Treating y as constant) (only those terms which are free from x)

Q: Check whether the equation is exact or not then find the solution.

$$(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$$

Sol

$$M(x,y) = 3x^2 + 2e^y$$

$$N(x,y) = 2xe^y + 3y^2$$

$$\frac{\partial M}{\partial y} = 2e^y$$

$$\frac{\partial N}{\partial x} = 2e^y$$

$$\text{Hence, } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\Rightarrow eqn is exact

$$\begin{aligned}\int M dx &= \int (3x^2 + 2e^y) dx \\ &= x^3 + 2xe^y\end{aligned}$$

$$\int N dy \times \int 3y^2 dy = y^3$$

$$\int (3x^2 + 2e^y) dx + \int 3y^2 dy = C$$

$$\Rightarrow x^3 + 2x e^y + y^3 = C$$

Q1- What about $e^x (\cos y dx + \sin y dy) = 0$

$$Y(0) = 0$$

Sol: $e^x \cos y dx - e^x \sin y dy = 0$

$$\frac{\partial N}{\partial y} = -\sin y e^x$$

$$\frac{\partial N}{\partial x} = -e^x \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \text{ex} \text{ is exact}$$

No

$$\int e^x \cos y dx + \int 0 dy = C$$

$$\Rightarrow e^x \cos y = C \quad \text{--- } ①$$

Given, $Y(0) = 0$

From ①

$$e^0 \cos 0 = C$$

$$\Rightarrow C = 1$$

Now $\Rightarrow ① \Leftrightarrow$

$$\boxed{e^x \cos y = 1}$$

$$\int (3x^2 - 2e^y) dx + \int 3y^2 dy = C$$

$$\Rightarrow x^3 + 2xe^y + y^3 = C$$

Q:- What about $e^x (\cos y dx + \sin y dy) = 0$

$$Y(0) = 0$$

Sol

$$e^x \cos y dx - e^x \sin y dy = 0$$

$$\frac{\partial N}{\partial y} = -\sin y e^x$$

$$\frac{\partial N}{\partial x} = -e^x \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\Rightarrow exact

No

$$\int e^x \cos y dx + \int 0 dy = C$$

$$\Rightarrow e^x \cos y = C \quad \text{--- } ①$$

$$\text{Given, } Y(0) = 0$$

From ①

$$e^0 \cos 0 = C$$

$$\Rightarrow C = 1$$

Now sol is

$$\boxed{\int e^x \cos y = 1}$$

Integrating Factor: $(1 \cdot F) / M$

Integrating factor is a function $f(x,y)$, which is multiplied with a non-exact equation, it will be exact.

Ex: $y dx - x dy = 0 \quad \text{--- (1)}$

The above eqn is not exact.

Let $f(x,y) = \cancel{\frac{1}{y^2}}$

Now multiply $f(x,y)$ in eqn (1) we get

$$\frac{y dx - x dy}{y^2} = 0$$

$$\Rightarrow \frac{1}{y} dx - \frac{x}{y^2} dy = 0 \quad \text{--- (2)}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

$$\frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\Rightarrow eqn (2) is exact.

$$\int \frac{1}{y} dx + \int 0 dy = C$$

$$\Rightarrow \frac{x}{y} = C \quad (\text{S. Defn})$$

$$\text{Let } g(x, y) = \frac{1}{\frac{1}{2}(y^2 - x^2)}$$

Now multiplying $g(x, y)$ in eqn ⑦ we get

$$\frac{2y}{y^2 - x^2} dy - \frac{2x}{y^2 - x^2} dx = 0 \quad \text{--- (3)}$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{2(y^2 - x^2) - 2xy^2}{(y^2 - x^2)^2} \\ &= \frac{2y^2 - 2x^2 - 4xy^2}{(y^2 - x^2)^2} \\ &= \frac{-2x^2 - 2y^2}{(y^2 - x^2)^2} \\ &= \end{aligned}$$

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{-2(y^2 - x^2) + 2x(y^2 - x^2)}{(y^2 - x^2)^2} \\ &= \frac{-2y^2 + 2x^2 - 4x^2}{(y^2 - x^2)^2} \\ &= \frac{-2x^2 + 2y^2}{(y^2 - x^2)^2}\end{aligned}$$

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

\Rightarrow eqn ③ is exact

How to find integrating factor?

1. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(y)$ then I.F. = $e^{\int f(y) dy}$

2. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(x)$ then I.F. = $e^{\int -g(x) dx}$

3. If M & N are homogenous functions of same degree and $Mx + Ny \neq 0$ then I.F. = $\frac{1}{Mx + Ny}$. But, if $Mx + Ny = 0$ then $\frac{1}{xy}, \frac{1}{x^2}, \frac{1}{y^2}$ may be taken as integrating factor.

$\underline{x} - \underline{x}$

Q1. Find the solution of the following diff. eqn

$$(5x^3 + 12x^2 + 6y^2)dx + 6xydy = 0$$

Sol: Given, $(5x^3 + 12x^2 + 6y^2)dx + \underbrace{6xydy}_N = 0$ — (1)

$$\frac{\partial M}{\partial y} = 12y, \quad \frac{\partial N}{\partial x} = 6y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad (\text{Not exact equation})$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 12y - 6y = 6y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{6y}{6xy} = \frac{1}{x} = f(x)$$

Now, $I.F = e^{\int f(x)dx} = e^{\int \frac{1}{x} dx} = e^{\ln x}$

$$\Rightarrow u = x$$

Multiply I.F in eqn (1)

$$(5x^4 + 12x^3 + 6xy^2)dx + \underbrace{6x^2ydy}_N' = 0$$

(Exact equation)

Solution: $\int M'dx + \int N'dy = C$

(Treating 'y'
as constant)

(Taking the term
which are rest
from x)

$$\Rightarrow \int (5x^4 + 12x^3 + 6xy^2) dx + \int 0 dy = c$$

$$\Rightarrow x^5 + 3x^4 + 3x^2y^2 = c$$

G selection

Q1:- Find solution for the following differential equation

$$(3x^2y^3e^y + y^3 + y^2) dx + (x^3y^3e^y - xy) dy = 0$$

Sol'
Given, $(3x^2y^3e^y + y^3 + y^2) dx + (x^3y^3e^y - xy) dy = 0 \quad \text{--- } (1)$

$$\frac{\partial M}{\partial y} = 9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y$$

$$\frac{\partial N}{\partial x} = 3x^2y^3e^y - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad (\text{not exact eqn})$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 9x^2y^2e^y + 3y^2 + 3y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \neq \frac{9x^2y^2e^y + 3y^2 + 3y}{9x^2y^3e^y + 3x^2y^3e^y}$$

$$= \frac{9x^2y^2e^y + 3y^2 + 3y}{3x^2y^3e^y + y^3 + y^2} = \frac{3(3x^2y^2e^y + y^2 + y)}{y(3x^2y^2e^y + y^2 + y)}$$

$$= \frac{3}{y} = g(y)$$

$$\text{Now, } I.F = \frac{\int g(y) dy}{e^{\int f(x) dx}} = e^{\int -\frac{3}{y} dy} = e^{-3 \ln y} = \frac{1}{y^3}$$

Now multiplying I.F in eqn ①

$$(3x^2 e^y + 1 + \frac{1}{y}) dx + (x^3 e^y - \frac{x}{y^2}) dy = 0$$

M' N' (encl of eqn.)

To get solution:

$$\int M' dx + \int N' dy = c$$

(Taking 'y' as constant) (Taking the term rest from x)

$$\Rightarrow \int (3x^2 e^y + 1 + \frac{1}{y}) dx + \int 0 dy = c$$

$$\Rightarrow \boxed{x^3 e^y + x + \frac{1}{y} = c}$$

∴ Solution

$x \rightarrow x$

Linear equation:

$$\text{In the form of: } \frac{dy}{dx} + p(x)y = q(x)$$

$$I.F = u = e^{\int p dx}$$

$$\text{To compute: } yu = \int q u dx \quad (\text{To find soln of diff. eqn})$$

$$\text{eg: } y' - y = e^{2x}$$

$$\text{soln } I.F = u = e^{\int p dx} = e^{\int 1 dx} = e^{-x}$$

$$yu = \int q u dx$$

$$\Rightarrow ye^{-x} = \int e^{2x} \cdot e^{-x} dx$$

$$\Rightarrow ye^{-x} = e^x + C$$

G solution of diff. eqn

Q1. Find the solution of the following differential equation

$$\frac{dy}{dx} + y \tan x = \sin 2x, \quad y(0) = 1$$

Sol Given, $\frac{dy}{dx} + y \underbrace{\tan x}_{P(x)} = \underbrace{\sin 2x}_{Q(x)} \quad \text{--- (1)}$
I.F. = $e^{\int P(x) dx}$ $= e^{\int \tan x dx} = e^{\ln \sec x} = \sec x = u$

~~Multiplying L.H.S. by I.F. we get~~

~~$\sec x \frac{dy}{dx} + y \sec x$~~

solution!- $y u = \int Q u dx$

$$\Rightarrow y \sec x = \int \sin 2x \sec x dx$$

$$\Rightarrow y \sec x = 2 \int \sin x dx$$

$$\Rightarrow y \sec x = -2 \cos x + C$$

$$\Rightarrow y \sec x + 2 \cos x = C \quad \text{--- (2)}$$

Given, $y(0) = 1$ (when $x=0 \Rightarrow y=1$)

from (2) $1 \times \sec 0 + 2 \times \cos 0 = C$

$$\Rightarrow 1 \times 1 + 2 \times 1 = C$$

$$\Rightarrow C = 3$$

New solution will be

$$y_{\text{green}} + 20051 = 3$$

Bernoulli's equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^\alpha \quad \alpha \in \mathbb{R} \quad (1)$$

$$u = y^{1-\alpha}$$

$$\frac{du}{dx} = (1-\alpha)y^{-\alpha} \frac{dy}{dx}$$

Dividing y^α in both sides in eqn ①

$$y^{-\alpha} \frac{dy}{dx} + P(x)y^{1-\alpha} = Q(x)$$

$$\Rightarrow \frac{1}{(1-\alpha)} \left(\frac{du}{dx} \right) + uP(x) = Q(x)$$

$$\Rightarrow \frac{du}{dx} + (1-\alpha)uP(x) = (1-\alpha)Q(x)$$

Apply procedure of linear equation

Q1 solve $y' + 4xy + 2y^3 = 0$

SOL

$$y' + 4xy = -2y^3 \quad (1)$$

$P(x) \quad Q(x)$

Let $u = y^{1-3} = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}y'$$

Dividing y^3 in eqn ① we get

$$y^{-3}y' + 4xy^{-2} = -x$$

$$\Rightarrow -\frac{1}{2} \frac{du}{dx} + 4xu = -x$$

$$\Rightarrow \frac{du}{dx} - 8xu = 2x \quad \text{--- (2)}$$

$$1.F = u = e^{\int -8x dx}$$

$$= e^{-4x^2}$$

solve for (2)

$$dx(1.F) = \int 2x e^{-4x^2} dx$$

$$\Rightarrow u \cdot e^{-4x^2} = \int 2x e^{-4x^2} dx$$

$$\Rightarrow u \cdot e^{-4x^2} = \frac{-1}{4} \int e^z dz$$

$$\Rightarrow u \cdot e^{-4x^2} = \frac{-1}{4} e^z + C$$

$$\Rightarrow u \cdot e^{-4x^2} = -\frac{1}{4} e^{-4x^2} + C$$

$$\Rightarrow u = -\frac{1}{4} + C e^{4x^2}$$

$$\Rightarrow \boxed{y^2 = C e^{4x^2} + -\frac{1}{4}}$$

put $-4x^2 = z$
 $\Rightarrow -8x dx = dz$
 $\Rightarrow 2x dx = -\frac{dz}{4}$

selection

Application 1st order differential equation :-

① The rate of change of temperature $\frac{dT}{dt}$ ($T(t)$ is the temperature) is directly proportional to the difference of temperatures (T) and temperature of surrounding (T_A).

That means $\frac{dT}{dt} \propto (T - T_A)$

Let the temperature of the room at $t=0$ is $66^\circ F$ at $t=2$, $63^\circ F$. If temperature of surrounding is $32^\circ F$. Then find the temperature of the room after 10 mins.

Sol:-

$$\frac{dT}{dt} \propto (T - T_A)$$

$$\Rightarrow \frac{dT}{dt} = K(T - T_A)$$

$$\Rightarrow \frac{dT}{T - T_A} = Kdt$$

$$\Rightarrow \ln(T - T_A) = kt + C$$

$$\Rightarrow T - T_A = e^{kt+C}$$

$$\Rightarrow T = e^{kt+C} + T_A$$

$$\text{or } T(t) = e^{kt+C} + T_A$$

$$\text{at } t=0, T=66$$

$$66 = e^{k \cdot 0 + C} + 32$$

$$\Rightarrow e^C = 34$$

$$\Rightarrow C = \ln 34$$

Now,

$$T(t) = e^{kt+C} + T_A$$

at $t=2$, $T=63$

$$63 = e^{K \cdot 2 + \ln 34} + 32$$

$$\Rightarrow 31 = 34 e^{2K} *$$

$$\Rightarrow e^{2K} = \frac{31}{34}$$

$$\Rightarrow K = \frac{1}{2} \ln \frac{31}{34}$$

Note,

$$T(t) = e^{\left(\frac{1}{2} \ln \frac{31}{34}\right)t + \ln 34} + T_A$$

Put $t=10$

$$\begin{aligned}\Rightarrow T(10) &= e^{\left(\frac{1}{2} \ln \frac{31}{34}\right) \times 10 + \ln 34} \\ &= e^{\ln \left(\frac{31}{34}\right)^5 + \ln 34} + 32 \\ &= 34 \cdot \left(\frac{31}{34}\right)^5 + 32 \\ &= \frac{(31)^5}{(34)^4} + 32\end{aligned}$$

Malthus law of population growth :-

Let $P(t)$ be the population of particular species at time t .
The rate of change of population at time t is proportional to the population present at that time.

$$\frac{dP(t)}{dt} \propto P(t)$$

At $t=0$ population of particular species is 5.3 million.
At $t=1$, $P(t) = 4.8$ million. Find population of the species
at $t=10$.

Sol:

$$\frac{dP(t)}{dt} \propto P(t)$$

$$\Rightarrow \frac{dP(t)}{dt} = k P(t)$$

$$\Rightarrow \frac{dP(t)}{P(t)} = k dt$$

$$\Rightarrow \ln(P(t)) = kt + c$$

$$\Rightarrow P(t) = e^{kt+c}$$

$$\text{at } t=0, P(0) = 5.3$$

$$5.3 = e^{k \cdot 0 + c}$$

$$\Rightarrow 5.3 = e^c$$

$$\Rightarrow c = \ln 5.3$$

$$\text{at } t=1, P(1) = 4.8$$

$$4.8 = e^{k \cdot 1 + \ln 5.3}$$

$$\Rightarrow 4.8 = 5.3 e^k$$

$$\Rightarrow k = \ln\left(\frac{4.8}{5.3}\right)$$

$N=0$

$$P(t) = e^{\left(\ln\left(\frac{4.8}{5.3}\right)\right)t + \ln 5.3}$$

~~$$\Rightarrow \cancel{P(t) = e^{\ln\left(\frac{4.8}{5.3}\right)t + \ln 5.3}}$$~~

$$\text{at } t=10 \quad 10 \ln\left(\frac{4.8}{5.3}\right) + C_0(5.3)$$

$$P(10) = e^{C_0(5.3)}$$

$$\Rightarrow P(10) = \left(\frac{4.8}{5.3}\right)^{10} \cdot (5.3)$$

voltage drop (E_L) across an inductor = $L \frac{dI}{dt}$
 //, //, E_C , //, a capacitor = $\frac{1}{C} Q$

$$E_R = RI$$

$$L \frac{dI}{dt} + RI = E(t)$$

$$\Rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right) I = \frac{1}{L} E(t)$$

then :-

If $L = 0.1 \text{ H}$, $R = 5 \Omega$, $E(t) = 12$ find ~~for the parallel circuit~~ diff. eq & current

Soln Diff. eq will be :-

$$\frac{dI}{dt} + \left(\frac{R}{L}\right) I = \frac{1}{L} E(t)$$

$$\Rightarrow \frac{dI}{dt} + \left(\frac{5}{0.1}\right) I = \frac{1}{0.1} \times 12$$

$$\Rightarrow \boxed{\frac{dI}{dt} + 50I = 120} \rightarrow \text{linear differential eq}$$

$$\text{I.F.} = u = e^{\int 50 dt} = e^{50t}$$

$$\text{solution:- } I \cdot u = \int 120 e^{50t} dt$$

$$\Rightarrow I \cdot e^{50t} = \int 120 e^{50t} dt$$

$$\Rightarrow \boxed{I \cdot e^{50t} = \frac{12}{5} e^{50t} + C}$$

↳ solution:

$$\text{Current, } I = \frac{12}{5} + C \cdot e^{-50t}$$

Q. If $E(t) = E_0 \sin \omega t$ then find the current? - 2

Sol:

P.E will be

$$L \frac{dI}{dt} + IR = E_0 \sin \omega t$$

$$\Rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right) I = \left(\frac{E_0}{L}\right) \sin \omega t$$

$$I \cdot M = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$$

$$I \cdot M = \int \frac{E_0}{L} \sin \omega t \cdot e^{\frac{R}{L} t} dt$$

$$\Rightarrow I \cdot e^{\frac{R}{L} t} = \frac{E_0}{L} \int \sin \omega t \cdot e^{\frac{R}{L} t} dt$$

$$\Rightarrow I e^{\frac{R}{L} t} = \frac{E_0}{L} \frac{e^{\frac{R}{L} t}}{\left(\frac{R}{L}\right)^2 + \omega^2} \times$$

$$\left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right)$$

+ C

$$\begin{aligned} & \int e^{ax} \sin bx dx \\ &= \frac{e^{ax}}{a^2+b^2} (\alpha \sin bt - b \cos bt) \end{aligned}$$

$$\begin{aligned} & \int e^{ax} \cos bx dx \\ &= \frac{e^{ax}}{a^2+b^2} (\alpha \cos bt + b \sin bt) \end{aligned}$$

$$\Rightarrow I = \frac{E_0 L}{R^2 + \omega^2 L^2} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + C e^{-\frac{R}{L} t}$$

DT - 02/02/2024

Second order homogeneous equation with constant coefficient

General form:-

$$y'' + ay' + by = 0$$

Solution

Step-1: Put $y'' = \lambda^2$, $y' = \lambda$, $y = 1$

Step-2: Characteristic equation becomes,

$$\lambda^2 + a\lambda + b = 0$$

Step-3: Solve the characteristic equation

(A) If $\lambda = \lambda_1, \lambda_2$

Then gen. soln,

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

(B) If $\lambda = \lambda_1 = \lambda_2$

Then general soln

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

(C) Let $\lambda = d \pm i\beta$

The general soln

$$y = e^{dx} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$Q:- y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

or put $y'' = \lambda^2, \quad y' = \lambda, \quad y = 1$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 2, 1$$

general soln,

$$y = C_1 e^{2x} + C_2 e^x \quad \text{--- (1)}$$

$$y(0) = 1$$

$$\text{From (1)} \quad 1 = C_1 e^0 + C_2 e^0$$

$$\Rightarrow C_1 + C_2 = 1 \quad \text{--- (2)}$$

From (1)

$$y' = 2C_1 e^{2x} + C_2 e^x$$

$$y'(0) = 1$$

$$1 = 2C_1 e^0 + C_2 e^0$$

$$\Rightarrow 2C_1 + C_2 = 1 \quad \text{--- (3)}$$

From (2) & (3)

$$C_1 = 0$$

$$C_2 = 1$$

Now,

$$y = 0 \cdot e^{2x} + 1 \cdot e^x$$

$$\Rightarrow \boxed{y = e^x} \quad \text{solution}$$

Q. Let $y = e^{2x}(C_1 \cos x + C_2 \sin x)$ be the solution of second order diff. & linear homogenous eq. Then Find eq?

Sol:

$$y = e^{2x}(C_1 \cos x + C_2 \sin x)$$

$$\alpha = 2, \beta = 1$$

$$\alpha \pm i\beta = 2 \pm i \quad 2+i, 2-i$$

$$y = e^{1x}(C_1 \cos x + C_2 \sin x)$$

root₁, root₂

when solution of quadratic eq

$$\begin{aligned} & \cancel{x^2 - (\alpha + \beta)x + \alpha\beta = 0} \\ \Rightarrow & \cancel{x^2 - 3x + 2 = 0} \\ \Rightarrow & \cancel{y'' - 3y' + 2y = 0} \quad \text{aff. eq} \end{aligned}$$

$$x^2 - (r_{\text{root1}} + r_{\text{root2}})x + (r_{\text{root1}} \times r_{\text{root2}}) = 0$$

$$\Rightarrow x^2 - 4x + 5 = 0$$

$$\Rightarrow \boxed{y'' - 4y' + 5y = 0} \quad \text{Diff. eq}$$

~~Cauchy Euler equation~~

$$\hookrightarrow x^2 y'' + axy' + by = 0$$

Characteristic equation: $m^2 + (a-1)m + b = 0$
(m is variable)

- ① If m_1 & m_2 are two different real roots then
general solution,

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

- ② If $m = m_1 = m_2$,

general solution,

$$y = (C_1 + C_2 \ln x) x^m$$

- ③ If $m = \alpha + i\beta$,

general solution,

$$y = x^\alpha (A \cos(\beta \ln x) + B \sin(\beta \ln x))$$

Q: $x^2 y'' - 2xy' + 2y = 0 \Rightarrow y(1) = 1.5, y'(1) = 1$

Find general solution,

s.t.

~~$m^2 + 0$~~

$$x^2 y'' - 2xy' + 2y = 0$$

$(x^2 y'' + axy' + by = 0)$

$$a = -3, b = 2$$

Characteristic equation:

$$m^2 + (-3+1)m + 2 = 0$$

$$\Rightarrow m^2 - 2m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\Rightarrow m = 2, 1$$

general solution

$$y = C_1 x^2 + C_2 x \quad \text{--- (1)}$$

$$y(1) = 1.5$$

$$\text{so, } 1.5 = C_1 \cdot 1^2 + C_2 \cdot 1$$

$$\Rightarrow C_1 + C_2 = 1.5 \quad \text{--- (2)}$$

$$\text{From (1)} \quad y = 2C_1 x + C_2 \quad \text{--- (3)}$$

$$y'(1) = 1$$

$$\text{so, } 1 = 2 \cdot C_1 \cdot 1 + C_2$$

$$\Rightarrow 2C_1 + C_2 = 1 \quad \text{--- (4)}$$

From (2) & (4)

$$C_1 = -0.5$$

$$C_2 = 2$$

Now general solution will be

$$y = -0.5x^2 + 2x$$

Q1. $(x^2 D^2 - 3xD + 4)y = 0$ Find general solution

Sol¹ $(x^2 D^2 - 3xD + 4)y = 0$ Remember
 $D^2y = y''$
 $Dy = y'$

$$\Rightarrow x^2 D^2 y - 3xDy + 4y = 0$$

$$\Rightarrow x^2 y'' - 3xy' + 4y = 0$$

$$a = -3, b = 4 \quad (x^2 y'' + axy' + by = 0)$$

Characteristic equation:

$$m^2 + (a-1)m + b = 0$$

$$\Rightarrow m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

general sol¹,

$$y = (C_1 + C_2 \ln x) x^m$$

$$\Rightarrow y = (C_1 + C_2 \ln x) x^2$$

Q1. $x^2 y'' + 3xy' + 4y = 0$, find solution.

Sol¹ $x^2 y'' + 3xy' + 4y = 0$ $(x^2 y'' + axy' + by = 0)$

$$a = 3, b = 4$$

Characteristic eqn:

$$m^2 + (a-1)m + b = 0$$

$$\Rightarrow m^2 + 3m + 4 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2}$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{-12}}{2}$$

$$m = -1 \pm i\sqrt{3}$$

$$\alpha = -1, \beta = \sqrt{3}$$

general soln

$$y = e^{-x} [A \cos(\sqrt{3} \ln x) + B \sin(\sqrt{3} \ln x)]$$

$$y = x^{-1} [\cancel{A \cos(\sqrt{3} \ln x)} + B \sin(\sqrt{3} \ln x)]$$

$$Q_1 \quad y'' + 4y' + 3y = 0$$

sol

$$y'' + 4y' + 3y = 0$$

$$\Rightarrow n \omega$$

~~$$\lambda^2 + 4\lambda + 3 = 0$$~~

$$\Rightarrow (\lambda + 3)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -3, -1$$

general soln

$$y = C_1 e^{-3x} + C_2 e^{-x}$$

Sum rule:-

If $r(x)$ is the sum of two or more functions in column 1 then take y_p as the sum of corresponding functions of column 2

Q:- $y'' + 2y' - 35y = 12e^{5x} + 37 \sin 2x$

Solution = ??

Ans $y'' + 2y' - 35y = 12e^{5x} + 37 \sin 2x \quad \text{--- (1)}$

To find y_c

$$y'' + 2y' - 35y = 0$$

$$\lambda^2 + 2\lambda - 35 = 0$$

$$\Rightarrow \lambda = -7, 5$$

$$y_c = C_1 e^{-7x} + C_2 e^{5x}$$

To find y_p

$$y_p(x) = 12e^{5x} + 37\sin 5x$$

$$y_p = Ce^{5x} + K \sin 5x + M \cos 5x$$

put $y = y_p$ in eqn ①

$$y_p'' + 2y_p' - 35y_p = 12e^{5x} + 37\sin 5x$$

$$\Rightarrow 25Ce^{5x} - 35K \sin 5x - 35M \cos 5x$$

$$+ 25Ce^{5x} + 5K \cos 5x + 5M \sin 5x$$

After modification,

$$y_p = Ce^{5x} + K \sin 5x + M \cos 5x$$

$$y_p' = 5Ce^{5x} + Ce^{5x} + 5K \cos 5x - 5M \sin 5x$$

$$y_p'' = 25Ce^{5x} + 5Ce^{5x} + 5Ce^{5x} - 25K \sin 5x$$

$$- 25M \cos 5x$$

putting $y = y_p$ in eqn ①

$$25Ce^{5x} + 5Ce^{5x} + 5Ce^{5x} - 25K \sin 5x - 25M \cos 5x$$

$$+ 2(5Ce^{5x} + Ce^{5x} + 5K \cos 5x - 5M \sin 5x)$$

$$- 35(Ce^{5x} + K \sin 5x + M \cos 5x) = 12e^{5x} + 37\sin 5x$$

$$\Rightarrow 12ce^{5x} + (-60k - 10m) \sin 5x + (10k - 60m) \cos 5x = 12e^{5x} + 37 \sin 5x$$

comparing both the sides we will get

$$12c = 12 \quad \& \quad -60k - 10m = 37 \\ \Rightarrow \boxed{c=1} \quad \& \quad 10k - 60m = 0$$

So, $\boxed{k = \frac{-3}{5}}$, $\boxed{m = \frac{1}{10}}$

Now,

$$\text{solution: } y = y_c + y_p$$

$$\Rightarrow y = C_1 e^{-7x} + C_2 e^{5x} + C_3 x e^{5x} + k \sin 5x + m \cos 5x$$

$$\Rightarrow \boxed{y = C_1 e^{-7x} + C_2 e^{5x} + C_3 x e^{5x} - \frac{3}{5} \sin 5x - \frac{1}{10} \cos 5x}$$

Solution of non-homogeneous eqn by variation of parameters

$$\rightarrow y'' + ay' + by = r(x) \quad \leftarrow$$

Step 1 Make it homogenous
and find

$$y_c = c_1 y_1 + c_2 y_2$$

Step 2 Find Wronskian by the formula

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

Step 3 $y_p = -y_1 \int \frac{y_2 r}{W} + y_2 \int \frac{y_1 r}{W}$

Q: $y'' + y = \sin x$, solution = ??

Sol

$$y'' + y = \sin x \quad \text{--- ①}$$

S1 Making homogeneous

$$y'' + y = 0$$

$$y \rightarrow \lambda \quad \lambda^2 + 1 = 0 \\ \Rightarrow \lambda = \pm i \quad (\equiv \alpha \pm i\beta)$$

$$y_c = e^{\alpha x} (A \sin \beta x + B \cos \beta x)$$

$$y_c = A \sin x + B \cos x \\ (\equiv Q_1 y_2 + Q_2 y_1)$$

Step

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

$$= \begin{vmatrix} \cancel{\sin x} & \cancel{\cos x} \\ \cos x & -\sin x \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

S-3

$$y_p = -y_1 \int \frac{y_2 r}{\omega} + y_2 \int \frac{y_1 r}{\omega}$$

$$= -\cos x \int \frac{\sin x \cdot \sin x dx}{1} + \sin x \int \frac{\cos x \sin x dx}{1}$$

$$= -\cos x \ln |\sin x| + \sin x \cdot x$$

$$= \cos x \ln |\cos x| + x \sin x$$

solution:

$$y = y_c + y_p$$

$$\Rightarrow y = A \sin x + B \cos x + \cos \ln |\cos x| + x \sin x$$

Q: $x^2y'' - 4xy' + 6y = 21x^{-4}$, solution?

Sol'

S-1 $x^2y'' - 4xy' + 6y = 0$ (making homogeneous)

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3$$

$$y_c = C_1 x^2 + C_2 x^3$$

S-2

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= 3x^4 - 2x^3 = x^4$$

$$y_p = -x^2 \int \frac{x^3 \cdot 21x^{-4}}{x^4} dx + x^3 \int \frac{x^2 \cdot 21x^{-4}}{x^4} dx$$

$$= -x^2 \int 21x^{-5} dx + x^3 \int 21x^{-6} dx$$

$$= -21x^2 x^{-4} + 21x^3 x^{-5}$$

$$= \frac{21}{4} x^{-2} - \frac{21}{5} x^{-2}$$

$$= x^{-2} \left(\frac{105 - 84}{20} \right)$$

$$= \frac{21}{20} x^{-2}$$

solution! $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 x^2 + c_2 x^3 + \frac{21}{20} x^{-2}}$$

Power series solution of differential equation :-

power series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)$

Q Radics of convergence of power series will be determined

by

$$R = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

or $R = \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}$

Q Circle of convergence: $|x-x_0| = R$

Q! Find radics of convergence

1. $\sum_{m=0}^{\infty} x^m m(m+1)$

2. $\sum_{m=0}^{\infty} \frac{x^{2m}}{m!}$

3. $\sum_{m=0}^{\infty} \frac{1}{3^m} (x-3)$

4. $\sum_{n=0}^{\infty} (n-2)(n+1) x^n$

Sofn

①

$$\sum_{m=0}^{\infty} 2^m m(m+1)$$

multiple of $m=1$

$$\therefore R = \lim_{m \rightarrow \infty} \left| \frac{a_m}{a_{m+1}} \right|$$

$$a_m = m(m+1)$$

$$a_{m+1} = (m+1)(m+2)$$

$$R = \lim_{m \rightarrow \infty} \left(\frac{a_m}{a_{m+1}} \right)$$

$$= \lim_{m \rightarrow \infty} \frac{m(m+1)}{(m+1)(m+2)}$$

$$\leq \lim_{m \rightarrow \infty} \frac{m}{m+2}$$

$$= \lim_{m \rightarrow \infty} \frac{1}{1 + \frac{2}{m}} = 1$$

②

$$\sum_{m=0}^{\infty}$$

$\frac{2^m}{m!} \rightarrow$ (multiple of $m=2$)

$$\Rightarrow R^2 = \lim_{m \rightarrow \infty} \frac{a_m}{a_{m+1}}$$

$$a_m = \frac{1}{m!}$$

$$a_{m+1} = \frac{1}{(m+1)!}$$

$$R^2 = \lim_{m \rightarrow \infty} \frac{a_m}{a_{m+1}}$$

$$\leq \lim_{m \rightarrow \infty} \frac{\frac{1}{m!}}{\frac{1}{(m+1)!}}$$

$$= \lim_{m \rightarrow \infty} (m+1) = \infty \Rightarrow R = \infty$$

$$③ \sum_{m=0}^{\infty} \frac{1}{3^m} (x-3)^{2m-1000} \quad (\text{multiple of } m = 2) \\ a_m = \frac{1}{3^m} \\ a_{m+1} = \frac{1}{3^{m+1}}$$

$$R^2 = \lim_{m \rightarrow \infty} \frac{a_m}{a_{m+1}} \\ = \lim_{m \rightarrow \infty} \frac{\frac{1}{3^m}}{\frac{1}{3^{m+1}}} \\ = 3$$

$$\Rightarrow R = \sqrt{3}$$

a circle of convergence: $|x-3| = \sqrt{3}$

$$④ \sum_{n=0}^{\infty} (n-2)(n+1) \quad (\text{multiple of } n=1) \\ \text{so } R = \lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}}$$

$$a_n = (n-2)(n+1)$$

$$a_{n+1} = (n-1)(n+2)$$

$$R = \lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \lim_{n \rightarrow \infty} \frac{(n-2)(n+1)}{(n-1)(n+2)} \\ = \lim_{n \rightarrow \infty} \frac{n^2 - n - 2}{n^2 + n - 2} \\ = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n} - \frac{2}{n^2}}{1 + \frac{1}{n} - \frac{2}{n^2}} \\ = 1$$

Conversion of power series (changing only m/n / i value)

e.g. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ ————— ①

If we want \rightarrow The series will start from $n=1$
then ($n=0 \rightarrow n=1$) that mean $n \rightarrow n-1$

Now in ① Replacing n by $n-1$

$$\sum_{n-1=0}^{\infty} (-1)^{n-1} \frac{x^{2(n-1)-1}}{(2(n-1)+1)!}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-3}}{(2n-3)!}$$

Q! Find power series solution of

$$y' - y = 0$$

Sol

S-1 Name co-efficient of highest order derivative part 'y'

$$y' - y = 0 \quad \text{--- } ①$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$\begin{aligned} y' &= \sum_{m=0}^{\infty} a_m m x^{m-1} \\ &= \sum_{m=1}^{\infty} a_m m x^{m-1} \end{aligned}$$

From ①

$$\sum_{m=1}^{\infty} a_m m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} a_{m+1} (m+1) x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} [a_{m+1} (m+1) - a_m] x^m = 0 \quad \text{--- } ②$$

Comparing Co-efficient of x^m in $(a_0 + a_1 x + a_2 x^2 + \dots)$

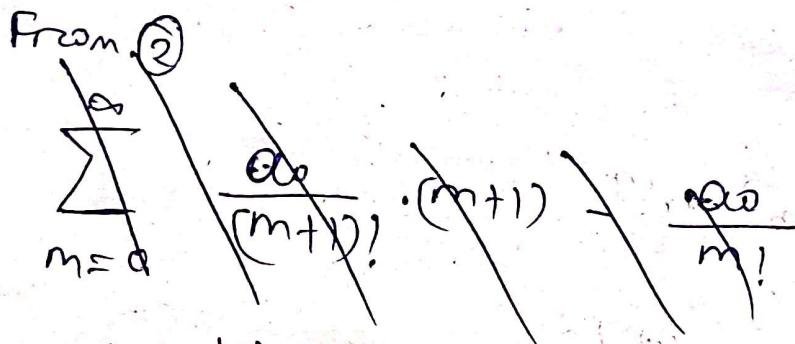
$$(m+1)a_{m+1} - a_m = 0$$

$$\Rightarrow a_{m+1} = \frac{a_m}{m+1}$$

For $m=0 \rightarrow a_1 = \frac{a_0}{1}$

For $m=1 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$

For $m=2 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2!} = \frac{a_0}{3!}$



Now solution:-

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + \frac{a_0 x}{1!} + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \dots$$

Q! Find power series solution

1. $y'' = y'$ 2. $y'' = 3x^3y$

Sol

① $y'' = y' \quad \text{--- } ①$

$\Rightarrow y'' - y' = 0$

Now,

$$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{--- } ② \quad (\text{Will be the solution})$$

$$\Rightarrow y' = \sum_{m=0}^{\infty} a_m m x^{m-1}$$

$$= \sum_{m=1}^{\infty} a_m m x^{m-1}$$

$$y'' = \sum_{m=1}^{\infty} a_m \cdot m \cdot (m-1) x^{m-2}$$

$$\Rightarrow y'' = \sum_{m=2}^{\infty} a_m \cdot m \cdot (m-1) x^{m-2}$$

From eq ①

$$\sum_{m=2}^{\infty} a_m \cdot m \cdot (m-1) x^{m-2} = \sum_{m=1}^{\infty} a_m \cdot m x^{m-1}$$

$$\Rightarrow \sum_{m=0}^{\infty} a_{m+2} \cdot (m+2)(m+1) x^m$$

$$= \sum_{m=0}^{\infty} a_{m+1} \cdot (m+1) x^m$$

$$\Rightarrow \sum_{m=0}^{\infty} \{a_{m+2}(m+2)(m+1) - a_{m+1}(m+1)\} x^m = 0$$

Comparing both the sides

$$a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) = 0$$

$$\Rightarrow (m+2)a_{m+2} = a_{m+1}$$

$$\Rightarrow a_{m+2} = \frac{a_{m+1}}{m+2}$$

$$\text{For } m=0 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_1}{2!}$$

$$\text{For } m=1 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{3 \cdot 2!} = \frac{a_1}{3!}$$

$$\text{For } m=2 \rightarrow a_4 = \frac{a_3}{4} = \frac{a_1}{4 \cdot 3!} = \frac{a_1}{4!}$$

Now solution.

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 x + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 x + a_1 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \dots$$

$$\Rightarrow \sum_{m=0}^{\infty} \{a_{m+2}(m+2)(m+1) - a_{m+1}(m+1)\}x^m = 0$$

Comparing both the sides

$$a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) = 0$$

$$\Rightarrow (m+2)a_{m+2} = a_{m+1}$$

$$\Rightarrow a_{m+2} = \frac{a_{m+1}}{m+2}$$

For $m=0$

$$\rightarrow a_2 = \frac{a_1}{2} = \frac{a_1}{2!}$$

For $m=1$

$$\rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{3 \cdot 2!} = \frac{a_1}{3!}$$

For $m=2$

$$\rightarrow a_4 = \frac{a_3}{4} = \frac{a_1}{4 \cdot 3!} = \frac{a_1}{4!}$$

Now solution:

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 x + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 x + a_1 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \dots$$

Legendre equation :-

$$\rightarrow (1-x^2) y'' - 2x y' + n(n+1)y = 0.$$

Solution :- $y = c_1 y_1(x) + c_2 y_2(x)$

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)}{6!} x^6 + \dots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 + \dots$$

For any integer 'n' (positive integer) one of the solution ($y_1(x)$) other will be series.

Polynomial solution of legendre eqn when multiplied with a constant is called legendre polynomial.

The formula for legendre polynomial explicitly can be written as

$$P_n(x) = \sum_{m=0}^n \frac{(-1)^m (2n-2m)!}{2^m m! (n-m)! (n-2m)!} x^{n-2m}$$

$$M = \begin{cases} n/2 \\ \frac{n-1}{2} \end{cases} \quad \text{which will be integer that will be taken}$$

when $n=0, m \leq 0$

$$P_0(x) = \sum_{m=0}^0 \frac{(-1)^m (2x^0 - 2m)!}{2^0 m! (0-m)! (0-2m)!} x^{0-2m}$$
$$= 1$$

¶ when $n=1, m \leq 0$

$$P_1(x) = \sum_{m=0}^0 \frac{(-1)^m (2x^1 - 2m)!}{2^1 m! (1-m)! (1-2m)!} x^{1-2m}$$
$$= x$$

Rodrigue's formula:-

$$\hookrightarrow P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Polynomial solution of legendary equation when multiplied with a constant ω called legendary polynomial

Proof:-

$$\text{Let } V = (x^2 - 1)^n$$

$$\frac{dv}{dx} = 2nx(x^2 - 1)^{n-1}$$

$$\Rightarrow \frac{dv}{dx} = 2nx \frac{(x^2 - 1)^n}{(x^2 - 1)}$$

$$\Rightarrow (x^2 - 1) \frac{dv}{dx} = 2nxV$$

$$\Rightarrow (1 - x^2) \frac{dv}{dx} + 2nxV = 0$$

Differentiating above eqn we get

$$(1 - x^2) \frac{d^2v}{dx^2} - 2x \frac{dv}{dx} + 2nx \frac{dv}{dx} + 2nv = 0$$

$$\Rightarrow (1 - x^2) \frac{d^2v}{dx^2} + (2n - 2)x \frac{dv}{dx} + 2nv = 0$$

Differentiating the above eqn n' times using Leibniz rule we get

$$(1-x^2) \frac{d^{n+2}v}{dx^{n+2}} - 2nx \frac{d^{n+1}v}{dx^{n+1}} - 2n(n-1) \frac{d^nv}{dx^n} + (2n-2) \left\{ nx \frac{d^{n+1}v}{dx^{n+1}} + n \frac{d^nv}{dx^n} \right\} + 2n \frac{d^nv}{dx^n} = 0$$

$$(1-x^2) \frac{d^{n+2}v}{dx^{n+2}} + \{-2nx + (2n-2)x\} \frac{d^{n+1}v}{dx^{n+1}} + \{-n(n+1) + (2n-2)n\} \frac{d^nv}{dx^n} = 0$$

$$(1-x^2) \frac{d^{n+2}v}{dx^{n+2}} - 2n \frac{d^{n+1}v}{dx^{n+1}} + n(n+1) \frac{d^nv}{dx^n} = 0$$

$$\text{Let } w = \frac{d^nv}{dx^n}$$

$$\Rightarrow (1-x^2) \frac{d^2w}{dx^2} - 2n \frac{dw}{dx} + n(n+1)w = 0$$

Polynomial of legendary equation
w. is a solution of legendary equation

Legendary polynomial = $P_n(x) = cw$

Note: $P_n(1) = 1 \quad \forall n \in N$

$N_2(O)$

$$P_n(x) = C \frac{d^n}{dx^n}(x)$$

$$= C \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$= C \left[\frac{d^n}{dx^n} \{ (x^2 + 1)^n (1 - x)^n \} \right]$$

$$= C \left\{ (x-1)^n \frac{d^n}{dx^n} (x+1)^n + n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (x+1)^n \right\}$$

$$P_n(1) = C \{ 0 + 0 + 0 + \dots + n! + 2^n \}$$

$\Rightarrow 1 = b \cdot c \cdot ? \cdot ?$

$$\Rightarrow C = \frac{1}{2^n n!}$$

ପଦ୍ମ

$$P_n(x) = c_n \omega$$

$$P_n(m) = \frac{1}{2^n n!} w$$

$$\Rightarrow P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Q: Show that $\int_{-1}^1 P_m(x) P_n(x) dx = 0$, if $m \neq n$

Sol'

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

Hence P_m & P_n both are the solution of the above equation.

Now,

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \quad \text{--- (2)}$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad \text{--- (3)}$$

Multiply P_n in eqn (2) and P_m in eqn (3), and subtract

$$(1-x^2)P_m''P_n - 2xP_m'P_n + m(m+1)P_mP_n = 0$$

$$(1-x^2)P_n''P_m - 2xP_n'P_m + n(n+1)P_nP_m = 0$$

(-) (+) (-)

$$(1-x^2)[P_m''P_n - P_n''P_m] - 2x[P_m'P_n - P_n'P_m]$$

$$+ [m(m+1) - n(n+1)] P_m P_n = 0 \quad \text{--- (4)}$$

Let $Z = (1-x^2)(P_m'P_n - P_n'P_m)$

$$\begin{aligned} Z' &= (1-x^2) (P_m''P_n + P_m'P_n' - P_n''P_m' - P_n'P_m) \\ &\quad - 2x(P_m'P_n - P_n'P_m) \end{aligned}$$

$$\Rightarrow \frac{d}{dx} [(1-x^2)(P_m'P_n - P_n'P_m)] = (1-x^2)(P_m''P_n - P_n''P_m) \\ - 2x(P_m'P_n - P_n'P_m).$$

From eqn(4) we get

$$\frac{d}{dx} [(1-x^2)(P_m' P_n - P_m P_n')] + (m-n)(m+n+1) P_m P_n = 0$$

Integrating the above eqn from -1 to $+1$ we get

$$\left[(1-x^2)(P_m' P_n - P_m P_n') \right]_1^0 + (m-n)(m+n+1) \int_{-1}^1 P_m P_n = 0$$

$$\Rightarrow \int_{-1}^1 P_m P_n = 0 \quad (\text{for } m \neq n)$$

$$\Rightarrow \boxed{\int_{-1}^1 P_m(x) P_n(x) dx = 0}$$

METHODS

Scalar point function :-

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$f(1, 3, -1) = 1 + 9 + 1 = 6$$

Vector point function :-

$$f(x, y, z) = xy\hat{i} + y^2z\hat{j} + z^2\hat{k}$$

$$f(1, -3, -1) = -3\hat{i} - 4\hat{j} + \hat{k}$$

scalar field :-

↪ A scalar point function when made equal to a constant
is called a scalar field.

$$\text{eg: } f(x, y, z) = x^2 + y^2 + z^2 = 4$$

Gradient of a scalar point function :-

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

where $\nabla \rightarrow$ nabla

$$\Rightarrow \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\text{eg: } f(x) = x^2 + y^2 + z^2$$

$$\text{grad } f = \nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Directional derivative of f at the point P in the direction of a vector \vec{b} is defined as:

$$D_{\vec{b}} f = \left[\text{grad } f \right]_P \cdot \frac{\vec{b}}{|\vec{b}|}$$

Q: Find directional derivative of $f = 2x^2 + 3y^2 + z^2$ at $P(3, 1, 3)$ in the direction of $\vec{v} - 2\vec{R}$.

Sol: $f = 2x^2 + 3y^2 + z^2, \vec{b} = \vec{v} - 2\vec{R}$

$$\text{grad } f = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}$$

$$\left[\text{grad } f \right]_{P(3, 1, 3)} = 8\hat{i} + 6\hat{j} + 6\hat{k}$$

$$D_{\vec{b}} f = \left[\text{grad } f \right]_{(3, 1, 3)} \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$= (8\hat{i} + 6\hat{j} + 6\hat{k}) \cdot \frac{(\vec{v} - 2\vec{R})}{\sqrt{1+4}}$$

$$= \frac{8 - 12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

Note

Q If f is a scalar point function having continuous 1st partial derivatives at point ' P ' then gradient of f is non-zero if it is not a zero vector then f increase maximum in the direction of ' P '.

Let $r(t)$ lies on the surface:

$$f(x, y, z) = c \quad \text{--- } ①$$

$$f(x(t), y(t), z(t)) = c \quad \text{--- } ②$$

$$\text{Let } r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

Differentiating eqn ② we get

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial x}{\partial t} \hat{i} + \frac{\partial y}{\partial t} \hat{j} + \frac{\partial z}{\partial t} \hat{k} \right) = 0$$

$$\Rightarrow \nabla f \cdot \frac{dr}{dt} = 0$$

$$\text{or } \nabla f \cdot r' = 0$$

∇f is a vector perpendicular to the surface

Laplace equation :-

$$Q. \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Let v be a vector point function then

Divergence of a vector point function

Q If \mathbf{v} has a vector point function

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v}$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Note

If $\operatorname{div} \mathbf{v} = 0 \Rightarrow$ condition of incompressibility

Q: ① $\operatorname{div}(\operatorname{grad} f) = \nabla^2 f$

② $\nabla f g = f \nabla g + g \nabla f$

③ Directional derivative = ??

If $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3,0,4)$ & in the direction of $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

④ $\nabla^2 f = ??$

where $f = \cos h^2 x - \sin h^2 y$

Solution:

① LHS = $\operatorname{div}(\operatorname{grad} f)$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \operatorname{grad} f$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right)$$

Divergence of a vector point function

Q If \mathbf{v} is a vector point function

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v}$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Note

If $\operatorname{div} \mathbf{v} = 0 \Rightarrow$ condition of incompressibility

Q: ① $\operatorname{div}(\operatorname{grad} f) = \nabla^2 f$

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If $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3,0,4)$ & in the direction of $\vec{\alpha} = \hat{i} + \hat{j} + \hat{k}$

④ $\nabla^2 f = ??$

where $f = \operatorname{cosh}^2 x - \operatorname{sinh}^2 y$

Solution:

① LHS = $\operatorname{div}(\operatorname{grad} f)$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \operatorname{grad} f$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right)$$

$$\begin{aligned}
 &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} + \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} + \frac{\partial}{\partial z} \cdot \frac{\partial f}{\partial z} \\
 &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
 &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f \\
 &= \nabla^2 f = \text{RHS}
 \end{aligned}$$

Q Let $f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$
 $g = g_x \hat{i} + g_y \hat{j} + g_z \hat{k}$

LHS = $\nabla f \cdot g = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \cdot (f_x g_x + f_y g_y + f_z g_z)$

Q ~~for~~ $\nabla f \cdot g = f \nabla g + g \nabla f$

$$\text{LHS.} = \nabla f \cdot g$$

$$\begin{aligned}
 &= \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) f g \\
 &= \frac{\partial (fg)}{\partial x} \hat{i} + \frac{\partial (fg)}{\partial y} \hat{j} + \frac{\partial (fg)}{\partial z} \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right) \hat{i} + \left(f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y} \right) \hat{j} \\
 &\quad + \left(f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z} \right) \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 &= f \left(\frac{\partial g}{\partial x} \hat{i} + \frac{\partial g}{\partial y} \hat{j} + \frac{\partial g}{\partial z} \hat{k} \right) + g \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \\
 &= f \nabla g + g \nabla f = \text{RHS}
 \end{aligned}$$

$$\textcircled{3} \text{ Given } f = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\text{grad } f = \frac{-x}{(x^2+y^2+z^2)^{3/2}} \hat{i} - \frac{y}{(x^2+y^2+z^2)^{3/2}} \hat{j} - \frac{z}{(x^2+y^2+z^2)^{3/2}} \hat{k}$$

$$\text{grad } f \Big|_{(3,0,4)} = \frac{-3}{125} \hat{i} - \frac{4}{125} \hat{k}$$

$$\begin{aligned} \text{directional derivative} &= \text{grad } f \Big|_{(3,0,4)} \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= \left(\frac{-3}{125} \hat{i} - \frac{4}{125} \hat{k} \right) \cdot \left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \right) \\ &= \frac{1}{\sqrt{3}} \times \frac{-7}{125} = \frac{-7}{125\sqrt{3}} \end{aligned}$$

i) Given, $f_{\alpha} = \cosh^2 x - \sinh^2 y$

$$\frac{\partial f}{\partial x} = 2 \cosh x \cdot \sinh x$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= 2 \sinh x \cdot \sinh x + 2 \cosh x \cdot \cosh x \\ &= 2 \sinh^2 x + 2 \cosh^2 x\end{aligned}$$

$$\frac{\partial f}{\partial y} = -2 \sinh y \cdot \cosh y$$

$$\frac{\partial^2 f}{\partial y^2} = -2 \cosh^2 y - 2 \sinh^2 y$$

$$\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial z^2} = 0$$

Now, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$$= 2\sinh^2 x + 2\cosh^2 x - 2\cosh^2 y - 2\sinh^2 y + 0$$

\therefore ②

$$= 2x \left(\frac{e^y + e^{-y}}{2} \right)^2 + 2x \left(\frac{e^y - e^{-y}}{2} \right)^2$$

$$- 2x \frac{(e^y - e^{-y})^2}{2^2} - 2x \left(\frac{e^y + e^{-y}}{2} \right)^2$$

$$= \frac{2(e^{2y} + e^{-2y})}{2} - 2 \frac{(e^{2y} - e^{-2y})}{2}$$

$$= e^{2y} + e^{-2y} - e^{2y} + e^{-2y}$$

Curd of a vector point function :-

Given $\mathbf{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

Now,

$$\text{curl } \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \nabla \times \mathbf{v}$$

$$= \hat{i}(x-y) - \hat{j}(y-z) + \hat{k}(z-x)$$
$$= \vec{0}$$

Q! If $\mathbf{v} = x^2y\hat{i} + y^3\hat{j} + z^2xy\hat{k}$

then $\text{curl } \mathbf{v} = ?$

Sol'

$$\text{curl } \mathbf{v} = \vec{\nabla} \times \mathbf{v}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^3 & z^2xy \end{vmatrix}$$

$$= \hat{i}(z^2x - 0) - \hat{j}(z^2y - 0) + \hat{k}(0 - x^2)$$
$$= z^2x\hat{i} - z^2y\hat{j} - x^2\hat{k}$$

Qn:- Find scalar point function f whose gradient is

$$yz\hat{i} + xz\hat{j} + xy\hat{k}$$

Sol:- Given $\text{grad } f = yz\hat{i} + xz\hat{j} + xy\hat{k}$ — (1)

$$\Rightarrow \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} — (2)$$

Comparing (1) & (2)

$$\frac{\partial f}{\partial x} = yz \quad (i) \qquad \frac{\partial f}{\partial y} = xz \quad (ii) \qquad \frac{\partial f}{\partial z} = xy \quad (iii)$$

$$\Rightarrow \partial f = yz \partial x$$

$$\Rightarrow f = yzx + C_1(y, z) — (3)$$

again $\frac{\partial f}{\partial y} = zx$

$$\frac{\partial f}{\partial x} = xz \frac{\partial f}{\partial y}$$

\Rightarrow Derivating eq (3) w.r.t y we get

$$\frac{\partial f}{\partial y} = zx + \frac{\partial}{\partial y} g(y, z) — (4)$$

Comparing eq (ii) & (4) we get

$$\frac{\partial}{\partial y} g(y, z) = 0$$

$$\Rightarrow \partial g(y, z) = 0 \quad \partial y$$

$$\Rightarrow g(y, z) = C_2(z)$$

From ④

$$f = yzx + Q(z) \quad \text{---} \textcircled{5}$$

$$\Rightarrow \frac{\partial f}{\partial z} = yx + \frac{\partial}{\partial z} Q(z) \quad \text{---} \textcircled{6}$$

Comparing eq ⑥ with eq (ii) we get

$$\frac{\partial}{\partial z} Q(z) = 0$$

$$\Rightarrow Q(z) = C$$

From ⑤

$$f = yzx + C$$

(Ans.)

$$\Rightarrow \frac{\partial f}{\partial x} = yz$$

Q:- Find the scalar point function whose gradient
is $x^2y\hat{i} + y^3\hat{j} + z^2y\hat{k}$

Soln Given, $\Rightarrow \text{grad } f = x^2y\hat{i} + y^3\hat{j} + z^2y\hat{k}$

$$\Rightarrow \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = x^2y\hat{i} + y^3\hat{j} + z^2y\hat{k}$$

Comparing both the sides,

$$\frac{\partial f}{\partial x} = x^2y \quad \text{(i)}, \quad \frac{\partial f}{\partial y} = y^3 \quad \text{(ii)}, \quad \frac{\partial f}{\partial z} = z^2y \quad \text{(iii)}$$

$$\Rightarrow \partial f = x^2y \partial x$$

$$\Rightarrow f = \frac{x^3}{3}y + G(y, z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x^3}{3} + \frac{\partial}{\partial y} G(y, z) = y^3 \quad (\text{From (ii)})$$

we can't compare both side
 Here we can't find the scalar point function

Q:- Prove that $\text{curl}(\text{grad } f) = \vec{0}$ where f has 2nd order continuous partial derivative

Sol:

$$\text{LHS} = \text{curl}(\text{grad } f)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) + \hat{j} \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) + \hat{k} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right).$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= \vec{0} = \text{RHS}$$

$$\text{Q1} \text{ If } f = xy^2, g = x^2y$$

Verify $\nabla(fg) = f \nabla g + g \nabla f$

$$\text{Q2} \text{ If } f = e^x \cos y, P(3, \pi, 0)$$

$$a = 2\hat{i} + 3\hat{j}, \text{ find } \nabla f$$

$$\text{Q3} \text{ If } f = x^2y z^2, \text{ verify } \operatorname{div}(\nabla f) = \nabla^2 f$$

$$\text{Q4} \text{ Find } f \text{ if } \nabla f = [y e^z, e^z, 1]$$

Soln

$$\text{Q1} \text{ Given } f = xy^2, g = x^2y$$

$$\begin{aligned} \text{LHS} &= \nabla(fg) = \nabla(xy^2 \cdot x^2y) \\ &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (x^3 y^3) \end{aligned}$$

$$(4) = 3x^2y^3 \hat{i} + 3x^3y^2 \hat{j}$$

$$\text{RHS} = f \nabla g + g \nabla f$$

$$= xy^2 \left\{ \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) x^2y \right\}$$

$$+ x^2y \left\{ \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) xy^2 \right\}$$

$$= xy^2 x (2xy\hat{i} + x^2\hat{j})$$

$$+ x^2y (y^2\hat{i} + 2xy\hat{j})$$

$$= 2x^3y^3 \hat{i} + x^3y^2 \hat{j} + x^2y^3 \hat{i} + 2x^3y^2 \hat{j}$$

$$= 3x^3y^3 \hat{i} + 3x^3y^2 \hat{j}$$

$$\therefore \text{LHS} = \text{RHS}$$

$$\textcircled{2} \quad \text{Given } f = e^x \cos y \quad P(2\pi, 0), \alpha = 2\hat{i} + 3\hat{j}$$

$$\begin{aligned}\text{grad } f &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) e^x \cos y \\ &= e^x \cos y \hat{i} - e^x \sin y \hat{j}\end{aligned}$$

$$\begin{aligned}\text{grad } f]_{(2\pi, 0)} &= [e^2 \cos \pi \hat{i} - e^2 \sin \pi \hat{j} \\ &= -e^2 \hat{i}\end{aligned}$$

$$\begin{aligned}D_{\alpha} f &= \text{grad } f]_{(2\pi, 0)} \cdot \frac{\vec{\alpha}}{|\vec{\alpha}|} \\ &= (-e^2 \hat{i}) \cdot \frac{(2\hat{i} + 3\hat{j})}{\sqrt{2^2 + 3^2}} \\ &\quad \checkmark \\ &= -\frac{2e^2}{\sqrt{13}}\end{aligned}$$

$$\textcircled{3} \quad \text{Given, } f = x^2 y^2 z^2$$

$$\text{LHS} = \text{div}(\text{grad } f)$$

$$= \text{div} \left\{ \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) x^2 y^2 z^2 \right\}$$

$$= \text{div} \cdot (2xyz^2 \hat{i} + x^2 z^2 \hat{j} + 2x^2 yz^2 \hat{k})$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (2xyz^2 \hat{i} + x^2 z^2 \hat{j} + 2x^2 yz^2 \hat{k})$$

$$= 2yz^2 + 2x^2y$$

$$RHS = \nabla^2 f$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial}{\partial x} (2xyz^2) + \frac{\partial}{\partial y} (x^2z^2) + \frac{\partial}{\partial z} (2x^2yz)$$

LHS = ~~$2yz^2 + 2x^2y$~~

LHS = RHS

④ Given $\nabla f = [ye^x, e^x, 1]$

$$\Rightarrow \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = ye^x i + e^x j + k$$

$$\frac{\partial f}{\partial x} = ye^x \quad \text{--- } ①$$

$$\frac{\partial f}{\partial y} = e^x \quad \text{--- } ②$$

$$\frac{\partial f}{\partial z} = 1 \quad \text{--- } ③$$

$$\Rightarrow f = ye^x + C_1(y, z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = e^x + \frac{\partial}{\partial y} g(y, z)$$

$$\Rightarrow e^x = e^x + \frac{\partial}{\partial y} g(y, z)$$

~~$\Rightarrow g(y, z) = 0$~~

$$\Rightarrow g(y, z) = C_2(z)$$

$$\Rightarrow f = g e^z + g(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(c_2(z)) \Rightarrow i = \frac{\partial}{\partial z}(c_2(z))$$

$$\Rightarrow \cancel{g e^z} \cancel{+ c_2(z)} \Rightarrow z + c = g(z)$$

Now
If $f = g e^z + z + c$

$\alpha \circ D$

Q1. If $f = xyz$, $\nabla f = x^2y$

Verify $\nabla(fg) = f \nabla g + g \nabla f$

Solⁿ

Given, $f = xyz$, $g = x^2y$

$$LHS = \nabla(fg)$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (xyz \cdot x^2y)$$

$$= \frac{\partial}{\partial x} (x^3y^2z) i + \frac{\partial}{\partial y} (x^3y^2z) j$$

$$+ \frac{\partial}{\partial z} (x^3y^2z) k$$

$$= 3x^2y^2z i + 2x^3y^2z j + x^3y^2k$$

$$RHS = f \nabla g + g \nabla f$$

$$= (xyz) \left\{ \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) x^2y \right\}$$

$$+ x^2y \left\{ \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) xyz \right\}$$

$$= (xyz) (2xy i + x^2 j + x^3 k)$$

$$+ x^2y (y^2 i + x^2 j + x^3 k)$$

$$= 2x^2y^2z i + x^3y^2z j + x^2y^2z i + x^3y^2k$$

$$= 3x^2y^2z i + 2x^3y^2z j + x^3y^2k$$

$$\therefore LHS = RHS$$