Module 3

* Partial Derivative :-

1) Find partial Derivative.

(a) $f(x, y, z, w) = x^2 e^{2y+3z} \cos 4w$.

(b) $f(r, \theta, z) = \dfrac{r(2-\cos 2\theta)}{r^2+z^2}$.

a) $\dfrac{\partial f}{\partial x} = f_x = 2x e^{2y+3z} \cos 4w$.

$= \dfrac{2f}{x}$.

$\dfrac{\partial f}{\partial y} = x^2 e^{3z} \cdot e^{2y} \cdot 2\cos 4w$.

$= 2f$

$\dfrac{\partial f}{\partial z} = f_z = x^2 e^{3z} \cdot e^{2y} 3\cos 4w$.

$= 3f$.

$\dfrac{\partial f}{\partial w} = x^2 e^{2y+3z} - \sin 4w$.

(b) $f(r, \theta, z)$.

$$\frac{\partial f}{\partial r} = (2-\cos 2\theta) \times (r^2+z^2)^{-1}$$

$$\Rightarrow \cancel{2-\cos 2\theta \times -1 (r^2+z^2)^{-2} \times 2r}$$

$$\Rightarrow \cancel{-\frac{2r(2-\cos 2\theta)}{(r^2+z^2)^2}}$$

$$\Rightarrow (2-\cos 2\theta) \times \frac{(r^2+z^2)-2r^2}{(r^2+z^2)^2} \quad 4.$$

$$\Rightarrow 2-\cos 2\theta + \frac{z^2-r^2}{(r^2+z^2)^2}$$

$$\frac{\partial f}{\partial \theta} = \frac{r}{r^2+z^2} \times \frac{\delta}{\delta\theta}(2-2\cos 2\theta).$$

$$\Rightarrow \frac{r}{r^2+z^2} \times 2\sin 2\theta.$$

$$\Rightarrow \frac{2\sin 2\theta\, r}{(r^2+z^2)}$$

$$\frac{\delta f}{\delta z} = r(2-\cos 2\theta)\frac{\delta}{\delta z}\frac{1}{r^2+z^2}$$

$$\Rightarrow r(2-\cos 2\theta) \times \frac{+2z}{(r^2+z^2)^2}.$$

$$\Rightarrow \frac{r(2-\cos 2\theta)\, 2z}{(r^2+z^2)^2}.$$

Q). $v(x, y\ z) = \cos 3x \cos 4y \sinh 5z$.

$$\frac{\delta^2 v}{\delta x^2}, \frac{\delta^2 v}{\delta y^2}, \frac{\delta^2 v}{\delta z^2}.$$

Ans).
$$\left\{ \begin{aligned} &\sinh = \frac{e^{x} + e^{-x}}{2}. \\ &(\sinh a)' = \cosh x \end{aligned} \right\}.$$

- $$\frac{\delta v}{\delta x} = -3\sin x \times - \cancel{4\sin y} \cos 4y \sinh 5z.$$

$$\frac{\delta^2 v}{\delta x^2} = -9\cos x \cos 4y \sinh 5z.$$

- $$\frac{\delta^2 v}{\delta y^2} = \cos 3x \ -4\sin 4x \ \sinh 5z.$$

$$\Rightarrow -4\cos 3x \sin 4y \sinh 5z.$$

$$\frac{\delta^2 v}{\delta y^2} = -16\cos 3x \cos 4y \sinh 5z.$$

- $$\frac{\delta^2 v}{\delta z^2} = 5\cos 3x \cos 4y \cancel{\sinh} \cosh 5z.$$

$$\frac{\delta^2 v}{\delta z^2} = 25\cos 3x \cos 4y \sinh 5z.$$

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} + \frac{\delta^2 v}{\delta z^2} = 0.$$

$\rightarrow$ laplase eq.

Q). If $u = e^{a\theta}\cos(a\ln r)$.

~~sh~~ show that

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0.$$

Ans) $U_r = \frac{\delta u}{\delta r}$.

$\delta_{rr} = \frac{\delta^2 u}{\delta r^2}$.

$u_r = e^{a\theta}(-\sin(a\ln r))a\frac{1}{r}$

$\Rightarrow \frac{a}{r}e^{a\theta}\sin(a\ln r)$.

$$U_{rr} = ae^{a\theta}\frac{\delta}{\delta r}\frac{(\sin(a\ln r))}{r} \qquad u'v - v'u.$$

$$\Rightarrow ae^{a\theta}\frac{\cos(a\ln r)\times\frac{a}{r}r^2 - 2r\times\sin(a\ln r)}{r^2}$$

$$\Rightarrow ae^{a\theta}\frac{ar\cos(a\ln r) - 2r\sin(a\ln r)}{r^2}.$$

$$\Rightarrow -ae^{a\theta}\frac{a\cos(a\ln r) - 2\sin(a\ln r)}{r}.$$

$$\Rightarrow -ae^{a\theta}\left\{\frac{1}{r}\cos(a\ln r)\frac{a}{r} - \sin(a\ln r)\frac{1}{r^2}\right\}.$$

* **Hessian Matrix :-**

Let $f : R^n \longrightarrow R$.

All 2nd order partial derivative exists.

$f(x_1, x_2, x_3)$

$f = x_1^2 + x_2^2 + x_3^2 + 4x_4.$

$f = \frac{1}{x_1^2} + x_2^2.$

$\frac{\delta f}{\delta x} = \frac{-2}{x_1^3}.$

$\frac{\delta^2 f}{\delta x^2} = 2.$

$$H_f = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 x_2} & \cdots & \frac{\delta^2 f}{\delta x_1 \delta x_n} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x^2} & \cdots & \frac{\delta^2 f}{\delta x_2 \delta x_n} \\ \frac{\delta^2 f}{\delta x_n \delta x_1} & \cdots & \cdots & \frac{\delta^2 f}{\delta x_n^2} \end{bmatrix}$$

Q). Find partial matrix.

$f = x_1^4 + 2x_2^2 x_3^3 + x_4^5 x_1^3 \cdot f = (x_1, x_2, x_3, x_4)$

Ans)
$$\begin{bmatrix} 4x_1^3 + 3x_4^5 x_1^2 & 4x_2 x_3^3 & 6x_2^2 x_3^2 & 5x_4^4 x_1^3 \\ 4x_1^3 + 3x_4^5 x_1^2 & 4x_2 x_3^3 & 6x_2^2 x_3^2 & 5x_4^4 x_1^3 \\ 4x_1^3 + 3x_4^5 x_1^2 & 4x_2 x_3^3 & 6x_2^2 x_3^2 & 5x_4^4 x_1^3 \\ 4x_1^3 + 3x_4^5 x_1^2 & 4x_2 x_3^3 & 6x_2^2 x_3^2 & 5x_4^7 x_1^3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12x_1^2 + 6x_1^5 x_1 & 0 & 0 & 15x_4^4 x_1^2 \\ 0 & 4x_3^3 & 12x_2 x_3^2 & 0 \\ 0 & 12x_3^2 x_2 & 12x_2^2 x_3 & 0 \\ 0 + 15x_4^4 x_1^2 & 0 & 0 & 20x_4^3 x_1^3 \end{bmatrix}$$

* Jacobian Matrix :-

$f : R^n \rightarrow R.$

$f = (f_1, f_2, f_3, \ldots \ldots f_n).$

$$J = \begin{pmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} & \cdots & \frac{\delta f_1}{\delta x_n} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} & \cdots & \frac{\delta f_2}{\delta x_n} \\ \frac{\delta f_n}{\delta x_1} & \frac{\delta f_n}{\delta x_2} & \cdots & \frac{\delta f_n}{\delta x_n} \end{pmatrix}$$

Q) Find Jacobian matrix of $x = r\cos\theta$
$y = r\sin\theta$.

$f(x, y)$.

Ans). $J(r, \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$.

Q). $f(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$.

Q) $x = r\cos\theta \sin\phi$
$v = r\sin\theta \sin\phi$
$w = r\cos\theta \cos\phi$

Ans). $J(x_1, x_2, x_3) = \begin{vmatrix} 2x_1 x_2 x_3 & x_2^2 x_3 & x_2 x_3^2 \\ x^2 x_3 & 2x_1 x_2 x_3 & x_1 x_3^2 \\ x_1^2 x_2 & x_1 x_2^2 & 2x_1 x_2 x_3 \end{vmatrix}$

=)

ns). $J(x_1, x_2, x_3) = \begin{bmatrix} 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 & x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2 & x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 \\ 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 & x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2 & x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 \\ 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 & x_1x_3 + 2x_1x_2x_3 + x_1x_3^2 & x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x_2x_3 & 2x_1x_3 + 2x_2x_3 + x_3^2 & 2x_1x_2 + x_2^2 + 2x_2x_3 \\ 2x_1x_3 + 2x_2x_3 + x_3^2 & 0 + 2x_1x_3 + 0 & x_2^2 + 2x_1x_2 + 2x_1x_3 \\ 2x_1x_2 + x_2^2 + 2x_2x_3 & x_1 + 2x_1x_2 + 2x_1x_3 & 0 + 0 + 2x_1x_2 \end{bmatrix}$

Ans) $\theta : J(r, \theta, \phi)$

$$\begin{bmatrix} \cos\theta \sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & +r\sin\theta\cos\phi \\ \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \end{bmatrix}$$

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* Eoctremum value of $f(x,y)$

steps 1) Set $\frac{\delta f}{\delta x} = 0, \quad \frac{\delta f}{\delta y} = 0.$

2) sum the simultaneous. eq to get the critical point (a,b)

3) find $r = \frac{\delta^2 f}{\delta x^2}, \quad s = \frac{\delta^2 f}{\delta x \delta y}, \quad t = \frac{\delta^2 f}{\delta y^2}$ at (a,b).

4) i) If $rt - s^2 > 0$ & $r < 0$ then the point (a,b) is the point of local maximum.

ii) $rt - s^2 > 0$ & $r > 0$ then the point (a,b) is the point of local minimum.

iii) $r = 0$ ~~$rt - s^2 < 0$~~ then (a,b) is called saddle point.

iv) $rt - s^2 = 0$ then the test fails.

→ Examine the following function for maximum :-

1) $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy.$

Ans) $\frac{\delta f}{\delta x} = 4x^3 - 4x + 4y.$

$\frac{\delta f}{\delta y} = 4y^3 - 4y + 4x.$

$x^3 - x + y = 0. \quad \text{---- (i)}$

$y^3 - y + x = 0 \quad \text{---- (ii)}.$

~~Multiplying eq (ii).~~ adding (i) & (ii)

$x^3 + y^3 = 0.$

$x = -y. \quad \text{---- (iii)}$

Putting eq (iii) in eq (i).

$$x^3 - x - x = 0.$$

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0.$$

$$x = 0, \ x = \sqrt{2}, \ x = -\sqrt{2}.$$

Putting this value in eq (iii)

$$y = 0, -\sqrt{2}, \sqrt{2}.$$

Critical points are, $(0,0)$

$(\sqrt{2}, -\sqrt{2})$

$(-\sqrt{2}, \sqrt{2})$ ... (iv)

$$r = f_{xx} = 12x^2 - 4.$$

$$t = f_{yy} = 12y^2 - 4$$

$$s = 4.$$

Now put value ~~of x~~ in $r\,t\,s$

at $(0,0)$

$r = -4$

$s = 4$

$t = -4.$

~~at (0,0).~~

$$rt - s^2 = 16 - 16 = 0.$$

The test fails

at $(\sqrt{2}, -\sqrt{2})$

$$r = 20, \ s = 4, \ t = 20.$$

$rt - s^2 = 400 - 16$

$\Rightarrow 384 \rightarrow 0$

$(\sqrt{2}, -\sqrt{2})$ is the point of local min.

Q). $f(x, y) = 2 - x^2 - y^2$.

Ans). $\frac{\delta f}{\delta x} = -2x$, $\frac{\delta f}{\delta y} = -2y$.

$-2x = 0$.

$-2y = 0$.

$x + y = 0$.

$x = -y$.

critical point $(0, 0)$.

$r = -2$.

$s = 0$

$t = -2$.

at $(0, 0)$

$rt - s^2 = 4 - 0 > 0$.

$(0, 0)$ point of local min.

Q). Find the local max. or min.

$f(x, y) = x^4 + y^4 - 4xy - 1$.

Ans) $\frac{\delta f}{\delta x} = 4x^3 - 4y$, $\frac{\delta f}{\delta y} = 4y^3 - 4x$.

$r = 12x^2$, $t = 12y^2$, $s = -4$.

$12x^2 = 0$ ---(i) $12y^2 = 0$ ---(ii)

$$x^2 + y^2 = 0$$
$$x = \pm y.$$

critical point $(0,0)$.

$$4x^3 - 4y = 0 \quad \cdots (i)$$
$$4y^3 - 4x = 0 \quad \cdots (ii)$$

$$x^3 - y = 0. \quad \cdots (i)$$
$$y^3 - x = 0. \quad \cdots (ii).$$

Multiplying eq (i) with y^2.

$$x^3y^2 - y^3 = 0$$
$$y^3 - x = 0$$

$$x^3y^2 - x = 0.$$
$$x(x^2y^2 - 1) = 0.$$
$$x^2y^2 = 1$$
$$x^2 = \frac{1}{y^2}.$$
$$x = \frac{1}{y}$$

$$x^3 - y^3 + x - y = 0.$$
$$(x-y)\{x^2 + xy + y^2 + 1\} = 0.$$
$$x^2 + xy + y^2 + 1 = 0, \quad x = y.$$
$$3x^2 + 1 = 0.$$

Ans) $\frac{\delta f}{\delta x} = 4x^3 - 4y$, $\frac{\delta f}{\delta y} = 4y^3 - 4x$ . .

$x^3 - y = 0$ . . . (i)

$y^3 - x = 0$ . . . (ii)

From (1) $x^3 = y$.

$x^3 + y^3 - y - x = 0$.

$(x+y)(x^2+y^2-xy) - y - x = 0$.

$x^6 - x = 0$.

$x(x^5 - 1) = 0$.

$x = 0, 1$

$y = 0, 1$

(-1 is not taken becz $x = y$).

critical point $(0,0)$ $(1,1)$ $(-1,-1)$.

~~at por~~ $r = 12x^2$, $t = 12x^2$, $s = -4$.

at $(0,0)$,

$rt - s^2 = 0 \cancel{-} 16$

$\Rightarrow -16 < 0$

~~local max.~~

$\cancel{rt - s^2 = 0}$. $r = 0$

~~No conclusion~~. Saddle point.

$144 - 16 = 138$

at $(1,1)$.

$rt - s^2 = 144 - 16$

$\Rightarrow 138 > 0$.

local min.

at $(-1,-1)$.

$rt - s^2 = 144 - 16$

$\Rightarrow 138 > 0$

local min.

~~A) Find the local max. or min.~~

→ Extrimum ~~Method~~ value of $f(x)$ under certain restriction

Lagrange method of ~~determi~~ undetermined

Let $f(x,y,z)$ whose EV to be determined under the constant $\phi(x,y,z) = a$.

Step 1 → set $F = f + \lambda\phi$.

Step 2 → set $f(x) = f(y) = f_z = 0$.

Step 3 → Sum simultaneously this

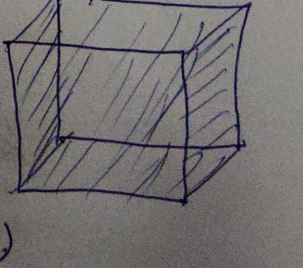
$$f_x = 0$$
$$f_y = 0$$
$$f_z = 0$$
$$\phi = 0$$

Step 4 → The sol. will give u extremum value. (max. or min).

Q). A rectangular box open at the top having V 32 $(ft)^3$. Find dim. of box requiring least material for its construction.

Ans) $f(x,y,z) = xy + 2yz + 2zx$.

this side open



$$xyz = 32$$

$$\phi = (xyz - 32 = 0.$$

$$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

$$F_x = y + 2z + \lambda yz = 0 \quad \cdots (i)$$

$F_y = 2z + x + \lambda xz = 0$ . . . (ii)

$F_z = 2y + 2x + \lambda xy = 0$ . . (iii)

$xyz = 32$ . . . . (iv)

$\phi = xyz$

$x \times (eq\ 1) - y(eq\ 2)$

$\Rightarrow 2zx - 2zy = 0$

$x = y$

$y \times eq(2) - z \times eq(iii)$

$\Rightarrow xy - 2xz = 0$

$y = 2z$

$x = y = 2z$

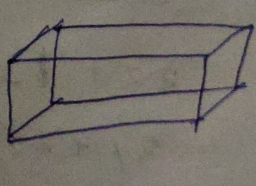
$2z * 2z * z = 32$

$z^3 = 8$

$z = 2$

Q). A rectangular box without lid is to be made of $12m^2$ Tin sheet, find max. value of such box

Ans). $f(x, y, z) = xy + 2yz + 2zx = 12$



$\phi = \cancel{xyz}$

$xy + 2yz + 2zx - 12$

$F = f + \lambda\phi$

$\Rightarrow xy + 2yz + 2zx + \lambda(xy + 2yz + 2zx - 12)$

$\Rightarrow$

$f(x) = y + 2z + \lambda y + \lambda 2z = 0 \quad \ldots (i)$

$f(y) = x + 2z + \lambda x + 2\lambda z = 0 \quad \ldots (ii)$

$f(z) = 2y + 2x + 2\lambda y + 2\lambda x = 0 \quad \ldots (iii)$

$xy + 2zx + 2zy + \lambda(xy + 2zx + 2zy - 12) = 0.$

$\phi = 0.$

$xy + 2yz + 2zx = 12.$

multiplying x in eq (i) & y in eq (ii)

$\cancel{yx} + 2zx + \cancel{\lambda yx} + \lambda 2zx = 0$

$\cancel{xy} + 2zy + \cancel{\lambda xy} + 2\lambda zy = 0.$

$2zx(\lambda + 1) - 2zy(\lambda + 1) = 0$

$2zx - 2zy = 0. \qquad \lambda = -1.$

$2x = +2y.$

$x = y.$

$2 \times$ eq (ii) $-$ eq (iii)

$\cancel{2x} + 4z + \cancel{2\lambda x} + 4\lambda z = 0.$

$2y + \cancel{2x} + 2\lambda y + \cancel{2\lambda x} = 0$

$4z(1+\lambda) - 2y(1+\lambda) = 0.$

$4z = 2y$

$y = 2z.$

$x = y$

$y = 2z$

$x = 2z.$

$\cancel{F = xy + 2yz + 2zx + \lambda(xy + 2yz + 2zx - 12) = 0.}$

$\cancel{4z^2 + 4z^2 + 4z^2 - 1(z^2 + 4z^2 + 4z^2 - 12) = 0.}$

$\cancel{4z^2 + 4z}$

$\cancel{12z^2 - z^2 +}$

$\phi = xy + 2yz + 2zx = 12.$

$\cancel{4z^2 + 4z^2 + 4z^2 = 12.}$

$\cancel{12z^2 = 12}$

$\cancel{z^2 = \frac{12}{9}}$

$z^2 = 1$

$z = 1.$

$y^2 + y^2 + y^2 = 12$

$3y^2 = 12.$

$y^2 = 4.$

$y = 2.$

$x = 2.$

$z = 1.$

3) Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.