Math 214 – Foundations of Mathematics Homework 3/27 Sam Harrington

(Due Thursday, April 10)

COMPLETED Proposition 8.47: $\mathbb{R}_{>0}$ has no upper bound.

Proof. Let $x \in \mathbb{R}_{>0}$.
For purpose of contradiction, assume $\mathbb{R}_{>0}$ has an upper bound. That is, there exists $n \in \mathbb{R}$ such that $x \leqslant n$ for all $x \in \mathbb{R}_{>0}$. – Def of upper bound
Take $n + 1$,
$n - n + 1 = 1, 1 \in \mathbb{R}_{>0} : n < n + 1$
Because $n > 1$ (if not $2 > n$ a contradiction that n is an upper bound($2 \in \mathbb{R}_{>0}$ by Ax 8.26 (i))), $n - 1 \in \mathbb{R}_{>0}$, and $2 \in \mathbb{R}_{>0}$
Since $n + 1 \in \mathbb{R}_{>0}$ (again by Ax 8.26 (i)), we have reached a contradiction that $x \leqslant n$ for all x.
Proving that $\mathbb{R}_{>0}$ has no upper bound □

A quick note, I realized abt half way through that it would have been easier to prove that $n + 1 \in \mathbb{R}_{>0}$ through the same process we showed $1 \in \mathbb{N}$

Proposition 8.50: If the sets $A, B \subseteq \mathbb{R}$ are bounded above and $A \subseteq B$, then

$$\sup(A) \leqslant \sup(B),$$

supposing $\sup(A)$ and $\sup(B)$ exist.

Proof. Let sets $A, B \subseteq \mathbb{R}$ be bounded above and $A \subseteq B$. Also let $\sup(A)$ and $\sup(B)$ exist.
Define $\sup(A) := k$ and $\sup(B) := j$.
For purpose of contradiction, assume $k > j$. Then,
Because $j, k \in \mathbb{R}$ - def of bdd above - there exists $z \in \mathbb{R}$ such that $j < z < k$ – THM 8.43.
Goal: show the set defined of z must contain some $a \in A$

> Assume that the set z does not contain some $a \in A$
> This means that there is no $a > \sup(B)$, so $a \leqslant \sup(B)$ i.e. it is an upper bound by def.
> This means $j \not< k$ a contradiction: Meaning the set defined of z must contain some $a \in A$

Because $k < z$, for all z, $z \notin B$.

> This can be proved through a quick contradiction: suppose $z \in B$
> If $z \in B$ then $z \leqslant k$ a contradiction.

Because $z \notin B$ but $z \in A$, We have reached contradiction that $A \subseteq B$ implying $\sup(A) \leqslant \sup(B)$

□