Math 214 – Foundations of Mathematics Homework 2/11 Sam Harrington

(Due Tuesday, February 25)

COMPLETED Proposition 2.23: Let $m, n \in \mathbb{N}$. If n is divisible by m, then $m \leq n$.

Proof. Let $m, n \in \mathbb{N}$. And suppose n is divisible by m Then,

$n = m * j$	definition of divisibility
$j \in \mathbb{N}$	Prop 2.11
$j >= 1$	Prop 2.20
$j = 1$ or $j > 1$	def. of divisibility

Case 1: $j = 1$

$$n = m * 1$$
$$n = m$$

Case 2: $j > 1$

$j - 1 \in \mathbb{N}$	def. of ¿
$m(j-1) \in \mathbb{N}$	Ax 2.1 (ii)
$n - m \in \mathbb{N}$	
$n > m \in \mathbb{N}$	def of ¿
$m \leq n$	def of ¿

□

Proposition 2.33: Let A be a nonempty subset of $\mathbb{Z}$ and $b \in \mathbb{Z}$, such that for each $a \in A$, $b \leq a$. Then, A has a smallest element.

Proof. Let $b \in \mathbb{Z}, A \in \mathbb{Z}, a \in A, b \leq a$ and let b be fixed Then we have two cases:
When for some $a, b = a$ and therefore $b \in A$

Since $b \leq a$ for each $a \in A$

Secondly: for $a \in A, b < a$ Let $B := \{a_i - b : a_i \in A\}$

$$B \subseteq \mathbb{N} \qquad\qquad \text{def. of } <$$

By THM 2.32 the set B has a least element.

Since, b is fixed, by the def. of smallest element: $\min(B) = a_j - b \leq a_n - b$

$$\begin{aligned} min(B) = a_j - b &\leq a_n - b \\ = a_j - b + b &\leq a_n - b + b && \text{Prop 2.7(i)} \\ = a_j &\leq a_n \end{aligned}$$

Since a_n is the of A, by definition of smallest element the $\min(A)$ is existing being a_j

In both cases, A has a smallest element □