Math 214 – Foundations of Mathematics   Homework 2/27   Sam Harrington

(Due Thursday, March 20)

Theorem 5.15: (DeMorgan's Laws) Given two subsets $A, B \subseteq X$, $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$.

Proof. Let $A, B \subseteq X$:
For the $(A \cap B)^c = A^c \cup B^c$ case:

$y \in (A \cap B)^c$	
is equivalent to $y \notin (A \cap B)$	def of set difference
iet $y \notin \{a : a \in A, a \in B\}$	def of intersect
iet $y \notin A$ or $y \notin B$	property of negation
iet $y \in A^c$ or $y \in B^c$	def of set difference
iet $y \in A^c \cup B^c$	def of union
$(A \cap B)^c \subseteq A^c \cup B^c$	
$x \in A^c \cup B^c$	
iet $x \in A^c$ or $x \in B^c$	def of union
iet $x \notin A$ or $x \notin B$	def of set difference
iet $x \notin \{a : a \in A, a \in B\}$	property of negation
iet $x \notin (A \cap B)$	def of intersect
iet $x \in (A \cap B)^c$	def of set difference
$A^c \cup B^c \subseteq (A \cap B)^c$	
$A^c \cup B^c = (A \cap B)^c$	

For the $(A \cup B)^c = A^c \cap B^c$ case:

$$\begin{aligned}
y &\in (A \cup B)^c & \\
\text{iet } y &\notin (A \cup B) & \text{def of set difference} \\
\text{iet } y &\notin \{a : a \in A \text{ or } a \in B\} & \text{def of union} \\
\text{iet } y &\notin A \text{ and } y \notin B & \text{property of negation} \\
\text{iet } y &\in A^c \text{ and } y \in B^c & \text{def of set difference} \\
\text{iet } y &\in A^c \cap B^c & \text{def of intersect} \\
(A \cup B)^c &\subseteq A^c \cap B^c & \\
x &\in A^c \cap B^c & \\
\text{iet } x &\in A^c \text{ and } x \in B^c & \text{def of intersect} \\
\text{iet } x &\notin A \text{ and } x \notin B & \text{def of set difference} \\
\text{iet } x &\notin \{a : a \in A \text{ or } a \in B\} & \text{property of negation} \\
\text{iet } x &\notin (A \cup B) & \text{def of union} \\
\text{iet } x &\in (A \cup B)^c & \text{def of set differnce} \\
A^c \cap B^c &\subseteq (A \cup B)^c & \\
A^c \cap B^c &= (A \cup B)^c &
\end{aligned}$$

□

Proposition 5.20(ii): Let A, B, and C be sets. Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Proof. Let A, B, C be sets:
Lets say y is an element of $A \times (B \cap C)$
By def of cart product, $y = (a, b)$ where $a \in A$ and $b \in (B \cap C)$
So, $b \in B$ and $b \in C$ - def on intersect
Since a is always $\in A$ we can derive $y = (a, b)$ where $a \in A$ and $b \in B$ and $y = (a, b)$ where $a \in A$ and $b \in C$
This creates two cart products: $y \in A \times B$ and $y \in A \times C$, we can then use the def of intersect to get $y \in (A \times B) \cap (A \times C)$
Since y is a generic element of $A \times (B \cap C)$

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C).$$

Starting with a generic element, y, of $(A \times B) \cap (A \times C)$.
Through the def of intersect, $y \in (A \times B)$ and $y \in (A \times C)$
We have two cart products, $y = (a, b)$ where $a \in A$, $b \in B$ and $b \in C$
Since b must be elements of B, C we can create the intersect $b \in (B \cap C)$

Now, we have $a \in A, b \in (B \cap C)$ which creates the cart product: $A \times (B \cap C)$
Since y was a generic element of $(A \times B) \cap (A \times C)$ we get:

$$
\begin{gathered}
(A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \\
(A \times B) \cap (A \times C) = A \times (B \cap C)
\end{gathered}
$$

$\square$

Proposition 6.5: Assume we are given an equivalence relation $\sim$ on a set A. For all $a_1, a_2 \in A$, either $[a_1] = [a_2]$ or $[a_1] \cap [a_2] = \varnothing$.

Proof. Let A be a set with an equivalence relation $\sim$ creating equivalence sets $[a]$
Begin by assuming $[a_1] \cap [a_2] \neq \varnothing$.

$$
\begin{aligned}
{[a_1] \cap [a_2]} &\neq \varnothing \\
\text{This implies there must be some } x \in [a_1] \text{ and } x &\in [a_2] \\
x \sim a_1 \text{ and } x &\sim a_2 && \text{def of eq class} \\
a_1 &\sim a_2 && \text{def of eq rel (iii) + (ii)} \\
{[a_1]} &= [a_2] && \text{Prop 6.4 (ii)}
\end{aligned}
$$

Proving either $[a_1] = [a_2]$ or $[a_1] \cap [a_2] = \varnothing$. $\square$