November 8, 2015 12:00 AM

$$\frac{\partial \eta}{\partial t} = \sum_{K=-\infty}^{\infty} e^{iKx} \frac{d\tilde{\eta}_K}{dt} \quad (3)$$

$$\frac{d^2\hat{\phi}_K(z)}{dz^2} - K^2 \hat{\phi}_K(z) = 0 \quad , \quad \phi_K(z) \sim e^{Kz} \quad (1)$$

$$\frac{\partial \phi(z=0)}{\partial t} = -g\eta_K \quad (2)$$

Initial Condition $\eta(x,t=0) = \eta_0(x) = \sum_{K=-\infty}^{\infty} e^{iKx}\tilde{\eta}_K(t=0)$

$$\eta(x,t) = \sum_{K=-\infty}^{\infty} e^{iKx} \hat{\eta}_K(t) \quad (4)$$

$$\phi(x,z,t) = \sum_{K=-\infty}^{\infty} e^{iKx} \hat{\phi}_K(z)\, \hat{\phi}_K(t) \quad (5)$$

Starting with $\eta(x,t)$,

$\frac{\partial \eta}{\partial t} = \frac{\partial \dot{\phi}}{\partial t}$, It's a wave, Known solution.

$\Rightarrow \eta(x,t) = C_1 \cos(\omega_K t) + C_2 \sin(\omega_K t)$, at rest.

$\frac{\partial \eta}{\partial t}(t=0) = 0 = -C_1 \cancelto{0}{\sin(0)} + C_2\cos(0)$

$\Rightarrow C_2 = 0.$

$\eta(x,t) = C_1 \cos(\omega_K t)$, use another IC.

$\eta(x,0) = \eta_0(x) = C_1 \cancelto{1}{\cos(0)}$

$\rightarrow C_1 = \eta_0(x)$, $\eta(x,t) = \eta_0(x)\cos(\omega_K t)$

$\therefore$ From ④, $\eta_k(x,t) = \sum_{-\infty}^{\infty} e^{ikx} \eta_0(x) \cos(\omega_k t)$

as expected.

Now for ϕ. Already know $\phi_k(z) \sim e^{kz}$

From ②, $\phi(z=0) = -g \int \eta_k \, dt$,

$$\phi_k(t) = -g \int \eta_0(x) \cos(\omega_k t) \, dt$$, pull $\eta_0(x)$ out.

$$= -g \eta_0(x) \left[\frac{1}{\omega_k} \sin(\omega_k t) \right]$$

Therefore, subbing $\phi_k(z) = e^{kz}$ and

$\phi_k(t) = -\dfrac{g\eta_0(x)}{\omega_k} \sin(\omega_k t)$ into ⑤,

we get $\phi(x,z,t) = -\sum_{-\infty}^{\infty} e^{kz} e^{ikx} \dfrac{g\eta_0(x)}{\omega_k} \sin(\omega_k t)$

as expected.

When $k=0 \Rightarrow \omega_k = 0$, so the cosines and sines become 1's and 0's which does not depend on time.