Quantum searching with continuous variables

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A fast quantum search algorithm for continuous variables is presented. The result is the quantum continuous variable analog of Grover's algorithm originally proposed for qubits. A continuous variable analog of the Hadamard (*i.e.*, Fourier transform) operation is used in conjunction with inversion about the average of quantum states to allow the approximate identification of an unknown quantum state in a way that gives a square-root speed-up over search algorithms using classical continuous variables. Also, we show that this quantum search algorithm is robust for a generalised Fourier transformation on continuous variables.

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Quantum systems can register and process information in ways that classical systems cannot. As a result, it is possible for quantum computers to perform certain computational tasks faster than any classical computer [1–9]. It is becoming increasingly clear that at the heart of quantum computation lies two basic quantum phenomena, one is quantum interference and the other quantum entanglement. The real upsurge of interest in this field came after Shor's [10] remarkable discovery of an algorithm for factoring large numbers [11]. Subsequently, a fast quantum search algorithm has been discovered by Grover [12], which takes $O(\sqrt{N})$ steps instead of N steps to search an unmarked item in a unsorted list of N entries. Later the optimality of this search algorithm was proved [13,14]. In particular, it was shown that this search algorithm can use almost any unitary transformation on qubit states [15]. Further, it has even been argued that this algorithm would work even without entanglement (though at a cost in resources) [16].

These algorithms are usually implemented on quantum systems whose observables have discrete spectra, such as a collection of two-level atoms, ions, or spin-$\frac{1}{2}$ particles (called qubits). However, there are other classes of quantum systems whose observables form continuous spectra. So, it is important to know how these algorithms can be generalised for continuous quantum variables? With the recent advances in our ability to manipulate continuous quantum information in teleportation [17], error-correction codes [18,19] and its feasibility of implementation using linear devices [20], it is natural to ask whether one can provide some quantum algorithms that might be implemented on a continuous variable quantum computer. In fact, the usefulness of quantum computation over continuous variables has been recently emphasised [21]. It has been shown that universal quantum computation over continuous variable is not only possible, but could be effected using simple non-linear operations with coupling provided solely by linear operations [21]. These operations form a universal set of quantum gates for continuous variables allowing 'quantum floating point' arithmetic. While discrete quantum computation can be thought of as the coherent manipulation of qubits, continuous quantum computation can be thought of as the manipulation of 'qunats', where the qunat (pronounced as 'Q nat') is the unit of continuous quantum information.

In this letter, we propose a fast quantum search algorithm with continuous variables. Here a continuous variable can be anything, *e.g.*, position, momentum, energy (unbounded) or amplitudes of the electromagnetic field. With the help of the Fourier transform (viewed as an active operation) on a continuous basis state (analogous to the Hadamard transform in the case of qubits) and a suitably generalised inversion operator, we construct a search operator which can be implemented on a continuous variable quantum computer. The inversion operator requires the projection operator for continuous basis states which we discuss. We show that the application of the compound operator takes $O(\sqrt{N})$ iterations to search an unmarked item in a list of N entries. Further, we generalise our quantum searching with continuous variables to one based on a generalised Fourier transformation and which still gives a square root reduction in the number of steps. This shows that the quantum search algorithm with continuous variables is robust to the choice of arbitrary Fourier transformations. We also discuss the robustness of the search algorithm if one uses strongly peaked normalisable states instead of ideal infinite-energy position eigenstates.

Here, we discuss how to perform a quantum search algorithm using continuous variables. First we need to map a conventional discrete search problem into a continuous variable context. Suppose we have a function $f(k) : K \to \{0, 1\}$ defined on a domain K with $k \in K = \{1, 2, \ldots, N\}$. This function has a non-zero value equal to 1 for some element $k = k_f$ and is 0 for all other elements in the set K. Our task is to discover the value of k_f given the ability to apply the function f

to inputs or superpositions of inputs, and given no further information about the function $f(k)$. In order to implement this in a quantum computer with *continuous variables* we require a collection of n qunats. The state vector of each qunat belongs to a Hilbert space of infinite dimension. Since we have an infinite number of basis states, we cannot map each basis state within the Hilbert space onto each entry in the set K. (This would be a many-to-one mapping.) One could avoid this problem by choosing a subspace of the full Hilbert space with N disjoint regions of the spectrum. However, we do not discuss this approach in detail here.

Let us consider a collection of n continuous variables whose Hilbert space is spanned by a basis of states $|x\rangle = |x_1, x_2, \ldots, x_n\rangle$, satisfying the orthogonality condition $\langle x|x'\rangle = \delta(x_1 - x_1')\cdots\delta(x_n - x_n') = \delta(x - x')$. For example, one can consider a compact region of the state space divided into N equal subvolumes, each with measure Δx^n, one for each member of the set K. Let x_f be the centre of the subvolume corresponding to k_f. In the context of this continuous variable embedding, executing the function f corresponds to adjoining an extra state to the system, originally in the state $|0\rangle$, and applying an operator $U_f : |x\rangle|0\rangle \to |x\rangle|1\rangle$ if x belongs to the region corresponding to k_f and $|x\rangle|0\rangle \to |x\rangle|0\rangle$ otherwise. Clearly, if one samples the region at random by applying the operator to a series of random points, it will take $O(N)$ calls of the operator to find k_f.

If one exploits the power of quantum superposition and entanglement, however, fewer function calls are required. Our approach is discussed below. Let us pick an initial state in the position basis such as $|x_i\rangle = |x_1, x_2, \ldots, x_n\rangle_i$ for a quantum computer with continuous spectrum at random. The final (target) state is given by $|x_f\rangle = |x_1, x_2, \ldots, x_n\rangle_f$. We need a suitable unitary operator, which can take the initial state to the final state. Just as we have the Hadamard transformation in discrete computation, one of the basic operations with continuous variables is the Fourier transformation between position and momentum variables in phase space. By defining the Fourier transformation as an active operation on n qunat states $|x\rangle$ we can write it as

$$\mathcal{F}|x\rangle = \frac{1}{\sqrt{\pi^n}} \int dy\, e^{2ixy} |y\rangle \ , \tag{1}$$

where $xy = x_1y_1 + \cdots + x_ny_n$, $|y\rangle = |y_1, y_2, \ldots, y_n\rangle$ and both x and y are in the position basis. This has been used by one of the present authors [18,20] in developing an error correction code for continuous variables. This Fourier transformation can be straightforwardly applied in physical situations. For example, when $|x\rangle$ represents quadrature eigenstate of a set of modes of the electromagnetic field, $\mathcal{F}|x\rangle$ is simply an eigenstate of the conjugate quadrature.

Suppose, we apply the unitary operator $\mathcal{F}$ to a basis state $|x_i\rangle$, then the relative amplitude of finding the system in the target state $|x_f\rangle$ is $\langle x_f|\mathcal{F}|x_i\rangle = \mathcal{F}_{fi} = e^{2ix_ix_f}/\sqrt{\pi^n}$. Therefore, the relative probability of finding the system in the final qunat states will be given by $|\mathcal{F}_{fi}|^2 = 1/\pi^n$. Hence, we have to repeat the experiment at least $1/|\mathcal{F}_{fi}|^2 = \pi^n$ times to successfully obtain the state $|x_f\rangle$. Here, we prove that search algorithm based on continuous variable can take $\sqrt{\pi^n}$ steps to reach the final state starting from an initial state. (Here, we may identify the number of entries N with π^n.)

The next operator we need is the unitary operator, which can invert the sign of a basis state $|x\rangle$. We can define the selective inversion operator for a continuous basis $|x\rangle$ as

$$I_x = 1 - 2P_{\Delta x} \ , \tag{2}$$

where $P_{\Delta x}$ is the projection operator for continuous variables. Unlike the discrete case we cannot define the projection operator for the basis $|x\rangle$ as $P_x = |x\rangle\langle x|$, because the operator P_x is an ill defined and it will not satisfy $P_x^2 = P_x$. The correct projection operator for continuous variables is defined [22] as

$$P_{\Delta x} = \int_{x_0 - \Delta x/2}^{x_0 + \Delta x/2} dx' |x'\rangle\langle x'| \ . \tag{3}$$

The reason for this definition is that we cannot project an arbitrary state which is represented in terms of continuous basis state onto a point to get the exact eigenvalue. There will be always a spread within an interval. We can only project a state around x_0 to a selectivity Δx of the measuring apparatus. It is not possible to design a device to make a perfectly selective measurement of a continuous variable. The interval $[x_1, x_2]$ cannot be narrowed down, because it will always contains an infinite number of eigenvalues [22]. Thus, if we have a wave packet the effect of projection is to truncate it around x_0 within an interval Δx. This operator satisfies $P_{\Delta x}^2 = P_{\Delta x}$ and $P_{\Delta x}|x\rangle = |x\rangle$ as expected. With the help of the above inversion operator we can construct a compound search operator $\mathcal{C}$ defined as

$$\mathcal{C} = -I_{x_i}\, \mathcal{F}^\dagger I_{x_f} \mathcal{F} \ . \tag{4}$$

It may be remarked that the selective inversion of the target state $|x_f\rangle$ can be achieved by attaching an ancilla qunat and considering the quantum XOR circuit for continuous variables [18]. If a quantum circuit exists that transforms $|x\rangle|a\rangle \to |x\rangle|f(x) + a\rangle$, then by choosing the ancilla state $|a\rangle = \mathcal{F}|\pi/2\rangle = \int dy\, e^{i\pi y}|y\rangle/\sqrt{\pi^n}$ we can selectively invert the state $|x\rangle$ for which $f(x) = 1$, *i.e.*, $|x\rangle\mathcal{F}|\pi/2\rangle \to -|x\rangle\mathcal{F}|\pi/2\rangle$.

Let us define a state $|\tilde{x}_f\rangle \equiv \mathcal{F}^\dagger|x_f\rangle$. We can show that the operator $\mathcal{C}$ can preserve the two-dimensional subspace spanned by the states $|x_i\rangle$ and $|\tilde{x}_f\rangle$. First, we show the action of $\mathcal{C}$ on $|x_i\rangle$. This can be expressed as

$$\mathcal{C}|x_i\rangle = |x_i\rangle - 4P_{\Delta x_i}\mathcal{F}^\dagger P_{\Delta x_f}\mathcal{F}|x_i\rangle + 2\mathcal{F}^\dagger P_{\Delta x_f}\mathcal{F}|x_i\rangle \ , \tag{5}$$

where $P_{\Delta x_i} = \int_{x_{i1}}^{x_{i2}} dx'_i \, |x'\rangle_{ii}\langle x'|$, ($x_{i1} = x_0 - \Delta x_i/2$, $x_{i2} = x_0 + \Delta x_i/2$) and likewise for $P_{\Delta x_f}$. Using these facts we can simplify the above equation to

$$\mathcal{C}|x_i\rangle = (1 - \frac{4}{\pi^n})|x_i\rangle + \frac{2}{\sqrt{\pi^n}} \int_{x_{f1}}^{x_{f2}} dx'_f \, e^{2ix_i x'_f} \mathcal{F}^\dagger |x'_f\rangle \; . \quad (6)$$

Similarly, we can evaluate the action of $\mathcal{C}$ on $|\tilde{x}_f\rangle$. It is given by

$$\mathcal{C}|\tilde{x}_f\rangle = |\tilde{x}_f\rangle - \frac{2}{\sqrt{\pi^n}} \int_{x_{i1}}^{x_{i2}} dx'_i \, e^{2ix'_i x_f} |x'_i\rangle \; . \quad (7)$$

Thus, the operator $\mathcal{C}$ creates superpositions of two qunat states just as Grover's operator creates superpositions of two qubit states. Once we understand the action of $\mathcal{C}$ on qunats we can obtain the total number of steps required in reaching the target state. Here, we use geometric structures from the projective Hilbert space of a quantum system to obtain the number of steps in the quantum searching. The projective Hilbert space admits a natural measure of distance called Fubini-Study distance [23]. This measures the shortest distance between any two (not necessarily normalized) states $|\psi_1\rangle$ and $|\psi_2\rangle$ whose projections on $\mathcal{P}$ are $\Pi(\psi_1)$ and $\Pi(\psi_2)$, respectively. This can be defined as

$$d^2(|\psi_1\rangle, |\psi_2\rangle) = 4\Bigg(1 - \Bigg|\Bigg\langle \frac{\psi_1}{||\psi_1||} \Bigg| \frac{\psi_2}{||\psi_2||} \Bigg\rangle\Bigg|^2\Bigg). \quad (8)$$

Here, the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ can be quantum states over continuous variables or over discrete variables. For unnormalisable states the above definition still works provided it is understood that the norm of the states can be made finite. In dealing with position eigenstates we can imagine that either the particle is moving in a finite space [22] so that position eigenstates do not diverge or one can use normalisable states having strong peaks around some value of the position axis.

During the quantum searching with continuous variables we want to reach a state $|\tilde{x}_f\rangle$ from an initial state $|x_i\rangle$. This means we have to travel a shortest distance between these states which is given by $d^2(|x_i\rangle, |\tilde{x}_f\rangle) = 4(1 - 1/\pi^n)$. One application of the operator $\mathcal{C}$ creates a state $|x_i\rangle^{(1)} = \mathcal{C}|x_i\rangle$. We calculate the shortest distance between the resulting state $|x_i\rangle^{(1)}$ and the initial state $|x_i\rangle$. We note that the overlap of these states is given by $\langle x_i|\mathcal{C}|x_i\rangle = (1 - 4/\pi^n)\langle x_i|x_i\rangle + 2\Delta x_f/\pi^n$. For large database search $N = \pi^n$ is very large and if we assume that the measuring device has a narrow selectivity, then Δx_f is also small. Hence, we can neglect the second term in the overlap (as it is a product of two small terms). Also note that term $\langle x_i|x_i\rangle$ is not normalised but nevertheless it cancels out in during calculation. With this idea in mind we can evaluate the shortest distance between these states which is given by $d^2(|x_i\rangle, |x_i^{(1)}\rangle) \approx 32/\pi^n$. Thus in one application of the search operator $\mathcal{C}$ we can move the initial basis a shortest distance $O(1/\sqrt{\pi^n})$. Therefore, to travel the full distance on the quantum state space we need N_s number of steps given by

$$N_s = \frac{d(|x_i\rangle, |\tilde{x}_f\rangle)}{d(|x_i\rangle, |x_i^{(1)}\rangle)} \approx O(\sqrt{\pi^n}) \; . \quad (9)$$

This shows that a quantum computer based on qunats can take $O(\sqrt{\pi^n})$ applications of $\mathcal{C}$ to reach the target state which otherwise would have taken $O(\pi^n)$ number of steps by the application of $\mathcal{F}$ on $|x_i\rangle$. Because the state is moving along a geodesic each application of $\mathcal{C}$ rotates the initial state in the right direction. This is our quantum search algorithm with continuous variables.

Instead of position eigenkets one can use strongly peaked normalisable state such as

$$|r_i\rangle = \frac{1}{(2\pi\epsilon)^{n/4}} \int dx \, \exp[-\frac{(x - x_i)^2}{4\epsilon^2}]|x\rangle \; . \quad (10)$$

When $\epsilon \to 0$, the state $|r_i\rangle$ becomes a position eigenstate $|x_i\rangle$. Our algorithm can be practically implemented with such states. One can see that the action of the search operator $\mathcal{C}$ on $|r_i\rangle$ gives

$$\mathcal{C}|r_i\rangle = (1 - \frac{4}{\pi^n})|r_i\rangle + \frac{2}{\sqrt{\pi^n}} \frac{1}{(2\pi\epsilon)^{n/4}} \times \int dx \int_{x_{f1}}^{x_{f2}} dx'_f e^{-(x-x_i)^2/4\epsilon^2 + 2ixx'_f} \mathcal{F}^\dagger |x'_f\rangle \; . \quad (11)$$

From the above formula one can see that a single application of search operator moves the initial state $|r_i\rangle$ a distance given by $d^2(|r_i\rangle, |r_i^{(1)}\rangle) \approx 48/\pi^n$. Now, if one defines the target state as

$$|r_f\rangle = \frac{1}{(2\pi\epsilon)^{n/4}} \int dx \exp[-\frac{(x - x_f)^2}{4\epsilon^2}]|x\rangle \; , \quad (12)$$

then one can check that the total (shortest) distance between the initial state $|r_i\rangle$ and the desired state $|\tilde{r}_f\rangle = \mathcal{F}^\dagger|r_f\rangle$ is $4[1 - 4O(\epsilon^2)/\pi^n]$. Hence, by using (9) the total number of steps required to reach the target state is $N_s = O(\sqrt{\pi^n})$.

Now we show that the quantum search algorithm over continuous variables is robust to some extent. Instead of the Fourier transform $\mathcal{F}$ if we replace it by a generalised Fourier transform (GFT) in the search operator $\mathcal{C}$, still the algorithm works, *i.e.*, we do get a square root reduction in the number of steps. We define a generalised Fourier transform as an active operation in the position basis $|x\rangle$ as

$$\mathcal{F}^{(\theta)}|x\rangle = (\frac{i}{\pi \sin\theta})^{n/2} \times \int dy \, exp\Big[-\frac{i}{\sin\theta}[(x^2 + y^2)\cos\theta - 2xy]\Big]|y\rangle \; . \quad (13)$$

The GFT with a flexible angle θ gives a physical change of the basis $|x\rangle$ by any desired amount [24]. Here, it should be mentioned that $\theta > \arcsin(1/\pi)$, since for smaller values of θ the assumption $\sin^n \theta \gg 1/\pi^n$ does not hold. The GFT for $\theta = 2\pi m$, m being an integer, corresponds to no change of basis. The GFT for $\theta = \pi/2$ corresponds to the Fourier transform defined in (1) (up to a constant phase shift equal to $n\pi/4$, n being the number of qunats). If we apply GFT to an initial basis $|x_i\rangle$ then by probability rules of quantum theory we have to perform at least $O[(\pi \sin\theta)^n]$ number of trials to reach a target state $|x_f\rangle$. We will prove that the generalised search operator acting on continuous variables will take $O[\sqrt{(\pi \sin\theta)^n}]$ steps to reach the final state.

The search operator with this GFT takes the form

$$\mathcal{C}^{(\theta)} = -I_{x_i} \mathcal{F}^{(\theta)^\dagger} I_{x_f} \mathcal{F}^{(\theta)} \; . \tag{14}$$

We can see that the action of the generalised search operator on the initial state $|x_i\rangle$ is given by

$$\mathcal{C}^{(\theta)}|x_i\rangle = \big(1 - \frac{4}{(\pi \sin\theta)^n}\big)|x_i\rangle + 2\sqrt{\frac{i^n}{(\pi \sin\theta)^n}} \tag{15}$$
$$\times \int_{x_{f1}}^{x_{f2}} dx'_f \exp\big\{-\frac{i}{\sin\theta}[(x_i^2 + x_f'^2)\cos\theta - 2x_i x_f]\big\}\mathcal{F}^\dagger |x'_f\rangle \, .$$

Similarly, the action of the generalised search operator $\mathcal{C}^{(\theta)}$ on $|\tilde{x}\rangle_f$ can be calculated. It is given by

$$\mathcal{C}^{(\theta)}|\tilde{x}\rangle_f = |\tilde{x}\rangle_f - 2\sqrt{\frac{(-i)^n}{(\pi \sin\theta)^n}} \tag{16}$$
$$\times \int_{x_{i1}}^{x_{i2}} dx'_i \exp\big\{\frac{i}{\sin\theta}[(x_i'^2 + x_f^2)\cos\theta - 2x'_i . x_f]\big\}|x'_i\rangle \, .$$

It can be seen that the generalised search operator creates linear superposition of qunat states in the search process. Now, we can calculate the Fubini-Study distances to know how many steps are needed to reach the target state. The shortest distance between the states $|x_i\rangle$ and $|\tilde{x}_f\rangle$ is $d^2(|x_i\rangle, |\tilde{x}_f\rangle) = 4[1 - 1/(\pi \sin\theta)^n]$. Notice that single application of the search operator $\mathcal{C}^{(\theta)}$ moves the initial state by a distance given by $d^2(|x_i\rangle, |x_i^{(1)}\rangle) = (32/\pi \sin\theta)^n$. Therefore, to travel a shortest distance $d(|x_i\rangle, |\tilde{x}_f\rangle)$ we need $N_s \approx O[\sqrt{(\pi \sin\theta)^n}]$ number of steps.

Thus, using a generalised Fourier transform we have proved that there is a square root reduction in the number of steps working with continuous variables. As expected for an angle $\theta = \pi/2$ we get back the original result with the search operator $\mathcal{C}$. This result is similar to the recent result of Grover [15], where the search algorithm for qubits has been generalised for arbitrary unitary transformations.

In conclusion, we have for the first time provided an efficient algorithm such as quantum searching to be implemented on a quantum computer with continuous variables. The key elements in this generalisation are the Fourier transformation and inversion operators which constitute the search operator for qunats in an infinite dimensional Hilbert space. We find that a square root speed up is possible with quantum computers based on qunats. Also, the continuous variable search is possible with almost any Fourier transformation. This may be practically implemented for any large data base search using linear and non-linear optical devices with the role for qunats being played by electromagnetic fields. It may well be that for large data base searches it is beneficial to use continuous quantum variables.

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