MATHEMATICAL PROGRAMMING

CO3

NON-LINEAR PROGRAMMING

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LINEAR PROGRAMMING	NONLINEAR PROGRAMMING
A method to achieve the best outcome in a mathematical model whose requirements are represented by linear relationships	A process of solving an optimization problem where the constraints or the objective functions are nonlinear
Helps to find the best solution to a problem using constraints that are linear	Helps to find the best solution to a problem using constraints that are nonlinear

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NLPP

Non Linear Programming Problem (NLPP)

An optimisation problem in which objective function z or some/all constraints are non linear {Higher power of x_1, x_2 ~~far~~ than one] is called NLPP.

Types :
- i) with no constraints
- ii) with Equality constraints → Lagrange's method
- iii) with inequality constraints. → Kuhn-Tucker conditions.

Example of Non-Linear Programming Problem $F(x) = 3(x_1)^2 + 1.4x_1x_2 + 2(x_2)^2$

QUADRATIC PROGRAMMING

- An NLP problem with non-linear objective function and linear constraints. Such an NLP problem is called quadratic programming problem.
- The general mathematical model of quadratic programming problem is as follows:

$$\text{Optimize (Max or Min) } Z = \left\{ \sum_{j=1}^{n} c_j\, x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j\, d_{jk}\, x_k \right\}$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i$$

and $x_j \ge 0$ for all i and j

CONT...

- In matrix notations, QP problem is written as:

$$\text{Optimize (Max or Min) } Z = \mathbf{cx} + \frac{1}{2}\mathbf{x}^T\,\mathbf{Dx}$$

subject to the constraints

$$\mathbf{Ax} \leq \mathbf{b}$$

and $\mathbf{x} \geq 0$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$; $\mathbf{c} = (c_1, c_2, \ldots, c_n)$; $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$

$\mathbf{D} = [d_{jk}]$ is an $n \times n$ symmetric matrix, i.e. $d_{jk} = d_{kj}$; $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix

Example of QUADRATIC Programming Problem:

Maximize $f(x) = 2x_1 + 3x_2 - x_1^2 - x_2^2$

subject to

$x_1 + x_2 \leq 2$

$2x_1 + x_2 \leq 3$, $x_1, x_2 \geq 0$.

CONT…

1. If the objective function in QP problem is of minimization, then matrix D is symmetric and positive definite.
2. if the objective function is of maximization, then matrix D is symmetric and negative-definite.
3. If matrix D is null, then the QP problem reduces to the standard LP problem.

DIFFERENCE BETWEEN LINEAR PROGRAMMING AND QUADRATIC PROGRAMMING

- **In linear programming (LP) problems, the objective and all of the constraints are linear functions of the decision variables.**
- **Example : $75\ x_1 + 50\ x_2 + 35\ x_3$**
- **LP problems are usually solved via the Simplex method.**

- ***In the quadratic programming (QP) problem, the objective is a quadratic function of the decision variables, and the constraints are all linear functions of the variables.***
- ***Example : $2\ x_1^2 + 3\ x_2^2 + 4\ x_1\ x_2$***
- ***QP*** **problems are usually solved via**
  - **Beale’s method,**
  - **Wolfe method,**
  - **Karush-Kuhn Tucker (KKT) Condition**

CONT…

1. **Non-Linear Programming** :
   1. **Quadratic programs** – Constrained quadratic programming problems,
      - **Beale’s method,**
      - **Wolfe method,**
      - **Karush-Kuhn Tucker (KKT) Conditions.**

Non-Linear Programming Problems

Lagrange Multipliers Method

Problems: For ONE Equality Constraint

For ONE Equality Constraint:

Lagrange Multipliers Method

If, f – *objective function;*
g – *consraint;*
λ – *Lagrangian Multiplier*

$$Let,\ L = f - \lambda \cdot g$$

$$Solve,\quad \frac{\partial L}{\partial x_1} = 0,\quad \frac{\partial L}{\partial x_2} = 0,\quad \frac{\partial L}{\partial \lambda} = 0$$

To find x_1, x_2 *and* λ, *we get* $X_0(x_1, x_2)$ *which is a point of maxima or minima*

Find Hessian Matrix as:

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 g}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 g}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 g}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 g}{\partial x_2^2} \end{vmatrix}$$

(i) *If* $\Delta_3 \geq 0 \Rightarrow Z$ *is Maximum*

(ii) *If* $\Delta_3 \leq 0 \Rightarrow Z$ *is Minimum*

Use the method of Lagrangian multipliers to solve the following NLP problem. Does the solution maximize or minimize the objective function?

Optimize $Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to the constraint $\quad g(x) = x_1 + x_2 + x_3 = 20$

$$x_1, x_2, x_3 \geq 0$$

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$L(x,\lambda) = f(x) - \lambda g(x) \qquad L(x,\lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20)$

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0 \qquad \frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda = 0$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 + 10 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda - 10}{4}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + 8 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda - 8}{2}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6x_3 + 6 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda - 6}{6}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 + x_3 - 20 = 0$$

$$x_1 + x_2 + x_3 = 20$$

$$\frac{\lambda - 10}{4} + \frac{\lambda - 8}{2} + \frac{\lambda - 6}{6} = 20$$

$$11\lambda - 90 = 240$$

$$\lambda = \frac{330}{11} = 30$$

Putting the values of x_1, x_2 and x_3 in the last equation $\partial L / \partial \lambda = 0$ and solving for λ, we get $\lambda = 30$. Substituting the value of λ in the other three equations, we get an extreme point: $(x_1, x_2, x_3) = (5, 11, 4)$.

$$x_1 = \frac{\lambda - 10}{4} \Rightarrow x_1 = 5$$

$$x_2 = \frac{\lambda - 8}{2} \Rightarrow x_2 = 11$$

$$x_2 = \frac{\lambda - 6}{6} \Rightarrow x_3 = 4$$

$$X_0(x_1, x_2, x_3) \Rightarrow X_0(5, 11, 4)$$

To prove the sufficient condition of whether the extreme point solution gives maximum or minimum value of the objective function we evaluate $(n-1)$ principal minors as follows:

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 g}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 g}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 g}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 g}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$\Delta_{n+1} = \Delta_{3+1} = \Delta_4$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 48$$

$f_{x_1} = \underline{4x_1 + 10} \rightarrow f_{x_1 x_1} = 0$

$$\text{Optimize } Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

$$g(x) = x_1 + x_2 + x_3 = 20$$

$f_{x_3} = 6x_3 + 6$

$f_{x_3^2} = 6$

Since, the sign of Δ_3 and Δ_4 are alternative, therefore extreme points: $(x_1, x_2, x_3) = (5, 11, 4)$ is local maximum. At this point the value of the objective function is $Z = 281$.

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 g}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 g}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 g}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 g}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$\Delta_{n+1} = \Delta_{3+1} = \Delta_4$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 48$$

$f_{x_1} = \underline{4x_1 + 10} \rightarrow f_{x_1 x_2} = 0$

$$\text{Optimize } Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

$$g(x) = x_1 + x_2 + x_3 = 20$$

$f_{x_3} = 6x_3 + 6$

$f_{x_3^2} = 6$

Since, the sign of Δ_3 and Δ_4 are alternative, therefore extreme points: $(x_1, x_2, x_3) = (5, 11, 4)$ is local maximum. At this point the value of the objective function is $Z = 281$.

KUHN-TUCKER CONDITIONS

1. **The necessary and sufficient Kuhn-Tucker conditions to get an optimal solution to the maximization QP problem subject to linear constraints.**

KUHN-TUCKER CONDITIONS WITH ONE CONSTRAINT

For ONE inequality Constraint:

Kuhn–Tucker Conditions

If, f – objective function; g – consraint; λ – Lagrangian Multiplier

$$Let,\ K = f - \lambda \cdot g$$

Find x_1, x_2 and λ by solving the eqautions, $\dfrac{\partial K}{\partial x_1} = 0,\quad \dfrac{\partial K}{\partial x_2} = 0,\quad \dfrac{\partial K}{\partial \lambda} = 0$

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KUHN-TUCKER CONDITIONS

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda \cdot g_{x_2} = 0$
3. $\lambda \cdot g = 0$
4. $g \leq 0$
5. $\lambda \geq 0$ (*if Z is max*)
6. $x_1, x_2 \geq 0$

NOTE:

For, Maximize: consider $\lambda \geq 0$

For, Minimize: consider $\lambda \leq 0$

In equation (3), either $\lambda = 0$ *or* $g = 0$.

From solving these different cases, & verify whether all above conditions are satisfied.

Solve the following NLPP using the Kuhn-Tucker method:

Maximize: $z = 2x_1^2 - 7x_2^2 + 12x_1x_2$ ← f

sub. to, $2x_1 + 5x_2 \leq 98$ ← g

$x_1, x_2 \geq 0$

$g \equiv 2x_1 + 5x_2 - 98$

$$Let, K = f - \lambda \cdot g = 2x_1^2 - 7x_2^2 + 12x_1x_2 - \lambda(2x_1 + 5x_2 - 98)$$

$$\frac{\partial K}{\partial x_1} = 4x_1 + 12x_2 - 2\lambda = 0 \Rightarrow 2x_1 + 6x_2 - \lambda = 0 \ldots\ldots (1)$$

$$\frac{\partial K}{\partial x_2} = -14x_2 + 12x_1 - 5\lambda = 0 \Rightarrow 12x_1 - 14x_2 - 5\lambda = 0 \ldots\ldots (2)$$

$$\lambda \cdot g = \lambda \cdot (2x_1 + 5x_2 - 98) = 0 \ldots\ldots (3)$$

$$g \leq 0 \Rightarrow (2x_1 + 5x_2 - 98) \leq 0 \ldots\ldots (4)$$

$$\lambda \geq 0 \ldots\ldots (5)$$

$$x_1, x_2 \geq 0 \ldots\ldots (6)$$

Now, $\lambda \cdot g = \lambda \cdot (2x_1 + 5x_2 - 98) = 0$

r ONE inequality Constraint:

f – objective function;
g – consraint;
λ – Lagrangian Multiplier

$$Let, K = f - \lambda \cdot g$$

nd x_1, x_2 and λ by solving the eqautions,

$$\frac{\partial K}{\partial x_1} = 0, \frac{\partial K}{\partial x_2} = 0, \frac{\partial K}{\partial \lambda} = 0$$

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda \cdot g_{x_2} = 0$
3. $\lambda \cdot g = 0$
4. $g \leq 0$
5. $\lambda \geq 0$ (*if Z is max*)
6. $x_1, x_2 \geq 0$

- *In equation (3), either $\lambda = 0$ or $g = 0$.*

From solving these different cases, & verify whether all above conditions are satisfied

Now, $\lambda \cdot g = \lambda \cdot (2x_1 + 5x_2 - 98) = 0$

Case 1: $If\ \boxed{\lambda = 0,}$

$$2x_1 + 6x_2 - \cancel{\lambda} = 0 \ldots\ldots (1)$$

$$12x_1 - 14x_2 - \cancel{5\lambda} = 0 \ldots\ldots (2)$$

$$\left.\begin{array}{l} x_1 + 3x_2 = 0 \\ 6x_1 - 7x_2 = 0 \end{array}\right\}$$

$$\Rightarrow x_1 = x_2 = 0$$

$$Z = 2x_1^2 - 7x_2^2 + 12x_1x_2$$

$$Z = 0$$

Hence, this case, does not created feasible solutio

Therefore, assumption of $\lambda = 0$ is not correct.

Therefore, we need to REJECT, these values at λ

Case 2: $If\ 2x_1 + 5x_2 - 98 = 0,$

$$2x_1 + 6x_2 - \lambda = 0 \times (5)$$

$$10x_1 + 30x_2 - \cancel{5\lambda} = 0$$

$$12x_1 - 14x_2 - \cancel{5\lambda} = 0$$

$$-2x_1 + 44x_2 = 0 \quad \& \quad 2x_1 + 5x_2 = 98$$

$$\Rightarrow x_1 = 44\ and\ x_2 = 2$$

$$2(44) + 6(2) - \lambda = 0 \Rightarrow \lambda = 100 \geq 0$$

These, values satisfies all the necessary conditions

The optimal solution is:

$$x_1 = 44,\ x_2 = 2$$

$$Z_{max} = 2(44)^2 - 7(2)^2 + 12(44 \times 2) = 4900$$

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Solve the following NLPP using the Kuhn-Tucker method:

$$Minimize: z = x_1^3 - 4x_1 - 2x_2 \rightarrow f$$

$$sub.to, \quad x_1 + x_2 \leq 1 \rightarrow g$$

$$x_1, x_2 \geq 0$$

$$Let, K = f - \lambda \cdot g = x_1^3 - 4x_1 - 2x_2 - \lambda(x_1 + x_2 - 1)$$

$$\frac{\partial K}{\partial x_1} = 3x_1^2 - 4 - \lambda = 0 \ldots\ldots(1)$$

$$\frac{\partial K}{\partial x_2} = -2 - \lambda = 0 \Rightarrow \boxed{\lambda = -2} \ldots\ldots(2)$$

$$\lambda \cdot g = \lambda \cdot (x_1 + x_2 - 1) = 0 \ldots\ldots(3)$$

$$g \leq 0 \Rightarrow (x_1 + x_2 - 1) \leq 0 \Rightarrow x_1 + x_2 \leq 1 \ldots\ldots(4)$$

$$\lambda \leq 0 \ldots\ldots(5)$$

$$x_1, x_2 \geq 0 \ldots\ldots(6)$$

Now, $\lambda \cdot g = \lambda \cdot (x_1 + x_2 - 1) = 0$

For ONE inequality Constraint:

f – *objective function;*
g – *consraint;*
λ – *Lagrangian Multiplier*

$$Let, K = f - \lambda \cdot g$$

Find x_1, x_2 *and* λ *by solving the eqautions,*

$$\frac{\partial K}{\partial x_1} = 0, \frac{\partial K}{\partial x_2} = 0, \frac{\partial K}{\partial \lambda} = 0$$

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda \cdot g_{x_2} = 0$
3. $\lambda \cdot g = 0$
4. $g \leq 0$
5. $\lambda \leq 0$ (*if Z is min*)
6. $x_1, x_2 \geq 0$

In equation (3), *either* $\lambda = 0$ *or* $g = 0$.

From solving these different cases, & verify whether all above conditions are satisfied

KUHN-TUCKER CONDITIONS

Now, $\lambda \cdot g = \lambda \cdot (x_1 + x_2 - 1) = 0$ *But as,* $\lambda = -2 \leq 0$

$$\rightarrow x_1 + x_2 - 1 = 0 \Rightarrow x_1 + x_2 = 1$$

from (1), $3x_1^2 - 4 - \lambda = 0 \Rightarrow 3x_1^2 - 4 - (-2) = 0$

$$3x_1^2 = 2 \quad \Rightarrow x_1 = \sqrt{\frac{2}{3}} = 0.8165$$

Since, $x_1 + x_2 = 1 \Rightarrow x_2 = 1 - \sqrt{\frac{2}{3}} = (1 - 0.8165)$

$$\Rightarrow \quad x_1 = \sqrt{2/3} \ \ and\ x_2 = (1 - 0.8165)$$

These, values satisfies all the necessary conditions

The optimal solution is:

$$x_1 = 0.8165 \ \ and\ x_2 = (1 - 0.8165)$$

$$Z_{min} = x_1^3 - 4x_1 - 2x_2 = (0.8165)^3 - 4(0.8165) - 2(1 - 0.8165) = -3.0887$$

For TWO inequality Constraints:

Kuhn–Tucker Conditions

If, f – objective function; g_1, g_2 – consraints; λ_1, λ_2 – Lagrangian Multiplier

$$Let, K = f - \lambda_1 g_1 - \lambda_2 g_2$$

Find x_1, x_2 and λ_1, λ_2 by solving the eqautions, $\frac{\partial K}{\partial x_1} = 0, \quad \frac{\partial K}{\partial x_2} = 0, \quad \frac{\partial K}{\partial \lambda_1} = 0, \quad \frac{\partial K}{\partial \lambda_2} = 0$

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda_1 \cdot g_{x_1} - \lambda_2 \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda_1 \cdot g_{x_2} - \lambda_2 \cdot g_{x_2} = 0$
3. $\lambda_1 \cdot g_1 = 0$
4. $\lambda_2 \cdot g_2 = 0$
5. $g_1 \leq 0$
6. $g_2 \leq 0$
7. $\lambda_1, \lambda_2 \geq 0$ (**For Z max**)
8. $x_1, x_2 \geq 0$

NOTE:

For, Maximize: consider $\lambda_1, \lambda_2 \geq 0$

For, Minimize: consider $\lambda_1, \lambda_2 \leq 0$

In equation (3), and (4) either $\lambda = 0$ or $g = 0$.

From solving these different cases, & verify whether all above conditions are satisfied.

EXAMPLE

Solve the following NLPP using the Kuhn-Tucker method:

$$Maximize: z = -2x_1^2 - 2x_2^2 + 12x_1 + 21\, x_2 + 2x_1x_2$$

$$sub.to, \quad x_2 \le 8\,; \;\; x_1 + x_2 \le 10 \qquad x_1, x_2 \ge 0$$

$$\to K = f - \lambda_1\, g_1 - \lambda_2\, g_2 = -2x_1^2 - 2x_2^2 + 12x_1 + 21\, x_2 + 2x_1x_2 - \lambda_1(x_2 - 8) - \lambda_2\,(x_1 + x_2 - 10)$$

$$\frac{\partial K}{\partial x_1} = -4x_1 + 12 + 2\,x_2 - \lambda_2 = 0 \Rightarrow -4x_1 + 2x_2 + 12 - \lambda_2 = 0 \ldots\ldots (1)$$

$$\frac{\partial K}{\partial x_2} = -4x_2 + 2x_1 + 21 - \lambda_1 - \lambda_2 = 0 \Rightarrow 2x_1 - 4x_2 + 21 - \lambda_1 - \lambda_2 = 0 \ldots\ldots (2)$$

$$\lambda_1 \cdot g_1 = \lambda_1(x_2 - 8) = 0 \ldots\ldots (3)$$

$$\lambda_2 \cdot g_2 = \lambda_2(\,x_1 + x_2 - 10) = 0 \ldots\ldots (4)$$

$$g_1 \le 0 \Rightarrow (x_2 - 8) \le 0 \ldots\ldots (5)$$

$$g_2 \le 0 \Rightarrow (\,x_1 + x_2 - 10) \le 0 \ldots\ldots (6)$$

$$\lambda_1, \lambda_2 \ge 0 \;(\text{For } Z \text{ max}) \ldots\ldots (7)$$

$$x_1, x_2 \ge 0 \ldots\ldots (8)$$

For TWO inequality Constraints:

f – objective function;
g_1, g_2 – consraints;
λ_1, λ_2 – Lagrangian Multiplier

$$Let,\, K = f - \lambda_1\, g_1 - \lambda_2\, g_2$$

Find x_1, x_2 and λ_1, λ_2

$$\frac{\partial K}{\partial x_1} = 0, \frac{\partial K}{\partial x_2} = 0, \frac{\partial K}{\partial \lambda_1} = 0, \frac{\partial K}{\partial \lambda_2} = 0$$

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda_1 \cdot g_{x_1} - \lambda_2 \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda_1 \cdot g_{x_2} - \lambda_2 \cdot g_{x_2} = 0$
3. $\lambda_1 \cdot g_1 = 0$
4. $\lambda_2 \cdot g_2 = 0$
5. $g_1 \le 0$
6. $g_2 \le 0$
7. $\lambda_1, \lambda_2 \ge 0$ (For Z max)
8. $x_1, x_2 \ge 0$

Case 1: $If\ \lambda_1 = 0\ and\ \lambda_2 = 0,$

$$-4x_1 + 2x_2 + 12 - \cancel{\lambda_2}^{0} = 0$$

$$2x_1 - 4x_2 + 21 - \cancel{\lambda_1}^{0} - \cancel{\lambda_2}^{0} = 0$$

$$-4x_1 + 2x_2 = -12$$

$$2x_1 - 4x_2 = -21$$

$$x_1 = \frac{15}{9}\ \ and\ \ x_2 = 9$$

$$g_1 \le 0 \Rightarrow (x_2 - 8) \le 0 \dots\dots (5) \times$$

$$9 - 8 = 1 \nleq 0$$

$$g_2 \le 0 \Rightarrow (x_1 + x_2 - 10) \le 0 \dots\dots (6) \times$$

$$(15/9 + 9 - 10) = 6.5 \nleq 0$$

Therefore, assumption of $\lambda_1 = 0\ and\ \lambda_2 = 0$ is not correct

Therefore, we need to REJECT these values at $\lambda_1 = 0\ and\ \lambda_2 = 0$

If $A \cdot B = 0$ either $A = 0$ or $B = 0$

Case 2: $If\ \lambda_1 \neq 0\ and\ \lambda_2 \neq 0,$

$$\lambda_1 \cdot g_1 = \overset{\neq 0}{\lambda_1}(x_2 - 8) = 0 \dots\dots (3)$$

$$\lambda_2 \cdot g_2 = \overset{\neq 0}{\lambda_2}(x_1 + x_2 - 10) = 0 \dots\dots (4)$$

$$x_2 = 8$$

$$x_1 + x_2 = 10$$

$$x_1 = 2\ \ and\ \ x_2 = 8$$

$$-4x_1 + 2x_2 + 12 - \lambda_2 = 0 \Rightarrow \lambda_2 = 20 \geq 0$$

$$2x_1 - 4x_2 + 21 - \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = -27 \le 0 \times$$

But, $\lambda_1, \lambda_2 \ge 0$ (For Z max) ...... (7)

Therefore, assumption of $\lambda_1 \neq 0\ and\ \lambda_2 \neq 0$, is not correct

Therefore, we need to REJECT these values at $\lambda_1 \neq 0\ and\ \lambda_2 \neq 0$,

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Case 3: $If\,\lambda_1 \neq 0\ and\ \lambda_2 = 0,$

$$\lambda_1 \cdot g_1 = \lambda_1(x_2 - 8) = 0 \ldots\ldots(3)$$

$$\Rightarrow x_2 = 8$$

$$-4x_1 + 2x_2 + 12 - \cancel{\lambda_2} = 0 \Rightarrow -2x_1 + x_2 = -6$$

$$\Rightarrow -2x_1 + 8 = 6$$
$$\Rightarrow x_1 = 7$$

$$\Rightarrow x_1 = 7\ and\ \ x_2 = 8$$

$$2x_1 - 4x_2 + 21 - \lambda_1 - \cancel{\lambda_2} = 0 \Rightarrow \lambda_1 = 3 \geq 0$$

$$g_2 \leq 0 \Rightarrow (x_1 + x_2 - 10) = 5 \leq 0 \ldots\ldots(6)$$

These, values are not satisfying all the necessary conditions

Therefore, we need to REJECT these values at $\lambda_1 \neq 0\ and\ \lambda_2 = 0,$

Case 4: $If\,\lambda_1 = 0\ and\ \lambda_2 \neq 0,$

$$\lambda_2 \cdot g_2 = \lambda_2(x_1 + x_2 - 10) = 0 \ldots\ldots(4)$$

$$x_1 + x_2 = 10 \quad (\times 2)$$

$$2x_1 - 4x_2 + 21 - \cancel{\lambda_1} - \lambda_2 = 0$$
$$\Rightarrow 2x_1 - 4x_2 + 21 - \cancel{\lambda_2} = 0$$
$$-4x_1 + 2x_2 + 12 - \cancel{\lambda_2} = 0$$

$$6x_1 - 6x_2 + 9 = 0$$

$$2x_1 - \cancel{2x_2} = -3$$
$$2x_1 + \cancel{2x_2} = 20$$

$$\Rightarrow x_1 = 17/4$$
$$\Rightarrow x_2 = 23/4$$

$$-4x_1 + 2x_2 + 12 - \lambda_2 = 0 \Rightarrow \lambda_2 = \frac{13}{2} \geq 0$$

These, values satisfies all the necessary conditions

The optimal solution is: $x_1 = 17/4,\ x_2 = 23/4$

$$Z_{max} = -2\left(\frac{17}{4}\right)^2 - 2\left(\frac{23}{4}\right)^2 + 12\left(\frac{17}{4}\right) + 21(23/4) + 2(17/4 \times 23/4) = 1734/16$$

BEALE'S METHOD

- In 1959, EML Beale developed a technique of solving QPP that does not use KT conditions to get the optimal solution.
- His technique involves partitioning of the variables in to basic and non-basic ones and using classical calculus results.
- At each iteration the objective function is expressed in terms of only non-basic variables.

CONT…

- **The algorithm is given below :**

$$\text{Optimize (Max or Min) } Z = \mathbf{cx} + \frac{1}{2}\mathbf{x}^T\,\mathbf{D}\mathbf{x}$$

subject to the constraints

$$\mathbf{A}\mathbf{x} \le \mathbf{b}$$

and $\mathbf{x} \ge 0$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$; $\mathbf{c} = (c_1, c_2, \ldots, c_n)$; $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$

$\mathbf{D} = [d_{jk}]$ is an $n \times n$ symmetric matrix, i.e. $d_{jk} = d_{kj}$; $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix

- **If the given problem is of minimization type, then we multiply the objective function by -1 to convert it into maximization type.**

CONT….

- **Step 1:** Introduce slack and / or surplus variables in the inequality constrains to get equalities i.e. $Ax = b$, $b>=0$.
- **Step 2:** Obtain a basic feasible solution of the problem.
- **Step 3:** Chose arbitrarily any "m" variables as basic and (n-m) variables as non-basic (keeping in mind basic feasible solution). Denote basic variables by $x_B = (x_{B1}, x_{B2}, x_{B3}, \ldots\ldots, x_{Bm})$ and non-basic variables by $x_{NB} = (x_{NB1}, x_{NB2}, x_{NB3}, \ldots\ldots, x_{NBn-m})$.

CONT….

- **Step 4:** Express each basic variable x_{Bi} entirely in terms of non-basic variables x_{NBi}'s(and u_i's if any) using the given as well as additional constraints, if any.
- **Step 5 :** Express the objective function f also entirely in terms of non- basic variables x_{NBi}'s(and u_i's if any) .

CONT….

Step VI. For incoming variable, examine the partial derivatives of $f(x)$ formulated above w.r.t non-basic variables at the point $x_{NB} = 0$ (and $u_i = 0$).

(*i*) If $\left(\dfrac{\partial f(x)}{\partial x_{NB_k}}\right)_{\substack{x_{NB}=0\\ u_i=0}} \leq 0$ for each $k = 1, 2, \ldots, n-m$ and $\left(\dfrac{\partial f(x)}{\partial u_i}\right)_{\substack{x_{NB}=0\\ u_i=0}} = 0$ for each i, then current basic feasible solution is optimal because by increasing x_{NB_k} or u_i there will be no improvement in f.

CONT....

(*ii*) If $\left(\dfrac{\partial f(x)}{\partial x_{NB_k}}\right)_{\substack{x_{NB}=0 \\ u_j=0}} > 0$ for at least one k, then choose the most +ve one. The corresponding non-basic variable will enter the basis (*i.e.* one of the non-basic variables which is currently at 0 level corresponding to the largest +ve value of $\dfrac{\partial f}{\partial x_{NB}}$ will be selected to enter the basis or for this it is preferable to go on increasing its value from 0 till a point where either any one of the present basic variables become negative or $\dfrac{\partial f(x)}{\partial x_{NB_j}}$ reduce to 0 and is about to become negative).

CONT….

(*iii*) If $\left(\frac{\partial f(x)}{\partial x_{NB_k}}\right)_{\substack{x_{NB}=0\\ u_i=0}} < 0$ for each $k = 1, 2, ..., n-m$ but $\left(\frac{\partial f(x)}{\partial u_i}\right)_{\substack{x_{NB}=0\\ u_i=0}} \neq 0$ for some $i = r$,

then introduce a new non-basic variable u_j defined by $u_j = \frac{1}{2}\frac{\partial f}{\partial u_i}$ and treat u_r as a basic variable (to be ignored later on).

Then, go to step IV

Step VII. For leaving basic variable, let $\boxed{x_{NB_i} = x_k}$ be the entering variable identified is Step VI(*ii*).

Compute Min $\left\{\frac{\alpha_{h0}}{|\alpha_{hk}|}, \frac{|\gamma_{k0}|}{|\gamma_{kk}|}\right\}$ for $\alpha_{hk} < 0$, $\gamma_{kk} < 0$ only for all basic variables x_h, where α_{h0} is the constant term, α_{hk} is the coefficient of x_k in the expression of basic x_h when expressed in terms of non-basic variables, γ_{k0} is constant term and γ_{kk} is the coefficient of x_k in $\frac{\partial f}{\partial x_k}$.

(*i*) If minimum ratio occurs for some $\frac{\alpha_{h0}}{|\alpha_{h0}|}$, the corresponding basic variable x_h will leave the basis.

... corresponds to non-basic

CONT….

(ii) If the minimum ratio occurs for some $\frac{|\gamma_{k0}|}{|\gamma_{kk}|}$, the exit criterion corresponds to non-basic variable. In this case, we introduce an additional non-basic variable, called free variable defined by

$$u_i = \frac{1}{2}\frac{\partial f}{\partial x_k} \quad (u_i \text{ is unrestricted})$$

and this relation becomes an additional constraint equation. It will lead to one additional equation and one more basic variable than before.

PROBLEM 1

$$\max f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

s.t.c. $\quad x_1 + 2x_2 \leq 4$

and $\quad x_1, x_2 \geq 0$

$$\max f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

stc. $x_1 + 2x_2 \leq 4$

and $x_1, x_2 \geq 0$

PROBLEM

- **Step 1: Introduce slack and / or surplus variables in the inequality constrains to get equalities i.e. Ax = b, b>=0.**

Write the given problem in standard form

$$\max f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

stc. $x_1 + 2x_2 + S_1 = 4$

$x_1, x_2, S_1 \geq 0$

PROBLEM

- **Step 2: Obtain a basic feasible solution of the problem.**
- **Step 3: Chose arbitrarily any "m" variables as basic and (n-m) variables as non-basic (keeping in mind basic feasible solution). Denote basic variables by $x_B = (x_{B1}, x_{B2}, x_{B3}, \ldots\ldots, x_{Bm})$ and non- basic variables by $x_{NB} = (x_{NB1}, x_{NB2}, x_{NB3}, \ldots\ldots, x_{NBn-m})$.**

Let us choose arbitrary S_1 as initial basic variable

i.e. $x_B = (S_1)$; $x_{NB} = (x_1, x_2)$

Working Rule

Express x_B & f in terms of x_{NB} as

To find the entering variable:

To find the leaving variable:

Max $f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

s.t. $x_1 + 2x_2 + s_1 = 4$;

$x_1, x_2, s_1 \geq 0$

$$x_B = (s_1)\,;\; x_{NB} = (x_1, x_2)$$

PROBLEM

- **Step 4: Express each basic variable x_{Bi} entirely in terms of non- basic variables x_{NBi}'s(and u_i's if any) using the given as well as additional constraints, if any.**
- **Step 5 : Express the objective function f also entirely in terms of non- basic variables x_{NBi}'s(and u_i's if any) .**

Iteration 1.

Express x_B and f in terms of x_{NB} as

$$S_1 = 4 - x_1 - 2x_2$$

and $f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

PROBLEM

- Incoming variable

To find the entering variable

Partial derivative of f w.r.to $x_{NB} = (x_1, x_2)$

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(4x_1 + 6x_2 - x_1^2 - 3x_2^2) = 4 - 2x_1$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}(4x_1 + 6x_2 - x_1^2 - 3x_2^2) = 6 - 6x_2$$

at the point $x_{NB} = (x_1, x_2) = 0$

$$\frac{\partial f}{\partial x_1} = 4 - 2\times 0 = 4$$

$$\frac{\partial f}{\partial x_2} = 6 - 6\times 0 = 6$$ ← most positive

PROBLEM

- Incoming variable / Entering variable is $\mathbf{x_2}$

since $\frac{\partial f}{\partial x_2}$ is the most positive.

So, x_2 entering variable in the basis.

Iteration 1:

Express x_B & f in terms of x_{NB} as

$$s_1 = 4 - x_1 - 2x_2$$

and $$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$$\frac{\partial f}{\partial x_1} = -2x_1 + 4 ; \quad \frac{\partial f}{\partial x_2} = -6x_2 + 6$$

At the point $x_{NB} = (x_1, x_2) = 0$,

$\frac{\partial f}{\partial x_1} = 4$ & $\frac{\partial f}{\partial x_2} = 6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

$$x_B = (s_1) ; \; x_{NB} = (x_1, x_2)$$

$$\boxed{x_B = (\quad , x_2); x_{NB} = (x_1, \quad)}$$

To find the leaving variable:

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PROBLEM

- **Leaving basic Variable :**

To find the leaving variable

$$\min\left\{\frac{\alpha_{h0}}{|\alpha_{hk}|}, \frac{\gamma_{k0}}{|\gamma_{kk}|}\right\} = \min\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{\gamma_{22}}\right\}$$

$$= \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\}$$

$$= \min\{2, 1\}$$

$$= 1 \text{ corresponding to } \frac{\gamma_{20}}{|\gamma_{22}|}$$

Rule: $\text{Min}\left\{\frac{\text{constant term}}{|\text{coeff of ENTERING Variable}|}\right\}$ x_2

$$s_1 = 4 - x_1 - 2x_2$$

$$\frac{\partial f}{\partial x_2} = -6x_2 + 6$$

Iteration I:

Express x_B & f in terms of x_{NB} as

$$\boxed{s_1 = 4 - x_1 - 2x_2}$$

and $\quad f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

$x_B = (s_1)$; $x_{NB} = (x_1, x_2)$

x_1

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$\frac{\partial f}{\partial x_1} = -2x_1 + 4$; $\boxed{\frac{\partial f}{\partial x_2} = -6x_2 + 6}$

At the point $x_{NB} = (x_1, x_2) = 0$,

$\frac{\partial f}{\partial x_1} = 4$ & $\frac{\partial f}{\partial x_2} = 6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

To find the leaving variable:

$$\text{Min}\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{22}|}\right\} = \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\}$$

Rule: $\text{Min}\left\{\frac{\text{constant term}}{|\text{coeff of ENTERING Variable}|}\right\}$

x_2

PROBLEM

- **Leaving basic Variable :**

Thus introduce a free non-basis

$$u_1 = \frac{1}{2}\frac{\partial f}{\partial x_2}$$

$$u_1 = \frac{1}{2}(6 - 6x_2)$$

$$\boxed{u_1 = 3 - 3x_2}$$

$x_B = (S_1, x_2)$; $x_{NB} = (x_1, u_1)$

Iteration 1:

Express x_B & f in terms of x_{NB} as

$x_B = (s_1)$; $x_{NB} = (x_1, x_2)$

$$s_1 = 4 - x_1 - 2x_2$$

and $$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

$x_B = (s_1, x_2)$; $x_{NB} = (x_1, u_1)$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$$\frac{\partial f}{\partial x_1} = -2x_1 + 4 ; \quad \frac{\partial f}{\partial x_2} = -6x_2 + 6$$

At the point $x_{NB} = (x_1, x_2) = 0$,

$\frac{\partial f}{\partial x_1} = 4$ & $\frac{\partial f}{\partial x_2} = 6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

To find the leaving variable:

$$\text{Min}\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{22}|}\right\} = \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\}$$

$$= 1 \text{ corresponding to } \frac{\gamma_{20}}{|\gamma_{22}|}.$$

Thus, introduce a free non-basic variable

$$u_1 = \frac{1}{2}\frac{\partial f}{\partial x_2} \qquad \text{i.e., } \boxed{u_1 = 3 - 3x_2}$$

PROBLEM

Iteration 2.

$$u_1 = 3 - 3x_2$$

$$\Rightarrow 3x_2 = 3 - u_1$$

$$\Rightarrow x_2 = \frac{3 - u_1}{3} \Rightarrow \boxed{x_2 = 1 - \frac{1}{3}u_1}$$

$$s_1 = 4 - x_1 - 2x_2$$

$$= 4 - x_1 - 2\left(1 - \frac{1}{3}u_1\right)$$

$$= 4 - x_1 - 2 + \frac{2}{3}u_1$$

$$\boxed{s_1 = 2 - x_1 + \frac{2}{3}u_1}$$

and $f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

$$= 4x_1 + 6\left(1 - \frac{1}{3}u_1\right) - x_1^2 - 3\left(\frac{3 - u_1}{3}\right)^2$$

PROBLEM

$$= 4x_1 + 6 - 2u_1 - x_1^2 - 3\left(\frac{9 - 6u_1 + u_1^2}{9}\right)$$

$$= 4x_1 + 6 - \cancel{2u_1} - x_1^2 - 3 + \cancel{2u_1} - \frac{1}{3}u_1^2$$

$$\boxed{f = 3 + 4x_1 - x_1^2 - \frac{1}{3}u_1^2}$$

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PROBLEM

To find the entering variable

Partial derivative of f w.r.to $x_{NB} = (x_1, u_1)$

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\left(3+4x_1-x_1^2-\frac{1}{3}u_1^2\right) = 4-2x_1$$

$$\frac{\partial f}{\partial u_1} = \frac{\partial}{\partial u_1}\left(3+4x_1-x_1^2-\frac{1}{3}u_1^2\right) = -\frac{2}{3}u_1$$

at the point $x_{NB} = (x_1, u_1) = 0$

$$\frac{\partial f}{\partial x_1} = 4-2\times 0 = 4 \leftarrow \text{most positive}$$

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}\times 0 = 0$$

PROBLEM

since $\frac{\partial f}{\partial x_1}$ is the most positive

so, x_1 entering variable in the basis

To find the leaving variable

$$\min\left\{\frac{\alpha_{10}}{|\alpha_{11}|}, \frac{\alpha_{20}}{|\alpha_{21}|}, \frac{\gamma_{10}}{|\gamma_{11}|}\right\} = \min\left\{\frac{2}{|-1|}, \frac{1}{|0|}, \frac{4}{|-2|}\right\}$$

$$= \min\left\{2, \frac{1}{0}, 2\right\}$$

$$= 2 \text{ corresponding to } \frac{\gamma_{10}}{|\gamma_{11}|}$$

$\frac{\alpha_{10}}{|\alpha_{11}|}$ $\boxed{s_1 = 2 - x_1 + \frac{2}{3}u_1}$

$\frac{\alpha_{20}}{|\alpha_{21}|}$ $\boxed{x_2 = 1 - \frac{1}{3}u_1}$

$\frac{\gamma_{10}}{|\gamma_{11}|}$ $\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\left(3 + 4x_1 - x_1^2 - \frac{1}{3}u_1^2\right) = 4 - 2x_1$

PROBLEM

Thus define a free non-basic variable

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial x_1}$$

$$u_2 = \frac{1}{2}(4-2x_1)$$

$$\boxed{u_2 = 2-x_1}$$

Now $x_B = (s_1, x_2, x_1)$; $x_{NB} = (u_1, u_2)$

Iteration 2:

$x_B = (s_1, x_2)\,;\; x_{NB} = (x_1, u_1)$

$x_2 = \frac{3-u_1}{3}$; $s_1 = 2 - x_1 + \frac{2}{3}u_1$

And $f = 3 - x_1^2 - \frac{u_1^2}{3} + 4x_1$

To find the entering variable:

Partial derivative of f w.r.t. x_1, u_1:

$$\frac{\partial f}{\partial x_1} = -2\,x_1 + 4 \;;\; \frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1$$

At the point $x_{NB} = (x_1, u_1) = 0$, we get

$$\frac{\partial f}{\partial x_1} = 4\,;\, \frac{\partial f}{\partial u_1} = 0$$

Thus, ENTERING variables is $\boxed{x_1.}$

To find the leaving variable:

$$Min\left\{\frac{\alpha_{10}}{|\alpha_{11}|}, \frac{\alpha_{20}}{|\alpha_{21}|}, \frac{\gamma_{10}}{|\gamma_{11}|}\right\} = Min\left\{\frac{2}{|-1|}, \frac{1}{|0|}, \frac{4}{|-2|}\right\}$$

$= 2$ corresponding to $\frac{\gamma_{10}}{|\gamma_{11}|}$.

Thus, define a free non-basic variable

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial x_1}$$

i.e., $\boxed{u_2 = -x_1 + 2}$

PROBLEM

Iteration 3. $\quad S_1, \underline{x_2}\ \underline{x_1} \quad u_1, u_2$

Express x_B and f in terms of x_{NB}

$$\boxed{x_1 = 2 - u_2}$$

$$\boxed{x_2 = 1 - \frac{1}{3}u_1}$$

$$S_1 = 2 - (2 - u_2) + \frac{2}{3}u_1$$

$$\boxed{S_1 = u_2 + \frac{2}{3}u_1}$$

and $\quad f = 3 + 4(2 - u_2) - (2 - u_2)^2 - \frac{1}{3}u_1^2$

$$= 3 + 8 - \cancel{4u_2} - 4 + \cancel{4u_2} - u_2^2 - \frac{1}{3}u_1^2$$

$$\boxed{f = 7 - u_2^2 - \frac{1}{3}u_1^2}$$

TO FIND ENTERING VARIABLE

differentiating f w.r.to $x_{NB} = (u_1, u_2)$

$$\frac{\partial f}{\partial u_1} = \frac{\partial}{\partial u_1}\left(7 - u_2^2 - \frac{1}{3}u_1^2\right) = -\frac{2}{3}u_1$$

$$\frac{\partial f}{\partial u_2} = \frac{\partial}{\partial u_2}\left(7 - u_2^2 - \frac{1}{3}u_1^2\right) = -2u_2$$

at point $x_{NB} = (u_1, u_2) = 0$ we get

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}\times 0 = 0$$

$$\frac{\partial f}{\partial u_2} = -2\times 0 = 0$$

Thus current x_B is optimal

Hence $x_1 = 2 - u_2 = 2 - 0 = 2$

$x_2 = 1 - \frac{1}{3}u_1 = 1 - \frac{1}{3}\times 0 = 1$

$S_1 = u_2 + \frac{2}{3}u_1 = 0 + \frac{2}{3}\times 0 = 0$

PROBLEM

and max $f = 7 - u_2^2 - \frac{1}{3}u_1^2$

$= 7 - 0 - \frac{1}{3} \times 0$

$= 7$

Iteration 3:

Express x_B & f in terms of x_{NB} as

$$x_B = (s_1, x_2, x_1); \quad x_{NB} = (u_1, u_2)$$

$$x_1 = 2 - u_2 \; ; \quad x_2 = 1 - \frac{1}{3}u_1 \; ;$$

$$s_1 = u_2 + \frac{2}{3}u_1 \quad f = 7 - u_2^2 - \frac{u_1^2}{3}$$

To find the entering variable:

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1 \; ; \quad \frac{\partial f}{\partial u_2} = -2u_2$$

At the point $x_{NB} = (u_1, u_2) = 0$, we get

$$\frac{\partial f}{\partial u_1} = 0 \quad ; \quad \frac{\partial f}{\partial u_2} = 0$$

Thus, current x_B is the optimal.

Answer:

$$x_1 = 2 \; ; x_2 = 1 \; ; s_1 = 0$$

& Max $f = 7$

EXAMPLE 1 COMPLETE PROBLEM SOLUTION

Working Rule

Express x_B & f in terms of x_{NB} as

To find the entering variable:

To find the leaving variable:

$$\text{Max } f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$
$$\text{s.t. } x_1 + 2x_2 + s_1 = 4\,;$$
$$x_1, x_2, s_1 \geq 0$$

$$x_B = (s_1)\,;\ x_{NB} = (x_1, x_2)$$

M = Number of constraints
N = Number of Variables
We choose (n-m) variables equal to 0
N-M = 3-1 = 2 Variables Zero.

Iteration I:

Express x_B & f in terms of x_{NB} as

$x_B = (s_1)$; $x_{NB} = (x_1, x_2)$

$$\boxed{s_1 = 4 - x_1 - 2x_2}$$

and $f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$\frac{\partial f}{\partial x_1} = -2x_1 + 4$; $\boxed{\frac{\partial f}{\partial x_2} = -6x_2 + 6}$

At the point $x_{NB} = (x_1, x_2) = 0$,

$\frac{\partial f}{\partial x_1} = 4$ & $\frac{\partial f}{\partial x_2} = 6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

To find the leaving variable:

$$\text{Min}\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{22}|}\right\} = \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\}$$

Rule: $\text{Min}\left\{\frac{constant\ term}{|coeff\ of\ ENTERING\ Variable|}\right\}$ x_2

Iteration 1:

Express x_B & f in terms of x_{NB} as

$x_B = (s_1)$; $x_{NB} = (x_1, x_2)$

$x_B = (S_1, x_2)$; $x_{NB} = (x_1, u_1)$

$$s_1 = 4 - x_1 - 2x_2$$

and $$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$$\frac{\partial f}{\partial x_1} = -2x_1 + 4; \quad \frac{\partial f}{\partial x_2} = -6x_2 + 6$$

At the point $x_{NB} = (x_1, x_2) = 0$,

$\frac{\partial f}{\partial x_1} = 4$ & $\frac{\partial f}{\partial x_2} = 6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

To find the leaving variable:

$$\text{Min}\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{22}|}\right\} = \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\}$$

$= 1$ corresponding to $\frac{\gamma_{20}}{|\gamma_{22}|}$.

Thus, introduce a free non-basic variable

$$u_1 = \frac{1}{2}\frac{\partial f}{\partial x_2} \quad \text{i.e.,} \quad \boxed{u_1 = 3 - 3x_2}$$

Iteration 2:

$x_2 = \frac{3-u_1}{3}$; $s_1 = 2 - x_1 + \frac{2}{3}u_1$

And $f = 3 - x_1^2 - \frac{u_1^2}{3} + 4x_1$

$$x_B = (s_1, x_2)\,;\; x_{NB} = (x_1, u_1)$$

To find the entering variable:

Partial derivative of f w.r.t. x_1, u_1:

$$\frac{\partial f}{\partial x_1} = -2\,x_1 + 4 \;;\; \frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1$$

At the point $x_{NB} = (x_1, u_1) = 0$, we get

$$\frac{\partial f}{\partial x_1} = 4\,;\, \frac{\partial f}{\partial u_1} = 0$$

Thus, ENTERING variables is $\boxed{x_1.}$

To find the leaving variable:

$$Min\left\{\frac{\alpha_{10}}{|\alpha_{11}|}, \frac{\alpha_{20}}{|\alpha_{21}|}, \frac{\gamma_{10}}{|\gamma_{11}|}\right\} = Min\left\{\frac{2}{|-1|}, \frac{1}{|0|}, \frac{4}{|-2|}\right\}$$

$= 2$ corresponding to $\frac{\gamma_{10}}{|\gamma_{11}|}$.

Thus, define a free non-basic variable

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial x_1}$$

i.e., $\boxed{u_2 = -x_1 + 2}$

Iteration 3:

Express x_B & f in terms of x_{NB} as

$x_B = (s_1, x_2, x_1)$; $x_{NB} = (u_1, u_2)$

$$x_1 = 2 - u_2 \; ; \quad x_2 = 1 - \frac{1}{3}u_1 \; ;$$

$$s_1 = u_2 + \frac{2}{3}u_1 \quad f = 7 - u_2^2 - \frac{u_1^2}{3}$$

To find the entering variable:

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1 \; ; \quad \frac{\partial f}{\partial u_2} = -2u_2$$

At the point $x_{NB} = (u_1, u_2) = 0$, we get

$$\frac{\partial f}{\partial u_1} = 0 \; ; \quad \frac{\partial f}{\partial u_2} = 0$$

Thus, current x_B is the optimal.

Answer:

$x_1 = 2 \; ; x_2 = 1 \; ; s_1 = 0$

& Max $f = 7$

PROBLEM 2

Example 2: Use Beale's method to solve QPP:

Minimize $f(x) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

s.t. $2x_1 + x_2 \geq 6$;

$x_1 - 4x_2 \geq 0$;

$x_1, x_2 \geq 0$

CONT……

Example 2: Use Beale's method to solve QPP:

Minimize $f(x) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

s.t. $2x_1 + x_2 \geq 6$;

$x_1 - 4x_2 \geq 0$;

$x_1, x_2 \geq 0$

Solution: Write in standard form:

$\boxed{\text{Maximize}}\ f(x) = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$

s.t. $2x_1 + x_2 - s_1 = 6$;

$x_1 - 4x_2 - s_2 = 0$;

$x_1, x_2, s_1, s_2 \geq 0$

Here, $m = 2,\ n = 4.$

Let us choose $x_B = (s_1, s_2)$ and $x_{NB} = (x_1, x_2)$

CONT..

Iteration 1:

Express x_B & f in terms of x_{NB} as

$$s_1 = -6 + 2x_1 + x_2$$
$$s_2 = x_1 - 4x_2$$

And $f = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$

$f = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$

s.t. $2x_1 + x_2 - s_1 = 6$;

$x_1 - 4x_2 - s_2 = 0$;

$x_1, x_2, s_1, s_2 \geq 0$

$x_B = (s_1, s_2)$ and $x_{NB} = (x_1, x_2)$

To find the entering variable:

Partial derivative of f w.r.t. x_1, x_2:

$\frac{\partial f}{\partial x_1} = 4 - 2x_1 + 2x_2$; $\frac{\partial f}{\partial x_2} = 2x_1 - 4x_2$

At the point $x_{NB} = (x_1, x_2) = 0$, we get

$\frac{\partial f}{\partial x_1} = 4$; $\frac{\partial f}{\partial x_2} = 0$

Since $\frac{\partial f}{\partial x_1}$ is the most positive, so x_1 is the ENTERING variables.

To find the leaving variable:

$$\text{Min}\left\{\frac{\alpha_{10}}{|\alpha_{11}|}, \frac{\alpha_{20}}{|\alpha_{21}|}, \frac{\gamma_{10}}{|\gamma_{10}|}\right\} = \min\left\{-\frac{6}{|2|}, \frac{0}{|1|}, \frac{4}{|-2|}\right\}$$

$= -3$ corresponding to s_1

Thus s_1 is the LEAVING variable.

Iteration 2:

$x_B = (s_2, x_1)$; $x_{NB} = (x_2, s_1)$

Express x_B & f in terms of x_{NB} as

$$x_1 = 3 + \frac{s_1}{2} - \frac{x_2}{2}$$

$$s_2 = x_1 - 4x_2 = 3 + \frac{s_1}{2} - \frac{9x_2}{2}$$

And $f = 9 + x_2 - s_1 + \frac{3}{2}x_2 s_1 - \frac{13}{4}x_2^2 - \frac{1}{4}s_1^2$

To find the entering variable:

Partial derivative of f w.r.t. x_1, x_2:

$$\frac{\partial f}{\partial x_2} = 1 + \frac{3}{2}s_1 - \frac{13}{2}x_2 ; \quad \frac{\partial f}{\partial s_1} = -1 + \frac{3}{2}x_2 - \frac{1}{2}s_1$$

At the point $x_{NB} = (x_2, s_1) = 0$, we get

$$\frac{\partial f}{\partial x_2} = 1 ; \quad \frac{\partial f}{\partial s_1} = -1$$

Thus, ENTERING variables is x_2.

To find the leaving variable:

Min $\left\{\frac{\alpha_{20}}{|\alpha_{22}|}, \frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{20}|}\right\} = \min\left\{\frac{3}{\left|-\frac{9}{2}\right|}, \frac{3}{\left|-\frac{1}{2}\right|}, \frac{1}{\left|-\frac{13}{2}\right|}\right\}$

$= \frac{2}{13}$ corresponding to $\frac{\gamma_{20}}{|\gamma_{20}|}$

Thus, introduce a free non-basic variable $u_1 = \frac{1}{2}\frac{\partial f}{\partial x_2}$ and get

$$u_1 = \frac{1}{2} + \frac{3}{4}s_1 - \frac{13}{4}x_2$$

Iteration 3:

Express x_B & f in terms of x_{NB} as

$$x_B = (s_2, x_1, x_2); \quad x_{NB} = (s_1, u_1)$$

$$x_2 = \frac{2}{13} + \frac{3}{13}s_1 - \frac{4}{13}u_1$$

$$x_1 = \frac{38}{13} - \frac{3}{26}s_1 + \frac{2}{13}u_1$$

$$s_2 = \frac{30}{13} - \frac{27}{26}s_1 + \frac{18}{13}u_1$$

And $f = 9 + x_2 - s_1 + \frac{3}{2}x_2 s_1 - \frac{13}{4}x_2^2 - \frac{1}{4}s_1^2$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (s_1, u_1)$ and set $x_{NB} = 0$, we get

$$\frac{\partial f}{\partial s_1} = -\frac{9}{13} \quad ; \quad \frac{\partial f}{\partial u_1} = 0$$

Thus, the current x_B is optimal.

Hence,

$$x_1 = \frac{38}{13}, \; x_2 = \frac{2}{13};$$

$$\text{Min } Z = \frac{116}{13}$$

WOLFE METHOD

It is used for solving the Quadratic programming problem (QPP)

The general form of the QPP is

$$f(x) = cx + \frac{1}{2} x^T Q\, x$$

$$\text{s.t. } Ax \le b \text{ and } x \ge 0$$ Linear

where Q is a symmetric matrix and b, c are the real vectors.

CONT…….

Important features:

1) The function $x^T Q x$ defines a quadratic form.

2) The constraints are assumed to be LINEAR, which ensures the convex solution space.

Kuhn-Tucker conditions for QPP

Consider a QPP

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) \leq 0, \quad x \geq 0$$

Convert all constraints into equality.

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) + S^2 = 0$$

$$x \geq 0$$

Slack Variable

For ONE Equality Constraint:

Lagrange Multipliers Method

If, f – *objective function;*
g – *consraint;*
λ – *Lagrangian Multiplier*

$$Let, L = f - \lambda \cdot g$$

$$Solve, \quad \frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

To find x_1, x_2 *and* λ, *we get* $X_0(x_1, x_2)$ *which is a point of maxima or minima*

Kuhn-Tucker conditions for QPP

Consider a QPP

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) \le 0, \quad x \ge 0$$

Convert all constraints into equality.

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) + S^2 = 0$$

$$x \ge 0$$

The Lagrangian function is

$$L(x, \lambda, S) = f(x) - \lambda(g(x) + S^2)$$

$$\frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \nabla f - \lambda \nabla g = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \qquad \Rightarrow g(x) + S^2 = 0$$

$$\frac{\partial L}{\partial S} = 0 \qquad \Rightarrow \lambda S = 0$$

$$\lambda, S, x \ge 0$$

Steps involved in Wolfe's Method

Step 1: Write all constraints in $\leq$ sign

Step 2: Convert all constraints into Equality by adding slack variables S_i^2 in the i^{th} constraints.

Step 3: Obtain Kuhn-Tucker conditions:

Construct the Lagrangian function

$$L = f(x) - \lambda(g(x) + S^2) - \mu(-x + S^2)$$

The necessary and sufficient conditions are:

$$\frac{\partial L}{\partial X} = 0; \frac{\partial L}{\partial \lambda} = 0; \frac{\partial L}{\partial \mu} = 0; \frac{\partial L}{\partial S} = 0$$

Take $s_i = S_i^2$, and derive the Kuhn-Tucker conditions.

Step 4: Construct the modified LPP using artificial variables.

Step 5: Solve by Simplex algorithm by Two-phase method.

Step 6: The optimal solution obtained in Step 5 is an optimal solution to the given QPP also.

WOLFE'S METHOD

Q⇒ Solve the following quadratic Programming problem by Wolfe's method.

$$\max z = 4x_1 + 2x_2 - x_1^2 - x_2^2 - 5$$

s.t.c. $\quad x_1 + x_2 \le 4$

$$x_1, x_2 \ge 0$$

Soln. Write all constraints in $\leq$ sign

$$x_1 + x_2 \leq 4$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

convert into equaility

$$\max z = 4x_1 + 2x_2 - x_1^2 - x_2^2 - 5$$

s.t. $x_1 + x_2 + S_1^2 = 4$

$$-x_1 + S_2^2 = 0$$

$$-x_2 + S_3^2 = 0$$

Construct the Lagrangian

$$L = (4x_1 + 2x_2 - x_1^2 - x_2^2 - 5) - \lambda_1(x_1 + x_2 + S_1^2 - 4)$$
$$- \lambda_2(-x_1 + S_2^2) - \lambda_3(-x_2 + S_3^2)$$

By Kuhn – Tucker conditions

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4 - 2x_1 - \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \boxed{2x_1 + \lambda_1 - \lambda_2 = 4}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2 - 2x_2 - \lambda_1 + \lambda_3 = 0$$

$$\Rightarrow \boxed{2x_2 + \lambda_1 - \lambda_3 = 2}$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 + s_1^2 - 4 = 0$$

$$\Rightarrow \boxed{x_1 + x_2 + s_1^2 = 4}$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow \boxed{-x_1 + s_1^2 = 0}$$

$$\frac{\partial L}{\partial \lambda_3} = 0 \Rightarrow \boxed{-x_2 + s_3^2 = 0}$$

Using Artificial Variables

$$\text{Max } Z = -A1 - A2$$

$$\text{Min } Z = +A1 + A2$$

s.t.c. $x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

One Condition

Construct the modified L.P.P.

$$\max Z = -A_1 - A_2$$

s.t.c. $2x_1 + \lambda_1 - \lambda_2 + A_1 = 4$

$2x_2 + \lambda_1 - \lambda_3 + A_2 = 2$

$x_1 + x_2 + S_1^2 = 4$

Construct the modified L.P.P.

$$\max Z = -A_1 - A_2$$

s.t.

$$2x_1 + \lambda_1 - \lambda_2 + A_1 = 4$$

$$2x_2 + \lambda_1 - \lambda_3 + A_2 = 2$$

$$x_1 + x_2 + S_1^2 = 4$$

Now using simplex table

	c_j		0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
-1	A_1	4	2	0	1	-1	0	1	0	0	$\min\{\frac{4}{2}, \frac{4}{1}\}$
-1	A_2	2	0	2	1	0	-1	0	1	0	$\min\{2, 4\}$
0	S_1^2	4	1	1	0	0	0	0	0	1	$= 2$
Z_j		-6	-2	-2	-2	1	1	-1	-1	0	
$Z_j - C_j$			-2	-2	-2	1	1	0	0	0	

A1 Leaving Variable

X1 Enter Variable

Now using simplex table

C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio
	C_j		0	0	0	0	0	-1	-1	0	
-1	A_1	4	2	0	1	-1	0	1	0	0	$\min\{\frac{4}{2}, \frac{4}{1}\}$
-1	A_2	2	0	2	1	0	-1	0	1	0	$\min\{2, 4\}$
0	S_1^2	4	1	1	0	0	0	0	0	1	$= 2$
Z_j		-6	-2	-2	-2	1	1	tri 1	-1	0	
$Z_j - C_j$			-2	-2	-2	1	1	0	0	0	

x_1 enters and A_1 leaves the basi

C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio
	C_j		0	0	0	0	0	-1	-1	0	
0	x_1	2	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\min\{\frac{2}{2}, \frac{2}{1}\}$
-1	A_2	2	0	2	1	0	-1	0	1	0	$\min\{1, 2\}$
0	S_1^2	2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	$= 1$
Z_j		-2	0	-2	-1	0	1	0	-1	0	
$Z_j - C_j$			0	-2	-1	0	1	1	0	0	

Leaving variable X1 should be unit matrix (1 , 0, 0)

multiply -1 * x1 of current table value + s_1^2

x_1 enters and A_1 leaves the basi

		C_j	0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
0	x_1	2	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\min\{\frac{2}{2}, \frac{2}{1}\}$
-1	A_2	2	0	2	1	0	-1	0	1	0	$\min\{1, 2\}$
0	S_1^2	2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	$= 1$
Z_j		-2	0	-2	-1	0	1	0	-1	0	
$Z_j - C_j$			0	-2	-1	0	1	1	0	0	

x_2 Enters and A_2 leaves the basis.

		C_j	0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
0	x_1	2	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	
0	x_2	1	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	
0	s_1^2	1	0	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	
z_j		0	0	0	0	0	0	0	0	0	
$z_j - c_j$			0	0	0	0	0	1	1	0	

multiply -1 * x2 + s_1^2

Here all $z_j - c_j \geq 0$

Thus optimum solution is

$x_1 = 2, \quad x_2 = 1$

$$\max z = 4x_1 + 2x_2 - x_1^2 - x_2^2 - 5$$

$$= 4 \times 2 + 2 \times 1 - 2^2 - 1^2 - 5$$

$$= 8 + 2 - 4 - 1 - 5$$

$$= 10 - 10$$

$$= 0$$

Problem 2-Wolefs Method

Q→(62) Apply Wolfe's method to solve QPP.

$$\max Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

s.t. $x_1 + 2x_2 \le 2$; $x_1, x_2 \ge 0$

Soln: Write all constraints in $\le$ sign

$$x_1 + 2x_2 \le 2$$

$$-x_1 \le 0$$

$$-x_2 \le 0$$

Convert into equality

$$\max Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

s.t. $x_1 + 2x_2 + S_1^2 = 2$

$$-x_1 + S_2^2 = 0$$

$$-x_2 + S_3^2 = 0$$

Construct the lagrangian function

$$L = (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2) - \lambda_1(x_1 + 2x_2 + s_1^2 - 2)$$
$$- \lambda_2(-x_1 + s_2^2) - \lambda_3(-x_2 + s_3^2)$$

By Kuhn Tucker condition

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4 - 4x_1 - 2x_2 - \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \boxed{4x_1 + 2x_2 + \lambda_1 - \lambda_2 = 4}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 6 - 2x_1 - 4x_2 - 2\lambda_1 + \lambda_3 = 0$$

$$\Rightarrow \boxed{2x_1 + 4x_2 + 2\lambda_1 - \lambda_3 = 6}$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + 2x_2 + s_1^2 - 2 = 0$$

$$\Rightarrow \boxed{x_1 + 2x_2 + s_1^2 = 2}$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow \boxed{-x_1 + s_2^2 = 0}$$

$$\frac{\partial L}{\partial \lambda_3} = 0 \Rightarrow \boxed{-x_2 + s_3^2 = 0}$$

(DEEMED TO BE UNIVERSITY)

Construct the modified LPP.

$$\max Z = -A_1 - A_2$$

s.t.

$$4x_1 + 2x_2 + \lambda_1 - \lambda_2 + A_1 = 4$$

$$2x_1 + 4x_2 + 2\lambda_1 - \lambda_3 + A_2 = 6$$

$$x_1 + 2x_2 + S_1^2 = 2$$

Now using simplex table

	C_j		0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
-1	A_1	4	4	2	1	-1	0	1	0	0	$\min\{\frac{4}{4}, \frac{6}{2}, \frac{2}{1}\}$
-1	A_2	6	2	4	2	0	-1	0	1	0	$= \min\{1, 3, 2\}$
0	S_1^2	2	1	2	0	0	0	0	0	1	$= 1$
Z_j		-10	-6	-6	-3	1	1	-1	-1	0	
$Z_j - c_j$			-6	-6	-3	1	1	0	0	0	

x_1 enters and A_1 leaves the basis.

	C_j		0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
0	x_1	1	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\min\left\{\frac{1}{1/2}, \frac{4}{3}, \frac{1}{3/2}\right\}$
-1	A_2	4	0	3	$\frac{3}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	0	$= \min\left\{2, \frac{4}{3}, \frac{2}{3}\right\}$
0	S_1^2	1	0	$\frac{3}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	1	
Z_j		-4	0	-3	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	-1	0	$= \frac{2}{3}$
$Z_j - c_j$			0	-3	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	0	

x_1 enters and S_1^2 leaves the basis.

		C_j	0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
0	x_1	$\frac{2}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{3}$	$\min\left\{\frac{\frac{2}{3}}{\frac{1}{3}}, \frac{2}{2}\right\}$
-1	A_2	2	0	0	2	0	-1	0	1	-2	$= \min\{2, 1\}$
0	x_2	$\frac{2}{3}$	0	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	0	$\frac{2}{3}$	$= 1$
Z_j		-2	0	0	-2	0	1	0	-1	2	
$Z_j - c_j$			0	0	-2	0	1	1	0	2	

↑

λ_1 enters and A_2 leaves the basis.

		C_j	0	0	0	0	0	-1	-1	0	
C_B	y_B	x_B	x_1	x_2	λ_1	λ_2	λ_3	A_1	A_2	S_1^2	Ratio.
0	x_1	$1/3$	1	0	0	$-1/3$	$1/6$	$1/3$	$1/12$	0	
0	λ_1	1	0	0	1	0	$-1/2$	0	$1/2$	-1	
0	x_2	$5/6$	0	1	0	$1/6$	$-1/12$	$-1/6$	$1/12$	$1/2$	
Z_j		0	0	0	0	0	0	0	0	0	
$Z_j - C_j$			0	0	0	0	0	1	1	0	

Since all $z_j - c_j \geq 0$

Thus optimum solution is

$$x_1 = \frac{1}{3}, \quad x_2 = \frac{5}{6}$$

and $\max z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

$$= 4\times\frac{1}{3} + 6\times\frac{5}{6} - 2\times\left(\frac{1}{3}\right)^2 - 2\times\frac{1}{3}\times\frac{5}{6} - 2\times\left(\frac{5}{6}\right)^2$$

$$= \frac{4}{3} + \frac{30}{6} - \frac{2}{9} - \frac{10}{18} - \frac{25}{18}$$

$$= \frac{24 + 90 - 4 - 10 - 25}{18}$$

$$= \frac{114 - 39}{18}$$

$$= \frac{75}{18}$$

$$= \frac{25}{6}$$

GEOMETRIC PROGRAMMING

- A **Geometric Program (GP)** is a type of **non-linear optimization problem** whose objective and constraints have a **particular form**.
- Geometric programming optimization problems are **typically not convex optimization problems**.

CONT..

- Geometric programming optimization problems can be **transformed from non-convex optimization problems to convex optimization problems given by their special properties.**
- The convexification for geometric programming optimization problems is implemented by a **mathematical transformation of the objective function and constraint functions** and a **change of decision variables.**

CONT…

- **Definition:**
  - The standard form of Geometric Programming optimization is to **minimize the objective function which must be posynomial.**
  - The inequality constraints can only have the form of a posynomial less than or equal to one, and the equality constraints can only have the form of a monomial equal to one.

CONT…

- Geometric programs have **several powerful properties:**
  1. **Unlike most non-linear optimization problems, large GPs can be solved extremely quickly.**
  2. **If there exists an optimal solution to a GP, it is guaranteed to be globally optimal.**
  3. **Modern GP solvers require no initial guesses or tuning of solver parameters.**
- These properties arise because GPs become convex optimization problems via a logarithmic transformation.

APPLICATIONS

- **Power Control :**
  - Transmitter power control in communication system is one of the typical applications of geometric programming.
  - The objective of the problem is to minimize the total power of the transmitters subject to transmitter power limits and the signal to interference and noise ratio (SINR) of many receiver/transmitter pairs.
- **Optimal Doping Profile**
- **Floor Planning**
- **Digital Gate Sizing**

CONT…

- GP objectives and inequalities are formed out of **monomials and posynomials.**
- **Monomial** is defined as:

$$f(x) = cx_1^{a1} x_2^{a2} \dots \dots xn^{an}$$

  - where c is a positive constant, $x_{1..n}$ are decision variables, and $a_{1..n}$ are real exponents.
- For example, taking x , y and z to be positive variables, the expressions:

$7x$, $4xy^2z$ $2x/y^2z^{0.3}$ $\sqrt{2XY}$

CONT…

- **Posynomial** is defined as a sum of monomials:

$$\mathbf{g}(x) = \sum_{k=1}^{k} c_k x_1^{a_1 k}\ x_2^{a_2 k}\ \ldots\ldots\ldots x_n^{a_n k}$$

  - **where c is a positive constant, $x_{1..n}$ are decision variables, and $a_{1..n}$ are real exponents.**

- **For example, the expressions**

$$x^2+2xy+1 \qquad\qquad 7xy+0.4(yz)^{-1/3}$$

where $c_k > 0$, This is a function of sum of monomial functions.

CONT…

- Monomials can be defined as the subset of posynomials having only one term.
- Using f_i to represent a monomial and

  g_i to represent a posynomial.
- GP in standard form is written as:

$$\text{Minimize} \quad g_0(x)$$

$$\text{Subject to} \quad f_i(x) = 1, \quad I = 1, \ldots\ldots\ldots, m$$

$$g_i(x) \leq 1, \quad I = 1, \ldots\ldots\ldots\ldots, n$$

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CONT…

- **Give the following example of a GP in standard form:**

$$
\begin{array}{ll}
\text{minimize} & x^{-1}y^{-1/2}z^{-1} + 2.3xz + 4xyz \\
\text{subject to} & (1/3)x^{-2}y^{-2} + (4/3)y^{1/2}z^{-1} \le 1 \\
 & x + 2y + 3z \le 1 \\
 & (1/2)xy = 1
\end{array}
$$

$$f(x_1, x_2, x_3) = x_1 x_2 x_3 + \frac{x_1}{x_2 x_3} + \frac{3x_2}{x_1 x_2}$$

Monomial $\longrightarrow$ $x_1/x_2 x_3 = x_1 x_2^{-1} x_3^{-1}$

Posynomial $\longrightarrow$ $x_1 x_2 x_3 + x_1 x_2^{-1} x_3^{-1} + 3 x_2 x_1^{-1} x_2^{-1}$

CO

$$f(x_1, x_2, x_3) = x_1 x_2 x_3 + \frac{x_1}{x_2 x_3} + \frac{3x_2}{x_1 x_2}$$

Monomial Single term.

$$\frac{x}{x_2 x_3} = x_1 x_2^{-1} x_3^{-1}$$

Posynomial $x_1 x_2 x_3 + x_1 x_2^{-1} x_3^{-1} + 3x_2 x_1^{-1} x_2^{-1}$.

Formal defination.

$$u_j = c_j x_1^{a_{1j}} x_2^{a_{2j}} \cdots x_n^{a_{nj}}$$

$$f(x) = u_1 + u_2 + \cdots + u_N.$$

CO

Formal defination.

$$u_j = c_j \, x_1^{a_{1j}} \, x_2^{a_{2j}} \cdots x_n^{a_{nj}}$$

$$f(x) = u_1 + u_2 + \cdots + u_N .$$

Polynomial.

General form.

$$\sum_{j=1}^{N} c_j \prod_{i=1}^{m} (x_i)^{a_{ij}}$$

$x_i \geq 0$.

$a_{ij} \rightarrow$ real

$c_j > 0$.

Geometric Programming

- Geometric programming (GP) was developed by R J Duffin, C Zener and E L Peterson for solving the class of optimization problems involving special functions called posynomials.
- It is a very efficient technique for certain highly non-linear and non-convex problems.
- This techinque is based on the arithmetic mean-geometric mean inequality and therefore called **Geometric Programming**.

CONT…

$u_1, u_2, u_3, \cdots, u_n \longrightarrow n$ - non-negative numbers.

$\delta_1, \delta_2, \ldots \delta_n > 0$ s.t. $\delta_1 + \delta_2 + \cdots + \delta_n = 1$.

$$AM \geq GM$$

$$\frac{\delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_n u_n}{\delta_1 + \delta_2 + \cdots + \delta_n} \geq u_1^{\delta_1} u_2^{\delta_2} \cdots u_n^{\delta_n}$$

$$\Rightarrow \sum_{i=1}^{n} \delta_i u_i \geq \prod_{i=1}^{n} u_i^{\delta_i}$$

$\delta_i = \frac{1}{n}$ for all i,

$$\sum_{i=1}^{n} \frac{u_i}{n} \geq \prod_{i=1}^{n} u_i^{\frac{1}{n}} \Rightarrow \frac{u_1 + u_2 + \cdots + u_n}{n} \geq (u_1 u_2 \ldots u_n)^{\frac{1}{n}}.$$

CONT...

$u_1, u_2, u_3, \dots, u_n \rightarrow n$ - non-negative numbers

$\delta_1, \delta_2, \dots \delta_n > 0$ s.t. $\delta_1 + \delta_2 + \dots + \delta_n = 1$

$AM \geq GM$

$$\frac{\delta_1 u_1 + \delta_2 u_2 + \dots + \delta_n u_n}{\delta_1 + \delta_2 + \dots + \delta_n} \geq u_1^{\delta_1} u_2^{\delta_2} \dots u_n^{\delta_n}$$

$$\Rightarrow \sum_{i=1}^{n} \delta_i u_i \geq \prod_{i=1}^{n} u_i^{\delta_i}$$

Let $U_i = \delta_i u_i \quad \forall i$

$$\boxed{\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i}}$$

The equality holds when

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \dots = \frac{U_n}{\delta_n} = K \text{ (say)}$$

$U_1 = K\delta_1, \quad U_2 = K\delta_2 \dots U_n = K\delta_n$

$$\sum_{i=1}^{n} U_i = K[\delta_1 + \delta_2 + \dots + \delta_n] = K \times 1 = K.$$

$$\prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i} = \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \dots \left(\frac{U_n}{\delta_n}\right)^{\delta_n}$$

$$= K^{\delta_1} K^{\delta_2} \dots K^{\delta_n}$$

$$= (K)^{\delta_1 + \dots + \delta_n}$$

$$= K.$$

Consider the following problem:

$$\text{(GP) Min } f(x) = \sum_{j=1}^{n} c_j u_j(x),$$

with $c_j > 0$ and $u_j(x)$ has the form

$$u_j(x) = \prod_{i=1}^{m} (x_i)^{a_{ij}} = x_1^{a_{1j}} x_2^{a_{2j}} \ldots x_m^{a_{mj}}$$

where a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ be real numbers and $x_i > 0, i = 1, 2, ..., m$.
Then $u_j(x)$ defined here is called posynomial.

For example: $f(x) = \frac{1}{2} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} x_3 + \frac{2}{3} x_1^{\frac{1}{2}} x_2^{-\frac{2}{3}}$
is a posynomial for $x_1, x_2, x_3 > 0$.

$$\text{Min } f(x) = x_1 + x_2 + \frac{1}{x_1 x_2}, \quad x_1, x_2 > 0.$$

$$= U_1 + U_2 + U_3$$

$$\geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}, \quad \delta_1+\delta_2+\delta_3 = 1$$

$$\delta_1, \delta_2, \delta_3 > 0.$$

$$= \left(\frac{x_1}{\delta_1}\right)^{\delta_1}\left(\frac{x_2}{\delta_2}\right)^{\delta_2}\left(\frac{1}{x_1 x_2 \delta_3}\right)^{\delta_3}$$

$$= x_1^{\delta_1-\delta_3}\, x_2^{\delta_2-\delta_3}\left(\frac{1}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}.$$

let $\delta_1 - \delta_3 = 0, \quad \delta_2 - \delta_3 = 0, \quad \delta_1 + \delta_2 + \delta_3 = 1$

$\Rightarrow \delta_1 = \delta_3, \quad \delta_2 = \delta_3 \Rightarrow \delta_1 = \delta_2 = \delta_3$

$$\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n}\left(\frac{U_i}{\delta_i}\right)^{\delta_i},$$

$$\sum_{i=1}^{n} \delta_i = 1$$

$$\delta_i > 0 \ \forall i$$

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$$\text{Min } f(x) = x_1 + x_2 + \frac{1}{x_1 x_2}, \quad x_1, x_2 > 0.$$

$$= U_1 + U_2 + U_3$$

$$\geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}, \quad \delta_1 + \delta_2 + \delta_3 = 1, \quad \delta_1, \delta_2, \delta_3 > 0.$$

$$= \left(\frac{x_1}{\delta_1}\right)^{\delta_1}\left(\frac{x_2}{\delta_2}\right)^{\delta_2}\left(\frac{1}{x_1 x_2 \delta_3}\right)^{\delta_3}$$

$$= x_1^{\delta_1 - \delta_3} x_2^{\delta_2 - \delta_3}\left(\frac{1}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3} = 3^{1/3}\, 3^{1/3}\, 3^{1/3} = 3$$

Let $\delta_1 - \delta_3 = 0, \quad \delta_2 - \delta_3 = 0, \quad \delta_1 + \delta_2 + \delta_3 = 1$

$\Rightarrow \delta_1 = \delta_3, \quad \delta_2 = \delta_3 \Rightarrow \delta_1 = \delta_2 = \delta_3 = \frac{1}{3}.$

$f(x) \geq 3. \Rightarrow \text{Min } f = 3$

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \frac{U_3}{\delta_3}$$

$$\Rightarrow \frac{x_1}{1/3} = \underbrace{\frac{x_2}{1/3} = \frac{1}{(x_1 x_2)\frac{1}{3}}}_{x_1 x_2^2 = 1}$$

$\Rightarrow x_1 = x_2; \quad x_1 x_2^2 = 1$

$x_1^3 = 1 \Rightarrow x_1 = 1 = x_2$

$$f = 5x_1 + 20x_2 + 10x_1^{-1}x_2^{-1}, \quad x_1, x_2 > 0.$$

$$= U_1 + U_2 + U_3$$

$$\geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}, \quad \delta_1+\delta_2+\delta_3 = 1, \quad \delta_i > 0 \ \forall i$$

$$= \left(\frac{5x_1}{\delta_1}\right)^{\delta_1}\left(\frac{20x_2}{\delta_2}\right)^{\delta_2}\left(\frac{10x_1^{-1}x_2^{-1}}{\delta_3}\right)^{\delta_3}$$

$$= x_1^{\delta_1-\delta_3}\, x_2^{\delta_2-\delta_3}\left(\frac{5}{\delta_1}\right)^{\delta_1}\left(\frac{20}{\delta_2}\right)^{\delta_2}\left(\frac{10}{\delta_3}\right)^{\delta_3}.$$

Let $\delta_1-\delta_3=0;\ \delta_2-\delta_3=0,\ \delta_1+\delta_2+\delta_3=1$

$\Rightarrow \delta_1=\delta_2=\delta_3=\frac{1}{3}.$

$$f \geq (15)^{1/3}(60)^{1/3}(30)^{1/3} = (15\times 60\times 30)^{1/3}$$

$$= (15\times 15\times 4\times 15\times 2)^{1/3}$$

$$= 30.$$

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \frac{U_3}{\delta_3}$$

$$\Rightarrow \frac{5x_1}{1/3} = \frac{20x_2}{1/3} = \frac{10x_1^{-1}x_2^{-1}}{1/3}$$

$\Rightarrow \boxed{x_1 = 4x_2}$ $\qquad 2x_2 = \frac{1}{x_1x_2}$

$2x_1x_2^2 = 1$

$8x_2^3 = 1$

$\Rightarrow x_2 = \frac{1}{2}$

$x_1 = 4\times\frac{1}{2} = 2$

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