Deep Learning from Ground Up

More Linear Algebra

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Overview

What is Statistics?

We've all heard the term

- So what is it?

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- Probability is the field of mathematics that deals with uncertain events. We need it for statistics because data is **inherently** uncertain.
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- We will enumerate some identities for multivariate probability distributions.

Distribution

What's a distribution now?

- A probability distribution is some function which assigns a probability measure to a set Ω.
- For every member in the set Ω, it gives us a value (between 0 and 1).
- For example, the function $f(x) = e^{a+bx}$, for $-1 \leq x \leq 1$ assigns a number to each value in the set {-1, 1}, but the numbers do not sum to one.
- However $P(x) = \frac{e^{a+bx}}{\int_{-1}^{1} e^{a+by}}$ does sum (integrate) to one for $-1 \leq x \leq 1$.

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- However $P(x) = \frac{e^{a+bx}}{\int_{-1}^{1} e^{a+by}}$ does sum (integrate) to one for $-1 \leq x \leq 1$.
- What's the value of $\int_y e^{a+by}$, for $-1 \leq y \leq 1$.
- The denominator is the called the *Normalizing Function* or sometimes the *Partition Function* (the term comes from statistical physics, google it!).

Probability and Uncertainty

Dealing with uncertainty

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- Regardless of what the actual process is, be it deterministic or non-deterministic, we deal with it by ascribing that uncertainty to our knowledge and actions.

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- Regardless of what the actual process is, be it deterministic or non-deterministic, we deal with it by ascribing that uncertainty to our knowledge and actions.
- We will talk of probability while dealing exclusively with data and there can always be some error in gathering data, or data itself may not represent the truth (that which we're seeking).
- We say in such cases that there is **inherent noise** in the observations and we refer to it as *observational noise*. It simply exists because the world can't be measured accurately enough.

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Probability and Models

Modelling the World

- We'll talk about what it means to be bayesian in a while, but first we'll mention what a model is.
- A model can be thought of as a mathematical formulation of an event or set of events, which aims to provide information on it or them.
- A model therefore aims to tell us *something* about those events. What that *something* is, depends on the events and the model.
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- For instance there can be a model of heat distribution across a room and it may not necessarily be a probability distribution.
- There are two questions we'll always want to know:
  1. How accurate is our model?
  2. How can we make a better model?
- Accuracy can be estimated perhaps by measuring some aspects of the world (recall assignments and Linear Algebra)
- But how do we make one? Well it depends on what we're trying to model.

Statistical Models

Modelling Data

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- In the most general sense, we want some function $f : \text{Some Data} \rightarrow \text{Rules}$, such that $f(\text{Rules}) = \text{All Similar Data}$
- In short we want to find the laws governing the data. Since we can't be absolutely certain of either the nature of the data (observational uncertainty) or the nature of the model (systematic error), there's always some uncertainty both in the function we model and data provided by that model.
- This is different than the *bias/variance dilemma* we hear of. (Which we'll get to in time.)

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- We say that the *data was generated by the distribution* and we try to find the *parameters* of the distribution.
- The simplest assumption we can make in case of discrete events is that all events are equally likely. This is known as the *Uniform* distribution.
- What is the distribution for a sequence of events, with no turn of a sequence affecting the other and both events being equally likely?

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- It's the normal distribution or the *Gaussian*.

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Odds and Ends

We'll define some terms and notations:

- We'll denote by $\mathbb{E}(x)$ (or sometimes $\mu(x)$ and call it mean) the expectation of a random variable x, and it's value will be calculated as $\mathbb{E}(x) = \sum xf(x)$ or $\int_\Omega xf(x)dx$.

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- $\mathbb{E}(x)$ is linear! $\mathbb{E}(x+y) = \mathbb{E}(x) + \mathbb{E}(y)$ for two rv's x and y.
- $\mathbb{E}(x)$ is called the first moment of x and the second moment (around $\mathbb{E}(x)$ or μ. We can leave x out if its clear from context) is, $\mathbb{E}(x-\mu)^2 = \sum \mathbb{E}(x^2) - \mathbb{E}^2(x)$ (prove. Homework) or, $\int_{\Omega} (x-\mu)^2 f(x)dx$, also called variance (around the mean) σ^2

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- In general for an arbitrary function $g, \mathbb{E}(g)$ can be defined by simply substituting g for x. Note that $\mathbb{E}$ depends on the variable and if g has as its domain variables other than x then $\mathbb{E}$ has to be specified, e.g., $\mathbb{E}_x(g) = \int (x^2 + y^2) f(x) dx$, for $g(x, y) = x^2 + y^2 =?$ (Classwork! hint: use linearity of Expectation)

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- In our case, we'll have to find them from data, hence they are a function of data (mean, variance both are functions of data)
- We say we perform *inference* for the parameters. We won't go into the details of inference here, however we'll talk about some concepts as required.
- Revisiting the Gaussian, from the formula $\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2} \frac{(x-\mu)}{\sigma^2}}$ we see that the distribution is completely specified by μ and σ^2, hence, we say that the Gaussian is parametrized by its mean and variance (also called location and scale in some contexts).
- In short we make an assumption that the data was generated by some distribution (say the Gaussian) and we try to estimate its parameters (μ and σ^2) from the data we have.

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- So, if we have some data $x_i = X$ and we assume that each data point is drawn uniformly from the population and is independent of the other. We say $\mathcal{L}(\theta_1, \theta_2|X) = ?$ (What would independence imply?)

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  We just equate the gradient to zero.
  $$\hat{\theta_1} = \frac{\partial \log \mathcal{L}(\theta_1,\theta_2|X)}{\partial\theta_1} = -\sum(\frac{x_i-\theta_1}{\theta_2}) = 0 \Rightarrow \hat{\theta_1} = \sum\frac{x_i}{n} = \bar{x}$$
- So $\hat{\theta_1} = \hat{\mu}$. Similarly we can show that $\hat{\theta_2} = \hat{\sigma^2} = \sum\frac{(x_i-\bar{x})^2}{n}$

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- So $\hat{\theta_1} = \hat{\mu}$. Similarly we can show that $\hat{\theta_2} = \hat{\sigma^2} = \sum \frac{(x_i - \bar{x})^2}{n}$
- So, say the data is $\{f(2,8), f(3,5), f(6,4), f(10,2)\}$, where $f(a,b) = \underbrace{a, a, ..., a}_{\text{b times}}$, if we assume it to be drawn from a Gaussian distribution, we can immediately arrive at the maximum likelihood estimate $\hat{\mu} = 3.8$ and $\hat{\sigma^2} = ?$ Homework!

Likelihood

A bit more about the Likelihood

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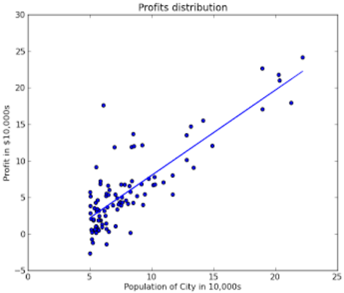
Linear Models

The Simplest Case

- A Linear Model makes the assumption that the data points are linearly related.



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The Simplest Case contd.

- Here we're talking about *Regression*. Regression means that we want to predict a numerical value for a *response* or *dependent* or *target* variable y_i corresponding to a set of *independent* or *input* or *explanatory* variables x_i.
- Although we'll mostly talk about one dimensional x, y, or perhaps two dimensional $\mathbf{x}$ and one dimensional y, both the input and the target variables can exist in multiple dimensions.

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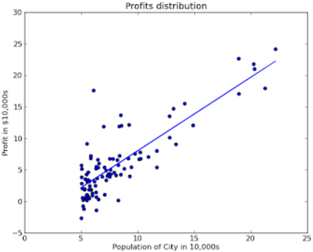
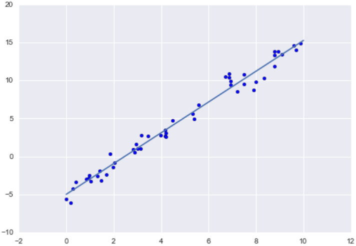
Which is a better fit?





Formulation

What is Linear anyway?

- We've already seen the concept of Linearity and its importance. Linear Models are the simplest models and make quite strong assumptions regarding the data.
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- It's the same formulation as previous for a straight line except X is now what's referred to as *augmented* vector and ϵ is the observational noise which can never (realistically) be zero.

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- So for a set of data points (y_i, x_i):

$$Y \equiv \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \dots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \dots & \dots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ or, } Y = X\beta + \epsilon$$

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- In most cases, you'll see a simple Ordinary Least Squares (OLS) Estimate.
- The simples perspective is to go with our assumptions and try to find values of parameters which give the best results.
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  Or, $\beta = \frac{1}{n} \sum^n y_i / x_i$, So simple!

Formulation

It's again in the assumptions we make

- In most cases, you'll see a simple Ordinary Least Squares (OLS) Estimate.
- The simples perspective is to go with our assumptions and try to find values of parameters which give the best results.
- For a straight line assumption, we can try to find a line which is the least average distance away from the data points. This is the geometrical perspective.
- $\sum d_i^2 = \sum (y_i - \beta x_i)^2$, where x_k, β are as defined above.
- For the minimal distance, we differentiate w.r.t. β, so, $\frac{\partial d_i^2}{\partial \beta} = 2 \sum x_i(y_i - \beta x_i)$, or, $\frac{1}{n} \sum^n \beta = \frac{1}{n} \sum^n y_i/x_i$
  Or, $\beta = \frac{1}{n} \sum^n y_i/x_i$, So simple!
- The response variable can now be written as $y_{new} = x_{new}\beta$
- However if we were to approach it from a statistical perspective, we would like to quantify the uncertainty in our estimated parameter.

Formulation

Slightly Deeper

- We make a slight augmentation and an assumption.
- First notice that:

$$X\beta = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \dots & \dots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \dots & \dots & \dots \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = X\beta \text{ for } X \text{ of degree } 2$$

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- So, again our model is $Y = X\beta + \epsilon,\ Y \sim \mathcal{N}(\mu, \mathbf{I}\sigma^2), \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$
  For OLS estimate we have to minimize $(Y - X\beta)^2$, again differentiating w.r.t. β we get:
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  $-X(Y - X\beta) = 0 \Rightarrow X^T Y = X^T X\beta \Rightarrow (X^T X)^{-1} X^T Y = \beta$
- The quantity $(X^T X)^{-1} X^T$ here is called *Moore Penrose Pseudoinverse* and can be thought of as equivalent of an inverse matrix for non-square matrices.

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[2] N. Reid. "Likelihood". In: *Journal of the American Statistical Association* 95.452 (2000), pp. 1335–1340.