APPLIED LINEAR ALGEBRA

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WORKSHEET-3

(i) Find the basis of span (S), where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

Soln:

- The basis is spanning set of vectorspace V which is linearly independent i.e it is the minimum spanning set

Linear dependence

- given span $(v_1, v_2, \ldots v_n)$ if $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 + \cdots c_n\bar{v}_n = 0$ if and only if $c_1 = c_2 = \cdots c_n = 0$ then vectors $\bar{v}_1, \bar{v}_2 \ldots \bar{v}_n$ are linearly independent and the span forms basis

- $$c_1\underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} + c_2\underset{v_2}{\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}} + c_3\underset{v_3}{\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}} + c_4\underset{v_4}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}} = 0 \quad \{ \because c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 + c_4\bar{v}_4 = 0 \}$$

$$\Rightarrow \begin{aligned} c_1 - c_2 + 2c_3 + c_4 &= 0 \\ 2c_1 - 2c_2 + 6c_3 + c_4 &= 0 \\ c_1 - c_2 - 2c_3 + 3c_4 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 6 & 1 \\ 1 & -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \bar{0}$$

- Using Gauss elimination method

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 6 & 1 \\ 1 & -1 & -2 & 3 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 6 & 1 \\ 0 & 0 & -4 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\simeq \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & 2 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\simeq \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} R_3 \to R_3 + 2R_2$$

pivot elements : a_{11} & a_{23}

$\Rightarrow$ free variables : c_2 & c_4

basic variables : c_1 & c_3

$\Rightarrow$ has infinitely many solutions & is linearly dependent

- c_2 & c_4 not having pivot elements implies they do not form basis for the given vectorspace i.e the corresponding vectors. $\bar{v}_2$ and $\bar{v}_4$ can be represented in terms of $\bar{v}_1$ and $\bar{v}_3$
- hence the span $(S) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \right\}$ is the minimum spanning set and the basis for the vectorspace V it spans.

② Determine whether each of the following sets is a basis for R^3

a) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \right\}$

b) $S = \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$

c) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 9 \\ 10 \end{bmatrix} \right\}$

Soln:

a) To prove linear independent and it spans R^3

Linear independence

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \bar{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \bar{0}$$

$$\Rightarrow c_1 + 2c_2 - 2c_3 = 0$$
$$c_2 + c_3 = 0$$
$$-c_1 - c_2 + 4c_3 = 0$$

- By Gauss elimination on homogenous systems

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} \simeq \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} R_3 \to R_3 + R_1$$

$$\simeq \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

Unique solution

$$c_1 + 2c_2 - 2c_3 = 0 \Rightarrow c_1 = 0$$
$$c_2 + c_3 = 0 \Rightarrow c_2 = 0$$
$$c_3 = 0$$

Hence, the three vectors are linearly independent

Linear Span

Let $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ be any vector in the vectorspace R^3

if $\bar{v}_1, \bar{v}_2, \bar{v}_3$ spans R^3 then every vector in R^3 can be expressed in terms of v_1, v_2, v_3

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$c_1 + 2c_2 - 2c_3 = \alpha$$
$$c_2 + c_3 = \beta$$
$$-c_1 - c_2 + 4c_3 = \gamma$$

Solving using Gauss elimination

$$M = \left[\begin{array}{ccc|c} 1 & 2 & -2 & \alpha \\ 0 & 1 & 1 & \beta \\ -1 & -1 & 4 & \gamma \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 1 & 2 & \gamma+\alpha \end{array}\right] R_3 \to R_3 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 0 & 1 & \gamma+\alpha-\beta \end{array}\right] R_3 \to R_3 - R_2$$

$$\Rightarrow c_1 + 2c_2 - 2c_3 = \alpha$$
$$c_2 + c_3 = \beta \Rightarrow c_2 = \beta - c_3 = \beta - (\gamma + \alpha - \beta) = 2\beta - \gamma - \alpha$$
$$c_3 = \gamma + \alpha - \beta$$

$$c_1 = \alpha + 2c_3 - 2c_2 = \alpha + 2(\gamma + \alpha - \beta) - 2(2\beta - \gamma - \alpha)$$
$$= \alpha - 2\beta + 4\gamma$$

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = (\alpha-2\beta+4\gamma)\begin{pmatrix}1\\0\\-1\end{pmatrix} + (2\beta-\alpha-\gamma)\begin{pmatrix}2\\1\\-1\end{pmatrix} + (\alpha-\beta+\gamma)\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

$\therefore S = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right\}$ is a basis for R^3

b) To prove linear independent & it spans R^3

\# Linear independence

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

$$\bar{v}_1 = \begin{pmatrix}1\\4\\7\end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix}2\\5\\8\end{pmatrix} \quad \bar{v}_3 = \begin{pmatrix}3\\6\\9\end{pmatrix}$$

$$c_1\begin{pmatrix}1\\4\\7\end{pmatrix} + c_2\begin{pmatrix}2\\5\\8\end{pmatrix} + c_3\begin{pmatrix}3\\6\\9\end{pmatrix} = \bar{0}$$

$$c_1 + 2c_2 + 3c_3 = 0$$
$$4c_1 + 5c_2 + 6c_3 = 0$$
$$7c_1 + 8c_2 + 9c_3 = 0$$

- By Gauss elimination method

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \approx \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 0 & -6 & -12\end{bmatrix} R_3 \to R_3 - 7R_1$$

$$\approx \begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & -6 & -12\end{bmatrix} R_2 \to R_2 - 4R_1$$

$$\approx \begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{bmatrix} R_3 \to R_3 - 2R_2$$

Number of pivot elements < Number of unknowns

$\Rightarrow$ Since system is homogenous it has non-trivial infinitely many solutions

$\Rightarrow$ Hence vectors are not linearly independent & S is not a basis for R^3

c) To prove linear independence and it spans R^3

Linear independence

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 + c_4\bar{v}_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \quad v_2 = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} \quad v_4 = \begin{pmatrix} -1 \\ 9 \\ 10 \end{pmatrix}$$

$$c_1\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} + c_4\begin{pmatrix} -1 \\ 9 \\ 10 \end{pmatrix} = \bar{0}$$

$$c_1 + 7c_2 + 3c_3 - c_4 = 0$$
$$2c_1 + 4c_2 + 8c_3 + 9c_4 = 0$$
$$5c_1 + 6c_3 + 10c_4 = 0$$

- By gauss elimination method

$$\begin{bmatrix} 1 & 7 & 3 & -1 \\ 2 & 4 & 8 & 9 \\ 5 & 0 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 & -1 \\ 2 & 4 & 8 & 9 \\ 0 & -35 & -9 & 5 \end{bmatrix} R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 7 & 3 & -1 \\ 0 & -10 & 2 & 11 \\ 0 & -35 & -9 & 5 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 7 & 3 & -1 \\ 0 & -10 & 2 & 11 \\ 0 & 0 & -16 & -\frac{335}{10} \end{bmatrix} R_3 \to R_3 - \frac{35}{10}R_2$$

pivots $= 3$

unknowns $= 4$ i.e not pivot for $\bar{v}_4$

Hence, the given vectors are not linearly independent as pivots < unknowns / vectors

- Only $\bar{v}_1, \bar{v}_2, \bar{v}_3$ form basis for R^3

$\Rightarrow$ S is not a basis for R^3

- Also it is a homogenous system with more unknowns i.e it has non-trivial solutions and hence linearly dependent

(3) Let P_4 be the vector space consisting of all polynomials of degree 4 or less with a real number co-efficients.
Let W be the subspace of P_4 by

$$W = \{p(x) \in P_4 \mid p(1) + p(-1) = 0 \text{ and } p(2) + p(-2) = 0\}$$

Find the basis of subspace W and determine the dimention of W

Soln:

let $p(x) \in P_4$ then $p(x)$ can be written as

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

- Vectors of subspace W

$p(1) + p(-1) = 0$

$$\Rightarrow (a_0 + a_1 + a_2 + a_3 + a_4) + (a_0 - a_1 + a_2 - a_3 + a_4) = 0$$

$$\Rightarrow 2(a_0 + a_2 + a_4) = 0$$

$$\Rightarrow a_0 + a_2 + a_4 = 0 \rightarrow (1)$$

$p(2) + p(-2) = 0$

$$\Rightarrow (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4) + (a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4) = 0$$

$$\Rightarrow a_0 + 4a_2 + 16a_4 = 0 \rightarrow (2)$$

From (1) & (2)

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 4 & 0 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using gauss elimination

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & a_4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 4 & 0 & 16 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 15 \end{bmatrix} \quad R_2 \rightarrow R_2 - R_1$$

$a_0, a_2 \rightarrow$ basic variables

$a_1, a_3, a_4 \rightarrow$ free variables let $a_1 = \alpha$

$a_3 = \beta$

$a_4 = \gamma$

$a_0 + a_2 + a_4 = 0 \Rightarrow a_0 = -\gamma + 5\gamma = 4\gamma$

$3a_2 + 15a_4 = 0 \Rightarrow a_2 = -5\gamma$

$\therefore$ Any polynomial in W can be represented as $(a_0, a_1, a_2, a_3, a_4)^T$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 4\gamma \\ \alpha \\ -5\gamma \\ \beta \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} \rightarrow (1)$$

$$\Rightarrow \text{basis: } S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \{ x, x^3, 4 - 5x^2 + x^4 \}$$

$\Rightarrow$ cardinality of basis is dimension. Hence dimension is 3

Note: Eqn (1) spans W from the equation and since they are not derived from each other. They are linearly independent

Verification:

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} = 0$$

$4c_3 = 0$

$c_1 = 0$

$-5c_3 = 0 \quad \Rightarrow c_1 = c_2 = c_3 = 0$

$c_2 = 0$

$c_3 = 0$

$\therefore$ The three vectors are basis for W

(4) Let V be the vectorspace of all $n \times n$ real matrices and $M \in V$ a fixed matrix. Define $W = \{A \in V \mid AM = MA\}$ The set W here is called the centralizer of M in V. Prove that W is a subspace of V

Soln:- <u>Criteria</u>

- The zero vector $\bar{0} \in V$ is in W
- If $\bar{x}, \bar{y} \in W$, then $\vec{x} + \vec{y} \in W$
- If $\bar{x} \in W$ and $c \in K$, then $c\vec{x} \in W$

Zero vector for matrices is a null matrix

$OM = MO = 0$

Hence zero vector $\bar{0} \in W$ also

Suppose A and $B \in W$ then

$A \in W \quad B \in W$

$A + B \in W$ or not

- for this it should satisfy condition $AM = MA$

$(A+B)M = AM + BM$ {Distributive property of matrices in addition}

$= MA + MB$ {$\because A \in W$ & $B \in W$; $MA = AM$ & $MB = BM$}

$= M(A+B)$

$\because (A+B)M = M(A+B)$ vector addition property satisfied

& $A+B \in W$

$cA \in W$ or not

$(cA)M = c(AM)$ {Associative property of matrices in multiplication}

$= c(MA)$ {$\because A \in W$ & $B \in W$; $MA = AM$ & $MB = BM$}

$= M(cA)$ {Commutative under multiplication with scalar $cM = Mc$}

$\because (cA)M = M(cA) \Rightarrow cA \in W$

i.e scalar multiplication property satisfied

Since three criteria of subspace satisfied
W is a subspace of V

⑤ Let V be the vector space of all 3x3 matrices. Let A be the matrix given below and we define

$$W = \{M \in V \mid AM = MA\}$$

Then W is a subspace of V. Determine which matrices are in the subspace W and find the dimension of W

a) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$, where a, b are distinct real numbers

b) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b & c are distinct real numbers

Soln:-

a) Let matrix M be $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

W is a subspace defined by

$$W = \{M \in V \mid AM = MA\}$$

where $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$

for matrix M to be in subspace W : AM = MA

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}\cdot a & a_{12}\cdot a & a_{13}\cdot a \\ a_{21}\cdot a & a_{22}\cdot a & a_{23}\cdot a \\ a_{31}\cdot b & a_{32}\cdot b & a_{33}\cdot b \end{bmatrix} = \begin{bmatrix} a\cdot a_{11} & a\cdot a_{12} & b\cdot a_{13} \\ a\cdot a_{21} & a\cdot a_{22} & b\cdot a_{23} \\ a\cdot a_{31} & a\cdot a_{32} & b\cdot a_{33} \end{bmatrix}$$

the condition satisfies only when
$a_{31} = a_{32} = a_{13} = a_{23} = 0$ as $\left\{ \begin{matrix} a_{13}\cdot a = a_{13}\cdot b & a_{31}\cdot a = a_{31}\cdot b \\ a_{23}\cdot a = a_{23}\cdot b & a_{32}\cdot a = a_{32}\cdot b \end{matrix} \right\}$

$$\Rightarrow M = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$ where $M \in W$ i.e all materices in W

this M can be written in terms of standard basis as

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{23} \end{bmatrix} = a_{11}\underbrace{\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}}_{V_1} + a_{12}\underbrace{\begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}}_{V_2} + a_{21}\underbrace{\begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}}_{V_3} + a_{22}\underbrace{\begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}}_{V_4}$$

$$+ a_{33}\underbrace{\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}}_{V_5}$$

$\therefore$ $\{V_1, V_2, V_3, V_4, V_5\}$ form basis for W. Hence dimension is 5. (V_1, V_2, V_3, V_4, V_5 are linearly independent & span W) for given A

b) Let matrix M be $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

W is a subspace defined by
$AM = MA$ where $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a\cdot a_{11} & a\cdot a_{12} & a\cdot a_{13} \\ b\cdot a_{21} & b\cdot a_{22} & b\cdot a_{23} \\ c\cdot a_{31} & c\cdot a_{32} & c\cdot a_{33} \end{bmatrix} = \begin{bmatrix} a\cdot a_{11} & b\cdot a_{12} & c\cdot a_{13} \\ a\cdot a_{21} & b\cdot a_{22} & c\cdot a_{23} \\ a\cdot a_{31} & b\cdot a_{32} & c\cdot a_{33} \end{bmatrix}$$

The above condition satisfies only when

$a\cdot a_{12} = b\cdot a_{12}$

$a\cdot a_{13} = c\cdot a_{13}$ $\Rightarrow a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$

$b\cdot a_{21} = a\cdot a_{21}$ i.e M is a diagonal matrix

$b\cdot a_{23} = c\cdot a_{23}$

$c\cdot a_{31} = a\cdot a_{31}$

$c\cdot a_{32} = b\cdot a_{32}$

$\Rightarrow M \in W$ where

$$M = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \quad \{\text{all matrices in } W\}$$

this M can be written in terms of standard basis as

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{V_1} + a_{22} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{V_2} + a_{33} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{V_3}$$

Hence V_1, V_2, V_3 are linear independent & span W therefore $\{V_1, V_2, V_3\}$ form basis for the W subspace for given A and the dimention of this basis is 3

(6) Let V be the vectorspace of all 2×2 matrices and let the subset S of V be defined by $S=\{A_1, A_2, A_3, A_4\}$ where

$$A_1=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix} \quad A_2=\begin{bmatrix}0 & -1\\ 1 & 4\end{bmatrix} \quad A_3=\begin{bmatrix}-1 & 0\\ 1 & -10\end{bmatrix} \quad A_4=\begin{bmatrix}3 & 7\\ -2 & 6\end{bmatrix}$$

Find a basis of the span span(S) consisting of vectors in S and find the dimension of span (S)

Soln:- Basis

- should be linearly independent vectors
- should span (S)

Linearly independent

if linearly independent

$c_1A_1+c_2A_2+c_3A_3+c_4A_4=0$ where 0 is NULL matrix

$$c_1\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}+c_2\begin{bmatrix}0 & -1\\ 1 & 4\end{bmatrix}+c_3\begin{bmatrix}-1 & 0\\ 1 & -10\end{bmatrix}+c_4\begin{bmatrix}3 & 7\\ -2 & 6\end{bmatrix}=\bar{0}$$

$c_1-c_3+3c_4=0$

$2c_1-c_2+7c_4=0$

$-c_1+c_2+c_3-2c_4=0$

$3c_1+4c_2-10c_3+6c_4=0$

$$\Rightarrow \underbrace{\begin{bmatrix}1 & 0 & -1 & 3\\ 2 & -1 & 0 & 7\\ -1 & 1 & 1 & -2\\ 3 & 4 & -10 & 6\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{bmatrix}}_{X}=0$$

by Gauss elimination

$$A=\begin{bmatrix}1 & 0 & -1 & 3\\ 2 & -1 & 0 & 7\\ -1 & 1 & 1 & -2\\ 3 & 4 & -10 & 6\end{bmatrix}\cong\begin{bmatrix}1 & 0 & -1 & 3\\ 2 & -1 & 0 & 7\\ -1 & 1 & 1 & -2\\ 0 & 4 & -7 & -3\end{bmatrix}_{R_4\to R_4-3R_1}\cong\begin{bmatrix}1 & 0 & -1 & 3\\ 2 & -1 & 0 & 7\\ 0 & 1 & 0 & 1\\ 0 & 4 & -7 & -3\end{bmatrix}_{R_3\to R_3+R_1}$$

$$\cong\begin{bmatrix}1 & 0 & -1 & 3\\ 0 & -1 & 2 & 1\\ 0 & 1 & 0 & 1\\ 0 & 4 & -7 & -3\end{bmatrix}_{R_2\to R_2-2R_1}$$

$$\approx \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_4 \to R_4 + 4R_2$$

$$\approx \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\approx \begin{array}{c} \begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} R_4 \to R_4 - \frac{R_3}{2}$$

basic variables: c_1, c_2, c_3

free variables: c_4

$\therefore$ not linearly independent as it has trivial infinitely many solutions

A_4 can be represented in terms of A_1, A_2, A_3

$$c_1 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 0 \\ 1 & -10 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$$

$$\begin{aligned} c_1 - c_2 &= 3 \\ 2c_1 - c_2 &= 7 \\ -c_1 + c_2 + c_3 &= -2 \\ 3c_1 + 4c_2 - 10c_3 &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 4 \\ c_2 &= 1 \\ c_3 &= 1 \end{aligned}$$

Hence $\{A_1, A_2, A_3\}$ span S and since they are linear independent (have pivots) they form basis for S and the dimension is 3.

⑦ Let P_2 be the vector space of all polynomials of degree 2 or less than 2 with real-coefficients. Let

$$S = \{1+x+2x^2, x+2x^2, -1, x^2\}$$

be the set of four vectors in P_2. Then find a basis of the subspace of span (S) among the vectors.

Soln: Basis:
- should be linearly independent
- must span S

Linearly independent

- General expression of a polynomial with degree 2

$$a_0 + a_1 x + a_2 x^2$$

can be represented in terms of vector as $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

- $S = \{1+x+2x^2, x+2x^2, -1, x^2\}$

$$\Rightarrow \left\{ \underset{V_1}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}, \underset{V_2}{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}, \underset{V_3}{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}, \underset{V_4}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

If linearly independent then

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 + c_4\bar{v}_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$$

$$= c_1\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow c_1 - c_3 = 0$$
$$c_1 + c_2 = 0$$
$$2c_1 + 2c_2 + c_4 = 0$$

$$\Rightarrow A = \underset{A}{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix}} \underset{X}{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}} = \bar{0}$$

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix} R_3 \to R_3 - 2R_1$$

$$\simeq \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix} R_2 \to R_2 - R_1$$

$$\simeq \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} R_3 \to R_3 - 2R_2$$

basic variables : c_1, c_2, c_4

free variables : c_3

$\therefore$ not linearly independent as it has trivial infinitely many solutions

- This is because $\bar{v}_3$ is dependent and can be expressed in terms of $\bar{v}_1, \bar{v}_2$ & $\bar{v}_4$ $\Rightarrow$ $\bar{v}_1, \bar{v}_2, \bar{v}_4$ span S
- Taking set $\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}$ we would get REF form as

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_4 \end{array} \end{array} \Rightarrow c_1 = c_2 = c_4 = 0$$

Hence set $\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}$ form basis for S and with dimension of 3

$$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ span } S$$

$\Rightarrow \{1 + x + 2x^2, x + 2x^2, x^2\}$ span S is the basis of S

- Verification $\bar{v}_3$ can be expressed in terms of $\bar{v}_1, \bar{v}_2$ & $\bar{v}_4$

$$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$c_1 = -1$

$c_1 + c_2 = 0 \qquad \Rightarrow c_1 = -1, c_2 = 1, c_3 = 0$

$2c_1 + 2c_2 + c_3 = 0$

⑧ Each of the following sets are not subspace of the specified vector space. For each set, give reason why it is not a subspace

b) let $M_{2\times2}$ be a vector space of all 2×2 real matrices then $S_6 = \{A \in M_{2\times2} \mid \det(A) = 0\}$ in the vector space $M_{2\times2}$

Ans: Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ whose det $\neq 0$ $\quad A+B \notin S_6$

hence doesn't satisfy closure of vector addition property

a) let $M_{2\times2}$ be the vector space of all 2×2 matrices

$S_6 = \{A \in M_{2\times2} \mid \det(A) \neq 0\}$ in vector space $M_{2\times2}$

Ans:

A zero vector in matrix of 2×2 is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ whose det $= 0$

$\Rightarrow$ zero vector $\notin S_6$

Thus S_6 is not a subspace since zero vector is not included in it

c) let $C[-1,1]$ be the vector space of all real continous functions defined on interval $[a,b]$

$$S_8 = \{f(x) \in C[-2,2] \mid f(-1)\,f(1) = 0\}$$

in vector space $C[-2,2]$

Ans: Consider the continous functions

$f(x) = x-1$ and $g(x) = x+1$

then checking if they belong to S_8

$f(1)\cdot f(-1) = (1-1)(-1-1) = 0$

$g(1)\,g(-1) = (1+1)(-1+1) = 0$

Considering sum $f(x)+g(x) = 2x = h(x)$ say

$h(1) \cdot h(-1) = 2(1) \cdot 2(-1)$

$= 2 \times -2 = -4 \neq 0$

Hence it doesn't satisfy closure on vector addition property &

$f(x) + g(x) \notin S_8$

thus S_8 is not a subspace