MAT3004 Applied Linear Algebra

Worksheet - I

1. Solve the following system of equations using Gauss Elimination.

a)
$$x + 2y + 3z = 4$$
$$5x + 6y + 7z = 8$$
$$9x + 10y + 11z = 12$$

$$(x, y, z) = \{(t-2, -2t+3, t), t \in R\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

$M = [A | b]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right]$$ — Augmented Matrix.

applying row operations

$R_2 \to R_2 - 5R_1$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 9R_1$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}.$$

$R_2 \to R_2 / -4$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -24 \end{bmatrix}$$

$R_3 \to R_3 + 8R_2$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Back substitution

$x + 2y + 3z = 4$ — ①

$y + 2z = 3$ — ②

from ②; $y = -2z + 3$.

substituting y in ①

$x + 2(-2z+3) + 3z = 4$

$x - 4z + 6 + 3z = 4$.

$x = z - 2$

we get,

x	y	z.
$z-2$	$-2z+3$	z.
⇓	↓	↓.

let $z = t$

$t-2 \quad -2t+3 \quad t$. $\quad t \in R$.

$(x, y, z) = (t-2, -2t+3, t)$ $t \in R$.

Pg 2

6).

$$2x_1 + 3x_2 + x_3 = 1$$
$$x_1 + x_2 + x_3 = 3$$
$$3x_1 + 4x_2 + 2x_3 = 4$$

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right]$$ — augmented matrix.

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 4 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -1 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Back substitution method,

$$x+y+z = 3 \quad \text{--- (1)}$$
$$y-z = -5 \quad \text{--- (2)}$$

from eqn (2) we get $y = z-5$

let $z = t$, $t \in R$

$\therefore\ z = t$

$y = t-5$

substituting values of y, z in (1) we get

$$x + t - 5 + t = 3$$

$$x = 8 - 2t$$

$\Rightarrow$

x	y	z	
$8-2t$	$-5+t$	t	, $t \in R$.

$$(x, y, z) = (8-2t, -5+t, t)$$

c)

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2x_2 + 6x_3 = 4$$
$$5x_1 - x_2 + 5x_3 = 6$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 10 & -7 & 0 & 7 \\ -3 & 2 & 6 & 4 \\ 5 & -1 & 5 & 6 \end{array}\right) = \text{Augmented matrix.}$$

$R_1 \rightarrow R_1/10.$

$$\begin{pmatrix} 1 & -7/10 & 0 & 7/10 \\ -3 & 2 & 6 & 4 \\ 5 & -1 & 5 & 6 \end{pmatrix}$$

$R_2 \rightarrow R_2 + 3R_1$

$$\begin{bmatrix} 1 & -7/10 & 0 & 7/10 \\ 0 & -1/10 & 6 & 61/10 \\ 5 & -1 & 5 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5R_1$

$$\begin{bmatrix} 1 & -7/10 & 0 & 7/10 \\ 0 & -1/10 & 6 & 61/10 \\ 0 & 5/2 & 5 & 5/2 \end{bmatrix}$$

$R_2 \rightarrow -10R_2$

$$\left[\begin{array}{cccc} 1 & -7/10 & 0 & 7/10 \\ 0 & 1 & -60 & -61 \\ 0 & 5/2 & 5 & 5/2 \end{array}\right].$$

$R_3 \rightarrow R_3 - 5/2\, R_2$.

$$\left[\begin{array}{cccc} 1 & -7/10 & 0 & 7/10 \\ 0 & 1 & -60 & -61 \\ 0 & 0 & 155 & 155 \end{array}\right]$$

$R_3 \rightarrow R_3/155$

$$\left[\begin{array}{cccc} 1 & -7/10 & 0 & 7/10 \\ 0 & 1 & -60 & -61 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Back Substitution :-

$x_1 - 7/10\, x_2 = 7/10$. — ①

$x_2 - 60x_3 = -61$ — ②

$x_3 = 1$. — ③.

from eqn 3 in eqn 2 $\Rightarrow$ $x_2 - 60 = -61$ $\Rightarrow$ $x_2 = -1$ — ④.

from eqn ④ in eqn 1 $\Rightarrow$. $x_1 + \frac{7}{10} = \frac{7}{10}$.

$x_1 = 0$

$(x_1, x_2, x_3) = \underline{\underline{(0, -1, 1)}}$;

d)

$$x_1 - x_2 + 2x_3 = 4$$

$$2x_1 + 3x_2 - x_3 = 1$$

$$7x_1 + 3x_2 + 4x_3 = 7$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 7 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 7 & 3 & 4 & 7 \end{array}\right) = \text{augmented matrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 7 & 3 & 4 & 7 \end{pmatrix}$$

$R_3 \to R_3 - 7R_1$

$$\begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 0 & 10 & -10 & -21 \end{pmatrix}$$

$\cancel{R_3 \to R_3 - 2R}$

$R_2 \to R_2/5$

$$\begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -7/5 \\ 0 & 10 & -10 & -21 \end{pmatrix}$$

$R_3 \to R_3 - 10R_2$

$$\begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -7/5 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

$R_3 \to R_3/-7$

$$\begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -7/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Back substitution;

$$x_1 - x_2 + 2x_3 = 4$$

$$x_2 - x_3 = -7/5$$

$$0 \neq 1$$

$\therefore$ No soltn. for the given system of equations.

2) Solve the following system of equations using Gauss Jordan Method:-

a)

$$6x + 8y + 6z + 3w = -3$$

$$6x - 8y + 6z - 3w = 3$$

$$8y - 6w = 6.$$

$$\begin{pmatrix} 6 & 8 & 6 & 3 \\ 6 & -8 & 6 & -3 \\ 0 & 8 & 0 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 6 & 8 & 6 & 3 & -3 \\ 6 & -8 & 6 & -3 & 3 \\ 0 & 8 & 0 & -6 & 6 \end{array}\right)$$ ⇒ augmented matrix

$R_1 \rightarrow R_1/6.$

$$\begin{pmatrix} 1 & 4/3 & 1 & 1/2 & -1/2 \\ 6 & -8 & 6 & -3 & 3 \\ 0 & 8 & 0 & -6 & 6 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 6R_1.$

$$\begin{pmatrix} 1 & 4/3 & 1 & 1/2 & -1/2 \\ 0 & -16 & 0 & -6 & -6 \\ 0 & 8 & 0 & -6 & 6 \end{pmatrix}$$

$R_2 \rightarrow R_2/-16.$

$$\begin{pmatrix} 1 & 4/3 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 3/8 & -3/8 \\ 0 & 8 & 0 & -6 & 6 \end{pmatrix}$$

$R_3 \to R_3 - 8R_2$.

$$\begin{pmatrix} 1 & 4/3 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 3/8 & -3/8 \\ 0 & 0 & 0 & -9 & 9 \end{pmatrix}$$

$R_3 \to R_3/-9$.

$$\begin{pmatrix} 1 & 4/3 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 3/8 & -3/8 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

$R_2 \to R_2 - \frac{3R_3}{8}$

$$\begin{pmatrix} 1 & 4/3 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_1 \to R_1 - 1/2\, R_3$.

$$\begin{pmatrix} 1 & 4/3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

$R_1 \to R_1 - 4/3 R_2$.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$x_1 + 0 + x_3 = 0$

$x_2 = 0$

$x_4 = -1$

$x_1 = -x_3$

Let $x_3 = t$

$x_1 = -t$

$(x_1, x_2, x_3, x_4) = (-t, 0, t, -1) \quad t \in R$

c). $x_1 + x_2 + x_3 + x_4 = 0$

$2x_1 + x_2 - x_3 + 3x_4 = 0$

$x_1 - 2x_2 + x_3 + x_4 = 0$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ 1 & -2 & 1 & 1 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & -3 & 0 & 0 & 0 \end{pmatrix}$$

$R_2 \to R_2 / -1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -3 & 0 & 0 & 0 \end{pmatrix}$$

$R_3 \rightarrow R_3 + 3R_2$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 9 & -3 & 0 \end{pmatrix}$$

$R_3 \rightarrow R_3/9$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 3R_3$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{pmatrix}$$

$R_1 \rightarrow R_1 - R_3$.

$$\begin{pmatrix} 1 & 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{pmatrix} \quad R_1 \rightarrow R_1 - R_2.$$

$$\begin{pmatrix} 1 & 0 & 0 & 4/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{pmatrix}$$

solving eqns; we get.

$$x_1 + \frac{4x_4}{3} = 0 \quad - (1)$$

$$x_2 = 0 \quad - (2)$$

$$x_3 - \frac{x_4}{3} = 0 - (3)$$

let $x_4 = t$.

$x_1 = -4t/3$ — from eqn (1)

$x_2 = 0$

$x_3 = t/3$ — from eqn (3)

	x_1	x_2	x_3	x_4
$\rightarrow$	$-\frac{4t}{3}$	0	$t/3$	t
$\rightarrow$	$-4t$	0	t	$3t$

$(x_1, x_2, x_3, x_4) = t(-4, 0, 1, 3)$. $t \in R$.

d).

$x_1 + 2x_2 + x_3 = -1$.

$-x_1 - x_2 + 2x_3 = 2$.

$2x_1 + 3x_2 = -2$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ -1 & -1 & 2 & 2 \\ 2 & 3 & 0 & -2 \end{array}\right) \Rightarrow \text{augmented matrix}$$

$R_2 \rightarrow R_2 + R_1$.

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 3 & 0 & -2 \end{pmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -2 & 0 \end{pmatrix}$$

$R_3 \to R_2 + R_3$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$R_2 \to R_2 - 3R_3$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

solving equs.

$x_1 = 2$.

$x_2 = -2$.

$x_3 = 1$

$(x_1, x_2, x_3) = (2, -2, 1)$.

b).
$x_1 + x_2 + x_3 = 0$

$x_1 - x_2 - x_3 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix} \Rightarrow \text{augmented matrix}$$

$R_2 \to R_2 - R_1$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{pmatrix}$$

$R_2 \to R_2/-2$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$R_1 \to R_1 - R_2$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

solving above eqns.

$x_1 + 0 + 0 = 0$.

$0 + x_2 + x_3 = 0$.

$x_1 = 0$. — (1)

$x_2 + x_3 = 0$. — (2)

from eqn 2,

$x_2 = -x_3$.

let $x_3 = t$, $t \in R$.

$x_2 = -t$.

$(x_1, x_2, x_3) = (0, -t, t)$, $t \in R$

(3) Consider a linear system whose augmented matrix is of the form $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right)$. For what values of a will the system have a unique solution.

$R_2 \to R_2 + R_1$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 2 & -2 & a & 3 \end{array}\right)$$

$R_3 \to R_3 - 2R_1$.

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & a-2 & 1 \end{pmatrix}$$

$R_3 \to R_3 + R_2.$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & \cancel{1}0 & a+2 & 4 \end{pmatrix}$$

$\cancel{a+2 \neq 0}$ $a+2=0$

$$\text{Rank}(A) = \begin{cases} 2, & a+2=0 \\ 3, & a+2 \neq 0 \end{cases}$$

$a \in R - \{-2\}.$

for a system to have unique solution, $R(A) = R(A|B) =$ no. of variab

so, Rank(A) must be 3.

for Rank (A) to be equal to 3 then accepted values of 'a' are $a+2 \neq 0$, i.e, $a \in R - \{-2\}$

(4) Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right)$$

a) Is it possible for the system to be inconsistent? Explain

b) For what values of β will the system have infinitely many solutions?

sol:-

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{pmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ 0 & 3 & \beta+1 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & \beta+1 & 0 \end{pmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \beta-2 & 0 \end{pmatrix} \quad \text{——} \ (1)$$

a). The given system is a homogeneous system. A homogeneous system is always consistent, if number of equations < number of unknowns then it has infinitely many solutions else it has a unique solution.

Even if $Rank(A) \neq Rank(A/B)$ the system has no solutions.

b) from eqn (1)

if $\beta - 2 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

x_1 & x_2 are basic variables & x_3 becomes free variable.

x_1 & x_2 can be represented in terms of $x_3 = \alpha$ say.

Hence system has infinetly many solutions. for $\beta - 2 = 0$

$\Rightarrow \beta = 2$