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CSE 3380: Linear Algebra for CSE

University of Texas at Arlington
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Assignment 5

You must show your work to receive credit.

Topics Covered

- Gram-Schmidt
- Least Squares
- Linear Models
- Eigenvectors and Eigenvalues
- Symmetric Matrices
- Singular Value Decomposition

1. Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \\ -2 \\ -5 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

2. Compute the QR factorization of the given matrix A using `scipy.linalg.qr`. Verify R by hand using the Q matrix that was computed. Save your script as `problem3.py`.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 3 & -2 \\ -2 & 0 & 6 \end{bmatrix}$$

3. Find the least squares solution given A and $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

4. Using the dataset `dataset1.txt`, available through Canvas, find the least squares solution using `np.linalg.lstsq`. You can load the data using `np.loadtxt`. After finding the least squares solution, plot the data and the solution using `matplotlib`.

5. Using the dataset `dataset2.txt`, available through Canvas, find the least squares solution using `np.linalg.lstsq`. You can load the data using `np.loadtxt`. After finding the least squares solution, plot the data and the solution using `matplotlib`.

6. By hand, find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$$

7. Find an orthonormal eigenbasis for the following matrix.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$$

8. Compute the Singular Value Decomposition for the given matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

9. Let $X \in \mathbb{R}^{N \times M}, P = X^T X,$ and $C = XX^T$. Show that if $\mathbf{v}_i$ is an eigenvector of P then $X\mathbf{v}_i$ is an eigenvector of C.

10. Use `np.linalg.eig` to calculate the eigenvalues and eigenvectors of the given matrix. Using `matplotlib`, plot the standard basis vectors, the vectors defined by the columns of A, and the calculated eigenvectors. Save your script as `problem2.py`.

$$A = \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

You may submit your work as either a scanned PDF OR you may take pictures of your homework solutions and combine them into a PDF. Compress the written part with the programming files into a single zip file. **Do not submit individual images. Rename your submission as LASTNAME_ID_A5.zip**.

1. Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \\ -2 \\ -5 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

$3 + -3 + 0$

$z \cdot v_1 = -8$

$\hat{z} = c_1 v_1 + c_2 v_2$

$z \cdot v_2 = -23$

$\left(\frac{z \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{z \cdot v_2}{v_2 \cdot v_2}\right) v_2$

$v_1 \cdot v_1 = 30$

$v_2 \cdot v_2 = 58$

$$= \frac{-8}{30}\begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix} + \frac{-23}{58}\begin{bmatrix} 2 \\ -5 \\ -2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -8/10 \\ 8/30 \\ -8/15 \\ 16/15 \end{bmatrix} + \begin{bmatrix} -46/58 \\ 115/58 \\ 46/58 \\ 115/58 \end{bmatrix} = \begin{bmatrix} -231/145 \\ 1957/870 \\ 113/435 \\ 2653/870 \end{bmatrix}$$

3. Find the least squares solution given A and $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T \cdot b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix} \Rightarrow \begin{matrix} 3x_1 + 3x_2 = 6 \\ 3x_2 + 11x_2 = 14 \end{matrix}$$

$$-1\begin{pmatrix} 3x_1 + 3x_2 = 6 \\ 3x_2 + 11x_2 = 14 \end{pmatrix} = \begin{matrix} -\cancel{3x_1} - 3x_2 = -6 \\ + \ \cancel{3x_1} + 11x_2 = 14 \end{matrix}$$

$$\frac{\cancel{8}x_2}{\cancel{8}} = \frac{8}{8}$$

$$\frac{\cancel{3}x_1}{\cancel{3}} - \cancel{0.75} = \frac{6.75}{3}$$
$$+\cancel{0.75}$$

$x_2 = 1$ $\quad$ $x_1 = 1$

6. By hand, find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$$

Eigen corresponding $\lambda = 1$

$Ax = x$

$4x_1 + x_3 = x_1 \Rightarrow 3x_1 = -x_3$

$-2x_1 + x_2 = x_2 \Rightarrow -2x_1 = 0$

$-2x_1 + x_3 = x_3 \Rightarrow -2x_1 = 0$

$$\begin{cases} x_3 = 0 \\ x_1 = 0 \\ x_2 = \text{free} \end{cases} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigen corresponding $\lambda = 2$

$4x_1 + x_3 = 2x_1 \Rightarrow 2x_1 = -x_3$

$-2x_1 + x_2 = 2x_2 \Rightarrow -2x_1 = x_2$

$-2x_1 + x_3 = 2x_3 \Rightarrow -2x_1 = x_3$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

Eigen corresponding $\lambda = 3$

$4x_1 + x_3 = 3x_1 \Rightarrow x_1 = -x_3$

$-2x_1 + x_2 = 3x_2 \Rightarrow -2x_1 = -2x_2$

$-2x_1 + x_3 = 3x_3 \Rightarrow -2x_1 = -2x_3$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\text{eigen space} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

7. Find an orthonormal eigenbasis for the following matrix.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & -2 & 2 \\ -2 & 4-\lambda & -4 \\ 2 & -4 & 4-\lambda \end{bmatrix} = 0$$

$(4-\lambda)(4-\lambda)$

$16 - 8x + x^2$

$$0 = 1-\lambda \begin{bmatrix} 4-\lambda & -4 \\ -4 & 4-\lambda \end{bmatrix} + 2$$

$$\begin{bmatrix} -2 & -4 \\ 2 & 4-\lambda \end{bmatrix} + 2 \begin{bmatrix} -2 & 4-\lambda \\ 2 & -4 \end{bmatrix}$$

$$0 = \lambda^2 - 8x - x^3 + 8x^2$$

$$0 = -\lambda^3 + 9x^2 - 8\lambda$$

$$0 = -\lambda^3 + 9x^2$$

$$0 = (\lambda - 9)\lambda^2 \qquad \lambda = 0, 9$$

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x - 2x_3 \qquad x_3 = \text{free}$

$x_2 = \text{free}$

$\lambda = 0, \quad x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

when $\lambda = 9 \quad \begin{bmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 0.5x_3 \\ x_2 &= -x_3 \\ x_3 &= \text{free} \end{aligned}$

when $\lambda = 0 \quad z_2 = v_2 - \dfrac{v_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$

let $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$v_2 \cdot v_1 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = -4$

$v_1 \cdot v_1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 5$

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2/5 \\ 4/5 \\ 1 \end{bmatrix}, \quad v_3 \begin{bmatrix} 0.5 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvector to othronomal eigenvector

$$\|v_1\| = \sqrt{2^2+1^2} = \sqrt{5}$$

$$v_3 = \frac{v_3}{\|v_3\|} = \sqrt{0.5^2+(-1)^2+1^2} = \sqrt{9/5}$$

$$\sqrt{2.25} \quad = 3/\sqrt{5}$$

$$V = \begin{bmatrix} 2/\sqrt{5} & -\frac{2\sqrt{5}}{15} & 1/2\sqrt{2.25} \\ 1/\sqrt{5} & \frac{4\sqrt{5}}{5} & -1/\sqrt{2.25} \\ 0 & \sqrt{5}/3 & 1/\sqrt{2.25} \end{bmatrix}$$

8. Compute the Singular Value Decomposition for the given matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}$$

1) Find Eigen value

$$\begin{bmatrix} 13-\lambda & 6 \\ 6 & 4-\lambda \end{bmatrix}$$

$$\Rightarrow (13-\lambda) \times (4-\lambda) - 6 \times 6 = 0$$

$$\Rightarrow (52 - 17\lambda + \lambda^2) - 36 = 0$$

$$\Rightarrow (\lambda^2 - 17\lambda + 16) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 16) = 0$$

$$\Sigma \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\lambda = 1, 16}$$

Eigen vector for $\lambda = 16$

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigen value for $\lambda = 1$

$$V_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad L : \sqrt{2^2+1^2} = \sqrt{5}$$

$$u_1 = \left(\frac{2}{\sqrt{5}} , \frac{1}{\sqrt{5}} \right)$$

$$V_2 L = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{4}$$

$$u_2 = \left(\frac{-5}{\sqrt{\frac{5}{4}}} , \frac{1}{\sqrt{\frac{5}{4}}} \right)$$

$$A' \cdot A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

Eigen values

$$\begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}$$

$$\Rightarrow (4-\lambda) \times (13-\lambda) - 6 \times 6 = 0$$

$$\Rightarrow (52 - 17\lambda + \lambda^2) - 36 = 0$$

$$\Rightarrow (\lambda^2 - 17\lambda + 16) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 16) = 0$$

eigen value : 1, 16

$$v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

egien value for $\lambda = 16$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad L = \sqrt{\left(\frac{1}{2}\right)^2, 1^2} = \sqrt{\frac{5}{4}}$$

$$\left(\frac{\frac{1}{2}}{\sqrt{5/4}}, \frac{1}{\sqrt{5/4}} \right)$$

$$\sqrt{-2^2 + 1^2} = \sqrt{5}$$

$$v_2 \begin{bmatrix} \frac{-2}{\sqrt{5/4}} & \frac{1}{\frac{\sqrt{5}}{4}} \end{bmatrix}$$

$$\Sigma \begin{bmatrix} \sqrt{16} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$u = [u_1, u_2] = \begin{bmatrix} 0.89442719 & -0.4472136 \\ 0.4472136 & 0.89442719 \end{bmatrix}$$

Using $v_i = \frac{1}{6} A^T \cdot u_i$

$$\therefore \begin{bmatrix} 0.4427134 & -0.8949318 \\ 0.8044718 & 0.4472358 \end{bmatrix}$$

9. Let $X \in \mathbb{R}^{N \times M}$, $P = X^T X$, and $C = XX^T$. Show that if $\mathbf{v}_i$ is an eigenvector of P then $X\mathbf{v}_i$ is an eigenvector of C.

$P = X^T X$, $\quad C = X X^T$

Given v_i is an eigen vector of P

$P v_i = a v_i$ for some a

Thus $X^T X v_i = \lambda v_i$

$\Rightarrow C X v_i = (X X^T)(X v_i)$

$= X(X^T X v_i)$

$= X(\lambda v_i)$

$\Rightarrow C(X v_i) = \lambda (X v_i)$

hence $X v_i$ is eigen vector of C