The Largest Empty Circle Inside a Convex Hull: A Classic Problem in Computational Geometry

1 Problem Definition

The Largest Empty Circle (LEC) problem within a convex hull represents the canonical form of this computational geometry challenge. Formally, given a set of points $P = \{p_1, p_2, \ldots, p_n\}$ in the Euclidean plane with $n \geq 3$, the objective is to find the largest circle $\mathcal{C}^* = (c^*, r^*)$ that satisfies:

(1) Contains no points from set P in its interior: $\mathrm{int}(\mathcal{C}^*) \cap P = \emptyset$

(2) Has its center within the convex hull of P: $c^* \in \mathrm{CH}(P)$

More precisely, we seek to solve:

$$r^* = \max\{r \geq 0 : \exists c \in \mathrm{CH}(P) \text{ such that } B(c, r) \cap P = \emptyset\} \tag{1}$$

where $B(c, r) = \{x \in \mathbb{R}^2 : \|x - c\| < r\}$ denotes the open ball of radius r centered at c.

This problem has applications in facility location theory, where the center represents an optimal location for placing a new facility that maximizes the distance to existing facilities while remaining within the feasible region.

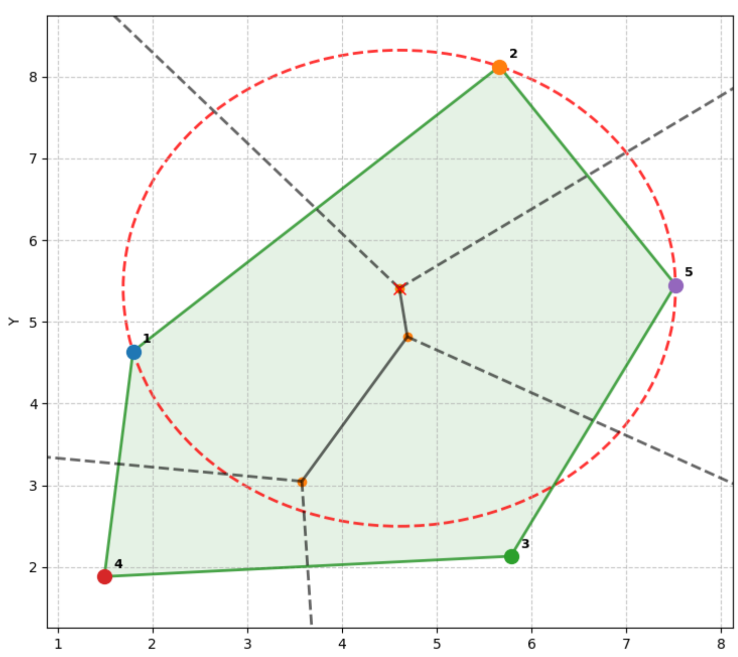

Figure 1: Largest Empty Circle (LEC) within a convex hull. The red dashed circle represents the LEC, with colored points showing the input set P. Gray lines indicate the Voronoi diagram, and the convex hull is shown with solid green edges.

2 Theoretical Foundation

2.1 Properties of the LEC Solution

The fundamental characterization of optimal solutions follows from geometric optimization principles.

Theorem 2.1 (Structural Characterization)**.** *The center c^* of the largest empty circle is located at one of the following positions:*

(1) A vertex v of the Voronoi diagram $Vor(P)$ such that $v \in CH(P)$

(2) An intersection point between a Voronoi edge and the boundary $\partial CH(P)$

Proof. Let $\rho(x) = \min_{p \in P} \|x - p\|$ be the distance from point x to the nearest point in P. The LEC problem is equivalent to maximizing $\rho(x)$ subject to $x \in \mathrm{CH}(P)$.

Suppose c^* achieves the maximum but lies neither at a Voronoi vertex nor at a Voronoi edge-boundary intersection. Then c^* lies in the interior of some Voronoi cell $V(p_k)$, so $\rho(c^*) = \|c^* - p_k\|$ with p_k being the unique nearest point.

Since c^* is not a Voronoi vertex, there exists a neighborhood where p_k remains the unique nearest point. Within this neighborhood, we can move in the direction $-\nabla\|x - p_k\|$ to increase the distance to p_k, contradicting optimality.

For points on $\partial\mathrm{CH}(P)$, the same argument applies unless the point lies at a Voronoi edge intersection, where the geometric constraint creates a critical point. □

This property constrains the search space to a finite set of candidate points, enabling efficient algorithms.

2.2 Mathematical Formulation

Let $\mathrm{CH}(P)$ denote the convex hull and $\mathrm{Vor}(P)$ the Voronoi diagram of P. The LEC problem admits equivalent formulations:

Geometric formulation:

$$c^* = \arg \max_{x \in \mathrm{CH}(P)} \min_{p \in P} \|x - p\| \tag{2}$$

Optimization formulation:

$$\begin{aligned} \text{maximize} \quad & r && (3)\\ \text{subject to} \quad & \|c - p_i\| \geq r, \quad i = 1, \ldots, n && (4)\\ & c \in \mathrm{CH}(P), \quad r \geq 0 && (5) \end{aligned}$$

The optimal radius is:

$$r^* = \min_{p \in P} \|c^* - p\| \tag{6}$$

3 Algorithm Development

3.1 Voronoi-Based Approach

The algorithm leverages the structural properties from Theorem 2.1:

Algorithm 1 Voronoi-Based LEC Algorithm

Compute convex hull $\mathrm{CH}(P)$ and Voronoi diagram $\mathrm{Vor}(P)$
Initialize candidate set $C \leftarrow \emptyset$
for all Voronoi vertices $v \in \mathrm{Vor}(P)$ **do**
    if $v \in \mathrm{CH}(P)$ **then**
        $C \leftarrow C \cup \{v\}$
    end if
end for
for all Voronoi edges $e \in \mathrm{Vor}(P)$ and boundary edges $\ell \in \partial\mathrm{CH}(P)$ **do**
    if $e \cap \ell \neq \emptyset$ **then**
        $C \leftarrow C \cup \{e \cap \ell\}$
    end if
end for
$(c^*, r^*) \leftarrow \arg\max_{c \in C} \min_{p \in P} \|c - p\|$
return (c^*, r^*)

Theorem 3.1 (Correctness)**.** *Algorithm 1 returns the optimal solution to the LEC problem.*

Proof. By Theorem 2.1, the optimal center must be in the candidate set C. The algorithm examines all such candidates and selects the one maximizing the minimum distance to points in P. □

3.2 Time Complexity Analysis

Theorem 3.2 (Optimal Complexity)**.** *The time complexity is $O(n \log n)$, which is optimal for the LEC problem.*

Proof. The complexity components are:

- Convex hull: $O(n \log n)$ (Graham scan)
- Voronoi diagram: $O(n \log n)$ (Fortune's algorithm)
- Candidate evaluation: $O(n)$

The $\Omega(n \log n)$ lower bound follows by reduction from the maximum gap problem. □

4 Geometric Properties and Constraints

4.1 Degeneracy Handling

Degenerate configurations require special treatment:

(1) **Collinear points:** Three or more collinear points cause Voronoi vertices to be undefined or at infinity

(2) **Cocircular points:** Four or more cocircular points create ill-defined Voronoi vertices

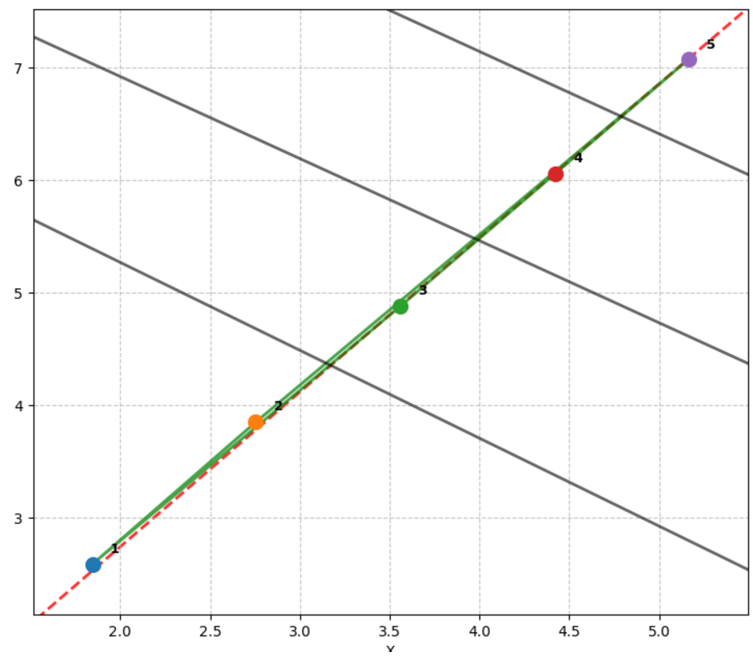

Figure 2: Degenerate case: Collinear points creating challenges for Voronoi diagram construction.

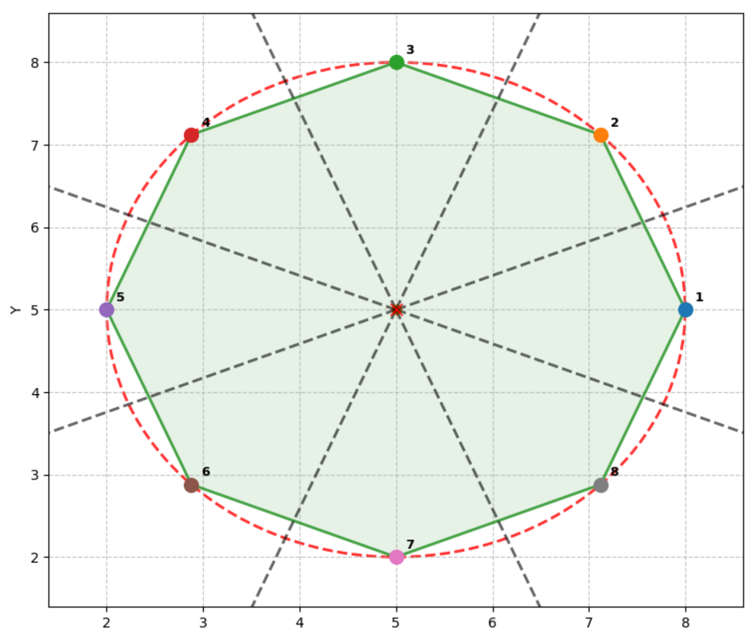

Figure 3: Degenerate case: Cocircular points affecting Voronoi diagram structure and creating numerical challenges.

Proposition 4.1 (Degeneracy Resolution)**.** *Symbolic perturbation techniques with exact arithmetic predicates ensure algorithmic robustness while preserving combinatorial structure.*

4.2 Boundary Considerations

When the optimal center lies on $\partial \mathrm{CH}(P)$:

(1) The circle is tangent to at least one convex hull edge

(2) The circle may have fewer than three boundary points

Lemma 4.2 (Boundary Optimization). *For each edge $e \in \partial CH(P)$, the optimal point on e maximizes:*

$$\max_{c \in e} \min_{p \in P} \|c - p\|$$

5 Mathematical Proof of Optimality

Lemma 5.1 (Voronoi Property). *For any point x, if $x \in V(p_i)$, then* $\arg\min_{p \in P} \|x - p\| = p_i$.

Theorem 5.2 (Global Optimality). *The function $f(x) = \min_{p \in P} \|x - p\|$ achieves its maximum over $CH(P)$ only at Voronoi vertices within $CH(P)$ or at Voronoi edge-boundary intersections.*

Proof. The function f is continuous and piecewise linear on Voronoi cells. Within each cell $V(p_i)$, $f(x) = \|x - p_i\|$ increases monotonically away from p_i. Therefore, maxima occur only at cell boundaries (Voronoi edges) or geometric constraints ($\partial \mathrm{CH}(P)$). □

6 Implementation Considerations

6.1 Numerical Stability

Critical implementation aspects include:

(1) **Intersection computation:** Robust geometric predicates for Voronoi edge-boundary intersections

(2) **Point location:** Epsilon-tolerant testing for points on convex hull boundary

Proposition 6.1 (Numerical Robustness). *Epsilon-based comparisons with adaptive precision provide guaranteed correctness while maintaining efficiency.*

6.2 Efficient Data Structures

(1) **Doubly-Connected Edge List (DCEL):** Efficient representation for Voronoi diagrams and convex hulls

(2) **Priority queue:** Optimal candidate selection

7 Extension to Weighted Points

Definition 7.1 (Weighted LEC). For points with positive weights $w_i > 0$, the weighted LEC maximizes:

$$c^* = \arg\max_{x \in \mathrm{CH}(P)} \min_{p_i \in P} \frac{\|x - p_i\|}{w_i} \tag{7}$$

This extension applies to non-uniform facility location where sites have varying importance or influence ranges.

8 Connection to Circle Packing

The LEC problem connects to fundamental circle problems:

(1) **Inscribed circle problem:** Largest circle inscribed in a polygon

(2) **Minimax circle problem:** Smallest enclosing circle

Theorem 8.1 (Geometric Duality). *These connections reflect duality principles where LEC and minimum enclosing circle problems exchange interior and exterior optimization roles.*

9 Example Calculation

Consider four points $P = \{(0,0), (0,3), (3,0), (3,3)\}$.

The convex hull forms rectangle $[0,3] \times [0,3]$. The Voronoi diagram has perpendicular bisectors as edges, with the unique interior vertex at $(1.5, 1.5)$. This point is equidistant from all corners with distance $1.5\sqrt{2} \approx 2.12$, confirming it as the LEC center.

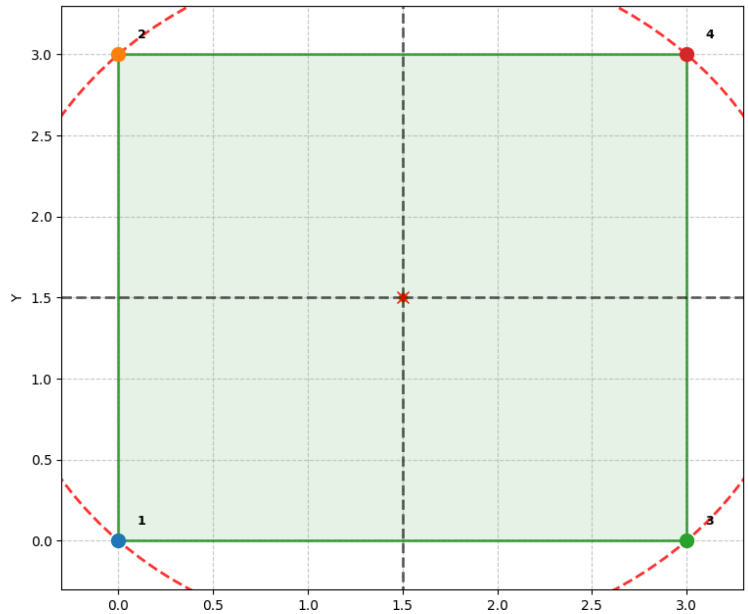

Figure 4: Complex LEC example showing Voronoi diagram (gray dashed lines), largest empty circle (red dashed), and the optimal center at a Voronoi vertex.

10 Practical Applications

The LEC problem has critical applications in:

(1) **Facility location:** Optimal placement of emergency services to maximize coverage within administrative boundaries

(2) **Wireless network design:** Positioning access points to maximize coverage while avoiding interference from existing installations

(3) **Urban planning:** Identifying locations for public facilities that maximize distance from existing structures within zoning constraints

Proposition 10.1 (Application Optimality)**.** *In each domain, the LEC solution provides mathematically optimal balance between coverage maximization and constraint satisfaction.*

These applications demonstrate the continued relevance of this classical problem in modern operations research and network optimization.