Graham Scan Algorithm for Convex Hulls in Voronoi Diagrams

Theoretical Description

The **Graham Scan** algorithm is a classical and efficient method to compute the **convex hull** of a finite set of points in the plane. The convex hull is the smallest convex polygon enclosing all the points. In the context of Voronoi diagrams, convex hulls are often used to bound the set of sites or to analyze the outer boundary of Voronoi cells.

The Graham Scan algorithm proceeds by first selecting a reference point, then sorting all points by the polar angle they make with this reference, and finally iteratively constructing the hull by maintaining a stack of candidate vertices and ensuring the polygon remains convex.

Key ideas:

- The point with the lowest y-coordinate (and if tied, the lowest x-coordinate) is chosen as the **pivot** P_0. This point is guaranteed to be on the hull.
- All other points are sorted by the **polar angle** they form with P_0, in counterclockwise order.
- Starting from P_0, points are processed in sorted order and added to a stack representing the hull vertices.
- At each step, the algorithm checks the orientation of the last three points on the stack:

$$\text{orientation}(P_{i-2}, P_{i-1}, P_i) = \begin{cases} > 0 & \text{counterclockwise turn} \\ = 0 & \text{collinear} \\ < 0 & \text{clockwise turn} \end{cases}$$

- If a **non-left turn** (clockwise or collinear) is detected, the middle point is removed from the hull (popped from the stack), ensuring the hull remains convex.

Orientation test: Given three points $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (x_c, y_c)$, the orientation is computed by the cross product:

$$\text{orientation}(A, B, C) = (x_b - x_a)(y_c - y_a) - (y_b - y_a)(x_c - x_a).$$

A positive value indicates a counterclockwise turn, negative indicates clockwise, and zero means collinear.

Algorithm

Algorithm 1 Graham Scan for Convex Hull

Input: Set of points $S = \{P_1, P_2, \ldots, P_n\}$
Output: Vertices of the convex hull in counterclockwise order
Find the pivot P_0 with the lowest y-coordinate (break ties by lowest x)
Sort points in $S \setminus \{P_0\}$ by polar angle with respect to P_0
Initialize stack H and push P_0, P_1 (first point in sorted order)
for $i = 2$ to $n - 1$ **do**
    while $|H| \geq 2$ and orientation(secondTop(H), top(H), P_i) ≤ 0 **do**
        Pop top of H
    end while
    Push P_i onto H
end for
return H ▷ Stack H contains hull vertices

Explanation:

- The sorting step arranges points so the hull can be traced by moving counterclockwise.
- The stack maintains the current convex polygon. Whenever adding a new point would cause a clockwise turn (non-left turn), the last point added is removed.
- This ensures the polygon remains convex and includes all hull vertices.

Complexity Analysis

- **Time Complexity:**
  - Finding the pivot: $O(n)$
  - Sorting points by polar angle: $O(n \log n)$
  - Single pass to construct hull with stack operations: $O(n)$

  Overall time complexity:

  $$\boxed{O(n \log n)}$$

- **Space Complexity:**

  $$O(n)$$

  for storing points and the stack.

Relation to Voronoi Diagrams

The convex hull computed by Graham Scan can be used in Voronoi diagram computations to:

- Identify the outer boundary of the set of sites.
- Limit the Voronoi diagram construction to the convex hull region.
- Analyze infinite Voronoi edges which occur outside the convex hull.