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# CS 412 - Introduction to Machine Learning
## Assignment 1

Q1. To prove: $A \perp B$ (i.e, A and B are independent)

| a | b | P(A = a, B = b) |
|---|---|---|
| 0 | 0 | 0.5 |
| 0 | 1 | 0.0 |
| 1 | 0 | 0.0 |
| 1 | 1 | 0.5 |

A:

| B \ A | 0 | 1 |
|---|---|---|
| 0 | 0.5 | 0 |
| 1 | 0 | 0.5 |

$P(A=0) = 0.5$
$P(A=1) = 0.5$
$P(B=0) = 0.5$
$P(B=1) = 0.5$

Conditions to prove $A \perp B$

$A \perp B \iff P(A,B) = P(A) \cdot P(B)$ .. (i)
$\iff P(A|B) = P(A)$ .. (ii)
$\iff P(B|A) = P(B)$ .. (iii)

Considering and substituting in (i)

$P(A,B) = P(A) \cdot P(B)$

1. $P(A=0, B=0) = 0.5$ .. Given (LHS) [Left hand side]
$P(A=0) = 0.5$
$P(B=0) = 0.5$
$P(A=0) \cdot P(B=0) = (0.5)(0.5)$
$= 0.25$ .. RHS [Right hand side]

$LHS \neq RHS$
$P(A=0, B=0) \neq P(A=0) P(B=0)$ .. (iv)

2. $P(A=0, B=1) = P(A=0) \cdot P(B=1)$

$0 \neq (0.5)(0.5)$

LHS $\neq$ RHS

$P(A=0, B=1) = P(A=0) \cdot P(B=1)$ .. (v)

3. $P(A=1, B=0) = P(A=1) \cdot P(B=0)$

$0 \neq (0.5)(0.5)$

$0 \neq 0.25$

~~$P(A=$~~

$P(A=1, B=0) \neq P(A=1) \cdot P(B=0)$ .. (vi)

4 $P(A=1, B=1) = P(A=1) \cdot P(B=1)$

$0.5 \neq (0.5)(0.5)$

$P(A=1, B=1) \neq P(A=1) \cdot P(B=1)$ .. (vii)

LHS $\neq$ RHS

As the conditioned failed for the given distribution.

.. from (iv)(v)(vi)(vii)

$\therefore$ A and B are <u>not</u> independent.

Q2. To prove : $A \perp B \mid C$ (ie A and B are conditionally independent given c)

| a | b | c | $P(A=a, B=b, C=c)$ | Substituting as |
|---|---|---|---|---|
| 0 | 0 | 0 | 0.056 | h |
| 0 | 0 | 1 | 0.120 | i |
| 0 | 1 | 0 | 0.224 | j |
| 0 | 1 | 1 | 0.120 | k |
| 1 | 0 | 0 | 0.024 | l |
| 1 | 0 | 1 | 0.180 | m |
| 1 | 1 | 0 | 0.180 | n |
| 1 | 1 | 1 | 0.096 | o |

A: Conditional independence asserts that,

$$A \perp B \mid C \iff P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

Rewriting,

$$P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

$$\frac{P(A, B, C)}{P(C)} = \frac{P(A, C)}{P(C)} \cdot \frac{P(B, C)}{P(C)}$$

$$P(A, B, C) = \frac{P(A, C) \cdot P(B, C)}{P(C)}$$

$$= \frac{\left(\Sigma_b P(A, b, C)\right)\left(\Sigma_a P(a, B, C)\right)}{\left(\Sigma_{a,b} P(a, b, C)\right)}$$

Instantiating A, B, C in all 8 ways to obtain the following 8 equations :

| | | | |
|---|---|---|---|
| P(F F F) | $h = (h+j)(h+l)/(h+j+l+n)$ | or | $hn = jl$ |
| P(F F T) | $i = (i+k)(i+m)/(i+k+m+o)$ | or | $io = km$ |
| P(F T F) | $j = (h+j)(j+n)/(h+j+l+n)$ | or | $jl = hn$ |
| P(F T T) | $k = (i+k)(k+o)/(i+k+m+o)$ | or | $km = io$ |
| P(T F F) | $l = (l+n)(h+l)/(h+j+l+n)$ | or | $jl = hn$ |
| P(T F T) | $m = (m+o)(i+m)/(i+k+m+o)$ | or | $km = io$ |
| P(T T F) | $n = (l+n)(j+n)/(h+j+l+n)$ | or | $hn = jl$ |
| P(T T T) | $o = (m+o)(k+o)/(i+k+m+o)$ | or | $io = km$ |

Substituting we get,

$hn = jl$ ..(i)

$io = km$ ..(ii)

(i) $hn = jl \iff (0.056)(0.180) \neq (0.224)(0.024)$

$0.010 \neq$ ~~0.248~~ $0.005376$

LHS $\neq$ RHS

(ii) $io = km \iff (0.120)(0.096) \neq (0.120)(0.180)$

$0.011 \neq 0.0216$

LHS $\neq$ RHS

$\therefore$ As $hn \neq jl$ and $io \neq km$

A and B are <u>not</u> conditionally independent given C.

Q3. Two binary random variables A and B

If $A \perp B$

$P(A=0, B=0) = 0.18$

$P(A=1, B=0) = 0.28$

$P(A=0, B=1) = ?$

A:

| a | b | $P(A=a, B=b)$ |
|---|---|---|
| 0 | 0 | 0.18 |
| 0 | 1 | n.a $x$ |
| 1 | 0 | 0.28 |
| 1 | 1 | n.a $y$ |

Determining values of $x, y$, considering $A \perp B$,

As,

$x + y + 0.18 + 0.28 = 1$ $\quad \therefore x \geq 0$

$x + y = 0.54$ $\quad y \geq 0$ .. (i)

>

| | | A | |
|---|---|---|---|
| | | 0 | 1 |
| B | 0 | 0.18 | 0.28 |
| | 1 | $x$ | $y$ |

$P(A=0) = 0.18 + x$ $\quad P(A=1) = 0.28 + y$

$P(B=0) = 0.46$ $\quad P(B=1) = x + y$

.. (ii)

> $A \perp B \iff P(A,B) = P(A) \cdot P(B)$

$P(A=0, B=1) = P(A=0) \cdot P(B=1)$

$x = (0.18 + x) \cdot (x + y)$

$x = 0.18x + 0.18y + x^2 + xy$

$x^2 + xy + 0.18y - 0.82x = 0$

$x^2 + x(0.54 - x) + 0.18(0.54 - x) - 0.82x = 0$

$$\therefore \cancel{x^2} + 0.54x - \cancel{x^2} + 0.0972 - 0.18x - 0.82x = 0$$

$$\therefore 0.46x = 0.0972$$

$$\therefore x = 0.211$$

From (1)

$$y = 0.54 - 0.211$$

$$y = 0.329$$

$$\therefore P(A=0, B=1) = (0.18 + 0.211)(0.211 + 0.329)$$
$$= (0.391)(0.54)$$
$$= 0.211 .$$

$$\therefore P(A=1, B=1) = y = 0.329$$
$$P(A=0, B=1) = 0.211$$