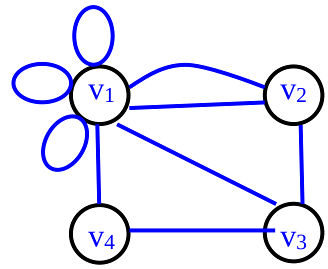
MA238 Fall 2020 Midterm 1 **Time: 80 minutes**
October 20, 2020 **No aids**
Instructor: Dr. Kathie Cameron **Similar assigned problems are noted in green**

1. [7 marks]

(a) Given the following adjacency matrix of a graph G with vertices v_1, v_2, v_3, and v_4, draw G. Similar to Problems 4 – 11, 17

[2 ½ marks]
- ½ for each error

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 3 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



(b) [1] If G is a simple undirected graph and A is its adjacency matrix, what is the sum of the entries in the column corresponding to vertex x? d(x) [or deg(x) or degree(x) or degree of x]
Similar to Problems 4 -29

[1] If G is a simple directed graph and A is its adjacency matrix, what is the sum of the entries in the row corresponding to vertex x? $d^{out}(x) – d^{in}(x)$ [or $d^+(x) – d^-(x)$] Similar to Problems 4 -29

(c) [2 ½ marks] If G is a graph with vertices $v_1, \ldots, v_n$, and with adjacency matrix A, and if the entry in row 6 and column 8 of A^{19} is 54, what does this tell us about G?
The number of walks of length 19 from v_6 to v_8 is 54 (may say: joining v_6 to v_8)
[½] [½] [1] [½] See Problems 7 and modules

2. [11 marks]

(a) For any graph G = (V, E), state the Handshaking Theorem which relates the degrees and E.
[1] The sum of the degrees is twice the number of edges OR

$$\sum_{v \in V} d(v) = 2|E|$$

Show how the formula of (a) simplifies when every vertex of G has degree q, where q is a nonnegative integer. Similar to Problems 3 -15
[1] q times the number of vertices is twice the number of edges OR

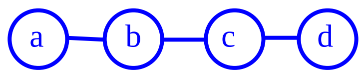
$q|V| = 2|E|$

(b) [1] How many edges does the complete graph K_n on n vertices have? $n(n-1) / 2$ or $\binom{n}{2}$
Problems 3 -15

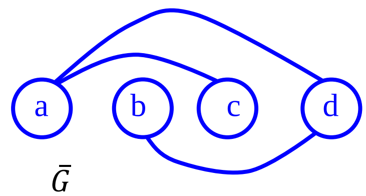
(c) Use (a) or (b) to show that your answer to (c) is correct.
The degree of each vertex of K_n is n-1. [1]
By (c), $n(n-1) = 2|E|$ [1]
Thus $|E| = n(n-1) / 2$

(d) A graph G is self-complementary if G is isomorphic to its complement $\bar{G}$.
How many edges does a self-complementary graph have? $n(n-1) / 4$ [1]
Explain your answer. Since G and $\bar{G}$ are isomorphic, they have the same number of edges. [½]
number of edges of G + number of edges of $\bar{G}$ = number of edges of K_n [½]
Thus number of edges of G is ½ $(n(n-1)/2) = n(n-1) / 4$

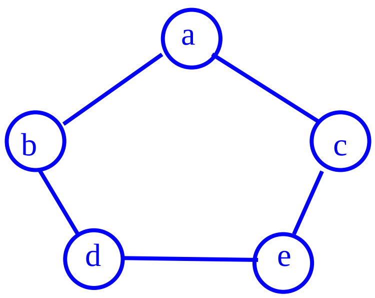
(e) [4] Draw a self-complementary graph with 4 vertices and one with 5 vertices. Draw the complement of each of your graphs, using the same vertex labels in both the graph and its complement, so it is clear that one is the complement of the other. This is Problems 5 – 50, 51



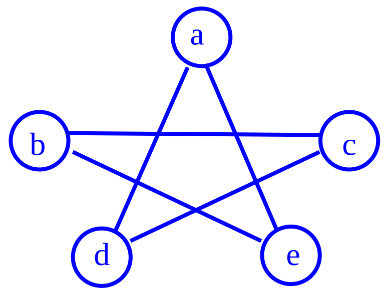
G is the path on four vertices



$\bar{G}$

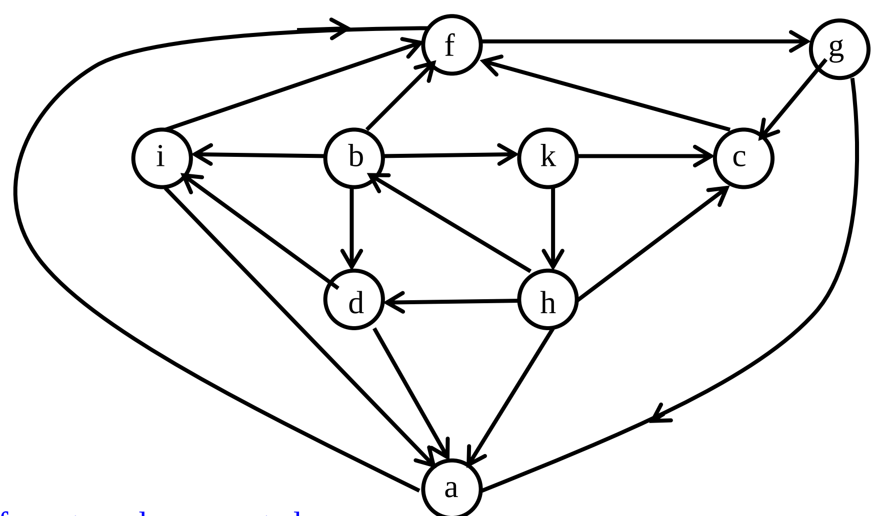


G is C_5

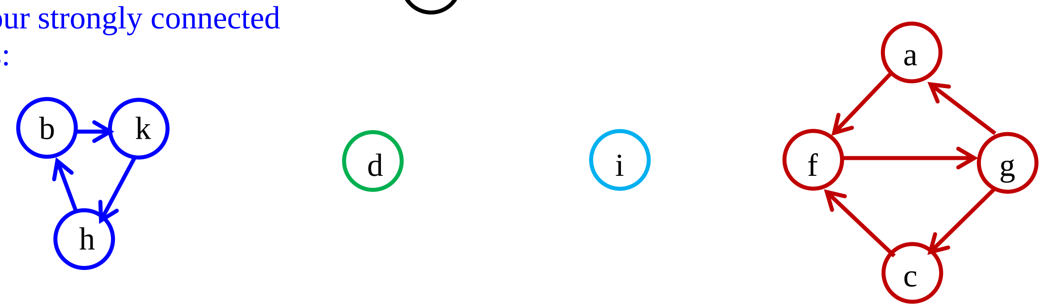


$\bar{G}$ is $\overline{C_5}$

3. [4 marks]
Draw the strongly connected components of the graph below. Similar to Problems 6 -15

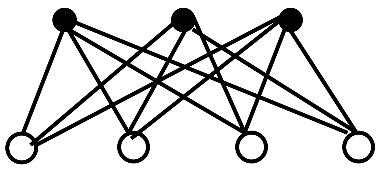


There are four strongly connected components:

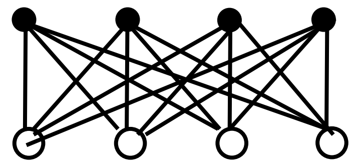


[1] [½] [½] [2]

4. [14 marks]
Recall that $K_{m,n}$ denotes the complete bipartite graph with a set B of m black vertices and a set W of n white vertices, and an edge joining each black vertex to each white vertex.
(a) Draw $K_{3,4}$ and $K_{4,4}$. [2 marks, 1 mark each] Problems 3- 20(c)



$K_{3,4}$



$K_{4,4}$.

(b) What are the degrees of vertices in $K_{m,n}$? Similar to Problems 3-37
The degree of each of the m black vertices (i.e. vertex of B) is n [1]
The degree of each of the n white vertices (i.e. vertex of W) is m [1]

(c) Complete the following:
(i) A graph G = (V,E) has an Euler circuit if and only if
G is connected [½] and every vertex of G has even degree [1]
(ii) A graph G = (V,E) has an Euler trail but no Euler circuit if and only if
G is connected [½] and G has exactly two vertices with odd degree [1]

(d) Determine the values of m and n for which $K_{m,n}$ has the following structure. Explain your answers briefly.
(i) a Hamilton cycle This is Problems 8 - 45
$m = n \geq 2$ [1 for m = n, ½ bonus for $\geq$ 2]

A Hamilton cycle must start in one part (say B) and then alternate between B and W, and then go back to the starting vertex. [1 for a reason]
So $m = |B| = |W| = n$
(Note however that $K_{1,1}$ has no cycle at all.)

(ii) an Euler circuit This is Problems 8 - 28

m and n must both be even [1]

Note that $K_{m,n}$ is always connected.

All degrees are even $\Leftrightarrow$ m and n are both even [1]

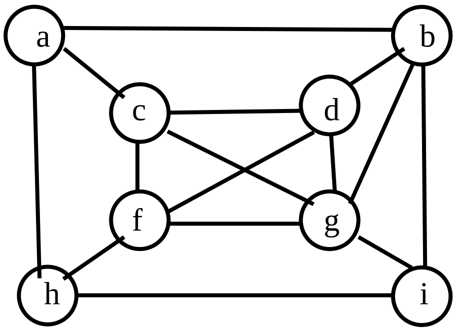
(iii) an Euler trail but no Euler circuit This is Problems 8 - 28

Either (m = n = 1) [½] or (m = 2 and n is odd) or (m is odd and n = 2) [1 ½]

Note that $K_{m,n}$ is always connected.

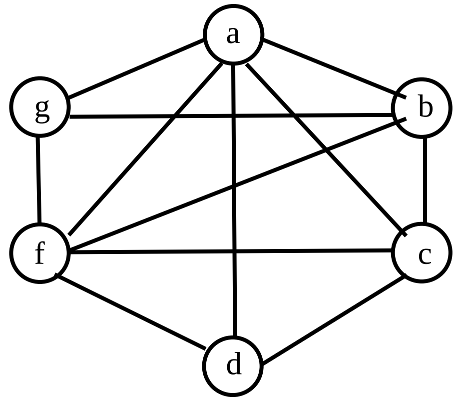
$K_{m,n}$ has exactly two odd-degree vertices
both in B $\Leftrightarrow$ |B| = 2 and there are an odd number of vertices in W
both in W $\Leftrightarrow$ |W| = 2 and there are an odd number of vertices in B
one in B, one in W $\Leftrightarrow$ |B| =|W| = 1 [1 for a reason]

5. [6 marks] Similar to Problems 9 – 5 to 9 and 20 to 23
For each of the following graphs, determine whether or not it is planar.
If it is planar, give a planar representation.
If it is not planar, explain why it is not planar.



G

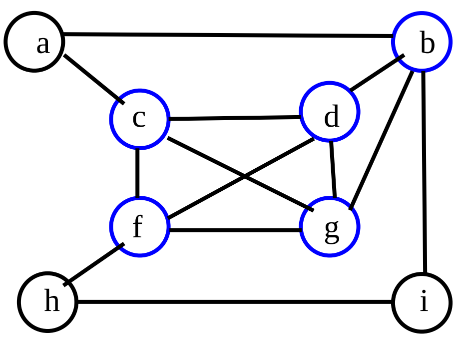
[4 marks: ½ for "not planar", 1.5 for correct subgraph,
1.5 for showing the subgraph is a homeomorph of K_5 or K_5
½ for using Kuratowski's Thm]



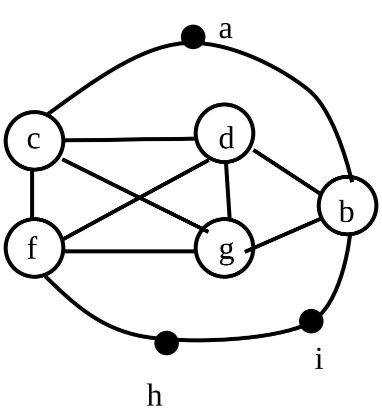
H

[2 marks: ½ for "planar"
1.5 for a planar representation]

G is not planar since
G contains the subgraph G' below which is
homeomorphic to K_5.
Thus by Kuratowski's Theorem, G is not planar.



G'



G'

Alternatively, G also contains the subgraph G” below which is homeomorphic to $K_{3,3}$, and thus again by Kuratowski’s Theorem, G is not planar.

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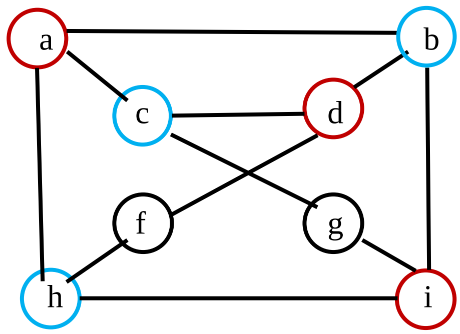
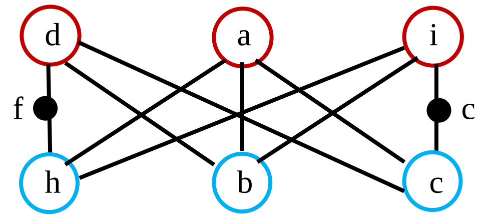


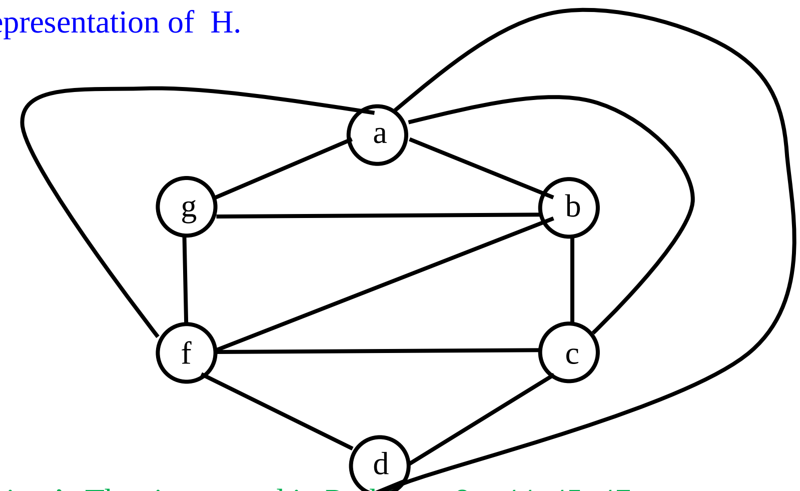
G”



G”

Graph H is planar. Here is a planar representation of H.

H:



6. [6 marks] Similar to Graph Models, Dirac’s Thm is covered in Problems 8 – 44, 45, 47
Model the following as a problem on a graph:

22 women are attending a banquet hosted by the Laurier Centre for Women in Science.
Each woman knows 11 others in the group.
Is it possible to seat all the women at a large round table so that each woman knows the two women she is seated between?

Give your answer by filling out the following information:
Construct a graph as follows: [½]
The vertices are the 22 women. [½]
There is an edge from vertex x to vertex y exactly when woman x knows woman y. [1]
Now state a theorem that you are using.
Dirac’s Theorem. [2 marks as below]
If G is a simple [½] graph with n vertices where $n \geq 3$ such that the degree of each vertex is at least[½] n/2 [½], **then** G has a Hamilton cycle. [½]
Now briefly explain how the theorem proves that the appropriate seating arrangement exists or does not exist.
Here the number n of vertices is 22, which is ≥ 3. The graph is simple.
The degree of each vertex (woman) is 11. [½]
$11 = \text{degree of each vertex} \geq 22/2 = n/2$ [½]
Thus Dirac’s Theorem applies and tells us that there is a Hamilton cycle. [½]
Use the cycle as the seating arrangement. [½]

7. [8 marks] From class modules and lectures

(a) In class we proved the following:

Lemma: Every simple planar graph has a vertex x with degree(x) $\leq$ 5.

Use the lemma and simple induction to prove that

If G = (V, E) is a simple planar graph, then G can be coloured with 6 colours.

Give your proof by filling in the blanks below and then completing the proof.

Proof (by induction on the number of vertices).

Base Case: The statement is clearly true for any simple graph with at most 6 vertices: just give each vertex a different colour. [1]

Induction Hypothesis:): Every simple planar graph with k vertices can be properly coloured with at most 6 colours. [1]

Now complete the proof.

We must prove true for graphs with k+1 vertices.

Let G be a simple planar graph with k+1 vertices. [½]
By the Lemma, G has a vertex x with degree(x) $\leq$ **5.** . [½]
The graph G-x is simple and planar and has k vertices, [½]
so by the IH . [½], G-x can be coloured with 6 colours. [½]
At most 5 colours will be used for the vertices joined to x . [½]
(since there are at most 5 such vertices). [½]
So there is a colour different from these which can be used for v. [½]
This gives a colouring of G with 6 colours. [½]

By the Principle of Mathematical Induction, the theorem holds. □

(b) Is 6 the best upper bound we know for the number of colours required to colour a simple planar graph? Explain your answer.

No. [½] The famous Four Colour Theorem says that every simple planar graph can be coloured with at most 4 colours. . [1]

8. [9 marks]

In this question, G = (V, E) is a simple planar graph, and R is the set of regions in a planar representation of G.

(a) Assuming G is connected, state Euler's Formula for G.

$|V| - |E| + |R| = 2$ [1]

(b) State a generalization of Euler's Formula to planar graphs G with k connected components.

$|V| - |E| + |R| = k + 1$ [1] This is Problems 9 - 18

(c) In class we proved:

Corollary: If G = (V, E) is a simple planar graph with at least 3 vertices, then $|E| \leq 3|V| - 6$.

Use the corollary to show that K_5 is not planar. Done in modules and in class

K_5 has 5 vertices and 10 edges . [½]. K_5 is also simple.
If K_5 were planar, then it would have to satisfy the inequality of the Corollary: $|E| \leq 3|V| - 6$ [½]
But $10 = |E| > 3|V| - 6 = 9$. [½]
Thus K_5 is not planar. [½]

(d) Prove the corollary as follows. Done in modules and in class

Copy out what is written below onto your paper and complete it by filling in the blanks and then finishing the argument.

Proof. Since G = (V, E) is a simple connected planar graph with at least 3 edges, each region F in a planar representation of G is bounded by at least 3 edges. [1]

So
$\sum_{F \in R}(\textit{number of edges in the boundary of region } F) \geq 3|R|$ [1]

Also

$$\sum_{F \in R}(\textit{number of edges in the boundary of region } F) = 2|E|$$

since each edge is in the boundary of two regions, [1]

so each edge is counted twice in the summation.

Now use Euler's Formula and some algebra to complete the proof.
[2]:
By Euler's Formula, since |V| -|E| +|R| = 2, we have 3|V| -3|E| +3|R| = 6. i.e. 3|R| = 6 -3|V| + 3|E|
By the above, 3 |R| $\leq$ 2 |E|,
so we get 6 -3|V| + 3|E| $\leq$ 2 |E| which simplifies and rearranges to |E| $\leq$ 3|V| -6