MA238 Winter 2021 Midterm
February 25, 2021
Instructor: Dr. Kathie Cameron

Time: 90 minutes
No aids – no calculators
2 pages, 8 questions, 56 marks

Numerical answers do not need to be simplified.

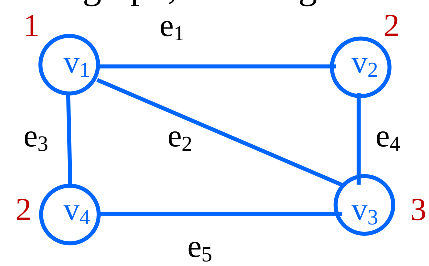
1. [7 marks]

(a) [3] If G is a graph with vertices $v_1, \ldots, v_n$, and with **adjacency matrix** A, and if the entry in row 29 and column 11 of A^{33} is 42, what does this tell us about G?

There are 42 walks of length 33 from vertex 29 to vertex 11 in G

(b) [3] Given the following **incidence matrix**, draw the graph, labelling the vertices and edges.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



(c) [1] Give a colouring of the vertices of the graph of (a) using as few colours as possible. Call the colours 1, 2, … Shown in red on the graph

2. [8 marks]

(a) [1] State the Handshaking Theorem which relates edges and degrees in any graph G = (V, E).

The sum of the degrees equals twice the number of edges

OR $\sum_{v \in V} d(v) = 2|E|$

(b) State Euler's Formula for a connected **planar** graph G = (V, E). If you use any terminology, say what it means.

[0.5] Let R be the set of regions in a planar representation of G.

[1] $|V| - |E| + |R| = 2$

(c) Suppose G is a simple connected planar graph with degree sequence

(7, 7, 6, 5, 4, 4, 4, 3, 3, 2, 2, 1).

(i) How many edges does G have? Show your work.

[1] $2|E| = \sum_{v \in V} d(v) = 7+ 7+ 6+ 5+ 4+ 4+ 4+ 3+ 3+ 2+ 2+ 1 = 48$

[1] So $|E| = 48/2 = 24$

(ii) How many edges does the complement $\bar{G}$ of G have? Show your work.

[0.5]G has 12 vertices.

[1] The number of edges of $\bar{G}$ is the number of edges of K_{12} – the number of edges of G

[1] = (12)(11)/2 – 24

= 66 – 24 = 42 (not required)

(iii) What does Euler's Formula tell you about G ?

[1] Euler's Formula tells us that in any planar representation of G, the number of regions is $|R| = 2 - |V| + |E| = 2 – 12 + 24 = 14$

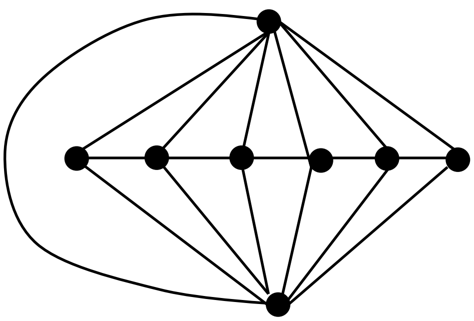
3. [3 marks] Let G be a simple graph with two connected components, one with 100 vertices and one with 200 vertices. What is the maximum number of edges of G? Explain your answer.

The graph with the maximum number of edges would be the the graph consisting of K_{100} and K_{200} with no edges between them. [1 for correct idea]
This graph has (100)(99) / 2 + (200)(199)/2 edges. [2]

4. [6 marks] We have shown that:

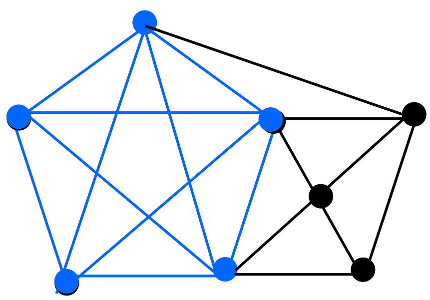
   If G = (V, E) is a simple connected planar graph, then $|E| \leq 3|V| - 6$.

   (a) [3] Draw an example of a simple connected **plane graph** with 8 vertices and 18 edges, or explain why this is not possible. There are many examples. One is shown.

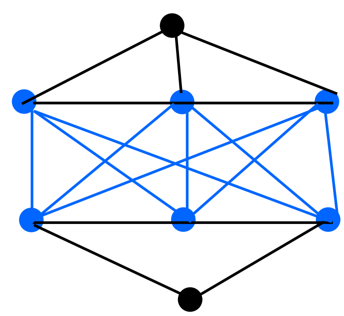


The problem asked for a plane graph. The answer must be drawn in the plane without crossings (which is the meaning of plane graph).

   (b) Draw an example of a simple connected **non-planar graph** with 8 vertices and 18 edges or explain why this is not possible. If your answer is a non-planar graph, give convincing evidence that it is not planar. There are many examples. Two are shown.



This graph is not planar because it has K_5 as a subgraph. K_5 is shown in blue.



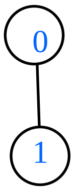
This graph is not planar because it has $K_{3,3}$ as a subgraph. $K_{3,3}$ is shown in blue.

The only convincing evidence for a graph to be non-planar is that it contain K_5 or $K_{3,3}$ or a homeomorph of one of them as a subgraph.
[2] for a valid graph
[1] for a valid reason why their graph is not planar.

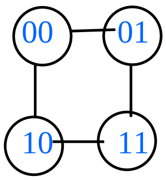
5. [3 marks] What is the chromatic number of each of the following graphs?
   (a) The complete graph K_{2021} 2021
   (b) The complete bipartite graph $K_{19, 54}$ 2
   (c) The cycle C_{135} 3

6. [10 marks]
   (a) [1] Define Euler circuit in a graph G = (V, E).
   An Euler circuit is a walk which starts and ends at the same vertex, goes through every vertex of G, and goes through every edge of G exactly once.
   Can say circuit or trail rather than walk. Path is not correct.
   (b) [2] State the Euler Circuit Theorem.
   A graph G = (V, E) has an Euler circuit $\boxed{\text{if and only if}}$ [0.5]
   G is connected [0.5] and every vertex has even degree. [1]
   (c) [3.5] Suppose graph G = (V, E) has an Euler circuit. Explain why the degree of each vertex is even. (Don't use the Euler Circuit Theorem in your answer.)
   Suppose the Euler circuit starts and ends at vertex s. [0.5]
   Each time the Euler circuit visits a vertex x other than s, it uses one edge to get to x and one to leave x [1], and it can not use these edges again. Thus the degree of x is even. . [0.5]
   For s, the first edge of the Euler circuit uses one edge meeting s and the last edge of the circuit uses one edge and every other visit uses two edges [1], so the degree of s is also even. [0.5]
   (d) [2] We used complete induction on the number of edges to prove the Euler Circuit Theorem. In the inductive step, we deleted the edges of a cycle and got a graph where each component has all vertices of even degree. Why do we need to use complete induction rather than simple induction?
   We need to use complete induction because we do not know how many edges each component has. There won't actually be a component with one fewer edge than the whole graph (except if a loop is removed), so we can't apply simple induction which would say that the theorem holds for graphs with one fewer edge than G.
   (e) [1.5] Complete the following: A graph has an Euler trail but no Euler circuit if and only if
   G is connected [0.5] and exactly two vertices have odd degree. .[1]

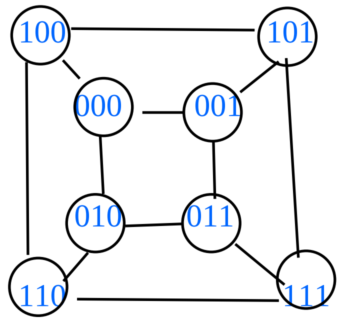
7. [12 marks] Recall that $Q_{,k}$ denotes the k-dimensional hypercube, which has a vertex for each of the 2^k 01-strings of length k (i.e. k-digit binary numbers).
Two vertices are joined by an edge if the corresponding strings differ in exactly one digit.
(a) [3] Draw Q_1, Q_2 and Q_3 and label their vertices with appropriate 01-strings.



Q_1



Q_2



Q_3

Answer the following questions where k is an arbitrary integer, not just for k = 1,2,3.

(b) What is the degree of each vertex of Q_k ? Explain briefly.
The degree of each vertex in Q_k is k. [1]
This is because there are k ways to change a single digit of a 01-string of length k, and this gives us the k vertices adjacent to a given vertex. [1]

(c) [1] Use the Handshaking Theorem to calculate the number of edges of Q_k .
$2|E| = \sum_{v \in V} d(v) = k \cdot 2^k$
Thus $|E| = k \cdot 2^{k-1}$

(d) For what values of k does Q_k have an Euler circuit?
[1] Q_k has an Euler circuit for all even values of k.

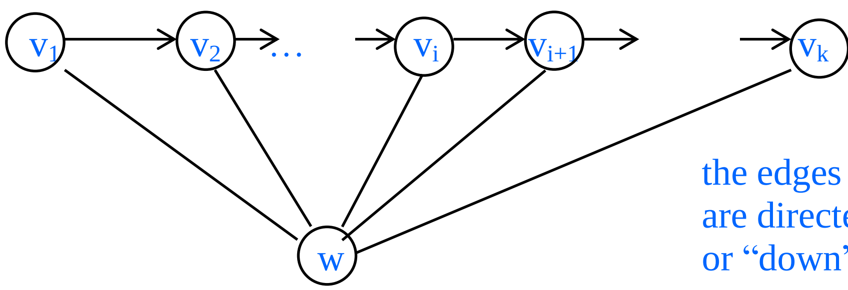
(e) For what values of k does Q_k have an Euler trail but no Euler circuit? Explain your answer.
[1] Only when k = 1.
[1] Since all vertices of Q_k have the same degree, the only way to have two odd-degree vertices is to have two vertices,
and Q_1 has exactly two vertices of degree 1.

(f) State Dirac's Theorem: Let G = (V,E) be a simple graph with n vertices where $n \geq 3$.
[1] If every vertex has degree $\geq n/2$, then G contains a Hamilton cycle.

(g) For what values of k does Dirac's Theorem tell us that Q_k has a Hamilton cycle? Explain your answer.
[1] For the degree k of each vertex to be at least half the number of vertices which is half of 2^k i.e. $k \geq 2^{k-1}$
[1] the only possibility is k = 2.

8. [11 marks] Recall that a tournament is a directed graph whose underlying undirected graph is a complete graph. Redei's Theorem says that every tournament has a directed Hamilton path.
(a) Prove Redei's Theorem.

[1] Suppose G is a tournament, P is a directed path with at least two vertices, and P is not a directed Hamilton path. Then there is a vertex, say w, which is not on P.
We will show how to find a longer directed path.



the edges incident to w
are directed but may go "up"
or "down"

[0.5] Since G is a tournament, w is joined to every vertex of P.
[1] If (w, v_1) is an edge, then w followed by P is a directed path including w and all of P.
[0.5] So assume (w, v_1) is not an edge. Thus (v_1, w) is an edge.
[1] If (v_k, w) is an edge, then P followed by w is a directed path including P and all of w.
[0.5] So assume (v_k, w) is not an edge. Thus (w, v_k) is an edge.
[0.5] Looking at the sequence of edges joining w to the vertices $v_1, v_2, \ldots, v_k$ of P, the first edge is (v_1, w) [that is, it goes "down"] and the last one is (w, v_k) [that is, it goes "up"].
[1] So the sequence must switch from "down" to "up" at some point.
[0.5] This means there is an i such that (v_i, w) and (w, v_{i+1}) are edges,
[1] But then $v_1, v_2, \ldots, v_i, w, v_{i+1}, , \ldots, v_k$ is a directed path containing P and w.
[0.5] We can continue to build a longer directed path until we have a directed Hamilton path.

(b) In a round-robin tournament, every team plays every other team exaclty once, and there are no ties.

(i) How can a round-robin tournament be modelled as a directed graph?

To answer a question like this, you must state what the vertices are and what the edges are.

[1] There is a vertex for each team
[1] And an edge directed from team a's vertex to team b's vertex exactly when team a beat team b.

(ii) A ranking of a round-robin tournament is an assignment of 1st place, 2nd place, etc. to the teams. What does Redei's Theorem tell us about rankings of a round-robin tournament?

[1] There is always a ranking of all the teams where the 1st place team beat the 2nd place team who beat the 3rd place team, etc.
Your answer must use Redei' Theorem.