MA238 Fall 2020 Midterm
October 20, 2020
Instructor: Dr. Kathie Cameron

Time: 80 minutes
No aids
3 pages, 8 questions, 65 marks

1. [7 marks]

(a) Given the following adjacency matrix of a graph G with vertices v_1, v_2, v_3, and v_4, draw G.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 3 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) If G is a simple undirected graph and A is its adjacency matrix, what is the sum of the entries in the column corresponding to vertex x?

(c) If G is a simple directed graph and A is its adjacency matrix, what is the sum of the entries in the row corresponding to vertex x?

(d) If G is a graph with vertices $v_1, \ldots, v_n$, and with adjacency matrix A, and if the entry in row 6 and column 8 of A^{19} is 54, what does this tell us about G?

2. [11 marks]

(a) For any graph G = (V, E), state the Handshaking Theorem which relates the degrees and E.

(b) Show how the formula of (a) simplifies when every vertex of G has degree q, where q is a nonnegative integer.

(c) How many edges does the complete graph K_n on n vertices have? ______________

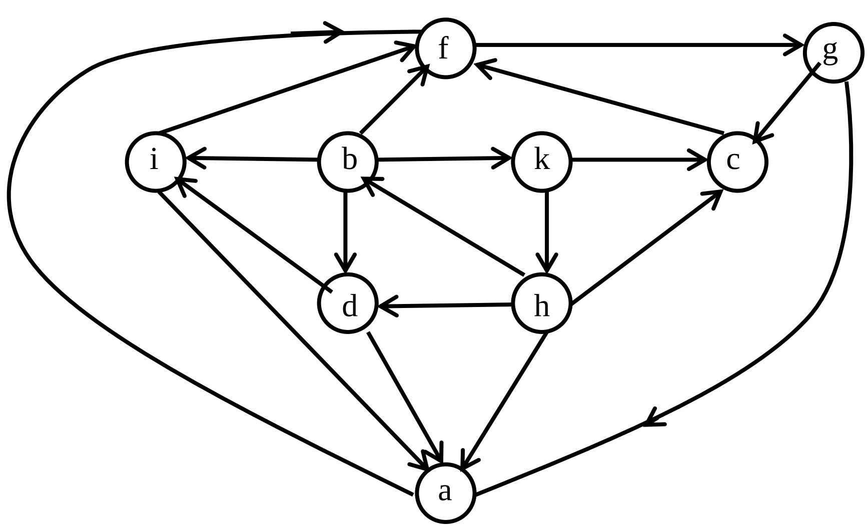
(d) Use (a) or (b) to show that your answer to (c) is correct.

(e) A graph G is self-complementary if G is isomorphic to its complement $\bar{G}$.
How many edges does a self-complementary graph have? ____________ Explain your answer.

(f) Draw a self-complementary graph with 4 vertices and one with 5 vertices. Draw the complement of each of your graphs, using the same vertex labels in both the graph and its complement, so it is clear that one is the complement of the other.

3. [4 marks]

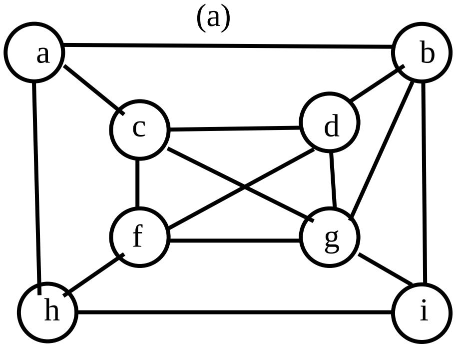
Draw the strongly connected components of the graph below.



4. [14 marks]
Recall that $K_{m,n}$ denotes the complete bipartite graph with a set B of m black vertices and a set W of n white vertices, and an edge joining each black vertex to each white vertex.
(a) Draw $K_{3,4}$ and $K_{4,4}$.
(b) What are the degrees of vertices in $K_{m,n}$?
(c) Complete the following:
   (i) A graph G = (V,E) has an Euler circuit if and only if _________
   (ii) A graph G = (V,E) has an Euler trail but no Euler circuit if and only if __________
(d) Determine the values of m and n for which $K_{m,n}$ has the following structure. Explain your answers briefly.
   (i) a Hamilton cycle
   (ii) an Euler circuit
   (ii) an Euler trail but no Euler circuit

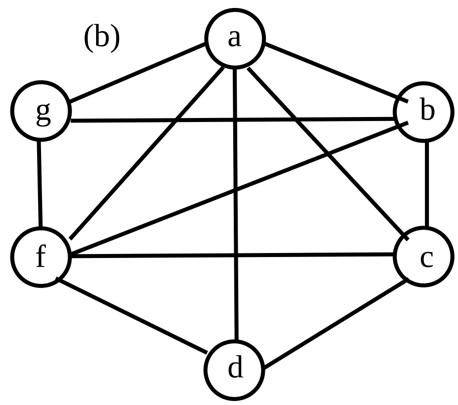
5. [6 marks]
For each of the following graphs, determine whether or not it is planar.
If it is planar, give a planar representation.
If it is not planar, explain why it is not planar.

(a)



G

(b)



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6. [6 marks]
Model the following as a problem on a graph:
   22 women are attending a banquet hosted by the Laurier Centre for Women in Science.
   Each woman knows 11 others in the group.
   Is it possible to seat all the women at a large round table so that each woman knows the two women she is seated between?

Give your answer by filling out the following information:
Construct a graph or directed graph (*write the correct one*) as follows:
The vertices are ____________________________
There is an edge fron vertex x to vertex y exactly when ____________________
Now state a theorem that you are using.
Theorem:____
Now briefly explain how the theorem proves that the appropriate seating arrangement exists or does not exist.

7. [8 marks]

(a) In class we proved the following:

Lemma: Every simple planar graph has a vertex x with degree(x) $\leq$ 5.

Use the lemma and simple induction to prove that

If G = (V, E) is a simple planar graph, then G can be coloured with 6 colours.

Give your proof by filling in the blanks below and then completing the proof.

Proof (by induction on the number of vertices).

Base Case: ____-

Induction Hypothesis:____

Now complete the proof.

(b) Is 6 the best upper bound we know for the number of colours required to colour a simple planar graph? Explain your answer.

8. [9 marks]

In this question, G = (V, E) is a simple planar graph, and R is the set of regions in a planar representation of G.

(a) Assuming G is connected, state Euler's Formula for G.

(b) State a generalization of Euler's Formula to planar graphs G with k connected components.

(c) In class we proved:

Corollary: If G = (V, E) is a simple planar graph with at least 3 vertices, then |E| $\leq$ 3|V| - 6.

Use the corollary to show that K_5 is not planar.

(d) Prove the corollary as follows.

Copy out what is written below onto your paper and complete it by filling in the blanks and then finishing the argument.

Proof. Since G = (V, E) is a simple connected planar graph with at least 3 edges, each region F in a planar representation of G is bounded by at least _______edges.

So

$$\sum_{F \in R} (number\ of\ edges\ in\ the\ boundary\ of\ region\ F) \geq \_\_\_\_\_\_\_\_$$

Also

$$\sum_{F \in R} (number\ of\ edges\ in\ the\ boundary\ of\ region\ F) = 2|E|$$

since ____

Now use Euler's Formula and some algebra to complete the proof.