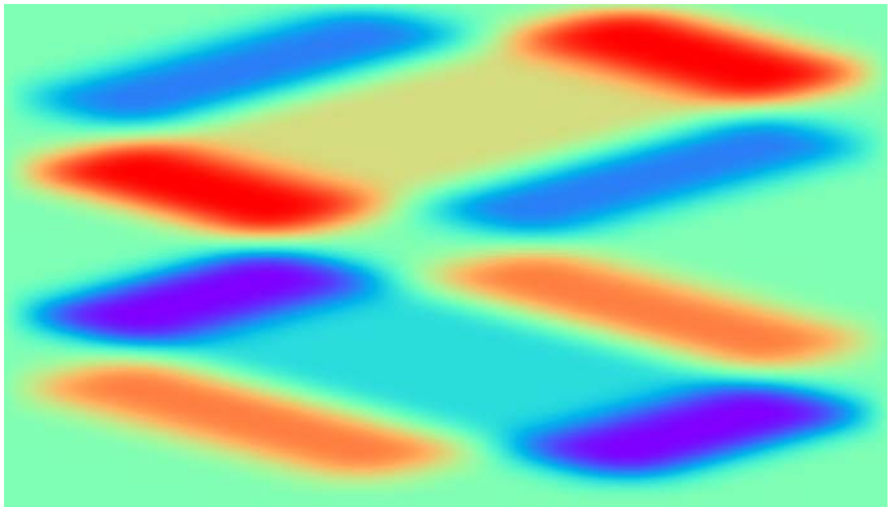
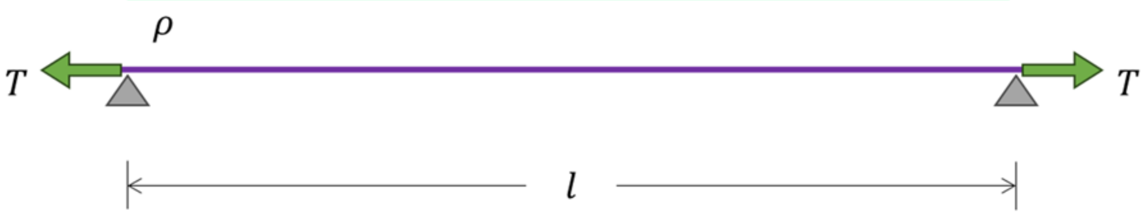




VIBRATION OF STRING IN TENSION

Samson Mano
saminnx@gmail.com
https://sites.google.com/site/samsoninfinite/
https://github.com/Samson-Mano

Wave Equation

The vibration of a string in tension is governed by the one-dimensional wave equation.

	$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$	1.0
	Where, c the speed of waves is given by, $$c = \sqrt{\frac{T}{\rho}}$$	2.0
	$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$	3.0
	where, $$0 \leq x \leq L$$ $$t \geq 0$$	
	End Boundary condition, $$y(0,t) = y(L,t) = 0 \ \forall \ t \geq 0$$	
	Initial Boundary condition, $$y(x,0) = f(x)$$ $$\frac{\partial y}{\partial x}(x,0) = g(x)$$	

Wave equation with an external force

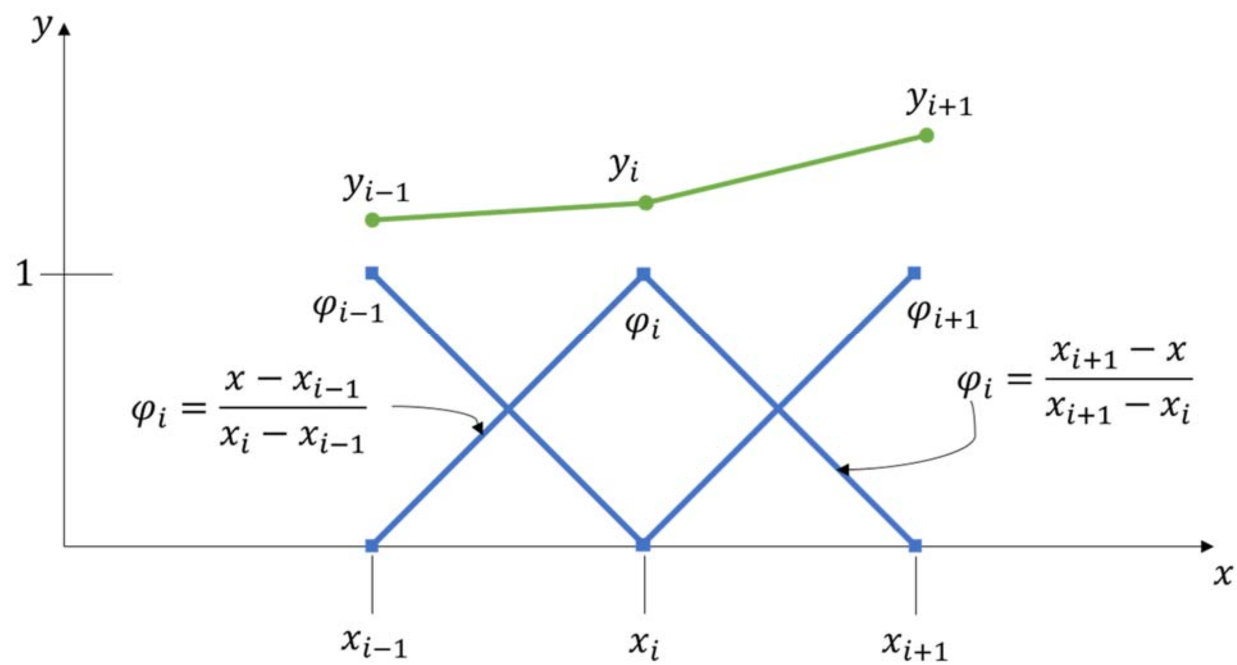
	$$\rho\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = \mathrm{F}(x,t)$$	1.0
	Where, $\mathrm{F}(x,t)$ is the external excitation force.	

Galerkin weighted residual method.

Galrekin method is a member of the larger class of methods known as the methods of weighted residuals. The weighted residual method is the method of converting a partial differential equation into a system of ordinary differential equation.

	For wave equation $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$ with exact solution $y(x,t)$.	
	Let $\tilde{y}(x,t)$ be an approximate solution, the residual R is the deviation from the exact solution. $$R(x,t) = \frac{\partial^2 \tilde{y}(x,t)}{\partial t^2} - c^2 \frac{\partial^2 \tilde{y}(x,t)}{\partial x^2}$$	
	Galerkin method choose a family of weighting function in evaluating the residual. The weighted average of the approximate solution should be zero, $$\int R(x,t) w_i(x) dx = 0$$	
	A good approximate solution to make the residual small is to consider x and t to be discrete variables. Then the approximate solution is of the form $$\tilde{y}(x,t) = \sum_i a_i(t)\varphi_i(x)$$	
	In Galerkin method the common chose for the weighting function is the basic function itself. This is called Bubnov-Galerkin method or sometimes just the Galerkin method. $$w_i(x) = \frac{\partial \tilde{y}(x,t)}{\partial a_i(t)} = \varphi_i(x)$$	

	Fourier-Galerkin method is a method of using Fourier expansion as the basis function.	

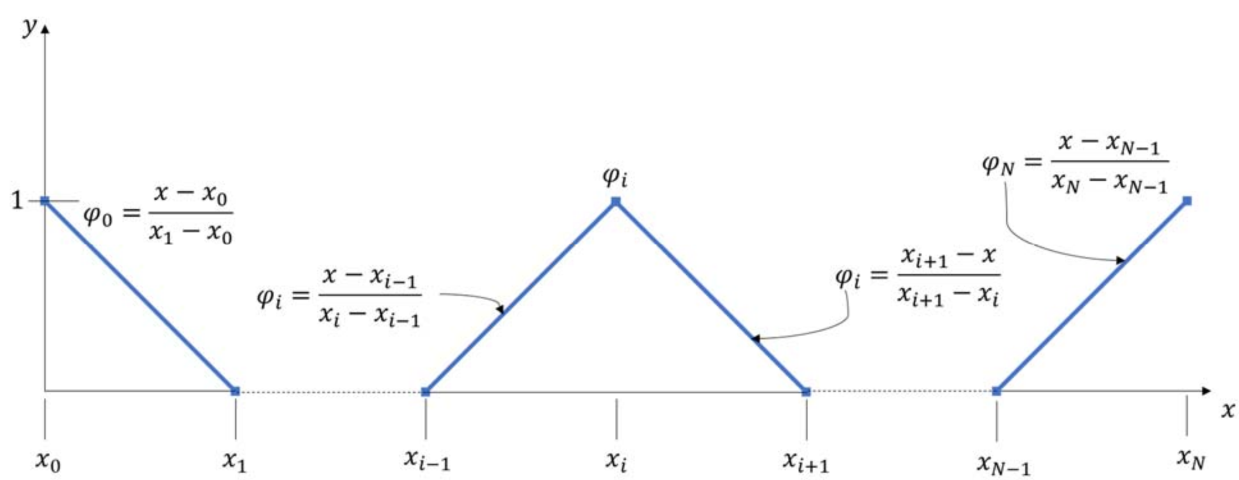


Linear Interpolation function

	$R(x,t) = \frac{\partial^2 \tilde{y}(x,t)}{\partial t^2} - c^2 \frac{\partial^2 \tilde{y}(x,t)}{\partial x^2}$	
	Applying the discrete sum of approximate solution $\tilde{y}(x,t) = \sum_i a_i(t)\varphi_i(x)$	
	$R(x,t) = \sum_i \frac{\partial^2 (a_i(t)\varphi_i(x))}{\partial t^2} - c^2 \frac{\partial^2 (a_i(t)\varphi_i(x))}{\partial x^2}$	

	$R(x,t) = \sum_i \varphi_i(x) \frac{d^2 a_i(t)}{dt^2} - c^2 a_i(t) \frac{d^2 \varphi_i(x)}{dx^2}$	
	Applying the above equation in the Galerkin expansion $\sum_i \int_0^1 \left(\varphi_i(x) \frac{d^2 a_i(t)}{dt^2} - c^2 a_i(t) \frac{d^2 \varphi_i(x)}{dx^2} \right) w_j(x) dx = 0$	
	$\sum_i \int_0^1 \left(\varphi_i(x) \varphi_j(x) \frac{d^2 a_i(t)}{dt^2} - c^2 a_i(t) \varphi_j(x) \frac{d^2 \varphi_i(x)}{dx^2} \right) dx = 0$	
	$\sum_i \frac{d^2 a_i(t)}{dt^2} \int_0^1 \varphi_i(x) \varphi_j(x) dx + \sum_i a_i(t) \int_0^1 \left(-c^2 \varphi_j(x) \frac{d^2 \varphi_i(x)}{dx^2} \right) dx = 0$	
	The second term can be integrated by parts, $\sum_i \frac{d^2 a_i(t)}{dt^2} \int_0^1 \varphi_i(x) \varphi_j(x) dx + \sum_i a_i(t) \int_0^1 \left(c^2 \frac{d\varphi_i(x)}{dx} \frac{d\varphi_j(x)}{dx} \right) dx - c^2 \varphi_j(x) \frac{d\varphi_i(x)}{dx} \Big\|_0^1 = 0$	
	Simplifying the above equation $\sum_i \frac{d^2 a_i}{dt^2} \int_0^1 \varphi_i \varphi_j dx + \sum_i a_i \int_0^1 \left(c^2 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \right) dx - c^2 \varphi_j \frac{d\varphi_i}{dx} \Big\|_0^1 = 0$	

	$A_{ij}\frac{d^2a}{dt^2}_i + B_{ij}a_i = 0$	
	where, $A_{ij} = \int_0^1 \varphi_i\varphi_j dx$	
	$B_{ij} = \int_0^1 \left(c^2\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\right)dx - c^2\varphi_j\frac{d\varphi_i}{dx}\Big\vert_0^1$	



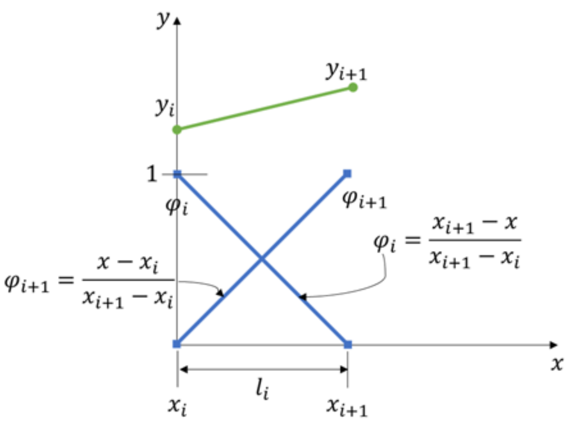
Piecewise linear basis interpolation function

	The piecewise linear basis function can be written as $\varphi_i(x) = \begin{cases} \frac{x - x_i}{x_i - x_{i-1}} \ \forall\, x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} \ \forall\, x_i \le x \le x_{i+1} \end{cases}$ for i =1,2,3 …… N-1	
	The functions for the ends are,	

	$$\varphi_0(x) = \frac{x_1 - x}{x_1 - x_0}\ \forall\, x_0 \le x \le x_1$$ $$\varphi_N(x) = \frac{x - x_N}{x_N - x_{N-1}}\ \forall\, x_{N-1} \le x \le x_N$$	

Finite element formulation of wave equation

	$$y_i(x) = \varphi_i(x) y_i + \varphi_{i+1}(x) y_{i+1}$$	Eq. C3
	Where $\varphi_i(x)$ and φ_{i+1} are the shape functions	



Shape function for a single element

	$$A_{ij} = \int_0^1 \varphi_i \varphi_j dx$$	
	For a single element, $$[A]_{i,i+1} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{(x_{i+1} - x)^2}{l^2} & \frac{(x_{i+1} - x)(x - x_i)}{l^2} \\ \frac{(x - x_i)(x_{i+1} - x)}{l^2} & \frac{(x - x_i)^2}{l^2} \end{bmatrix} dx$$	

	$[A] = \begin{bmatrix} \frac{\int_{x_i}^{x_{i+1}} (x_{i+1}^2 - 2x_{i+1}x + x^2)dx}{l^2} & \frac{\int_{x_i}^{x_{i+1}} (x_{i+1}x - x_{i+1}x_i - x^2 + xx_i)dx}{l^2} \\ sym & \frac{\int_{x_i}^{x_{i+1}} (x^2 - 2x_i x + x_i^2)dx}{l^2} \end{bmatrix}$	
	$[A_{0,0}] = \frac{1}{l^2}\int_0^l (l^2 - 2lx + x^2)dx$ $= \frac{1}{l^2}\left(l^2x - lx^2 + \frac{x^3}{3}\right)_0^l = \frac{l}{3}$	
	$[A_{0,1}] = \frac{1}{l^2}\int_0^l (lx - x^2)dx$ $= \frac{1}{l^2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right)_0^l = \frac{l}{6}$	
	$[A_{1,1}] = \frac{1}{l^2}\int_0^l x^2dx$ $= \frac{1}{l^2}\left(\frac{x^3}{3}\right)_0^l = \frac{l}{3}$	
	The A matrix for single element is given by $[A] = \begin{bmatrix} \frac{l}{3} & \frac{l}{6} \\ \frac{l}{6} & \frac{1}{3} \end{bmatrix}$	
	Similarly	

	$$B_{ij} = \int_0^1 \left(c^2 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \right) dx - c^2 \varphi_j \frac{d\varphi_i}{dx} \Big|_0^1$$	
	$$[B]_{i,i+1} = \int_{x_i}^{x_{i+1}} c^2 \begin{bmatrix} \dfrac{\frac{d(x_{i+1}-x)}{dx}\frac{d(x_{i+1}-x)}{dx}}{l^2} & \dfrac{\frac{d(x_{i+1}-x)}{dx}\frac{d(x-x_i)}{dx}}{l^2} \\ \dfrac{\frac{d(x-x_i)}{dx}\frac{d(x_{i+1}-x)}{dx}}{l^2} & \dfrac{\frac{d(x-x_i)}{dx}\frac{d(x-x_i)}{dx}}{l^2} \end{bmatrix} dx - c^2 \left(\begin{bmatrix} \frac{(x_{i+1}-x)}{l} & \frac{(x_{i+1}-x)}{l} \\ \frac{(x-x_i)}{l} & \frac{(x-x_i)}{l} \end{bmatrix} \begin{bmatrix} \dfrac{\frac{d(x_{i+1}-x)}{dx}}{l} & \dfrac{\frac{d(x_{i+1}-x)}{dx}}{l} \\ \dfrac{\frac{d(x-x_i)}{dx}}{l} & \dfrac{\frac{d(x-x_i)}{dx}}{l} \end{bmatrix} \right)_0^1$$	
	$$[B]_{i,i+1} = \int_{x_i}^{x_{i+1}} c^2 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} dx - c^2 \left(\begin{bmatrix} \frac{(x_{i+1}-x)}{l} & \frac{(x_{i+1}-x)}{l} \\ \frac{(x-x_i)}{l} & \frac{(x-x_i)}{l} \end{bmatrix} \begin{bmatrix} \frac{-1}{l} & \frac{-1}{l} \\ \frac{1}{l} & \frac{1}{l} \end{bmatrix} \right)_0^1$$	
	$$[B]_{i,i+1} = \int_{x_i}^{x_{i+1}} c^2 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} dx$$	
	The B matrix for single element is given by $$[B] = c^2 \begin{bmatrix} \frac{1}{l} & \frac{-1}{l} \\ \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$	
	Substituting in the original equation	

	$$A_{ij}\frac{d^2a}{dt^2}_i + B_{ij}a_i = 0$$	
	For a single element, $$\begin{bmatrix}\frac{l}{3} & \frac{l}{6}\\ \frac{l}{6} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\frac{d^2a}{dt^2}_i\\ \frac{d^2a}{dt^2}_{i+1}\end{bmatrix} + c^2\begin{bmatrix}\frac{1}{l} & \frac{-1}{l}\\ \frac{-1}{l} & \frac{1}{l}\end{bmatrix}\begin{bmatrix}a_i\\ a_{i+1}\end{bmatrix} = 0$$	

Modal Superposition Solution of String in Tension

	The natural frequency and mode shapes of string in tension is well known. $$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$$	
	The radian frequency of the nth normal mode is given by. $$\omega_n = \frac{n\pi c}{L}$$	
	The mode shape of the nth normal mode is given by. $$\phi_n = \sin\left(\frac{n\pi x}{L}\right)$$	

Generalized Solution

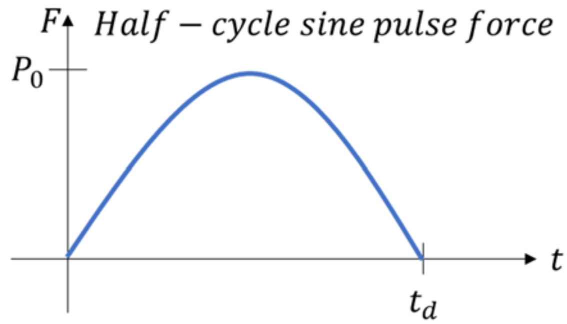
	The solution for the partial differential wave equation can be obtained by the modal superimposed solution of the ordinary differential equation of individual nodes. $$\ddot{a}_i + \omega_n^2 a_i = 0$$	
	The solution is given by. $$y_i(x,t) = \sum_{i=1}^{N} \phi_i \, a_i(t)$$	
	Let the following be the modal matrix, where the column matrix contains the individual modes shapes. $$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{pmatrix}$$	
	$$y(x,t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_N(t) \end{pmatrix}$$	
	Initial Boundary condition, $$y(x,0) = f(x)$$ $$\frac{\partial y}{\partial x}(x,0) = g(x)$$	
	Transform the initial condition to the modal co-ordinates, $$a(x,0) = \Phi^{-1} f(x)$$ $$\frac{\partial a}{\partial x}(x,0) = \Phi^{-1} g(x)$$	

	The solution to the initial condition is, $$a(t)_i = a_0 \cos \omega_n t + \frac{\partial a}{\partial x_0} \frac{1}{\omega_n} \sin \omega_n t$$	

Forced Response Solution

	$$\rho \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = \mathrm{F}(x,t)$$	1.0
	Where, $\mathrm{F}(x,t)$ is the external excitation force.	
	$$\mathrm{y} = [\Phi][a]$$	
	$$\frac{\partial^2 y}{\partial t^2} = [\Phi] \frac{d^2 a}{dt^2}$$	2.0
	$$\frac{\partial^2 y}{\partial x^2} = -\frac{n^2 \pi^2}{L^2} [\Phi][a]$$	3.0
	Substitute in the PDE gives, $$\rho[\Phi] \frac{d^2 a}{dt^2} + T \frac{n^2 \pi^2}{L^2} [\Phi] a = \mathrm{F}(x,t)$$	
	Divide by $\rho[\Phi]$ gives, $$\frac{d^2 a}{dt^2} + \frac{n^2 \pi^2 c^2}{L^2} a = \left(\frac{1}{\rho}\right) [\Phi]^{-1} \mathrm{F}(x,t)$$	
	The equation becomes (for a particular mode), $$\frac{d^2 a}{dt^2}_i + \omega_i^2 a_i = f(t)_i$$	

1) Solution to Half – Sine pulse force



The solution to half – Sine pulse force of form as shown below.

$$\mathrm{F}(x,t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_N(t) \end{pmatrix} sin(\omega_d t)$$

Where ω_d is the angular frequency of the pulse force time period t_d.

After modal transformation

$$f(t)_i = f_i sin(\omega_d t)$$

Where f_i is the modal amplitude.

Now, the modal solution a_i is given by,

$$a_i(t)_{t\le t_d} = \frac{f_i}{\omega_i^2}\left(\frac{\left(\sin\left(\frac{\pi t}{t_d}\right) - \frac{\pi}{\omega_i t_d}\sin(\omega_i t)\right)}{\left(1 - \frac{\pi^2}{\omega_i^2 t_d^2}\right)}\right)$$

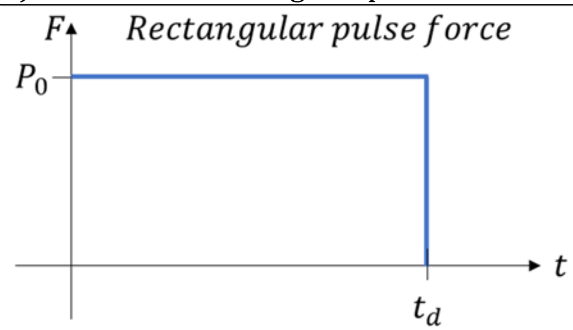
$$a_i(t)_{t>t_d} = \frac{f_i}{\omega_i^2}\frac{\left(\frac{2\pi}{\omega_i t_d}\right)}{\left(\frac{\pi^2}{\omega_i^2 t_d^2} - 1\right)}\left(\sin\left[\omega_i\left(t - \frac{t_d}{2}\right)\right]\cos\left(\frac{\omega_i t_d}{2}\right)\right)$$

The above solutions are not valid for $\frac{t_d}{T_n} = 0.5$ resonant case. Below solution is for the resonant case

$$a_i(t)_{t\le t_d} = \frac{f_i}{2\omega_i^2}\left(\sin(\omega_i t) - \omega_i t\cos(\omega_i t)\right)$$

$$a_i(t)_{t>t_d} = \frac{f_i \pi}{2\omega_i^2}\left(\cos(\omega_i t - \pi)\right)$$

2) Solution to Rectangular pulse force

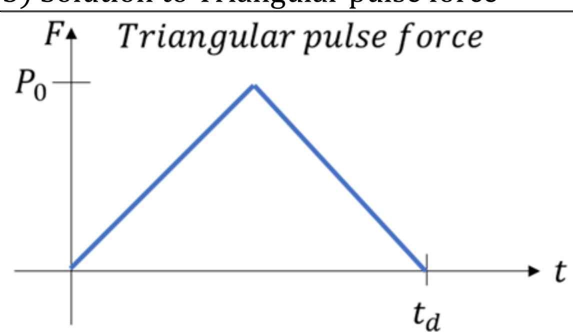


The modal solution a_i is given by,

$$a_i(t)_{t\le t_d} = \frac{f_i}{\omega_i^2}\left[1 - \cos\omega_i t\right]$$

$$a_i(t)_{t>t_d} = \frac{f_i}{\omega_i^2}\left(\cos\omega_i(t - t_d) - \cos\omega_i t\right)$$

3) Solution to Triangular pulse force



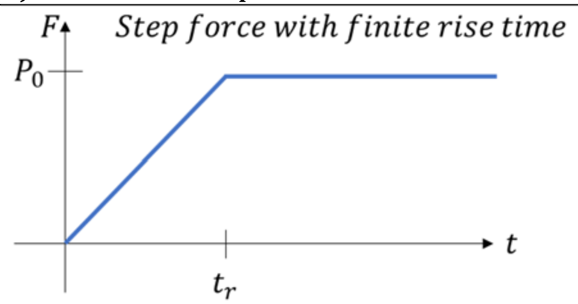
The modal solution a_i is given by,

$$a_i(t)_{t\le\frac{t_d}{2}} = \frac{2f_i}{\omega_i^2}\left[\frac{t}{t_d} - \frac{\sin\omega_i t}{t_d\omega_i}\right]$$

$$a_i(t)_{t_d>t>\frac{t_d}{2}} = \frac{2f_i}{\omega_i^2}\left(1 - \frac{t}{t_d} + \frac{1}{t_d\omega_i}\left[2\sin\omega_i\left(t - \frac{t_d}{2}\right) - \sin\omega_i t\right]\right)$$

$$a_i(t)_{t>t_d} = \frac{2f_i}{\omega_i^2}\left(\frac{1}{t_d\omega_i}\left(2\sin\omega_i\left(t-\frac{t_d}{2}\right) - \sin[\omega_i(t-t_d)] - \sin\omega_i t\right)\right)$$

4) Solution to Step Force with finite rise

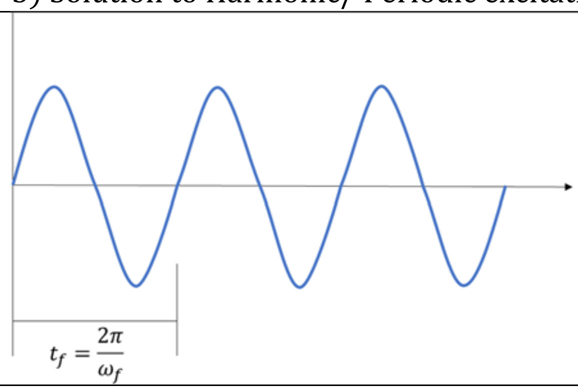


The modal solution a_i is given by,

$$a_i(t)_{t\le t_d} = \frac{f_i}{\omega_i^2}\left[\frac{t}{t_d} - \frac{\sin\omega_i t}{t_d\omega_i}\right]$$

$$a_i(t)_{t>t_d} = \frac{f_i}{\omega_i^2}\left(1 + \frac{1}{\omega_i t_d}[\sin\omega_i(t-t_d) - \sin\omega_i t]\right)$$

5) Solution to Harmonic/ Periodic excitation



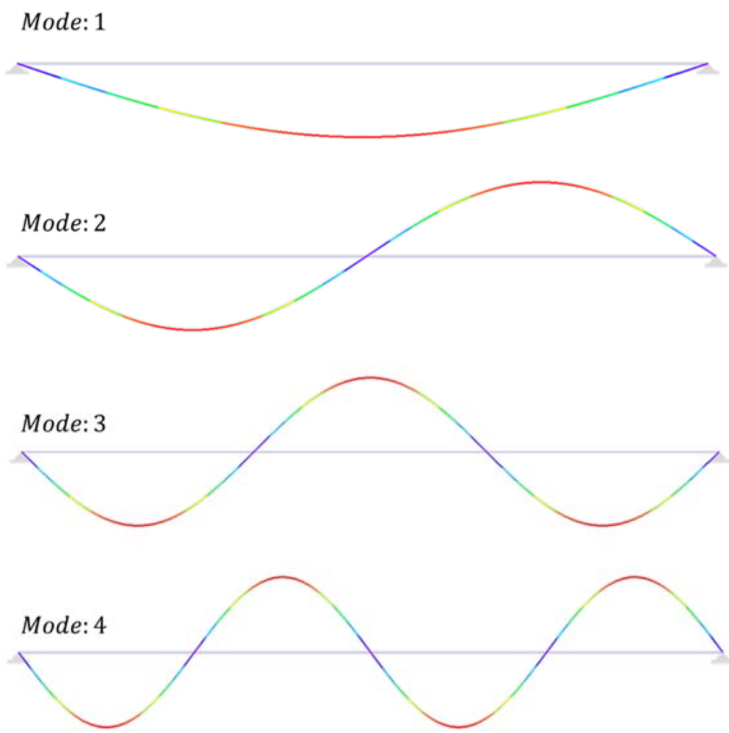
The modal solution a_i is given by,

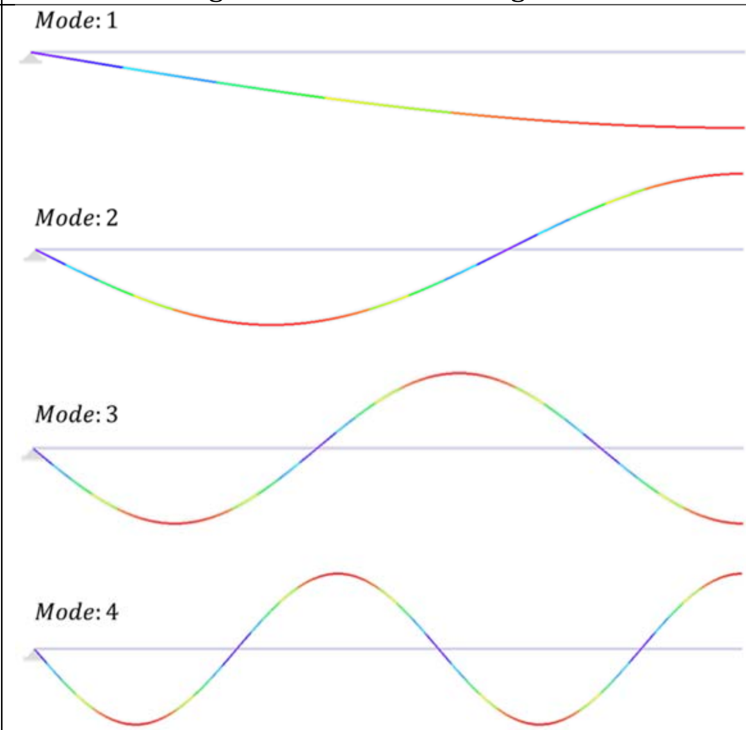
$$a_i(t) = \frac{f_i}{\omega_i^2}\left[\frac{1}{1-\left({}^{\omega_f}/_{\omega_i}\right)}\right]\left(\sin(\omega_f t) - \left({}^{\omega_f}/_{\omega_i}\right)\sin(\omega_i t)\right)$$

Resonant modal solution when $\left({}^{\omega_f}/_{\omega_i}\right) = 1.0$

	$$a_i(t) = \frac{f_i}{2\omega_i^2}\left(\omega_i t \cos(\omega_i t) - \sin(\omega_i t)\right)$$	

Natural frequency and mode shapes of various configurations

	Model 1: String in Fixed – Fixed Configuration	
	 	
	Angular natural frequency $$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} = \frac{\pi c}{L}\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ N \end{pmatrix}$$	
	Mode shape $$\Phi = \sin\frac{n\pi x}{L} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \sin\frac{\pi x}{L} & \sin\frac{2\pi x}{L} & \cdots & \sin\frac{N\pi x}{L} \\ \vdots & \vdots & \ddots & \vdots \\ \sin\frac{\pi L}{L} & \sin\frac{2\pi L}{L} & \cdots & \sin\frac{N\pi L}{L} \end{pmatrix}$$	

	Mode shape inverse $\Phi^{-1} = \frac{2}{(N+1)} \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{pmatrix}$	
	Model 2: String in Fixed – Free Configuration	
		
	Angular natural frequency $\omega_n = \frac{(2n-1)\pi}{2L} \sqrt{\frac{T}{\rho}} = \frac{\pi c}{2L} \begin{pmatrix} 1 \\ 3 \\ 5 \\ \vdots \\ 2N-1 \end{pmatrix}$	
	Mode shape	

$$\Phi = \sin\frac{(2n-1)\pi x}{2L} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{pmatrix}$$

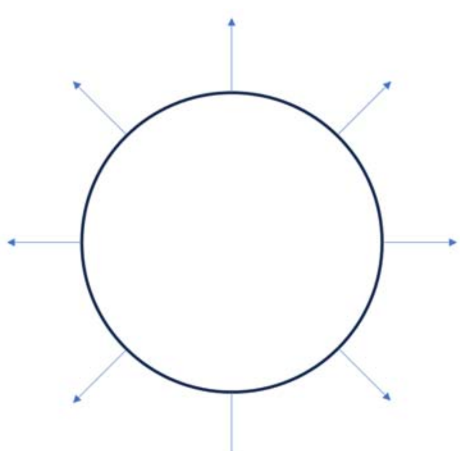
$$= \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \sin\frac{\pi x}{2L} & \sin\frac{3\pi x}{2L} & \cdots & \sin\frac{(2N-1)\pi x}{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \sin\frac{\pi L}{2L} & \sin\frac{3\pi L}{2L} & \cdots & \sin\frac{(2N-1)\pi L}{2L} \end{pmatrix}$$

Mode shape inverse

$$\Phi^{-1} = \frac{2}{(N+1)} \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \sin\frac{\pi(2x+1)}{2L} & \sin\frac{2\pi(2x+1)}{2L} & \cdots & 0.5\sin\frac{N\pi(2x+1)}{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \sin\frac{\pi(2L+1)}{2L} & \sin\frac{2\pi(2L+1)}{2L} & \cdots & 0.5\sin\frac{N\pi(2L+1)}{2L} \end{pmatrix}$$

Model 3: Circular String Free – Free Configuration

Physically achieve this configuration is tedious. Assume a circular string is spun around the center and the centrifugal force is acting as tension along the circular string trying to expand the string.
The other way is to imagine the string is kept circularly taut by springs normal to the string away from its center.
Both the case needs inertial force or spring force to keep the string in tension. This external force will be responsible for the wave motion of the circular string.



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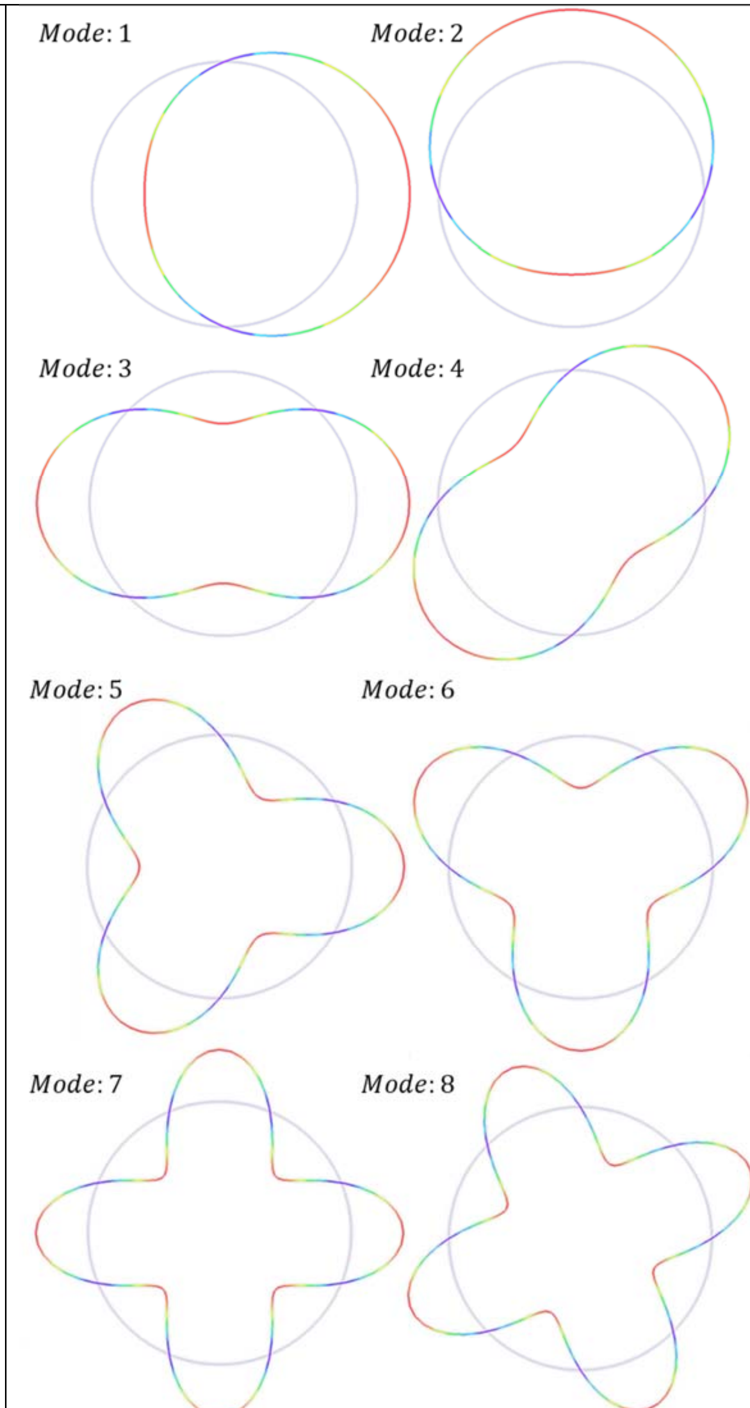
<tr><td></td><td>
</td><td></td></tr>
<tr><td></td><td>Angular natural frequency

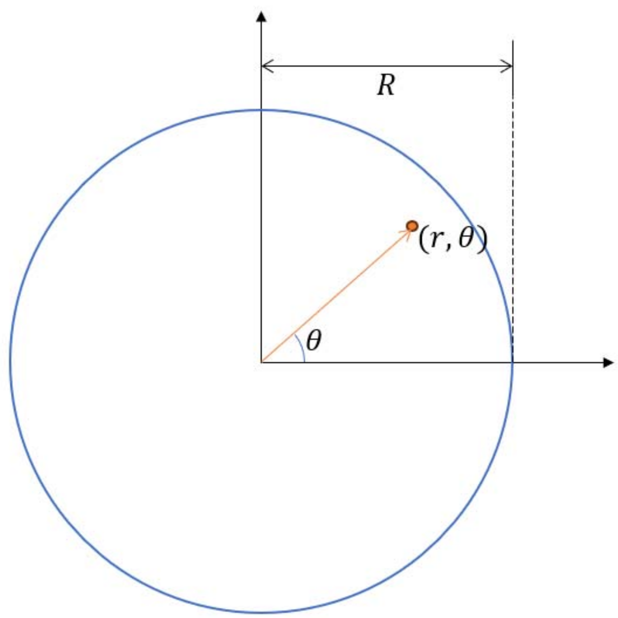
$$\omega_n = \begin{cases} \Phi_{2n-1} = \dfrac{n\pi}{L}\sqrt{\dfrac{T}{\rho}} \\ \\ \Phi_{2n} = \dfrac{n\pi}{L}\sqrt{\dfrac{T}{\rho}} \end{cases} = \frac{\pi c}{L}\begin{pmatrix}1\\1\\2\\2\\\vdots\\N\end{pmatrix}$$</td><td></td></tr>
</table>

Mode shape

$$\Phi = \begin{cases} \Phi_{2n-1} = r\cos(n\theta) \\ \Phi_{2n} = r\sin(n\theta) \end{cases} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{pmatrix}$$

$$= \begin{pmatrix} \vdots & \vdots & \vdots & \cdots & \vdots \\ r\cos\theta & r\sin\theta & r\cos 2\theta & \cdots & r\sin N\theta \ or \ r\cos N\theta \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

Model 4: Circular membrane vibration mode



For a circular membrane with radius R and a uniform radial tensile force T, the natural frequencies are,

$$\omega_{mn} = \frac{k_{mn}}{R}\sqrt{\frac{T}{\mu}}$$

The mode shape is given by the following relation, where Jm is the Bessel function of order m and kmn is the root of Bessel function.

$$\phi_{mn}(r,\theta) = J_m\left(k_{mn}\frac{r}{R}\right) * \cos m\theta$$

Bessel function Jm in integral form

$$J_m(x) = \frac{1}{\pi}\int_0^{\pi} cos(x sin\theta - m\theta)d\theta \,, x > 0, m \in Z$$

The roots of the Bessel functions Jm is given in the following table.

Zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1336	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.796	16.2235	17.616	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Numerical method for calculating the Bessel function.

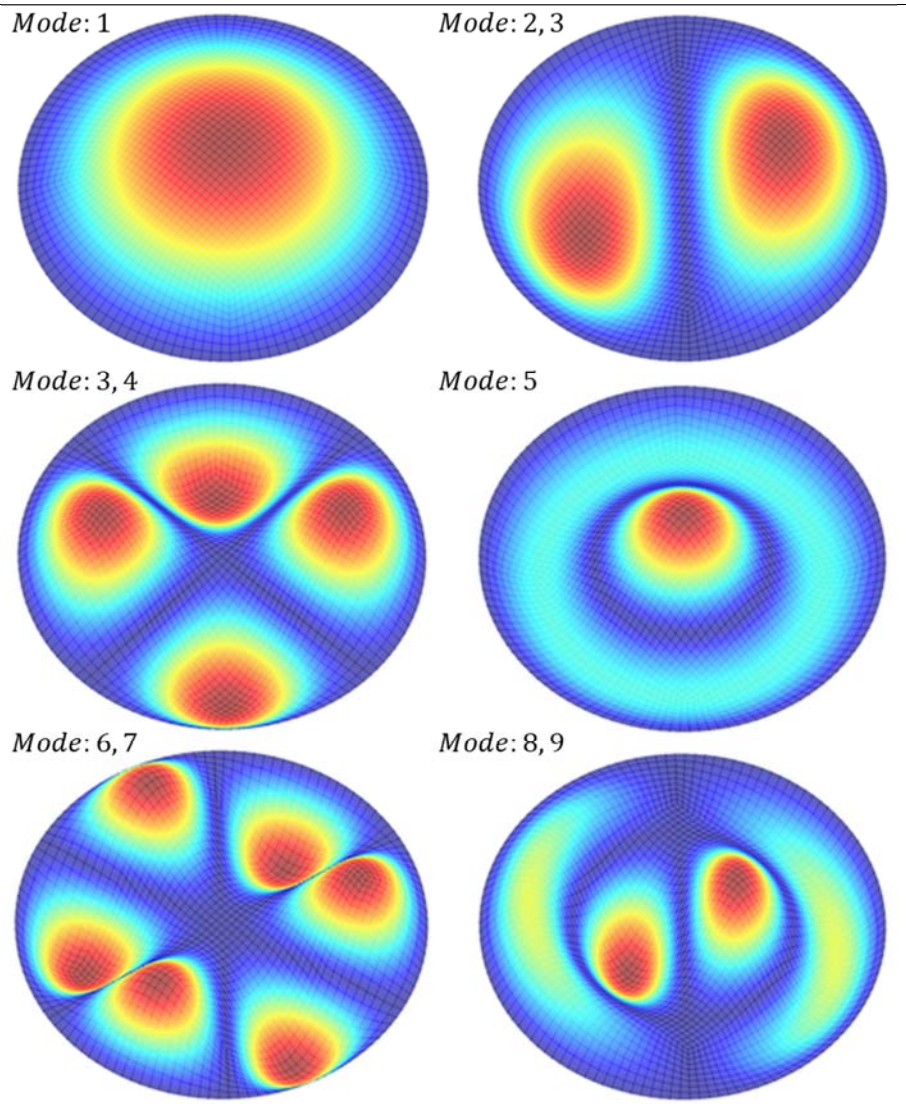
If m is odd

$$J_m(x) \approx \frac{1}{n}\sum_{k=0}^{n-1} \sin\left(x \sin\left(\frac{\pi}{2n}\left(k+\frac{1}{2}\right)\right)\right) \sin\left(\frac{m\pi}{2n}\left(k+\frac{1}{2}\right)\right)$$

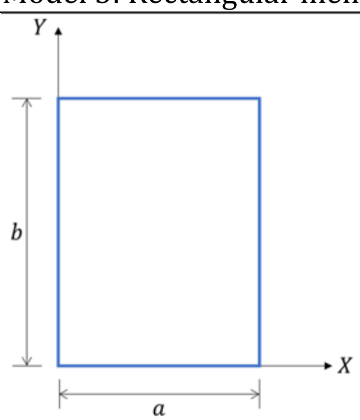
If m is even

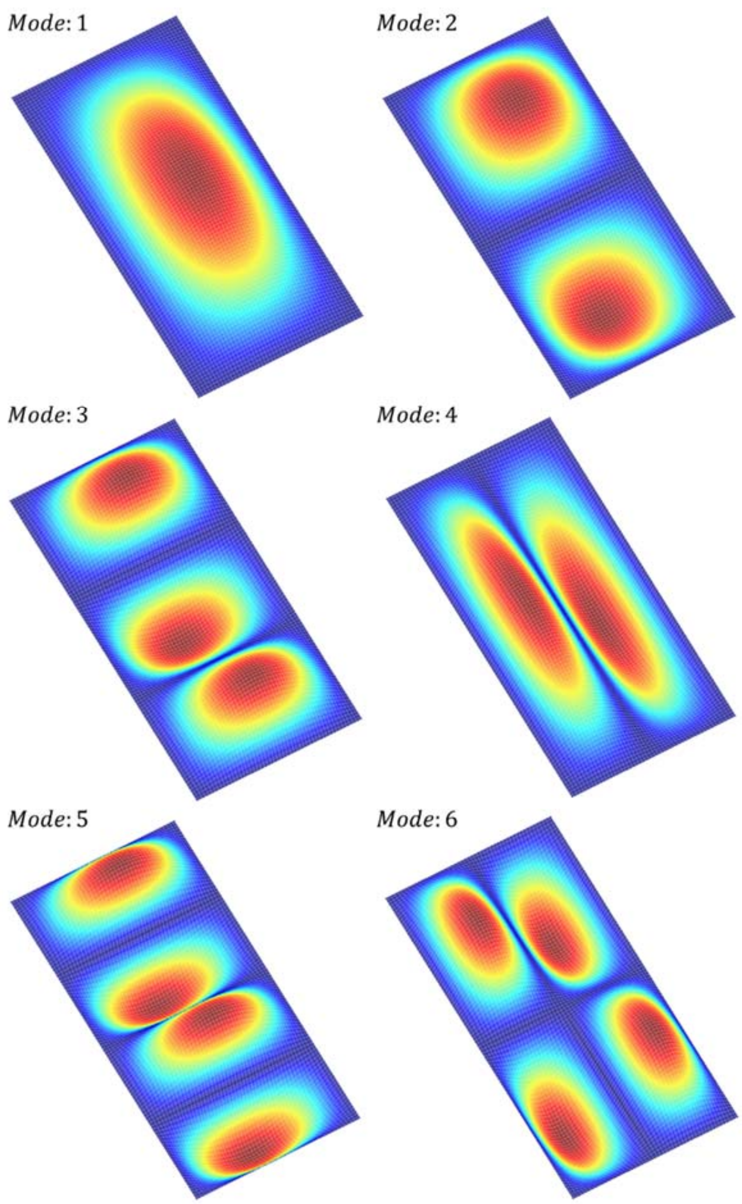
$$J_m(x) \approx \frac{1}{n}\sum_{k=0}^{n-1} \cos\left(x \cos\left(\frac{\pi}{2n}\left(k+\frac{1}{2}\right)\right)\right) \cos\left(\frac{m\pi}{2n}\left(k+\frac{1}{2}\right)\right)$$

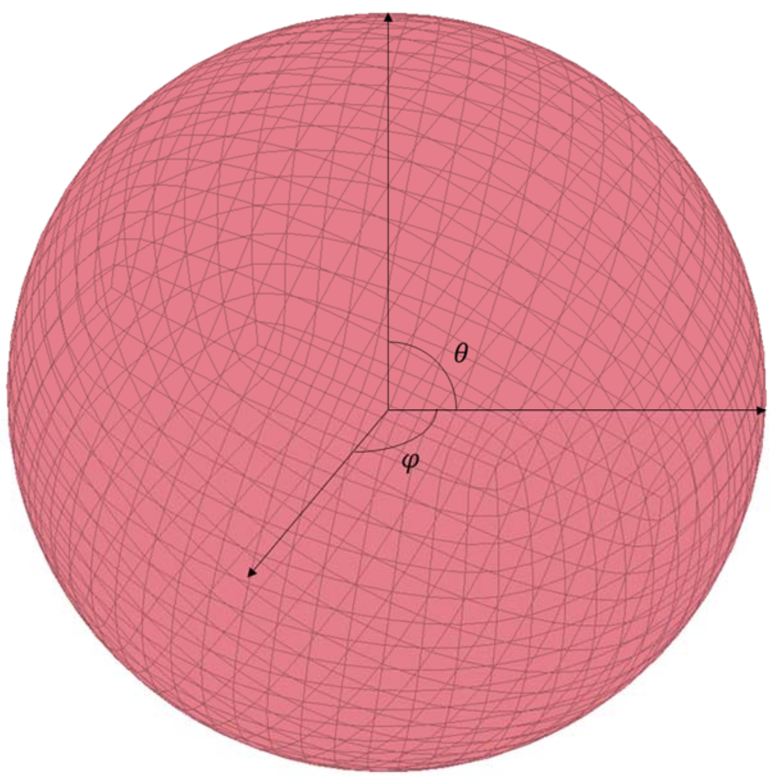
Mode shapes:



Model 5: Rectangular membrane vibration mode



	For a rectangular membrane with width a, height b and uniform rectangular tensile force T, the natural frequencies are, $$\omega_{mn} = \sqrt{\frac{T}{\mu}}\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$$	
	The mode shape is given by the following relation, $$\phi_{mn}(x,y) = \sin\left(\frac{\pi m}{a}x\right)\sin\left(\frac{\pi n}{b}y\right)$$	
	Mode shapes: 	

	Hypothesis: The inverse of the mode vector, denoted as $[\Phi]^{-1}$, can be represented as the product of a scalar factor and the transpose of the mode vector, expressed as $f^*[\Phi]^T$. $$[\Phi]^{-1} = f * [\Phi]^T$$ This finding holds significant utility as it reduces computation time, particularly within the context of 3D global models characterized by non-ordered nodes. In such scenarios, the eigen vectors matrix is also non-ordered. Consequently, attempting to invert a non-ordered eigen vectors matrix would yield erroneous results. Conversely, utilizing the transpose of the eigen vector ensures accurate calculations even in cases where the eigen vectors matrix lacks ordering.	
	Model 6: Spherical Harmonics	
		

	$$Y_m^l = \begin{cases} P_m^l(\cos\theta)\sin\lvert m\rvert\varphi\,, & m < 0 \\ P_m^l(\cos\theta)\cos m\varphi\,, & m \geq 0 \end{cases}$$	
	The first few Associated Legendre polynomials $P_m^l(\cos\theta)$ are	
	$$P_1^0 = \cos\theta$$ $$P_1^1 = -\sin\theta$$	
	$$P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$$ $$P_2^1 = -3\cos\theta\sin\theta$$ $$P_2^2 = 3\sin^2\theta$$	
	$$P_3^0 = \frac{1}{2}\cos\theta\,(5\cos^2\theta - 3)$$ $$P_3^1 = -\frac{3}{2}\sin\theta\,(5\cos^2\theta - 1)$$ $$P_3^2 = 15\cos\theta\sin^2\theta$$ $$P_3^3 = -15\sin^3\theta$$	