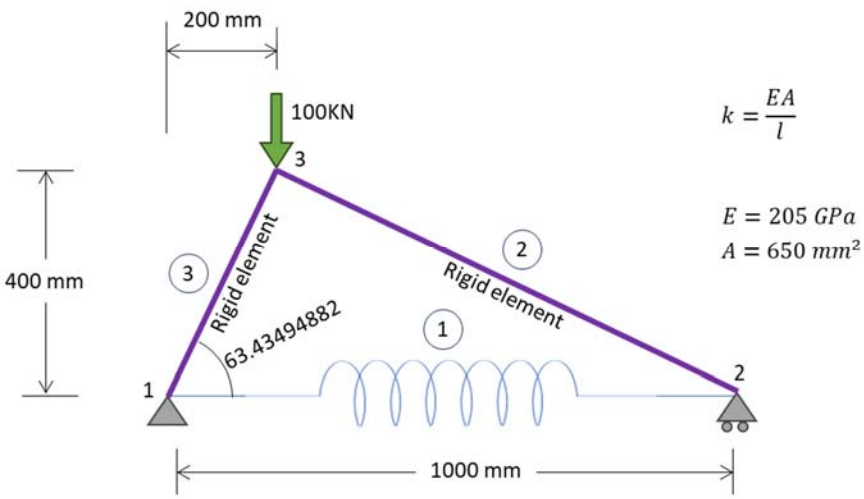

MULTI-POINT CONSTRAINTS

Formulation of rigid body elements

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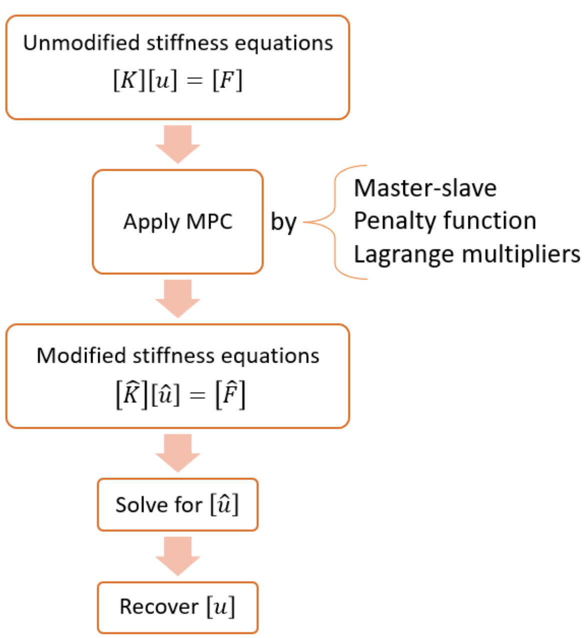
What is a Multipoint Constraint?

Multi-Point Constraints (MPCs) in finite element analysis encompass a unique category of elements designed to establish constraint relationships between multiple nodes. A single-point constraints sets a single degree of freedom to a known value (often zero) while a multipoint constraint (MPC) imposes a relationship among two or more degree of freedoms. These elements serve as an important tool for representing fundamental interactions, such as rigid and distributed connections within a structural system. The most common MPCs are, the RBE2 elements (kinematic couplings) and RBE3 elements (distributed couplings). In finite element analysis, MPCs rank as one of the most frequently used elements, in all sorts of engineering problems.

In my professional experience, the incorporation of MPCs has become nearly ubiquitous in finite element problem-solving, making them the fourth most prevalent finite element type, following shell, beam, and solid elements. Despite their widespread application, it is noteworthy that the detailed formulation of these multipoint constraints is often inadequately covered in mainstream textbooks. This document aims to bridge this gap by delving into the theoretical underpinnings and mathematical formulations of MPCs, shedding light on a critical aspect of finite element analysis.

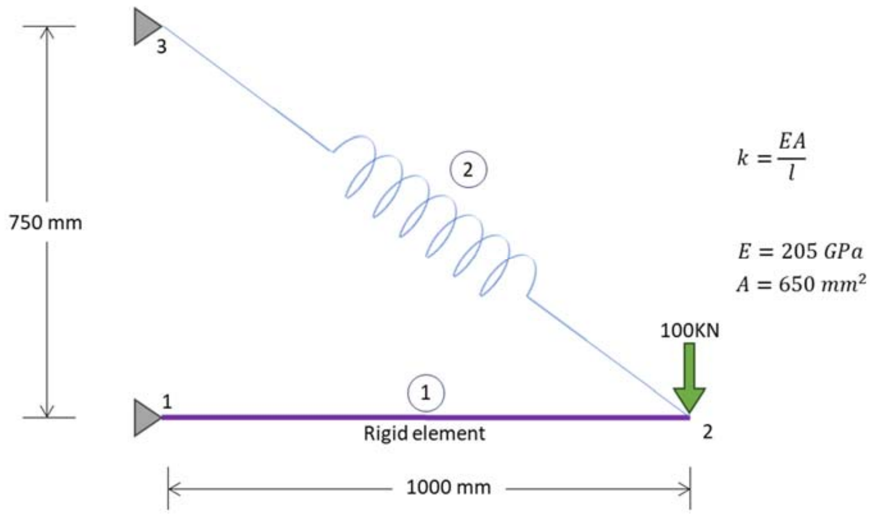
Implementation

The schematics of MPCs application is presented in the below figure (IFEM Couse Carlos A Felippa).



Example 1: Master – Slave MPC

Consider the below truss- model with MPCs.



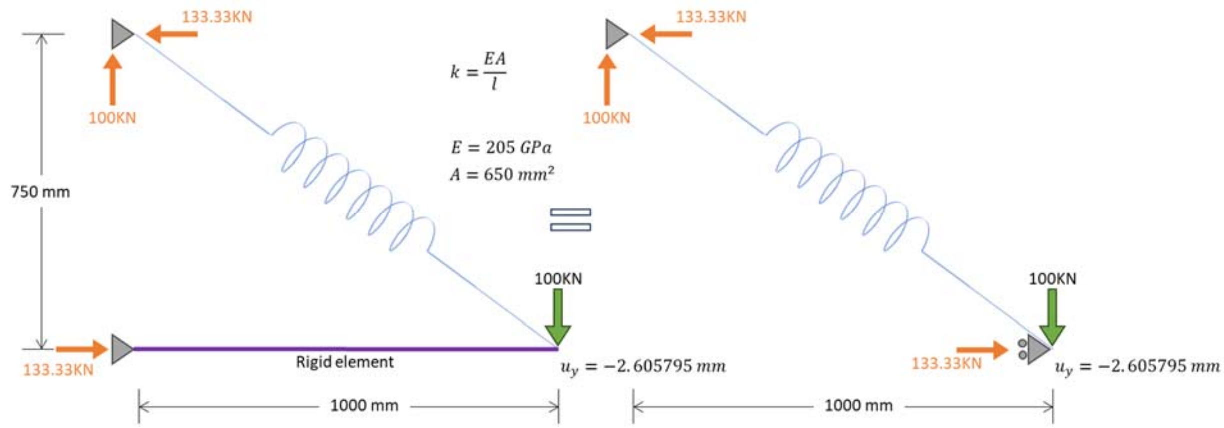
	Element stiffness matrix $$[K_e] = [L^T][k][L]$$	
	The direction cosines, $$l = \frac{x_2 - x_1}{l_e}, m = \frac{y_2 - y_1}{l_e}$$	
	Direction cosine transformation matrix, $$[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$	
	Element stiffness matrix $$[K_e] = [L^T][k][L] = \frac{EA}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$	1.0
	For element 1, $l = 1, m = 0$	

	$$[K_e] = \frac{EA}{l_e}\begin{bmatrix}1 & 0\\0 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$	
	$$[K_e] = k_1\begin{bmatrix}1 & 0 & -1 & 0\\0 & 0 & 0 & 0\\-1 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$	
	For element 2, $l = -0.8, m = 0.6$ $$[K_e] = \frac{EA}{1250}\begin{bmatrix}-0.8 & 0\\0.6 & 0\\0 & -0.8\\0 & 0.6\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}-0.8 & 0.6 & 0 & 0\\0 & 0 & -0.8 & 0.6\end{bmatrix}$$	
	$$[K_e] = k_2\begin{bmatrix}0.64 & -0.48 & -0.64 & 0.48\\-0.48 & 0.36 & 0.48 & -0.36\\-0.64 & 0.48 & 0.64 & -0.48\\0.48 & -0.36 & -0.48 & 0.36\end{bmatrix}$$	
	Global stiffness matrix $$[K_g] = \begin{bmatrix}k_1 & 0 & -k_1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0\\-k_1 & 0 & k_1 + 0.64k_2 & -0.48k_2 & -0.64k_2 & 0.48k_2\\0 & 0 & -0.48k_2 & 0.36k_2 & 0.48k_2 & -0.36k_2\\0 & 0 & -0.64k_2 & 0.48k_2 & 0.64k_2 & -0.48k_2\\0 & 0 & 0.48k_2 & -0.36k_2 & -0.48k_2 & 0.36k_2\end{bmatrix}$$	
	$$[u] = \begin{bmatrix}u_{1x}\\u_{1y}\\u_{2x}\\u_{2y}\\u_{3x}\\u_{3y}\end{bmatrix}$$	

	$$[F] = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ -100 \\ 0.0 \\ 0.0 \end{bmatrix}$$	
	The unmodified stiffness equation is given by $$[K][u] = [F]$$	2.0
	The idea behind applying the Multi Point Constraint by master – slave method is to establish a relationship between the displacements of selected nodes in the model, designating one set of nodes as "master" and another set as "slave." The master nodes dictate the behavior of the slave nodes through a transformation matrix, ensuring that the displacements are consistent with the imposed constraints. In this example, Node 2 and Node 1 are rigidly connected through Rigid Element 1, resulting in an equal x displacement between them.	
	$$[u] = [T][\hat{u}]$$	3.0
	Applying the transformation matrix to the transfer the x displacement through rigid element 1 gives $$\begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix}$$	
	Substituting Eqn 3.0 in the global stiffness equation Eqn 2.0 gives $$[K][T][\hat{u}] = [F]$$	
	Pre multiply by [T] transpose to make the matrices size consistent throughout the stiffness equation gives, $$[T^T][K][T][\hat{u}] = [T^T][F]$$	4.0
	Now the modified stiffness equation is given by,	

	$$[\hat{K}][\hat{u}] = [\hat{F}]$$	
	Modified stiffness $\hat{K}$ matrix $$[\hat{K}] = \begin{bmatrix} 0.64k_2 & 0 & -0.48k_2 & -0.64k_2 & 0.48k_2 \\ 0 & 0 & 0 & 0 & 0 \\ -0.48k_2 & 0 & 0.36k_2 & 0.48k_2 & -0.36k_2 \\ -0.64k_2 & 0 & 0.48k_2 & 0.64k_2 & -0.48k_2 \\ 0.48k_2 & 0 & -0.36k_2 & -0.48k_2 & 0.36k_2 \end{bmatrix}$$	
	Modified force $\hat{F}$ matrix $$[\hat{F}] = \begin{bmatrix} 0.0 \\ 0.0 \\ -100 \\ 0.0 \\ 0.0 \end{bmatrix}$$	
	Solving the stiffness matrix gives us (unit is mm) $$[\hat{u}] = \begin{bmatrix} 0.0 \\ 0.0 \\ -2.605795 \\ 0.0 \\ 0.0 \end{bmatrix}$$	

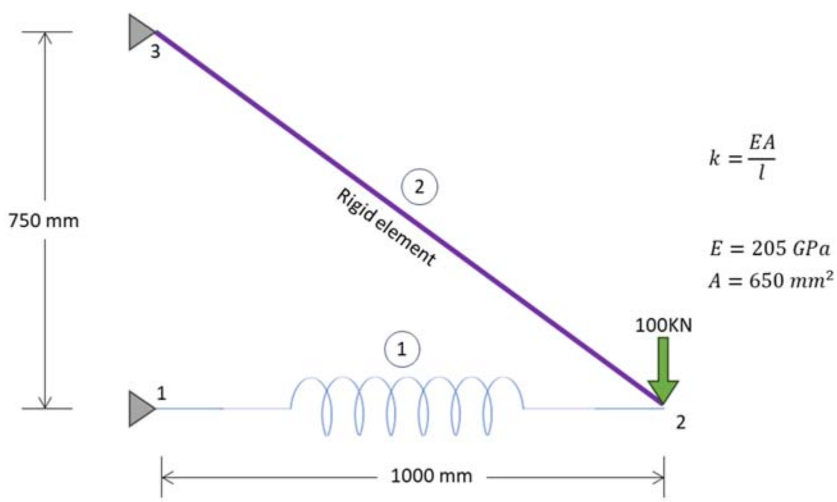
The rigid element model is equivalent to having a roller support restricting x translation as shown in the below.



If the inclined element is rigid, then finding a suitable transformation matrix is more difficult. Implementation of this method is also not easy.

Example 2: Penalty MPC

Inclined rigid element.

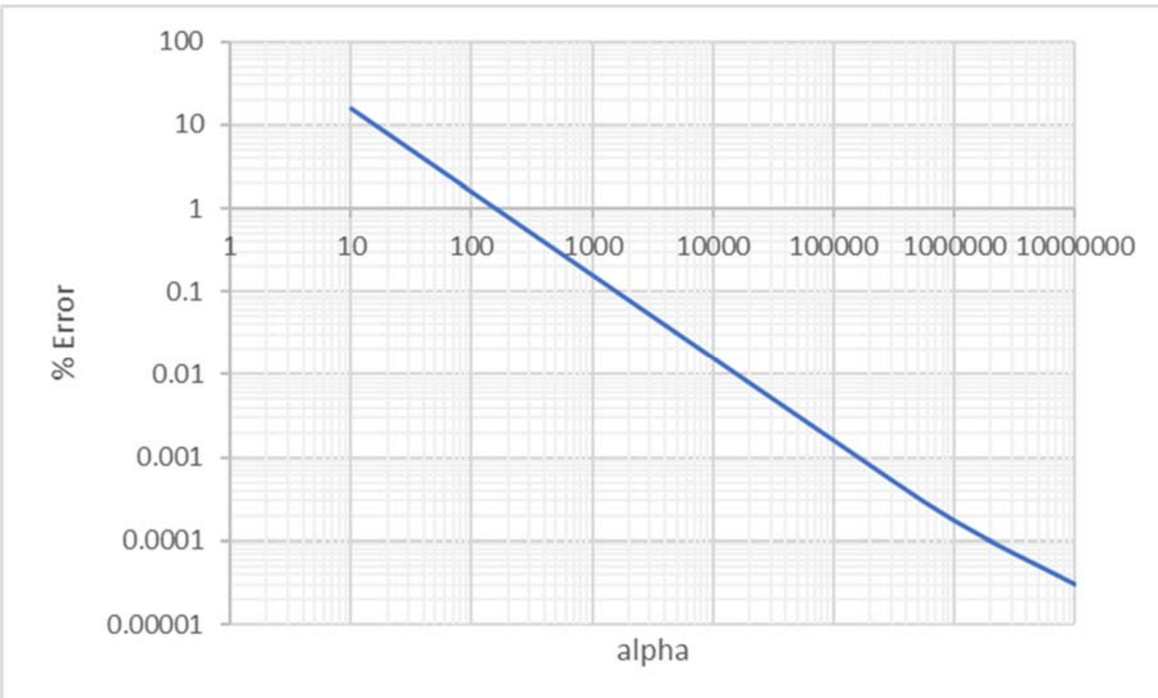


	In this example the global unmodified stiffness equation is the same as the previous.	
	Global stiffness matrix	

	$$[K_g] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + 0.64w & -0.48w & -0.64w & 0.48w \\ 0 & 0 & -0.48w & 0.36w & 0.48w & -0.36w \\ 0 & 0 & -0.64w & 0.48w & 0.64w & -0.48w \\ 0 & 0 & 0.48w & -0.36w & -0.48w & 0.36w \end{bmatrix}$$	1.0
	$$[u] = \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix}$$	
	In the above stiffness equation w is infinity because element 2 is by definition, rigid. In application, making w a very large number reduces the constraint violation between the rigidly connected node but increases the solution error.	
	After applying the boundary conditions, $$\begin{bmatrix} k_1 + 0.64w & -0.48w \\ -0.48w & 0.36w \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{2y} \end{bmatrix} = \begin{bmatrix} 0.0 \\ -100 \end{bmatrix}$$	2.0
	$$k_1 = \frac{EA}{l_e} = 133.25\ KN/mm$$	
	Solving the above equation gives $$\begin{bmatrix} (133.25 + 0.64w)\ u_{2x} - 0.48w\ u_{2y} \\ -0.48w\ u_{2x} + 0.36w\ u_{2y} \end{bmatrix} = \begin{bmatrix} 0.0 \\ -100 \end{bmatrix}$$	
	$$(133.25 + 0.64w)\ u_{2x} - 0.48w\ u_{2y} = 0.0$$ $$-0.48w\ u_{2x} + 0.36w\ u_{2y} = -100$$	3.0 & 4.0
	Solving manually gives the results as	
	$$u_{2x} = -1.000625\ mm$$	
	$$u_{2y} = -1.334167\ mm$$	

	But when implementing in computers, w needs to have some value, the following table shows various values for w (max(k) * alpha) and the results of u_{2y} compared to exact value	

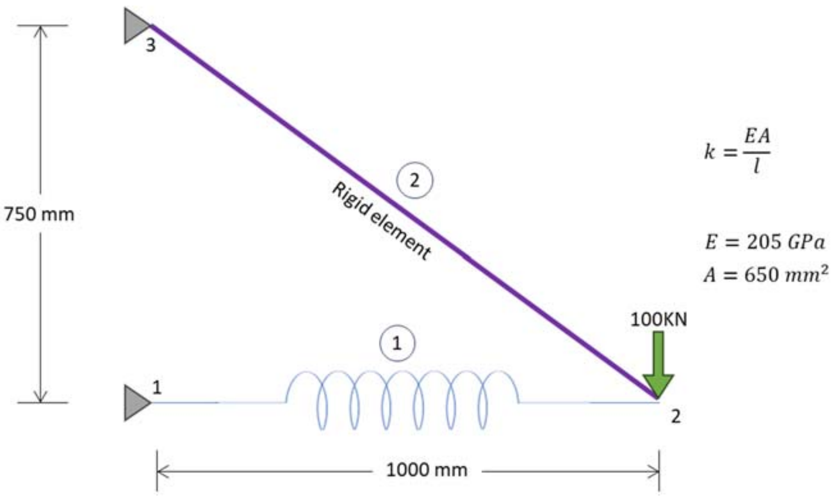
alpha	10	10^2	10^3	10^4	10^5	10^6	10^7
u_{2y}	1.542631	1.355014	1.336251824	1.33437565	1.334188034	1.334169	1.334167
u_{2y} - exact	0.208464	0.020847	0.002084824	0.00020865	2.10342E-05	2.27E-06	3.96E-07
% Error	15.62502	1.562514	0.1562641	0.01563908	0.001576578	0.00017	2.97E-05



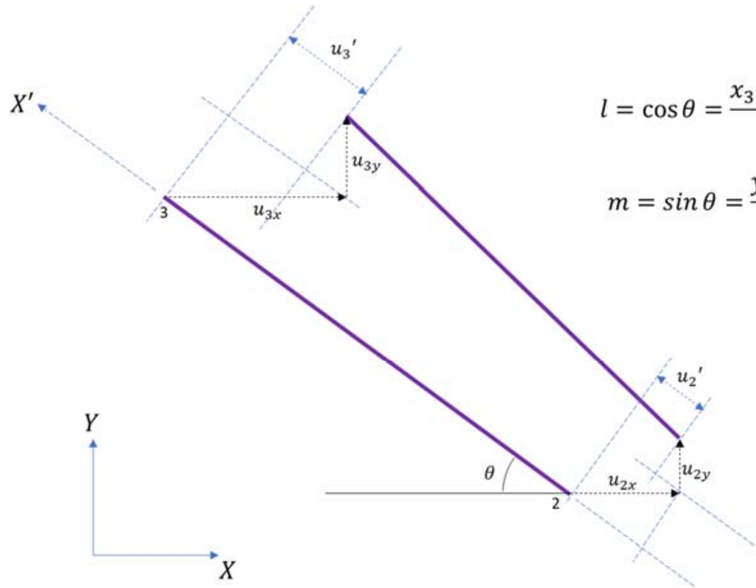
	The stiffness matrix approaches singularity as w approaches infinity. For w > 10^{16} the computer will not be able to distinguish K from singularity matrix when using standard low-level programming languages. For the above example a value of 10^8 yields a solution with error of order 10^{-8}.	

Example 3.1: Lagrange Multiplier MPC

Inclined rigid element.



	In this example the global unmodified stiffness equation is the same as the previous.	
	Global stiffness equation $\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + 0.64k_2 & -0.48k_2 & -0.64k_2 & 0.48k_2 \\ 0 & 0 & -0.48k_2 & 0.36k_2 & 0.48k_2 & -0.36k_2 \\ 0 & 0 & -0.64k_2 & 0.48k_2 & 0.64k_2 & -0.48k_2 \\ 0 & 0 & 0.48k_2 & -0.36k_2 & -0.48k_2 & 0.36k_2 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{bmatrix}$	1.0
	The rigid element facilitates the displacement along the length is equal which gives the following expression. $u_{2x}l + u_{2y}m - u_{3x}l - u_{3y}m = 0$	
	The below picture shows why the above expression is valid	



$$l = \cos\theta = \frac{x_3 - x_2}{l_e}$$

$$m = \sin\theta = \frac{y_3 - y_2}{l_e}$$

$$u_2' = u_{2x} l + u_{2y} m$$
$$u_3' = u_{3x} l + u_{3y} m$$
$$for\ rigid\ element\ u_2' = u_3'$$

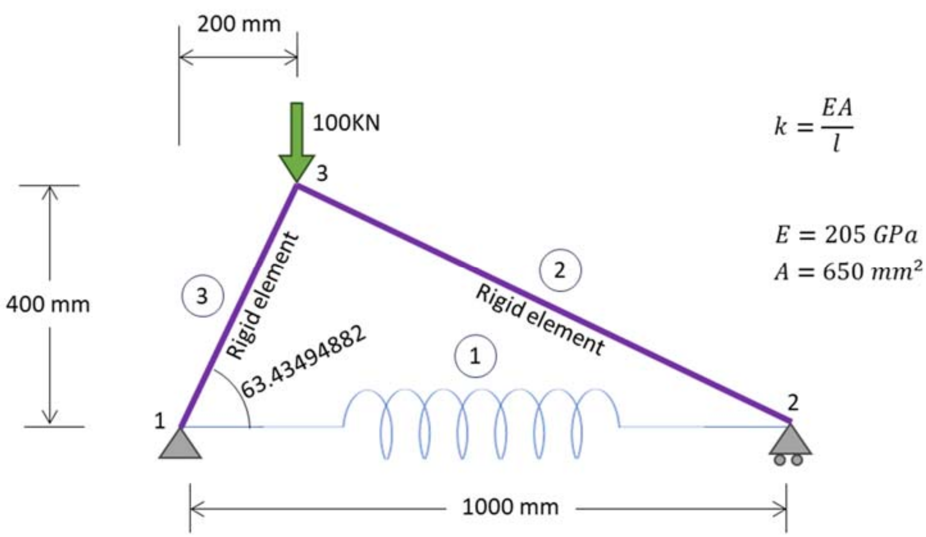
	Appending the constraints to the global stiffness equation gives, $$\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1+0.64k_2 & -0.48k_2 & -0.64k_2 & 0.48k_2 \\ 0 & 0 & -0.48k_2 & 0.36k_2 & 0.48k_2 & -0.36k_2 \\ 0 & 0 & -0.64k_2 & 0.48k_2 & 0.64k_2 & -0.48k_2 \\ 0 & 0 & 0.48k_2 & -0.36k_2 & -0.48k_2 & 0.36k_2 \\ 0 & 0 & l & m & -l & -m \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ 0 \end{bmatrix}$$	2.0
	For element 2, $l = -0.8, m = 0.6$ $$\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1+0.64k_2 & -0.48k_2 & -0.64k_2 & 0.48k_2 \\ 0 & 0 & -0.48k_2 & 0.36k_2 & 0.48k_2 & -0.36k_2 \\ 0 & 0 & -0.64k_2 & 0.48k_2 & 0.64k_2 & -0.48k_2 \\ 0 & 0 & 0.48k_2 & -0.36k_2 & -0.48k_2 & 0.36k_2 \\ 0 & 0 & -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ 0 \end{bmatrix}$$	3.0
	Below is the lagrange multiplier-augmented system. Its coefficient matrix, which is symmetric, is called the bordered stiffness matrix. The process	

	by which λ is appended to the vector of original unknowns is called adjunction.	
	$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1+0.64k_2 & -0.48k_2 & -0.64k_2 & 0.48k_2 & -0.8 \\ 0 & 0 & -0.48k_2 & 0.36k_2 & 0.48k_2 & -0.36k_2 & 0.6 \\ 0 & 0 & -0.64k_2 & 0.48k_2 & 0.64k_2 & -0.48k_2 & 0.8 \\ 0 & 0 & 0.48k_2 & -0.36k_2 & -0.48k_2 & 0.36k_2 & -0.6 \\ 0 & 0 & -0.8 & 0.6 & 0.8 & -0.6 & 0 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ \lambda \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ 0 \end{bmatrix}$$	4.0
	MPC – Lagrange Multiplier method multiplier – augmented system, bordered equation is, $$\begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} U \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ Q \end{bmatrix}$$	5.0
	The global system equation and matrix form of MPC equations is of the form. $$[K][U] = [F]$$ $$[C][U] - [Q] = 0$$	6.0
	Applying the essential boundary condition to bordered equation is, $$\begin{bmatrix} k_1+0.64k_2 & -0.48k_2 & -0.8 \\ -0.48k_2 & 0.36k_2 & 0.6 \\ -0.8 & 0.6 & 0.0 \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{2y} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0.0 \\ -100.0 \\ 0 \end{bmatrix}$$	
	In the above matrix equation, the value of k1 is a known quantity and k2 is infinity because of rigid element.	
	Lets chose k2 or w = 1000 * max (k) = 133250 $$\begin{bmatrix} 133.25+0.64(133250) & -0.48(133250) & -0.8 \\ -0.48(133250) & 0.36(133250) & 0.6 \\ -0.8 & 0.6 & 0.0 \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{2y} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0.0 \\ -100.0 \\ 0.0 \end{bmatrix}$$	9.0

	The result is very precise. $$\begin{bmatrix} u_{2x} \\ u_{2y} \\ \lambda \end{bmatrix} = \begin{bmatrix} -1.00062539 \\ -1.33416718 \\ -166.6666667 \end{bmatrix}$$	10.0

Example 3.2: Lagrange Multiplier MPC

Inclined rigid element.

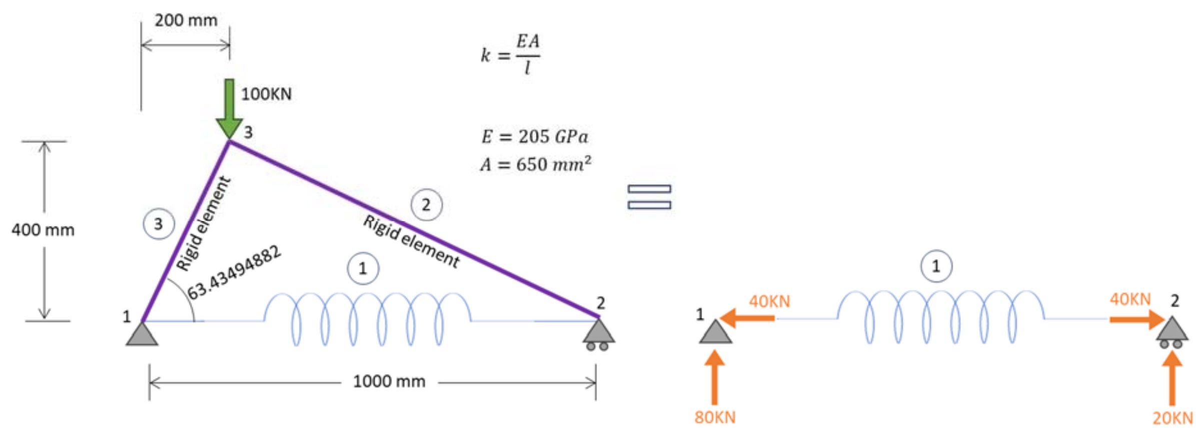


	Element stiffness matrix $$[K_e] = [L^T][k][L]$$	
	The direction cosines, $$l = \frac{x_2 - x_1}{l_e}, m = \frac{y_2 - y_1}{l_e}$$	
	Direction cosine transformation matrix, $$[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$	

	Element stiffness matrix $$[K_e] = [L^T][k][L] = \frac{EA}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$	1.0
	For element 1, $l = 1, m = 0$ $$[K_e] = \frac{EA}{l_e}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$	
	$$[K_e] = k_1\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$	2.0
	For element 2, $l = -0.894427191, m = 0.447213595$ $$[K_e] = \frac{EA}{l_e}\begin{bmatrix} -0.894427191 & 0 \\ 0.447213595 & 0 \\ 0 & -0.894427191 \\ 0 & 0.447213595 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ $$\begin{bmatrix} -0.894427191 & 0.447213595 & 0 & 0 \\ 0 & 0 & -0.894427191 & 0.447213595 \end{bmatrix}$$	
	$$[K_e] = k_2\begin{bmatrix} 0.8 & -0.4 & -0.8 & 0.4 \\ -0.4 & 0.2 & 0.4 & -0.2 \\ -0.8 & 0.4 & 0.8 & -0.4 \\ 0.4 & -0.2 & -0.4 & 0.2 \end{bmatrix}$$	3.0
	For element 3, $l = -0.447213595, m = -0.894427191$ $$[K_e] = \frac{EA}{l_e}\begin{bmatrix} -0.447213595 & 0 \\ -0.894427191 & 0 \\ 0 & -0.447213595 \\ 0 & -0.894427191 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ $$\begin{bmatrix} -0.447213595 & -0.894427191 & 0 & 0 \\ 0 & 0 & -0.447213595 & -0.894427191 \end{bmatrix}$$	

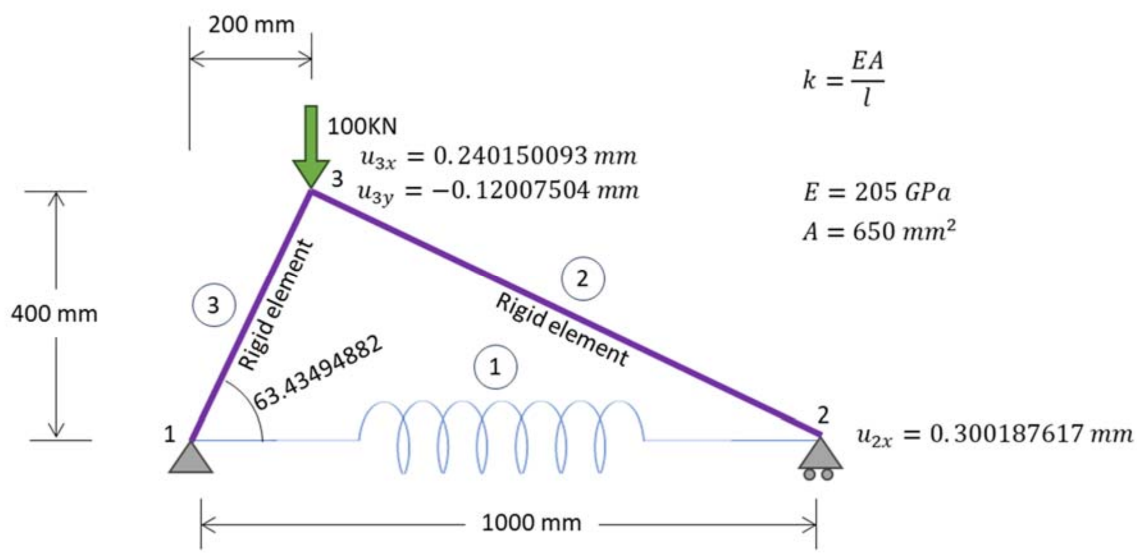
	$$[K_e] = k_3 \begin{bmatrix} 0.2 & 0.4 & -0.2 & -0.4 \\ 0.4 & 0.8 & -0.4 & -0.8 \\ -0.2 & -0.4 & 0.2 & 0.4 \\ -0.4 & -0.8 & 0.4 & 0.8 \end{bmatrix}$$	4.0
	Global stiffness matrix $$[K_g] = \begin{bmatrix} k_1 + 0.2k_3 & 0.4k_3 & -k_1 & 0 & -0.2k_3 & -0.4k_3 \\ 0.4k_3 & 0.8k_3 & 0 & 0 & -0.4k_3 & -0.8k_3 \\ -k_1 & 0 & k_1 + 0.8k_2 & -0.4k_2 & -0.8k_2 & 0.4k_2 \\ 0 & 0 & -0.4k_2 & 0.2k_2 & 0.4k_2 & -0.2k_2 \\ -0.2k_3 & -0.4k_3 & -0.8k_2 & 0.4k_2 & 0.8k_2 + 0.2k_3 & -0.4k_2 + 0.4k_3 \\ -0.4k_3 & -0.8k_3 & 0.4k_2 & -0.2k_2 & -0.4k_2 + 0.4k_3 & 0.2k_2 + 0.8k_3 \end{bmatrix}$$	5.0
	Appending the constraints to the global stiffness equation gives,	
	$$\begin{bmatrix} k_1 + 0.2k_3 & 0.4k_3 & -k_1 & 0 & -0.2k_3 & -0.4k_3 \\ 0.4k_3 & 0.8k_3 & 0 & 0 & -0.4k_3 & -0.8k_3 \\ -k_1 & 0 & k_1 + 0.8k_2 & -0.4k_2 & -0.8k_2 & 0.4k_2 \\ 0 & 0 & -0.4k_2 & 0.2k_2 & 0.4k_2 & -0.2k_2 \\ -0.2k_3 & -0.4k_3 & -0.8k_2 & 0.4k_2 & 0.8k_2 + 0.2k_3 & -0.4k_2 + 0.4k_3 \\ -0.4k_3 & -0.8k_3 & 0.4k_2 & -0.2k_2 & -0.4k_2 + 0.4k_3 & 0.2k_2 + 0.8k_3 \\ 0 & 0 & -0.89442719 & 0.44721359 & 0.89442719 & -0.44721359 \\ -0.44721359 & -0.89442719 & 0 & 0 & 0.44721359 & 0.89442719 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ 0 \\ 0 \end{bmatrix}$$	6.0
	Adjunction process: Lagrange multiplier augmented bordered stiffness matrix	

	$\begin{bmatrix} k_1+0.2k_3 & -0.4k_3 & -k_1 & 0 & -0.2k_3 & 0.4k_3 & 0 & -0.44721359 \\ -0.4k_3 & 0.8k_3 & 0 & 0 & 0.4k_3 & -0.8k_3 & 0 & -0.89442719 \\ -k_1 & 0 & k_1+0.8k_2 & -0.4k_2 & -0.8k_2 & 0.4k_2 & -0.89442719 & 0 \\ 0 & 0 & -0.4k_2 & 0.2k_2 & 0.4k_2 & -0.2k_2 & 0.44721359 & 0 \\ -0.2k_3 & 0.4k_3 & -0.8k_2 & 0.4k_2 & 0.8k_2+0.2k_3 & -0.4k_2-0.4k_3 & 0.89442719 & 0.44721359 \\ 0.4k_3 & -0.8k_3 & 0.4k_2 & -0.2k_2 & -0.4k_2-0.4k_3 & 0.2k_2+0.8k_3 & -0.44721359 & 0.89442719 \\ 0 & 0 & -0.89442719 & 0.44721359 & 0.89442719 & -0.44721359 & 0 & 0 \\ -0.44721359 & -0.89442719 & 0 & 0 & 0.44721359 & 0.89442719 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ 0 \\ 0 \end{bmatrix}$	
	After applying the essential boundary condition. The matrix reduces to	
	$\begin{bmatrix} k_1+0.8k_2 & -0.8k_2 & 0.4k_2 & -0.89442719 & 0 \\ -0.8k_2 & 0.8k_2+0.2k_3 & -0.4k_2+0.4k_3 & 0.89442719 & 0.44721359 \\ 0.4k_2 & -0.4k_2+0.4k_3 & 0.2k_2+0.8k_3 & -0.44721359 & 0.89442719 \\ -0.89442719 & 0.89442719 & -0.44721359 & 0 & 0 \\ 0 & 0.44721359 & 0.89442719 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{3x} \\ u_{3y} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ -100.0 \\ 0 \\ 0 \end{bmatrix}$	7.0
	Applying the stiffness values $k_1 = 133.25, k_2, k_3 = w = \max(k) * 1000$	
	$\begin{bmatrix} 133.25+0.8(133250) & -0.8(133250) & 0.4(133250) & -0.89442719 & 0 \\ -0.8(133250) & 0.8(133250)+0.2(133250) & -0.4(133250)+0.4(133250) & 0.89442719 & 0.44721359 \\ 0.4(133250) & -0.4(133250)+0.4(133250) & 0.2(133250)+0.8(133250) & -0.44721359 & 0.89442719 \\ -0.89442719 & 0.89442719 & -0.44721359 & 0 & 0 \\ 0 & 0.44721359 & 0.89442719 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{3x} \\ u_{3y} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ -100.0 \\ 0 \\ 0 \end{bmatrix}$	
	$\begin{bmatrix} 1066133.25 & -1066000 & 533000 & -0.89442719 & 0 \\ -1066000 & 1332500 & 0.0 & 0.89442719 & 0.44721359 \\ 533000 & 0.0 & 1332500 & -0.44721359 & 0.89442719 \\ -0.89442719 & 0.89442719 & -0.44721359 & 0 & 0 \\ 0 & 0.44721359 & 0.89442719 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{3x} \\ u_{3y} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ -100.0 \\ 0 \\ 0 \end{bmatrix}$	8.0
	Solving the above equation gives	
	$\begin{bmatrix} u_{2x} \\ u_{3x} \\ u_{3y} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.300187617261 \\ 0.240150093755 \\ -0.120075047012 \\ 44.721359549992 \\ -89.442559099983 \end{bmatrix}$	9.0



	Exact solution is given by, $$k_1 u_{2x} = F_x$$	
	$$133.25\ u_{2x} = 40$$	
	$$u_{2x} = 0.300187617$$	1.0
	Rigid element 2	
	$$l_2 u_{2x} + m_2 u_{2y} = l_2 u_{3x} + m_2 u_{3y}$$	
	$$-0.894427191 u_{2x} + 0.4472135955 u_{2y} = -0.894427191 u_{3x} + 0.4472135955 u_{3y}$$	
	Substituting u_{2x} and $u_{2y} = 0.0$ $$-0.894427191 u_{3x} + 0.4472135955 u_{3y} = -0.268495967279524$$	2.0
	Rigid element 3	
	$$l_3 u_{3x} + m_3 u_{3y} = l_3 u_{1x} + m_3 u_{1y}$$	

	$$-0.4472135955u_{3x} - 0.894427191u_{3y} = -0.4472135955u_{1x} - 0.894427191u_{1y}$$	
	Substituting $u_{1x} = 0.0$ and $u_{1y} = 0.0$ $$-0.4472135955u_{3x} - 0.894427191u_{3y} = 0.0$$	3.0
	Solving eqn 2.0 and 3.0 $$u_{3x} = 0.240150093$$	
	$$u_{3y} = -0.12007504$$	



Generalized Eigen Value problem – Lagrange multiplier MPC

Below is the free vibration generalized eigen value problem of a dynamic system without damping.

	Generalized eigen value problem of mass – spring system $$[K][\phi] = [\omega^2][M][\phi]$$	1.0
	MPC – Lagrange Multiplier method, multiplier – augmented system, bordered equation is, $$\begin{bmatrix} [K] & [C^T] \\ [C] & [0] \end{bmatrix} \begin{bmatrix} [\phi] \\ [\lambda] \end{bmatrix} = [\omega^2] \left(\begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [\phi] \\ [\lambda] \end{bmatrix} \right)$$	
	$$\left(\begin{bmatrix} [K] & [C^T] \\ [C] & [0] \end{bmatrix} - [\omega^2] \begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \right) \begin{bmatrix} [\phi] \\ [\lambda] \end{bmatrix} = 0$$	2.0
	The solution to the eigen value problem defined in eqn 2 is complicated by the presence of many null elements in the involved matrices. We can use change of variables to overcome this difficulty. $$[\phi^*] = [M]^{1/2}[\phi]$$ $$[\lambda^*] = (1/Lv)[\lambda]$$ $$where\ [M]^{1/2}\ is\ the\ square\ root\ of\ mass\ matrix$$ $$Lv\ is\ a\ large\ value$$	
	Change of variable $$\begin{bmatrix} [\phi^*] \\ [\lambda^*] \end{bmatrix} = \begin{bmatrix} [M]^{1/2} & [0] \\ [0] & [1/Lv] \end{bmatrix} \begin{bmatrix} [\phi] \\ [\lambda] \end{bmatrix}$$	3.0
	$$\begin{bmatrix} [\phi] \\ [\lambda] \end{bmatrix} = \begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix} \begin{bmatrix} [\phi^*] \\ [\lambda^*] \end{bmatrix}$$	4.0
	Substitute the eqn 4.0 in eqn 2.0 $$\left(\begin{bmatrix} [K] & [C^T] \\ [C] & [0] \end{bmatrix} - [\omega^2] \begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \right) \begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix} \begin{bmatrix} [\phi^*] \\ [\lambda^*] \end{bmatrix} = 0$$	5.0

	Pre-multiply the eqn 5.0, by the below $$\begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix}$$	
	$$\begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix} \left(\begin{bmatrix} [K] & [C^T] \\ [C] & [0] \end{bmatrix} - [\omega^2] \begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \right) \begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix} \begin{bmatrix} [\phi^*] \\ [\lambda^*] \end{bmatrix} = 0$$	6.0
	On a side note, Is the following expression valid? $$\lim_{x \to \infty} \left(\frac{1}{x^2} \right) = 0$$	
	Then, $$if\ Lv \to \infty\ then, \qquad \frac{1}{Lv^2}[I] = [0]$$	7.0
	Substitute eqn 7.0 in eqn 6.0 Mass matrix null lower block. $$\begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix} \left(\begin{bmatrix} [K] & [C^T] \\ [C] & [0] \end{bmatrix} - [\omega^2] \begin{bmatrix} [M] & [0] \\ [0] & \frac{1}{Lv^2}[I] \end{bmatrix} \right) \begin{bmatrix} [M]^{-1/2} & [0] \\ [0] & [Lv] \end{bmatrix} \begin{bmatrix} [\phi^*] \\ [\lambda^*] \end{bmatrix} = 0$$	
	Simplifying the above equation gives the below standard eigen value problem. $$\left(\begin{bmatrix} [M]^{-1/2}[K][M]^{-1/2} & Lv[M]^{-1/2}[C^T] \\ Lv[C][M]^{-1/2} & [0] \end{bmatrix} - [\omega^2] \begin{bmatrix} [I] & [0] \\ [0] & [I] \end{bmatrix} \right) \begin{bmatrix} [\phi^*] \\ [\lambda^*] \end{bmatrix} = 0$$	8.0