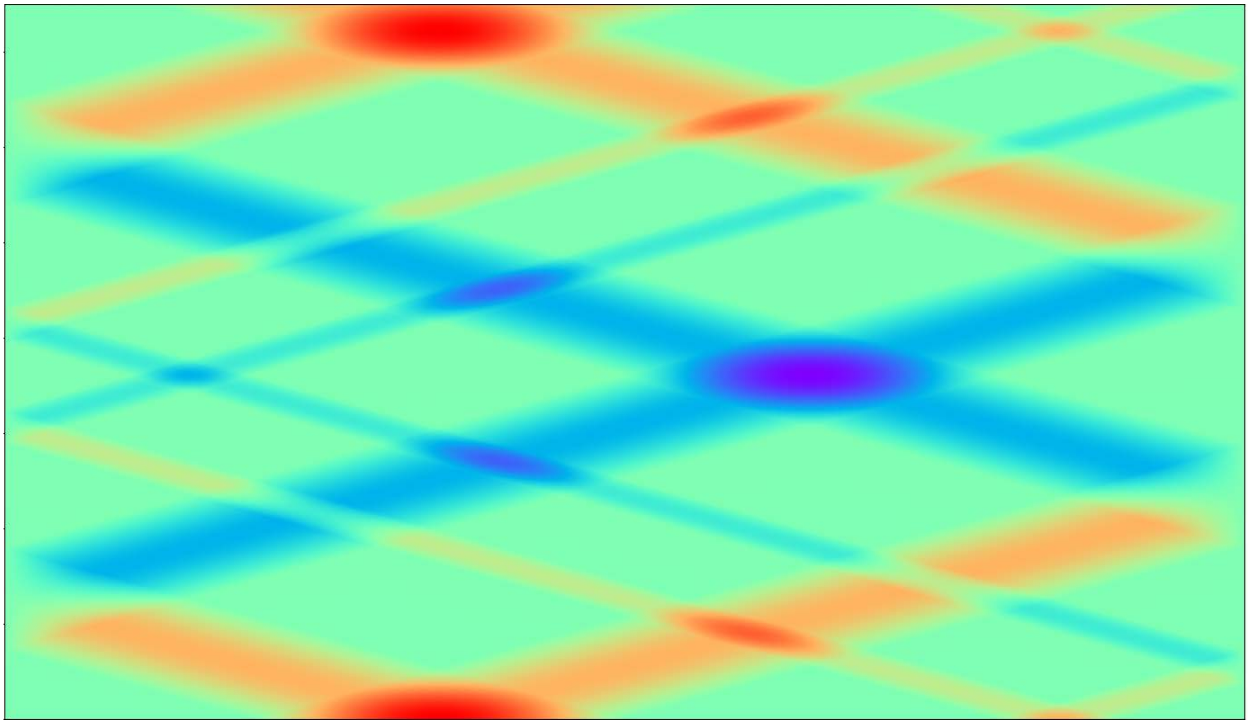


EM WAVE PROPAGATION

Modelling oscillation of charged particle

Samson Mano
saminnx@gmail.com
https://sites.google.com/site/samsoninfinite/
https://github.com/Samson-Mano

Introduction

Charged particle

A charged particle is a particle with an electric charge. Electric charge (q) is the physical property of mater that causes it to experience a force when placed in an electromagnetic field. Electric charge can be positive or negative (commonly carried by protons and electrons respectively, by convention). Like charges repel each other and unlike charges attract each other. An object with no net charge is referred to as electrically neutral.

Coulomb's law can be stated as a simple mathematical expression. The scalar form gives the magnitude of the vector of the electrostatic force F between two point charges q1 and q2, but not its direction. If r is the distance between the charges, the magnitude of the force is

	Columb's law: $$\lvert F\rvert = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$	1.0

where ε0 is the electric constant. If the product q1q2 is positive, the force between the two charges is repulsive; if the product is negative, the force between them is attractive.

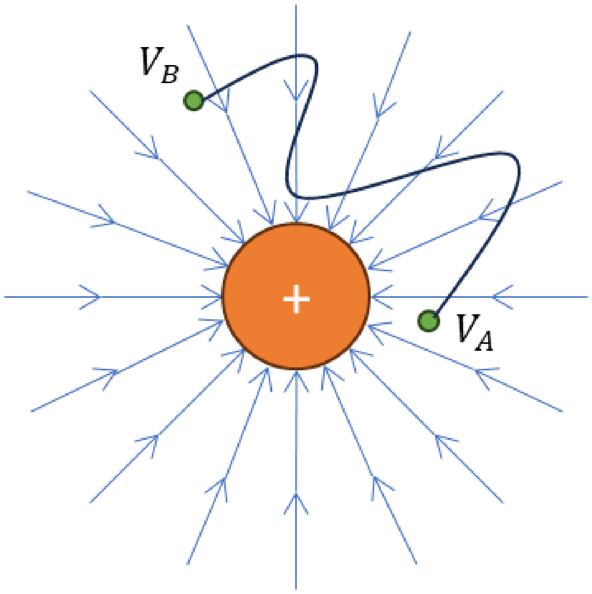
Electric field

An electric field is a vector field that associates to each point in space the Coulomb force experienced by a unit test charge. In the simplest case, the field is considered to be generated solely by a single source point charge.

	Electric field due to stationary point charge is: $$E(r) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$ Where $\hat{r}$ is the radial unit vector.	2.0

Electric potential

The electric potential, or voltage, is the difference in potential energy per unit charge between two locations in an electric field.

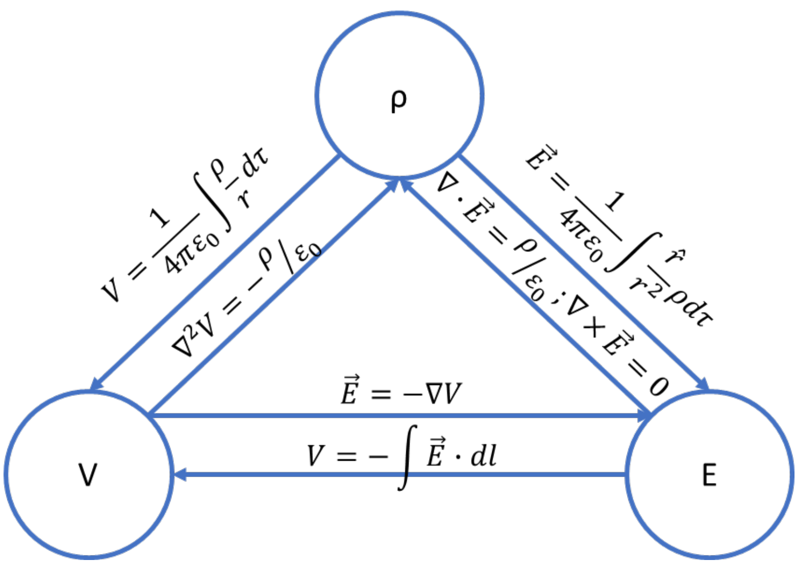


A charge q moving in a constant electric field E experience a force F = qE from that field. Gauss law states that the word done by the force is path independent.

	The difference in potential energy per unit charge between two locations in an electric field is given by, $$V_B - V_A = \int_A^B (\nabla V) \cdot dl = -\int_A^B E \cdot dl$$	
	$$E = -\nabla V$$	3.0
	The electric potential at point r is given by, $$V(r) = -\int_\infty^r \vec{E} \cdot \vec{dl}$$	
	$$= -\int_\infty^r E_r(dr)$$	
	$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathfrak{r}}$$	3.1

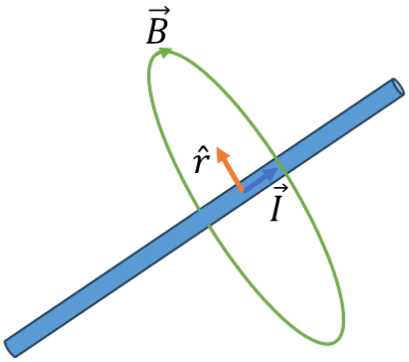
	Where r is the distance from the charge. The path independency is due to the fact in electrostatics $\nabla \times E = 0$.	
	The potential of collection of charges is given by, invoking superposition principle. $$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$	
	Or for a continuous distribution $$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathfrak{r}}dq$$	
	For a volume charge the potential is given by $$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{\mathfrak{r}}d\tau$$ Where ρ is the charge density	3.2

Electrostatics summary



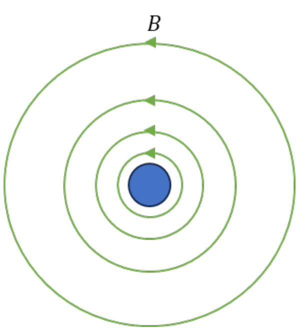
Magnetic field – Biot-Savart law

Biot-savart law is the describes the relationship between magnetic field and the electric current generating it. The Biot-Savart law allows calculating the value of magnetostatic field at any point in space, due to electrical current.



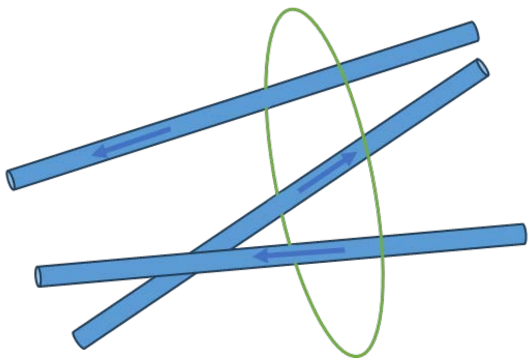
	Magnetic field due to electric current is: $$B(r) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{r}}{r^2}dl = \frac{\mu_0 I}{4\pi}\int \frac{\overrightarrow{dl}\times\hat{r}}{r^2}$$ Where $\hat{r}$ is the radial unit vector.	4.0

The magnetic field of an infinite straight wire is the integral of B around a circular path of radius s, centered at the wire is,



	$$\oint B\cdot dl = \oint \frac{\mu_0 I}{2\pi s}dl = \frac{\mu_0 I}{2\pi s}\oint dl = \mu_0 I$$	

For a bundle of straight wires shown below, the line integral will be



	$$\oint B \cdot dl = \mu_0 I_{enc} = \mu_0 \int J \cdot da$$	

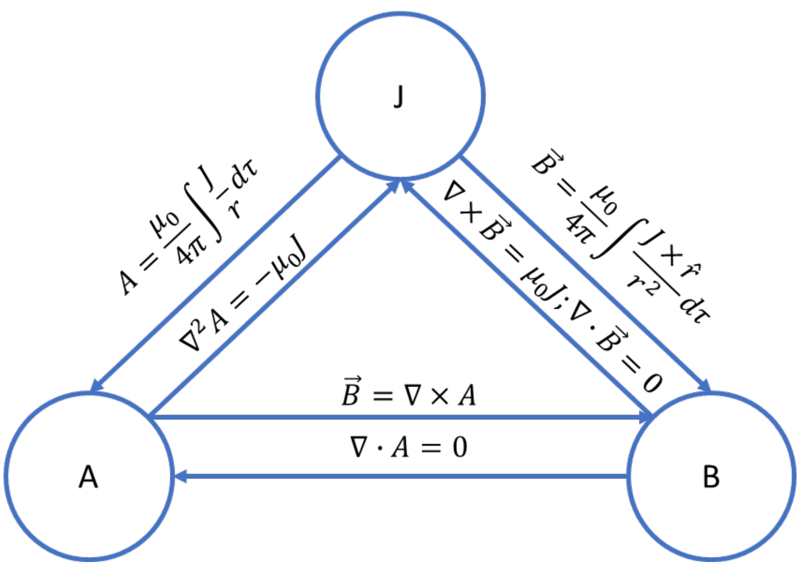
Where I_{enc} is the total current enclosed by the integration path. If the flow of charge is represented by a volume current density J

	Applying stokes theorem. $$\int (\nabla \times B) \cdot da = \mu_0 \int J \cdot da$$	
	$$\nabla \times B = \mu_0 J$$	4.1
	The biot-savart law for general case of a volume current is given by, $$B(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r') \times \hat{r}}{r^2} d\tau$$	4.2

Magnetic vector potential

	Since $\nabla \cdot B = 0$ the magnetic vector potential can be described as $$B = \nabla \times A$$	5.1
	Applying to the curl of B $$\nabla \times B = \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 J$$	
	Given $\nabla \cdot A = 0$ $$\nabla^2 A = -\mu_0 J$$	5.2
	The magnetic vector potential is given by, $$A(r) = \frac{\mu_0}{4\pi}\int \frac{J(r')}{\mathfrak{r}} d\tau$$	5.3

Magnetostatics summary



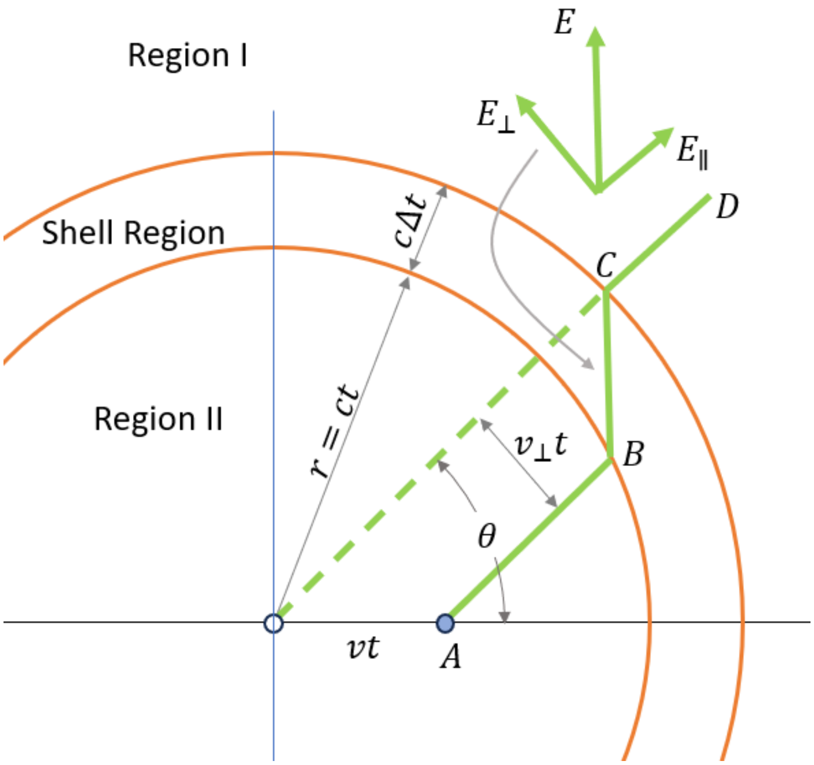
Accelerating charged particle

Accelerating charges produce changing electric and magnetic fields. Changing electric fields produce magnetic fields and changing magnetic fields produce electric fields. This interplay between induced electric and magnetic fields leads to propagating electromagnetic waves. Electromagnetic waves can propagate through free space.

Assume a charge q located near the origin is accelerating. It therefore produces electromagnetic radiation. At some position r in space and at some time t, the electric field of the electromagnetic wave produced by the accelerating charge is given by

	Electric field due to accelerating point charge is: $$E(r,t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{rc^2}a_{\perp}\left(t-\frac{r}{c}\right)$$ Where e_r is the radial unit vector.	6.0

Derivation of the Larmor's formula



Consider a charged particle which is at rest until time t = 0, experiences an acceleration a for an infinitesimal time interval Δt, and then continues to move with uniform velocity.

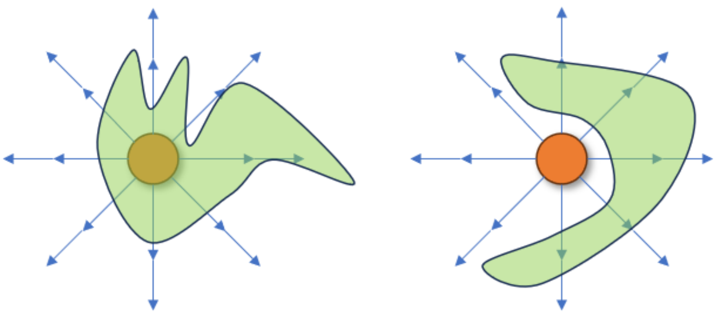
In Region I the electric field lines have not yet received the information about the acceleration, so they will look as if the particle is at rest, and nothing has changed. However, the field lines in Region II have received the information that the particle has accelerated, so they will match up to the final position of the particle (point A).

	Solving for the acceleration yields $$\vec{v}_f = \vec{v}_i + \vec{a}\Delta t$$ $$\vec{v} = 0 + \vec{a}\Delta t$$ $$\vec{a} = \frac{\vec{v}}{\Delta t}$$	7.0

	The kink in the electric field lines in the electromagnetic waves that propagates through space. $$\frac{E_\perp}{E_\parallel} = \frac{vt\sin\theta}{c\Delta t}$$	7.1
	$$E_\perp = \frac{vt\sin\theta}{c\Delta t}\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$	
	$$= \frac{vt\sin\theta}{c\Delta t}\frac{q}{4\pi\varepsilon_0}\frac{1}{(ct)r}$$	
	$$= \frac{\left(\frac{v}{\Delta t}\right)t\sin\theta}{c}\frac{q}{4\pi\varepsilon_0}\frac{1}{(ct)r}$$	
	$$E_\perp = \frac{q}{4\pi\varepsilon_0}\frac{1}{rc^2}\lvert a\rvert\sin\theta$$	7.2
	In Polar coordinates $$E_\perp = \frac{q}{4\pi\varepsilon_0}\frac{1}{rc^2}a_\perp\left(t-\frac{r}{c}\right)$$	7.3
	Where, $$a_\perp\left(t-\frac{r}{c}\right) = \left\lvert a\left(t-\frac{r}{c}\right)\right\rvert\sin\theta$$	7.4
	In vector format $$a_\perp\left(t-\frac{r}{c}\right) = \frac{1}{r^2}\left(\vec{r}\times(\vec{r}\times\vec{a})\right)$$	

Maxwell's Equation

I) Gauss law for Electric field



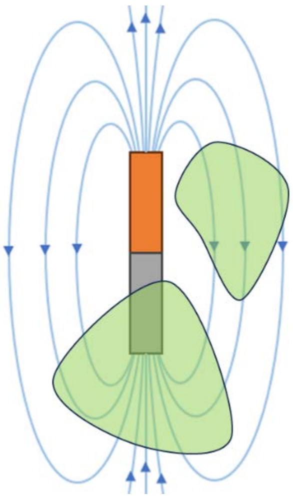
Encloses a net charge | Does not contain a charge

The Gauss theorem connects the 'flow' of electric field lines (flux) to the charges within the enclosed surface in simple terms. The net charge in the volume contained by a closed surface is exactly proportional to the net flux through the closed surface.

	$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ $\rho - Charge\ density, \varepsilon_0 - Permitivity\ of\ free\ space$	8.0

The net electric flow stays 0 if no charges are contained by a surface. The number of electric field lines entering the surface equals the number of field lines exiting the surface.

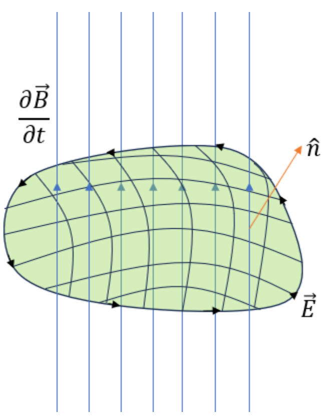
II) Gauss law for magnetic field



The net magnetic flux through any closed surface is equal to zero. There are no magnetic charges. Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes.

	$\vec{\nabla} \cdot \vec{B} = 0$	8.1

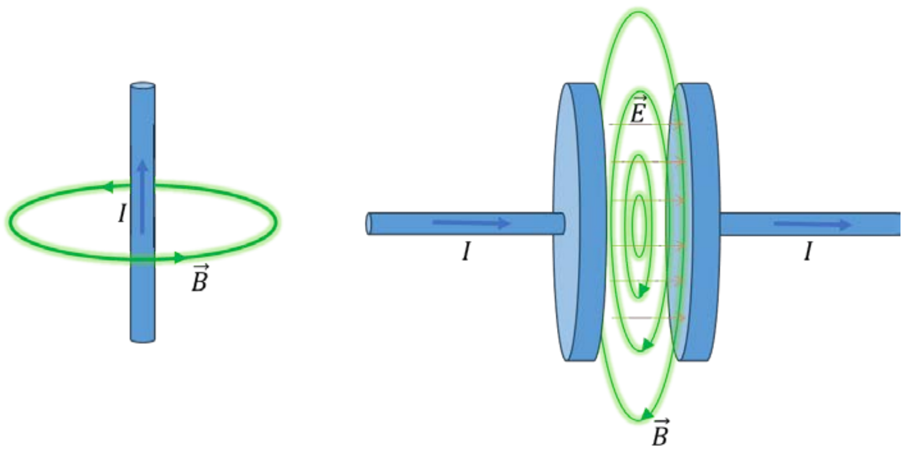
III) Differential form of Faraday's law



A circulating electric field is produced by a magnetic field that changes with time.

	$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$	8.2

IV) Ampere – Maxwell’s law

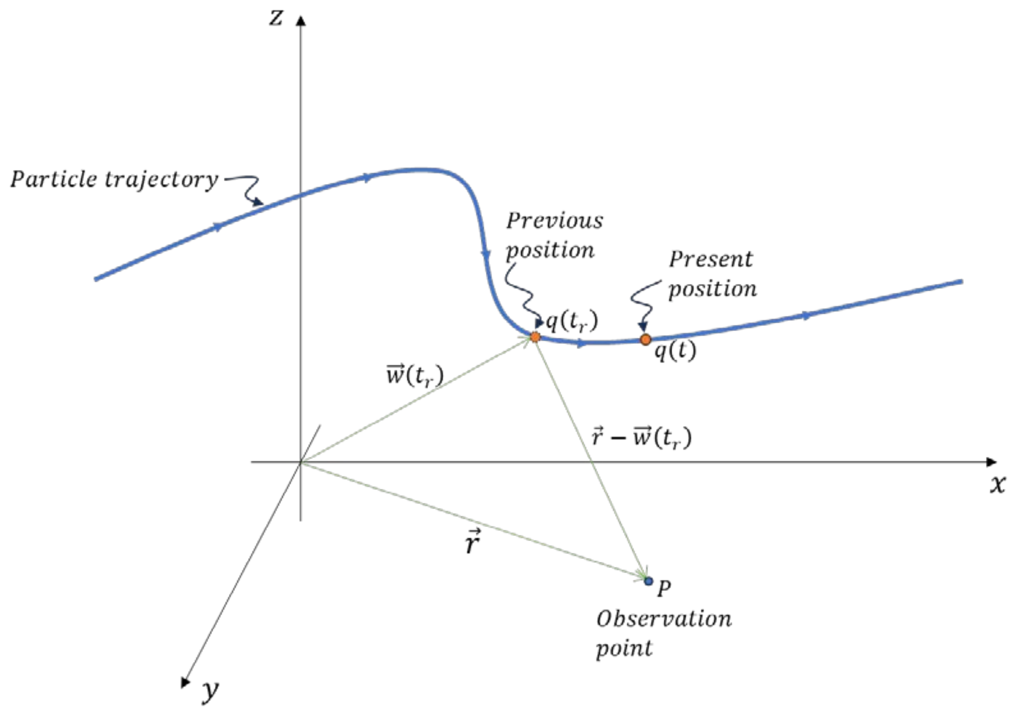


An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path that bounds that surface.

	A circulating magnetic field is produced by an electric current and by an electric field that changes with time. $$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$ $\mu_0 - Magnetic\ permeability\ of\ free\ space$ $\vec{J} - Eletric\ current\ density\ in\ amperes\ per\ square\ meter$ Electric current density is proportional to the electric field.	8.3

Delayed Potential

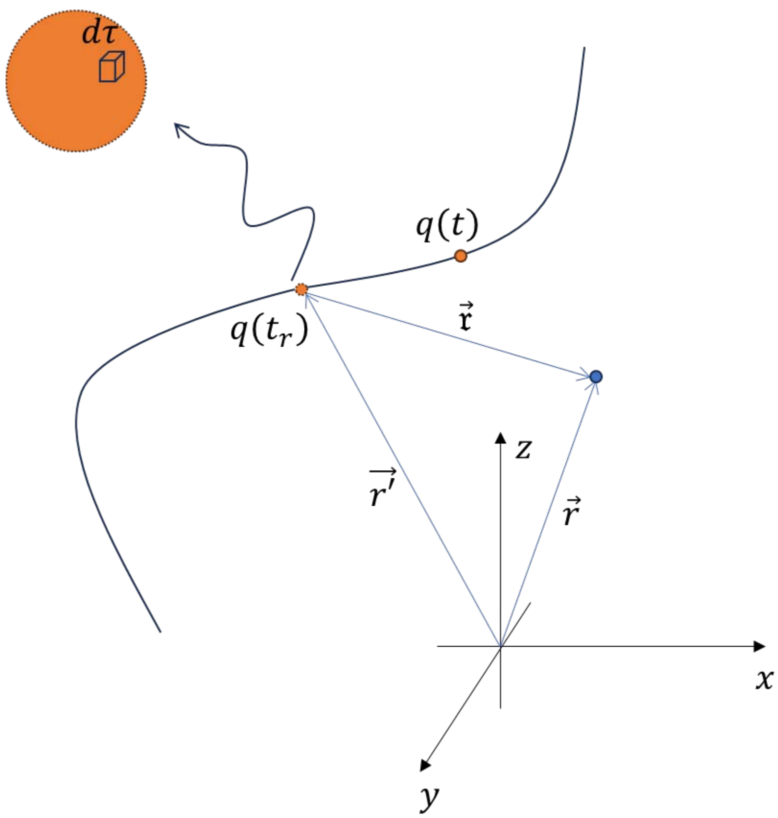
	In the static case, the Poisson's equation of Electric scalar potential and magnetic vector potential, $$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$	9.0
	$$\nabla^2 A = -\mu_0 J$$	
	With the familiar solution $$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{\mathfrak{r}}\, d\tau$$	9.1
	$$A(r) = \frac{\mu_0}{4\pi}\int \frac{J(r')}{\mathfrak{r}}\, d\tau$$	9.2



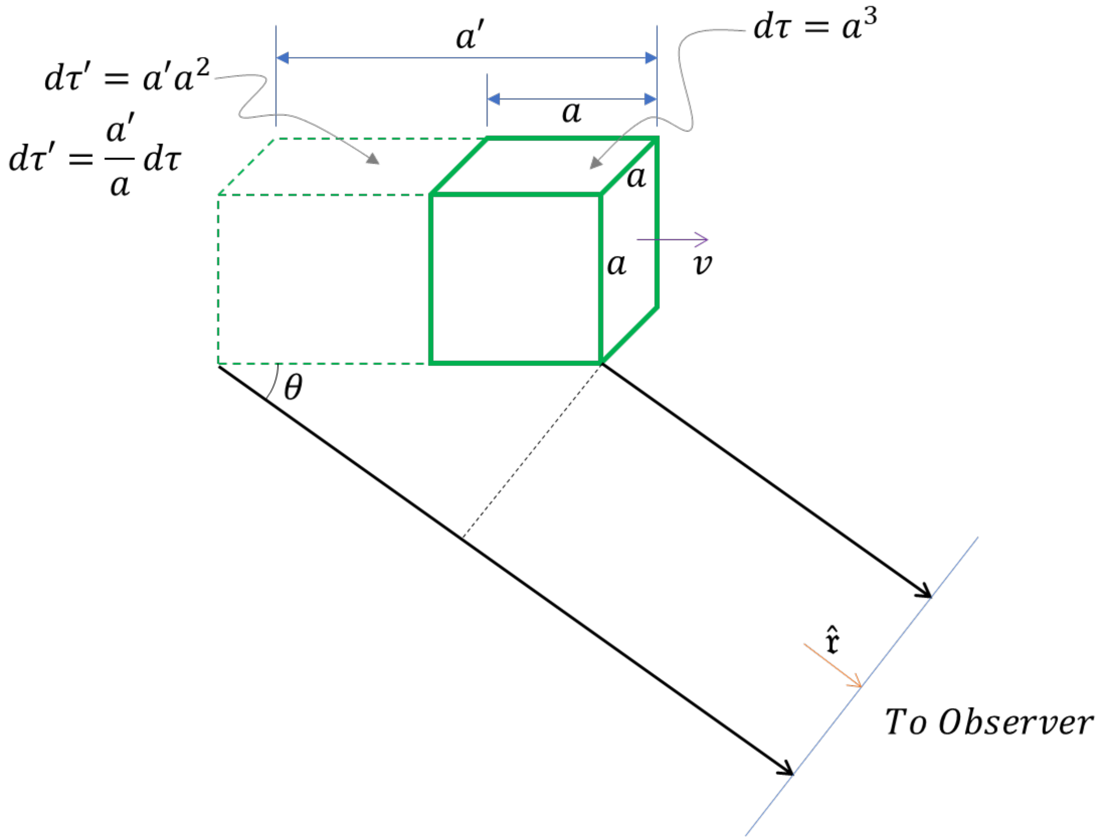
The delayed time t_r is the time when the electromagnetic field began to propagate from the point $q(t_r)$ where it is emitted to an observer at P. The objective is to calculate the delayed time t_r from the current time t. The present location of the particle is q(t), when the signal is reached to P the observer will notice the signal is coming from the particle at $q(t_r)$.

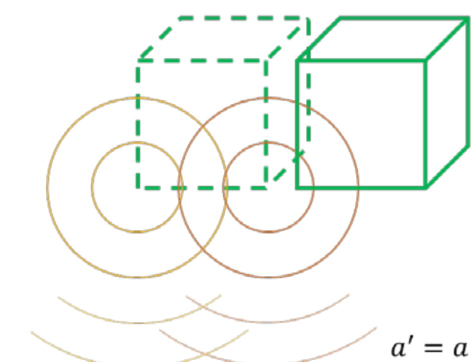
	So, between the time t and tr ($\Delta t = t - t_r$) the wave traveled the distance between $q(t_r)$ and P at the speed c. $$|\vec{r} - \vec{w}(t_r)| = c\Delta t = c(t - t_r)$$	
	The delayed time is given by, $$t_r = t - \frac{|\vec{r} - \vec{w}(t_r)|}{c}$$	10.0

Lienard – Wiechert Potential

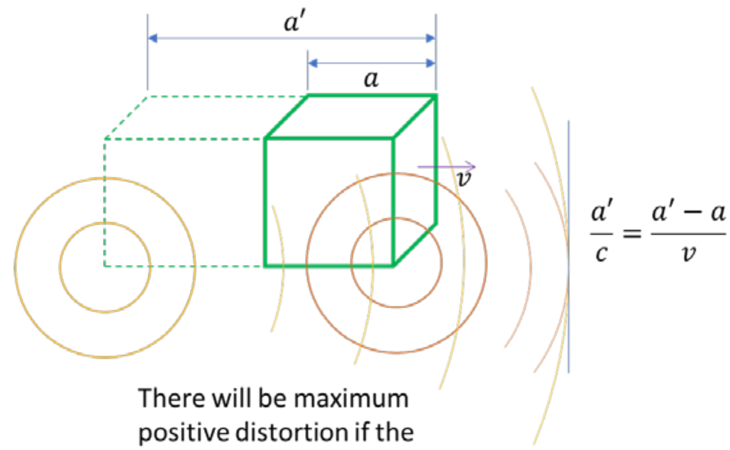


	The delayed time for a charge source shown above is given by, $$t_r = t - \frac{\mathfrak{r}}{c}$$	11.0
	The non-static scalar and magnetic vector potentials are given by, $$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t_r)}{\mathfrak{r}}\,d\tau'$$	11.1
	The current density of a rigid object is ρv $$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r',t_r)}{\mathfrak{r}}\,d\tau' = \frac{\mu_0}{4\pi}\int \frac{\rho(r',t_r)v(t_r)}{\mathfrak{r}}\,d\tau'$$	
	For a point charge q moving at a velocity v, the integral can be written as the sum of charge densities and $\mathfrak{r}_i$ is the distance to individual element. $$\int \frac{\rho(r',t_r)}{\mathfrak{r}}\,d\tau \approx \sum_i \frac{\rho_i \Delta V_i}{\mathfrak{r}_i}$$	
	Since the $\mathfrak{r}_i \gg a$ where a is the size of the charge density. The distance to the center of the charge can be taken as $\mathfrak{r}$ so it can be taken out of the integral. So, we only need to find the integral of the following which is not equal to q. $$\int \rho(r',t_r)\,d\tau' \neq q$$	
	The reason the integral of charge densities is not equal to the total charge is due to the distortion effect of moving charge. The potential from the charge is distorted due to its motion. The moving charge potential is higher if its moving towards the observer and lower if its moving away from the observer.	





There will be no distortion if the dimension is perpendicular to the observer because the length information reaches the observer at the same time



There will be maximum positive distortion if the dimension is parallel to the observer (the charge is moving towards the observer)

	The figures above are an attempt to explain the geometrical distortion effects due to the velocity of particle and the information delay in reaching the observer. The interval it takes from the length information to travel to the observer the volume itself moves a distance. $$\frac{a' \cos\theta}{c} = \frac{a' - a}{v}$$ Note that the length distortion is nothing to do with special relativity or Lorentz contraction. The length of the charge does not charge. The potential increase due to charge motion is reminiscent of Doppler effect.	11.2
	$$a' = \frac{a}{1 - \frac{v}{c} cos\ \theta}$$	11.3
	The distortion is only in the direction of the velocity, so the integral volume is. $$d\tau' = \frac{d\tau}{1 - \hat{\mathfrak{r}}\frac{v}{c}}$$	
	The dynamic scalar potential is given by, $$V(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v})}$$	11.4
	The magnetic vector potential is given by, $$A(r,t) = \frac{\mu_0}{4\pi}\frac{v}{\mathfrak{r}}\int \rho(r', t_r)\, d\tau' = \frac{\mu_0 v}{4\pi}\frac{qc}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v})}$$	11.5
	Vacuum permittivity, permittivity of free space, the electric constant ε_0 is related to magnetic constant, vacuum permeability, permeability of free space by the following equation,	11.6

	$$\varepsilon_0 = \frac{1}{\mu_0 c^2}$$	
	$$A(r,t) = \frac{v}{c^2} V(r,t)$$	11.7
	The above equations are the famous **Lienard-Wiechert** potentials for a moving point charge.	

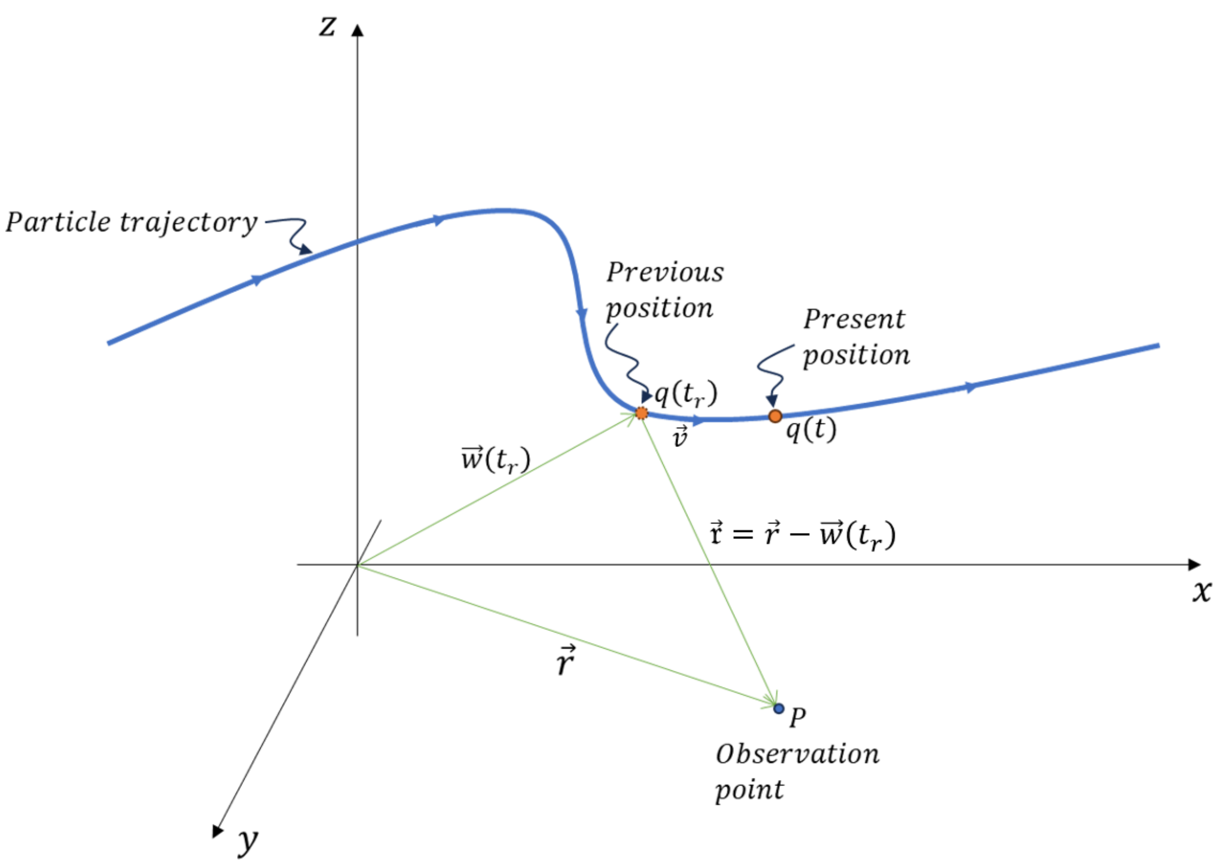
Fields of a moving point charge

	Differential form of Faraday's law Eqn 8.2 $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$	
	Applying the magnetic vector potential to the equation. $$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$	12.0
	$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$	
	$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$	12.1
	Here is a quantity whose curl vanishes; therefore, it can be written as the gradient of a scalar. $$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \mathrm{V}$$	12.2
	Substituting the above equation in the Gauss law Eqn 8.0 $$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$	

	$$\vec{\nabla}\cdot\left(-\nabla \mathrm{V}-\frac{\partial\vec{A}}{\partial t}\right)=\frac{\rho}{\varepsilon_0}$$	
	$$\nabla^2 \mathrm{V}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=-\frac{\rho}{\varepsilon_0}$$	12.3
	Substituting the Ampere – Maxwells equation Eqn 8.3 $$\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$	
	$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\mathrm{V}-\frac{\partial\vec{A}}{\partial t}\right)\right)$$	12.4
	$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\left(\mu_0\vec{J}-\mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)$$	
	Using the below vector identity $$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\nabla(\vec{\nabla}\cdot\vec{A})-\nabla^2 A$$	12.5
	$$\nabla(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}=\left(\mu_0\vec{J}-\mu_0\varepsilon_0\nabla\frac{\partial V}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)$$	
	$$\left(\nabla^2 A-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\nabla\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$$	12.6
	Equation 12.3 and Equation 12.6 contains all the information of Maxwell's Equation.	

Electric and magnetic fields of a point charge in arbitrary motion, using Lienard-Wiechert potentials.

	Re-writing the Lienard-Wiechert potential Eqn 11.4 & 11.7 $$V(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(\vec{𝔯}c - \vec{𝔯}\cdot\vec{v})}$$ $$A(r,t) = \frac{v}{c^2}V(r,t)$$	
	Re-writing the electric field equation Eqn 12.2 $$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$$	13.0
	Magnetic field equation Eqn 5.1 $$\vec{B} = \vec{\nabla}\times\vec{A}$$	13.1



	The vertical from the delayed position to the observation position and the velocity of the charge is given by, $$\vec{\mathfrak{r}} = \vec{r} - \vec{w}(t_r)$$ $$v = \dot{w}(t_r)$$	13.2
	Delayed time equation Eqn 10.0 $$\vec{\mathfrak{r}} = \lvert\vec{r} - \vec{w}(t_r)\rvert = c(t - t_r)$$	13.3
	To solve the Eqn 13.0, Let start by finding the gradient of V $$\nabla \mathrm{V} = \nabla\left(\frac{1}{4\pi\varepsilon_0}\frac{qc}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}}\cdot\vec{v})}\right)$$	13.4
	$$\nabla \mathrm{V} = \frac{qc}{4\pi\varepsilon_0}\frac{-1}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}}\cdot\vec{v})^2}\nabla(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}}\cdot\vec{v})$$	13.5
	Finding below gradient $$\nabla(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}}\cdot\vec{v}) = \mathrm{c}\nabla\vec{\mathfrak{r}} - \nabla(\vec{\mathfrak{r}}\cdot\vec{v})$$	13.6
	First term $$\mathrm{c}\nabla\vec{\mathfrak{r}} = c\nabla(c(t - t_r))$$	
	$$= -c^2\nabla t_r$$	13.7
	First term again, $$\mathrm{c}\nabla\vec{\mathfrak{r}} = c\nabla\big(\vec{r} - \vec{w}(t_r)\big)$$	
	$$= c\nabla\big(\vec{r} - \vec{w}(t_r)\big)$$	
	$$= c\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\big(\vec{r} - \vec{w}(t_r)\big)$$	

	First term Gradient of $\vec{r}$ $$\nabla\vec{r} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$	
	This is a second order tensor, $$\nabla\vec{r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$	
	Second term Gradient of $\vec{w}(t_r)$ $$\nabla\vec{w}(t_r) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left(w_x\hat{x} + w_y\hat{y} + w_z\hat{z}\right)$$	
	Second order tensor, $$\nabla\vec{w}(t_r) = \begin{bmatrix} \frac{\partial w_x}{\partial x} & \frac{\partial w_x}{\partial y} & \frac{\partial w_x}{\partial z} \\ \frac{\partial w_y}{\partial x} & \frac{\partial w_y}{\partial y} & \frac{\partial w_y}{\partial z} \\ \frac{\partial w_z}{\partial x} & \frac{\partial w_z}{\partial y} & \frac{\partial w_z}{\partial z} \end{bmatrix}$$	
	Substituting $$\mathrm{c}\nabla\vec{\mathrm{r}} = c\begin{bmatrix} 1 - \frac{\partial w_x}{\partial x} & \frac{\partial w_x}{\partial y} & \frac{\partial w_x}{\partial z} \\ \frac{\partial w_y}{\partial x} & 1 - \frac{\partial w_y}{\partial y} & \frac{\partial w_y}{\partial z} \\ \frac{\partial w_z}{\partial x} & \frac{\partial w_z}{\partial y} & 1 - \frac{\partial w_z}{\partial z} \end{bmatrix}$$	
	So does this mean, $$\begin{bmatrix} 1 - \frac{\partial w_x}{\partial x} & \frac{\partial w_x}{\partial y} & \frac{\partial w_x}{\partial z} \\ \frac{\partial w_y}{\partial x} & 1 - \frac{\partial w_y}{\partial y} & \frac{\partial w_y}{\partial z} \\ \frac{\partial w_z}{\partial x} & \frac{\partial w_z}{\partial y} & 1 - \frac{\partial w_z}{\partial z} \end{bmatrix} = -c\nabla t_r$$	

	Second term (applying product rule) $\nabla(\vec{\mathfrak{r}}\cdot\vec{v}) = (\vec{\mathfrak{r}}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{\mathfrak{r}} + \vec{\mathfrak{r}}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{\mathfrak{r}})$	13.8
	Term 1 of Eqn 13.8 $(\vec{\mathfrak{r}}\cdot\nabla)\vec{v} = \left(\vec{\mathfrak{r}}_x\frac{\partial}{\partial x} + \vec{\mathfrak{r}}_y\frac{\partial}{\partial y} + \vec{\mathfrak{r}}_z\frac{\partial}{\partial z}\right)\vec{v}(t_r)$	
	$= \left(\vec{\mathfrak{r}}_x\frac{\partial\vec{v}(t_r)}{\partial x} + \vec{\mathfrak{r}}_y\frac{\partial\vec{v}(t_r)}{\partial y} + \vec{\mathfrak{r}}_z\frac{\partial\vec{v}(t_r)}{\partial z}\right)$	
	$= \left(\vec{\mathfrak{r}}_x\frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial x} + \vec{\mathfrak{r}}_y\frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial y} + \vec{\mathfrak{r}}_z\frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial z}\right)$	
	$(\vec{\mathfrak{r}}\cdot\nabla)\vec{v} = \vec{a}(\vec{\mathfrak{r}}\cdot\nabla\mathrm{t}_r)$	13.9
	Term 2 of Eqn 13.8 $(\vec{v}\cdot\nabla)\vec{\mathfrak{r}} = \left(\vec{v}_x\frac{\partial}{\partial x} + \vec{v}_y\frac{\partial}{\partial y} + \vec{v}_z\frac{\partial}{\partial z}\right)\left(\vec{r} - \vec{w}(t_r)\right)$	
	Note that $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ $= \left(\vec{v}_x\frac{\partial\vec{r}}{\partial x} + \vec{v}_y\frac{\partial\vec{r}}{\partial y} + \vec{v}_z\frac{\partial\vec{r}}{\partial z}\right) - \left(\vec{v}_x\frac{\partial\vec{w}(t_r)}{\partial x} + \vec{v}_y\frac{\partial\vec{w}(t_r)}{\partial y} + \vec{v}_z\frac{\partial\vec{w}(t_r)}{\partial z}\right)$	
	$= \vec{v} - \left(\vec{v}_x\frac{\partial\vec{w}}{\partial t_r}\frac{\partial t_r}{\partial x} + \vec{v}_y\frac{\partial\vec{w}}{\partial t_r}\frac{\partial t_r}{\partial x} + \vec{v}_z\frac{\partial\vec{w}}{\partial t_r}\frac{\partial t_r}{\partial x}\right)$	
	$(\vec{v}\cdot\nabla)\vec{\mathfrak{r}} = \vec{v} - \vec{v}(\vec{v}\cdot\nabla\mathrm{t}_r)$	13.10
	Cross product Term 3 of Eqn 13.8	

	$\nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$	
	$= \left(\frac{\partial v_z}{\partial t_r}\frac{\partial t_r}{\partial y} - \frac{\partial v_y}{\partial t_r}\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial t_r}\frac{\partial t_r}{\partial z} - \frac{\partial v_z}{\partial t_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial t_r}\frac{\partial t_r}{\partial x} - \frac{\partial v_x}{\partial t_r}\frac{\partial t_r}{\partial y}\right)\hat{z}$	
	$\nabla \times \vec{v} = -\vec{a} \times \nabla \mathrm{t}_r$	13.11
	Cross product Term 4 of Eqn 13.8 $\nabla \times \vec{\mathfrak{r}} = \nabla \times \left(\vec{r} - \vec{w}(t_r)\right)$	
	$= \nabla \times \vec{r} - \nabla \times \vec{w}(t_r)$	
	Substituting $\nabla \times \vec{r} = 0$ $\nabla \times \vec{\mathfrak{r}} = -\nabla \times \vec{w}$	
	$= -\left(\left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}\right)\hat{x} + \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x}\right)\hat{y} + \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)\hat{z}\right)$	
	$= -\left(\left(\frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial y} - \frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(\frac{\partial w_x}{\partial t_r}\frac{\partial t_r}{\partial z} - \frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial x} - \frac{\partial w_x}{\partial t_r}\frac{\partial t_r}{\partial y}\right)\hat{z}\right)$	
	$\nabla \times \vec{\mathfrak{r}} = \vec{v} \times \nabla \mathrm{t}_r$	13.12
	Substituting Eqn 13.9, 13.10, 13.11, 13.12 in Eqn 13.8 $\nabla(\vec{\mathfrak{r}} \cdot \vec{v}) = \vec{a}(\vec{\mathfrak{r}} \cdot \nabla \mathrm{t}_r) + \vec{v} - \vec{v}(\vec{v} \cdot \nabla \mathrm{t}_r) - \vec{\mathfrak{r}} \times (\vec{a} \times \nabla \mathrm{t}_r) + \vec{v} \times (\vec{v} \times \nabla \mathrm{t}_r)$	13.13
	Triple product vector identity	

	$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$	
	Using the above identity $(\vec{\mathfrak{r}} \cdot \vec{a})\nabla \mathrm{t}_r = \vec{a}(\vec{\mathfrak{r}} \cdot \nabla \mathrm{t}_r) - \vec{\mathfrak{r}} \times (\vec{a} \times \nabla \mathrm{t}_r)$	13.14
	$\vec{v}^2 \nabla \mathrm{t}_r = \vec{v}(\vec{v} \cdot \nabla \mathrm{t}_r) - \vec{v} \times (\vec{v} \times \nabla \mathrm{t}_r)$	13.15
	Substituting Eqn 13.14 & 13.15 in 13.13	
	$\nabla(\vec{\mathfrak{r}} \cdot \vec{v}) = \vec{v} + (\vec{\mathfrak{r}} \cdot \vec{a})\nabla \mathrm{t}_r - \vec{v}^2 \nabla \mathrm{t}_r$	13.16
	Substituting Eqn 13.16 in 13.5 $\nabla \mathrm{V} = \frac{qc}{4\pi\varepsilon_0} \frac{1}{(\mathfrak{r}c - \vec{\mathfrak{r}} \cdot \vec{v})^2} \left(\vec{v} + \left(c^2 - \vec{v}^2 + (\vec{\mathfrak{r}} \cdot \vec{a})\right)\nabla \mathrm{t}_r\right)$	13.17
	We need to find the $\nabla \mathrm{t}_r$ $\nabla \vec{\mathfrak{r}} = -c \nabla t_r$	
	$-c\nabla t_r = \nabla\sqrt{\vec{\mathfrak{r}} \cdot \vec{\mathfrak{r}}} = \frac{1}{\sqrt{\vec{\mathfrak{r}} \cdot \vec{\mathfrak{r}}}} \nabla(\vec{\mathfrak{r}} \cdot \vec{\mathfrak{r}})$	13.18
	$= \frac{1}{\vec{\mathfrak{r}}}[(\vec{\mathfrak{r}} \cdot \nabla)\vec{\mathfrak{r}} + \vec{\mathfrak{r}} \times (\nabla \times \vec{\mathfrak{r}})]$	13.19
	Term 1, $(\vec{\mathfrak{r}} \cdot \nabla)\vec{\mathfrak{r}} = \vec{\mathfrak{r}} - \vec{v}(\vec{\mathfrak{r}} \cdot \nabla t_r)$	13.20
	Term 2, $\vec{\mathfrak{r}} \times (\nabla \times \vec{\mathfrak{r}}) = \vec{\mathfrak{r}} \times (\vec{v} \times \nabla t_r)$	13.21
	Substituting 13.20 & 13.21 in 13.19 $-c\nabla t_r = \frac{1}{\vec{\mathfrak{r}}}[\vec{\mathfrak{r}} - \vec{v}(\vec{\mathfrak{r}} \cdot \nabla t_r) + \vec{\mathfrak{r}} \times (\vec{v} \times \nabla t_r)]$	
	Using vector identity	

	$$-c\nabla t_r = \frac{1}{\vec{\mathfrak{r}}}[\vec{\mathfrak{r}} - (\vec{\mathfrak{r}} \cdot \vec{v})\nabla t_r]$$	
	Therefore, $$\nabla t_r = \frac{-\vec{\mathfrak{r}}}{\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v}}$$	13.22
	Substituting Eqn 13.22 in Eqn 13.17 $$\nabla \mathrm{V} = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v})^2}\left(\vec{v} - \left(c^2 - \vec{v}^2 + (\vec{\mathfrak{r}} \cdot \vec{a})\right)\frac{\vec{\mathfrak{r}}}{\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v}}\right)$$	
	$$\nabla \mathrm{V} = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v})^3}\left((\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v})\vec{v} - \left(c^2 - \vec{v}^2 + (\vec{\mathfrak{r}} \cdot \vec{a})\right)\vec{\mathfrak{r}}\right)$$	
	Second part of solving Eqn 13.0 is finding, $$\frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\vec{v}}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{qc}{(\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v})}\right)$$	13.23
	$$= \frac{qc}{4\pi\varepsilon_0 c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}}{\vec{\mathfrak{r}}c - \vec{\mathfrak{r}} \cdot \vec{v}}\right)$$	13.24
	In order to solve the above differentiation, we need to first solve $\frac{\partial t_r}{\partial t}$ $$\frac{\partial \vec{\mathfrak{r}}}{\partial t} = \frac{\partial(c(t - t_r))}{\partial t}$$	
	$$\frac{\partial \vec{\mathfrak{r}}}{\partial t_r}\frac{\partial t_r}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right)$$	
	$$\frac{\partial \vec{\mathfrak{r}}}{\partial t_r}\frac{\partial t_r}{\partial t} + c\frac{\partial t_r}{\partial t} = c$$	
		13.25

	$$\frac{\partial t_r}{\partial t} = \frac{c}{c + \frac{\partial \vec{\mathfrak{r}}}{\partial t_r}}$$	
	The other term is $$\frac{\partial \vec{\mathfrak{r}}}{\partial t_r} = \frac{\partial \lvert \vec{r} - \vec{w}(t_r) \rvert}{\partial t_r}$$	
	$$= \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial x}\frac{\partial x}{\partial t_r}\hat{x} + \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial y}\frac{\partial y}{\partial t_r}\hat{y} + \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial z}\frac{\partial z}{\partial t_r}\hat{z}$$	
	$$= \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial x} v_x \hat{x} + \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial y} v_y \hat{y} + \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial z} v_z \hat{z}$$	
	$$= \left(\frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial x}\hat{x} + \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial y}\hat{y} + \frac{\partial \lvert \vec{\mathfrak{r}} \rvert}{\partial z}\hat{z} \right) \cdot \vec{v}$$	
	$$= \nabla \lvert \vec{\mathfrak{r}} \rvert \cdot \vec{v}$$	
	$$\frac{\partial \vec{\mathfrak{r}}}{\partial t_r} = -\frac{\vec{\mathfrak{r}} \cdot \vec{v}}{\lvert \mathfrak{r} \rvert}$$	13.26
	Applying Eqn 13.26 in 13.25 $$\frac{\partial t_r}{\partial t} = \frac{c}{c - \frac{\vec{\mathfrak{r}} \cdot \vec{v}}{\lvert \mathfrak{r} \rvert}}$$	
	$$\frac{\partial t_r}{\partial t} = \frac{\mathfrak{r} c}{\mathfrak{r} c - \vec{\mathfrak{r}} \cdot \vec{v}}$$	13.27
	The above equation can be written as an operator, $$\frac{\partial}{\partial t} = \frac{\mathfrak{r} c}{\mathfrak{r} c - \vec{\mathfrak{r}} \cdot \vec{v}} \frac{\partial}{\partial t_r}$$	13.28

	Let's apply the above operator to Eqn 13.24 $$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}}{\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v}}\right)=\frac{\mathfrak{r}/c}{\mathfrak{r}c-\vec{\mathfrak{r}}\cdot\vec{v}}\frac{\partial}{\partial t_r}\left(\frac{\vec{v}}{\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v}}\right)$$	
	$$\frac{\mathfrak{r}/c}{\mathfrak{r}c-\vec{\mathfrak{r}}\cdot\vec{v}}\frac{\partial}{\partial t_r}\left(\frac{\vec{v}}{\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v}}\right) = \frac{\mathfrak{r}/c}{\mathfrak{r}c-\vec{\mathfrak{r}}\cdot\vec{v}}\frac{(\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})\frac{\partial\vec{v}}{\partial t_r}-\vec{v}\frac{\partial(\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})}{\partial t_r}}{(\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})^2}$$	
	$$=\frac{\mathfrak{r}}{c}\frac{1}{(\mathfrak{r}c-\vec{\mathfrak{r}}\cdot\vec{v})^3}\left((\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})\frac{\partial\vec{v}}{\partial t_r}-\vec{v}\frac{\partial(\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})}{\partial t_r}\right)$$	13.29
	Solving the first differential term of Eqn 13.29 $$\frac{\partial\vec{v}}{\partial t_r}=\vec{\mathrm{a}}$$	13.30
	Solving the second term, $$\frac{\partial(\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})}{\partial t_r}=\nabla(\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})\frac{1}{\nabla t_r}$$	13.31
	Applying Eqn 13.30 & 13.31 in 13.24 $$\frac{\partial A}{\partial t}=\frac{qc}{4\pi\varepsilon_0}\frac{1}{(\mathfrak{r}c-\vec{\mathfrak{r}}\cdot\vec{v})^3}\left((\vec{\mathfrak{r}}c-\vec{\mathfrak{r}}\cdot\vec{v})\left(-\vec{v}+{}^{\mathfrak{r}\vec{a}}/_{c}\right)+\vec{v}\left(c^2-\vec{v}^2+(\vec{\mathfrak{r}}\cdot\vec{a})\right)\frac{\mathfrak{r}}{c}\right)$$	
	Combining these results and introducing the vector $$\vec{u}=c\hat{\mathfrak{r}}-\vec{v}$$	
	The electric field is given by, $$E(\vec{r},t)=\frac{q}{4\pi\varepsilon_0}\frac{\mathfrak{r}}{(\vec{\mathfrak{r}}\cdot\vec{u})^3}\left((c^2-v^2)\vec{u}+\vec{\mathfrak{r}}\times(\vec{u}\times\vec{a})\right)$$	

	$$B(\vec{r},t) = \frac{1}{c}\hat{\mathfrak{r}} \times E(\vec{r},t)$$	

Inputs for implementation

The inputs for implementation are the path of the particle, frequency of oscillation and number of cycles. Path options will be closed curve or open curve. The path extend will be scaled to unit square.