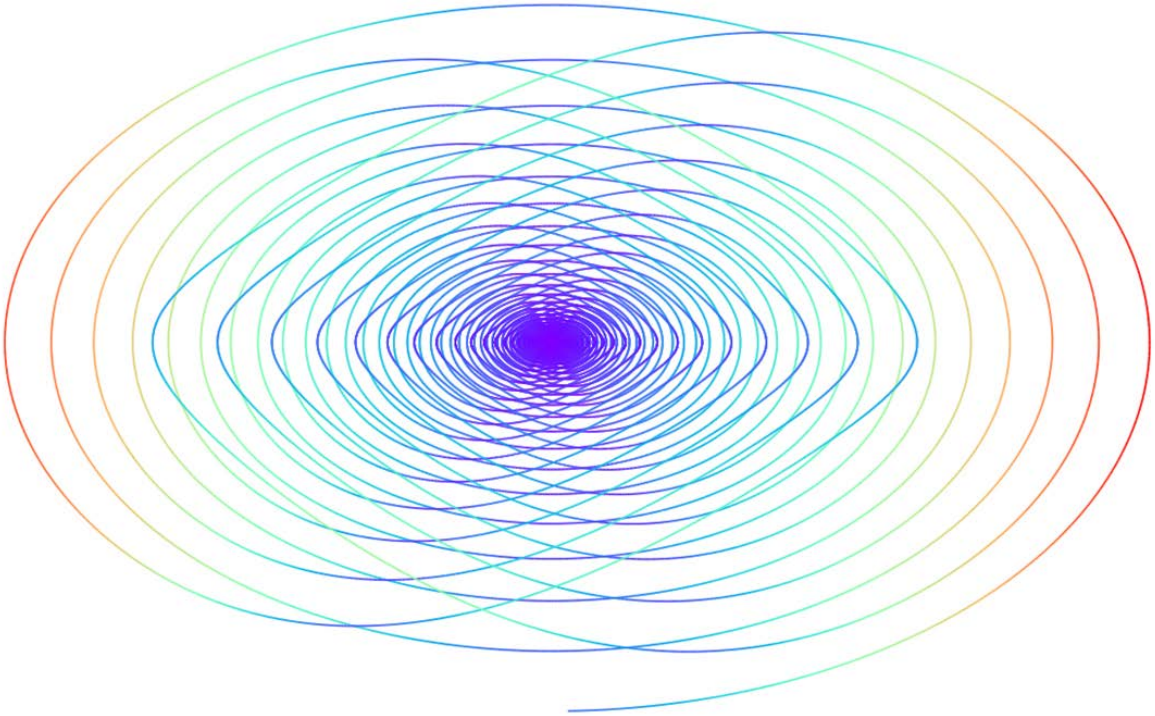

PULSE RESPONSE ANALYSIS

Analytical solution for MDOF
system's response to pulse force

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Contents

PART - I 3

Response of Single Degree-of-Freedom systems 3

Equation of motion 3

Free vibration of un-damped system 5

Arbitrary, step and pulse excitation response of un-damped system 7

Response to half – cycle sine pulse force 11

Response of Multi Degree-of-Freedom system 18

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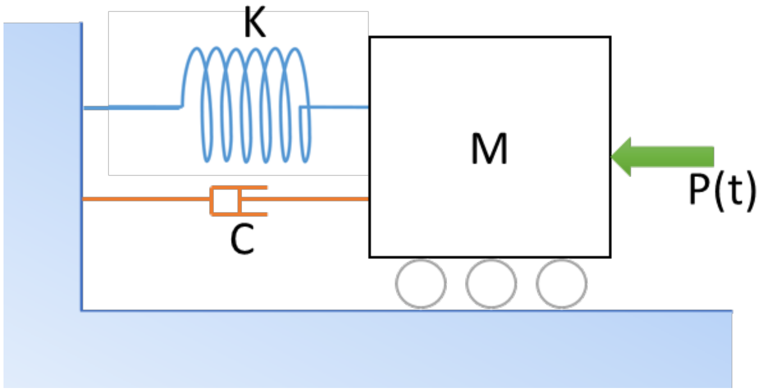
PART - I

Response of Single Degree-of-Freedom systems

Equation of motion



A single degree of freedom system is a system whose motion is defined by a single variable x (t). The archetype single degree of freedom is a spring – mass – damper system, where the mass can move in only one direction (see figure above).

Equation of Motion

This system has potential energy stored in the spring (while the mass is in motion) $PE_{spring}(x)$, Kinetic energy of the moving mass $KE_{mass}(x, dx/dt)$ and the external forces (dissipative forces D(t) and the applied force P(t)), which are

	$PE_{spring}(x) = \frac{1}{2}k\left(x(t)\right)^2$	
	$KE_{mass}\left(x, \frac{dx}{dt}\right) = \frac{1}{2}m\left(\frac{dx(t)}{dt}\right)^2$	
	$E\left(x, \frac{dx}{dt}\right) = -c\left(\frac{dx}{dt}\right) + P(t)$	
	Lagrangian L = KE – PE	

	$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$	
	Lagrangian equation of motion is given by	
	$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} - E = 0$$	
	$$\frac{d}{dt}(m\dot{x}) - (-kx) - (-c\dot{x} + P(t)) = 0$$	
	$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = P(t)$$	
	$$\boldsymbol{m\ddot{x} + c\dot{x} + kx = P(t)}$$	

Or

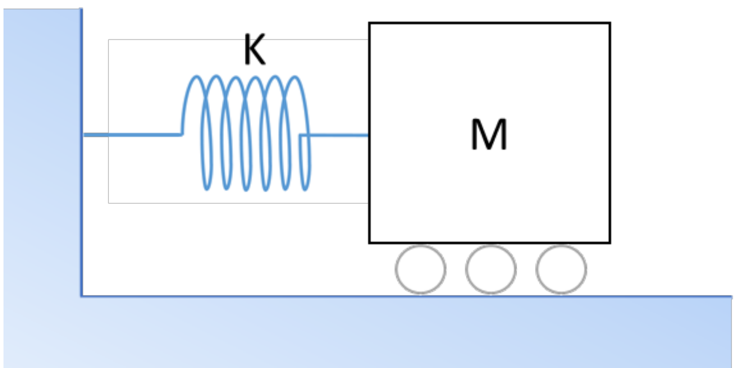
Simply by balancing the forces on the mass, we can arrive at the equation of motion. The forces acting on the mass are

	$$f_I = m\left(\frac{d^2x}{dt^2}\right)\ Internal\ Force$$	
	$$f_D = c\left(\frac{dx}{dt}\right) Dissipative\ Force$$	
	$$f_R = kx\ Restoring\ Force$$	
	$$f_E = P(t)\ External\ Force$$	
	$$\sum F = 0 => f_I + f_D + f_R = f_E$$	

	$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = P(t)$$	
	$$\boldsymbol{m\ddot{x} + c\dot{x} + kx = P(t)}$$	

Free vibration of un-damped system

The equation of motion is a second order ordinary differential equation. First, we solve the linear homogenous ordinary differential equation (free vibration). Consider the system is un-damped and has no external force as shown below



Now the equation of motion is

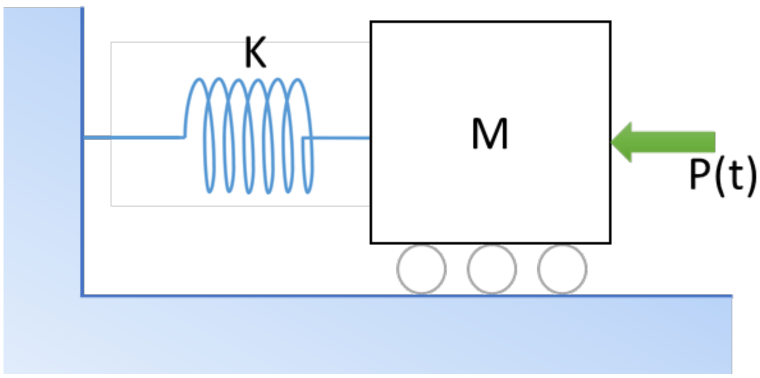
	$$m\ddot{x} + kx = 0$$	
	$$\ddot{x} + \frac{k}{m}x = 0$$	
	Let $\omega_n^2 = k/m$, where ω_n is the natural frequency	
	$$\ddot{x} + {\omega_n}^2 x = 0$$	
	Solution of linear homogenous second order differential equation is of the form,	
	$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$	

	$= C_1(\cos\omega_n t + i\sin\omega_n t) + C_2(\cos\omega_n t - i\sin\omega_n t)$	
	$= (C_1 + C_2)\cos\omega_n t + i(C_1 - C_2)\sin\omega_n t$	
	Let the initial conditions be at t = 0, x(t) = u_0 and dx (0)/dt = du_0/dt (some initial displacement and some velocity)	
	$u_0 = C_1 + C_2$	
	$\frac{du_0}{dt} = i\omega_n(C_1 - C_2)$	
	The solution is	
	Displacement: $\boldsymbol{x(t) = u_0 \cos\omega_n t + \frac{\dot{u}_0}{\omega_n}\sin\omega_n t}$	
	Velocity: $\boldsymbol{\dot{x}(t) = u_0 \cos\omega_n t + \frac{\dot{u}_0}{\omega_n}\sin\omega_n t}$	
	Acceleration: $\boldsymbol{\ddot{x}(t) = u_0 \cos\omega_n t + \frac{\dot{u}_0}{\omega_n}\sin\omega_n t}$	
	The amplitude A and phase angle φ of the motion is	
	$A = \sqrt{\left({u_0}^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2\right)}$	

	$$\emptyset = \tan^{-1}\left(\frac{-\left(\frac{\dot{u}_0}{\omega_n}\right)}{u_0}\right)$$	

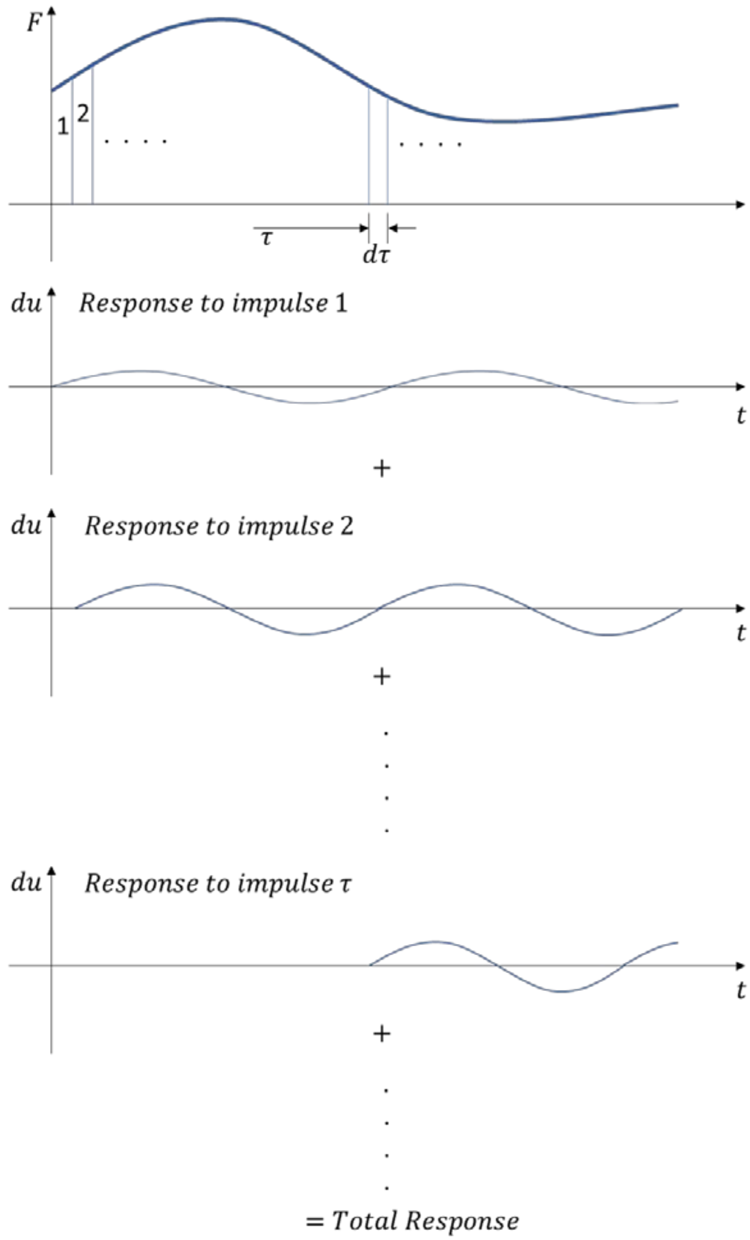
Arbitrary, step and pulse excitation response of un-damped system

Figure below shows the single degree of freedom system without damping and a forcing function P(t).



Let's assume the forcing function to be harmonic of the form

Schematic explanation of convolution integral



Momentum equation

Momentum is the product of an object's mass and velocity

	$P = mv$	

Impulse equation

Impulse is the product of the average net force acting on an object and its duration. It is the force – time integral.

	$J = \bar{F}\Delta t$	
	$J = \int F\, dt$	

Impulse – Momentum

Impulse – momentum theorem states that the change in momentum of an object equals the impulse applied to it. Impulse – momentum theorem is equivalent to Newton's second law of motion.

	$J = \Delta P$	
	$\bar{F}\Delta t = m\Delta v$	

For infinitesimally short duration, unit impulse at $t = \tau$ imparts to the mass, m the velocity is

	$1 = m\dot{u}(\tau)$	
	$\dot{u}(\tau) = \frac{1}{m}$	
	Displacement is zero prior to and up to impulse	
	$u(\tau) = 0$	

Free vibration of SDF system is given by

	$u(t) = u_0 \cos \omega_n t + \frac{\dot{u}_0}{\omega_n} \sin \omega_n t$	

Unit impulse causes free vibration of SDF system due to the initial displacement and initial velocity given by

$$h(t-\tau) = u(t) = \frac{1}{m\omega_n} \sin \omega_n (t-\tau)$$

For,

$$t \geq \tau$$

A force F(t) varying arbitrarily with time can be represented as a sequence of infinitesimally short impulses. The response of a linear dynamic system to one of there impulses, the one at time t of magnitude p(t)dt, is this magnitude times the unit impulse response function.

$$du(t) = [p(\tau)d\tau]h(t-\tau)$$

The response of the system at time t is the sum of the responses to all impulses up to that time, Thus.

$$u(t) = \int_0^t p(\tau)h(t-\tau)d\tau$$

This is known as the convolution integral, a general result that applies to any linear dynamic system.

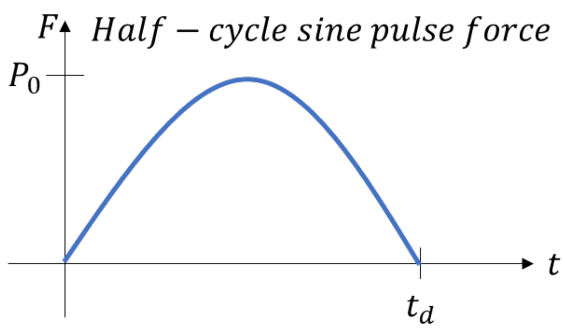
Duhamel integral for undamped system

$$u(t) = \frac{1}{m\omega_n} \int_0^t p(\tau)\ \sin[\omega_n (t-\tau)]\ d\tau$$

Duhamel integral for damped system

	$$u(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}\ \sin[\omega_D(t-\tau)]\ d\tau$$	

Response to half – cycle sine pulse force



	$$u(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\ \sin[\omega_n(t-\tau)]\ d\tau$$	
	For step force	
	$$p(\tau) = P_0 \sin\frac{\pi t}{t_d}$$	
	For $t<t_d$	
	$$u(t) = \frac{P_0}{m\omega_n}\int_0^t \sin\left(\frac{\pi\tau}{t_d}\right)\ \sin[\omega_n(t-\tau)]\ d\tau$$	
	$$u(t) = \frac{P_0}{m\omega_n}\left[\frac{\sin\left[\frac{\pi\tau}{t_d}+\omega_n\tau-\omega_n t\right]}{2\left(\frac{\pi}{t_d}+\omega_n\right)} - \frac{\sin\left[\frac{\pi\tau}{t_d}-\omega_n\tau+\omega_n t\right]}{2\left(\frac{\pi}{t_d}-\omega_n\right)}\right]_0^t$$	
	$$u(t) = \frac{P_0}{m\omega_n}\left(\frac{\frac{\pi}{t_d}\sin(\omega_n t) - \omega_n \sin\left(\frac{\pi t}{t_d}\right)}{\left(\frac{\pi^2}{t_d^2}-\omega_n^2\right)}\right)$$	

	$u(t) = \frac{P_0}{m\omega_n}\left(\frac{(-\omega_n)\left(\sin\left(\frac{\pi t}{t_d}\right) - \frac{\pi}{\omega_n t_d}\sin(\omega_n t)\right)}{(-\omega_n^2)\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)$	
	$u(t) = \frac{P_0}{m\omega_n^2}\left(\frac{\left(\sin\left(\frac{\pi t}{t_d}\right) - \frac{\pi}{\omega_n t_d}\sin(\omega_n t)\right)}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)$	
	$u(t)_{t \le t_d} = \frac{P_0}{k}\left(\frac{\left(\sin\left(\frac{\pi t}{t_d}\right) - \frac{\pi}{\omega_n t_d}\sin(\omega_n t)\right)}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)$	
	For $t>t_d$, the free vibration due to the rectangular pulse force at $t= t_d$ contributes to the response for $t > t_d$	
	$u(t)_{t>t_d} = u(t_d)\cos[\omega_n(t - t_d)] + \frac{\dot{u}(t_d)}{\omega_n}\sin[\omega_n(t - t_d)]$	
	$u(t_d) = \frac{P_0}{k}\left(\frac{\left(-\frac{\pi}{\omega_n t_d}\sin(\omega_n t_d)\right)}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)$	
	$\dot{u}(t_d) = \frac{P_0}{k}\left(-\frac{\pi}{t_d}\right)\left(\frac{(1 + \cos(\omega_n t_d))}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)$	
	Substituting in $u(t)_{t>t_d}$	

	$u(t)_{t>t_d} = \frac{P_0}{k}\left(\frac{\left(-\frac{\pi}{\omega_n t_d}\sin(\omega_n t_d)\right)}{\left(1-\frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)\cos[\omega_n(t-t_d)] - \frac{P_0}{k}\left(\frac{\pi}{\omega_n t_d}\right)\left(\frac{(1+\cos(\omega_n t_d))}{\left(1-\frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)\sin[\omega_n(t-t_d)]$	
	$u(t)_{t>t_d} = -\frac{\frac{P_0}{k}\left(\frac{\pi}{\omega_n t_d}\right)}{\left(1-\frac{\pi^2}{\omega_n^2 t_d^2}\right)}\ (\sin(\omega_n t_d)\cos[\omega_n(t-t_d)] +\ (1+\cos(\omega_n t_d))\sin[\omega_n(t-t_d)])$	
	$u(t)_{t>t_d} = \frac{P_0}{k}\frac{\left(\frac{\pi}{\omega_n t_d}\right)}{\left(\frac{\pi^2}{\omega_n^2 t_d^2}-1\right)}\ (\sin(\omega_n t_d)\cos[\omega_n(t-t_d)] +\ (1+\cos(\omega_n t_d))\sin[\omega_n(t-t_d)])$	
	Simplifying using trigonometric identities (compound – angle formulae)	
	$u(t)_{t>t_d} = \frac{P_0}{k}\frac{\left(\frac{\pi}{\omega_n t_d}\right)}{\left(\frac{\pi^2}{\omega_n^2 t_d^2}-1\right)}\ (\sin[\omega_n(t-t_d)] + \sin\omega_n t)$	
	Simplifying using trigonometric identities (sum and product formulae)	
	$u(t)_{t>t_d} = \frac{P_0}{k}\frac{\left(\frac{2\pi}{\omega_n t_d}\right)}{\left(\frac{\pi^2}{\omega_n^2 t_d^2}-1\right)}\ \left(\sin\left[\omega_n\left(t-\frac{t_d}{2}\right)\right]\cos\left(\frac{\omega_n t_d}{2}\right)\right)$	

The above solution no longer valid for $t_d/T_n = 0.5$. The below solution is the same as resonant vibration at harmonic free vibration response.

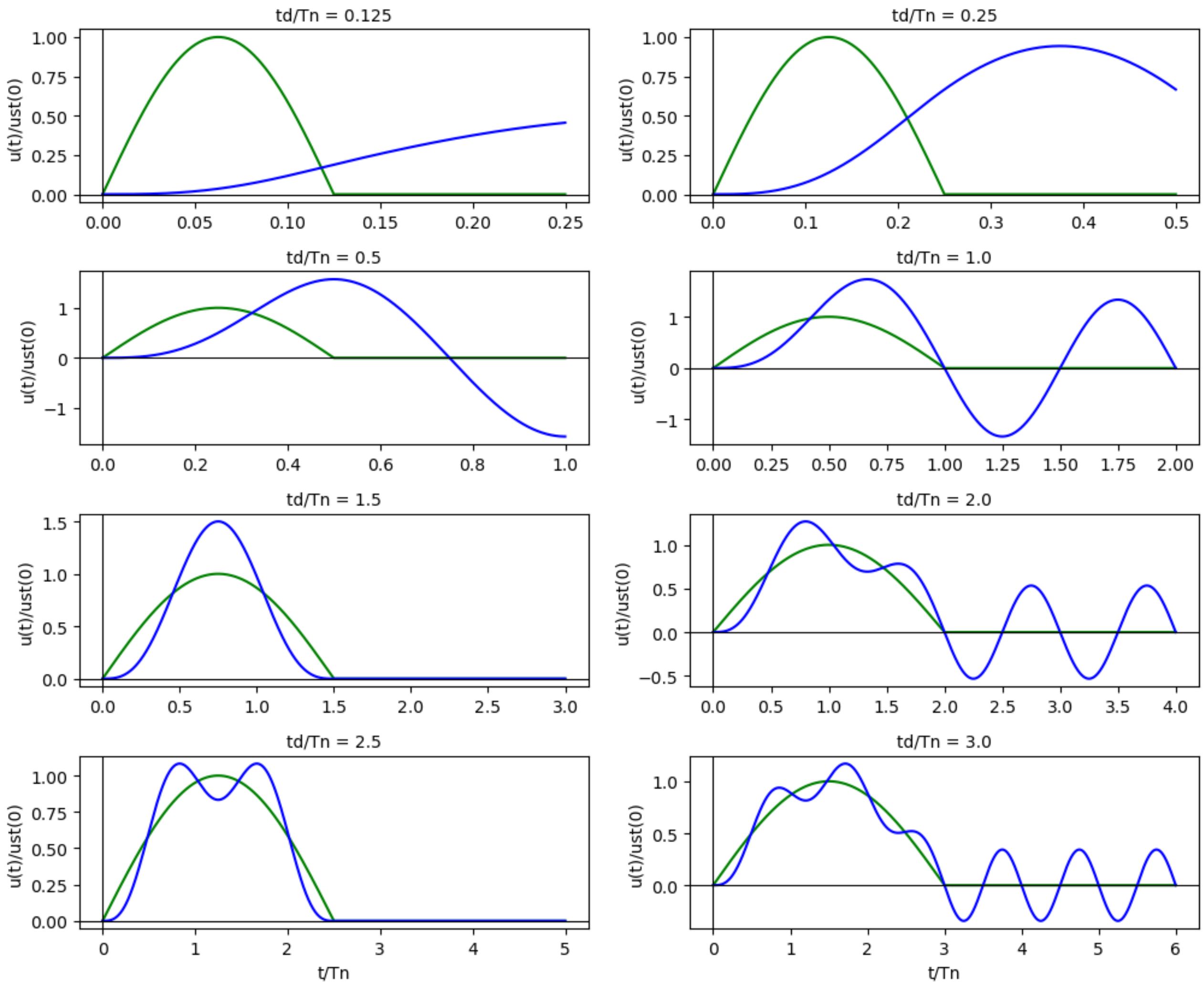
	$$u(t)_{t\le t_d} = \frac{P_0}{2k}(\sin(\omega_n t) - \omega_n t\cos(\omega_n t))$$	
	$$u(t)_{t>t_d} = \frac{P_0\pi}{2k}(\cos(\omega_n t - \pi))$$	

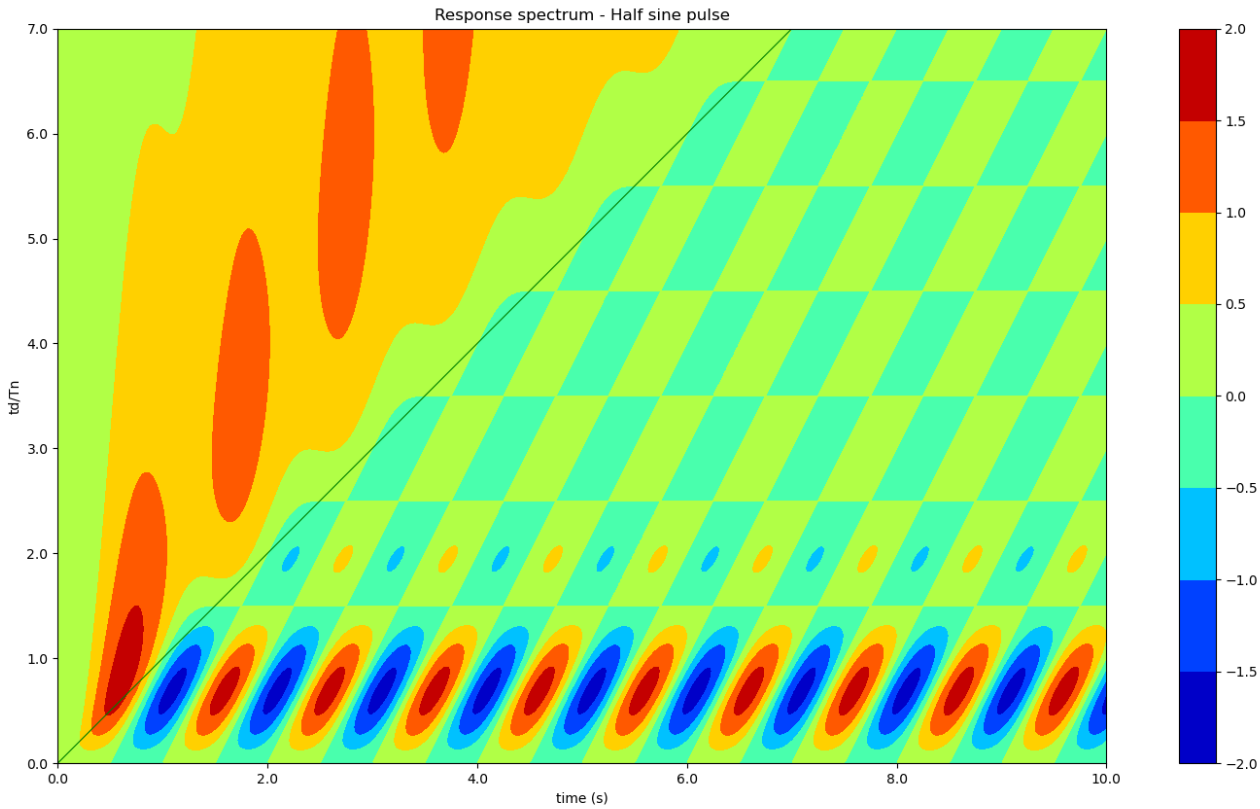
Summary of solution

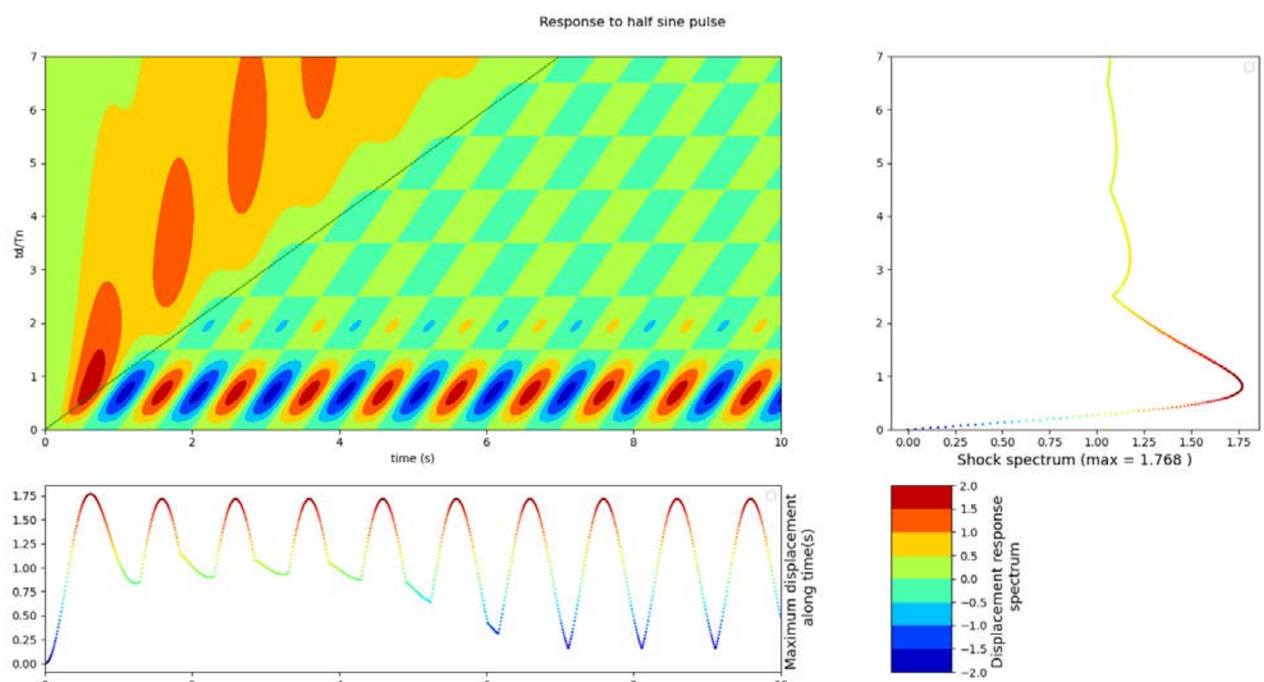
	<u>Analytical solution</u>	
	Displacement (t<= t_d): $$\boldsymbol{u(t)_{t\le t_d} = \frac{P_0}{k}\left(\frac{\left(\sin\left(\frac{\pi t}{t_d}\right) - \frac{\pi}{\omega_n t_d}\sin(\omega_n t)\right)}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)}$$	
	Velocity (t<= t_d): $$\boldsymbol{\dot{u}(t)_{t\le t_d} = \frac{P_0}{k}\left(\frac{\left(\left(\frac{\pi}{t_d}\right)\cos\left(\frac{\pi t}{t_d}\right) - \frac{\pi}{t_d}\cos(\omega_n t)\right)}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)}$$	
	Acceleration (t<= t_d): $$\boldsymbol{\ddot{u}(t)_{t\le t_d} = \frac{P_0}{k}\left(\frac{\left(-\left(\frac{\pi}{t_d}\right)^2\sin\left(\frac{\pi t}{t_d}\right) + \frac{\pi\omega_n}{t_d}\sin(\omega_n t)\right)}{\left(1 - \frac{\pi^2}{\omega_n^2 t_d^2}\right)}\right)}$$	
	Displacement (t> t_d): $$\boldsymbol{u(t)_{t>t_d} = \frac{P_0}{k}\frac{\left(\frac{2\pi}{\omega_n t_d}\right)\cos\left(\frac{\omega_n t_d}{2}\right)}{\left(\frac{\pi^2}{\omega_n^2 t_d^2} - 1\right)}\left(\sin\left[\omega_n\left(t - \frac{t_d}{2}\right)\right]\right)}$$	
	Velocity (t> t_d):	

	$\dot{u}(t)_{t>t_d} = \frac{P_0}{k} \frac{\left(\frac{2\pi}{\omega_n t_d}\right)\cos\left(\frac{\omega_n t_d}{2}\right)}{\left(\frac{\pi^2}{\omega_n^2 t_d^2} - 1\right)} \left(\omega_n \cos\left[\omega_n\left(t - \frac{t_d}{2}\right)\right]\right)$	
	Acceleration ($t>t_d$): $\ddot{u}(t)_{t>t_d} = \frac{P_0}{k} \frac{\left(\frac{2\pi}{\omega_n t_d}\right)\cos\left(\frac{\omega_n t_d}{2}\right)}{\left(\frac{\pi^2}{\omega_n^2 t_d^2} - 1\right)} \left(-\omega_n^2 \sin\left[\omega_n\left(t - \frac{t_d}{2}\right)\right]\right)$	
	<u>Special case solution (when $t_d/T_n = 0.5$)</u>	
	Displacement ($t<= t_d$): $u(t)_{t\le t_d} = \frac{P_0}{2k}(\sin(\omega_n t) - \omega_n t \cos(\omega_n t))$	
	Velocity ($t<=t_d$): $\dot{u}(t)_{t\le t_d} = \frac{P_0}{2k}(\omega_n^2 t \sin(\omega_n t))$	
	Acceleration ($t<=t_d$): $\ddot{u}(t)_{t\le t_d} = \frac{P_0}{2k}\omega_n^2(\sin(\omega_n t) + \omega_n t \cos(\omega_n t))$	
	Displacement ($t>t_d$): $u(t)_{t>t_d} = \frac{P_0 \pi}{2k}(\cos(\omega_n t - \pi))$	
	Velocity ($t>t_d$): $\dot{u}(t)_{t>t_d} = \frac{P_0 \pi}{2k}(\omega_n \sin(\omega_n t))$	
	Acceleration ($t>t_d$): $\ddot{u}(t)_{t>t_d} = \frac{P_0 \pi}{2k}(\omega_n^2 \cos(\omega_n t))$	

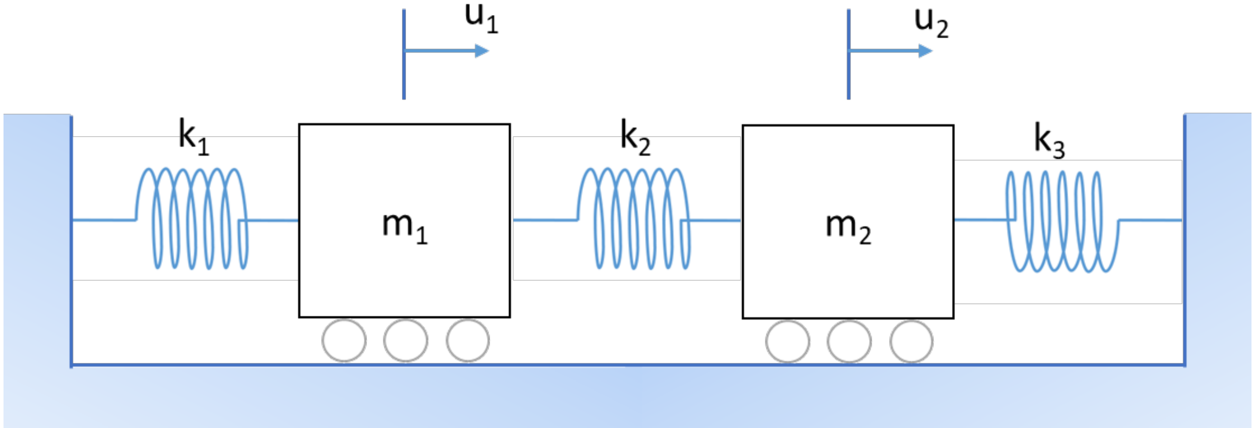
Response to half - sine pulse force



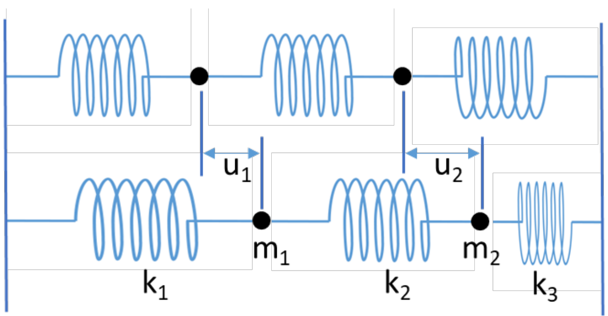

Response spectrum - Half sine pulse
td/Tn
time (s)


Response to half sine pulse
td/Tn
time (s)
Shock spectrum (max = 1.768)
Maximum displacement along time(s)
Displacement response spectrum

Response of Multi Degree-of-Freedom system



At node:1 & 2



	At node 1: $$m_1\ddot{u}_1 = -k_1u_1 - k_2(u_1 - u_2) + p_1$$	
	$$m_1\ddot{u}_1 = -k_1u_1 - k_2u_1 + k_2u_2 + p_1$$	
	$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = p_1$$	(1)
	At node 2: $$m_2\ddot{u}_2 = -k_2(u_2 - u_1) - k_3u_2 + p_2$$	
	$$m_2\ddot{u}_2 = -k_2u_2 + k_2u_1 - k_3u_2 + p_2$$	
	$$m_2\ddot{u}_2 - k_2u_1 + (k_2 + k_3)u_2 = p_2$$	(2)

	Eqn (1) & (2) in matrix form:	
	$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$	(3)
	Free vibration form of the above eqn.	
	$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$	
	$$[M]\ddot{u} + [K]u = 0$$	(4)
	The solution is of the form:	
	$$u = \phi e^{i\omega t}, \ddot{u} = -\omega^2 \phi e^{i\omega t}$$	
	Substitution to eqn (4)	
	$$[K]\phi - \omega^2 [M]\phi = 0$$	
	$$[k - \omega_n^2 m]\phi_n = 0$$	
	Characteristic Equation of the matrix eigenvalue problem	
	$$det[k - \omega_n^2 m] = 0$$	
	Modal and spectral matrices (ω_n and ϕn$(n = 1,2, \ldots, N)$)	
	Modal matrix: $$\Phi = [\phi_{jn}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix}$$	
	Spectral matrix: $$\Omega^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2 \end{bmatrix}$$	
	Then	

	$k\phi_n = m\phi_n\omega_n^2$	
	$=> K\Phi = M\Phi\Omega^2$	
	$\Phi^{\mathrm{T}}K\Phi = \Phi^{\mathrm{T}}M\Phi\Omega^2$	
	$\Phi^{\mathrm{T}}M\Phi = \mathrm{I}$	
	$\Phi^{\mathrm{T}}K\Phi = \Omega^2$	
	$k\phi_n = m\phi_n\omega_n^2$	
	Pre-multiply by ϕ_r^T	
	$\phi_r^T k\phi_n = \phi_r^T m\phi_n\omega_n^2$	
	Orthogonality of modes => when ($\mathrm{n} \neq r$)	
	$\phi_r^T k\phi_n = 0$	
	$\phi_r^T m\phi_n = 0$	
	The general solution to the eqn(3) can be written as a linear superposition of the n modes, each multiplied by a general time varying amplitude q_i.	
	$$u(t) = \sum_{i=1}^{N} q_i(t)\phi_i$$	
	$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{bmatrix} = q_1(t)\begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{N1} \end{bmatrix} + q_2(t)\begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{N2} \end{bmatrix} + q_3(t)\begin{bmatrix} \phi_{13} \\ \phi_{23} \\ \vdots \\ \phi_{N3} \end{bmatrix} + \cdots + q_N(t)\begin{bmatrix} \phi_{1N} \\ \phi_{2N} \\ \vdots \\ \phi_{NN} \end{bmatrix}$$	

	Premultiply the eqn (3) with Φ^{T}	
	$\Phi^{\mathrm{T}}[M]\ddot{u} + \Phi^{\mathrm{T}}[K]u = \Phi^{\mathrm{T}}p(t)$	
	$\Phi^{\mathrm{T}}[M]\Phi\ddot{\mathrm{Q}} + \Phi^{\mathrm{T}}[K]\Phi\mathrm{Q} = \Phi^{\mathrm{T}}p(t)$	
	$I\ddot{\mathrm{Q}} + \Omega^2\mathrm{Q} = \Phi^{\mathrm{T}}p(t)$	
	$I\ddot{\mathrm{Q}} + \Omega^2\mathrm{Q} = R(t)$	
	The i-th typical in equation above can be written as	
	$\ddot{\mathrm{q}}_i + \omega_{\mathrm{i}}^2\mathrm{q}_{\mathrm{i}} = r_i(t)$	
	Where the generalized load vector	
	$r_i(t) = [\phi_{1i} \quad \phi_{2i} \quad \cdots \quad \phi_{Ni}]\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}$	
	For multiple loads	
	$\ddot{\mathrm{q}}_i + \omega_{\mathrm{i}}^2\mathrm{q}_{\mathrm{i}} = \sum_{n=1}^{N} \phi_{ni} * p_n$	
	Using the principle of superposition, the solution is the summation of all response to the individual loads the loads	
	$\ddot{\mathrm{q}}_{ni} + \omega_{\mathrm{i}}^2\mathrm{q}_{\mathrm{ni}} = \phi_{ni} * p_n$	
	$\mathrm{q}_{\mathrm{i}} = \sum_{n=1}^{N} q_{ni}$	
	The conclusion is we can analytically solve multi degree of freedom system with initial condition and pulse force using modal superposition method.	

Linear acceleration method for solving Multi Degree-of-Freedom system.

	$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$	(3)
	Initial calculation:	
	$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1(0) \\ \ddot{u}_2(0) \end{pmatrix} = \begin{pmatrix} p_1(0) \\ p_2(0) \end{pmatrix} - \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$	
	$\begin{pmatrix} \ddot{u}_1(0) \\ \ddot{u}_2(0) \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1}\left(\begin{pmatrix} p_1(0) \\ p_2(0) \end{pmatrix} - \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}\right)$ Select Δt	
	$[\hat{K}] = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix} + (6/\Delta t^2)\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$	
	$[a] = (6/\Delta t)\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$	
	$[b] = 3\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$	
	Calculation for each time step i	
	$\begin{pmatrix} \Delta p_1 \\ \Delta p_2 \end{pmatrix}_i = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{i+1} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_i$	
	$\begin{pmatrix} \Delta\hat{p}_1 \\ \Delta\hat{p}_2 \end{pmatrix}_i = \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \end{pmatrix}_i + [a]\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}_i + [b]\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix}_i$	
	ΔDisplacement $\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}_i = [\hat{K}]^{-1}\begin{pmatrix} \Delta\hat{p}_1 \\ \Delta\hat{p}_2 \end{pmatrix}_i$	
	ΔVelocity	

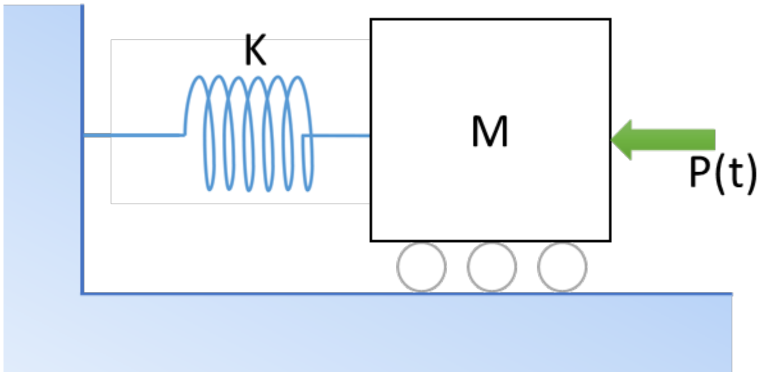
	$$\begin{pmatrix}\Delta\dot{u}_1\\ \Delta\dot{u}_2\end{pmatrix}_i = (3/\Delta t)\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}_i - 3\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix}_i - 0.5\Delta t\begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix}_i$$	
	ΔAcceleration $$\begin{pmatrix}\Delta\ddot{u}_1\\ \Delta\ddot{u}_2\end{pmatrix}_i = (6/\Delta t^2)\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}_i - (6/\Delta t)\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix}_i - 3\begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix}_i$$	
	Solution at step i	
	Displacement $$\begin{pmatrix}u_1\\ u_2\end{pmatrix}_{i+1} = \begin{pmatrix}u_1\\ u_2\end{pmatrix}_i + \begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}_i$$	
	Velocity $$\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix}_{i+1} = \begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix}_i + \begin{pmatrix}\Delta\dot{u}_1\\ \Delta\dot{u}_2\end{pmatrix}_i$$	
	Acceleration $$\begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix}_{i+1} = \begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix}_i + \begin{pmatrix}\Delta\ddot{u}_1\\ \Delta\ddot{u}_2\end{pmatrix}_i$$	

Central difference method for solving Multi Degree-of-Freedom system.

	$$\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}\begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix} + \begin{pmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2+k_3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}p_1(t)\\ p_2(t)\end{pmatrix}$$	(3)
	Initial calculation:	
	$$\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}\begin{pmatrix}\ddot{u}_1(0)\\ \ddot{u}_2(0)\end{pmatrix} = \begin{pmatrix}p_1(0)\\ p_2(0)\end{pmatrix} - \begin{pmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2+k_3\end{pmatrix}\begin{pmatrix}u_1(0)\\ u_2(0)\end{pmatrix}$$	
	$$\begin{pmatrix}\ddot{u}_1(0)\\ \ddot{u}_2(0)\end{pmatrix} = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}^{-1}\left(\begin{pmatrix}p_1(0)\\ p_2(0)\end{pmatrix} - \begin{pmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2+k_3\end{pmatrix}\begin{pmatrix}u_1(0)\\ u_2(0)\end{pmatrix}\right)$$ Select Δt	
	$$\begin{pmatrix}u_1\\ u_2\end{pmatrix}_{-1} = \begin{pmatrix}u_1(0)\\ u_2(0)\end{pmatrix} - \Delta t\begin{pmatrix}\dot{u}_1(0)\\ \dot{u}_2(0)\end{pmatrix} + 0.5\Delta t^2\begin{pmatrix}\ddot{u}_1(0)\\ \ddot{u}_2(0)\end{pmatrix}$$	

	$[a] = (1/\Delta t^2)\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$	
	$[b] = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} - (2/\Delta t^2)\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$	
	Calculation for each time step i	
	$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix}_i = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_i - [a]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{i-1} - [b]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_i$	
	$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{i+1} = [a]^{-1}\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix}_i$	
	$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}_i = \frac{1}{2\Delta t}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{i+1} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{i-1}\right)$	
	$\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix}_i = \frac{1}{\Delta t^2}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{i+1} + 2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_i - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{i-1}\right)$	

One dimensional mass-spring system analytical and numerical result comparison



Mass M = 2

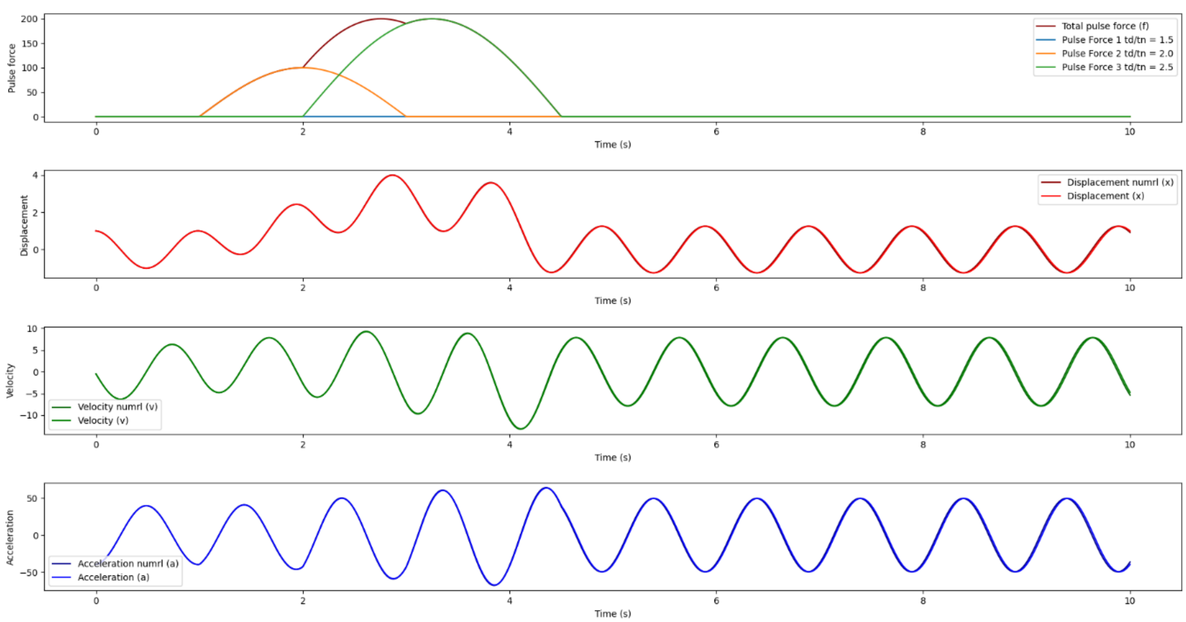
Stiffness K = 78.956835

Initial displacement U0 =1.0

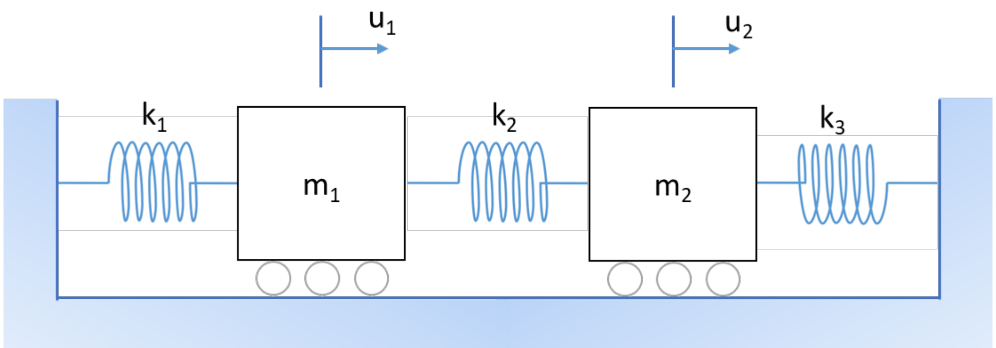
Initial velocity V0 = -0.5

Pulse force 1 = 100, start time = 1.0, end time = 3.0

Pulse force 2 = 200, start time = 2.0, end time = 4.5



Two dimensional mass-spring system analytical and numerical result comparison



Mass m1 = 20, Mass m2 = 30

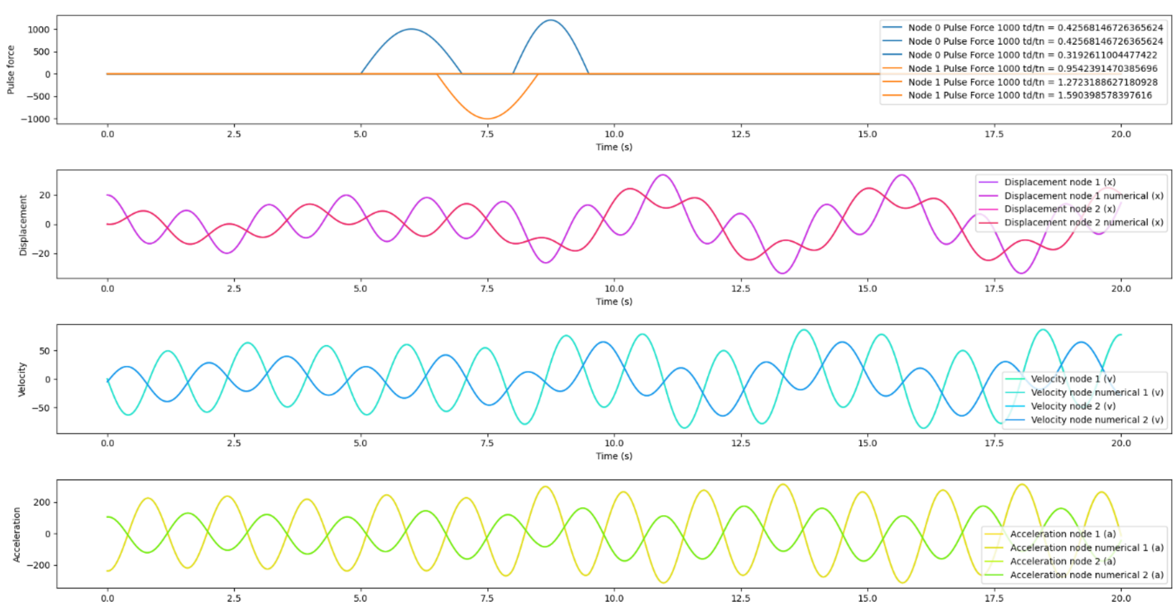
Stiffness k1 = 78.95683521, Stiffness k2 = 157.9136704, Stiffness k3 = 19.7392088

Initial displacement u1(0) = 20.0, Initial velocity v1(0) = 0.0

Initial displacement u2(0) = 0.0, Initial velocity v2(0) = -5.0

Pulse force 1 at node 1 = 1000.0 (5.0, 7.0), Pulse force 2 at node 1 = 1200.0 (8.0, 9.5)

Pulse force 1 at node 2 = -1000.0 (6.5, 8.5)



Analytical solution for the mdof system with pulse force matches with numerical solution.