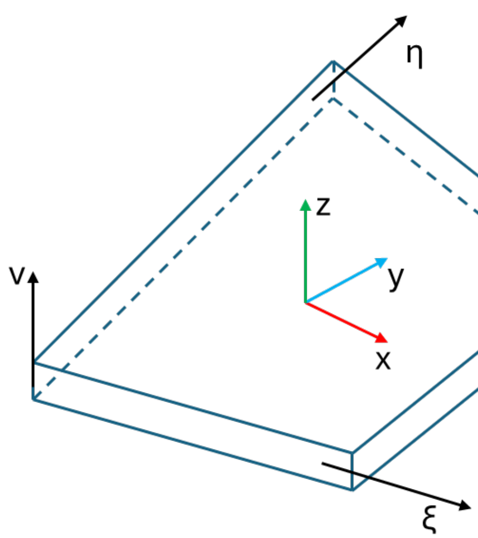
Material properties of the element

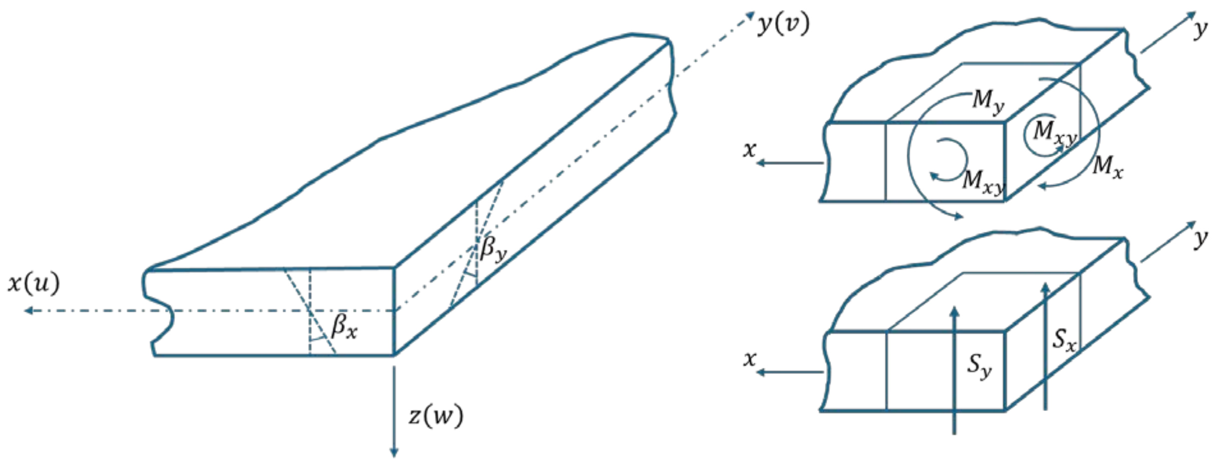
E	Young's modulus (force/length2) for membrane stiffness of isotropic Material
ν	Poisson's ratio
ρ	Material mass density
h	Thickness of the plate



Finite Element Formulation of Kirchoff plate

Assumption of Kirchoff's plate theory

1) The mid plane of the plate is a neutral plane meaning, the mid plane of the plate remains free of in-plane stress/ strain.
2) Plane section remain plane. The line elements lying perpendicular to the middle surface of the plate remain perpendicular to the middle surface during deformation.
3) Line elements lying perpendicular to the mid-surface do not change length during deformation, therefor vertical strain is zero.



Based on these assumptions, the displacement in x, y and z directions are

$$u = -z\frac{\partial w}{\partial x}$$

$$v = -z\frac{\partial w}{\partial y}$$

$$w = w(x, y)$$

And the strain displacement relations are

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = -z \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

It should be noted that the transverse shear deformation is ignored in this theory,

$$\begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = 0$$

For isotropic material, the elasticity matrix is given by

$$\varepsilon_x = \frac{1}{E}\sigma_x - \frac{v}{E}\sigma_y$$

$$\varepsilon_y = \frac{1}{E}\sigma_y - \frac{v}{E}\sigma_x$$

$$\gamma_{xy} = \frac{1+v}{E}\tau_{xy}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{bmatrix}$$

Therefore, the elasticity matrix is given by,

$$[E] = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}$$

The moment stress relations are,

$$[M] = \int_{-h/2}^{h/2} z[\sigma]dz = \int_{-h/2}^{h/2} z[E][\varepsilon]dz = \int_{-h/2}^{h/2} z^2[E][\kappa]dz = \frac{h^3}{12}[E][\kappa]$$

$$[M] = [D_b][\kappa]$$

Where,

$$[D_b] = \frac{h^3}{12}[E]$$

Kirchoff plate theory accounts for the bending deformation but the transverse shear deformation is neglected. The derivation of the finite element expressions is based on the principal of minimum potential energy. The strain energy U in the plate can be written as

$$U = \frac{1}{2}\iiint_{\Omega} [\sigma]^T[\varepsilon]\, dxdydz = \frac{1}{2}\iiint_{\Omega} [\varepsilon]^T[E][\varepsilon]\, dxdydz$$

Substituting,

$$[\varepsilon] = z[\kappa]$$

The strain energy U becomes,

$$U = \frac{1}{2}\iiint_{\Omega} z^2[\kappa]^T[E][\kappa]\, dxdydz$$

Integrating through thickness h in z direction yields

$$U = \frac{1}{2}\iint_{A} \frac{h^3}{6}[\kappa]^T[E][\kappa]dxdy$$

$$= \frac{1}{2}\iint_{A} [\kappa]^T[D_b][\kappa]dxdy$$

Assuming the displacement w within the plate element is interpolated from the elemental nodal degree of freedoms.

$$w = [N][d]$$

Then the curvature of the plate becomes,

$$[\kappa] = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2\frac{\partial^2}{\partial x \partial y} \end{bmatrix} w = -\begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2\frac{\partial^2}{\partial x \partial y} \end{bmatrix} [N][d]$$

Where the strain displacement B matrix is given by,

$$[B] = -\begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2\frac{\partial^2}{\partial x \partial y} \end{bmatrix} [N]$$

This substitution yields,

$$[U] = \frac{1}{2}\iint_A [\kappa]^T [D_b][\kappa] dxdy$$

$$[U] = \frac{1}{2}\iint_A [d]^T [B]^T [D_b][B][d] dxdy$$

$$= \frac{1}{2}[d]^T \left(\iint_A [B]^T [D_b][B] dxdy \right) [d]$$

$$= \frac{1}{2}[d]^T[K][d]$$

The element stiffness matrix is given by

$$[K] = \iint_A [B]^T[D_b][B]dxdy$$

Three node 9 DOF CKZ (Cheung, King and Zienkiewicz) Triangle element CTRIA3

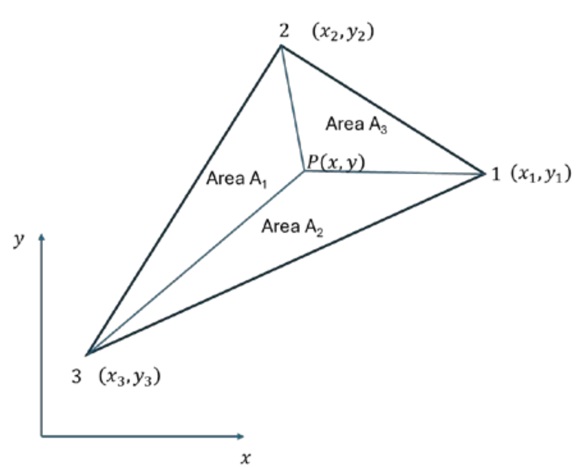
Let us first consider the 3 – noded triangle. The obvious choice of nodal variables gives a total of nine DOFs (w_i, $\partial w/\partial x_i$ and ∂w ∂y_i at each node). A complete cubic polynomial has ten terms and, hence, a problem arises when choosing the term to be dropped out.

To avoid arbitrary removal of single term from the cubic polynomial Bazeley et al. [BCIZ] developed a 3-noded plate triangle with 9 DOFs with so called Area co-ordinates. The element was subsequently modified by Cheung, King and Zienkiewicz [CKZ] (termed hereafter CKZ element).

For any point P inside an arbitrary triangle shown below, the point P(x,y) can be given by the area co-ordinates L_1, L_2, L_3.

$$x = L_1x_1 + L_2x_2 + L_3x_3$$

$$y = L_1y_1 + L_2y_2 + L_3y_3$$



The area co-ordinates are given by

$$L_1 = \frac{A_1}{A}; L_2 = \frac{A_2}{A}; L_3 = \frac{A_3}{A}$$

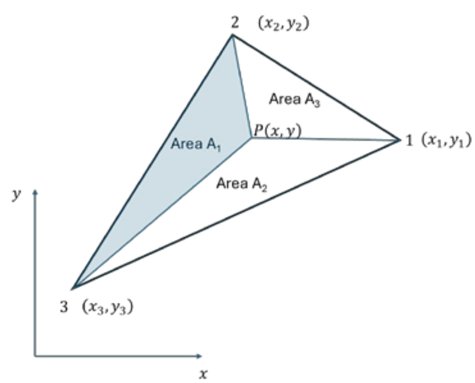
$$L_1 + L_2 + L_3 = 1$$

The area of triangle is given by

$$A = \frac{1}{2} det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A = \frac{1}{2}\left((x_2y_3 - x_3y_2) + (x_1y_2 - x_1y_3) + (y_1x_3 - y_1x_2)\right)$$

Area A1 is given by



$$A_1 = \frac{1}{2} det \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A_1 = \frac{1}{2}\left((x_2y_3 - x_3y_2) + x(y_2 - y_3) + y(x_3 - x_2)\right)$$

$$A_1 = \frac{1}{2}(a_1 + b_1x + c_1y)$$

$a_1 = x_2y_3 - x_3y_2$	$a_2 = x_3y_1 - x_1y_3$	$a_3 = x_1y_2 - x_2y_1$
$b_1 = y_2 - y_3$	$b_2 = y_3 - y_1$	$b_3 = y_1 - y_2$
$c_1 = x_3 - x_2$	$c_2 = x_1 - x_3$	$c_3 = x_2 - x_1$

Now the area co-ordinates are given by,

$$L_1 = \frac{(a_1 + b_1x + c_1y)}{2A}$$

$$L_2 = \frac{(a_2 + b_2x + c_2y)}{2A}$$

$$L_3 = \frac{(a_3 + b_3x + c_3y)}{2A}$$

The starting point is an incomplete cubic expansion of the deflection using area coordinates as

$$w = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4\left(L_1^2 L_2 + \frac{L_1L_2L_3}{2}\right) + \alpha_5\left(L_2^2 L_1 + \frac{L_1L_2L_3}{2}\right) + \alpha_6\left(L_2^2 L_3 + \frac{L_1L_2L_3}{2}\right) + \alpha_7\left(L_3^2 L_2 + \frac{L_1L_2L_3}{2}\right) + \alpha_8\left(L_3^2 L_1 + \frac{L_1L_2L_3}{2}\right) + \alpha_9\left(L_1^2 L_3 + \frac{L_1L_2L_3}{2}\right)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial w}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial w}{\partial L_3}\frac{\partial L_3}{\partial x}$$

$$\frac{\partial w}{\partial L_1} = \alpha_1 + \alpha_4\left(2L_1L_2 + \frac{L_2L_3}{2}\right) + \alpha_5\left(L_2^2 + \frac{L_2L_3}{2}\right) + \alpha_6\left(\frac{L_2L_3}{2}\right) + \alpha_7\left(\frac{L_2L_3}{2}\right) + \alpha_8\left(L_3^2 + \frac{L_2L_3}{2}\right) + \alpha_9\left(2L_1L_3 + \frac{L_2L_3}{2}\right)$$

$$\frac{\partial w}{\partial L_2} = \alpha_2 + \alpha_4\left(L_1^2 + \frac{L_1L_3}{2}\right) + \alpha_5\left(2L_2L_1 + \frac{L_1L_3}{2}\right) + \alpha_6\left(2L_2L_3 + \frac{L_1L_3}{2}\right) + \alpha_7\left(L_3^2 + \frac{L_1L_3}{2}\right) + \alpha_8\left(\frac{L_1L_3}{2}\right) + \alpha_9\left(\frac{L_1L_3}{2}\right)$$

$$\frac{\partial w}{\partial L_3} = \alpha_3 + \alpha_4\left(\frac{L_1L_2}{2}\right) + \alpha_5\left(\frac{L_1L_2}{2}\right) + \alpha_6\left(L_2^2 + \frac{L_1L_2}{2}\right) + \alpha_7\left(2L_3L_2 + \frac{L_1L_2}{2}\right) + \alpha_8\left(2L_3L_1 + \frac{L_1L_2}{2}\right) + \alpha_9\left(L_1^2 + \frac{L_1L_2}{2}\right)$$

Differentiation of the area co-ordinates gives the following

Area co-ordinates	Differentiation w.r.t x	Differentiation w.r.t y
$L_1 = \frac{(a_1 + b_1x + c_1y)}{2A}$	$\frac{\partial L_1}{\partial x} = \frac{b_1}{2A}$	$\frac{\partial L_1}{\partial y} = \frac{c_1}{2A}$
$L_2 = \frac{(a_2 + b_2x + c_2y)}{2A}$	$\frac{\partial L_2}{\partial x} = \frac{b_2}{2A}$	$\frac{\partial L_2}{\partial y} = \frac{c_2}{2A}$
$L_3 = \frac{(a_3 + b_3x + c_3y)}{2A}$	$\frac{\partial L_3}{\partial x} = \frac{b_3}{2A}$	$\frac{\partial L_3}{\partial y} = \frac{c_3}{2A}$

At Node 1,

$x = x_1$, $y = y_1$, $w = w_1$, $dw/dx = (dw/dx)_1$, $dw/dy = (dw/dy)_1$ and $L_1 = 1$, $L_2 = 0$, $L_3 = 0$

$$w_1 = \alpha_1$$

$$\frac{\partial w}{\partial x}_1 = \alpha_1\left(\frac{b_1}{2A}\right) + (\alpha_2 + \alpha_4)\left(\frac{b_2}{2A}\right) + (\alpha_3 + \alpha_9)\left(\frac{b_3}{2A}\right)$$

$$\frac{\partial w}{\partial y}_1 = \alpha_1\left(\frac{c_1}{2A}\right) + (\alpha_2 + \alpha_4)\left(\frac{c_2}{2A}\right) + (\alpha_3 + \alpha_9)\left(\frac{c_3}{2A}\right)$$

At Node 2,

$x = x_2$, $y = y_2$, $w = w_2$, $dw/dx = (dw/dx)_2$, $dw/dy = (dw/dy)_2$ and $L_1 = 0$, $L_2 = 1$, $L_3 = 0$

$$w_2 = \alpha_2$$

$$\frac{\partial w}{\partial x}_2 = (\alpha_1 + \alpha_5)\left(\frac{b_1}{2A}\right) + \alpha_2\left(\frac{b_2}{2A}\right) + (\alpha_3 + \alpha_6)\left(\frac{b_3}{2A}\right)$$

$$\frac{\partial w}{\partial y}_2 = (\alpha_1 + \alpha_5)\left(\frac{c_1}{2A}\right) + \alpha_2\left(\frac{c_2}{2A}\right) + (\alpha_3 + \alpha_6)\left(\frac{c_3}{2A}\right)$$

At Node 3,

$x = x_3$, $y = y_3$, $w = w_3$, $dw/dx = (dw/dx)_3$, $dw/dy = (dw/dy)_3$ and $L_1 = 0$, $L_2 = 0$, $L_3 = 1$

$$w_3 = \alpha_3$$

$$\frac{\partial w}{\partial x}_3 = (\alpha_1 + \alpha_8)\left(\frac{b_1}{2A}\right) + (\alpha_2 + \alpha_7)\left(\frac{b_2}{2A}\right) + \alpha_3\left(\frac{b_3}{2A}\right)$$

$$\frac{\partial w}{\partial y}_3 = (\alpha_1 + \alpha_8)\left(\frac{c_1}{2A}\right) + (\alpha_2 + \alpha_7)\left(\frac{c_2}{2A}\right) + \alpha_3\left(\frac{c_3}{2A}\right)$$

In matrix form,

$$
\begin{bmatrix} w_1 \\ \frac{\partial w}{\partial x}_1 \\ \frac{\partial w}{\partial y}_1 \\ w_2 \\ \frac{\partial w}{\partial x}_2 \\ \frac{\partial w}{\partial y}_2 \\ w_3 \\ \frac{\partial w}{\partial x}_3 \\ \frac{\partial w}{\partial y}_3 \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} & \frac{b_2}{2A} & 0 & 0 & 0 & 0 & \frac{b_3}{2A} \\
\frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} & \frac{c_2}{2A} & 0 & 0 & 0 & 0 & \frac{c_3}{2A} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} & 0 & \frac{b_1}{2A} & \frac{b_3}{2A} & 0 & 0 & 0 \\
\frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} & 0 & \frac{c_1}{2A} & \frac{c_3}{2A} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} & 0 & 0 & 0 & \frac{b_2}{2A} & \frac{b_1}{2A} & 0 \\
\frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} & 0 & 0 & 0 & \frac{c_2}{2A} & \frac{c_1}{2A} & 0
\end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix}
$$

Now we established w, in terms of α, the next step is to eliminate the α terms with an interpolating shape function.

$$
\begin{bmatrix} w_1 \\ \frac{\partial w}{\partial x}_1 \\ \frac{\partial w}{\partial y}_1 \\ w_2 \\ \frac{\partial w}{\partial x}_2 \\ \frac{\partial w}{\partial y}_2 \\ w_3 \\ \frac{\partial w}{\partial x}_3 \\ \frac{\partial w}{\partial y}_3 \end{bmatrix} = [A][\alpha]
$$

$$[\alpha] = [A]^{-1} \begin{bmatrix} w_1 \\ \frac{\partial w}{\partial x}_1 \\ \frac{\partial w}{\partial y}_1 \\ w_2 \\ \frac{\partial w}{\partial x}_2 \\ \frac{\partial w}{\partial y}_2 \\ w_3 \\ \frac{\partial w}{\partial x}_3 \\ \frac{\partial w}{\partial y}_3 \end{bmatrix}$$

Note that,

$$w = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4 \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2} \right) + \alpha_5 \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2} \right) + \alpha_6 \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2} \right) + \alpha_7 \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2} \right) + \alpha_8 \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2} \right) + \alpha_9 \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2} \right)$$

$$w = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2} \right) \\ \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2} \right) \\ \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2} \right) \\ \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2} \right) \\ \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2} \right) \\ \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2} \right) \end{bmatrix}^T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix}$$

Substitute α,

$$
w = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \end{bmatrix}^T [A]^{-1} \begin{bmatrix} w_1 \\ \frac{\partial w}{\partial x}_1 \\ \frac{\partial w}{\partial y}_1 \\ w_2 \\ \frac{\partial w}{\partial x}_2 \\ \frac{\partial w}{\partial y}_2 \\ w_3 \\ \frac{\partial w}{\partial x}_3 \\ \frac{\partial w}{\partial y}_3 \end{bmatrix}
$$

$$
w = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} & \frac{b_2}{2A} & 0 & 0 & 0 & 0 & \frac{b_3}{2A} \\ \frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} & \frac{c_2}{2A} & 0 & 0 & 0 & 0 & \frac{c_3}{2A} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} & 0 & \frac{b_1}{2A} & \frac{b_3}{2A} & 0 & 0 & 0 \\ \frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} & 0 & \frac{c_1}{2A} & \frac{c_3}{2A} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} & 0 & 0 & 0 & \frac{b_2}{2A} & \frac{b_1}{2A} & 0 \\ \frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} & 0 & 0 & 0 & \frac{c_2}{2A} & \frac{c_1}{2A} & 0 \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ \frac{\partial w}{\partial x}_1 \\ \frac{\partial w}{\partial y}_1 \\ w_2 \\ \frac{\partial w}{\partial x}_2 \\ \frac{\partial w}{\partial y}_2 \\ w_3 \\ \frac{\partial w}{\partial x}_3 \\ \frac{\partial w}{\partial y}_3 \end{bmatrix}
$$

$$w = [N]\begin{bmatrix} w_1 \\ \dfrac{\partial w}{\partial x}_1 \\ \dfrac{\partial w}{\partial y}_1 \\ w_2 \\ \dfrac{\partial w}{\partial x}_2 \\ \dfrac{\partial w}{\partial y}_2 \\ w_3 \\ \dfrac{\partial w}{\partial x}_3 \\ \dfrac{\partial w}{\partial y}_3 \end{bmatrix}$$

Where N is the shape function,

$$[N] = \begin{bmatrix} N_1 \\ \bar{N}_1 \\ \bar{\bar{N}}_1 \\ N_2 \\ \bar{N}_2 \\ \bar{\bar{N}}_2 \\ N_3 \\ \bar{N}_3 \\ \bar{\bar{N}}_3 \end{bmatrix}^T = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \dfrac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_1 + \dfrac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_3 + \dfrac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_2 + \dfrac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_1 + \dfrac{L_1 L_2 L_3}{2}\right) \\ \left(L_1^2 L_3 + \dfrac{L_1 L_2 L_3}{2}\right) \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{b_1}{2A} & \dfrac{b_2}{2A} & \dfrac{b_3}{2A} & \dfrac{b_2}{2A} & 0 & 0 & 0 & 0 & \dfrac{b_3}{2A} \\ \dfrac{c_1}{2A} & \dfrac{c_2}{2A} & \dfrac{c_3}{2A} & \dfrac{c_2}{2A} & 0 & 0 & 0 & 0 & \dfrac{c_3}{2A} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{b_1}{2A} & \dfrac{b_2}{2A} & \dfrac{b_3}{2A} & 0 & \dfrac{b_1}{2A} & \dfrac{b_3}{2A} & 0 & 0 & 0 \\ \dfrac{c_1}{2A} & \dfrac{c_2}{2A} & \dfrac{c_3}{2A} & 0 & \dfrac{c_1}{2A} & \dfrac{c_3}{2A} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{b_1}{2A} & \dfrac{b_2}{2A} & \dfrac{b_3}{2A} & 0 & 0 & 0 & \dfrac{b_2}{2A} & \dfrac{b_1}{2A} & 0 \\ \dfrac{c_1}{2A} & \dfrac{c_2}{2A} & \dfrac{c_3}{2A} & 0 & 0 & 0 & \dfrac{c_2}{2A} & \dfrac{c_1}{2A} & 0 \end{bmatrix}^{-1}$$

Using matrixcalc.org the following inverse of A matrix is calculated

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{b_1c_3 - b_3c_1}{b_3c_2 - b_2c_3} & \frac{-2Ac_3}{(b_3c_2 - b_2c_3)} & \frac{2Ab_3}{(b_3c_2 - b_2c_3)} & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \frac{b_2c_3 - b_3c_2}{b_3c_1 - b_1c_3} & \frac{-2Ac_3}{(b_3c_1 - b_1c_3)} & \frac{2Ab_3}{(b_3c_1 - b_1c_3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b_1c_2 - b_2c_1}{b_3c_1 - b_1c_3} & \frac{2Ac_1}{(b_3c_1 - b_1c_3)} & \frac{-2Ab_1}{(b_3c_1 - b_1c_3)} & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \frac{b_1c_3 - b_3c_1}{b_2c_1 - b_1c_2} & \frac{2Ac_1}{(b_2c_1 - b_1c_2)} & \frac{-2Ab_1}{(b_2c_1 - b_1c_2)} \\ -1 & 0 & 0 & 0 & 0 & 0 & \frac{b_3c_2 - b_2c_3}{b_2c_1 - b_1c_2} & \frac{-2Ac_2}{(b_2c_1 - b_1c_2)} & \frac{2Ab_2}{(b_2c_1 - b_1c_2)} \\ \frac{b_2c_1 - b_1c_2}{b_3c_2 - b_2c_3} & \frac{2Ac_2}{(b_3c_2 - b_2c_3)} & \frac{-2Ab_2}{(b_3c_2 - b_2c_3)} & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

One other identity to reduce the above matrix further is the inverse of area matrix

$$2[A] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$\frac{1}{2}[A]^{-1}$$

$$= \begin{bmatrix} \frac{x_3y_2 - x_2y_3}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} & \frac{-x_3y_1 + x_1y_3}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} & \frac{x_2y_1 - x_1y_2}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} \\ \frac{-y_2 + y_3}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} & \frac{y_1 - y_3}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} & \frac{-y_1 + y_2}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} \\ \frac{x_2 - x_3}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} & \frac{-x_1 + x_3}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} & \frac{x_1 - x_2}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} \end{bmatrix}$$

$$= \frac{1}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} \begin{bmatrix} x_3y_2 - x_2y_3 & x_1y_3 - x_3y_1 & x_2y_1 - x_1y_2 \\ y_3 - y_2 & y_1 - y_3 & y_2 - y_1 \\ x_2 - x_3 & x_3 - x_1 & x_1 - x_2 \end{bmatrix}$$

$$= \frac{1}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} \begin{bmatrix} x_3y_2 - x_2y_3 & x_1y_3 - x_3y_1 & x_2y_1 - x_1y_2 \\ y_3 - y_2 & y_1 - y_3 & y_2 - y_1 \\ x_2 - x_3 & x_3 - x_1 & x_1 - x_2 \end{bmatrix}$$

$$= \frac{-1}{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$$\frac{1}{2}[A]^{-1} = \frac{-1}{\mathrm{x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3}} \begin{bmatrix} \mathrm{a_1} & \mathrm{a_2} & \mathrm{a_3} \\ \mathrm{b_1} & \mathrm{b_2} & \mathrm{b_3} \\ \mathrm{c_1} & \mathrm{c_2} & \mathrm{c_3} \end{bmatrix}$$

$$\frac{1}{2}[A]^{-1} = \frac{1}{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)} \begin{bmatrix} \mathrm{a_1} & \mathrm{a_2} & \mathrm{a_3} \\ \mathrm{b_1} & \mathrm{b_2} & \mathrm{b_3} \\ \mathrm{c_1} & \mathrm{c_2} & \mathrm{c_3} \end{bmatrix}$$

$a_1 = x_2y_3 - x_3y_2$	$a_2 = x_3y_1 - x_1y_3$	$a_3 = x_1y_2 - x_2y_1$
$b_1 = y_2 - y_3$	$b_2 = y_3 - y_1$	$b_3 = y_1 - y_2$
$c_1 = x_3 - x_2$	$c_2 = x_1 - x_3$	$c_3 = x_2 - x_1$

Note that

$$det \begin{bmatrix} \mathrm{a_1} & \mathrm{a_2} & \mathrm{a_3} \\ \mathrm{b_1} & \mathrm{b_2} & \mathrm{b_3} \\ \mathrm{c_1} & \mathrm{c_2} & \mathrm{c_3} \end{bmatrix} = 1$$

The reason is It's a change-of-basis matrix from physical triangle coordinates to reference coordinates, and such a matrix has determinant = 1 if the transformation preserves orientation and area.

Therefore,

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 2A$$

This implies the following,

$$b_1c_3 - b_3c_1 = 2A$$

$$b_3c_2 - b_2c_3 = 2A$$

etc,

This will reduce the inverse [A] matrix to the following,

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & c_3 & b_3 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -c_3 & -b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & c_1 & b_1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -c_1 & -b_1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & c_2 & b_2 \\ 1 & -c_2 & -b_2 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Now the shape functions can be found.

$$N_1 = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$N_1 = L_1 + \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) - \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) - \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) + \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2}\right)$$

$$N_1 = L_1 + L_1^2 L_2 + L_1^2 L_3 - L_2^2 L_1 - L_3^2 L_1$$

$$\overline{N}_1 = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -c_2 \end{bmatrix}$$

$$\overline{\overline{N}}_1 = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2}\right) \\ \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2}\right) \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -b_2 \end{bmatrix}$$

Shape functions are shown below,

$$N_1 = L_1 + L_1^2 L_2 + L_1^2 L_3 - L_2^2 L_1 - L_3^2 L_1$$

$$\overline{N}_1 = c_3 \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2} \right) - c_2 \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2} \right)$$

$$\overline{\overline{N}}_1 = b_3 \left(L_1^2 L_2 + \frac{L_1 L_2 L_3}{2} \right) - b_2 \left(L_1^2 L_3 + \frac{L_1 L_2 L_3}{2} \right)$$

$$N_2 = L_2 + L_2^2 L_3 + L_2^2 L_1 - L_3^2 L_2 - L_1^2 L_2$$

$$\overline{N}_2 = c_1 \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2} \right) - c_3 \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2} \right)$$

$$\overline{\overline{N}}_2 = b_1 \left(L_2^2 L_3 + \frac{L_1 L_2 L_3}{2} \right) - b_3 \left(L_2^2 L_1 + \frac{L_1 L_2 L_3}{2} \right)$$

$$N_3 = L_3 + L_3^2 L_1 + L_3^2 L_2 - L_1^2 L_3 - L_2^2 L_3$$

$$\overline{N}_3 = c_2 \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2} \right) - c_1 \left(L_3^2 L_2 + \frac{L_1 L_2 L_3}{2} \right)$$

$$\overline{\overline{N}}_3 = b_2 \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2} \right) - b_1 \left(L_3^2 L_1 + \frac{L_1 L_2 L_3}{2} \right)$$

Note that the curvature of the plate is given by,

$$[\kappa] = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = -\begin{bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} w = -\begin{bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} [N][d]$$

Where the strain displacement B matrix is given by,

$$[B] = -\begin{bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} [N]$$

For three noded triangle, the strain displacement matrix is given by

$$[B] = -\begin{bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} \begin{bmatrix} N_1 & \overline{N}_1 & \overline{\overline{N}}_1 & N_2 & \overline{N}_2 & \overline{\overline{N}}_2 & N_3 & \overline{N}_3 & \overline{\overline{N}}_3 \end{bmatrix}$$

$$[B] = -\begin{bmatrix} \dfrac{\partial^2 N_1}{\partial x^2} & \dfrac{\partial^2 \overline{N}_1}{\partial x^2} & \dfrac{\partial^2 \overline{\overline{N}}_1}{\partial x^2} & \dfrac{\partial^2 N_2}{\partial x^2} & \dfrac{\partial^2 \overline{N}_2}{\partial x^2} & \dfrac{\partial^2 \overline{\overline{N}}_2}{\partial x^2} & \dfrac{\partial^2 N_3}{\partial x^2} & \dfrac{\partial^2 \overline{N}_3}{\partial x^2} & \dfrac{\partial^2 \overline{\overline{N}}_3}{\partial x^2} \\ \dfrac{\partial^2 N_1}{\partial y^2} & \dfrac{\partial^2 \overline{N}_1}{\partial y^2} & \dfrac{\partial^2 \overline{\overline{N}}_1}{\partial y^2} & \dfrac{\partial^2 N_2}{\partial y^2} & \dfrac{\partial^2 \overline{N}_2}{\partial y^2} & \dfrac{\partial^2 \overline{\overline{N}}_2}{\partial y^2} & \dfrac{\partial^2 N_3}{\partial y^2} & \dfrac{\partial^2 \overline{N}_3}{\partial y^2} & \dfrac{\partial^2 \overline{\overline{N}}_3}{\partial y^2} \\ 2\dfrac{\partial^2 N_1}{\partial x \partial y} & 2\dfrac{\partial^2 \overline{N}_1}{\partial x \partial y} & 2\dfrac{\partial^2 \overline{\overline{N}}_1}{\partial x \partial y} & 2\dfrac{\partial^2 N_2}{\partial x \partial y} & 2\dfrac{\partial^2 \overline{N}_2}{\partial x \partial y} & 2\dfrac{\partial^2 \overline{\overline{N}}_2}{\partial x \partial y} & 2\dfrac{\partial^2 N_3}{\partial x \partial y} & 2\dfrac{\partial^2 \overline{N}_3}{\partial x \partial y} & 2\dfrac{\partial^2 \overline{\overline{N}}_3}{\partial x \partial y} \end{bmatrix}$$

The first derivative of shape function is given by the following

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial N_1}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial N_1}{\partial L_3}\frac{\partial L_3}{\partial x}$$

The second derivative of shape function is,

$$\frac{\partial^2 N_1}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial N_1}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial N_1}{\partial L_3}\frac{\partial L_3}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_1}\frac{\partial L_1}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_2}\frac{\partial L_2}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_3}\frac{\partial L_3}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_1}\right)\frac{\partial L_1}{\partial x} + \frac{\partial L_1^2}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_2}\right)\frac{\partial L_2}{\partial x} + \frac{\partial L_2^2}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_3}\right)\frac{\partial L_3}{\partial x} + \frac{\partial L_3^2}{\partial x^2}$$

Note that,

$$\frac{\partial L_1^2}{\partial x^2} = 0 \; etc$$

Therefore,

$$\frac{\partial^2 N_1}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_1}\right)\frac{\partial L_1}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_2}\right)\frac{\partial L_2}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial N_1}{\partial L_3}\right)\frac{\partial L_3}{\partial x}$$

$$= \left(\frac{\partial^2 N_1}{\partial L_1^2}\frac{\partial L_1}{\partial x} + \frac{\partial^2 N_1}{\partial L_1 \partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial^2 N_1}{\partial L_1 \partial L_3}\frac{\partial L_3}{\partial x}\right)\frac{\partial L_1}{\partial x} + \left(\frac{\partial^2 N_1}{\partial L_2 \partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial^2 N_1}{\partial L_2^2}\frac{\partial L_2}{\partial x} + \frac{\partial^2 N_1}{\partial L_2 \partial L_3}\frac{\partial L_3}{\partial x}\right)\frac{\partial L_2}{\partial x} + \left(\frac{\partial^2 N_1}{\partial L_3 \partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial^2 N_1}{\partial L_3 \partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial^2 N_1}{\partial L_3^2}\frac{\partial L_3}{\partial x}\right)\frac{\partial L_3}{\partial x}$$

$$= \frac{\partial^2 N_1}{\partial L_1^2}\left(\frac{\partial L_1}{\partial x}\right)^2 + \frac{\partial^2 N_1}{\partial L_2^2}\left(\frac{\partial L_2}{\partial x}\right)^2 + \frac{\partial^2 N_1}{\partial L_3^2}\left(\frac{\partial L_3}{\partial x}\right)^2 + 2\frac{\partial^2 N_1}{\partial L_1 \partial L_2}\left(\frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial x}\right) + 2\frac{\partial^2 N_1}{\partial L_2 \partial L_3}\left(\frac{\partial L_2}{\partial x}\frac{\partial L_3}{\partial x}\right) + 2\frac{\partial^2 N_1}{\partial L_3 \partial L_1}\left(\frac{\partial L_3}{\partial x}\frac{\partial L_1}{\partial x}\right)$$

Which can be written in a matrix form as below,

$$\frac{\partial^2 N_1}{\partial x^2} = \begin{bmatrix} \left(\frac{\partial L_1}{\partial x}\right)^2 & \left(\frac{\partial L_2}{\partial x}\right)^2 & \left(\frac{\partial L_3}{\partial x}\right)^2 & 2\left(\frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial x}\right) & 2\left(\frac{\partial L_2}{\partial x}\frac{\partial L_3}{\partial x}\right) & 2\left(\frac{\partial L_3}{\partial x}\frac{\partial L_1}{\partial x}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 N_1}{\partial L_1^2} \\ \frac{\partial^2 N_1}{\partial L_2^2} \\ \frac{\partial^2 N_1}{\partial L_3^2} \\ \frac{\partial^2 N_1}{\partial L_1 \partial L_2} \\ \frac{\partial^2 N_1}{\partial L_2 \partial L_3} \\ \frac{\partial^2 N_1}{\partial L_3 \partial L_1} \end{bmatrix}$$

This identity could be expanded to other derivatives as below.

$$\begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} \\ \frac{\partial^2 N_1}{\partial y^2} \\ \frac{\partial^2 N_1}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial L_1}{\partial x}\right)^2 & \left(\frac{\partial L_2}{\partial x}\right)^2 & \left(\frac{\partial L_3}{\partial x}\right)^2 & 2\left(\frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial x}\right) & 2\left(\frac{\partial L_2}{\partial x}\frac{\partial L_3}{\partial x}\right) & 2\left(\frac{\partial L_3}{\partial x}\frac{\partial L_1}{\partial x}\right) \\ \left(\frac{\partial L_1}{\partial y}\right)^2 & \left(\frac{\partial L_2}{\partial y}\right)^2 & \left(\frac{\partial L_3}{\partial y}\right)^2 & 2\left(\frac{\partial L_1}{\partial y}\frac{\partial L_2}{\partial y}\right) & 2\left(\frac{\partial L_2}{\partial y}\frac{\partial L_3}{\partial y}\right) & 2\left(\frac{\partial L_3}{\partial y}\frac{\partial L_1}{\partial y}\right) \\ \left(\frac{\partial L_1}{\partial x}\frac{\partial L_1}{\partial y}\right) & \left(\frac{\partial L_2}{\partial x}\frac{\partial L_2}{\partial y}\right) & \left(\frac{\partial L_3}{\partial x}\frac{\partial L_3}{\partial y}\right) & \frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial y} + \frac{\partial L_1}{\partial y}\frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial x}\frac{\partial L_3}{\partial y} + \frac{\partial L_2}{\partial y}\frac{\partial L_3}{\partial x} & \frac{\partial L_3}{\partial x}\frac{\partial L_1}{\partial y} + \frac{\partial L_3}{\partial y}\frac{\partial L_1}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 N_1}{\partial L_1^2} \\ \frac{\partial^2 N_1}{\partial L_2^2} \\ \frac{\partial^2 N_1}{\partial L_3^2} \\ \frac{\partial^2 N_1}{\partial L_1 \partial L_2} \\ \frac{\partial^2 N_1}{\partial L_2 \partial L_3} \\ \frac{\partial^2 N_1}{\partial L_3 \partial L_1} \end{bmatrix}$$

Let the B matrix be

$$[B] = \begin{bmatrix} B_1 & \bar{B}_1 & \bar{\bar{B}}_1 & B_2 & \bar{B}_2 & \bar{\bar{B}}_2 & B_3 & \bar{B}_3 & \bar{\bar{B}}_3 \end{bmatrix}$$

Where B1 matrix is shown below,

$$[B_1] = -\begin{bmatrix} \dfrac{\partial^2 N_1}{\partial x^2} \\ \dfrac{\partial^2 N_1}{\partial y^2} \\ 2\dfrac{\partial^2 N_1}{\partial x \partial y} \end{bmatrix}$$

So, the idea is to find the B matrix is quite simple once the product first derivative of L matrix and second derivative of N matrices are formed. The N matrices and L matrices are presented in this writeup. The c++ code triCKZ_element.cpp contains the full formulation of B matrix (including the membrane stiffness).

The stiffness matrix formulation presented in this document accounts solely for the bending stiffness, represented by a 9×9 matrix. In addition to this, the membrane stiffness—representing the in-plane stiffness contribution of the shell—also contributes a 9×9 matrix. Together, these form a complete 18×18 stiffness matrix for the shell element.

The derivation of the membrane stiffness matrix is beyond the scope of this document. For a detailed formulation, the reader is referred to the following paper:

"A Triangular Membrane Element with Rotational Degree of Freedom"

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