Chapter 5
Finite Element Method

5.1 Introduction

With the wide spread use of modern computers, the finite element method becomes a powerful technique for the approximate solution of problems in mechanics. Basically, the energy method rather than the governing differential equations is used to solve the problems approximately. In this chapter, two types of plate theories, Kirchhoff theory and Mindlin theory, are reviewed. As a consequent, formulations for plate elements based on these two plate theories are presented.

5.2 Kirchhoff Plate Theory

The assumptions of Kirchhoff Plate Theory have been given in Chapter one. Based on those assumptions, the displacements in the x and y directions are

$$u = -z\frac{\partial w}{\partial x}, \qquad v = -z\frac{\partial w}{\partial y} \tag{5.1}$$

and the strain-displacement relations are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix} = -z\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = z\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{5.2}$$

It should be noted that the transverse shear deformation is ignored in this theory (i.e. $\gamma_{xz} = \gamma_{yz} = 0$).

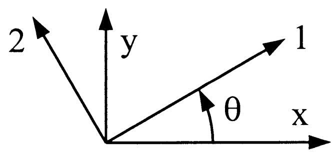

Fig. 5.1

For plates composed of an orthotropic material, the stress-strain relations in the material coordinates (1,2,3) as shown in Fig. 5.1 can be written as

$$\{\sigma'\} = [E']\{\varepsilon'\} \tag{5.3}$$

where $\{\sigma'\} = \{\sigma_1, \sigma_2, \tau_{12}\}^T$, $\{\varepsilon'\} = \{\varepsilon_1, \varepsilon_2, \gamma_{12}\}^T$, and

$$[E'] = \frac{1}{1-\nu_{12}\nu_{21}}\begin{bmatrix} E_{11} & \nu_{12}E_{11} & 0 \\ \nu_{21}E_{22} & E_{22} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{5.4}$$

Let $\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{xy}\}^T$ and $\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T$ be the stresses and strains in the global coordinates (x,y,z). From the coordinate transformation, it is known that

$$\{\varepsilon'\} = [T]\{\varepsilon\} \tag{5.5}$$

$$\{\sigma\} = [T]^T\{\sigma'\} \tag{5.6}$$

where

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{5.7}$$

Then we obtain the stress-strain relations in the global coordinates as

$$\{\sigma\} = [T]^T\{\sigma'\} = [T]^T[E']\{\varepsilon'\} = [T]^T[E'][T]\{\varepsilon\}$$

or

$$\{\sigma\} = [E]\{\varepsilon\} \tag{5.8}$$

where

$$[E] = [T]^T[E'][T] \tag{5.9}$$

If the plates are composed of an isotropic material, the transformation of [E'] matrix into [E] matrix is not necessary and [E] is given as

$$[E] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{5.10}$$

Let us define $\{M\} = \{M_x, M_y, M_{xy}\}^T$ and $\{\kappa\} = \{\kappa_x, \kappa_y, \kappa_{xy}\}^T$. In Chapter one, it has been shown that the moment-stress relations are

$$\{M\} = \int_{-h/2}^{h/2} z\{\sigma\}dz = \int_{-h/2}^{h/2} z[E]\{\varepsilon\}dz = \int_{-h/2}^{h/2} z^2[E]\{\kappa\}dz = \frac{h^3}{12}[E]\{\kappa\}$$

Thus

$$\{M\} = [D]_b\{\kappa\} \tag{5.11}$$

where

$$[D]_b = \frac{h^3}{12}[E] \tag{5.12}$$

5.3 Formulation of the Finite Element Method for Kirchhoff Plate Theory

A conventional approach for the derivation of the finite element expressions is based on the principal of minimum potential energy. Let Ω be the entire volume of the plate. Since transverse shear deformation is neglected in the Kirchhoff theory, strain energy U in the plate is determined entirely by in-plane stresses $\{\sigma\}$ and strains $\{\varepsilon\}$ as follows

$$U = \frac{1}{2}\iiint_{\Omega} \{\sigma\}^T \{\varepsilon\} dxdydz = \frac{1}{2}\iiint_{\Omega} \{\varepsilon\}^T [E] \{\varepsilon\} dxdydz = \frac{1}{2}\iiint_{\Omega} z^2 \{\kappa\}^T [E] \{\kappa\} dxdydz$$

The integration through thickness h in the z direction yields

$$U = \frac{1}{2}\iint_{A} \{\kappa\}^T [D]_b \{\kappa\} dxdy \tag{5.13}$$

where A is the total surface area of the plate. Assuming the displacement w within the i-th plate element is interpolated from the elemental nodal degrees of freedoms $\{d\}_i$, we have

$$w = [N]_w \{d\}_i \tag{5.14}$$

Then the curvatures of the plate become

$$\{\kappa\} = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} w = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} [N]_w \{d\}_i = [B]\{d\}_i \tag{5.15}$$

where

$$[B] = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} [N]_w \tag{5.16}$$

The substitution of Eq. (5.15) into Eq. (5.13) yields

$$U = \frac{1}{2}\sum_{i=1}^{n} \iint_{A_i} \{d\}_i^T [B]^T [D]_b [B] \{d\}_i dxdy = \frac{1}{2}\sum_{i=1}^{n} \{d\}_i^T \left(\iint_{A_i} [B]^T [D]_b [B] dxdy \right) \{d\}_i$$

where n is the total number of element and A_i is the surface area of the i-th element. Let $[k]_i$ be the stiffness matrix of the i-th element, then

$$[k]_i = \iint_{A_i} [B]^T [D]_b [B] dxdy \tag{5.17}$$

Hence, the strain energy of the plate can be written as

$$U = \frac{1}{2}\sum_{i=1}^{n} \{d\}_i^T [k]_i \{d\}_i \tag{5.18}$$

From Chapter four, the potential of external forces for a plate subjected to lateral load q(x,y) can be written as

$$V = -\iint_A qw dxdy \tag{5.19}$$

Substituting Eq. (5.14) into the above expression, we obtain

$$V = -\sum_{i=1}^{n} \iint_{A_i} q[N]_W \{d\}_i dxdy = -\sum_{i=1}^{n} \iint_{A_i} q\{d\}_i^T [N]_W^T dxdy = -\sum_{i=1}^{n} \{d\}_i^T \iint_{A_i} q[N]_W^T dxdy$$

Let $\{r\}_i$ be the nodal load vector of the i-th element. Then

$$\{r\}_i = \iint_{A_i} q[N]_W^T dxdy \tag{5.20}$$

Hence, we obtain

$$V = -\sum_{i=1}^{n} \{d\}_i^T \{r\}_i \tag{5.21}$$

After the strain energy and the potential of lateral load are obtained, the total potential energy Π of the plate is

$$\Pi = U + V = \frac{1}{2}\sum_{i=1}^{n} \{d\}_i^T [k]_i \{d\}_i - \sum_{i=1}^{n} \{d\}_i^T \{r\}_i$$

or

$$\Pi = \frac{1}{2}\{u\}^T [K]\{u\} - \{u\}^T \{R\} \tag{5.22}$$

where $\{u\}$ is the nodal degrees of freedoms for the entire plate. The $[K]$ and $\{R\}$ are the structural stiffness matrix and the structural nodal load vector of the plate given as follow

$$[K] = \sum_{i=1}^{n} [k]_i \tag{5.23}$$

$$[R] = \sum_{i=1}^{n} \{r\}_i \tag{5.24}$$

Applying the principle of minimum potential energy to Eq. (5.22), we get

$$\delta\Pi = \frac{1}{2}\left\{\delta\{u\}^T [K]\{u\} + \{u\}^T [K]\delta\{u\}\right\} - \delta\{u\}^T \{R\} = \delta\{u\}^T [K]\{u\} - \delta\{u\}^T \{R\} = 0$$

Rearranging the above equation, we obtain the finite element expression for plate as

$$[K]\{u\} = \{R\} \tag{5.25}$$

5.4 Kirchhoff Plate Element

Although many types of Kirchhoff elements are available (different displacement assumption, different geometry), we will consider only the rectangular plate element with 12 degrees of freedoms. This was the earliest successful element developed, and although it is not ideal, very good results can be achieved with it.

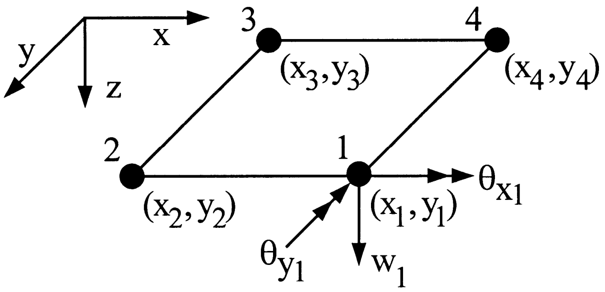

Fig. 5.2

Let us consider the four-noded rectangular plate element shown in Fig. 5.2. There are three degrees of freedoms $(w_j, \theta_{xj}, \theta_{yj})$ per node. Hence, the vector $\{d\}_i$ for the element nodal degrees of freedoms becomes

$$\{d\}_i = \{w_1, \theta_{x1}, \theta_{y1}, w_2, \theta_{x2}, \theta_{y2}, w_3, \theta_{x3}, \theta_{y3}, w_4, \theta_{x4}, \theta_{y4}\}^T \tag{5.26}$$

The deflection w within the element can be approximated by the following polynomial expression as

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \tag{5.27a}$$

or

$$w = \{L\}\{a\} \tag{5.27b}$$

where

$$\{L\} = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3\} \tag{5.27c}$$

$$\{a\} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}^T \tag{5.27d}$$

The rotations of the plate can be related to the deflection as

$$\theta_x = \frac{\partial w}{\partial y} = a_3 + a_5x + 2a_6y + a_8x^2 + 2a_9xy + 3a_{10}y^2 + a_{11}x^3 + 3a_{12}xy^2 \tag{5.28a}$$

$$\theta_y = \frac{\partial w}{\partial x} = a_2 + 2a_4x + a_5y + 3a_7x^2 + 2a_8xy + a_9y^2 + 3a_{11}x^2y + a_{12}y^3 \tag{5.28b}$$

Inserting the elemental local coordinates of the j-th node into Eqs. (5.27) and (5.28), we obtain

$$\begin{Bmatrix} w_j \\ \theta_{xj} \\ \theta_{yj} \end{Bmatrix} = \begin{bmatrix} 1 & x_j & y_j & x_j^2 & x_jy_j & y_j^2 & x_j^3 & x_j^2y_j & x_jy_j^2 & y_j^3 & x_j^3y_j & x_jy_j^3 \\ 0 & 0 & 1 & 0 & x_j & 2y_j & 0 & x_j^2 & 2x_jy_j & 3y_j^2 & x_j^3 & 3x_jy_j^2 \\ 0 & 1 & 0 & 2x_j & y_j & 0 & 3x_j^2 & 2x_jy_j & y_j^2 & 0 & 3x_j^2y_j & y_j^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \dots \\ a_{12} \end{Bmatrix} \quad (5.29)$$

where j = 1, 2, 3, and 4. Thus, we can express $\{d\}_i$ as

$$\{d\}_i = [C]\{a\} \quad (5.30)$$

The [C] is a 12 by 12 matrix given in Table 5.1. The vector {a} can be related to $\{d\}_i$ by

$$\{a\} = [C]^{-1}\{d\}_i \quad (5.31)$$

Table 5.1

$$[C] = \begin{bmatrix}
1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & x_1^3 & x_1^2y_1 & x_1y_1^2 & y_1^3 & x_1^3y_1 & x_1y_1^3 \\
0 & 0 & 1 & 0 & x_1 & 2y_1 & 0 & x_1^2 & 2x_1y_1 & 3y_1^2 & x_1^3 & 3x_1y_1^2 \\
0 & 1 & 0 & 2x_1 & y_1 & 0 & 3x_1^2 & 2x_1y_1 & y_1^2 & 0 & 3x_1^2y_1 & y_1^3 \\
1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & x_2^3 & x_2^2y_2 & x_2y_2^2 & y_2^3 & x_2^3y_2 & x_2y_2^3 \\
0 & 0 & 1 & 0 & x_2 & 2y_2 & 0 & x_2^2 & 2x_2y_2 & 3y_2^2 & x_2^3 & 3x_2y_2^2 \\
0 & 1 & 0 & 2x_2 & y_2 & 0 & 3x_2^2 & 2x_2y_2 & y_2^2 & 0 & 3x_2^2y_2 & y_2^3 \\
1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 & x_3^3 & x_3^2y_3 & x_3y_3^2 & y_3^3 & x_3^3y_3 & x_3y_3^3 \\
0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & x_3^2 & 2x_3y_3 & 3y_3^2 & x_3^3 & 3x_3y_3^2 \\
0 & 1 & 0 & 2x_3 & y_3 & 0 & 3x_3^2 & 2x_3y_3 & y_3^2 & 0 & 3x_3^2y_3 & y_3^3 \\
1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 & x_4^3 & x_4^2y_4 & x_4y_4^2 & y_4^3 & x_4^3y_4 & x_4y_4^3 \\
0 & 0 & 1 & 0 & x_4 & 2y_4 & 0 & x_4^2 & 2x_4y_4 & 3y_4^2 & x_4^3 & 3x_4y_4^2 \\
0 & 1 & 0 & 2x_4 & y_4 & 0 & 3x_4^2 & 2x_4y_4 & y_4^2 & 0 & 3x_4^2y_4 & y_4^3
\end{bmatrix}$$

Substituting the above expression into Eq. (5.27b), we obtain

$$w = \{L\}\{a\} = \{L\}[C]^{-1}\{d\}_i = [N]_W\{d\}_i \quad (5.32)$$

$$[N]_W = \{L\}[C]^{-1} \quad (5.33)$$

The approximate curvatures at any point in the element can be determined by differentiating the expression for w as follows

$$\{\kappa\} = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{2\partial^2}{\partial x \partial y} \end{Bmatrix} w = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{2\partial^2}{\partial x \partial y} \end{Bmatrix} \{L\}\{a\} = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{2\partial^2}{\partial x \partial y} \end{Bmatrix} \{L\}[C]^{-1}\{d\}_i = [H][C]^{-1}\{d\}_i = [B]\{d\}_i \quad (5.34)$$

where

$$[H] = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} \{L\} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4x & -4y & 0 & -6x^2 & -6y^2 \end{bmatrix} \tag{5.35}$$

$$[B] = [H][C]^{-1} \tag{5.36}$$

Finally, the stiffness matrix $[k]_i$ and the nodal load vector $\{r\}_i$ for the i-th element are

$$[k]_i = \iint_{A_i} [B]^T [D]_b [B] dxdy = [[C]^{-1}]^T \left(\iint_{A_i} [H]^T [D]_b [H] dxdy \right) [C]^{-1} \tag{5.37}$$

$$\{r\}_i = \iint_{A_i} q[N]_w^T dxdy = [[C]^{-1}]^T \left(\iint_{A_i} q\{L\}^T dxdy \right) \tag{5.38}$$

5.5 Mindlin Plate Theory

The assumptions for the Mindlin plate theory are all the same as those for the Kirchhoff plate theory except the assumption (4) in Section 1.4. In the Mindlin plate theory, a line that is straight and normal to the midsurface before loading is assumed to remain straight but not necessarily normal to the midsurface after loading. Thus transverse shear deformation is allowed. The motion of a point not on the midsurface is not governed by slopes $\partial w / \partial x$ and $\partial w / \partial y$ as in the Kirchhoff theory. Rather, its motion is dependent on rotations β_x and β_y of lines that were normal to the midsurface of the undeformed plate (Fig. 5.3).

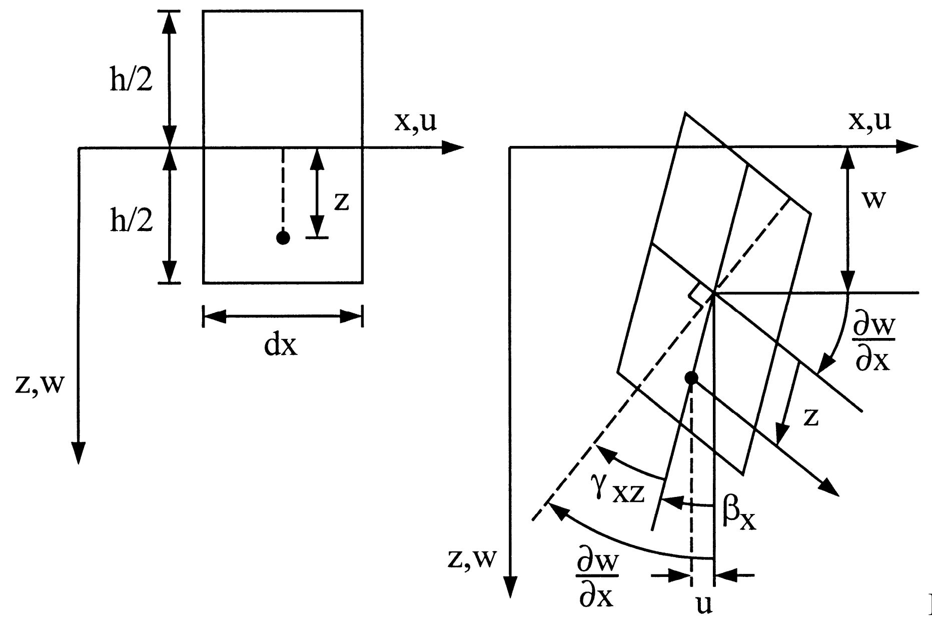

Fig. 5.3

Since there is no membrane stretching, the displacements of the plate are

$$u = -z\beta_x \tag{5.39a}$$

$$v = -z\beta_y \tag{5.39b}$$

and the inplane strain-displacement relations are

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = z \begin{Bmatrix} -\dfrac{\partial \beta_x}{\partial x} \\ -\dfrac{\partial \beta_y}{\partial y} \\ -\dfrac{\partial \beta_x}{\partial y} - \dfrac{\partial \beta_y}{\partial x} \end{Bmatrix} = z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = z\{\kappa\} \tag{5.40}$$

where $\kappa_x = -\partial\beta_x / \partial x$, $\kappa_y = -\partial\beta_y / \partial y$, $\kappa_{xy} = -\partial\beta_x / \partial y - \partial\beta_y / \partial x$ and $\{\kappa\} = \{\kappa_x, \kappa_y, \kappa_{xy}\}^T$. The transverse shear strains are assumed to be constant through the thickness of the plate. Thus

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{Bmatrix} \tag{5.41}$$

The inplane stress-strain relations are the same as those for Kirchhoff theory.

$$\{\sigma\} = [E]\{\varepsilon\}$$

where [E] is given by Eqs. (5.9) and (5.4) for orthotropic materials and by Eq. (5.10) for isotropic materials.

For orthotropic materials, the transverse shear stress-strain relations in the material coordinate (1,2,3) (Fig. 5.1) are

$$\{\tau'\} = [G']\{\gamma'\} \tag{5.42}$$

where $\{\tau'\} = \{\tau_{13}, \tau_{23}\}^T$, $\{\gamma'\} = \{\gamma_{13}, \gamma_{23}\}^T$, and

$$[G'] = \begin{bmatrix} G_{13} & 0 \\ 0 & G_{23} \end{bmatrix} \tag{5.43}$$

Let $\{\tau\} = \{\tau_{xz}, \tau_{yz}\}^T$ and $\{\gamma\} = \{\gamma_{xz}, \gamma_{yz}\}^T$ be the transverse shear stresses and strains in the global coordinates (x, y, z). From the coordinate transformation, it is known that

$$\{\gamma'\} = [T_s]\{\gamma\} \tag{5.44}$$

$$\{\tau\} = [T_s]^T\{\tau'\} \tag{5.45}$$

where

$$[T_s] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5.46}$$

Then we obtain the transverse shear stress-strain relations in the global coordinates as

$$\{\tau\} = [T_s]^T\{\tau'\} = [T_s]^T[G']\{\gamma'\} = [T_s]^T[G'][T_s]\{\gamma\} = [G]\{\gamma\}$$

where

$$[G] = [T_s]^T [G'][T_s] \tag{5.47}$$

If the plates are composed of an isotropic material, the transformation of $[G']$ matrix into $[G]$ matrix is not necessary and $[G]$ is given as

$$[G] = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} = \begin{bmatrix} \dfrac{E}{2(1+\nu)} & 0 \\ 0 & \dfrac{E}{2(1+\nu)} \end{bmatrix} \tag{5.48}$$

It has been shown in Chapter one that the transverse shear forces $\{V\} = \{V_x, V_y\}^T$ can be obtained by integrating the transverse shear stresses through the thickness as

$$\{V\} = \int_{-h/2}^{h/2} \{\tau\} dz = \int_{-h/2}^{h/2} [G]\{\gamma\} dz = [G]\{\gamma\} \int_{-h/2}^{h/2} dz = [D]_s \{\gamma\} \tag{5.49}$$

where

$$[D]_s = \frac{h}{\alpha}[G] \tag{5.50}$$

and α is a shear correction factor to account for the actual nonuniform distribution of τ_{xz} and τ_{yz}. In the Mindlin plate theory, except the definition of $\{\kappa\}$ is different from that of the Kirchhoff theory, the moment-curvature relations are the same as those for the Kirchhoff theory.

$$\{M\} = \int_{-h/2}^{h/2} z\{\sigma\} dz = \int_{-h/2}^{h/2} z[E]\{\varepsilon\} dz = \int_{-h/2}^{h/2} z^2 [E]\{\kappa\} dz = \frac{h^3}{12}[E]\{\kappa\} = [D]_b \{\kappa\}$$

Combining the bending moments and the transverse shear forces together, we obtain the following relations

$$\begin{Bmatrix} \{M\} \\ \{V\} \end{Bmatrix} = \begin{bmatrix} [D]_b & [0] \\ [0]^T & [D]_s \end{bmatrix} \begin{Bmatrix} \{\kappa\} \\ \{\gamma\} \end{Bmatrix} \tag{5.51}$$

or

$$\{\underline{M}\} = [D]\{\underline{k}\} \tag{5.52}$$

where $\{\underline{M}\} = \{M_x, M_y, M_{xy}, V_x, V_y\}^T$, $\{\underline{\kappa}\} = \{\kappa_x, \kappa_y, \kappa_{xy}, \gamma_{xz}, \gamma_{yz}\}^T$ and

$$[D] = \begin{bmatrix} [D]_b & [0] \\ [0]^T & [D]_s \end{bmatrix} \tag{5.53a}$$

$$[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.53b}$$

5.6 Formulation of the Finite Element Method for Mindlin Plate Theory

Mindlin plate theory accounts for the bending deformation and for transverse shear deformation. Hence, the strain energy U in the plate can be written as

$$U = \frac{1}{2}\iiint_{\Omega}\left[\{\sigma\}^T\{\varepsilon\} + \{\tau\}^T\{\gamma\}\right]dxdydz = \frac{1}{2}\iiint_{\Omega}\left[\{\varepsilon\}^T[E]\{\varepsilon\} + \{\gamma\}^T[G]\{\gamma\}\right]dxdydz$$

$$= \frac{1}{2}\iiint_{\Omega}\left[z^2\{\kappa\}^T[E]\{\kappa\} + \{\gamma\}^T[G]\{\gamma\}\right]dxdydz$$

The integration through thickness h in the z direction yields

$$U = \frac{1}{2}\iint_{A}\left[\{\kappa\}^T[D]_b\{\kappa\} + \{\gamma\}^T[D]_s\{\gamma\}\right]dxdy = \frac{1}{2}\iint_{A}\{\underline{\kappa}\}^T[D]\{\underline{\kappa}\}dxdy \tag{5.54}$$

Assuming that w, β_x and β_y within the i-th plate element are interpolated from the elemental nodal degrees of freedoms $\{d\}_i$ by the following expression

$$\begin{Bmatrix} w \\ \beta_x \\ \beta_y \end{Bmatrix} = [N]\{d\}_i \tag{5.55}$$

Then we have

$$\{\underline{\kappa}\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial\beta_x}{\partial x} \\ -\dfrac{\partial\beta_y}{\partial y} \\ -\dfrac{\partial\beta_x}{\partial y} - \dfrac{\partial\beta_y}{\partial x} \\ \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{Bmatrix} = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & -\dfrac{\partial}{\partial y} \\ 0 & -\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & -1 & 0 \\ \dfrac{\partial}{\partial y} & 0 & -1 \end{bmatrix} \begin{Bmatrix} w \\ \beta_x \\ \beta_y \end{Bmatrix} = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & -\dfrac{\partial}{\partial y} \\ 0 & -\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & -1 & 0 \\ \dfrac{\partial}{\partial y} & 0 & -1 \end{bmatrix} [N]\{d\}_i$$

or

$$\{\underline{\kappa}\} = [B]\{d\}_i \tag{5.56}$$

where

$$[B] = [\partial][N], \qquad [\partial] = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & -\dfrac{\partial}{\partial y} \\ 0 & -\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & -1 & 0 \\ \dfrac{\partial}{\partial y} & 0 & -1 \end{bmatrix} \tag{5.57}$$

The substitution of Eq. (5.56) into Eq. (5.54) yields

$$U=\frac{1}{2}\sum_{i=1}^{n}\iint_{A_i}\{d\}_i^T[B]^T[D][B]\{d\}_i\,dxdy=\frac{1}{2}\sum_{i=1}^{n}\{d\}_i^T\left(\iint_{A_i}[B]^T[D][B]dxdy\right)\{d\}_i$$

$$=\frac{1}{2}\sum_{i=1}^{n}\{d\}_i^T[k]_i\{d\}_i$$

where $[k]_i$ is the stiffness matrix for the i-th element given as follows

$$[k]_i=\iint_{A_i}[B]^T[D][B]dxdy \tag{5.58}$$

The potential of external load q(x,y) for a plate is

$$V=-\iint_A qwdxdy$$

Similar to the Kirchhoff theory, we can assume that w within the i-th plate element can be interpolated from the elemental nodal degrees of freedoms $\{d\}_i$ by the following expression

$$w=[N]_W\{d\}_i$$

Hence, the potential of the load becomes

$$V=-\sum_{i=1}^{n}\iint_{A_i}q[N]_W\{d\}_i dxdy=-\sum_{i=1}^{n}\iint_{A_i}q\{d\}_i^T[N]_W^T dxdy$$

$$=-\sum_{i=1}^{n}\{d\}_i^T\iint_{A_i}q[N]_W^T dxdy=-\sum_{i=1}^{n}\{d\}_i^T\{r\}_i$$

where $\{r\}_i$ is the nodal load vector of the i-th element as before

$$\{r\}_i=\iint_{A_i}q[N]_W^T dxdy$$

Since the expressions for the strain energy of the plate and for the potential of the external load are similar to those of the Kirchhoff theory, we will obtain the same finite element expression as before,

$$[K]\{u\}=\{R\}$$

where $\{u\}$ is the nodal degrees of freedoms for the entire plate and

$$[K]=\sum_{i=1}^{n}[k]_i$$

$$\{R\}=\sum_{i=1}^{n}\{r\}_i$$

5.7 Mindlin Plate Elements

Mindlin plate elements account for bending deformation and transverse shear deformation. Accordingly, they may be used to analyze thick plates as well as thin plates.

Typical Mindlin plate elements may have 4 nodes, 8 nodes, 9 nodes, 12 nodes and 16 nodes as shown in Fig. 5.4.

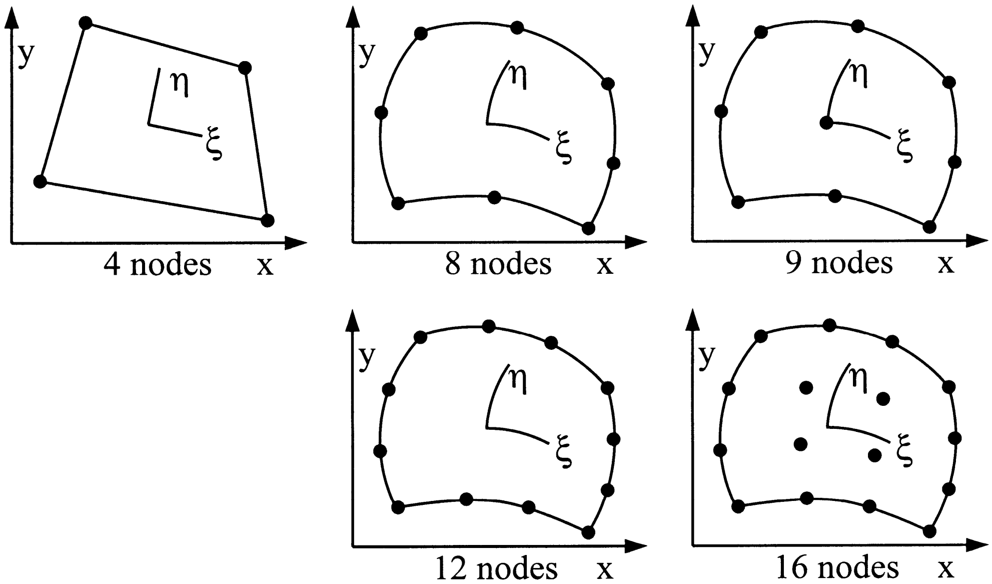

Fig. 5.4

We will use the four-noded plate element as an example to illustrate the formulation of Mindlin plate element. Let us consider the plate element shown in Fig. 5.5. The coordinates of a point in the element can be expressed by using the natural coordinates (ξ, η) as

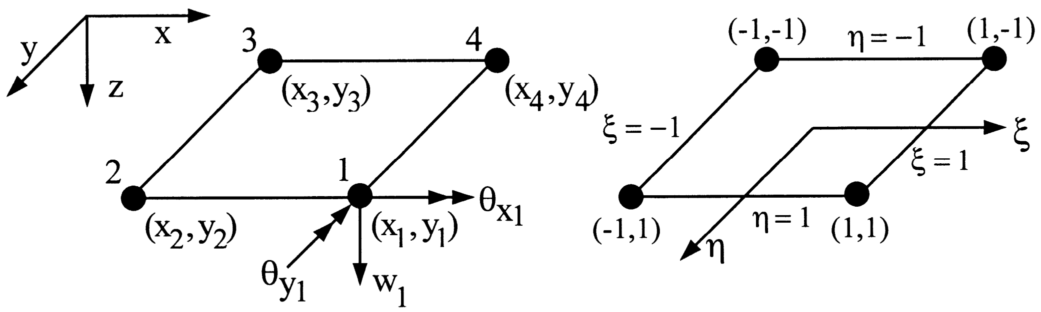

Fig. 5.5

$$x = \sum_{j=1}^{4} N_j x_j = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \tag{5.59a}$$

$$y = \sum_{j=1}^{4} N_j y_j = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \tag{5.59b}$$

where N_i are shape functions given as

$$N_1(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta) \tag{5.60a}$$

$$N_2(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta) \tag{5.60b}$$

$$N_3(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta) \tag{5.60c}$$

$$N_4(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta) \tag{5.60d}$$

The element has three degrees of freedoms $(w_j, \theta_{xj}, \theta_{yj})$ per node. The same shape functions N_i can be used to interpolate w, β_x and β_y from nodal values of these quantities. Hence,

$$w = \sum_{j=1}^{4} N_j w_j = N_1 w_1 + N_2 w_2 + N_3 w_3 + N_4 w_4 \tag{5.61a}$$

$$\beta_x = \sum_{j=1}^{4} N_j \theta_{yj} = N_1 \theta_{y1} + N_2 \theta_{y2} + N_3 \theta_{y3} + N_4 \theta_{y4} \tag{5.61b}$$

$$\beta_y = \sum_{j=1}^{4} N_j \theta_{xj} = N_1 \theta_{x1} + N_2 \theta_{x2} + N_3 \theta_{x3} + N_4 \theta_{x4} \tag{5.61c}$$

Since the shape functions in Eqs. (5.59) and (5.61) are identical, the element is an isoparametric plate element. Let us define $\{d\}_i = \{w_1, \theta_{x1}, \theta_{y1}, w_2, \theta_{x2}, \theta_{y2}, w_3, \theta_{x3}, \theta_{y3}, w_4, \theta_{x4}, \theta_{y4}\}^T$. Then Eq. (5.61) can be written in a matrix form as

$$\begin{Bmatrix} w \\ \beta_x \\ \beta_y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \end{bmatrix} \{d\}_i = [N]\{d\}_i \tag{5.62}$$

where the matrix for shape functions [N] is

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \end{bmatrix} \tag{5.63}$$

Since there are two coordinate systems (x, y) and (ξ, η), the transformation relations between the derivatives of these two systems are

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{5.64}$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{5.65}$$

where [J] is called the Jacobian matrix. The derivative terms in the Jacobian matrix can be obtained by differentiating Eq. (5.59) with respect to ξ and η. This yields

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{j=1}^{4} \frac{\partial N_j}{\partial \xi} x_j = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4 \tag{5.66a}$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \sum_{j=1}^{4} \frac{\partial N_j}{\partial \xi} y_j = \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 + \frac{\partial N_3}{\partial \xi} y_3 + \frac{\partial N_4}{\partial \xi} y_4 \tag{5.66b}$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{j=1}^{4} \frac{\partial N_j}{\partial \eta} x_j = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4 \tag{5.66c}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \sum_{j=1}^{4} \frac{\partial N_j}{\partial \eta} y_j = \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 \tag{5.66d}$$

Let matrix $[\Gamma]$ be the inverse of $[J]$. Then

$$[\Gamma] = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = [J]^{-1} = \frac{1}{\det[J]} \begin{bmatrix} J_{22} & -J_{21} \\ -J_{12} & J_{11} \end{bmatrix} \tag{5.67}$$

where $\det[J]$ is the determinant of the Jacobian matrix and

$$\det[J] = J_{11}J_{22} - J_{21}J_{12} \tag{5.68}$$

The [B] matrix can be obtained by substituting Eq. (5.63) into Eq. (5.57). Hence,

$$[B] = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & -\dfrac{\partial}{\partial y} \\ 0 & -\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & -1 & 0 \\ \dfrac{\partial}{\partial y} & 0 & -1 \end{bmatrix} \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \end{bmatrix} \tag{5.69a}$$

or

$$[B] = \begin{bmatrix} 0 & 0 & -\dfrac{\partial N_1}{\partial x} & 0 & 0 & -\dfrac{\partial N_2}{\partial x} & 0 & 0 & -\dfrac{\partial N_3}{\partial x} & 0 & 0 & -\dfrac{\partial N_4}{\partial x} \\ 0 & -\dfrac{\partial N_1}{\partial y} & 0 & 0 & -\dfrac{\partial N_2}{\partial y} & 0 & 0 & -\dfrac{\partial N_3}{\partial y} & 0 & 0 & -\dfrac{\partial N_4}{\partial y} & 0 \\ 0 & -\dfrac{\partial N_1}{\partial x} & -\dfrac{\partial N_1}{\partial y} & 0 & -\dfrac{\partial N_2}{\partial x} & -\dfrac{\partial N_2}{\partial y} & 0 & -\dfrac{\partial N_3}{\partial x} & -\dfrac{\partial N_3}{\partial y} & 0 & -\dfrac{\partial N_4}{\partial x} & -\dfrac{\partial N_4}{\partial y} \\ \dfrac{\partial N_1}{\partial x} & 0 & -N_1 & \dfrac{\partial N_2}{\partial x} & 0 & -N_2 & \dfrac{\partial N_3}{\partial x} & 0 & -N_3 & \dfrac{\partial N_4}{\partial x} & 0 & -N_4 \\ \dfrac{\partial N_1}{\partial y} & -N_1 & 0 & \dfrac{\partial N_2}{\partial y} & -N_2 & 0 & \dfrac{\partial N_3}{\partial y} & -N_3 & 0 & \dfrac{\partial N_4}{\partial y} & -N_4 & 0 \end{bmatrix} \tag{5.69b}$$

The derivatives of shape functions in $[B]$ are determined as

$$\frac{\partial N_j}{\partial x} = \frac{\partial N_j}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_j}{\partial \eta}\frac{\partial \eta}{\partial x} = \Gamma_{11}\frac{\partial N_j}{\partial \xi} + \Gamma_{12}\frac{\partial N_j}{\partial \eta} \tag{5.70a}$$

$$\frac{\partial N_j}{\partial y} = \frac{\partial N_j}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_j}{\partial \eta}\frac{\partial \eta}{\partial y} = \Gamma_{21}\frac{\partial N_j}{\partial \xi} + \Gamma_{22}\frac{\partial N_j}{\partial \eta} \tag{5.70b}$$

From Eq. (5.58), the stiffness matrix for the i-th plate element is

$$[k]_i = \iint_{A_i} [B]^T[D][B]dxdy = \int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\det[J]d\xi d\eta \tag{5.71}$$

where $dxdy = \det[J]d\xi d\eta$. The nodal load vector of the i-th element is

$$\{r\}_i = \iint_{A_i} q[N]_w^T dxdy = \int_{-1}^{1}\int_{-1}^{1} q[N]_w^T \det[J]d\xi d\eta \tag{5.72}$$

and

$$[N]_w = \{N_1, 0, 0, N_2, 0, 0, N_3, 0, 0, N_4, 0, 0\} \tag{5.73}$$

It should be noted that closed-form expressions for integrals of terms in Eqs. (5.71) and (5.72) would be lengthy. Accordingly, numerical integration such as Gauss quadrature (Cook, et al., 2002) should be used to evaluate the integrals.

5.8 Reduced and Selective Integration for Mindlin Plate Element

The stiffness matrix [k] of a Mindlin plate element can usually be obtained by the following expression

$$[k] = \int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\det[J]d\xi d\eta \tag{5.74}$$

From Eq. (5.53a), it is known that the [D] matrix can be decomposed into $[D]_b$ and $[D]_s$ matrices associated with the bending curvatures and transverse shear strains, respectively. The [B] matrix can also be separated into two sub-matrices $[B]_b$ and $[B]_s$, in which $[B]_b$ matrix is obtained by eliminating rows 4 and 5 of [B] in Eq. (5.56) and $[B]_s$ matrix is obtained by eliminating rows 1, 2 and 3 of [B]. Consequently, the stiffness matrix [k] can be separated into a bending stiffness $[k]_b$ and a transverse shear stiffness $[k]_s$.

$$[k] = \int_{-1}^{1}\int_{-1}^{1}[B]_b^T[D]_b[B]_b\det[J]d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1}[B]_s^T[D]_s[B]_s\det[J]d\xi d\eta = [k]_b + [k]_s \tag{5.75}$$

where

$$[k]_b = \int_{-1}^{1}\int_{-1}^{1}[B]_b^T[D]_b[B]_b\det[J]d\xi d\eta \tag{5.76a}$$

$$[k]_s = \int_{-1}^{1}\int_{-1}^{1}[B]_s^T[D]_s[B]_s\det[J]d\xi d\eta \tag{5.76b}$$

When the thickness of a plate is thin, the Mindlin plate element may produce shear locking phenomenon [Cook, et al., 2002] by employing a full integration technique (Fig. 5.6), i.e. the $[k]_s$ matrix may approach infinite. The locking problems may be avoided by adopting a uniformly reduced integration on $[k]_b$ and $[k]_s$ matrices, or by applying a selective integration rule, i.e., full integration on $[k]_b$ and reduced integration on $[k]_s$ [Pugh, Hinton and Zienkiewicz, 1978; Hughes, Cohen, and Haroun, 1978]. These two integration rules are also indicated in Fig. 5.6 for various plate elements.

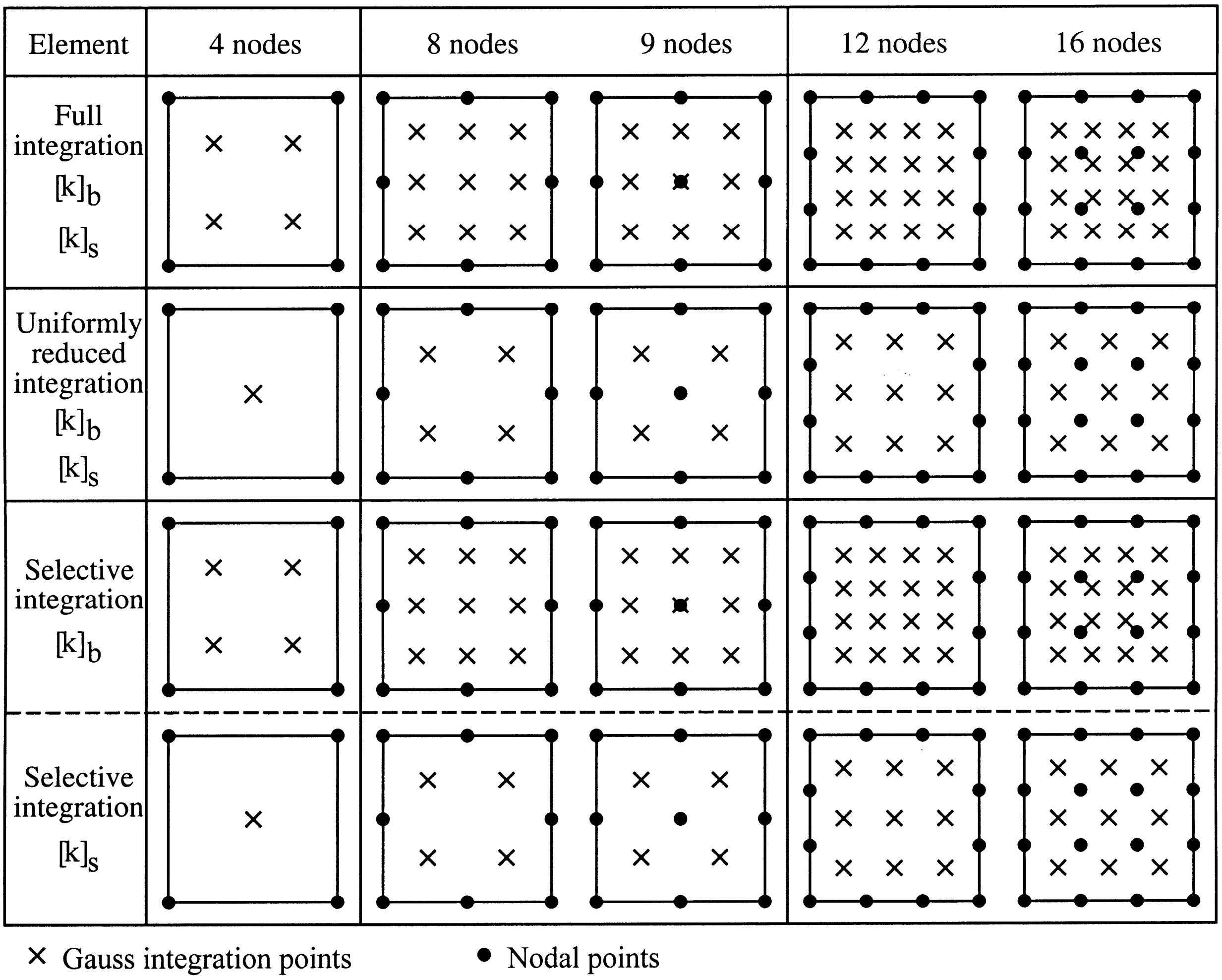

Fig. 5.6

5.9 Boundary Conditions

Consider a smooth portion of the plate boundary as shown in Fig. 5.7(a) and a local coordinate system (n, t, z) attached to it, where t denotes the tangential direction and n the outward normal direction. In the finite element analysis of plate problems, the boundary conditions involve the constrains on the nodal degrees of freedoms (w, θ_n, θ_t) only. The most common boundary conditions in practice are given as follow:

Clamped

$$w = 0, \qquad \theta_n = 0, \qquad \theta_t = 0 \tag{5.77}$$

Simply supported

$$w = 0, \qquad \theta_n = 0 \tag{5.78}$$

Free

No need to specified.

Sliding

$$\theta_t = 0 \tag{5.79}$$

Symmetric

$$\theta_t = 0$$

Antisymmetric

$$w = 0, \qquad \theta_n = 0 \tag{5.80}$$

When the boundary conditions in terms of (x, y, z) coordinate are required as shown in Fig. 5.7(b), the following transformations for rotations can be used.

$$\theta_n = \theta_x \cos\beta - \theta_y \sin\beta \tag{5.81a}$$

$$\theta_t = \theta_x \sin\beta + \theta_y \cos\beta \tag{5.81b}$$

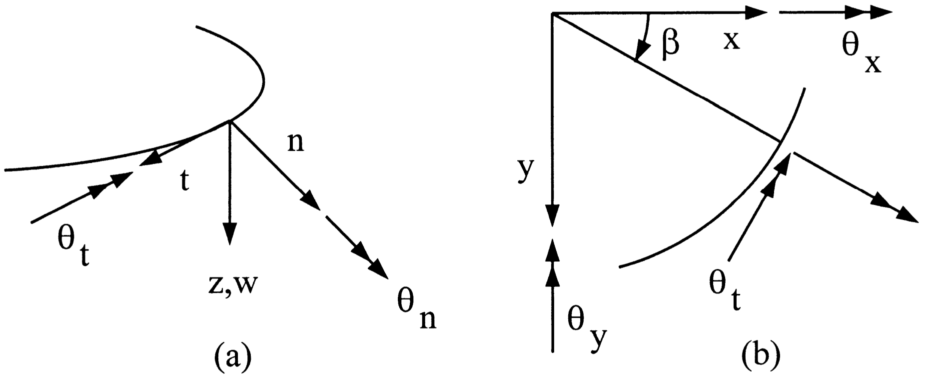

Fig. 5.7

5.10 References

[5.1] Cook, R. D., Malkus, D. S., Plesha, M. E. and Witt, R. J., Concepts and Applications of Finite Element Analysis, Fourth Edition, Wiley, 2002.

[5.2] Hughes, T. J. R., Cohen, M., and Haroun, M., "Reduced and Selective Integration Techniques in the Finite Element Analysis of Plates," Nuclear Engineering and Design, Vol. 46, 1978, pp. 203-222.

[5.3] Pugh, E. D. L., Hinton, E., and Zienkiewicz, O. C., "A Study of Quadrilateral Plate Bending Elements with 'Reduced' Integration," International Journal for Numerical Methods in Engineering, Vol. 12, 1978, pp. 1059-1079.