derivatives of w with respect to the different side directions meeting at each node to be defined as nodal variable. Obviously this is quite difficult to generalize for practical purposes.

A development of the BFS element to include the continuity of higher derivatives was outlined in [Sp].

The derivation of conforming plate quadrilaterals starting from triangular plate elements will be studied in Section 5.6.

5.5 TRIANGULAR THIN PLATE ELEMENTS

Triangular plate elements are of interest for the analysis of plates with irregular shapes. Their formulation, however, has the same difficulties for satisfying conformity as the rectangular elements previously studied. Some of the more popular non-conforming and conforming Kirchhoff plate triangles are presented next.

5.5.1 Non-conforming thin plate triangles

Let us consider first the 3-noded triangle. The obvious choice of nodal variables gives a total of nine DOFs (w_i, $\left(\frac{\partial w}{\partial x}\right)_i$ and $\left(\frac{\partial w}{\partial y}\right)_i$ at each node). A complete cubic polynomial has ten terms and, hence, a problem arises when choosing the term to be dropped out. A number of authors have proposed different elements on the basis of the term omitted. Unfortunately all of them require substantial manipulation for ensuring conformity.

Adini and Clough [AC] omitted the xy term in the cubic expansion, i.e.

$$w(x,y) = a_1+a_2x+a_3y+a_4x^2+a_6y^2+a_7x^3+a_8x^2y+a_9xy^2+a_{10}y^3 \quad (5.52)$$

This simple criterion yields a poor element which is unable to reproduce constant torsion curvature $\left(\frac{\partial^2 w}{\partial x \partial y}\right)$ states. In addition, the element does not satisfy the C^1 continuity requirement.

Tocher and Kapur [TK] grouped the terms a_8 and a_9 of the cubic polynomial as

$$\begin{aligned} w(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + \\ + a_7x^3 + a_8(x^2y + xy^2) + a_9y^3 \end{aligned} \quad (5.53)$$

This element does not respect the continuity of the normal rotation along the sides. Also, matrix $\mathbf{A}$ of Eq.(5.33b) becomes singular when the sides of the triangle are parallel to the x, y axes [TK,To].

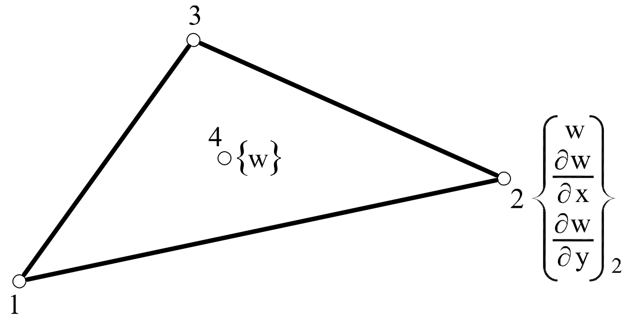


Fig. 5.17 4-noded plate triangle with 10 degrees of freedom [HK]

Harvey and Kelsey [HK] obtained a complete cubic deflection field by adding a fourth central node with a single deflection variable (Figure 5.17). The internal DOF can be eliminated by static condensation. The resulting element does not satisfy the continuity requirement for the normal rotation and it has poor convergence. The performance of this element can be substantially improved by imposing the continuity requirement using Lagrange multipliers. Harvey and Kelsey [HK] showed that the enhanced element satisfies the patch test and it converges monotonically to the exact solution. Further details can be found in [Ya].

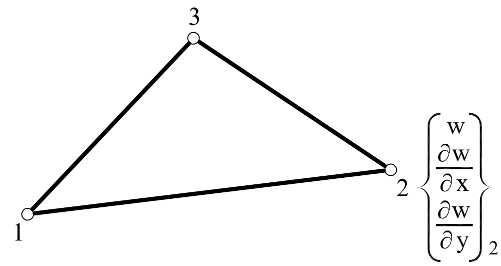
Bazeley *et al.* [BCIZ] developed a 3-noded plate triangle with 9 DOFs. The element was subsequently modified by Cheung, King and Zienkiewicz [CKZ] (termed hereafter CKZ element). The starting point is an incomplete cubic expansion of the deflection using area coordinates as

$$
\begin{aligned}
w = {} & a_1L_1 + a_2L_2 + a_3L_3 + a_4\left(L_1^2L_2 + \frac{L_1L_2L_3}{2}\right) + a_5\left(L_2^2L_1 + \frac{L_1L_2L_3}{2}\right) \\
& + a_6\left(L_2^2L_3 + \frac{L_1L_2L_3}{2}\right) + a_7\left(L_3^2L_2 + \frac{L_1L_2L_3}{2}\right) + \\
& + a_8\left(L_3^2L_1 + \frac{L_1L_2L_3}{2}\right) + a_9\left(L_1^2L_3 + \frac{L_1L_2L_3}{2}\right)
\end{aligned}
\tag{5.54}
$$

The bracketed terms guarantee the reproduction of an arbitrary curvature field (including that of constant curvature) for a zero value of the nodal deflections.

Following a similar procedure as for the MZC rectangle, Eq.(5.54) can be written in the form

$$
w = \sum_{i=1}^{3}\left(N_iw_i + \bar{N}_i\left(\frac{\partial w}{\partial x}\right)_i + \bar{\bar{N}}_i\left(\frac{\partial w}{\partial y}\right)_i\right)
\tag{5.55}
$$



Shape functions :

$$N_1 = L_1 + L_1^2 L_2 + L_1^3 L_3 - L_1 L_2^2 - L_1 L_3^2$$
$$\bar{N}_1 = c_3 (L_1^2 L_2 + L) - c_2 (L_3 L_1^2 + L)$$
$$\bar{\bar{N}}_1 = b_3 (L_1^2 L_2 + L) - b_2 (L_3 L_1^2 + L)$$
$$N_2 = L_2 + L_2^2 L_3 + L_2^2 L_1 - L_2 L_3^2 - L_2 L_1^2$$
$$\bar{N}_2 = c_1 (L_2^2 L_3 + L) - c_3 (L_1 L_2^2 + L)$$
$$\bar{\bar{N}}_2 = b_1 (L_2^2 L_3 + L) - b_3 (L_1 L_2^2 + L)$$
$$N_3 = L_3 + L_3^2 L_1 + L_3^2 L_2 - L_3 L_1^2 - L_3 L_2^2$$
$$\bar{N}_3 = c_2 (L_3^2 L_1 + L) - c_1 (L_2 L_3^2 + L)$$
$$\bar{\bar{N}}_3 = b_2 (L_3^2 L_1 + L) - b_1 (L_2 L_3^2 + L)$$

$$L = \frac{L_1 L_2 L_3}{2} \qquad b_i = y_j - y_m \qquad c_i = x_m - x_j$$

Fig. 5.18 Shape functions for the 3-noded CKZ plate triangle [CKZ]

The shape functions N_i , $\bar{N}_i$ and $\bar{\bar{N}}_i$ are given in Figure 5.18. The stiffness matrix for this element can be found in [CKZ].

The CKZ triangle violates the continuity requirement for the normal rotation and hence is non conforming. However, it converges in a monotonic manner and this has contributed to its popularity [CKZ].

Bazeley *et al.* [BCIZ] proposed a correction to the deflection field leading to a linear distribution of the normal rotation along the sides. This modification does not improve the CKZ element substantially and the performance of the original form is sometimes superior.

Different authors have tried to enhance the behaviour of the CKZ triangle so that it passes the patch test [BN,FB2,KA]. A simple proposal was due to Specht [Sp] who achieved conformity by adding 4th degree terms to the cubic expansion (5.54) as

$$\begin{aligned} w = {} & a_1 L_1 + a_2 L_2 + a_3 L_3 + a_4 L_1 L_2 + a_5 L_2 L_3 + a_6 L_1 L_3 + \\ & + a_7 \left[L_1^2 L_2 + \frac{L}{2} (3(1-\gamma_3)) L_1 - (1+3\gamma_3) L_2 + (1+3\gamma_3) L_3 \right] + \\ & + a_8 \left[L_2^2 L_3 + \frac{L}{2} (3(1-\gamma_1)) L_2 - (1+3\gamma_1) L_3 + (1+3\gamma_1) L_1 \right] + \\ & + a_9 \left[L_3^2 L_1 + \frac{L}{2} (3(1-\gamma_2)) L_3 - (1+3\gamma_2) L_1 + (1+3\gamma_2) L_2 \right] \end{aligned} \tag{5.56}$$

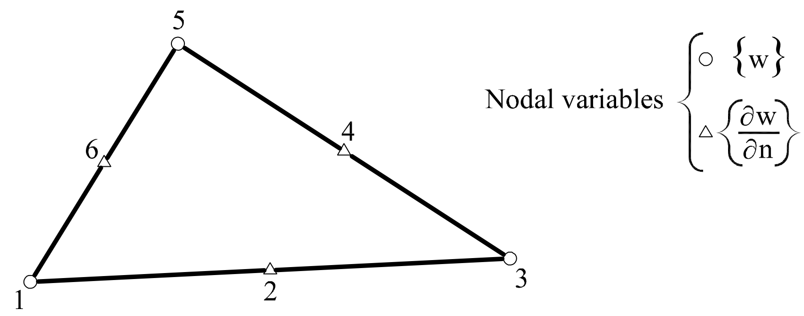


Fig. 5.19 Morley plate triangle of constant curvature

with $L = L_1L_2L_3$, $\gamma_1 = \frac{l_3^2-l_2^2}{l_1^2}$; $\gamma_2 = \frac{l_1^2-l_3^2}{l_2^2}$; $\gamma_3 = \frac{l_2^2-l_1^2}{l_3^2}$ and l_1, l_2 and l_3 are the element sides. The element passes all patch tests and performs excellently [Sp,TZSC,ZT2].

Morley 6 DOFs constant curvature triangle

Morley proposed a simple non-conforming 6-noded triangle with just 6 DOFs [Mo,Mo2]. The element uses a complete quadratic expansion of the deflection in terms of the three corner deflection values and the three normal rotations at the mid-sides (Figure 5.19). The Morley triangle has constant curvature and bending moment fields. It satisfies the patch test and converges despite the violation of the C^1 continuity requirement, which demands a cubic deflection field. The element stiffness can be explicitly obtained by

$$\mathbf{K}^{(e)} = A^{(e)}\mathbf{B}^T\mathbf{D}_b\mathbf{B} \tag{5.57}$$

with

$$\mathbf{B} = \frac{1}{A^{(e)}}[\mathbf{G}_1, \mathbf{G}_2] \tag{5.58a}$$

$$\mathbf{G}_1 = -\begin{bmatrix} C_4S_4 - C_6S_6 & C_5S_5 - C_4S_4 & C_6S_6 - C_5S_5 \\ -C_4S_4 + C_6S_6 & -C_5S_5 + C_4S_4 & -C_6S_6 + C_5S_5 \\ -C_4^2 + S_4^2 + C_6^2 - S_6^2 & -C_5^2 + S_5^2 + C_4^2 - S_4^2 & -C_6^2 + S_6^2 + C_5^2 S_5^2 \end{bmatrix} \tag{5.58b}$$

$$\mathbf{G}_2 = \left[\mathbf{G}_2^4, \mathbf{G}_2^5, \mathbf{G}_2^6\right] \quad , \quad \mathbf{G}_2^k = -\begin{bmatrix} C_k^2 l_k \\ S_k^2 l_k \\ 2S_kC_kl_k \end{bmatrix} \quad , \quad k = 4, 5, 6 \tag{5.58c}$$

where $C_k = y_{ji}/l_k$, $S_k = -x_{ji}/l_k$, $x_{ji} = x_j - x_i$, $y_{ji} = y_j - y_i$ and $l_k = (x_{ji}^2 + y_{ji}^2)^{1/2}$ is the lengh of side k. The constant bending moment field is given by

$$\hat{\boldsymbol{\sigma}}_b = \mathbf{D}_b \mathbf{B} \mathbf{a}^{(e)} \tag{5.59a}$$

with

$$\mathbf{a}^{(e)} = \left[w_1, w_2, w_3, \left(\frac{\partial w}{\partial n}\right)_4, \left(\frac{\partial w}{\partial n}\right)_5, \left(\frac{\partial w}{\partial n}\right)_6 \right]^T \tag{5.59b}$$

The equivalent nodal force vector due to a uniformly distributed loading $f_z = q$ is

$$\mathbf{f}^{(e)} = q\frac{A^{(e)}}{3}[1,1,1,0,0,0]^T \tag{5.60}$$

Nodal point loads are assumed to act at the corner nodes only. Full details on the derivation of the Morley triangle can be found in [Wo].

The Morley triangle is so far the simplest Kirchhoff plate triangle involving deflections and rotations as variables. Its simplicity is comparable to that of the constant strain triangle for plane elasticity problems. Despite its slow convergence, the Morley triangle enjoys big popularity for analysis of plates and shells. A thin plate triangle with identical features as the Morley triangle can be derived starting from the Reissner-Mindlin TLLL triangle (Section 6.8.3) using a Discrete Kirchhoff approach (see Section 6.8.3).

5.5.2 Conforming thin plate triangles

Satisfying the conformity requirements in triangles is a challenging task. The technique of using the curvatures as additional nodal variables is cumbersome and it also makes the extension of the elements for shell analysis difficult. A more successful alternative is to guarantee the continuity of the normal rotation along the sides using additional mid-side variables. Some of these elements are described next.

A conforming plate triangle emerges as a modification of the CKZ element previously described. The shape functions of Eq.(5.54) define a quadratic variation of the normal rotation along each side which can not be uniquely described by the two end values. A solution to this problem is adding three additional mid-side variables which coincide with the normal rotation to each side (Figure 5.20) [ZT]. This suffices to define a complete cubic variation of $\frac{\partial w}{\partial n}$ along each side and conformity is thus satisfied.

Clough and Tocher [CT] developed another conforming triangle starting from an idea of Hsieh in correspondence with Clough [Ya] (denoted here as HCT element). The shape functions are obtained by dividing the element into three inner triangular subdomains as shown in Figure 5.21a.

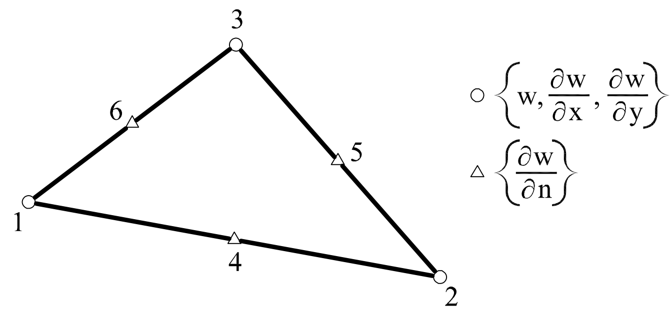


Fig. 5.20 Conforming 12 DOFs triangle

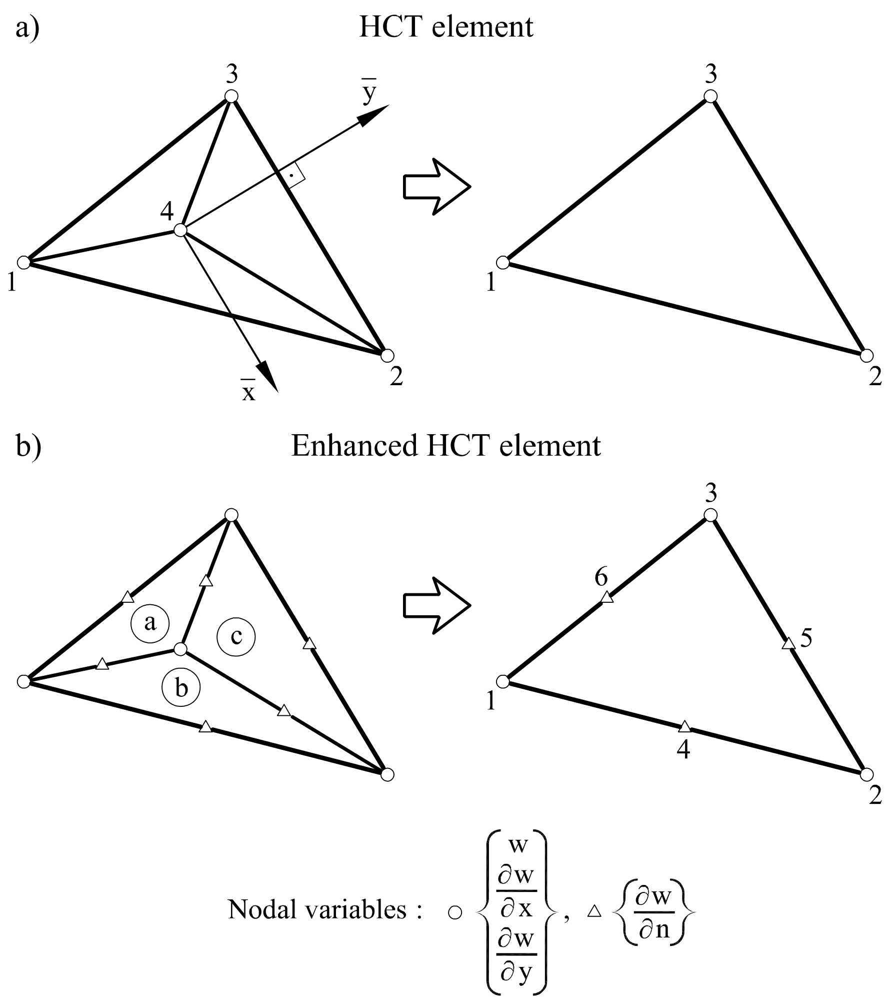
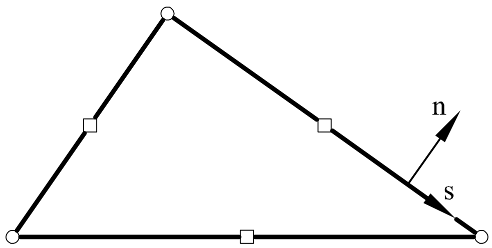


Fig. 5.21 HCT conforming thin plate triangles

A nine-term incomplete cubic expansion is written in the local axes $\bar{x}$, $\bar{y}$ for every triangular subdomain $4ij$ with $\bar{y}$ chosen orthogonal to the side ij. Thus, for the triangle 423 (Figure 5.21) we write

$$w_A = C_1 + C_2\bar{x} + C_3\bar{y} + C_4\bar{x}^2 + C_5\bar{y}^2 + C_6\bar{x}\bar{y} + C_7\bar{x}^3 + C_8\bar{x}\bar{y}^2 + C_9\bar{y}^3 \quad (5.61)$$



$$\circ \begin{Bmatrix} w \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} \quad \square \begin{Bmatrix} w \\ \frac{\partial w}{\partial n} \\ \frac{\partial^2 w}{\partial n \partial s} \end{Bmatrix}$$

Fig. 5.22 18 DOFs conforming plate triangle proposed by Irons [Ir]

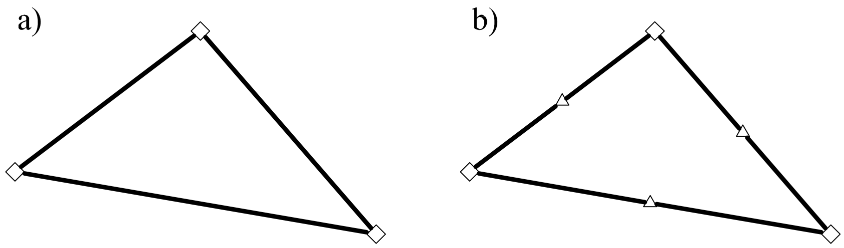
with $\bar{y}$ being orthogonal to side 23. Similar expressions are used for triangles 412 and 431. The omission of the term $\bar{x}^2\bar{y}$ in Eq.(5.61) guarantees that the normal rotation varies linearly along the external sides, whereas the deflection varies quadratically. The local stiffness matrices for each triangular subdomain are transformed to global axes for assembly purposes. The three DOFs of the central node are eliminated by imposing continuity of the normal rotation at the mid-point of the inner sides (three conditions). Further details can be found in [CMPW,CT,Ya]. The HCT element is conforming and it has 9 DOFs as the CKZ element. However, it has a slightly stiffer behaviour.

The performance of the HCT element can be enhanced by starting from three triangular subdomains where, in addition to the standard corner variables, a normal rotation variable is introduced at each mid-side point. This defines a quadratic variation of the rotation along the sides. After eliminating the internal variables a 12 DOFs plate triangle similar to the modified CKZ element is obtained (Figure 5.18b) [Ga2].

A drawback of elements with mid-side normal rotations as variables is that they involve a different number of DOFs per node. To overcome this problem Irons [Ir] proposed a 18 DOFs quartic triangle where the deflection and the curvature $\frac{\partial^2 w}{\partial n \partial s}$ are added as mid-side variables (Figure 5.22).

Other authors have proposed different 3-noded conforming plate triangles based on cubic and quartic expansions of the normal rotation $\frac{\partial w}{\partial n}$ along the sides. Cowper *et al.* [CKLO] proposed a 18 DOFs plate triangle with w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, $\frac{\partial^2 w}{\partial x \partial y}$ at each node (Figure 5.23a). The shape functions omit three terms of a complete quintic polynomial (which has 21 terms) preserving a cubic variation of $\frac{\partial w}{\partial n}$ along the three sides.

This element can be enhanced by adding three mid-side nodes with the normal rotation as variable [AFS,Be,Ir] (Figure 5.23b). The shape



$$\diamond \left\{ w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \frac{\partial^2 w}{\partial x \partial y} \right\}$$

$$\triangle \left\{ \frac{\partial w}{\partial n} \right\}$$

Fig. 5.23 18 and 21 DOFs conforming plate triangles

functions are now complete quintic polynomials (21 terms) and $\frac{\partial w}{\partial n}$ has a quartic variation along the sides.

More information on these elements can be found in [Ya,ZT2] in addition to the previously quoted references. Despite their accuracy, the practical acceptance of thin plate triangles with curvatures as nodal variables has been limited. The main reason for this is the intrinsic difficulties for their extension to shell analysis.

5.6 CONFORMING THIN PLATE QUADRILATERALS OBTAINED FROM TRIANGLES

One of the first conforming plate quadrilaterals derived from triangles is due to Fraejis de Veubeke [FdV,FdV2]. The element was later developed by Sander [San]. The starting point is the splitting of the quadrilateral into four inner triangles as shown in Figure 5.24a. A complete 10-term cubic polynomial is used to approximate the deflection within each subdomain and, thus, the total number of initial variables is 40. After eliminating the internal variables the DOFs are reduced to 16, i.e. the three standard corner variables and the normal rotations at the mid-side points. The element is conforming and it can be distorted to arbitrary quadrilateral shapes. The mid-side normal rotations are eliminated by imposing a linear variation of $\frac{\partial w}{\partial n}$ along each side. This, however, does not improve the performance of the element [FdV2].

A second thin plate quadrilateral was developed by Clough and Felippa [CF] almost at the same time as the Fraejis de Veubeke element described