Material properties of the element

|  |  |
| --- | --- |
| E | Young's modulus (force/length2) for membrane stiffness of isotropic  Material |
| **n** | Poisson's ratio |
| **r** | Material mass density |
| h | Thickness of the plate |

A drawing of a square with arrows

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## Finite Element Formulation of Kirchoff plate

Assumption of Kirchoff’s plate theory

1. The mid plane of the plate is a neutral plane meaning, the mid plane of the plate remains free of in-plane stress/ strain.
2. Plane section remain plane. The line elements lying perpendicular to the middle surface of the plate remain perpendicular to the middle surface during deformation.
3. Line elements lying perpendicular to the mid-surface do not change length during deformation, therefor vertical strain is zero.

Blue lines on a black background

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Based on these assumptions, the displacement in x, y and z directions are

And the strain displacement relations are

It should be noted that the transverse shear deformation is ignored in this theory,

For isotropic material, the elasticity matrix is given by

Therefore, the elasticity matrix is given by,

The moment stress relations are,

Where,

Kirchoff plate theory accounts for the bending deformation but the transverse shear deformation is neglected. The derivation of the finite element expressions is based on the principal of minimum potential energy. The strain energy U in the plate can be written as

Substituting,

The strain energy U becomes,

Integrating through thickness h in z direction yields

Assuming the displacement w within the plate element is interpolated from the elemental nodal degree of freedoms.

Then the curvature of the plate becomes,

Where the strain displacement B matrix is given by,

This substitution yields,

The element stiffness matrix is given by

### Three node 9 DOF CKZ (Cheung, King and Zienkiewicz) Triangle element CTRIA3

Let us first consider the 3 – noded triangle. The obvious choice of nodal variables gives a total of nine DOFs (wi, ­ ∂w/∂xi and ∂w ∂yi at each node). A complete cubic polynomial has ten terms and, hence, a problem arises when choosing the term to be dropped out.

To avoid arbitrary removal of single term from the cubic polynomial Bazeley et al. [BCIZ] developed a 3-noded plate triangle with 9 DOFs with so called Area co-ordinates. The element was subsequently modified by Cheung, King and Zienkiewicz [CKZ] (termed hereafter CKZ element).

For any point P inside an arbitrary triangle shown below, the point P(x,y) can be given by the area co-ordinates L1, L2, L3.

A blue line drawing of a triangular object

AI-generated content may be incorrect.

The area co-ordinates are given by

The area of triangle is given by

Area A1 is given by

A blue triangle with a black background

AI-generated content may be incorrect.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

Now the area co-ordinates are given by,

The starting point is an incomplete cubic expansion of the deflection using area coordinates as

Differentiation of the area co-ordinates gives the following

|  |  |  |
| --- | --- | --- |
| Area co-ordinates | Differentiation w.r.t x | Differentiation w.r.t y |
|  |  |  |
|  |  |  |
|  |  |  |

At Node 1,

x = x1, y = y1, w = w1, dw/dx = (dw/dx)1, dw/dy = (dw/dy)1 and L1 = 1, L2 = 0, L3 = 0

At Node 2,

x = x2, y = y2, w = w2, dw/dx = (dw/dx)2, dw/dy = (dw/dy)2 and L1 = 0, L2 = 1, L3 = 0

At Node 3,

x = x3, y = y3, w = w3, dw/dx = (dw/dx)3, dw/dy = (dw/dy)3 and L1 = 0, L2 = 0, L3 = 1

In matrix form,

Now we established w, in terms of α, the next step is to eliminate the α terms with an interpolating shape function.

Note that,

Substitute α,

Where N is the shape function,

Using matrixcalc.org the following inverse of A matrix is calculated

One other identity to reduce the above matrix further is the inverse of area matrix

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

Note that

The reason is It’s a change-of-basis matrix from physical triangle coordinates to reference coordinates, and such a matrix has determinant = 1 if the transformation preserves orientation and area.

Therefore,

This implies the following,

etc,

This will reduce the inverse [A] matrix to the following,

Now the shape functions can be found.

Shape functions are shown below,

Note that the curvature of the plate is given by,

Where the strain displacement B matrix is given by,

For three noded triangle, the strain displacement matrix is given by

The first derivative of shape function is given by the following

The second derivative of shape function is,

Note that,

Therefore,

Which can be written in a matrix form as below,

This identity could be expanded to other derivatives as below.

Let the B matrix be

Where B1 matrix is shown below,

So, the idea is to find the B matrix is quite simple once the product first derivative of L matrix and second derivative of N matrices are formed. The N matrices and L matrices are presented in this writeup. The c++ code triCKZ\_element.cpp contains the full formulation of B matrix.

## Finite Element formulation of the Mindlin plate

The in-plane strain – displacement relations are

The transverse shear strains are assumed to be constant through the thickness of the plate.

Stresses and the curvatures/ Twist in a linear elastic plate, from Hooke’s law

The inplane stress – strain relations are

The transverse shear stress – strain relations are

Where α is the shear correction factor (constant α = 12/ π2 or 6/5)

For [D] – elasticity matrix.

Stress strain relation is given by,

Mindlin plate theory accounts for the bending deformation and for transverse shear deformation. Hence, the strain energy U in the plate can be written as

Substituting,

The strain energy U becomes,

Integrating through thickness t in z direction yields

Assuming that w, βx and βy within the i-th plate element are interpolated from the elemental nodal degrees of freedoms [d]i by the following expression.

Where,

Below is the B matrix formulation for elements having n Nodes.

Where K is the stiffness matrix of the element given as follows,

## 3 Node Triangle Element CTRIA3

A blue and green lines on a black background

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Shape functions of 3-Noded triangle element

Kronecker delta property

The shape functions of triangle element at their natural coordinate system ξ & η

Note: Multiplied by -12, (-1 [d][N]) [Ds](-1[d][N])

A point (x,y) inside the triangle with known exterior points (x1,y1) (x2,y2) & (x3,y3), through Local area (natural) co-ordinates is given by

Substituting the shape functions,

Differentiating w.r.t element co-ordinates

Similar to vertex co-ordinates, displacements inside the triangle with known nodal displacements (u1,v1)(u2,v2) & (u3,v3) is given by

Differentiating w.r.t element co-ordinates