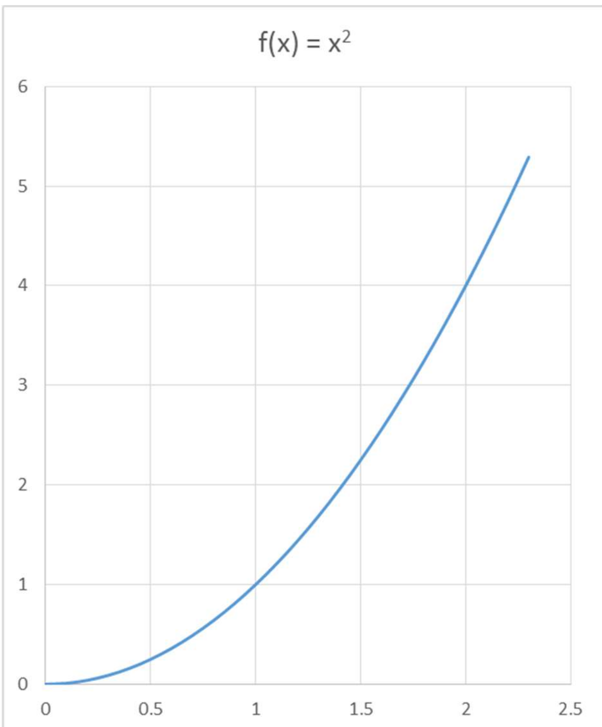
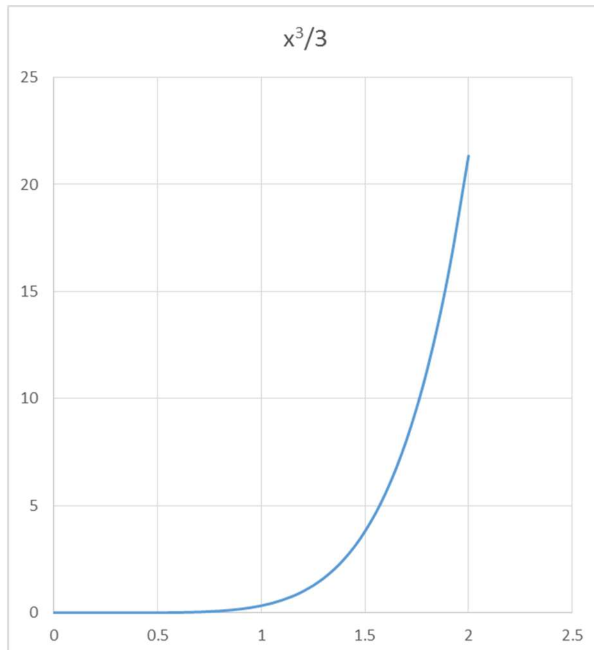
Line Integral

INTEGRATION

Integration in mathematics is the process of finding area of a function f(x) plotted as a graph. For example take a function f(x) = x^2 plotted in a graph below.





$$\int x^2\,dx = \frac{x^3}{3}$$

If we find the cumulative area falling under x^2 and plot in a new graph it would be one that of right $x^3/3$. Also integration of x^n is $x^{n+1}/n+1$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

Proof by Fermat's method can be found in http://math2.org/math/integrals/more/x%5En.htm

Note: Integration is the inverse of differentiation.

The above all are very fundamental and basic concepts.

Example:

$$\int_a^b x^2 dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3}$$

Area under x^2

- From 0 to 1 is 1/3.
- From 1 to 2 is 7/3 etc.,

Now let's assume x is defined by a function t ie., x = g(t) = 3t

- So if we move from 1 to 5 in t, then x will go from 3 to 15

dx/dt = 3 (the rate of change of x in t) Also length of t is (5-1) 4 and length of x is 3 x 4 = 12 (15-3) , Below is the length explanation

$$\int_3^{15} dx = \int_1^5 \frac{dx}{dt} dt = 3\int_1^5 dt = 12$$

=> dx = 3dt & $x^2 = (3t)^2$

$$\int_1^5 27\, t^2 dt = \left[27\frac{t^3}{3}\right]_1^5 = 27\frac{5^3 - 1^3}{3} = 1116$$

Same can be expressed in x, note the bounds

$$\int_3^{15} x^2 dx = \left[\frac{x^3}{3}\right]_3^{15} = \frac{15^3 - 3^3}{3} = 1116$$

The above example is to emphasize on the rate of change of x on t, which is important while defining f(x) through other parameters like t such as f(x) = f(g(t))

So (chain rule),

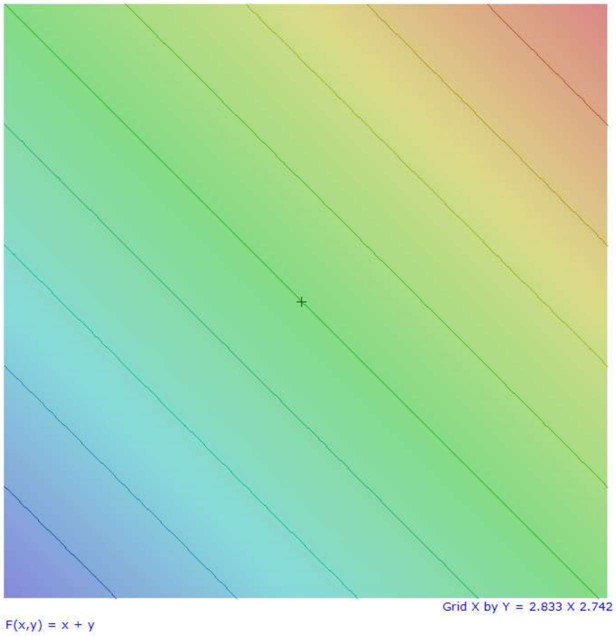
$$\int f(x)\, dx = \int f(g(t)) \frac{dx}{dt} dt$$

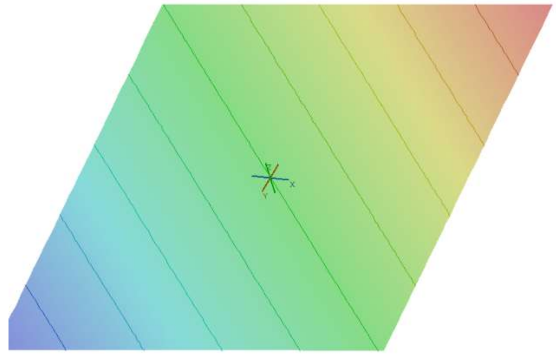
dx/dt describes the rate of change of x in t

Line Integral

Line integral is the integration dealing with multiple variables. If our function is defined in f(x,y) or f(x,y,z) and line integration is the concept to find area along a path curve C.

Say, f(x,y) = x+y below is the scalar plot





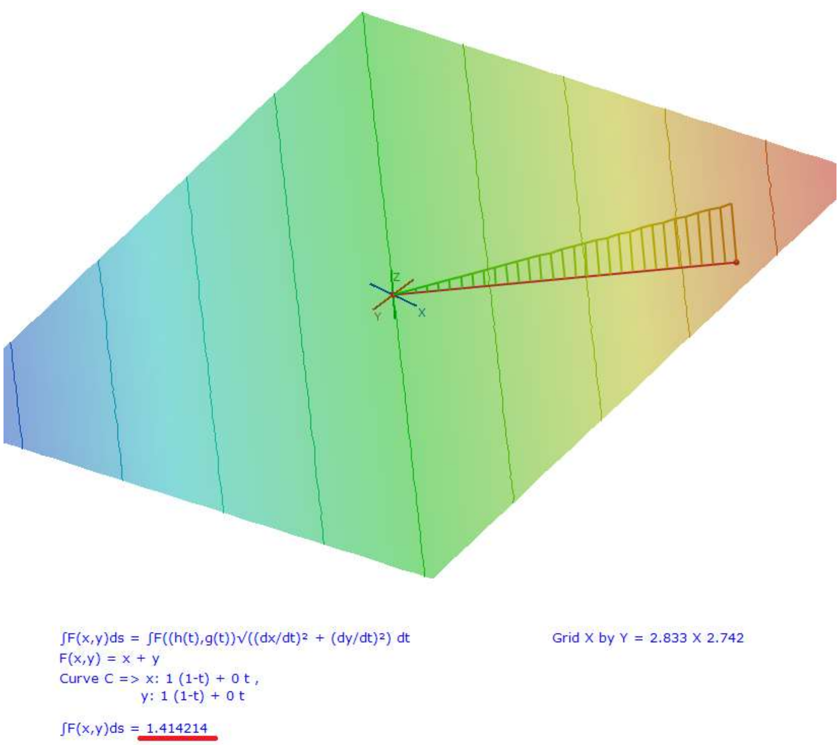
3D Plot (note that

Now from the 3D plot we can easily visualize the slope of f(x,y) = x + y. let's find the area this function will make along a line from A(0,0) to B(1,1)

At A(0,0) => f(x,y) =0 & B(1,1) => f(x,y) = 2

Length AB = √2 = 1.41421..

Area of the triangle = 0.5 x 2 x √2 = √2 (Note f(x,y) is a linear function)



Let's generalize our understanding. Line integral of a function f(x,y) along a path C is defined as

$$Line\ integral = \int_C f(x,y)ds$$

Analytically we solve line integral by parameterizing the path C. Length of the path C (Arc Length is defined as

$$Arc\ Length = \int_a^b ds\ \ where\ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

The arc length concept is same as the parameterization of one dimensional functions y=f(x).

$$Line\ integral = \int_C f(x,y)ds = \int_a^b f(h(t),g(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example:

F(x,y) = x + y, path C is a line segment from A(2,2) to B(3,3). Find the line integral $\int_C f(x,y)ds$

Solution:

Parameterize the path AB

Line segment from point A to B is defined as A(1-t)+Bt for t = 0 to 1

x = 2(1-t)+3t = 2+t

dx/dt = dy/dt = 1

y = 2(1-t)+3t = 2+t

x+y = 4+2t

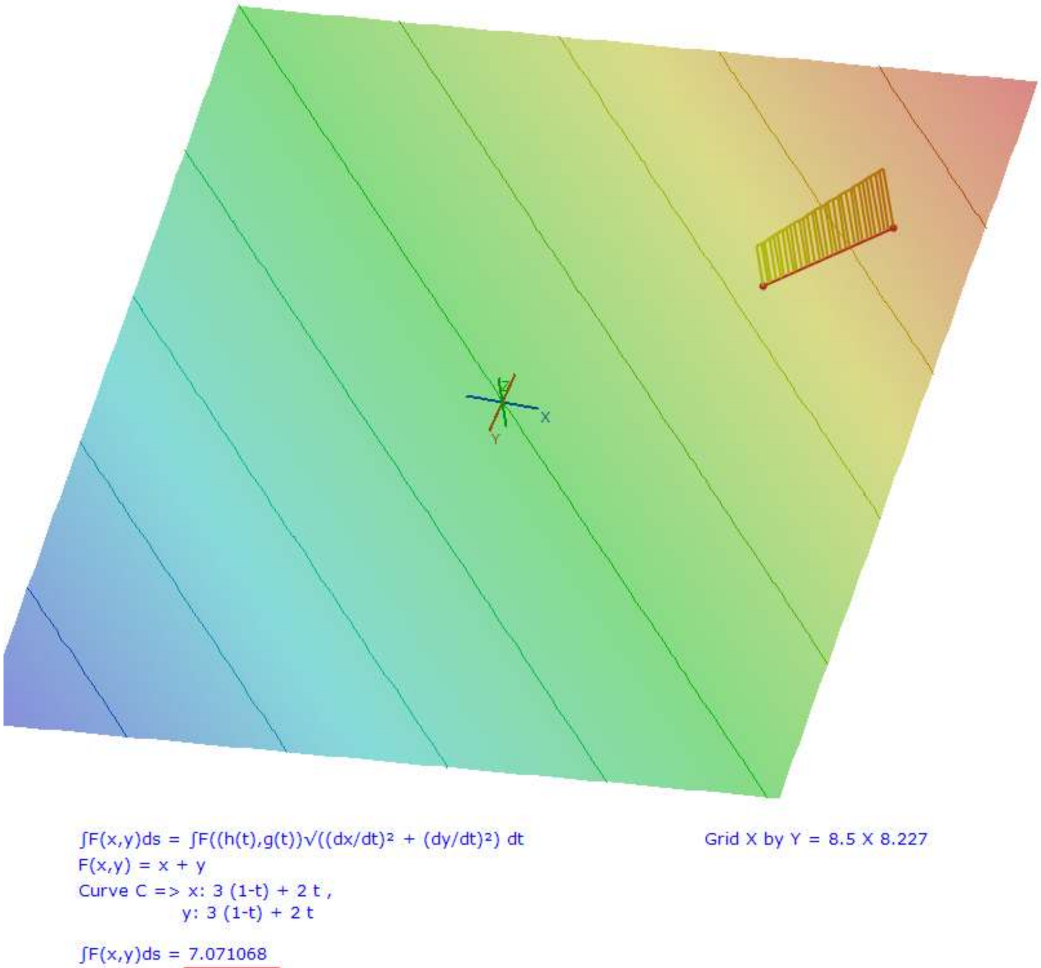
$$Line\ integral = \int_C f(x,y)ds = \int_a^b f(h(t),g(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$line\ integral = \int_C (x+y)ds = \int_0^1 (4+2t)\sqrt{(1)^2+(1)^2}\,dt = \int_0^1 (4+2t)\sqrt{2}\,dt$$

$$= \sqrt{2}\left[4t + \frac{2t^2}{2}\right]_0^1 = 5\sqrt{2}$$

Alternatively, at A(2,2) => f(x,y) = 4 & B(3,3) => f(x,y) = 6

Area of the quadrilateral with sides A(2,2,0), Af(2,2,4), Bf(3,3,6), C(3,3,0) is $5\sqrt{2}$



Below is another example of line integral with

$F(x,y) = 3x^2+5y^2$

Path Unit circle $x^2+y^2 = 1$

Solution:

Circle can be parameterized to

$x = \cos(t)$

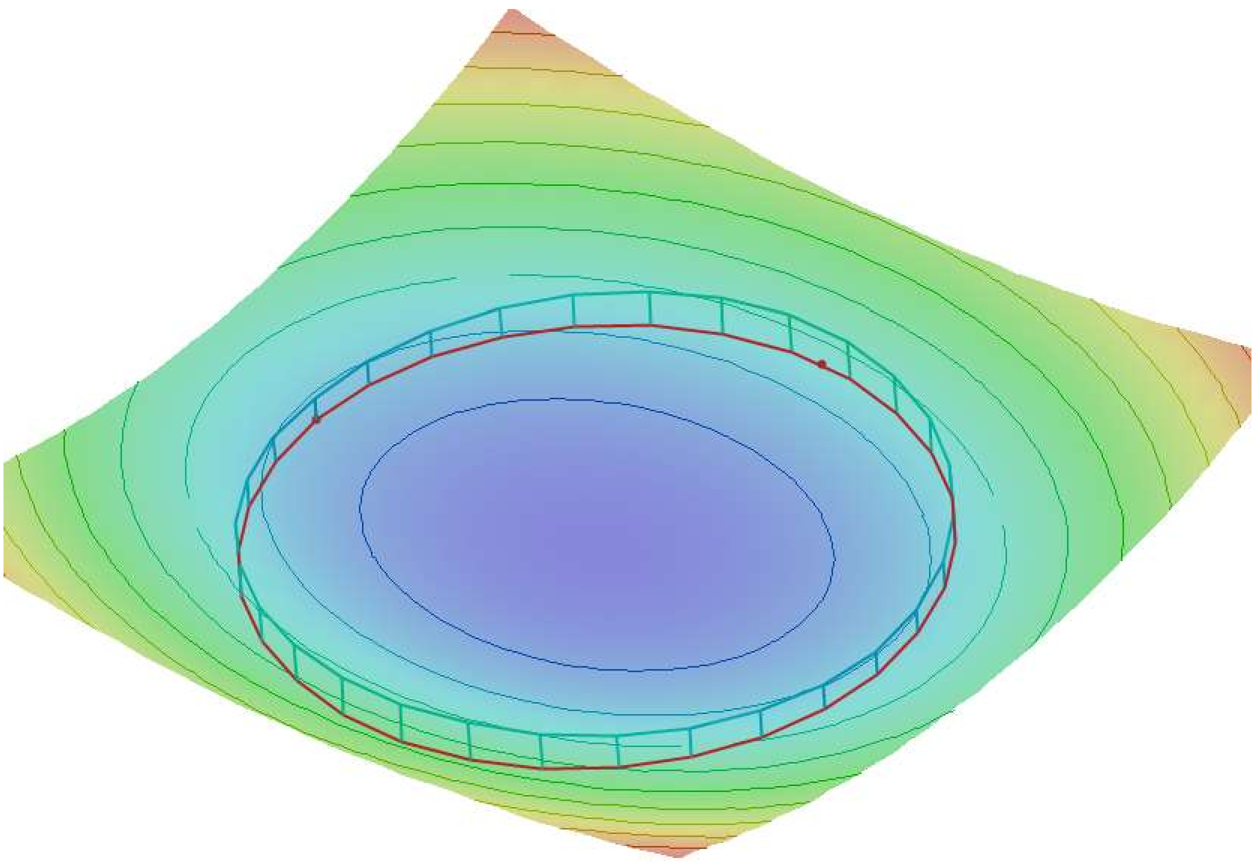
$y = \sin(t)$

$3x^2+5y^2 = 3 * \cos^2(t) + 5 * \sin^2(t)$

$$line\ integral = \int_C f(x,y)ds = \int_a^b f(h(t),g(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$line\ integral = \int_C (3x^2\ +\ 5y^2)ds = \int_0^{2\pi} (3 * cos^2(\text{t})\ +\ 5 *\ sin^2(\text{t}))\sqrt{(1)^2 + (1)^2}\,dt$$

$$= \ [4t]\begin{matrix}2\pi \\ 0\end{matrix} = 8\pi$$



∫F(x,y)ds = ∫F((h(t),g(t))√((dx/dt)² + (dy/dt)²) dt

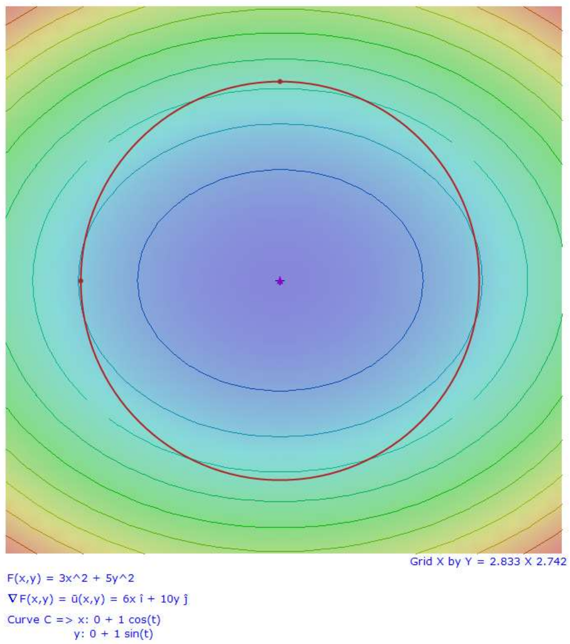
F(x,y) = 3x^2 + 5y^2

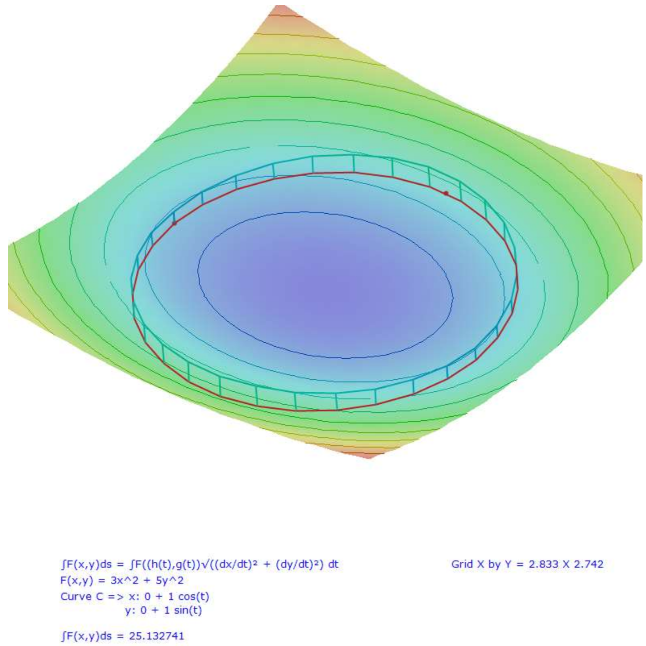
Curve C => x: 0 + 1 cos(t)

y: 0 + 1 sin(t)

∫F(x,y)ds = 25.132741

Grid X by Y = 2.833 X 2.742





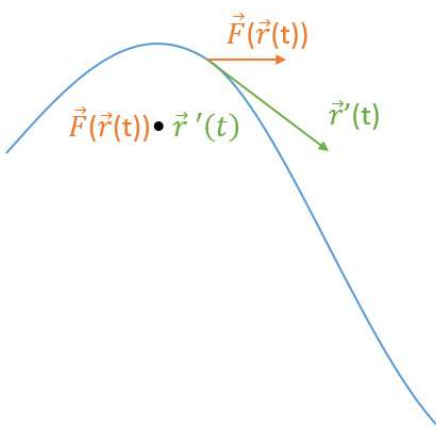
VECTOR LINE INTEGRAL

Vector line integral is the evaluation of line integral in a vector field.

The line integral for a path R(x,y) in a 2D vector field F(x,y) = P(x,y)I + Q(x,y)j is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)dt$$

In physics line integral is the work done on a particle travelling on a curve C inside a force field F



Example:

The force field F is <6x, 10y> and our path is a triangle A(0,0), B(1,0) & C(0,1). Find the work done on a particle by moving it along the triangle path.

Solution

The triangle sides can be parameterized as individual lines C1, C2 & C3

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C1} \vec{F} \cdot d\vec{r} + \int_{C2} \vec{F} \cdot d\vec{r} + \int_{C3} \vec{F} \cdot d\vec{r}$$

C1: x = t; y = 0

$$\int_{C1} < 6x, 10y > \cdot\ d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)dt$$

$$= \int_0^1 < 6t, 0 > \cdot < 1,0 > dt$$

$$= \int_0^1 6t\, dt$$

= 3

C2: x = 1- t; y = t

$$\int_{C1} < 6x, 10y > \cdot\ d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)dt$$

$$= \int_0^1 < 6(1-t), 10t > \cdot < -1,1 > dt$$

$$= \int_0^1 (-6(1-t) + 10t)\, dt$$

= 2

C3: x = 0; y = 1-t

$$\int_{C1} < 6x, 10y > \cdot \, d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)dt$$

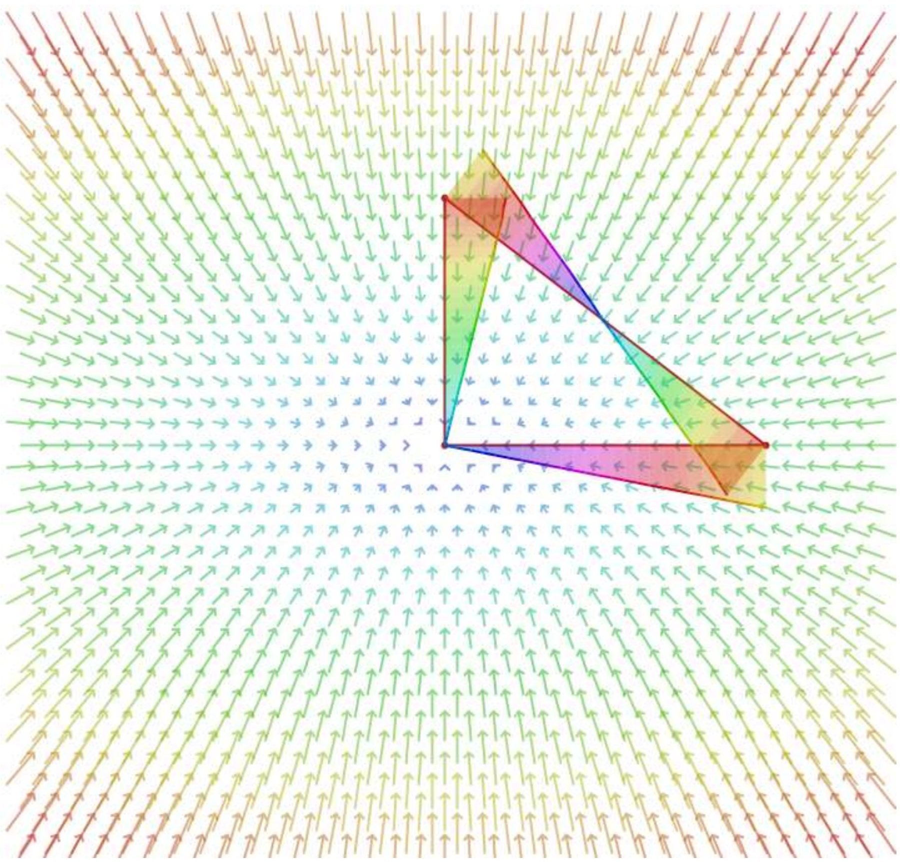
$$= \int_0^1 < 0, 10(1-t) > \cdot < 0, -1 > dt$$

$$= \int_0^1 (-10(1-t))\, dt$$

= - 5

$$\oint_c \vec{F} \cdot \, d\vec{r} \;\; = 3 + 2 - 5 = 0$$

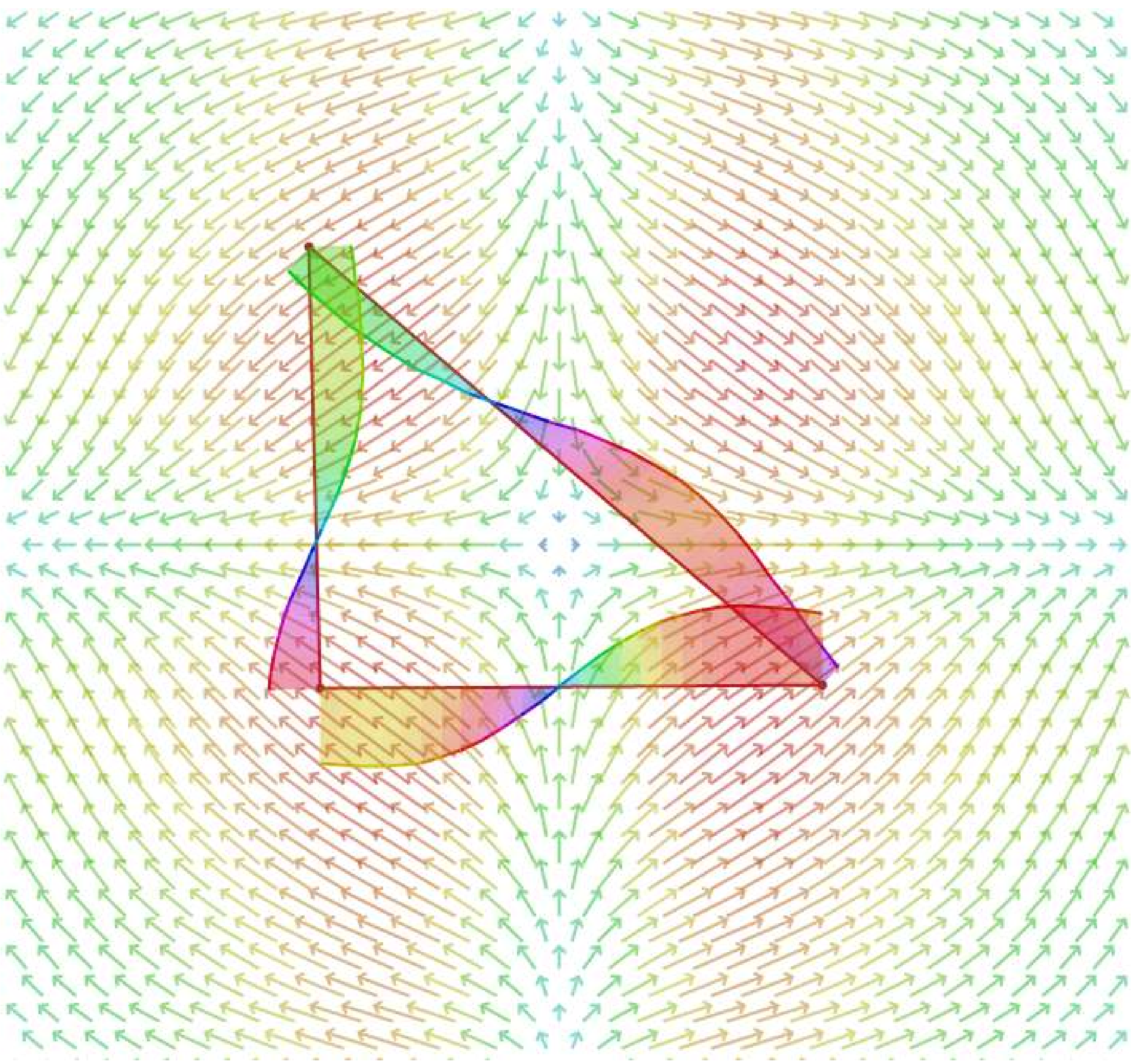
See the below picture

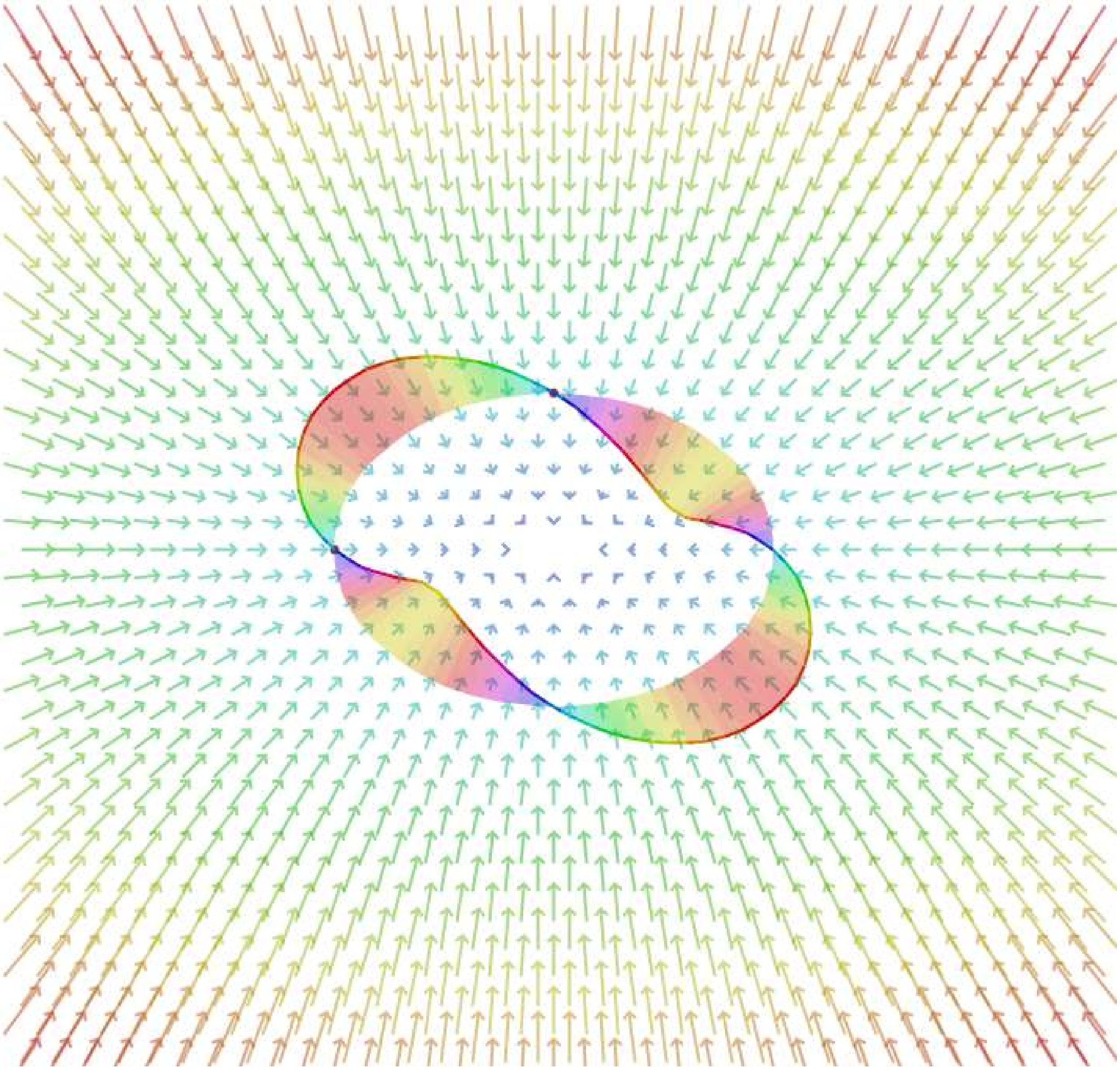


Conservative Vector field

The vector field <6x,10y> is the gradient of the function (scalar potential) $3x^2+5y^2$ which is called a conservative field. If we calculate the line integral in opposite direction the results will be still zero (-3-2+5)

Also from the picture we can clearly see that there is no circulation (vanishing curl) inside the bounds formed by the triangle which implies (according to Green's theorem) that the line integral along the boundary should be zero





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