

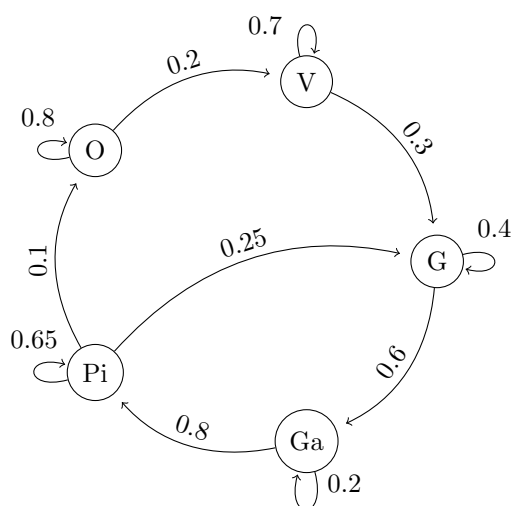
# Project 2024 - Markov Chains

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## Five states Mediterranean ecosystems

Q1-/ Draw the graph associated to this model.



Q2-/ Supposing that we are in state Pi, what is the probability of a trajectory of the type Pi-G-Ga? And of a trajectory of the type Pi-O-V?

Since we have a weighted graph,

- $P(Pi - G - Ga) = P(Pi \cap G \cap Ga | Pi) = P(Pi | Pi) * P(G | Pi) * P(Ga | G) = 0.65 * 0.25 * 0.6 = 0.0975$
- $P(Pi - O - V) = P(Pi \cap O \cap V | Pi) = P(Pi | Pi) * P(O | Pi) * P(V | O) = 0.65 * 0.1 * 0.2 = 0.013$

Q3-/ Give an example of a trajectory with a vanishing probability

We can take any trajectory for which we have a zero probability to go from a vertex to another in the transition probability matrix.

For example we see that the trajectory from garrigue (Ga) to oakgrove (O) has a vanishing probability

Q4-/ Using a computer, compute the distribution of the ecosystem for  $n=20$

$X_0$  is a uniform distribution, then all possible values of  $X$  are equally likely so we have  $X_0 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$ .

Then we multiply  $n=20$  times the transition probability matrix by  $P X_0$ . We can use a for loop to do

$$P * X_0 \text{ 20 times or directly do } P^{20} * X_0 \text{ which gives } X_{20} = \begin{bmatrix} 0.1752 \\ 0.1166 \\ 0.2044 \\ 0.1533 \\ 0.3505 \end{bmatrix}.$$

**Q5-/ Plot the evolution of the ecosystem's distribution as a function of  $n$**

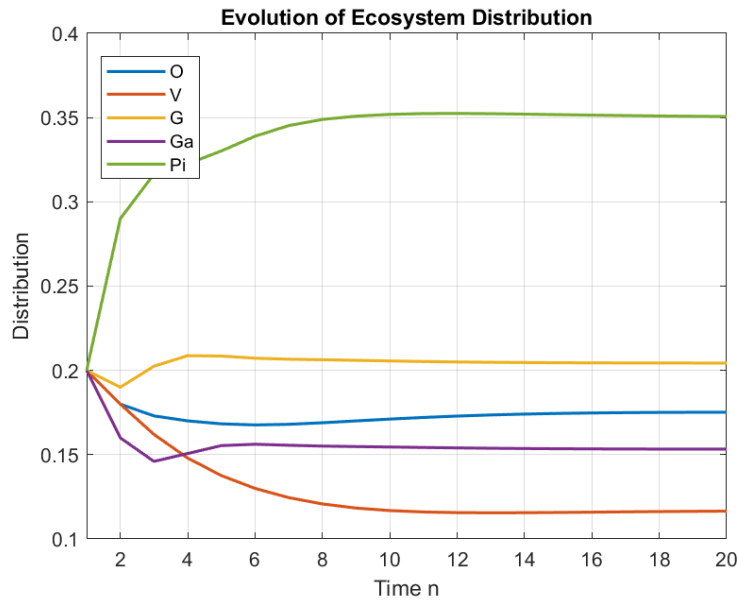


Figure 1: Evolution of the ecosystem's distribution as a function of  $n$

We can see that:

- Over time, each population seems to approach a stable equilibrium, a limit.
- The Pine forest (Pi) population grows significantly more than the other populations.
- The Grass (G) population decreases slightly before returning to its initial state.
- Garrigue (Ga), oakgrove (O) and Vineyards and orchards (V) populations are decreasing.

**Q6-/ Using your Linear Algebra course (and MATLAB for the computations), diagonalize  $P$ . In other words, find matrices  $A$ ,  $A^{-1}$ , and  $D$  (with  $D$  diagonal) such that  $P = ADA^{-1}$ . What do the matrices  $A$ ,  $A^{-1}$ , and  $D$  represent in terms of eigenvalues and eigenstates?**

**A more complex model for a more diverse ecosystem**

**Q9-/ Give the groups of states that all communicate with each other. Explain how this can be seen on the matrix.**

Here the groups of states that all communicate with each other :

## Communicating State Pairs

1 - 3	1 - 5	2 - 3
3 - 4	3 - 5	9 - 10
9 - 11	10 - 11	13 - 14
13 - 15	13 - 16	13 - 17
13 - 18	14 - 15	14 - 16
14 - 17	14 - 18	14 - 19
14 - 20	15 - 16	15 - 17
15 - 19	15 - 20	16 - 17
16 - 18	16 - 19	16 - 20
17 - 18	17 - 19	17 - 20
18 - 19	18 - 20	19 - 20

### Steps to verify communications :

- Let i for rows and j for columns
- We have to avoid all the redundant values like (i,j) if (j,i) is already considered.
- We have to see if there exists a direct or indirect transition from state (i to j) or (j to i)

### Q10-/ Start from the distribution $X_0$

**(a) Compute the distribution  $X_n$  of the system for  $n = 2$ ,  $n = 10$  and  $n = 50$**

We will use the transition formula:

$$X_n = P^n X_0,$$

where  $P$  is the given transition matrix.

For  $n = 2$ ,

$$X_2 = (0.11 \quad 0.105 \quad 0.515 \quad 0.12 \quad 0.15 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

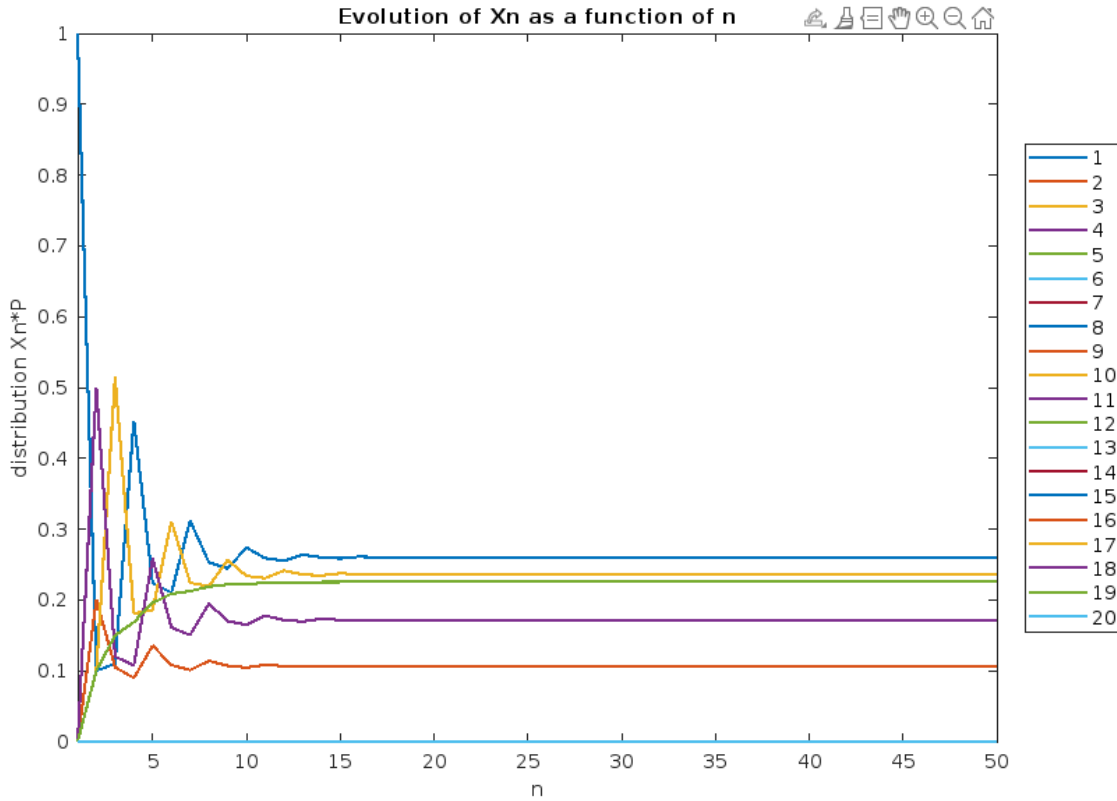
For  $n = 10$ ,

$$X_{20} = (0.26 \quad 0.1064 \quad 0.2363 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

For  $n = 50$ ,

$$X_{50} = (0.2601 \quad 0.1064 \quad 0.2362 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

(b) Plot  $X_n$  as a function of  $n$



(c) Does the sequence  $(X_n)_{n \in \mathbb{N}}$  admit a limiting distribution ?

By looking at the graph, we can see that the sequence seems to admit a limiting distribution. To be sure let's take  $n = 999$  and let's compare  $X_{1000}$  with  $X_{50}$ :

$$X_{50} = (0.2601 \quad 0.1064 \quad 0.2362 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

$$X_{999} = (0.2601 \quad 0.1064 \quad 0.2362 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

We clearly see that the values remain the same. Hence, the sequence  $(X_n)_{n \in \mathbb{N}}$  admits a limiting distribution.

**Q11-/ Now start from the distribution  $Y_0$**

**(a) Compute the distribution  $Y_n$  of the system for  $n = 2$ ,  $n = 10$  and  $n = 50$**

Similarly, we will use the transition formula:

$$Y_n = P^n Y_0,$$

where  $P$  is the given transition matrix.

For  $n = 2$ ,

$$Y_2 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

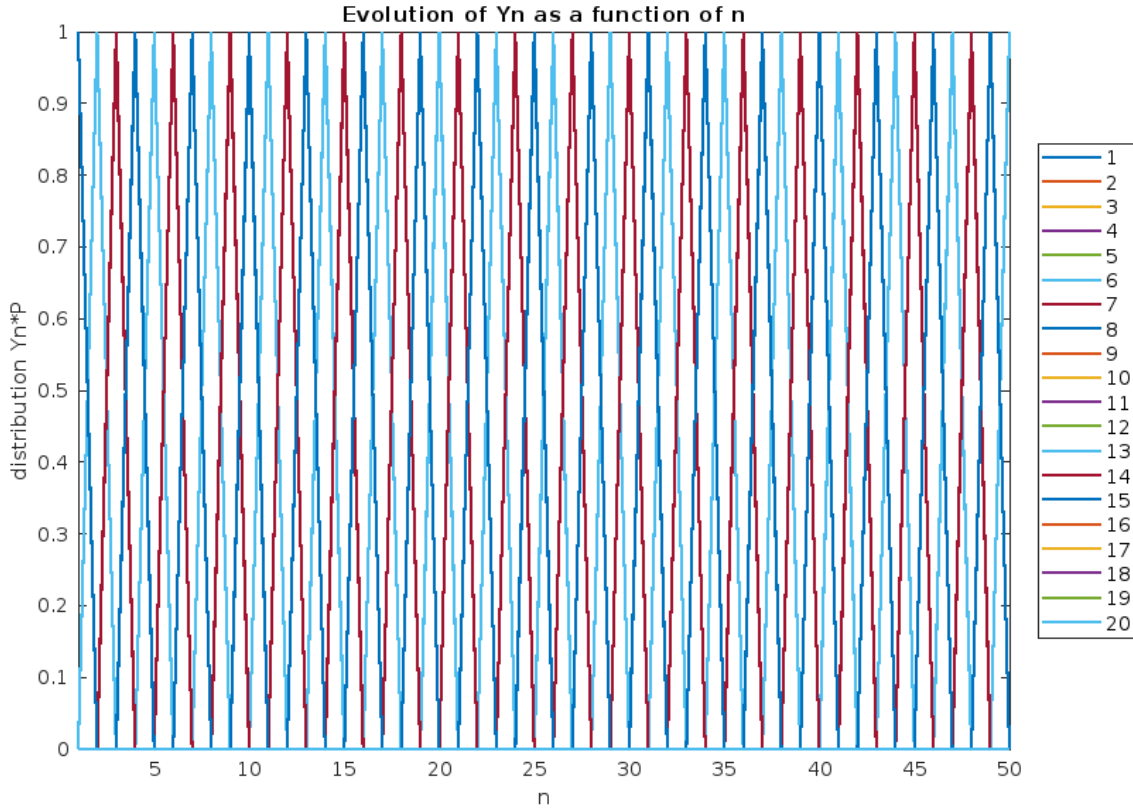
For  $n = 10$ ,

$$Y_{20} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

For  $n = 50$ ,

$$Y_{50} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

(b) Plot  $Y_n$  as a function of  $n$



(c) Does the sequence  $(Y_n)_{n \in \mathbb{N}}$  admit a limiting distribution ?

By looking at the graph, we can see that the sequence does not admit a limiting distribution. To be sure let's take  $n = 999$  and let's compare  $X_{1000}$  with  $X_{50}$ :

$$Y_{50} = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T,$$

$$Y_{999} = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T,$$

We clearly see that the value does not remain the same, in fact it switches between the 7th row and the 8th row. Hence, the sequence  $(Y_n)_{n \in \mathbb{N}}$  is unstable at  $+\infty$  so it doesn't admit a limiting distribution.

**Q12-/ Same questions starting from the initial distributions  $Z_0$  and  $W_0$**

(a) Compute the distribution  $Y_n$  of the system for  $n = 2$ ,  $n = 10$  and  $n = 50$

Similarly, we will use the transition formula:

$$Y_n = P^n Y_0,$$

where  $P$  is the given transition matrix.

## Question 12

For  $Z_n$ :

$$n = 2 : Z_2 = [0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.1, 0.03, 0.03, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 10 : Z_{10} = [0.0432, 0.0167, 0.0388, 0.0282, 0.0412, 2.43 \times 10^{-4}, 7.29 \times 10^{-5}, 2.187 \times 10^{-5}, 6.561 \times 10^{-6}, 1.9683 \times 10^{-6}, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 50 : Z_{50} = [0.484, 0.0197, 0.0442, 0.0324, 0.0417, 2.954 \times 10^{-25}, 8.8629 \times 10^{-26}, 2.6589 \times 10^{-26}, 7.9766 \times 10^{-27}, 2.393 \times 10^{-27}, 2.393 \times 10^{-27}, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

For  $W_n$ :

$$n = 2 : W_2 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.53, 0.155, 0.1, 0.105, 0.085, 0.145, 0.04, 0.0825]^T$$

$$n = 10 : W_{10} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.0525, 0.0101, 0.0056, 0.0056, 0.0053, 0.0075, 0.0024, 0.0025]^T$$

$$n = 50 : W_{50} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.0817, 4.4996 \times 10^{-9}, 2.5013 \times 10^{-9}, 2.4982 \times 10^{-9}, 2.3432 \times 10^{-9}, 3.3164 \times 10^{-9}, 1.0574 \times 10^{-9}, 1.1181 \times 10^{-9}]^T$$

## Question 13

### (a) Compute of $\alpha_n$ distribution

For  $\alpha_n$ :

$$n = 2 : \alpha_2 = [0, 0, 0, 0, 0.2, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 10 : \alpha_{10} = [0.0589, 0.0235, 0.0528, 0.0392, 0.0531, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 50 : \alpha_{50} = [0.0654, 0.0267, 0.0597, 0.0437, 0.0563, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

### (b) Plot of $\alpha_n$ as a function of $n$

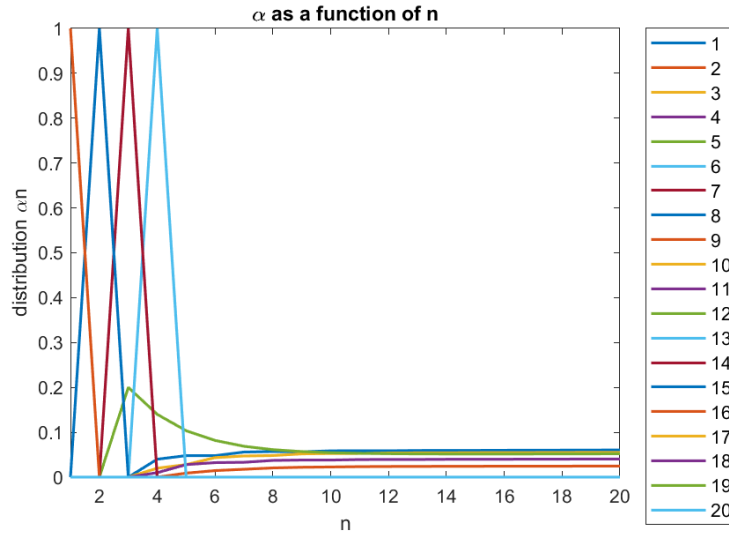


Figure 2:  $\alpha$  as a function of  $n$

(c) Linear relation between the limiting distributions of  $(\alpha_n)_{n \in \mathbb{N}}$ ,  $(X_n)_{n \in \mathbb{N}}$ , and  $(W_n)_{n \in \mathbb{N}}$

## Back to coral reefs

### Question 14

We know that global warming starts at  $+0\text{C}^\circ$  of the current global temperature.

So we will define  $0\text{C}^\circ$  as  $X_0$ .

Then, let  $X_i$  be the state for which the global warming is at  $i \times 0.2\text{C}^\circ$ .  $i \in \mathbb{Z}$

For example, at state  $X_{-2}$ , the global warming temperature will be  $-0.4\text{C}^\circ$ .

All these states are related to each other with the relations :

- $+0.2\text{C}^\circ$ , probability of 50%
- $-0.2\text{C}^\circ$ , probability of 20%
- remain the same, probability of 30%

From this we can create a graph representing these states (It is an infinite graph)

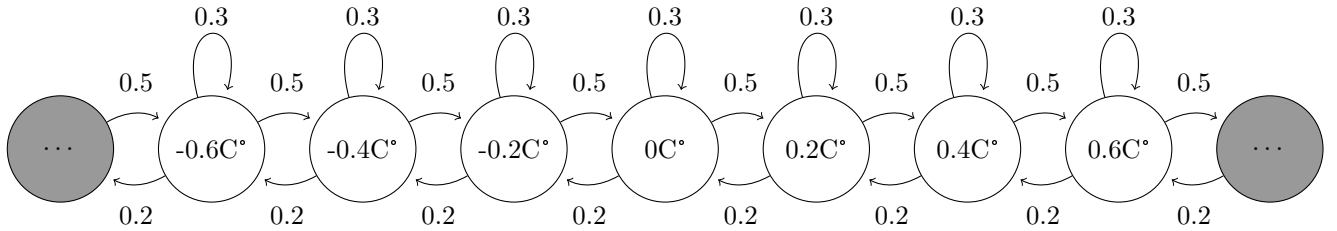


Figure 3: Markov chain of global warming possibilities

And we can create the corresponding (infinite) Markov matrix.

$$\frac{1}{10} \times \begin{pmatrix} 3 & 5 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 3 & 5 & & 0 & 0 & 0 \\ 0 & 2 & 3 & \ddots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & 3 & 5 & 0 \\ 0 & 0 & 0 & & 2 & 3 & 5 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 3 \end{pmatrix}_k$$

It can be said that the probabilities on the borders do not add up to 1, but this is inevitable as we cannot create a truly infinite matrix.

We are just estimating the matrix around the state  $X_0$ , so we will take a matrix of size  $k = 2 \times Y + 1$ ,  $Y$  being the number of decades since the start of global warming.

### Question 15

It is said that the coral reefs will die if the temperature gets to  $+0.6\text{C}^\circ$ .

This corresponds to the state  $X_3$

So The average lifespan (or the life expectancy) of the coral reefs will be :

The decade at which the probability of being in a state where  $P(X_{i \geq 3})$  is more than 50%

Using matlab to make a code that calculates this probability, depending on the number of decades elapsed.

We obtain : at 9 decades  $P(X_{i \geq 3}) = 0.5465$

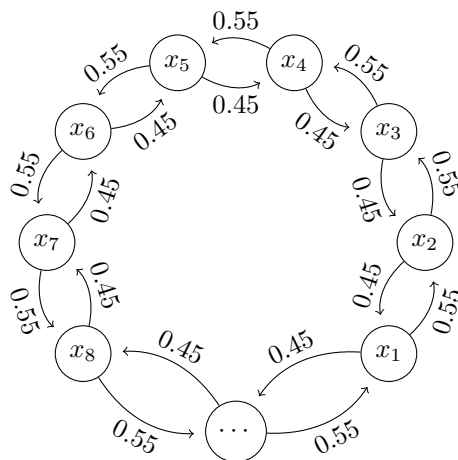
This means that in this new case, the coral reefs are expected to live around 90 more years.

## Question 16

Using probabilities, we can calculate the esperance

## To go Further

## Question 17



These dots represent the 991 missing nodes  
The relation is the same for all of these nodes.