

Project 2024 - Markov Chains

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Five states Mediterranean ecosystems

Q1-/ Draw the graph associated to this model.

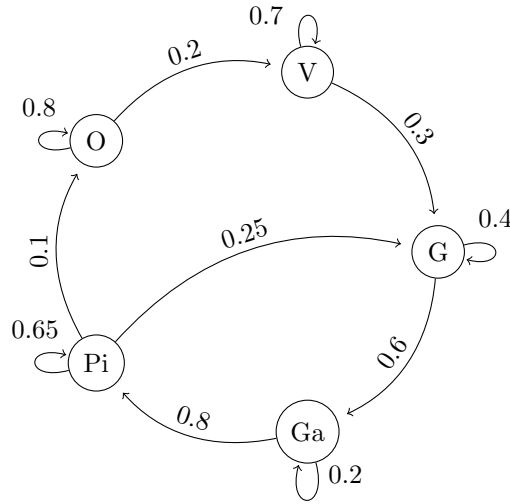


Figure 1: Trouver un titre ptdr

Q2-/ Supposing that we are in state Pi, what is the probability of a trajectory of the type Pi-G-Ga? And of a trajectory of the type Pi-O-V?

Since we have a weighted graph,

- $P(Pi - G - Ga) = P(Pi \cap G \cap Ga | Pi) = P(Pi | Pi) * P(G | Pi) * P(Ga | G) = 0.65 * 0.25 * 0.6 = 0.0975$
- $P(Pi - O - V) = P(Pi \cap O \cap V | Pi) = P(Pi | Pi) * P(O | Pi) * P(V | O) = 0.65 * 0.1 * 0.2 = 0.013$

Q3-/ Give an example of a trajectory with a vanishing probability

We can take any trajectory for which we have a zero probability to go from a vertex to another in the transition probability matrix.

For example we see that the trajectory from garrigue (Ga) to oakgrove (O) has a vanishing probability

Q4-/ Using a computer, compute the distribution of the ecosystem for n=20

X_0 is a uniform distribution, then all possible values of X are equally likely so we have $X_0 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$.

Then we multiply $n=20$ times the transition probability matrix by $P X_0$. We can use a for loop to do

$$P * X_0 \text{ 20 times or directly do } P^{20} * X_0 \text{ which gives } X_{20} = \begin{bmatrix} 0.1752 \\ 0.1166 \\ 0.2044 \\ 0.1533 \\ 0.3505 \end{bmatrix}.$$

Q5-/ Plot the evolution of the ecosystem's distribution as a function of n

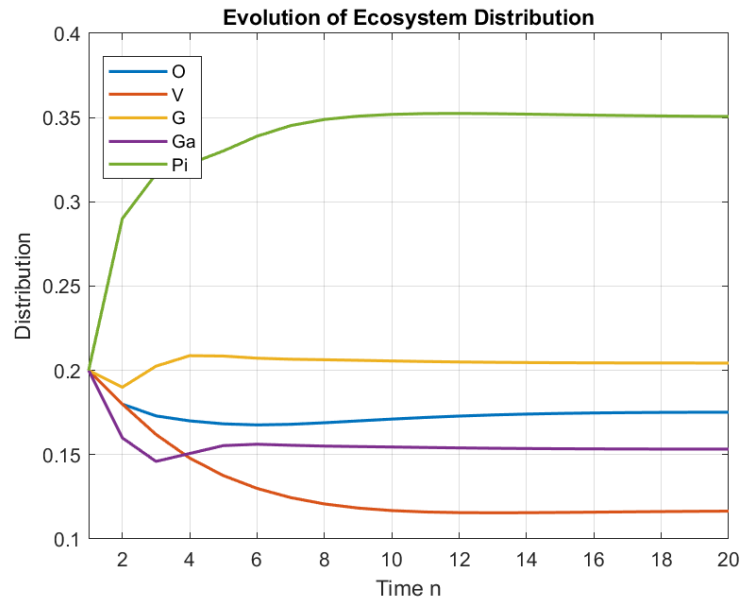


Figure 2: Evolution of the ecosystem's distribution as a function of n

We can see that:

- Over time, each population seems to approach a stable equilibrium, a limit.
- The Pine forest (Pi) population grows significantly more than the other populations.
- The Grass (G) population decreases slightly before returning to its initial state.
- Garrigue (Ga), oakgrove (O) and Vineyards and orchards (V) populations are decreasing.

Q6-/ Using your Linear Algebra course (and MATLAB for the computations), diagonalize P . In other words, find matrices A , A^{-1} , and D (with D diagonal) such that $P = ADA^{-1}$. What do the matrices A , A^{-1} , and D represent in terms of eigenvalues and eigenstates?

A more complex model for a more diverse ecosystem

Q9-/ Give the groups of states that all communicate with each other. Explain how this can be seen on the matrix.

Here the groups of states that all communicate with each other :

Communicating State Pairs

1 - 3	1 - 5	2 - 3
3 - 4	3 - 5	9 - 10
9 - 11	10 - 11	13 - 14
13 - 15	13 - 16	13 - 17
13 - 18	14 - 15	14 - 16
14 - 17	14 - 18	14 - 19
14 - 20	15 - 16	15 - 17
15 - 19	15 - 20	16 - 17
16 - 18	16 - 19	16 - 20
17 - 18	17 - 19	17 - 20
18 - 19	18 - 20	19 - 20

Steps to verify communications :

- Let i for rows and j for columns
- We have to avoid all the redundant values like (i,j) if (j,i) is already considered.
- We have to see if there exists a direct or indirect transition from state (i to j) or (j to i)

Q10-/ Start from the distribution X_0

(a) Compute the distribution X_n of the system for $n = 2$, $n = 10$ and $n = 50$

We will use the transition formula:

$$X_n = P^n X_0,$$

where P is the given transition matrix.

For $n = 2$,

$$X_2 = (0.11 \quad 0.105 \quad 0.515 \quad 0.12 \quad 0.15 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

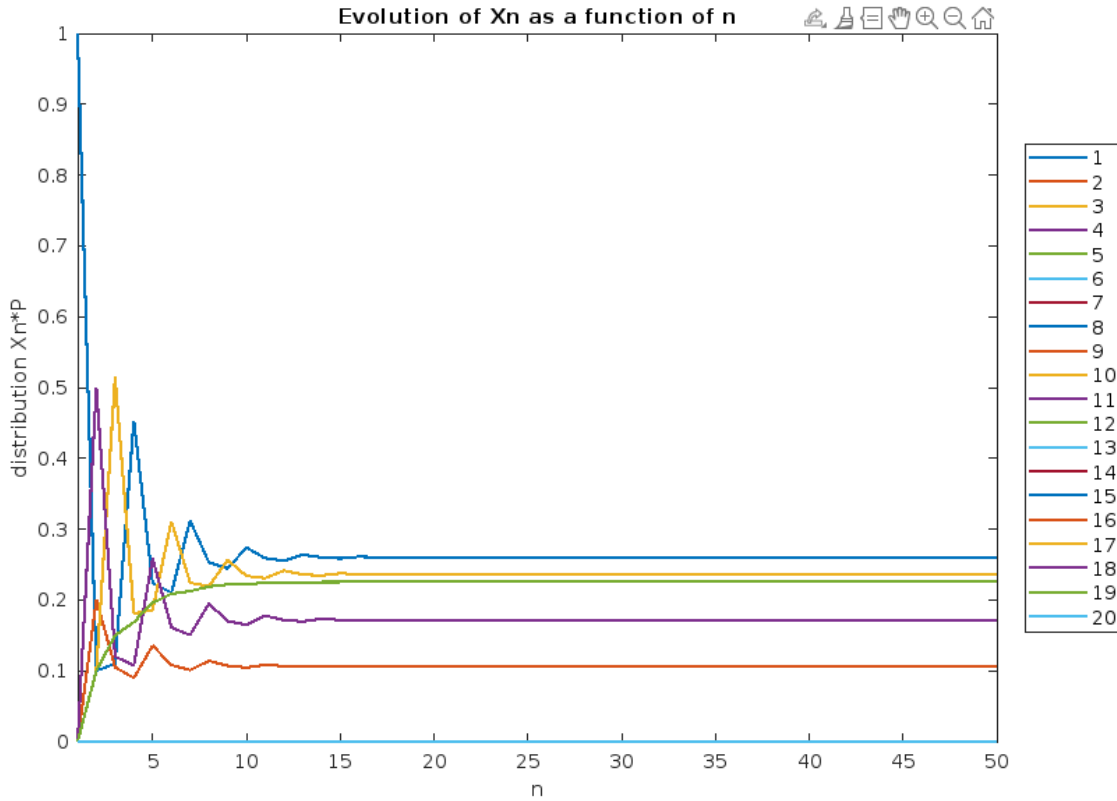
For $n = 10$,

$$X_{20} = (0.26 \quad 0.1064 \quad 0.2363 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

For $n = 50$,

$$X_{50} = (0.2601 \quad 0.1064 \quad 0.2362 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

(b) Plot X_n as a function of n



(c) Does the sequence $(X_n)_{n \in \mathbb{N}}$ admit a limiting distribution ?

By looking at the graph, we can see that the sequence seems to admit a limiting distribution. To be sure let's take $n = 999$ and let's compare X_{1000} with X_{50} :

$$X_{50} = (0.2601 \quad 0.1064 \quad 0.2362 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

$$X_{999} = (0.2601 \quad 0.1064 \quad 0.2362 \quad 0.1717 \quad 0.2256 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

We clearly see that the values remain the same. Hence, the sequence $(X_n)_{n \in \mathbb{N}}$ admits a limiting distribution.

Q11-/ Now start from the distribution Y_0

(a) Compute the distribution Y_n of the system for $n = 2$, $n = 10$ and $n = 50$

Similarly, we will use the transition formula:

$$Y_n = P^n Y_0,$$

where P is the given transition matrix.

For $n = 2$,

$$Y_2 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

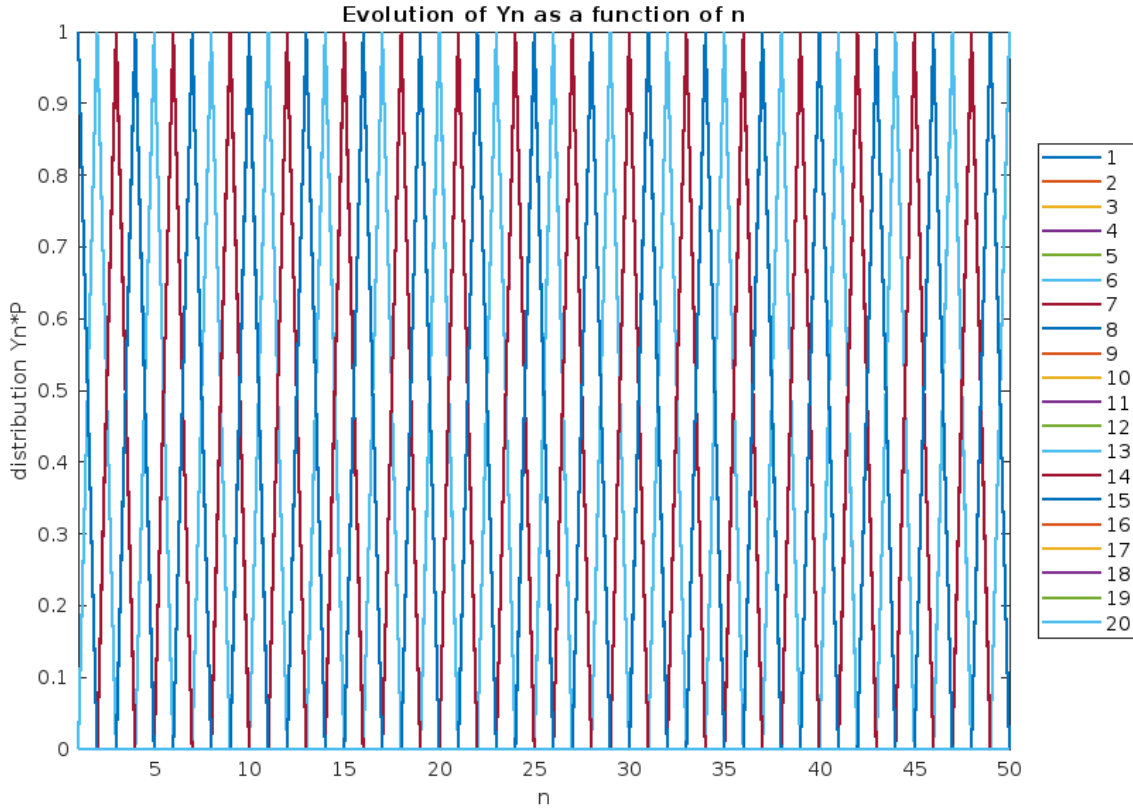
For $n = 10$,

$$Y_{20} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

For $n = 50$,

$$Y_{50} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

(b) Plot Y_n as a function of n



(c) Does the sequence $(Y_n)_{n \in \mathbb{N}}$ admit a limiting distribution ?

By looking at the graph, we can see that the sequence does not admit a limiting distribution. To be sure let's take $n = 999$ and let's compare X_{1000} with X_{50} :

$$Y_{50} = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T,$$

$$Y_{999} = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T,$$

We clearly see that the value does not remain the same, in fact it switches between the 7th row and the 8th row. Hence, the sequence $(Y_n)_{n \in \mathbb{N}}$ is unstable at $+\infty$ so it doesn't admit a limiting distribution.

Question 12

For Z_n :

$$n = 2 : Z_2 = [0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.1, 0.03, 0.03, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 10 : Z_{10} = [0.0432, 0.0167, 0.0388, 0.0282, 0.0412, 2.43 \times 10^{-4}, 7.29 \times 10^{-5}, 2.187 \times 10^{-5}, \\ 6.561 \times 10^{-6}, 1.9683 \times 10^{-6}, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 50 : Z_{50} = [0.484, 0.0197, 0.0442, 0.0324, 0.0417, 2.954 \times 10^{-25}, 8.8629 \times 10^{-26}, \\ 2.6589 \times 10^{-26}, 7.9766 \times 10^{-27}, 2.393 \times 10^{-27}, 2.393 \times 10^{-27}, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

For W_n :

$$n = 2 : W_2 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.53, 0.155, 0.1, 0.105, 0.085, 0.145, 0.04, 0.0825]^T$$

$$n = 10 : W_{10} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.0525, 0.0101, 0.0056, 0.0056, 0.0053, 0.0075, 0.0024, 0.0025]^T$$

$$n = 50 : W_{50} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.0817, 4.4996 \times 10^{-9}, 2.5013 \times 10^{-9}, 2.4982 \times 10^{-9}, \\ 2.3432 \times 10^{-9}, 3.3164 \times 10^{-9}, 1.0574 \times 10^{-9}, 1.1181 \times 10^{-9}]^T$$

Question 13

(a) Compute of α_n distribution

For α_n :

$$n = 2 : \alpha_2 = [0, 0, 0, 0, 0.2, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 10 : \alpha_{10} = [0.0589, 0.0235, 0.0528, 0.0392, 0.0531, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

$$n = 50 : \alpha_{50} = [0.0654, 0.0267, 0.0597, 0.0437, 0.0563, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

(b) Plot of α_n as a function of n

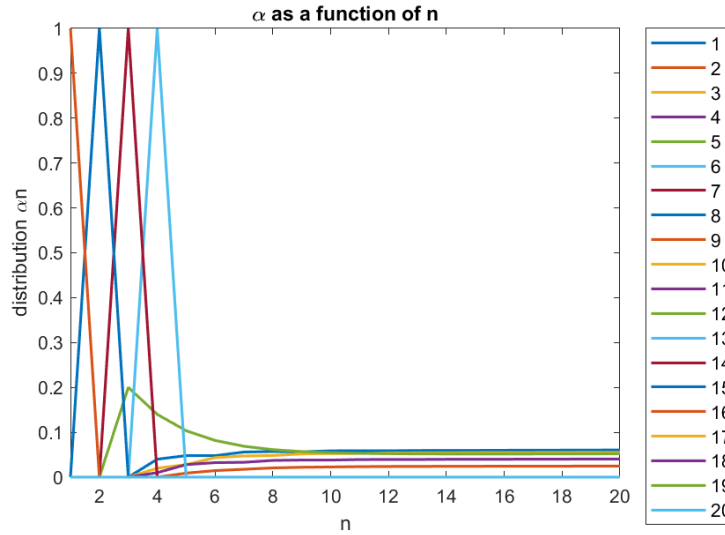


Figure 3: α as a function of n

(c) Linear relation between the limiting distributions of $(\alpha_n)_{n \in \mathbb{N}}$, $(X_n)_{n \in \mathbb{N}}$, and $(W_n)_{n \in \mathbb{N}}$