Project 2024 - Markov Chains

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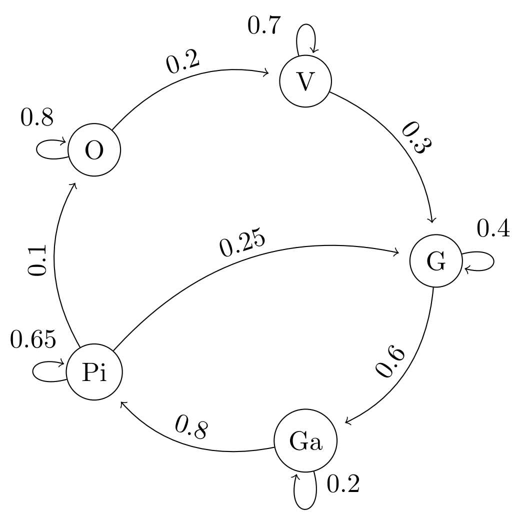
Five states Mediterranean ecosystems

Q1-/ Draw the graph associated to this model.



Q2-/ Supposing that we are in state Pi, what is the probability of a trajectory of the type Pi-G-Ga? And of a trajectory of the type Pi-O-V?

Since we have a weighted graph,

- $\mathrm{P}(Pi-G-Ga) = \mathrm{P}(Pi \cap G \cap Ga|Pi) = \mathrm{P}(Pi|Pi) * \mathrm{P}(G|Pi) * \mathrm{P}(Ga|G) = 0.65 * 0.25 * 0.6 = 0.0975$
- $\mathrm{P}(Pi-O-V) = \mathrm{P}(Pi \cap O \cap V|Pi) = \mathrm{P}(Pi|Pi) * \mathrm{P}(O|Pi) * \mathrm{P}(V|O) = 0.65 * 0.1 * 0.2 = 0.013$

Q3-/ Give an example of a trajectory with a vanishing probability

We can take any trajectory for witch we have a zero probability to go from a vertex to another in the transition probability matrix.

For example we see that the trajectory from garrigue (Ga) to oakgrove (O) has a vanishing probability

Q4-/ Using a computer,compute the distribution of the ecosystem for n=20

X_0 is a uniform distribution, then all possible values of X are equally likely so we have $X_0 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$.

Then we multiply n=20 times the transition probability matrix by P X_0 . We can use a for loop to do P*X_0 20 times or directly do $P^{20} * X_0$ witch gives $X_{20} = \begin{bmatrix} 0.1752 \\ 0.1166 \\ 0.2044 \\ 0.1533 \\ 0.3505 \end{bmatrix}$.

Q5-/ Plot the evolution of the ecosystem's distribution as a function of n

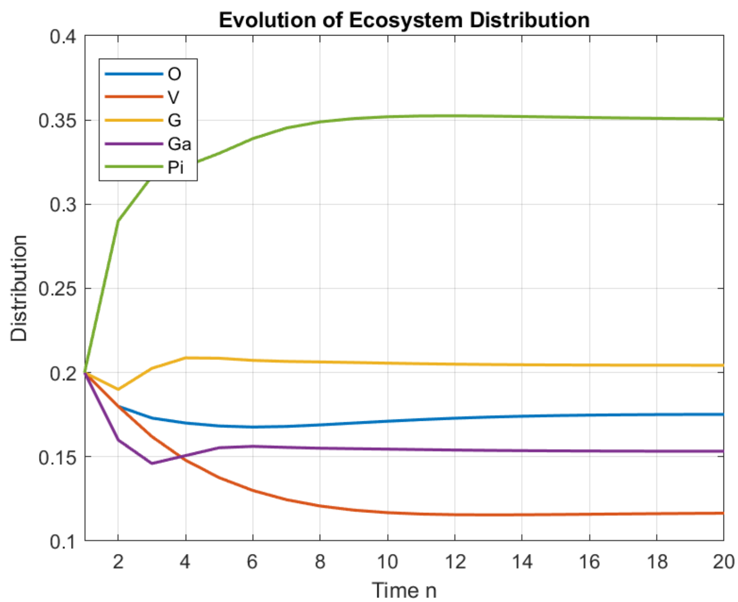


Figure 1: Evolution of the ecosystem's distribution as a function of n

We can see that:

- Over time, each population seems to approach a stable equilibrium, a limit.
- The Pine forest (Pi) population grows significantly more than the other populations.
- The Grass (G) population decreases slightly before returning to its initial state.
- Garrigue (Ga), oakgrove (O) and Vineyards and orchards (V) populations are decreasing.

Q6-/ Using your Linear Algebra course (and MATLAB for the computations), diagonalize P. In other words, find matrices A, A^{-1}, and D (with D diagonal) such that $P = ADA^{-1}$. What do the matrices A, A^{-1}, and D represent in terms of eigenvalues and eigenstates?

A more complex model for a more diverse ecosystem

Q9-/ Give the groups of states that all communicate with each other. Explain how this can be seen on the matrix.

Here the groups of states that all communicate with each other :

Communicating State Pairs

1 - 3	1 - 5	2 - 3
3 - 4	3 - 5	9 - 10
9 - 11	10 - 11	13 - 14
13 - 15	13 - 16	13 - 17
13 - 18	14 - 15	14 - 16
14 - 17	14 - 18	14 - 19
14 - 20	15 - 16	15 - 17
15 - 19	15 - 20	16 - 17
16 - 18	16 - 19	16 - 20
17 - 18	17 - 19	17 - 20
18 - 19	18 - 20	19 - 20

Steps to verify communications :

- Let i for rows and j for columns
- We have to avoid all the redundant values like (i,j) if (j,i) is already considered.
- We have to see if there exists a direct or indirect transition from state (i to j) or (j to i)

Q10-/ Start from the distribution X_0

(a) Compute the distribution X_n of the system for $n = 2$, $n = 10$ and $n = 50$

We will use the transition formula:

$$X_n = P^n X_0,$$

where P is the given transition matrix.

For $n = 2$,

$$X_2 = \begin{pmatrix} 0.11 & 0.105 & 0.515 & 0.12 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

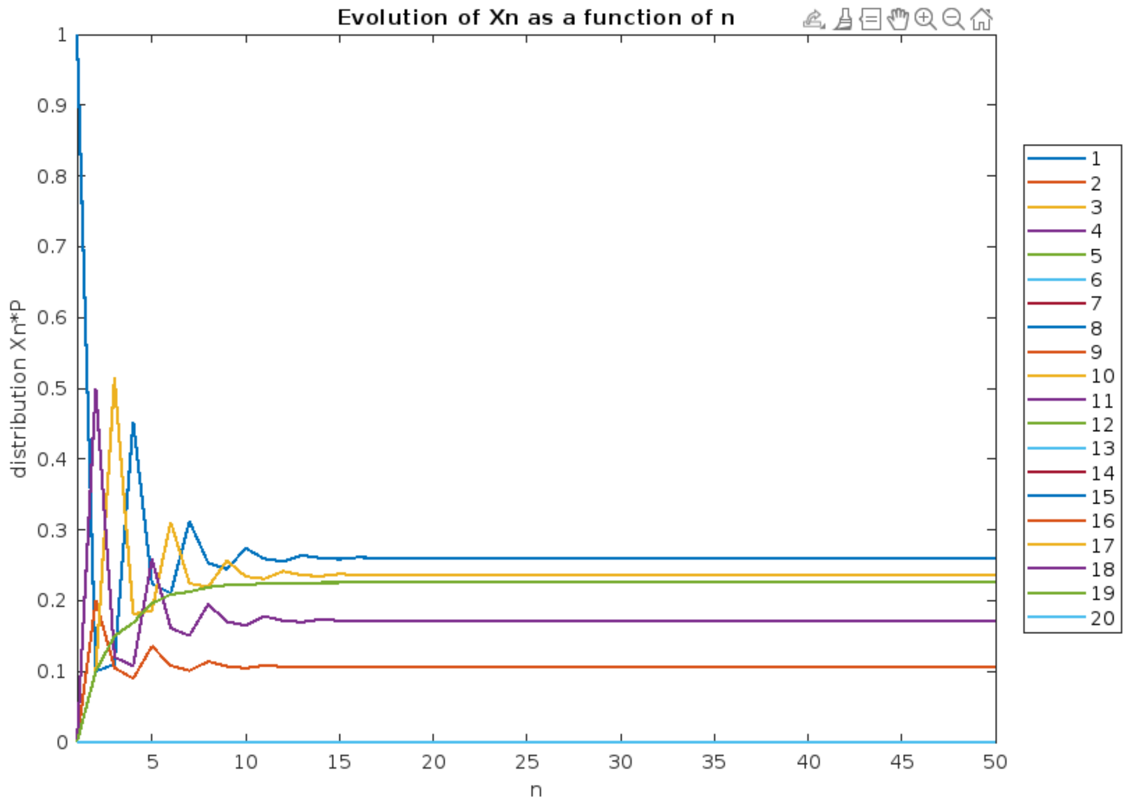
For $n = 10$,

$$X_{20} = \begin{pmatrix} 0.26 & 0.1064 & 0.2363 & 0.1717 & 0.2256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

For $n = 50$,

$$X_{50} = \begin{pmatrix} 0.2601 & 0.1064 & 0.2362 & 0.1717 & 0.2256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

(b) Plot X_n as a function of n



(c) Does th sequence $(X_n)_{n\in\mathbb{N}}$ admit a limiting distribution ?

By looking at the graph, we can see that the sequence seems to admit a limiting distribution. To be sure let's take n = 999 and lets compare X_{1000} with X_{50}:

$$X_{50} = \begin{pmatrix}0.2601 & 0.1064 & 0.2362 & 0.1717 & 0.2256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T,$$

$$X_{999} = \begin{pmatrix}0.2601 & 0.1064 & 0.2362 & 0.1717 & 0.2256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T,$$

We clearly see that the value remain the same. Hence, the sequence $(X_n)_{n\in\mathbb{N}}$ admit a limiting distribution.

Q11-/ Now start from the distribution Y_0

(a) Compute the distribution Y_n of the system for $n = 2$, $n = 10$ and $n = 50$

Simmilarly, we will use the transition formula:

$$Y_n = P^n Y_0,$$

where P is the given transition matrix.

For $n = 2$,

$$Y_2 = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T,$$

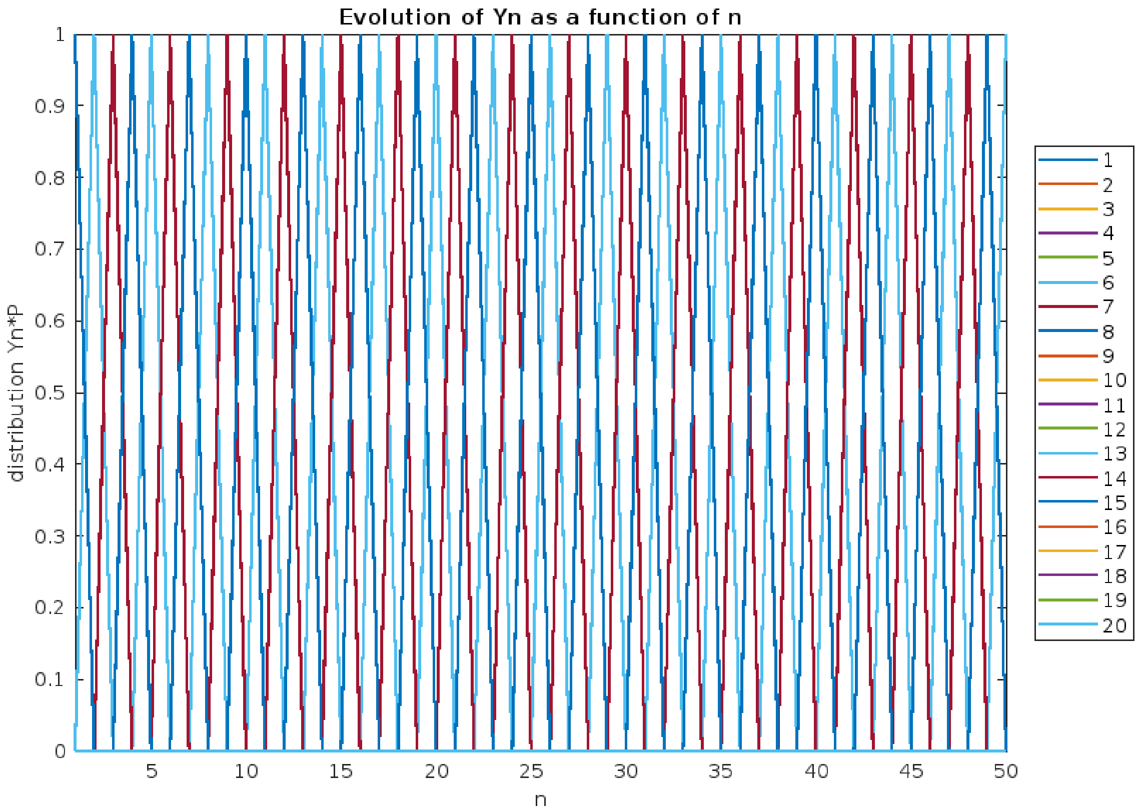
For $n = 10$,

$$Y_{20} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T,$$

For $n = 50$,

$$Y_{50} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T,$$

(b) Plot Y_n as a function of n



(c) Does th sequence $(Y_n)_{n\in\mathbb{N}}$ admit a limiting distribution ?

By looking at the graph, we can see that the sequence does not admit a limiting distribution. To be sure let's take n = 999 and lets compare X_{1000} with X_{50}:

$$Y_{50} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

$$Y_{999} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

We clearly see that the value does not remain the same, in fact it switches between the 7th row and the 8th row. Hence, the sequence $(Y_n)_{n\in\mathbb{N}}$ is unstable at +infinity so it doesn't admit a limiting distribution.

Q12-/ Same questions strating from the initial distributions Z_0 and W_0

(a) Compute the distribution Y_n of the system for $n = 2$, $n = 10$ and $n = 50$

Simmilarly, we will use the transition formula:

$$Y_n = P^n Y_0,$$

where P is the given transition matrix.

Question 12

For Z_n:

$$n = 2: \;\; Z_2 = [0, 0, 0, 0, 0, 0, 0, 0.6, 0.1, 0.03, 0.03, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$
$$n = 10: \;\; Z_{10} = [0.0432, 0.0167, 0.0388, 0.0282, 0.0412, 2.43 \times 10^{-4}, 7.29 \times 10^{-5}, 2.187 \times 10^{-5}, 6.561 \times 10^{-6}, 1.9683 \times 10^{-6}, 0, 0, 0, 0, 0, 0, 0, 0]^T$$
$$n = 50: \;\; Z_{50} = [0.484, 0.0197, 0.0442, 0.0324, 0.0417, 2.954 \times 10^{-25}, 8.8629 \times 10^{-26}, 2.6589 \times 10^{-26}, 7.9766 \times 10^{-27}, 2.393 \times 10^{-27}, 2.393 \times 10^{-27}, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

For W_n:

$$n = 2: \;\; W_2 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.53, 0.155, 0.1, 0.105, 0.085, 0.145, 0.04, 0.0825]^T$$
$$n = 10: \;\; W_{10} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.0525, 0.0101, 0.0056, 0.0056, 0.0053, 0.0075, 0.0024, 0.0025]^T$$
$$n = 50: \;\; W_{50} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.0817, 4.4996 \times 10^{-9}, 2.5013 \times 10^{-9}, 2.4982 \times 10^{-9}, 2.3432 \times 10^{-9}, 3.3164 \times 10^{-9}, 1.0574 \times 10^{-9}, 1.1181 \times 10^{-9}]^T$$

Question 13

(a) Compute of α_n distribution

For α_n:

$$n = 2: \;\; \alpha_2 = [0, 0, 0, 0, 0.2, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$
$$n = 10: \;\; \alpha_{10} = [0.0589, 0.0235, 0.0528, 0.0392, 0.0531, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$
$$n = 50: \;\; \alpha_{50} = [0.0654, 0.0267, 0.0597, 0.0437, 0.0563, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$$

(b) Plot of α_n as a function of n

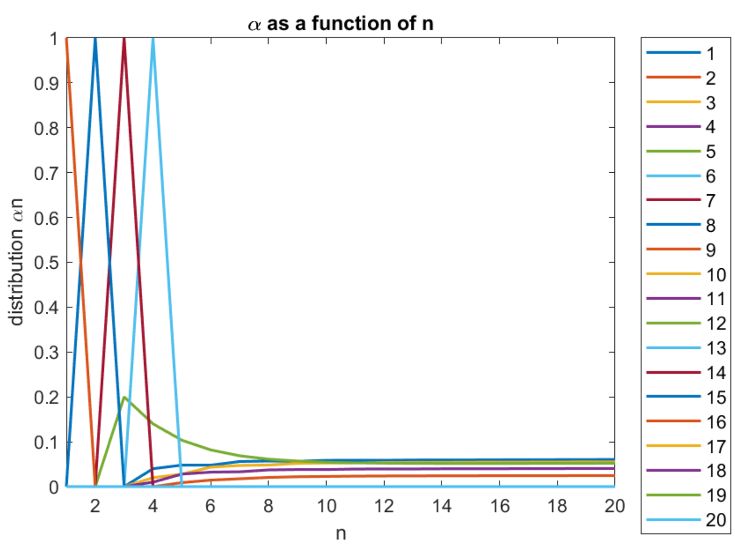


Figure 2: α as a function of n

(c) Linear relation between the limiting distributions of $(\alpha_n)_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}}$, and $(W_n)_{n\in\mathbb{N}}$

Back to coral reefs

Question 14

We know that global warming starts at +0C° of the current global temperature.
So we will define 0C° as X_0.
Then, let X_i be the state for which the global warming is at $i \times 0.2C$. $i \in \mathbb{Z}$
For example, at state X_{-2}, the global warming temperature will be -0.4C°.

All these states are related to each other with the relations :

- +0.2C°, probability of 50%
- -0.2C°, probability of 20%
- remain the same, probability of 30%

From this we can create a graph representing these states (It is an infinite graph)

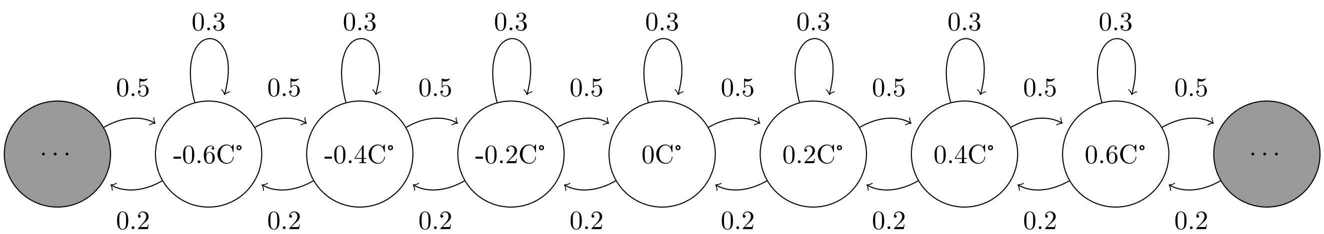


Figure 3: Markov chain of global warming possibilities

And we can create the corresponding (infinite) Markov matrix.

$$\frac{1}{10} \times \begin{pmatrix} 3 & 5 & 0 & \cdots & \cdots & 0 & 0 \\ 2 & 3 & 5 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 3 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 3 & 5 & 0 \\ 0 & 0 & \cdots & 0 & 2 & 3 & 5 \\ 0 & 0 & \cdots & \cdots & 0 & 2 & 3 \end{pmatrix}_k$$

It can be said that the probabilities on the borders do not add up to 1, but this is inevitable as we cannot create a truly infinite matrix.
We are just estimating the matrix around the state X_0, so we will take a matrix of size $k = 2 \times Y + 1$, Y beeing the number of decades since the start of global warming.

Question 15

It is said that the coral reefs will die if the temperature gets to +0.6C°.
This corresponts to the state X_3 So The average lifespan (or the life expectancy) of the coral reefs will be :
The decade at which the probability of beeing in a state where $P(X_{i\geq 3})$ is more than 50%

Using matlab to make a code that calculates this probability, depending on the number of decades elapsed. We obtain : at 9 decades $P(X_{i\geq 3}) = 0.5465$

This means that in this new case, the coral reefs are expected to live around 90 more years.

Question 16

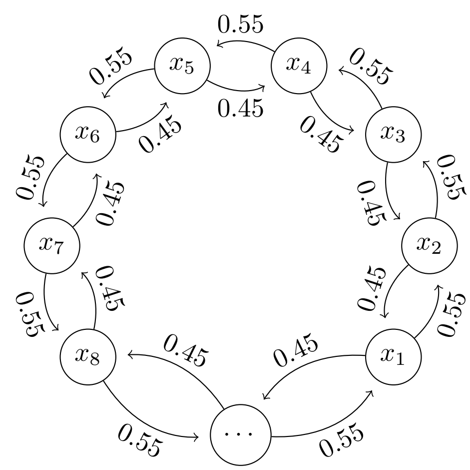
Using probability laws, we can calculate the expected value :

$$\begin{array}{rrcl} & E(\mathbb{X}) & = & 0.2 \times P(+0.2C) + (-0.2) \times P(-0.2C) + 0 \times P(\pm 0) \\ \Leftrightarrow & E(\mathbb{X}) & = & 0.2 \times 0.5 - 0.2 \times 0.2 + 0 \times 0.3 \\ \Leftrightarrow & E(\mathbb{X}) & = & 0.6 \end{array}$$

We know the expected value is how much the random variable will change after each event
So After N events the average change will be $N \times E$.
Here, $E > 0$ So we can have $N \times E(X) = 0.6$,$N > 0$
0.6 being the stopping factor, because the corals will die.
So we can search for N in this expression, $N = \frac{0.6}{E(X)} = 10$
So the theorical average lifespan is 10 decades, 100 years or a century.

To go Further

Question 17



These dots represent the 991 missing nodes
The relation is the same for all of these nodes.