Bootstrap confidence bands for Local Projections: a Monte Carlo study

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Abstract

This thesis investigates the application of bootstrap methods for estimating confidence bands for Impulse Response Functions within the Local Projections (LP) framework, a method increasingly employed in empirical macroeconomics. LP provides a flexible alternative to traditional Vector Autoregression models, offering robustness to misspecification and simplifying inference through direct regressions. While bootstrap methods are frequently used in practical applications to construct confidence intervals for LP estimates, there is limited theoretical work on the consistency and validity of bootstrap procedures in the general LP framework. To address this gap, this thesis conducts Monte Carlo simulations across various data-generating processes to evaluate the performance of bootstrap confidence intervals. Our results show that bootstrap methods achieve coverage rates close to nominal levels across both univariate and multivariate settings, often outperforming asymptotic techniques. We interpret these results as evidence of bootstrap consistency and presence of asymptotic refinements in those scenarios. However, we also identify potential coverage issues caused by small misspecification in finite samples. These findings contribute to the literature by providing new insights into the theoretical foundations of bootstrap methods in LP settings and their practical implications for macroeconomic research.

Contents

1 Introduction

The estimation of Impulse Response Functions (henceforth, IRF) has always been the main focus in empirical macroeconomics applications (Ramey, 2016; Nakamura and Steinsson, 2018). The seminal paper by Sims (1980), which has more than 20 thousands citations on Google Scholar, laid the groundwork for classical IRF estimation, using simple Vector Autoregression (VAR) models (even though, Frisch (1933) already explored similar concepts). Building on his work, a substantial body of research has focused on developing new techniques for identifying structural shocks from reduced-form parameters. Initially, this was achieved through AB-models (e.g., Blanchard and Perotti, 2002), followed by the adoption of alternative methods, such as sign restrictions (Granziera et al., 2018), identification through heteroskedasticity (Lanne and Lütkepohl, 2008; Bacchiocchi and Fanelli, 2015), and the use of external instruments (e.g., Stock and Watson, 2018). Local Projections (LP), introduced by Jordà (2005), offer an alternative approach to IRF estimation by bypassing the need to estimate reduced-form parameters through VAR models. Instead, they rely on direct regressions of future outcomes on present values. This simple method has been shown to be more flexible, robust, and intuitive compared to the classical VAR-based approach (Jordà, 2023; Jordà and Taylor, 2024).

A growing body of literature has explored the theoretical and practical benefits of LP over VAR models, particularly in handling persistent data and long-horizon estimations. For instance, Montiel Olea and Plagborg-Møller (2021) demonstrated that LP effectively handle highly persistent data and the challenges associated with long forecast horizons. By incorporating lags into the LP model, this method remains valid for both stationary and non-stationary processes, thereby simplifying inference compared to VAR-based approaches. Importantly, the two estimation strategies (LP and VAR) are not unrelated; Plagborg-Møller and Wolf (2021) established the relation between the LP approach and VAR-based methods. Other studies address practical estimation issues; for example, Lusompa (2023) developed a FGLS estimation strategy that accounts for autocorrelation in LP errors, resulting in more efficient estimates than OLS with heteroskedasticity and autocorrelation robust standard errors (Newey and West, 1986). In a recent study, Montiel Olea et al. (2024) highlighted the primary strength of the LP approach over its VAR counterpart: robustness to misspecification. They find that in locally misspecified frameworks, LP provides reliable estimates even in situations where VAR methods have been proven to be problematic. As evidence of this strength, Li et al. (2024) conducted a simulation study of LP for structural IRF across thousands of data-generating processes, designed to reflect the characteristics of U.S. macroeconomic data. They found that LP techniques outperform vector autoregressions in a wide range of scenarios.

Furthermore, LP have been employed in various contexts, effectively combining their simplicity with different econometric models. For example, Gonçalves et al. (2022) and Gonçalves et al. (2024) applied LP within a state-dependencies framework; Cloyne et al. (2023) utilized LP to compute the Kitagawa-Blinder-Oaxaca decomposition to estimate the fiscal multiplier in a heterogeneous agent model; Dube et al. (2023) incorporated LP into a Difference-in-Differences model. This latter application illustrates how local projections serve as a bridge between the realms of

micro and macroeconometrics. Additionally, recent advancements incorporate Bayesian methods and regularization techniques. For instance, Ferreira et al. (2023) proposed a Bayesian Local Projections (BLP) approach, which addresses the bias-variance trade-off by incorporating informative priors, thus improving the estimation of IRF (see also Barnichon and Brownlees, 2019). Similarly, Tanaka (2020) suggested a roughness penalty prior to enhance the efficiency of LP in small samples, demonstrating its utility in forecasting and policy analysis.

Despite their advantages, LP have been criticized for several issues. For instance, their relative inefficiency compared to VAR, particularly in finite samples. Barnichon and Brownlees (2019) and Kilian and Kim (2011) discussed the efficiency trade-off inherent in LP, noting that while they impose fewer parametric assumptions, this flexibility can lead to higher variance in the estimated IRF, especially at longer horizons. Addressing this efficiency concern, Brugnolini (2018) and Herbst and Johannsen (2024) proposed improvements such as bias correction and refined lag-selection methods to mitigate small-sample bias and improve the precision of LP estimates.

All the above mentioned works rely on LP to make inference about some population parameters, therefore LP's inferential theory is another research theme. Xu (2023) provided a general theory, initially developed by Montiel Olea and Plagborg-Møller (2021), about the asymptotic properties of the LP estimator. They proved that a root of the LP estimator is asymptotically distributed as a standard normal distribution. Moreover, they established that the confidence intervals constructed from this asymptotic distribution maintain correct asymptotic coverage.

In many structural macroeconometric applications, bootstrap confidence bands are commonly used for LP estimates. However, general results on bootstrap consistency and asymptotic refinements in the LP framework are lacking. Velez (2023) is the only study to investigate the theoretical aspects of bootstrap IRF within the LP context, specifically in the univariate AR(1) case. They proposed an LP-residual bootstrap scheme, proving its uniform consistency and asymptotic validity across a broad class of AR(1) processes. Additionally, they demonstrated that this method yields asymptotic refinements, though over a limited class of AR(1) processes.

This thesis contributes to fill this gap, by giving some Monte-Carlo simulations evidence about bootstrap methods effectiveness under different DGPs. This work is organized as follows. In section 2 we summarize the LP method, its properties and variations. Section 3 presents a review of bootstrap theory. In section 4 we show results of the Monte-Carlo simulations. Section 5 concludes. We find that bootstrap confidence bands have coverage close to the desired nominal level in several frameworks, both in univariate and multivariate scenarios. Moreover, we note that these procedures outperform the asymptotic ones in terms of coverage. We interpret these results as evidence of bootstrap consistency and presence of asymptotic refinements in certain scenarios. Lastly, we show how some misspecified settings can create coverage issues for the LP confidence intervals in finite samples.

2 Local Projections

This section is organized as follows: subsection 2.1 displays the concept of IRF and summarizes the classical VAR approach; subsection 2.2 presents the underlying idea of LP; subsection 2.3 shows how to estimates the parameters of interest using the LP idea; subsection 2.4 exhibits the lag-augmented version of LP and its properties; subsection 2.5 proves the robustness of LP to misspsecification; subsection 2.6 concludes with some issues regarding this approach.

2.1 Impulse Response Function and VAR estimation

Take a stationary n-variate VAR(k) process $\boldsymbol{y}_t$:

$$\boldsymbol{y}_t = \sum_{i=1}^{k} A_i \boldsymbol{y}_{t-i} + \boldsymbol{\varepsilon}_t \tag{1}$$

and take the definition of impulse responses function:

$$\text{IR}(t, h, \boldsymbol{d}) = E(\boldsymbol{y}_{t+h} | \boldsymbol{\varepsilon}_t = \boldsymbol{d}; X_t) - E(\boldsymbol{y}_{t+h} | \boldsymbol{\varepsilon}_t = \boldsymbol{0}; X_t)$$

where $X_t \equiv (\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \dots)'$, and $\boldsymbol{d}$ are the relevant experimental shocks, i.e. the mixture of shocks we are interested in. Intuitively, the IRF can be seen as the function that provides the difference between the average "treated" scenario (where relevant experimental shocks are introduced at time t) and the average "control" scenario (where the shocks at time t are set to zero for all variables), after h periods have passed since the shocks occurred. Typically, we are interested in multiple experimental shocks, hence we construct a matrix D, whose columns represent different shock vectors $\boldsymbol{d}$. For instance, in SVAR B-models, we aim to identify a matrix B such that

$$\boldsymbol{\varepsilon}_t = B \boldsymbol{u}_t$$

where $\boldsymbol{\varepsilon}_t$ represents the VAR innovations, and $\boldsymbol{u}_t$ represents the structural shocks. In this context $D = B$, as the i-th column of B indicates the shock in $\boldsymbol{\varepsilon}_t$ which is equivalent to a unitary shock in i-th structural shock $u_{i,t}$. Notice, if we want the IRF of all variables w.r.t. a shock in the i-th VAR innovation, then $\boldsymbol{d} = \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the selection vector.[1]

We can rewrite (1) through recursive substitution as:

$$\boldsymbol{y}_{t+h} = \sum_{i=1}^{k} \mathcal{C}_i^{h+1} \boldsymbol{y}_{t-i} + \sum_{j=0}^{h} \mathcal{C}_1^j \boldsymbol{\varepsilon}_{t+h-j} \tag{2}$$

where $\mathcal{C}_i^{h+1}$ is the i-th $n \times n$ block of the first n rows of $\mathcal{C}^{h+1}$, and $\mathcal{C}$ is the companion matrix. Then, the IRF for a VAR(k) are:

$$\text{IR}(t, h, \boldsymbol{d}) = \mathcal{C}_1^h \boldsymbol{d} \tag{3}$$

and we can estimate the IRF by plugging in (3) an estimator of the companion matrix. The most intuitive way to achieve this is to estimate the system as a VAR, and compute the associated

[1] Henceforth, we analyze the IRF w.r.t. the VAR innovations, without loss of generality.

companion matrix $\hat{\mathcal{C}}$. Different estimation techniques (GLS, ML, GMM, etc.) will lead to different $\hat{\mathcal{C}}$:

$$\hat{\mathcal{C}} = \begin{pmatrix} \hat{A}_1 & \hat{A}_2 & \dots & \hat{A}_{k-1} & \hat{A}_k \\ I_n & [0]_n & \dots & [0]_n & [0]_n \\ [0]_n & I_n & \dots & [0]_n & [0]_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_n & [0]_n & \dots & I_n & [0]_n \end{pmatrix}$$

The simplest strategy is to estimate the parametric matrices in (1) via OLS. Notice, while computing the s.e. of $\hat{\mathcal{C}}$ can be achieved through straightforward econometrics formulae, the estimator of the IRF $\hat{\mathcal{C}}_1^h$ is a non-linear transformation of the OLS estimate. Therefore we have to rely on deltha-method approximations to calculate the standard errors. Also, the presence of unit roots could complicate even further the inference process, as the limiting distribution of the estimator would be not normal in this case.

2.2 Local Projections idea

A different possible IRF estimation strategy is to estimate $\mathcal{C}_1^{h+1}$ directly from (2), in separate regressions for each h.[2] Rewrite (2) as:

$$\boldsymbol{y}_{t+h} = \sum_{i=1}^{k} \Gamma_i \boldsymbol{y}_{t-i} + \boldsymbol{\nu}_{t+h} \tag{4}$$

where $\Gamma_i = \mathcal{C}_i^{h+1}$. Then, we could directly estimate $\mathcal{C}_1^{h+1}$, by estimating Γ_1 through a linear model of $\boldsymbol{y}_{t+h}$ on $X_t \equiv (\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \dots, \boldsymbol{y}_{t-k})$, without the need to estimate all the equations jointly. Under the assumption of serial uncorrelation of the VAR innovations $\boldsymbol{\varepsilon}_t$ (and stationarity of y_t), the regressors X_t are uncorrelated with the LP errors, since all the terms of the MA are neither contemporaneous nor preceding the regressors, therefore this seems to be a valid strategy. Notice that the LP error term $\boldsymbol{\nu}_{t+h}$ is:

$$\boldsymbol{\nu}_{t+h} = \sum_{j=0}^{h} \mathcal{C}_1^j \boldsymbol{\varepsilon}_{t+h-j}$$

which is an MA(h) of the VAR innovations, whose order increases with the IRF horizon.

2.3 LP estimation and inference

The most straightforward estimation strategy is to estimate all the Γ_i in (4) by OLS. Since $\mathcal{C}_1^{h+1} = \Gamma_1$, and as the regressors are uncorrelated with the errors (under the assumption of stationarity of $\boldsymbol{y}_t$):

$$\hat{\Gamma}_1^{\text{OLS}} \xrightarrow{p} \mathcal{C}_1^{h+1}$$

[2] The sample size decreases linearly as h increases, thus we can estimate the IRF from $h = 0$ to $h = T - 1$, where T is the sample's length net of the number of lags of the VAR.

Moreover, the OLS estimator is also asymptotically normal in this case. However, the MA nature of the LP errors creates autocorrelation among the residuals, hence employing heteroskedasticity and autocorrelation robust (HAR) standard errors estimators when conducting inference is required.

Alternatively, FGLS procedures have been proposed by the literature. For instance, Lusompa (2023) shows that the autocorrelation process of LP can be written as a VMA process of the Wold errors, and that the corrected FGLS estimator leads to more efficient estimates, even when the DGP is not a VAR.

2.4 Lag-augmentation of the LP estimator

The validity of the LP approach is sensitive to persistence in the data, since the limiting distributions are non-normal when we are close to the unit root case. Also, even if we are in the stationary region, we still need to use HAR s.e. to account for the presence of serial correlation in the LP regressions' residuals.

A possible estimation strategy that could overcome the problem is the following. Assuming an AR(1) for simplicity[3], we can rewrite (2):

$$y_{t+h} = \rho^h y_t + \sum_{j=0}^{h-1} \rho^j \varepsilon_{t+h-j}$$

and $y_t = \rho y_{t-1} + \varepsilon_t$, thus:

$$y_{t+h} = \rho^h \varepsilon_t + \rho^{h+1} y_{t-1} + \sum_{j=0}^{h-1} \rho^j \varepsilon_{t+h-j} \tag{5}$$

Notice that, by linearly regressing y_{t+h} on ε_t and y_{t-1}, the OLS estimator of the coefficient on ε_t would be asymptotically normal. The intuition behind this result is that, even if y_{t+h} is non-stationary (i.e. $\rho = 1$), ε_t remains stationary, and we also control for the potentially non-stationary term y_{t-1}. However, the non-observability of ε_t in empirical applications makes this method impractical.

To overcome these issues Montiel Olea and Plagborg-Møller (2021) proposed the *lag-augmented* Local Projections. The procedure consists in adding as additional control variable y_{t-1} in the LP regressions. This idea is not unheard in the literature: for cointegrated VAR processes Toda and Yamamoto (1995) and Dolado and Lütkepohl (1996) showed the benefits of inserting additional lags of the dependent variable in the estimation. Moreover, Inoue and Kilian (2020) showed that the distribution of the parameters of interest uniformly converges to the normal. Therefore, for an AR(1), we would estimate the following model:

$$y_{t+h} = \beta_h y_t + \gamma_h y_{t-1} + \nu_{t+h} \tag{6}$$

OLS estimation leads to:

$$\begin{pmatrix} \hat{\beta}_h \\ \hat{\gamma}_h \end{pmatrix} = \left(\sum_{t=2}^{T-h} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} \begin{pmatrix} y_t & y_{t-1} \end{pmatrix} \right)^{-1} \left(\sum_{t=2}^{T-h} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} y_{t+h} \right) \tag{7}$$

[3]Yet, the following results still hold for a VAR(k).

Notice that the coefficient of y_t in the feasible lag-augmented regression (6) is equal to the coefficient of ε_t in (5). This is why we should expect the limit distribution of $\hat{\beta}_h$ to be normal even in near-unit-root cases. Indeed, Montiel Olea and Plagborg-Møller (2021) and Xu (2023) find that the lag-augmented LP is an asymptotically normal estimator even in high persistence scenarios.

This estimator has another advantage: HAR standard errors are not required to make inference. Using the Eicker-Huber-White s.e. (White, 1980) of $\hat{\beta}_h$ is sufficient:

$$\hat{s}_h = \frac{\sqrt{\sum_{t=1}^{T-h} \hat{\nu}_{h,t}^2 \hat{\varepsilon}_{h,t}^2}}{\sum_{t=1}^{T-h} \hat{\varepsilon}_{h,t}^2} \tag{8}$$

where $\hat{\nu}_{h,t}$ are the lag-augmented LP regressions' residuals:

$$\hat{\nu}_{h,t} = y_{t+h} - \hat{\beta}_h y_t - \hat{\gamma}_h y_{t-1}$$

and $\hat{\varepsilon}_{h,t}$ are the residualized regressor of interest:

$$\hat{\varepsilon}_{h,t} = y_t - \hat{\rho}_h y_{t-1}, \quad \text{where } \hat{\rho}_h = \frac{\sum_{t=1}^{T-h} y_t y_{t-1}}{\sum_{t=1}^{T-h} y_{t-1}^2}$$

This idea works because, although the lag-augmented LP regressions' residuals $\hat{\nu}_{h,t}$ present autocorrelation, the regression scores $\hat{\nu}_{h,t}\hat{\varepsilon}_t$ are serially uncorrelated.[4]

2.5 Robustness of the LP estimator

We showed that using LP or the classical VAR approach should lead to equivalent IRF (see also Plagborg-Møller and Wolf, 2021). However, the former approach delivers several advantages, starting from simpler standard erros, that do not rely on the delta-method, or its flexibility, for instance when dealing with nonlinear DGPs (Jordà, 2005).

In a recent work Montiel Olea et al. (2024) proved one of the main strengths of the LP estimator: its robustness to misspecification. Assume a univariate, stationary local-to-AR(1) model:[5]

$$y_t = \rho y_{t-1} + [1 + T^{-\zeta}\alpha(L)]\varepsilon_t \tag{9}$$

where $\alpha(L)$ is a potentially infinite-order lag polynomial. If we were to estimate (9) as an AR(1) model, we would obtain a reasonably good approximation. However, in finite samples, the model would be misspecified, since the MA component tends to 0 at a rate of $T^{-\zeta}$. The IRF of y_{t+h} w.r.t. ε_t according to (9) is:

$$\theta_{h,T} = \rho^h + T^{-\zeta} \sum_{l=1}^{h} \rho^{h-l}\alpha_l \tag{10}$$

Notice that the first term ρ^h is the usual AR(1) IRF, while the second part comes from the MA component. Then, the AR estimator $\hat{\delta}_h$ would be:

$$\hat{\delta}_h = \hat{\rho}^h = \left(\frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} \right)^h$$

[4]Under the assumption of strict stationarity of ε_t and $E(\varepsilon_t|\{\varepsilon_s\}_{s\neq t}) = 0$.

[5]Yet, the following results still hold for multivariate process.

and the (lag-augmented) LP estimator $\hat{\beta}_h$ would be the one in (7). We know that, given the presence of the MA term, the misspecification should pollute the asymptotic coverage of conventional confidence intervals of both estimators. However, Montiel Olea et al. (2024) proved that the limiting distribution of lag-augmented LP estimator is robust to the presence of local misspecification (under some mild conditions). Also, they showed that the AR estimator is more fragile, since its bias becomes asymptotically negligible only when $\zeta > 1/2$, whereas the LP estimator requires $\zeta > 1/4$, and MA terms of order $T^{-\zeta}$ with $\zeta \in (1/4, 1/2)$ can be detected with probability 1 asymptotically by conventional AR model specification test.

2.6 Drawbacks of the LP approach

The main drawback of the LP approach involves the usual bias-variance trad-eoff: even tough the LP consistently estimates the IRF in a wide spectrum of scenarios, under a VAR DGP the classical VAR approach is more efficient. This results in wider confidence bands for the LP estimates. This phenomena amplifies when we use the classical version of LP, since HAR standard errors lead to broader C.I. especially at distant horizons.

Among the reasons of this, we can find the sample sizes of the LP regressions: in finite samples, as we increase h, the number of observations decreases linearly. This results in higher variance of the true distribution of LP estimates, that generates two problems: (i) wider confidence bands; (ii) less reliability of the estimates.

3 Bootstrap Methods for Local Projections

This section is organized as follows: subsection 3.1 reviews the concept of bootstrap, why it works and how to apply it; subsection 3.2 explains what asymptotic refinements are, why we should expect bootstrap procedures to deliver them and why they are a useful property for inference; subsection 3.3 displays how bootstrap is used in practice and different possible schemes; subsection 3.4 summarizes the current literature's theoretical results on LP bootstrap.

3.1 The bootstrap

As defined by Horowitz (2001) in his seminal paper, *the bootstrap is a method for estimating the distribution of an estimator or test statistic by re-sampling one's data.*

Suppose that the c.d.f. of a statistic T_n we are interested in depends on the c.d.f. F of the data X:

$$G_n(x, F) = P(T_n < x; F)$$

where n is the sample size. Since F is usually unknown, then $G_n(x, F)$ is not available. Therefore, econometricians have relied on asymptotic distribution theory to make inference. In this framework, the finite sample distributions of the statistics are approximated by their asymptotic counterparts, i.e. we assume that $n \to \infty$, and compute T and $G(x, F)$ such as:

$$T_n \xrightarrow{d} T, \quad G_n(x, F) \to G(x, F)$$

If $G(x, F)$ does not depend on F, T_n is said to be asymptotically pivotal, and $G(x, F) = G(x)$ can be used to make inference on T_n. However, many problems could arise when we approximate $G_n(x, F)$ with its asymptotic counterpart. Firstly, in many cases computing analytically the asymptotic distribution is challenging, sometimes even impossible. Another issue is the goodness of the approximation: if the sample size n is small relative to the parameters space, applying asymptotic theory could lead to erroneous inference (e.g. finite sample bias, test could be severely undersized, ...).

Efron (1979) suggested to replace F with an estimator $\hat{F}_n$ (which put mass $1/n$ at each observation). Then, we can make inference on T_n by approximating $G_n(x, F)$ with $G_n(x, \hat{F}_n)$, rather than using $G(x)$. Yet, we are still using an approximation, therefore the inference is not exact. However, if $\hat{F}_n$ satisfies:

$$|\hat{F}_n(x) - F(x)| \xrightarrow{p} 0$$

uniformly for all x, then we should expect that:

$$|G_n(x, \hat{F}_n) - G_n(x, F_n)| \xrightarrow{p} 0 \text{ as } n \to \infty$$

for all x, or that:

$$|G_n(x, \hat{F}_n) - G(x, F)| \xrightarrow{p} 0 \text{ as } n \to \infty$$

This property is called bootstrap consistency and implies that bootstrap and asymptotic inference are equivalent.

However, calculating exact bootstrap distributions is mathematically challenging, thus Monte-Carlo simulations are usually used to approximate the distribution. Therefore bootstrapping is a simulation based method, hence the mathematical complexity is reduced, eliminating one of the problems of asymptotic inference. Tough, it requires higher computational power than using analytical formulae (yet, with modern computers this is not an issue).

3.2 Asymptotic refinements

We have seen that bootstrap consistency implies equivalence between bootstrap and asymptotic inference. Hence, there seems to be no gain in using bootstrap methods rather than asymptotic ones (other than reduced complexity in the mathematics). However, another benefit of bootstrap inference are asymptotic refinements. Suppose we are interested in a statistic such that $T_n \xrightarrow{d} N(0, 1)$, then:

$$G_n(x) \to G(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-\frac{t^2}{2}} dt$$

Provided the existence of the Edgeworth expansion, we can use it to rewrite:

$$G_n(x) = G(x) + g_1(x) n^{-\frac{1}{2}} + O(n^{-1}) \tag{11}$$

Assuming that the same expansion holds for the bootstrap statistic distribution $G_n^*(x) \coloneqq G_n(x, \hat{F}_n)$:

$$G_n^*(x) = G(x) + g_1^*(x) n^{-\frac{1}{2}} + O_p(n^{-1}) \tag{12}$$

Then, we can take difference between (11) and (12):

$$G_n^*(x) - G_n(x) = \big(g_1^*(x) - g_1(x)\big)n^{-\frac{1}{2}} + O_p(n^{-1})$$

and since g_1^* is usually distant $O_p(n^{-\frac{1}{2}})$ from g_1, we can write:

$$G_n^*(x) - G_n(x) = O_p(n^{-\frac{1}{2}})n^{-\frac{1}{2}} + O_p(n^{-1}) = O_p(n^{-1})$$

While, for the asymptotic distribution:

$$G_n(x) - G(x) = O(n^{-\frac{1}{2}})$$

Hence, for some statistics, the bootstrap distribution converges to the true distribution faster than the latter converges to its asymptotic counterpart. When we deal with time series data (especially macroeconomic variables) the sample size might be small w.r.t. the number of parameters we want to estimate, therefore bootstrap methods might come in handy, as they make inference more reliable. Notice that the gain in convergence applies to asymptotically pivotal statistics (for symmetric asymptotic pivotal distribution we have $O_p(n^{-\frac{3}{2}})$), while for non-pivotal ones (as the OLS) the speed of convergence is $O_p(n^{-\frac{1}{2}})$, hence there is no benefit.

3.3 Bootstrap techniques

Bootstrap techniques differ based on the choice of $\hat{F}_n$. The simplest technique in the linear regression world is the fixed-regressor bootstrap. Following, we explain a possible fixed-regressor bootstrap scheme for the 90% confidence interval of the OLS:

1. Fit the OLS in the original data, get the residuals and center them:

$$\hat{\beta} = \frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2}, \qquad \hat{\varepsilon}_i = y_i - \hat{\beta} x_i, \qquad \tilde{\varepsilon}_i = \hat{\varepsilon}_i - \bar{\hat{\varepsilon}}$$

   where $\bar{\hat{\varepsilon}}$ is the mean of the uncentered residuals $\hat{\varepsilon}_i$.

2. Generate B bootstrap samples using the statistics computed at the first point:

$$y_{i,b}^* = \hat{\beta} x_i + \varepsilon_{i,b}^*$$

   where $\varepsilon_{i,b}^*$ is a random draw from the empirical c.d.f. of the centered residuals $\tilde{\varepsilon}_i$.

3. Compute the T-ratio of interest in each bootstrap sample (notice that in the bootstrap samples, the true value of the parameter is $\hat{\beta}$):

$$T_{n,b}^* = \frac{\hat{\beta}_b^* - \hat{\beta}}{\hat{\text{se}}(\hat{\beta}_b^*)}$$

   where $\hat{\text{se}}(\cdot)$ is an estimator of the s.e. of $\hat{\beta}$.

4. Extract the 5-th ($\hat{q}_{T_n,0.05}$) and 95-th ($\hat{q}_{T_n,0.95}$) percentiles of the distribution of $T^*_{n,b}$, and compute the confidence interval of $\hat{\beta}$ as:

$$\left[\hat{\beta} + \hat{q}_{T_n,0.05} \cdot \hat{\text{se}}(\hat{\beta}), \hat{\beta} + \hat{q}_{T_n,0.95} \cdot \hat{\text{se}}(\hat{\beta})\right]$$

Notice that we bootstrap the T-statistic because it is asymptotically pivotal (as its asymptotic distribution is the standard normal in the classical econometrics framework), therefore this procedure delivers asymptotic refinements. We should also highlight that this is not only a fixed-regressor bootstrap, yet it is also an i.i.d. non-parametric unrestricted bootstrap. Following, we list other possible bootstrap techniques, by explaining how they should be applied to our example.

Firstly, we can differentiate parametric and non-parametric bootstrap schemes. In the former case, we would have resampled the errors from a distribution which depends on the data through a vector of parameters (e.g. we could have extracted B bootstrap residuals samples from a normal distribution with zero mean and variance equal to the residuals' one). However, we would have borne the risk of misspecification, that would have invalidated the bootstrap method. Instead, in non-parametric bootstrap techniques the unknown distribution F is replaced by the empirical distribution function, as in our case:

$$\hat{F}_n \coloneqq \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \leq x)$$

Another difference we should highlight is the one between restricted and unrestricted methods. In the former we would have imposed the null hypothesis $\beta = \beta_0$ while resampling, i.e., in the second step, we would have generated the bootstrap samples as:

$$y^*_{i,b} = \beta_0 x_i + \varepsilon^*_{i,b}$$

This method is particularly useful when we want to test the hypothesis $\beta = \beta_0$, as we can compute an approximated distribution of the test statistics and use it as alternative possibility to the asymptotic distribution.

We continue by explaining the concept of recursive bootstrap, which is an alternative to the fixed-regressor bootstrap, as it is the most common procedure when handling time series data. Assuming that the data are generated by an AR(1) process, then:

$$y_t = \rho y_{t-1} + \varepsilon_t$$

The idea is to incorporate the data dynamics while generating the bootstrap samples, hence in the second step we create the samples recursively:

$$y^*_{t,b} = \hat{\rho} y^*_{t-1,b} + \varepsilon^*_{t,b}$$

where $y^*_{0,b}$ is usually set equal to 0.

Lastly, as alternative to the i.i.d. bootstrap we present the wild bootstrap, proposed by Gonçalves and Kilian (2004). The difference is in the residuals resampling phase, where instead of extracting them from the empirical distribution, they are generated as follows:

$$\varepsilon^*_{i,b} = z\tilde{\varepsilon}_i$$

where z is a random draw from a standard normal distribution. Notice that by independence between z and $\tilde{\varepsilon}_i$, this implies:

$$E(\varepsilon^*_{i,b}) = E(\tilde{\varepsilon}_i)E(z) = 0, \qquad V(\varepsilon^*_{i,b}) = V(\tilde{\varepsilon}_i)V(z) = V(\tilde{\varepsilon}_i)$$

Therefore, mean and variance of the original residuals are preserved. Moreover, the variance of the i-th residual is also preserved. That gives the intuition behind the robustness to heteroskedasticity of this procedure.

3.4 Bootstrap and Local Projections

To the best of our knowledge, the only work that provides a theoretical framework on bootstrap techniques paired with the Local Projections estimator is Velez (2023), in which the author proposed the LP-residual bootstrap. It also proved that the confidence intervals produced by this technique are uniformly asymptotically valid, and that this procedure provides asymptotic refinements.

The LP-residual bootstrap procedure is defined as follows. Assume an AR(1) DGP, of parameter ρ, then:

1. Estimate ρ by OLS in the original data and compute the centered residuals $\tilde{\varepsilon}_t$:

$$\hat{\rho} = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{i=2}^{T} y_{t-1}^2}, \qquad \tilde{\varepsilon}_t = \hat{\varepsilon}_t - \bar{\hat{\varepsilon}}$$

   where $\hat{\varepsilon}_t$ are the residuals $y_t - \hat{\rho} y_{t-1}$ and $\bar{\hat{\varepsilon}}$ is their mean.

2. Generate recursively a bootstrap sample of size T:

$$y^*_{t,b} = \hat{\rho} y^*_{t-1,b} + \varepsilon^*_{t,b}$$

   where the initial value $y^*_{0,b}$ is equal to 0 and $\varepsilon^*_{t,b}$ is a random draw from the empirical distribution of the centered residuals.

3. Compute the lag-augmented LP $\hat{\beta}^*_{h,b}$ and its HC standard errors $\hat{s}^*_{h,b}$ as in (7) and (8) on each bootstrap sample, then compute the T-statistic:

$$T^*_{h,b} = \frac{\hat{\beta}^*_{h,b} - \hat{\rho}}{\hat{s}^*_{h,b}}$$

4. Compute the 90% C.I. of the lag-augmented LP estimator $\hat{\beta}_h$ using the 5-th ($\hat{q}_{T_h,0.05}$) and 95-th ($\hat{q}_{T_h,0.95}$) percentile of the distribution of the bootstrap T-statistic:

$$\left[\hat{\beta}_h + \hat{q}_{T_h,0.05} \cdot \hat{s}_h, \hat{\beta}_h + \hat{q}_{T_h,0.95} \cdot \hat{s}_h\right]$$

Therefore, in terms of the definitions given above, the LP-residual bootstrap is an i.i.d. recursive non-parametric unrestricted bootstrap.

4 Monte Carlo Simulations

In this section, the results of Monte Carlo simulations experiments under different conditions are shown. These help us understand the finite sample properties of bootstrap techniques in the LP framework. Firstly, we examine simple univariate AR(1) cases. Then, we increase the dimensionality of the setting using bivariate VAR(3) with different kinds of innovations.

4.1 Univariate case

Simulation We begin by describing the simulation procedure used in all the following settings:

1. Generate a sample of the given DGP of length T. Hence T is the size of the time series, not the number of rows of the regression, since the latter changes depending on h.
2. Estimate the IRF $\hat{\beta}_h$ via LP, using both the classical version and the lag-augmented one.
3. Estimate the s.e. of the IRF using the asymptotic formulae. For the classical approach we use the Newey-West HAR s.e. (Newey and West, 1986):

$$\hat{s}_h = \frac{\sqrt{\sum_{t=h+1}^{T} \hat{\varepsilon}_t^2 y_{t-h}^2 + 2\sum_{\nu=1}^{h}(1-\frac{\nu}{h+1})\sum_{t=h+\nu+1}^{T} y_{t-h}\hat{\varepsilon}_t\hat{\varepsilon}_{t-\nu}y_{t-h-\nu}}}{\sum_{t=h+1}^{T} y_{t-h}^2}$$

   where $\hat{\varepsilon}_t$ are defined as the residual of the LP regression. Notice, that we have to control for autocorrelation in the residuals up to lag h, as shown in (2). Whereas, for the lag-augmented LP we use the HC s.e. described in (8).
4. Compute the asymptotic C.I. as $\hat{\beta}_h \pm \Phi^{-1}(0.95)\hat{s}_h$, where $\Phi^{-1}(\cdot)$ is the inverse p.d.f. of the standard normal distribution.
5. Compute the recursive bootstrap C.I. and the wild bootstrap C.I.
6. Check whether the true value of the IRF (ρ^h) lies inside the different C.I.

Bootstrap techniques In the second-to-last point of the simulation, we compute the bootstrap C.I., and we now proceed to describe our bootstrap scheme, which follows the LP-residual bootstrap proposed by Velez (2023):

1. Estimate an AR(1) on the data.
2. Using $\hat{\rho}$, generate recursively B bootstrap samples:

$$y_{t,b}^* = \hat{\rho} y_{t-1,b}^* + \varepsilon_{t,b}^*$$

3. In each bootstrap sample, estimate the IRF by LP and the T-statistic as following:

$$T_{h,b}^* = \frac{\hat{\beta}_{h,b}^* - \hat{\rho}}{\hat{s}_{h,b}^*}$$

Since we are generating these samples as AR(1), then $\hat{\rho}^h$ is indeed the true value of the IRF in the bootstrap data.

4. Compute the 5-th and 95-th percentiles of the bootstrap distribution of the T-statistic, denoted by $\hat{q}_{T_h,0.05}$ and $\hat{q}_{T_h,0.95}$, and compute the C.I. as

$$\left[\hat{\beta}_h + \hat{s_h} \cdot \hat{q}_{T_h,0.05}, \hat{\beta}_h + \hat{s_h} \cdot \hat{q}_{T_h,0.95}\right]$$

Simulation Results We show the results for the simulations in different settings. In each framework, results are shown for different levels of persistence ρ of the true DGP, and using different IRF horizons. In all the experiments, we use 1000 bootstrap repetitions, 2500 simulations and $T = 100$.

Each table shows both the coverage probabilities for the different methods and the average length of the C.I. The columns of the tables correspond to the six different methodologies used: LP refers to the classical local projections asymptotic C.I., LP_b refers to the classical local projections recursive bootstrap C.I., LP_w refers to the classical local projections wild bootstrap C.I., LPLA refers to the lag-augmented local projections asymptotic C.I., LPLA_b refers to the lag-augmented local projections recursive bootstrap C.I., and LPLA_w refers to the lag-augmented local projections wild bootstrap C.I.

Notice that the theoretical results in Velez (2023) only apply with the usage of the recursive bootstrap paired with the lag-augmented version of LP.

All results are shown in appendix A.1. Also, appendix A.2 shows results of the simulations using $T = 25$.

I.I.D. Gaussian innovations Firstly, we focus on the following a AR(1) DGP

$$\begin{cases} y_t = \rho y_{t-1} + \varepsilon_t \\ \varepsilon_t \sim N(0,1) \end{cases}$$

Table 5 contains the results. All methods perform well in presence of low persistence and at short horizons. However, as we increase persistence or IRF horizon, bootstrap methods overperform their asymptotic counterparts, especially w.r.t. the classical LP, whose coverage significantly shrinks at very distant horizons. In the random walk case the bootstrap methods seem to undercover by 10 percentage points, and in the lag-augmented case their performance is as good as the asymptotic interval's one, whereas in the classical LP approach the gain in coverage is as significant as in the stationary cases. However, the wild bootstrap severely undercover in the unit root case for $h = 1$. Yet, it is known that this bootstrap technique is not robust to unit roots (Gonçalves and Kilian, 2004). Interestingly, the undercoverage is an issue only in the short horizon, as the estimates gets closer to 0.9 for higher horizons. We can also notice that the asymptotic lag-augmented LP C.I. provide high coverage even in the presence of unit roots, while the asymptotic classical LP C.I. do not, probably as a consequence of the robustness of the former methodology to unit roots,

as discussed above. Also, the wild bootstrap lag-augmented LP C.I. do not show the same issue as the classical LP, possibly for the same property.

Finally, the results are consistent with the theoretical framework developed by Velez (2023).

Conditionally heteroskedastic innovations In the following simulations we use as DGP an AR(1) with conditionally heteroskedastic innovations, using a GARCH(1,1) process (Bollerslev, 1986):

$$\begin{cases} y_t = \rho y_{t-1} + \varepsilon_t \\ \varepsilon_t = \sigma_t z_t \\ \sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \delta \sigma_{t-1}^2 \\ z_t \sim N(0,1) \end{cases}$$

For the simulations we set $\omega = 0.1$, $\alpha = 0.3$ and $\delta = 0.7$.

The results of the simulations (table 6) are substantially identical to the i.i.d.'s ones. Thus we can highlight again the gain in coverage obtainable through the usage of bootstrap techniques at farther IRF horizons and with more persistent data. As before, the bootstrap techniques displays issues only when pairing the wild bootstrap with the classical LP in the unit-root case.

Again, the findings are consistent with the theoretical predictions by Velez (2023), whose framework allows for the presence of conditionally heteroskedastic innovations.

Unconditionally heteroskedastic innovations Then, we move to an unconditionally heteroskedastic framework, using an AR(1) DGP which presents a structural break in the innovations' variance (Cavaliere et al., 2014):

$$\begin{cases} y_t = \rho y_{t-1} + \varepsilon_t \\ \varepsilon_t \sim N(0, \sigma_t^2) \end{cases}$$

where $\sigma_t^2 = 1$ for $t < T\tau$, while $\sigma_t^2 = k$ for $t \geq T\tau$. In our simulations, we use $k = 4$ and $\tau = 0.5$. Notice, the presence of non-covariance-stationary innovations is not included in the theoretical framework developed by Velez (2023). Therefore, there are no results about bootstrap consistency and asymptotic refinements in this scenario.

As in the other settings, table 7 shows coverage generally close to the desired nominal level for the bootstrap methods, with a clear gain w.r.t. the asymptotic counterparts. Generally, it seems that the results are not affected by the presence of heteroskedasticity. That is probably due to the usage of heteroskedastic-consistent s.e. in both the classical and the lag-augmented approach. Yet, the wild bootstrap performance is better than the recursive one, as predictable from the properties of the former, particularly at short horizons.

Local-to-AR(1) Lastly, we add a small form of misspecification, using a local-to-AR(1) DGP:

$$\begin{cases} y_t = \rho y_{t-1} + [1 + T^{-\zeta}\alpha(L)]\varepsilon_t \\ \varepsilon_t \sim N(0,1) \end{cases}$$

Notice that the true value of the IRF is not ρ^h, as in the previous cases, yet the one in (10). In the simulations we set $\zeta = 1/3$. For the innovations, we use two different sets of coefficients. The first polynomial we choose is:

$$\alpha_{\text{inv}}(L) = -L + 0.5L^2 - 0.5L^3 - 0.7L^4 + 0.6L^5 + 0.3L^6 + 0.4L^7 + 0.8L^8 - 0.5L^9$$

therefore, all the roots of $[1+T^{-\zeta}\alpha_{\text{inv}}(L)]\varepsilon_t$ lie outside of the unit circle (figure 1), thus the process is invertible. Whereas, for the second set we have:

$$\alpha_{\text{non}}(L) = -L + 1.5L^2 - 0.8L^3 - L^4 + 1.2L^5 + 2L^6 + 1.2L^7 + 0.8L^8 - L^9$$

and in this case some roots of $[1 + T^{-\zeta}\alpha_{\text{non}}(L)]\varepsilon_t$ lie inside of the unit circle (figure 2). However, no root lies on the unit circle, thus there exists an invertible MA describing the same process.

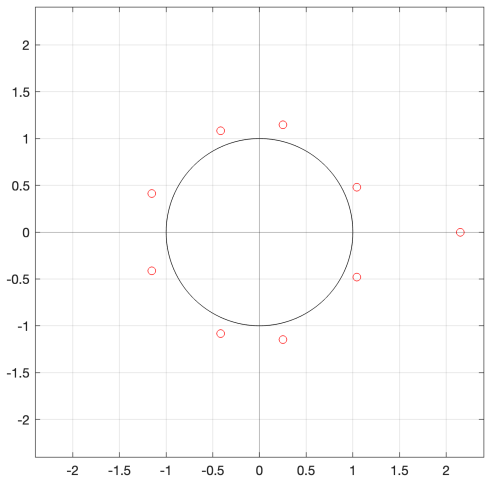

Figure 1: $[1 + T^{-\zeta}\alpha_{\text{inv}}(L)]\varepsilon_t$ roots

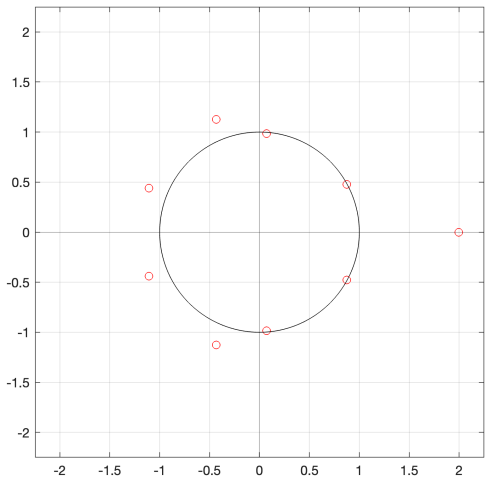

Figure 2: $[1 + T^{-\zeta}\alpha_{\text{non}}(L)]\varepsilon_t$ roots

Table 1 shows results for the invertible case. As in the previous cases, the bootstrap techniques leads to more reliable confidence intervals, with respect to their asymptotic counterparts. The lag-augmented LP estimates seem to be similar to the previous cases ones. However, the classical LP seems to not be robust to small misspecification for increased level of persistence, even though bootstrap techniques reach good coverage at longer horizons. These results are evidence supporting the idea that the double-robustness property of lag-augmented LP (Montiel Olea et al., 2024) holds even in small finite samples (even for $T = 25$, table 11).

To check for possible presence of bias in estimation, table 2 shows the results for the same case, yet using the wrong IRF formula, i.e. ignoring the part that comes from the MA process:

$$\theta_{h,T} - \rho^h = T^{-\zeta} \sum_{l=1}^{h} \rho^{h-l}\alpha_l$$

As predictable, the coverage is generally low, except at longer horizons. Yet, when we increase h the difference between the AR IRF and the Local-to-AR(1) ones is small, since $\alpha_l = 0$ if $l > 9$. However, it is interesting how the classical LP C.I. seem to perform better at higher level of persistence and short horizons, as if this estimator is on average closer to the wrongly computed IRF in these cases.

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.854	0.880	0.896	0.860	0.874	0.886	0.313	0.334	0.347	0.323	0.337	0.342
6	0.821	0.871	0.878	0.856	0.868	0.876	0.325	0.368	0.375	0.337	0.349	0.357
12	0.808	0.875	0.876	0.852	0.863	0.870	0.331	0.404	0.407	0.349	0.363	0.371
36	0.672	0.885	0.884	0.858	0.882	0.887	0.300	0.557	0.555	0.407	0.434	0.445
60	0.444	0.877	0.880	0.834	0.877	0.886	0.235	0.716	0.721	0.516	0.580	0.599
						$\rho = 0.5$						
1	0.814	0.847	0.857	0.850	0.870	0.874	0.309	0.329	0.340	0.321	0.336	0.341
6	0.791	0.854	0.861	0.867	0.884	0.893	0.334	0.383	0.392	0.343	0.356	0.365
12	0.808	0.882	0.885	0.864	0.877	0.882	0.350	0.432	0.436	0.358	0.373	0.382
36	0.662	0.880	0.886	0.844	0.867	0.876	0.315	0.595	0.593	0.421	0.449	0.463
60	0.469	0.874	0.879	0.809	0.857	0.864	0.245	0.762	0.768	0.530	0.597	0.621
						$\rho = 0.95$						
1	0.103	0.118	0.123	0.845	0.865	0.872	0.126	0.130	0.121	0.316	0.335	0.340
6	0.573	0.810	0.817	0.851	0.832	0.845	0.381	0.543	0.537	0.558	0.628	0.659
12	0.699	0.853	0.855	0.798	0.762	0.781	0.590	0.982	0.975	0.739	0.842	0.888
36	0.491	0.859	0.850	0.822	0.860	0.881	0.716	2.146	2.092	0.970	1.118	1.191
60	0.278	0.838	0.828	0.732	0.855	0.883	0.522	3.322	3.240	1.110	1.465	1.606
						$\rho = 1$						
1	0.031	0.036	0.040	0.843	0.862	0.860	0.069	0.070	0.053	0.315	0.333	0.334
6	0.405	0.603	0.604	0.829	0.811	0.824	0.252	0.368	0.329	0.620	0.717	0.755
12	0.624	0.833	0.825	0.782	0.715	0.739	0.467	0.851	0.793	0.925	1.156	1.228
36	0.346	0.810	0.794	0.690	0.644	0.666	0.828	3.564	3.379	1.618	2.421	2.633
60	0.175	0.803	0.786	0.599	0.666	0.688	0.783	8.770	8.389	2.201	4.186	4.652

Table 1: Local-to-AR(1), $T = 100$, 2500 simulations, 1000 bootstrap replications

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.316	0.356	0.390	0.364	0.397	0.416	0.313	0.334	0.347	0.323	0.337	0.342
6	0.752	0.810	0.815	0.791	0.812	0.820	0.325	0.368	0.375	0.337	0.349	0.357
12	0.808	0.875	0.876	0.852	0.863	0.870	0.331	0.404	0.407	0.349	0.363	0.371
36	0.672	0.885	0.884	0.858	0.882	0.887	0.300	0.557	0.555	0.407	0.434	0.445
60	0.444	0.877	0.880	0.834	0.877	0.886	0.235	0.716	0.721	0.516	0.580	0.599
						$\rho = 0.5$						
1	0.415	0.436	0.451	0.323	0.337	0.330	0.309	0.329	0.340	0.321	0.336	0.341
6	0.597	0.688	0.700	0.754	0.774	0.782	0.334	0.383	0.392	0.343	0.356	0.365
12	0.806	0.882	0.885	0.863	0.877	0.883	0.350	0.432	0.436	0.358	0.373	0.382
36	0.662	0.880	0.886	0.844	0.867	0.876	0.315	0.595	0.593	0.421	0.449	0.463
60	0.469	0.874	0.879	0.809	0.857	0.864	0.245	0.762	0.768	0.530	0.597	0.621
						$\rho = 0.95$						
1	0.723	0.698	0.611	0.292	0.292	0.286	0.126	0.130	0.121	0.316	0.335	0.340
6	0.751	0.890	0.888	0.692	0.634	0.655	0.381	0.543	0.537	0.558	0.628	0.659
12	0.699	0.867	0.865	0.823	0.796	0.809	0.590	0.982	0.975	0.739	0.842	0.888
36	0.493	0.860	0.850	0.824	0.862	0.886	0.716	2.146	2.092	0.970	1.118	1.191
60	0.278	0.839	0.828	0.734	0.856	0.883	0.522	3.322	3.240	1.110	1.465	1.606
						$\rho = 1$						
1	0.774	0.704	0.095	0.279	0.272	0.264	0.069	0.070	0.053	0.315	0.333	0.334
6	0.711	0.842	0.817	0.586	0.522	0.550	0.252	0.368	0.329	0.620	0.717	0.755
12	0.617	0.826	0.812	0.758	0.690	0.716	0.467	0.851	0.793	0.925	1.156	1.228
36	0.347	0.803	0.787	0.678	0.635	0.651	0.828	3.564	3.379	1.618	2.421	2.633
60	0.174	0.800	0.782	0.592	0.658	0.681	0.783	8.770	8.389	2.201	4.186	4.652

Table 2: Local-to-AR(1) with wrong IRF, $T = 100$, 2500 simulations, 1000 bootstrap replications

Finally, for the non-invertible case, table 3 displays undercoverage issues for all methods. However, we can notice two facts: (i) classical LP shows severe problems in high persistence scenario and at short horizons; (ii) bootstrap techniques still overperform their asymptotic counterparts. For farther horizons the estimates get closer to 90% for the bootstrap methods, yet this is linked with the closeness of AR IRF and Local-to-AR(1) ones in those case, once again.

Interestingly, table 4 shows better results than the previously commented table for the classical LP, while the lag-augmented LP C.I. performance is similar in both tables. Therefore, it seems that former estimator is generally closer to the AR(1) IRF, rather than the true IRF.

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	$LPLA_b$	$LPLA_w$	LP	LP_b	LP_w	LPLA	$LPLA_b$	$LPLA_w$
						$\rho = 0$						
1	0.577	0.616	0.633	0.516	0.542	0.546	0.329	0.351	0.364	0.318	0.333	0.337
6	0.562	0.623	0.642	0.596	0.614	0.633	0.315	0.353	0.363	0.314	0.326	0.335
12	0.794	0.866	0.872	0.828	0.840	0.850	0.336	0.408	0.411	0.341	0.355	0.362
36	0.673	0.887	0.885	0.815	0.839	0.843	0.314	0.576	0.574	0.402	0.428	0.438
60	0.459	0.869	0.870	0.789	0.835	0.838	0.238	0.719	0.721	0.505	0.566	0.581
						$\rho = 0.5$						
1	0.386	0.434	0.458	0.596	0.630	0.650	0.285	0.302	0.310	0.317	0.333	0.338
6	0.714	0.782	0.794	0.481	0.494	0.525	0.330	0.390	0.401	0.338	0.351	0.364
12	0.774	0.869	0.868	0.802	0.824	0.837	0.396	0.502	0.507	0.381	0.396	0.409
36	0.624	0.866	0.866	0.811	0.837	0.851	0.363	0.708	0.703	0.447	0.478	0.496
60	0.424	0.859	0.865	0.774	0.826	0.846	0.277	0.902	0.908	0.564	0.640	0.672
						$\rho = 0.95$						
1	0.015	0.019	0.022	0.587	0.650	0.668	0.097	0.099	0.086	0.318	0.336	0.340
6	0.030	0.042	0.022	0.561	0.520	0.537	0.366	0.526	0.507	0.635	0.730	0.769
12	0.271	0.444	0.395	0.609	0.554	0.572	0.623	1.093	1.071	0.957	1.155	1.224
36	0.455	0.864	0.850	0.776	0.796	0.812	0.763	2.685	2.593	1.275	1.600	1.715
60	0.240	0.855	0.850	0.708	0.838	0.868	0.559	4.483	4.319	1.452	2.106	2.316
						$\rho = 1$						
1	0.006	0.006	0.007	0.615	0.666	0.665	0.053	0.055	0.035	0.317	0.335	0.329
6	0.003	0.010	0.001	0.564	0.532	0.551	0.240	0.355	0.296	0.716	0.831	0.873
12	0.016	0.051	0.002	0.546	0.480	0.513	0.476	0.900	0.800	1.209	1.537	1.636
36	0.222	0.664	0.593	0.584	0.502	0.536	0.891	4.250	3.954	2.240	3.641	3.958
60	0.163	0.723	0.686	0.556	0.580	0.600	0.856	10.985	10.477	3.080	6.553	7.191

Table 3: Local-to-AR(1) with non-invertible innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.712	0.749	0.768	0.784	0.808	0.822	0.329	0.351	0.364	0.318	0.333	0.337
6	0.143	0.188	0.188	0.106	0.119	0.126	0.315	0.353	0.363	0.314	0.326	0.335
12	0.794	0.866	0.872	0.828	0.840	0.850	0.336	0.408	0.411	0.341	0.355	0.362
36	0.673	0.887	0.885	0.815	0.839	0.843	0.314	0.576	0.574	0.402	0.428	0.438
60	0.459	0.869	0.870	0.789	0.835	0.838	0.238	0.719	0.721	0.505	0.566	0.581
						$\rho = 0.5$						
1	0.803	0.828	0.837	0.705	0.718	0.714	0.285	0.302	0.310	0.317	0.333	0.338
6	0.020	0.039	0.041	0.078	0.093	0.099	0.330	0.390	0.401	0.338	0.351	0.364
12	0.775	0.869	0.869	0.801	0.824	0.838	0.396	0.502	0.507	0.381	0.396	0.409
36	0.624	0.866	0.866	0.811	0.837	0.851	0.363	0.708	0.703	0.447	0.478	0.496
60	0.424	0.859	0.865	0.774	0.826	0.846	0.277	0.902	0.908	0.564	0.640	0.672
						$\rho = 0.95$						
1	0.808	0.854	0.827	0.681	0.684	0.684	0.097	0.099	0.086	0.318	0.336	0.340
6	0.682	0.900	0.902	0.729	0.859	0.888	0.366	0.526	0.507	0.635	0.730	0.769
12	0.703	0.887	0.885	0.797	0.903	0.913	0.623	1.093	1.071	0.957	1.155	1.224
36	0.474	0.888	0.877	0.812	0.860	0.875	0.763	2.685	2.593	1.275	1.600	1.715
60	0.244	0.858	0.849	0.703	0.851	0.879	0.559	4.483	4.319	1.452	2.106	2.316
						$\rho = 1$						
1	0.849	0.844	0.193	0.678	0.680	0.666	0.053	0.055	0.035	0.317	0.335	0.329
6	0.759	0.905	0.872	0.740	0.844	0.856	0.240	0.355	0.296	0.716	0.831	0.873
12	0.668	0.852	0.821	0.774	0.870	0.881	0.476	0.900	0.800	1.209	1.537	1.636
36	0.351	0.838	0.815	0.751	0.762	0.776	0.891	4.250	3.954	2.240	3.641	3.958
60	0.165	0.817	0.791	0.639	0.715	0.732	0.856	10.985	10.477	3.080	6.553	7.191

Table 4: Local-to-AR(1) with non-invertible innovations and wrong IRF, $T = 100$, 2500 simulations, 1000 bootstrap replications

4.2 Multivariate case

Before moving in the multivariate scenario, we highlight the lack of theoretical results in this framework, as Velez (2023) theory regards the univariate AR(1) case.

Simulations and bootstrap techniques The simulation procedure and the bootstrap techniques are the same described for the univariate case, using the corrected multivariate formulae. The parameter of interest is the IRF of the first variable to the first innovation.

Simulation Results We report simulations results in different settings and for different levels of persistence. The four levels are denoted in the tables by the maximum eigenvalue's absolute value of the companion matrix, and are the following:

- $\max|\lambda_i| = 0$:
$$\boldsymbol{y}_t = \boldsymbol{\varepsilon}_t, \quad \text{with } A_1 = A_2 = A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
- $\max|\lambda_i| = 0.63555$:
$$\boldsymbol{y}_t = \begin{pmatrix} 0.3 & 0.1 \\ -0.4 & 0.1 \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} -0.02 & 0.05 \\ -0.3 & 0.3 \end{pmatrix} \boldsymbol{y}_{t-2} + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.04 \end{pmatrix} \boldsymbol{y}_{t-3} + \boldsymbol{\varepsilon}_t$$
- $\max|\lambda_i| = 0.93656$:
$$\boldsymbol{y}_t = \begin{pmatrix} 0.3 & 0.1065 \\ -0.4 & 0.7 \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} -0.3 & 0.7 \\ -0.5 & 0.1 \end{pmatrix} \boldsymbol{y}_{t-2} + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.3 \end{pmatrix} \boldsymbol{y}_{t-3} + \boldsymbol{\varepsilon}_t$$
- $\max|\lambda_i| = 1$[6]:
$$\boldsymbol{y}_t = \boldsymbol{y}_{t-1} + \boldsymbol{\varepsilon}_t, \quad \text{with } A_2 = A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Furthermore, figures 3 and 4 show the VAR roots for the second and the third persistence levels.

All results are shown in appendix A.3.

I.I.D. Gaussian innovations As above, we start with the simplest case, in which the DGP is the following:
$$\begin{cases} \boldsymbol{y}_t = A_1\boldsymbol{y}_{t-1} + A_2\boldsymbol{y}_{t-2} + A_3\boldsymbol{y}_{t-3} + \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, I_2) \end{cases}$$

Table 15 contains the results. Generally, all techniques deliver good coverage in the stationary region at short horizons. However, asympotitc C.I. (especially the classical LP C.I.) display undercoverage issues at increased h. In the random walk scenario, all methods show coverage far from

[6]In this case, we use one lag in the estimations, while three lags are used in the others, as using more lags than one would be equal to lag-augmentation

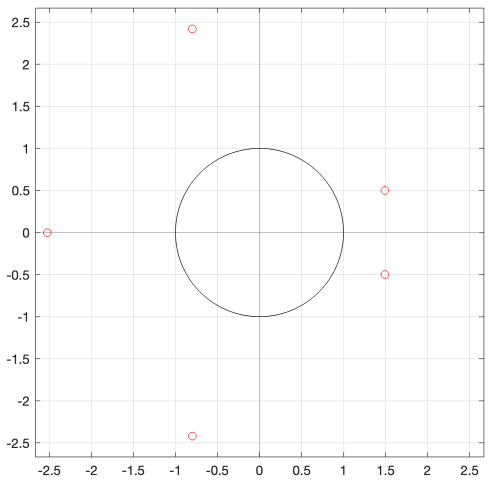

Figure 3: VAR roots for the second persistence level

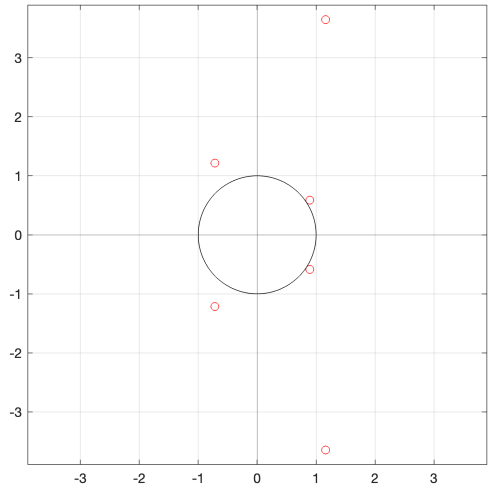

Figure 4: VAR roots for the third persistence level

the desired nominal level. Yet, bootstrap confidence bands keeps overperforming their asymptotic counterparts, especially at higher horizons and in the classical LP approach. Lastly, as noted in the univariate case, the wild bootstrap paired with classical LP displays severe coverage issues for very short horizons, yet the problems seem to vanish as h increases, once again.

Conditionally heteroskedastic innovations To allow for conditional heteroskedasticity in the innovations, we use a multivariate GARCH(1,1) process (Bauwens et al., 2006 for a survey on MGARCH processes). In order to increase the dimensionality of conditional heteroskedasticity processes, we need to choose a specific model. In the simulations we use a CCC-MGARCH(1,1), where CCC stands for constant conditional correlation. In the bivariate case, the CCC-MGARCH(1,1) process is described by the following equations:

$$\begin{cases} \boldsymbol{\varepsilon}_t = \Sigma_t^{\frac{1}{2}} \boldsymbol{z}_t \\ \Sigma_t = \Sigma_t^{\frac{1}{2}} \Sigma_t^{\frac{1}{2}} = \begin{pmatrix} \sigma_{1,t} & 0 \\ 0 & \sigma_{2,t} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} \sigma_{1,t} & 0 \\ 0 & \sigma_{2,t} \end{pmatrix} \\ \sigma_{i,t}^2 = \omega_i + \alpha_i \varepsilon_{i,t-1}^2 + \delta_i \sigma_{i,t-1}^2 \\ \boldsymbol{z}_t \sim i.i.d.(\mathbf{0}, I_2) \end{cases} \tag{13}$$

We can notice that the two process $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ evolve similarly to two separate GARCH(1,1) processes, yet the presence of ρ in the second equation, i.e. the conditional correlation, which is constant for all t in the CCC case, creates a link between the trajectories of the two conditional variances, hence generally:

$$\Sigma_t^{\frac{1}{2}} \neq \begin{pmatrix} \sigma_{1,t} & 0 \\ 0 & \sigma_{2,t} \end{pmatrix}$$

therefore the evolution of the two processes is actually interrelated.

In our simulations we used the following DGP:

$$\boldsymbol{y}_t = A_1\boldsymbol{y}_{t-1} + A_2\boldsymbol{y}_{t-2} + A_3\boldsymbol{y}_{t-3} + \boldsymbol{\varepsilon}_t$$

where $\boldsymbol{\varepsilon}_t$ evolves as described in (13), with $\rho = 0$, $\omega_1 = \omega_2 = 0.1$, $\alpha_1 = \alpha_2 = 0.2$, $\delta_1 = \delta_2 = 0.7$, and $\boldsymbol{z}_t \sim N(\mathbf{0}, I_2)$.

Table 16 contains the results, which are substantially identical to the previous case's ones. Bootstrap methods seem to often have coverage close to the wanted nominal level, and they seem to overperform the asymptotic bounds. Once again, in the random walk case bootstrap techniques show little undercoverage, nevertheless they seem to perform better than their asymptotic counterparts.

Unconditionally heteroskedastic innovations Similarly to the univariate case, to insert unconditional heteroskedasticty in the framework, we use innovations which present a structural break in the covariance matrix. Thus, the DGP evolves as follows:

$$\begin{cases} \boldsymbol{y}_t = A_1\boldsymbol{y}_{t-1} + A_2\boldsymbol{y}_{t-2} + A_3\boldsymbol{y}_{t-3} + \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Sigma_t) \end{cases}$$

where:

$$\Sigma_t = \begin{cases} I_2, & \text{if } t < T\tau \\ kI_2, & \text{if } t \geq T\tau \end{cases}$$

As done above, in the simulations we set $\tau = 0.5$ and $k = 4$. Table 17 shows the results, and they are perfectly aligned with the previous cases' ones. All methods display coverage close to 90% in the stationary region and at short horizon, with a net gap between bootstrap and asymptotic techniques when h rises. In the unit root case undercoverage is a common issue, yet the bootstrap methods seem still to outperform their asymptotic counterparts, especially for the classical LP approach. As expected, the wild bootstrap C.I. often display higher coverage than the recursive bootstrap ones.

Local-to-VAR(3) Lastly, we focus on a misspecified DGP, using a Local-to-VAR(3):

$$\begin{cases} \boldsymbol{y}_t = A_1\boldsymbol{y}_{t-1} + A_2\boldsymbol{y}_{t-2} + A_3\boldsymbol{y}_{t-3} + [1 + T^{-\zeta}\boldsymbol{\alpha}(L)]\boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, I_2) \end{cases}$$

Notice that the presence of the MA term changes the IRFs formula, and the VAR one is not valid anymore. In general, to compute VARMA IRF, it is useful to rewrite the system in a VAR(1) representation (like the companion form of a VAR(k)). Take a n-variate VARMA(p, q), and define

the following items:

$$\underset{n(p+q)\times 1}{\boldsymbol{Y}_t} = \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \\ \boldsymbol{u}_t \\ \boldsymbol{u}_{t-1} \\ \vdots \\ \boldsymbol{u}_{t-q+1} \end{pmatrix}, \qquad \underset{n(p+q)\times 1}{\boldsymbol{U}_t} = \begin{pmatrix} \boldsymbol{u}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{u}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}, \qquad \underset{n(p+q)\times n(p+q)}{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where the two parts of $\boldsymbol{U}_t$ are respectively $np \times 1$ and $nq \times 1$, and:

$$\underset{np\times np}{A_{11}} = \begin{pmatrix} A_1 & A_2 & \dots & A_{p-1} & A_p \\ I_n & [0]_n & \dots & [0]_n & [0]_n \\ [0]_n & I_n & \dots & [0]_n & [0]_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_n & [0]_n & \dots & I_n & [0]_n \end{pmatrix}, \qquad \underset{nq\times nq}{A_{12}} = \begin{pmatrix} M_1 & M_2 & \dots & M_q \\ [0]_n & [0]_n & \dots & [0]_n \\ \vdots & \vdots & \ddots & \vdots \\ [0]_n & [0]_n & \dots & [0]_n \end{pmatrix},$$

$$\underset{nq\times np}{A_{21}} = [0]_{nq\times np}, \qquad \underset{np\times nq}{A_{22}} = \begin{pmatrix} [0]_n & [0]_n & \dots & [0]_n & [0]_n \\ I_n & [0]_n & \dots & [0]_n & [0]_n \\ [0]_n & I_n & \dots & [0]_n & [0]_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_n & [0]_n & \dots & I_n & [0]_n \end{pmatrix}$$

where A_i is the i-th matrix of VAR coefficients, M_i is the i-th matrix of VMA coefficients, and $[0]_{a\times b}$ is a $a \times b$ matrix of zeros (if the subscript presents only one number, then the matrix is squared). Then, we can write:

$$\boldsymbol{Y}_t = A\boldsymbol{Y}_{t-1} + \boldsymbol{U}_t$$

hence, $\boldsymbol{Y}_t$ is a VAR(1). The system is stable if and only if $\boldsymbol{y}_t$ is stable, thus if all the eigenvalues of A_{11} are inside the unit circle (or if the roots of the characteristic polynomial are outside). If $\boldsymbol{Y}_t$ is stable, then:

$$\boldsymbol{Y}_t = \sum_{i=0}^{\infty} A^i \boldsymbol{U}_{t-i}$$

and we can go back to the original system:

$$\boldsymbol{y}_t = \sum_{i=0}^{\infty} JA^i H \boldsymbol{u}_{t-i}$$

where:

$$\underset{n\times n(p+q)}{J} = \begin{pmatrix} I_n & [0]_n & \dots & [0]_n \end{pmatrix}, \qquad \underset{n(p+q)\times n}{H} = \begin{pmatrix} I_n \\ [0]_n \\ \vdots \\ I_n \\ [0]_n \\ \vdots \\ [0]_n \end{pmatrix}$$

Thus, the IRF of $\boldsymbol{y}_{t+h}$ w.r.t. $\boldsymbol{\varepsilon}_t$ is $\Theta_h = JA^hH$. Notice that, in a local-to-VAR(3), since $M_i = T^{-\zeta}\alpha_i$, where α_i is the i-th coefficient matrix of $\boldsymbol{\alpha}(L)$, then $\Theta_{h,T}$ depends also on the sample size T. Therefore, if we assume stability of $\boldsymbol{y}_t$, as the sample size grows, the local-to-VAR(3) IRF get closer to the VAR IRF, because the misspecification becomes more negligible.

Another important property of VARMA processes is invertibility. A VARMA process is said to be invertible if and only if its innovations are an invertible process, i.e. if all VMA roots lie outside of the unit circle. Notice, if a VMA is non-invertible and no root lies on the unit circle, then it exists an equivalent invertible VMA describing the same process. In the simulations we use both an invertible VARMA and a non-invertible one (figures 5 and 6), $\boldsymbol{\alpha}_{\text{inv}}(L)$ and $\boldsymbol{\alpha}_{\text{non}}(L)$ can be found in appendix A.3 (equations 14 and 15).

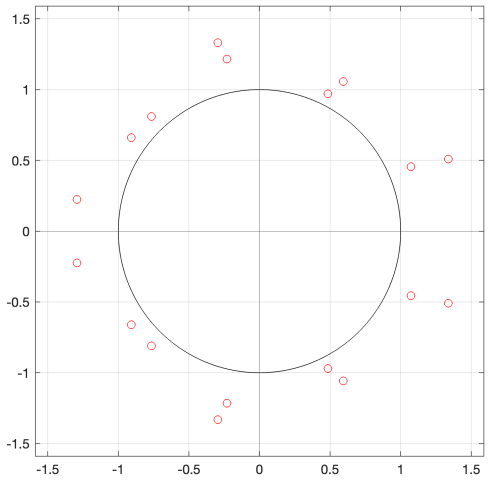

Figure 5: $[1 + T^{-\zeta}\boldsymbol{\alpha}_{\text{inv}}(L)]\boldsymbol{\varepsilon}_t$ roots

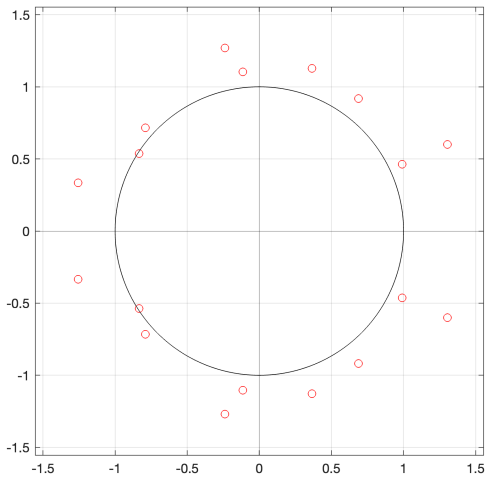

Figure 6: $[1 + T^{-\zeta}\boldsymbol{\alpha}_{\text{non}}(L)]\boldsymbol{\varepsilon}_t$ roots

Table 18 shows the results for the invertible case. Pairing LP and bootstrap techniques seems to lead to high coverage of the true IRF, even in the unit root case. However, undercoverage seems to be an issue at lower horizons. Nevertheless, asymptotic methods are generally overperformed by their bootstrap counterparts. Table 19 shows coverage for the VAR IRF parameters. All methods display better coverage for the VAR IRF than for the true IRF. Hence, this could signal bias of the LP estimator in finite sample for the true IRF.

Table 20 shows the results for the non-invertible case. Here, while all techniques display undercoverage issues, asymptotic methods performs better than bootstrap for short horizons ($h =$

1), especially for the classical LP estimator, while at increased horizons the latter techniques show better coverage, outperforming the former one. Table 21 displays higher coverage for the VAR IRF parameters, which can be once again attributed to the presence of bias. Notice, in most cases, confidence intervals for the LP (especially the bootstrapped ones) perform well, do not showing coverage far from 90%, while the classical VAR estimator would have probably leaded to worse bounds, in terms of coverage. If that is the case, then the LP approach would still be a better alternative than the classical one.

4.3 Comment

Simulations evidence is clearly in favor of bootstrap consistency and presence of asymptotic refinements for all non-misspecified frameworks in the stationary region. Nevertheless, in unit root scenarios bootstrap method generally overperfom their asymptotic counterpart, particularly in the lag-augmented approach. Whereas, results in misspecified settings shows more opaque evidence, even though bootstrap confidence bands, which still show undercoverage issues, seem to outperform large-samples approximations.

The results also display some properties already discovered and discussed by the literature about the comparison between classical LP and their lag-augmented version. In particular, the former methodology shows greater coverage issues than the first. Furthermore, its C.I. are usually very wide, resulting in impractical estimates, and the presence of unit roots and/or increasing the IRF horizon enhance these problems. However, the lag-augmented LP approach displays broader C.I. at short horizons, therefore choosing which approach is the best is a decision that has to be make based on the actual research question the user has in mind.

5 Conclusions

Local projections are a valid alternative to classical VAR estimation, and recent advancements shows that the LP approach might even be more useful than initially thought. Although the general theoretical framework behind LP bootstrap remains unknown, the current literature's results confirm the idea of bootstrap consistency and presence of asymptotic refinements, only for AR(1) processes though. After a review of the approach and a summary on bootstrap techniques, we gave evidence about the validity and strength of bootstrap procedures in the LP framework through Monte-Calro simulations experiments, under different settings. Our evidence seems to support the idea of bootstrap validity and presence of asymptotic refinements even in uncondionally heteroskedastic and/or multivariate scenarios. Unit roots, which need to be treated carefully, seems not to create issues in terms of those two properties. However, evidence in misspecified scenarios are not clear, even though bootstrap techniques seems to still outperform the asymptotic ones.

Future research should focus on generalizing the theoretical results about LP-residual bootstrap by Velez (2023), and understanding why misspecification in finite samples can produce problems, to make the LP approach even more useful and trustworthy.

A Simulations results

A.1 Univariate case

I.I.D. Gaussian innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.871	0.895	0.908	0.887	0.900	0.906	0.316	0.337	0.351	0.322	0.336	0.341
6	0.863	0.903	0.904	0.889	0.901	0.905	0.317	0.355	0.360	0.331	0.343	0.350
12	0.838	0.906	0.908	0.886	0.895	0.900	0.312	0.378	0.380	0.342	0.357	0.364
36	0.683	0.902	0.898	0.885	0.904	0.912	0.279	0.512	0.512	0.403	0.430	0.440
60	0.481	0.901	0.909	0.860	0.899	0.904	0.222	0.666	0.675	0.504	0.566	0.580
						$\rho = 0.5$						
1	0.872	0.893	0.900	0.884	0.897	0.905	0.276	0.293	0.299	0.324	0.340	0.345
6	0.842	0.903	0.914	0.892	0.901	0.915	0.374	0.446	0.456	0.381	0.395	0.407
12	0.821	0.904	0.905	0.880	0.893	0.905	0.377	0.481	0.486	0.394	0.409	0.423
36	0.667	0.890	0.893	0.889	0.909	0.926	0.335	0.656	0.658	0.460	0.492	0.511
60	0.459	0.907	0.912	0.858	0.898	0.916	0.264	0.867	0.883	0.577	0.657	0.693
						$\rho = 0.95$						
1	0.870	0.870	0.823	0.871	0.885	0.885	0.117	0.120	0.108	0.323	0.342	0.346
6	0.745	0.851	0.847	0.840	0.836	0.848	0.430	0.617	0.605	0.698	0.794	0.834
12	0.679	0.834	0.835	0.834	0.823	0.832	0.609	1.045	1.039	0.872	1.022	1.079
36	0.497	0.890	0.888	0.854	0.883	0.898	0.694	2.242	2.210	1.115	1.324	1.415
60	0.292	0.890	0.887	0.766	0.894	0.923	0.522	3.696	3.619	1.308	1.793	1.967
						$\rho = 1$						
1	0.873	0.827	0.120	0.881	0.888	0.886	0.066	0.067	0.047	0.322	0.341	0.338
6	0.727	0.816	0.755	0.826	0.818	0.827	0.289	0.424	0.367	0.779	0.904	0.949
12	0.640	0.810	0.782	0.793	0.764	0.779	0.484	0.897	0.825	1.093	1.379	1.464
36	0.356	0.827	0.812	0.727	0.683	0.702	0.818	3.680	3.510	1.887	2.930	3.189
60	0.177	0.819	0.809	0.655	0.721	0.735	0.764	9.029	8.754	2.593	5.149	5.739

Table 5: AR(1) with i.i.d. Gaussian innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

Conditionally heteroskedastic innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.873	0.905	0.926	0.879	0.899	0.910	0.371	0.398	0.423	0.379	0.398	0.412
6	0.843	0.887	0.902	0.892	0.904	0.914	0.332	0.373	0.388	0.357	0.372	0.382
12	0.833	0.894	0.906	0.891	0.903	0.906	0.309	0.375	0.384	0.349	0.365	0.371
36	0.687	0.906	0.903	0.874	0.894	0.900	0.274	0.503	0.498	0.402	0.430	0.436
60	0.467	0.893	0.902	0.853	0.896	0.907	0.236	0.711	0.714	0.536	0.605	0.617
						$\rho = 0.5$						
1	0.874	0.896	0.910	0.855	0.876	0.886	0.321	0.341	0.356	0.382	0.402	0.419
6	0.827	0.899	0.911	0.882	0.894	0.909	0.389	0.466	0.489	0.411	0.427	0.444
12	0.804	0.896	0.901	0.876	0.892	0.900	0.365	0.466	0.478	0.403	0.420	0.433
36	0.667	0.901	0.898	0.873	0.905	0.916	0.336	0.661	0.651	0.465	0.500	0.514
60	0.444	0.894	0.903	0.852	0.898	0.910	0.272	0.899	0.899	0.621	0.712	0.743
						$\rho = 0.95$						
1	0.878	0.878	0.818	0.867	0.888	0.894	0.127	0.130	0.119	0.384	0.407	0.422
6	0.731	0.841	0.843	0.831	0.833	0.842	0.438	0.628	0.625	0.775	0.887	0.952
12	0.669	0.821	0.827	0.830	0.818	0.830	0.606	1.040	1.043	0.913	1.078	1.162
36	0.506	0.898	0.888	0.838	0.878	0.895	0.710	2.310	2.223	1.106	1.327	1.421
60	0.283	0.898	0.890	0.773	0.900	0.924	0.549	4.013	3.821	1.374	1.914	2.066
						$\rho = 1$						
1	0.859	0.825	0.130	0.876	0.886	0.884	0.069	0.071	0.051	0.383	0.407	0.413
6	0.724	0.816	0.744	0.834	0.830	0.849	0.292	0.428	0.377	0.879	1.024	1.100
12	0.637	0.810	0.782	0.790	0.768	0.786	0.490	0.906	0.837	1.168	1.479	1.611
36	0.350	0.825	0.804	0.720	0.680	0.701	0.842	3.778	3.581	1.936	3.024	3.352
60	0.182	0.819	0.801	0.644	0.707	0.733	0.849	10.113	9.763	2.718	5.456	5.988

Table 6: AR(1) with GARCH(1,1) innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

Unconditionally heteroskedastic innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.863	0.897	0.915	0.879	0.898	0.908	0.357	0.385	0.407	0.367	0.387	0.397
6	0.853	0.892	0.903	0.880	0.892	0.894	0.357	0.402	0.424	0.383	0.400	0.412
12	0.790	0.877	0.893	0.881	0.893	0.899	0.347	0.422	0.453	0.407	0.427	0.441
36	0.646	0.874	0.890	0.863	0.880	0.894	0.373	0.689	0.726	0.580	0.623	0.644
60	0.487	0.899	0.900	0.864	0.898	0.901	0.441	1.339	1.345	1.026	1.165	1.180
						$\rho = 0.5$						
1	0.883	0.905	0.916	0.876	0.891	0.897	0.315	0.336	0.349	0.366	0.387	0.398
6	0.809	0.879	0.894	0.874	0.892	0.900	0.416	0.500	0.533	0.441	0.462	0.482
12	0.790	0.869	0.898	0.895	0.908	0.920	0.418	0.535	0.580	0.466	0.489	0.512
36	0.634	0.887	0.904	0.860	0.886	0.903	0.450	0.887	0.930	0.662	0.716	0.757
60	0.486	0.914	0.918	0.860	0.900	0.910	0.535	1.781	1.803	1.172	1.357	1.410
						$\rho = 0.95$						
1	0.858	0.857	0.750	0.862	0.878	0.880	0.137	0.141	0.120	0.369	0.392	0.403
6	0.718	0.824	0.824	0.834	0.840	0.844	0.486	0.698	0.703	0.794	0.913	0.980
12	0.629	0.818	0.834	0.833	0.823	0.832	0.685	1.176	1.245	0.997	1.182	1.268
36	0.512	0.884	0.887	0.812	0.853	0.894	0.967	3.249	3.129	1.500	1.842	1.979
60	0.277	0.867	0.860	0.766	0.890	0.907	1.001	7.610	7.334	2.519	3.618	3.807
						$\rho = 1$						
1	0.850	0.817	0.112	0.874	0.884	0.874	0.084	0.086	0.057	0.368	0.392	0.390
6	0.689	0.786	0.720	0.824	0.815	0.830	0.354	0.523	0.460	0.893	1.042	1.116
12	0.598	0.790	0.796	0.778	0.750	0.767	0.595	1.115	1.072	1.254	1.590	1.717
36	0.319	0.834	0.820	0.744	0.717	0.765	1.127	5.256	4.769	2.376	3.845	3.900
60	0.179	0.849	0.846	0.692	0.775	0.813	1.334	16.539	15.301	4.092	8.775	7.980

Table 7: AR(1) with heteroskedastic innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

A.2 Univariate case, with T=25

I.I.D. Gaussian innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.820	0.915	0.930	0.838	0.910	0.920	0.574	0.711	0.741	0.620	0.737	0.749
5	0.759	0.909	0.908	0.841	0.901	0.900	0.573	0.878	0.866	0.674	0.800	0.819
10	0.618	0.907	0.901	0.813	0.904	0.910	0.503	1.152	1.120	0.794	1.059	1.091
15	0.438	0.898	0.894	0.738	0.899	0.918	0.415	1.550	1.522	0.949	1.578	1.645
20	0.252	0.900	0.873	0.504	0.900	0.884	0.356	2.929	2.470	1.191	5.128	4.167
						$\rho = 0.5$						
1	0.832	0.914	0.921	0.822	0.900	0.889	0.515	0.629	0.641	0.614	0.735	0.755
5	0.725	0.901	0.896	0.833	0.895	0.906	0.627	1.056	1.037	0.767	0.920	0.968
10	0.559	0.897	0.883	0.796	0.894	0.910	0.553	1.444	1.367	0.887	1.213	1.285
15	0.393	0.903	0.890	0.724	0.903	0.917	0.461	2.055	1.947	1.052	1.828	1.987
20	0.220	0.897	0.852	0.490	0.904	0.884	0.377	3.650	2.852	1.230	5.500	4.633
						$\rho = 0.95$						
1	0.824	0.855	0.603	0.830	0.878	0.861	0.291	0.343	0.271	0.613	0.746	0.759
5	0.573	0.808	0.758	0.748	0.784	0.800	0.720	1.655	1.450	1.223	1.748	1.985
10	0.358	0.851	0.790	0.703	0.818	0.830	0.787	3.579	3.073	1.571	2.701	3.131
15	0.208	0.858	0.795	0.586	0.830	0.860	0.679	6.460	5.391	1.828	4.235	5.038
20	0.132	0.877	0.756	0.404	0.896	0.856	0.623	11.985	7.714	2.216	12.271	11.056
						$\rho = 1$						
1	0.815	0.834	0.126	0.826	0.878	0.865	0.239	0.284	0.199	0.613	0.747	0.746
5	0.554	0.808	0.679	0.738	0.755	0.779	0.663	1.608	1.320	1.343	1.995	2.278
10	0.324	0.832	0.734	0.661	0.739	0.758	0.826	4.247	3.550	1.889	3.599	4.278
15	0.180	0.822	0.760	0.554	0.777	0.796	0.824	9.256	7.723	2.391	6.431	7.698
20	0.126	0.824	0.702	0.379	0.855	0.822	0.868	20.522	13.489	3.164	20.538	18.348

Table 8: AR(1) with i.i.d. Gaussian innovations, $T = 25$, 2500 simulations, 1000 bootstrap replications

Conditionally heteroskedastic innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.805	0.904	0.928	0.808	0.892	0.915	0.635	0.788	0.840	0.686	0.817	0.848
5	0.745	0.892	0.895	0.824	0.889	0.896	0.626	0.966	0.976	0.766	0.912	0.944
10	0.626	0.911	0.910	0.812	0.900	0.905	0.627	1.455	1.413	0.972	1.302	1.338
15	0.460	0.910	0.902	0.752	0.913	0.906	0.607	2.318	2.183	1.356	2.275	2.321
20	0.280	0.912	0.873	0.530	0.899	0.867	0.571	4.845	3.934	1.922	8.509	6.840
						$\rho = 0.5$						
1	0.820	0.897	0.907	0.825	0.888	0.892	0.573	0.701	0.722	0.690	0.828	0.875
5	0.705	0.892	0.886	0.833	0.897	0.911	0.674	1.154	1.146	0.868	1.049	1.113
10	0.558	0.897	0.885	0.778	0.885	0.898	0.670	1.815	1.689	1.078	1.491	1.572
15	0.388	0.909	0.892	0.730	0.908	0.920	0.616	2.879	2.601	1.493	2.639	2.797
20	0.227	0.904	0.830	0.511	0.915	0.878	0.577	5.915	4.325	2.083	9.638	7.388
						$\rho = 0.95$						
1	0.806	0.840	0.485	0.814	0.874	0.862	0.326	0.389	0.306	0.690	0.840	0.886
5	0.555	0.824	0.767	0.754	0.792	0.805	0.775	1.803	1.578	1.370	1.962	2.253
10	0.350	0.862	0.795	0.702	0.801	0.815	0.936	4.410	3.630	1.897	3.370	3.833
15	0.225	0.880	0.813	0.588	0.853	0.868	0.921	9.140	7.115	2.506	6.222	6.904
20	0.130	0.868	0.736	0.395	0.896	0.859	0.986	20.673	12.108	3.336	21.838	17.464
						$\rho = 1$						
1	0.800	0.818	0.132	0.804	0.868	0.852	0.275	0.331	0.233	0.687	0.836	0.870
5	0.520	0.800	0.664	0.718	0.752	0.784	0.722	1.774	1.452	1.499	2.230	2.587
10	0.322	0.819	0.724	0.657	0.741	0.760	0.932	4.860	3.949	2.171	4.183	4.797
15	0.193	0.829	0.742	0.571	0.786	0.805	1.038	11.930	9.277	3.024	8.454	9.154
20	0.123	0.842	0.711	0.390	0.873	0.830	1.165	28.995	17.082	4.472	31.239	24.244

Table 9: AR(1) with GARCH(1,1) innovations, $T = 25$, 2500 simulations, 1000 bootstrap replications

Unconditionally heteroskedastic innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.799	0.892	0.919	0.814	0.897	0.918	0.614	0.773	0.827	0.671	0.804	0.844
5	0.708	0.862	0.880	0.815	0.880	0.896	0.640	0.992	1.045	0.808	0.969	1.020
10	0.602	0.907	0.894	0.772	0.880	0.886	0.749	1.745	1.692	1.167	1.576	1.589
15	0.426	0.906	0.900	0.720	0.900	0.910	0.841	3.249	3.140	1.863	3.170	3.299
20	0.241	0.914	0.878	0.499	0.906	0.884	0.703	6.068	4.985	2.262	10.428	8.479
						$\rho = 0.5$						
1	0.809	0.892	0.904	0.794	0.866	0.880	0.563	0.696	0.722	0.676	0.814	0.874
5	0.674	0.872	0.882	0.819	0.882	0.896	0.695	1.186	1.229	0.913	1.118	1.207
10	0.550	0.899	0.879	0.764	0.880	0.896	0.837	2.245	2.071	1.318	1.847	1.933
15	0.413	0.917	0.900	0.716	0.906	0.921	0.916	4.284	3.942	2.095	3.778	4.033
20	0.212	0.912	0.862	0.487	0.907	0.879	0.760	7.686	5.794	2.527	12.115	9.471
						$\rho = 0.95$						
1	0.807	0.854	0.298	0.792	0.859	0.852	0.342	0.407	0.310	0.674	0.822	0.882
5	0.529	0.808	0.746	0.718	0.767	0.801	0.844	1.988	1.749	1.388	2.026	2.317
10	0.351	0.866	0.776	0.701	0.829	0.849	1.166	5.646	4.421	2.187	3.998	4.365
15	0.215	0.865	0.799	0.596	0.864	0.892	1.258	12.754	10.036	3.316	8.450	8.776
20	0.130	0.891	0.741	0.399	0.894	0.853	1.151	24.486	14.133	4.063	25.847	20.165
						$\rho = 1$						
1	0.790	0.827	0.117	0.819	0.882	0.865	0.301	0.362	0.253	0.678	0.828	0.869
5	0.505	0.798	0.675	0.712	0.744	0.778	0.805	1.978	1.658	1.505	2.286	2.645
10	0.306	0.840	0.735	0.663	0.769	0.794	1.182	6.318	4.870	2.496	5.024	5.553
15	0.195	0.844	0.780	0.568	0.834	0.870	1.384	16.129	12.326	3.912	11.546	11.302
20	0.110	0.848	0.724	0.380	0.884	0.836	1.429	36.279	20.211	5.153	38.798	27.450

Table 10: AR(1) with heteroskedastic innovations, $T = 25$, 2500 simulations, 1000 bootstrap replications

Local-to-AR(1) with invertible innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.789	0.879	0.881	0.773	0.857	0.856	0.543	0.668	0.691	0.607	0.724	0.735
5	0.692	0.861	0.856	0.791	0.850	0.864	0.561	0.884	0.876	0.687	0.815	0.850
10	0.571	0.882	0.872	0.763	0.874	0.885	0.520	1.241	1.180	0.810	1.086	1.142
15	0.390	0.883	0.870	0.703	0.885	0.905	0.427	1.690	1.635	0.966	1.626	1.719
20	0.244	0.902	0.878	0.518	0.892	0.867	0.350	3.017	2.655	1.203	5.247	4.349
						$\rho = 0.5$						
1	0.740	0.852	0.877	0.786	0.868	0.884	0.565	0.698	0.722	0.603	0.718	0.727
5	0.702	0.865	0.860	0.767	0.835	0.848	0.550	0.863	0.851	0.665	0.792	0.820
10	0.563	0.880	0.864	0.767	0.875	0.889	0.506	1.196	1.135	0.795	1.066	1.105
15	0.379	0.892	0.872	0.683	0.872	0.888	0.418	1.638	1.589	0.944	1.587	1.685
20	0.229	0.895	0.865	0.489	0.890	0.853	0.356	3.066	2.660	1.162	5.104	4.251
						$\rho = 0.95$						
1	0.394	0.497	0.513	0.791	0.869	0.866	0.353	0.421	0.379	0.593	0.718	0.740
5	0.464	0.812	0.808	0.780	0.828	0.839	0.621	1.285	1.194	0.924	1.247	1.393
10	0.358	0.822	0.790	0.675	0.792	0.811	0.704	2.715	2.366	1.280	2.042	2.316
15	0.224	0.837	0.810	0.598	0.827	0.853	0.624	4.874	4.150	1.590	3.377	3.938
20	0.141	0.876	0.800	0.408	0.884	0.853	0.581	9.321	6.691	1.918	9.930	8.519
						$\rho = 1$						
1	0.292	0.364	0.375	0.790	0.866	0.864	0.287	0.342	0.278	0.589	0.714	0.723
5	0.372	0.765	0.759	0.762	0.792	0.814	0.574	1.280	1.120	0.998	1.416	1.609
10	0.310	0.823	0.784	0.644	0.732	0.753	0.717	3.238	2.763	1.509	2.655	3.077
15	0.196	0.834	0.786	0.561	0.757	0.780	0.763	7.497	6.335	2.022	4.917	5.778
20	0.118	0.840	0.752	0.375	0.830	0.782	0.792	16.196	11.455	2.613	15.207	13.202

Table 11: Local-to-AR(1), $T = 25$, 2500 simulations, 1000 bootstrap replications

Local-to-AR(1) with invertible innovations, using wrong IRF

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.477	0.613	0.655	0.529	0.635	0.671	0.543	0.668	0.691	0.607	0.724	0.735
5	0.652	0.838	0.831	0.708	0.775	0.786	0.561	0.884	0.876	0.687	0.815	0.850
10	0.571	0.882	0.872	0.763	0.874	0.885	0.520	1.241	1.180	0.810	1.086	1.142
15	0.390	0.883	0.870	0.703	0.885	0.905	0.427	1.690	1.635	0.966	1.626	1.719
20	0.244	0.902	0.878	0.518	0.892	0.867	0.350	3.017	2.655	1.203	5.247	4.349
						$\rho = 0.5$						
1	0.527	0.614	0.630	0.478	0.581	0.562	0.565	0.698	0.722	0.603	0.718	0.727
5	0.684	0.862	0.855	0.762	0.837	0.850	0.550	0.863	0.851	0.665	0.792	0.820
10	0.563	0.880	0.862	0.764	0.878	0.889	0.506	1.196	1.135	0.795	1.066	1.105
15	0.380	0.892	0.872	0.683	0.872	0.888	0.418	1.638	1.589	0.944	1.587	1.685
20	0.229	0.895	0.865	0.489	0.890	0.853	0.356	3.066	2.660	1.162	5.104	4.251
						$\rho = 0.95$						
1	0.593	0.608	0.366	0.423	0.476	0.446	0.353	0.421	0.379	0.593	0.718	0.740
5	0.514	0.783	0.768	0.550	0.554	0.578	0.621	1.285	1.194	0.924	1.247	1.393
10	0.367	0.836	0.800	0.692	0.819	0.833	0.704	2.715	2.366	1.280	2.042	2.316
15	0.224	0.844	0.817	0.604	0.833	0.861	0.624	4.874	4.150	1.590	3.377	3.938
20	0.147	0.879	0.801	0.409	0.886	0.854	0.581	9.321	6.691	1.918	9.930	8.519
						$\rho = 1$						
1	0.652	0.643	0.077	0.432	0.466	0.444	0.287	0.342	0.278	0.589	0.714	0.723
5	0.494	0.768	0.726	0.487	0.460	0.501	0.574	1.280	1.120	0.998	1.416	1.609
10	0.315	0.818	0.773	0.630	0.714	0.736	0.717	3.238	2.763	1.509	2.655	3.077
15	0.196	0.828	0.782	0.558	0.748	0.774	0.763	7.497	6.335	2.022	4.917	5.778
20	0.120	0.837	0.746	0.375	0.828	0.778	0.792	16.196	11.455	2.613	15.207	13.202

Table 12: Local-to-AR(1) with wrong IRF, $T = 25$, 2500 simulations, 1000 bootstrap replications

Local-to-AR(1) with non-invertible innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.488	0.600	0.625	0.502	0.598	0.598	0.572	0.707	0.736	0.602	0.716	0.724
5	0.520	0.717	0.723	0.593	0.686	0.702	0.527	0.813	0.807	0.638	0.758	0.780
10	0.582	0.878	0.863	0.753	0.869	0.872	0.495	1.144	1.091	0.782	1.044	1.076
15	0.402	0.882	0.873	0.711	0.879	0.894	0.413	1.568	1.527	0.959	1.600	1.674
20	0.250	0.894	0.868	0.491	0.894	0.868	0.347	2.892	2.593	1.126	4.872	3.961
						$\rho = 0.5$						
1	0.430	0.580	0.612	0.543	0.678	0.707	0.514	0.628	0.639	0.595	0.713	0.724
5	0.700	0.886	0.884	0.717	0.777	0.794	0.573	0.956	0.945	0.691	0.829	0.874
10	0.538	0.865	0.850	0.745	0.866	0.880	0.535	1.388	1.305	0.855	1.165	1.226
15	0.372	0.879	0.864	0.663	0.873	0.892	0.434	1.915	1.804	1.009	1.746	1.884
20	0.213	0.894	0.854	0.480	0.890	0.858	0.357	3.412	2.891	1.209	5.385	4.317
						$\rho = 0.95$						
1	0.190	0.268	0.282	0.567	0.726	0.729	0.277	0.328	0.256	0.599	0.728	0.733
5	0.560	0.864	0.837	0.676	0.697	0.724	0.651	1.498	1.296	1.115	1.597	1.815
10	0.209	0.644	0.485	0.450	0.498	0.527	0.788	3.675	3.095	1.658	2.922	3.412
15	0.174	0.738	0.659	0.484	0.667	0.695	0.752	7.557	6.291	2.060	4.963	5.898
20	0.117	0.814	0.706	0.365	0.836	0.776	0.723	14.897	10.226	2.452	14.215	12.590
						$\rho = 1$						
1	0.146	0.204	0.213	0.592	0.752	0.742	0.225	0.269	0.185	0.599	0.729	0.718
5	0.525	0.856	0.807	0.661	0.670	0.704	0.597	1.454	1.172	1.213	1.811	2.073
10	0.172	0.580	0.284	0.422	0.431	0.473	0.864	4.601	3.717	1.993	3.905	4.612
15	0.165	0.686	0.559	0.445	0.581	0.602	0.884	10.491	8.629	2.746	7.806	9.242
20	0.107	0.728	0.618	0.323	0.750	0.662	0.986	25.300	17.568	3.410	23.601	20.626

Table 13: Local-to-AR(1) with non-invertible innovations, $T = 25$, 2500 simulations, 1000 bootstrap replications

Local-to-AR(1) with non-invertible innovations, using wrong IRF

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\rho = 0$						
1	0.730	0.832	0.859	0.804	0.874	0.889	0.572	0.707	0.736	0.602	0.716	0.724
5	0.593	0.781	0.771	0.696	0.776	0.779	0.527	0.813	0.807	0.638	0.758	0.780
10	0.582	0.878	0.863	0.753	0.869	0.872	0.495	1.144	1.091	0.782	1.044	1.076
15	0.402	0.882	0.873	0.711	0.879	0.894	0.413	1.568	1.527	0.959	1.600	1.674
20	0.250	0.894	0.868	0.491	0.894	0.868	0.347	2.892	2.593	1.126	4.872	3.961
						$\rho = 0.5$						
1	0.792	0.866	0.872	0.730	0.816	0.799	0.514	0.628	0.639	0.595	0.713	0.724
5	0.586	0.824	0.824	0.762	0.830	0.852	0.573	0.956	0.945	0.691	0.829	0.874
10	0.540	0.866	0.850	0.745	0.865	0.880	0.535	1.388	1.305	0.855	1.165	1.226
15	0.372	0.879	0.864	0.663	0.873	0.892	0.434	1.915	1.804	1.009	1.746	1.884
20	0.213	0.894	0.854	0.480	0.890	0.858	0.357	3.412	2.891	1.209	5.385	4.317
						$\rho = 0.95$						
1	0.766	0.825	0.595	0.728	0.781	0.764	0.277	0.328	0.256	0.599	0.728	0.733
5	0.560	0.864	0.837	0.676	0.697	0.724	0.651	1.498	1.296	1.115	1.597	1.815
10	0.304	0.876	0.813	0.701	0.842	0.858	0.788	3.675	3.095	1.658	2.922	3.412
15	0.206	0.862	0.804	0.598	0.848	0.871	0.752	7.557	6.291	2.060	4.963	5.898
20	0.127	0.872	0.764	0.388	0.887	0.838	0.723	14.897	10.226	2.452	14.215	12.590
						$\rho = 1$						
1	0.760	0.802	0.176	0.725	0.783	0.761	0.225	0.269	0.185	0.599	0.729	0.718
5	0.542	0.856	0.790	0.648	0.649	0.683	0.597	1.454	1.172	1.213	1.811	2.073
10	0.300	0.866	0.780	0.691	0.792	0.809	0.864	4.601	3.717	1.993	3.905	4.612
15	0.167	0.838	0.761	0.586	0.792	0.811	0.884	10.491	8.629	2.746	7.806	9.242
20	0.109	0.827	0.733	0.348	0.852	0.783	0.986	25.300	17.568	3.410	23.601	20.626

Table 14: Local-to-AR(1) with non-invertible innovations and wrong IRF, $T = 25$, 2500 simulations, 1000 bootstrap replications

A.3 Multivariate case

I.I.D. Gaussian innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max \lvert\lambda_i\rvert = 0$						
1	0.872	0.996	0.996	0.864	0.999	0.995	0.320	0.359	0.374	0.326	0.365	0.375
6	0.859	0.901	0.902	0.877	0.898	0.896	0.316	0.360	0.366	0.336	0.355	0.359
12	0.822	0.897	0.896	0.881	0.900	0.903	0.311	0.385	0.388	0.347	0.370	0.374
36	0.684	0.898	0.904	0.868	0.906	0.914	0.277	0.534	0.532	0.409	0.457	0.462
60	0.450	0.900	0.898	0.816	0.904	0.905	0.223	0.740	0.725	0.510	0.637	0.637
						$\max \lvert\lambda_i\rvert = 0.63555$						
1	0.873	0.861	0.884	0.869	0.868	0.888	0.320	0.437	0.431	0.325	0.441	0.433
6	0.847	0.914	0.914	0.890	0.919	0.926	0.332	0.382	0.388	0.353	0.375	0.380
12	0.831	0.902	0.901	0.869	0.891	0.897	0.327	0.406	0.409	0.367	0.390	0.396
36	0.664	0.899	0.900	0.852	0.890	0.897	0.294	0.567	0.565	0.431	0.481	0.488
60	0.453	0.900	0.895	0.833	0.904	0.899	0.234	0.779	0.766	0.544	0.682	0.686
						$\max \lvert\lambda_i\rvert = 0.93656$						
1	0.872	0.884	0.943	0.871	0.887	0.942	0.319	0.411	0.413	0.326	0.413	0.415
6	0.855	0.834	0.831	0.872	0.834	0.834	0.596	0.741	0.765	0.650	0.749	0.769
12	0.819	0.923	0.926	0.872	0.928	0.939	0.669	0.849	0.863	0.774	0.851	0.870
36	0.643	0.895	0.892	0.854	0.904	0.908	0.638	1.261	1.263	0.956	1.112	1.144
60	0.413	0.897	0.891	0.790	0.884	0.896	0.489	1.716	1.690	1.143	1.525	1.563
						$\max \lvert\lambda_i\rvert = 1$						
1	0.694	0.526	0.356	0.865	0.859	0.851	0.089	0.097	0.087	0.323	0.347	0.343
6	0.513	0.550	0.489	0.728	0.632	0.652	0.358	0.578	0.551	0.746	0.913	0.954
12	0.428	0.570	0.534	0.627	0.518	0.544	0.551	1.127	1.096	0.990	1.323	1.404
36	0.273	0.679	0.665	0.541	0.511	0.541	0.747	3.275	3.231	1.499	2.379	2.584
60	0.179	0.779	0.772	0.525	0.621	0.655	0.720	7.267	7.145	2.015	3.906	4.303

Table 15: VAR(3) with i.i.d. Gaussian innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

Conditionally heteroskedastic innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max\|\lambda_i\| = 0$						
1	0.853	0.999	0.998	0.832	0.999	0.998	0.411	0.477	0.507	0.411	0.475	0.500
6	0.823	0.871	0.892	0.882	0.912	0.916	0.357	0.410	0.437	0.400	0.425	0.436
12	0.782	0.880	0.892	0.872	0.896	0.897	0.323	0.403	0.423	0.396	0.423	0.432
36	0.650	0.915	0.899	0.855	0.895	0.906	0.320	0.619	0.600	0.517	0.581	0.585
60	0.432	0.912	0.902	0.797	0.892	0.892	0.338	1.137	1.086	0.843	1.059	1.052
						$\max\|\lambda_i\| = 0.63555$						
1	0.833	0.853	0.888	0.842	0.872	0.904	0.408	0.558	0.559	0.413	0.564	0.559
6	0.822	0.882	0.896	0.884	0.920	0.923	0.381	0.442	0.468	0.429	0.461	0.474
12	0.799	0.873	0.894	0.856	0.871	0.886	0.351	0.438	0.459	0.423	0.452	0.464
36	0.632	0.903	0.895	0.860	0.889	0.894	0.351	0.681	0.668	0.524	0.592	0.600
60	0.398	0.901	0.901	0.822	0.900	0.903	0.363	1.225	1.166	0.864	1.099	1.090
						$\max\|\lambda_i\| = 0.93656$						
1	0.861	0.925	0.973	0.834	0.914	0.962	0.401	0.528	0.537	0.417	0.536	0.546
6	0.817	0.820	0.840	0.844	0.851	0.863	0.732	0.918	0.982	0.827	0.970	1.016
12	0.761	0.890	0.921	0.842	0.922	0.936	0.793	1.013	1.079	0.977	1.092	1.143
36	0.665	0.899	0.888	0.833	0.882	0.892	0.811	1.613	1.633	1.321	1.563	1.606
60	0.372	0.872	0.862	0.783	0.874	0.887	0.799	2.846	2.777	1.952	2.671	2.668
						$\max\|\lambda_i\| = 1$						
1	0.691	0.544	0.357	0.839	0.834	0.850	0.109	0.117	0.106	0.422	0.456	0.470
6	0.492	0.540	0.494	0.724	0.669	0.692	0.398	0.634	0.621	0.901	1.111	1.200
12	0.427	0.577	0.550	0.639	0.542	0.572	0.595	1.193	1.186	1.139	1.532	1.672
36	0.290	0.692	0.672	0.531	0.523	0.547	0.902	3.921	3.805	1.729	2.819	3.012
60	0.186	0.786	0.762	0.537	0.653	0.668	1.073	10.919	10.270	2.792	5.734	5.727

Table 16: VAR(3) with CCC-MGARCH(1,1) innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

Unconditionally heteroskedastic innovations

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max\|\lambda_i\| = 0$						
1	0.859	0.996	0.994	0.856	0.994	0.992	0.357	0.407	0.431	0.363	0.411	0.432
6	0.809	0.858	0.879	0.870	0.896	0.904	0.350	0.401	0.421	0.383	0.406	0.419
12	0.798	0.883	0.898	0.853	0.880	0.888	0.354	0.441	0.468	0.406	0.433	0.448
36	0.640	0.880	0.888	0.842	0.889	0.893	0.384	0.744	0.765	0.590	0.663	0.678
60	0.453	0.893	0.893	0.807	0.899	0.895	0.440	1.472	1.439	1.047	1.315	1.311
						$\max\|\lambda_i\| = 0.63555$						
1	0.859	0.858	0.895	0.844	0.860	0.890	0.357	0.489	0.487	0.364	0.493	0.488
6	0.849	0.902	0.914	0.851	0.891	0.902	0.371	0.430	0.450	0.403	0.430	0.443
12	0.793	0.873	0.890	0.860	0.880	0.893	0.367	0.457	0.486	0.430	0.459	0.475
36	0.632	0.885	0.898	0.860	0.898	0.908	0.402	0.777	0.799	0.620	0.697	0.714
60	0.446	0.910	0.907	0.818	0.902	0.906	0.466	1.564	1.529	1.095	1.377	1.380
						$\max\|\lambda_i\| = 0.93656$						
1	0.862	0.900	0.960	0.860	0.887	0.954	0.355	0.461	0.470	0.364	0.462	0.471
6	0.824	0.816	0.831	0.848	0.828	0.842	0.657	0.821	0.875	0.731	0.849	0.887
12	0.775	0.898	0.914	0.858	0.913	0.927	0.730	0.932	1.002	0.885	0.981	1.029
36	0.637	0.886	0.898	0.830	0.881	0.895	0.838	1.666	1.730	1.335	1.566	1.624
60	0.428	0.898	0.898	0.780	0.893	0.904	0.959	3.406	3.345	2.261	3.050	3.095
						$\max\|\lambda_i\| = 1$						
1	0.714	0.572	0.376	0.836	0.843	0.834	0.103	0.112	0.099	0.365	0.395	0.394
6	0.545	0.591	0.524	0.710	0.626	0.640	0.404	0.649	0.632	0.829	1.018	1.090
12	0.458	0.610	0.601	0.631	0.519	0.539	0.625	1.268	1.277	1.087	1.448	1.563
36	0.282	0.739	0.728	0.564	0.570	0.610	1.001	4.479	4.255	1.867	3.019	3.063
60	0.176	0.815	0.813	0.586	0.735	0.779	1.076	11.478	10.684	3.121	6.428	6.006

Table 17: VAR(3) with heteroskedastic innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

Local-to-VAR(3) with invertible innovations

$$\boldsymbol{\alpha}_{\text{inv}}(L) = \begin{pmatrix} 0.4 & 0.5 \\ -1 & 0.7 \end{pmatrix} L + \begin{pmatrix} 0.3 & -0.3 \\ -0.9 & 1 \end{pmatrix} L^2 + \begin{pmatrix} -0.5 & 0.1 \\ -0.7 & 0.8 \end{pmatrix} L^3 + \begin{pmatrix} 1.2 & 0 \\ -0.8 & 0.6 \end{pmatrix} L^4 - \begin{pmatrix} 0.4 & -0.3 \\ -0.3 & 0.5 \end{pmatrix} L^5 + \begin{pmatrix} 0 & 0.5 \\ 1 & 0 \end{pmatrix} L^6 - \begin{pmatrix} 0.1 & -0.2 \\ 0.2 & 0.1 \end{pmatrix} L^7 + \begin{pmatrix} 0.1 & -0.1 \\ 0.2 & 0.1 \end{pmatrix} L^8 + \begin{pmatrix} 0.9 & 0.4 \\ -0.4 & 0.3 \end{pmatrix} L^9 \quad (14)$$

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max \lvert\lambda_i\rvert = 0$						
1	0.836	0.967	0.971	0.856	0.978	0.980	0.318	0.360	0.373	0.318	0.358	0.368
6	0.852	0.900	0.903	0.869	0.889	0.894	0.317	0.364	0.369	0.335	0.354	0.358
12	0.810	0.888	0.890	0.858	0.880	0.884	0.320	0.397	0.401	0.354	0.376	0.381
36	0.667	0.895	0.896	0.852	0.885	0.891	0.291	0.562	0.560	0.417	0.466	0.472
60	0.429	0.884	0.883	0.809	0.893	0.890	0.228	0.761	0.747	0.532	0.665	0.666
						$\max \lvert\lambda_i\rvert = 0.63555$						
1	0.837	0.785	0.794	0.817	0.738	0.750	0.319	0.470	0.458	0.318	0.465	0.452
6	0.853	0.914	0.920	0.878	0.919	0.930	0.336	0.391	0.397	0.358	0.384	0.390
12	0.815	0.901	0.904	0.848	0.874	0.882	0.344	0.428	0.430	0.381	0.405	0.410
36	0.674	0.906	0.904	0.835	0.868	0.870	0.309	0.596	0.593	0.448	0.503	0.509
60	0.459	0.892	0.887	0.803	0.886	0.892	0.246	0.824	0.806	0.570	0.717	0.720
						$\max \lvert\lambda_i\rvert = 0.93656$						
1	0.842	0.850	0.932	0.829	0.824	0.920	0.322	0.459	0.457	0.323	0.458	0.454
6	0.788	0.632	0.628	0.840	0.666	0.670	0.720	1.116	1.150	0.791	1.114	1.136
12	0.836	0.892	0.902	0.891	0.912	0.917	0.861	1.164	1.196	1.040	1.204	1.240
36	0.639	0.888	0.885	0.853	0.902	0.909	0.854	1.740	1.753	1.329	1.585	1.641
60	0.430	0.888	0.887	0.784	0.910	0.922	0.644	2.344	2.300	1.545	2.128	2.195
						$\max \lvert\lambda_i\rvert = 1$						
1	0.004	0.003	0.003	0.714	0.691	0.678	0.074	0.081	0.072	0.321	0.344	0.335
6	0.112	0.142	0.112	0.612	0.542	0.564	0.332	0.549	0.519	0.802	0.986	1.028
12	0.094	0.176	0.132	0.532	0.429	0.451	0.558	1.194	1.154	1.145	1.560	1.647
36	0.212	0.561	0.532	0.452	0.415	0.448	0.832	4.026	3.949	1.832	3.061	3.307
60	0.166	0.710	0.697	0.444	0.553	0.584	0.820	9.064	8.974	2.473	5.129	5.590

Table 18: Local-to-VAR(3), $T = 100$, 2500 simulations, 1000 bootstrap replications

Local-to-VAR(3) with invertible innovations, using wrong IRF

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max \lvert\lambda_i\rvert = 0$						
1	0.818	0.992	0.990	0.787	0.984	0.977	0.318	0.360	0.373	0.318	0.358	0.368
6	0.852	0.900	0.903	0.869	0.889	0.894	0.317	0.364	0.369	0.335	0.354	0.358
12	0.810	0.888	0.890	0.858	0.880	0.884	0.320	0.397	0.401	0.354	0.376	0.381
36	0.667	0.895	0.896	0.852	0.885	0.891	0.291	0.562	0.560	0.417	0.466	0.472
60	0.429	0.884	0.883	0.809	0.893	0.890	0.228	0.761	0.747	0.532	0.665	0.666
						$\max \lvert\lambda_i\rvert = 0.63555$						
1	0.798	0.952	0.958	0.806	0.944	0.954	0.319	0.470	0.458	0.318	0.465	0.452
6	0.834	0.920	0.931	0.836	0.905	0.914	0.336	0.391	0.397	0.358	0.384	0.390
12	0.820	0.903	0.901	0.842	0.877	0.884	0.344	0.428	0.430	0.381	0.405	0.410
36	0.674	0.906	0.904	0.835	0.868	0.870	0.309	0.596	0.593	0.448	0.503	0.509
60	0.459	0.892	0.887	0.803	0.886	0.892	0.246	0.824	0.806	0.570	0.717	0.720
						$\max \lvert\lambda_i\rvert = 0.93656$						
1	0.794	0.982	0.995	0.792	0.963	0.984	0.322	0.459	0.457	0.323	0.458	0.454
6	0.427	0.986	0.990	0.487	0.983	0.984	0.720	1.116	1.150	0.791	1.114	1.136
12	0.836	0.929	0.936	0.893	0.944	0.953	0.861	1.164	1.196	1.040	1.204	1.240
36	0.654	0.908	0.909	0.852	0.919	0.921	0.854	1.740	1.753	1.329	1.585	1.641
60	0.431	0.888	0.887	0.786	0.912	0.922	0.644	2.344	2.300	1.545	2.128	2.195
						$\max \lvert\lambda_i\rvert = 1$						
1	0.757	0.648	0.500	0.824	0.837	0.828	0.074	0.081	0.072	0.321	0.344	0.335
6	0.578	0.642	0.580	0.744	0.764	0.775	0.332	0.549	0.519	0.802	0.986	1.028
12	0.493	0.640	0.604	0.712	0.714	0.731	0.558	1.194	1.154	1.145	1.560	1.647
36	0.277	0.700	0.680	0.564	0.584	0.601	0.832	4.026	3.949	1.832	3.061	3.307
60	0.180	0.782	0.772	0.533	0.672	0.696	0.820	9.064	8.974	2.473	5.129	5.590

Table 19: Local-to-VAR(3) with wrong IRF, $T = 100$, 2500 simulations, 1000 bootstrap replications

Local-to-VAR(3) with non-invertible innovations

$$\boldsymbol{\alpha}_{\text{non}}(L) = \begin{pmatrix} 1.5 & 0.5 \\ -1 & 0.7 \end{pmatrix} L + \begin{pmatrix} 0.3 & -0.3 \\ -1 & 1 \end{pmatrix} L^2 + \begin{pmatrix} -0.5 & 0.1 \\ -0.7 & 0.8 \end{pmatrix} L^3 + \begin{pmatrix} 1.2 & 0 \\ -1 & 0.6 \end{pmatrix} L^4 - \begin{pmatrix} 0.4 & -0.3 \\ -0.3 & 0.5 \end{pmatrix} L^5 + \begin{pmatrix} 1 & 0.5 \\ 1 & 1.5 \end{pmatrix} L^6 - \begin{pmatrix} 0.1 & -0.2 \\ 0.2 & 0.2 \end{pmatrix} L^7 + \begin{pmatrix} 0.1 & -0.1 \\ 0.2 & 0.1 \end{pmatrix} L^8 + \begin{pmatrix} 1 & 1.5 \\ -0.4 & 0.3 \end{pmatrix} L^9 \quad (15)$$

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max \lvert\lambda_i\rvert = 0$						
1	0.629	0.432	0.456	0.690	0.498	0.530	0.314	0.383	0.386	0.320	0.389	0.390
6	0.724	0.790	0.801	0.667	0.703	0.710	0.321	0.369	0.374	0.334	0.353	0.358
12	0.812	0.889	0.890	0.857	0.879	0.882	0.325	0.404	0.407	0.362	0.384	0.390
36	0.656	0.904	0.900	0.834	0.873	0.877	0.294	0.567	0.565	0.426	0.477	0.482
60	0.430	0.894	0.890	0.799	0.886	0.890	0.230	0.766	0.751	0.543	0.680	0.681
						$\max \lvert\lambda_i\rvert = 0.63555$						
1	0.635	0.400	0.378	0.621	0.374	0.368	0.317	0.538	0.509	0.322	0.541	0.512
6	0.820	0.925	0.931	0.811	0.891	0.898	0.354	0.412	0.419	0.375	0.402	0.409
12	0.818	0.892	0.895	0.887	0.906	0.909	0.361	0.451	0.453	0.406	0.431	0.437
36	0.661	0.886	0.890	0.851	0.897	0.905	0.324	0.628	0.625	0.477	0.535	0.543
60	0.436	0.887	0.885	0.802	0.890	0.894	0.254	0.855	0.838	0.604	0.762	0.768
						$\max \lvert\lambda_i\rvert = 0.93656$						
1	0.645	0.362	0.418	0.656	0.378	0.452	0.325	0.522	0.495	0.320	0.504	0.478
6	0.785	0.572	0.549	0.828	0.617	0.605	0.725	1.192	1.223	0.801	1.184	1.201
12	0.833	0.869	0.869	0.887	0.880	0.892	0.822	1.136	1.168	0.983	1.151	1.187
36	0.666	0.896	0.900	0.847	0.904	0.911	0.799	1.599	1.604	1.213	1.418	1.461
60	0.420	0.906	0.904	0.776	0.890	0.906	0.603	2.170	2.128	1.453	1.978	2.035
						$\max \lvert\lambda_i\rvert = 1$						
1	0.000	0.000	0.000	0.406	0.365	0.347	0.065	0.072	0.063	0.316	0.339	0.326
6	0.001	0.002	0.001	0.518	0.452	0.470	0.327	0.547	0.509	0.890	1.094	1.131
12	0.011	0.026	0.019	0.460	0.373	0.398	0.570	1.253	1.194	1.342	1.845	1.940
36	0.133	0.427	0.388	0.399	0.374	0.402	0.854	4.375	4.285	2.203	3.844	4.136
60	0.131	0.637	0.624	0.415	0.524	0.552	0.875	10.499	10.270	2.986	6.614	7.145

Table 20: Local-to-VAR(3) with non-invertible innovations, $T = 100$, 2500 simulations, 1000 bootstrap replications

Local-to-VAR(3) with non-invertible innovations, using wrong IRF

	Coverage						Average length					
h	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w	LP	LP_b	LP_w	LPLA	LPLA_b	LPLA_w
						$\max \lvert\lambda_i\rvert = 0$						
1	0.337	0.874	0.848	0.293	0.834	0.796	0.314	0.383	0.386	0.320	0.389	0.390
6	0.630	0.720	0.728	0.722	0.760	0.765	0.321	0.369	0.374	0.334	0.353	0.358
12	0.812	0.889	0.890	0.857	0.879	0.882	0.325	0.404	0.407	0.362	0.384	0.390
36	0.656	0.904	0.900	0.834	0.873	0.877	0.294	0.567	0.565	0.426	0.477	0.482
60	0.430	0.894	0.890	0.799	0.886	0.890	0.230	0.766	0.751	0.543	0.680	0.681
						$\max \lvert\lambda_i\rvert = 0.63555$						
1	0.332	0.998	0.995	0.362	0.995	0.994	0.317	0.538	0.509	0.322	0.541	0.512
6	0.736	0.805	0.810	0.808	0.846	0.852	0.354	0.412	0.419	0.375	0.402	0.409
12	0.818	0.892	0.896	0.884	0.906	0.906	0.361	0.451	0.453	0.406	0.431	0.437
36	0.661	0.886	0.890	0.851	0.897	0.905	0.324	0.628	0.625	0.477	0.535	0.543
60	0.436	0.887	0.885	0.802	0.890	0.894	0.254	0.855	0.838	0.604	0.762	0.768
						$\max \lvert\lambda_i\rvert = 0.93656$						
1	0.356	0.994	0.989	0.318	0.996	0.987	0.325	0.522	0.495	0.320	0.504	0.478
6	0.476	0.993	0.992	0.508	0.989	0.991	0.725	1.192	1.223	0.801	1.184	1.201
12	0.836	0.928	0.930	0.897	0.935	0.941	0.822	1.136	1.168	0.983	1.151	1.187
36	0.675	0.918	0.919	0.852	0.913	0.921	0.799	1.599	1.604	1.213	1.418	1.461
60	0.419	0.905	0.903	0.776	0.891	0.906	0.603	2.170	2.128	1.453	1.978	2.035
						$\max \lvert\lambda_i\rvert = 1$						
1	0.776	0.711	0.559	0.565	0.673	0.652	0.065	0.072	0.063	0.316	0.339	0.326
6	0.617	0.693	0.617	0.588	0.791	0.810	0.327	0.547	0.509	0.890	1.094	1.131
12	0.492	0.662	0.610	0.651	0.799	0.817	0.570	1.253	1.194	1.342	1.845	1.940
36	0.290	0.716	0.696	0.572	0.644	0.663	0.854	4.375	4.285	2.203	3.844	4.136
60	0.184	0.796	0.781	0.521	0.703	0.734	0.875	10.499	10.270	2.986	6.614	7.145

Table 21: Local-to-VAR(3) with non-invertible innovations and wrong IRF, $T = 100$, 2500 simulations, 1000 bootstrap replications

List of Tables

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