

Практическое задание к Уроку 6. Производная функции одной переменной. Часть 1

1. Найти производные указанных функций:

1) $y = x^3 \log_2 x$

$$y' = (x^3 \log_2 x)' = (x^3)' \cdot \log_2 x + x^3 \cdot (\log_2 x)' = 3x^2 \cdot \log_2 x + \frac{x^2}{\ln 2} = x^2 \left(3 \log_2 x + \frac{1}{\ln 2} \right)$$

2) $y = -10 \operatorname{arctg} x + 7e^x$

$$\begin{aligned} y' &= -10' \cdot \operatorname{arctg} x + (-10) \cdot (\operatorname{arctg} x)' + 7 \cdot e^{x'} = \\ &= 0 \cdot \operatorname{arctg} x - 10 \cdot \left(-\frac{1}{1+x^2} \right) + 7 \cdot e^x = \frac{10}{1+x^2} + 7e^x \end{aligned}$$

3) $y = \frac{1}{\sqrt[3]{x^2}} - \frac{2}{x^3} + \sqrt{7} \cdot x = x^{-\frac{2}{3}} - 2 \cdot x^{-3} + 7^{\frac{1}{2}} \cdot x$

$$\begin{aligned} y' &= \left(x^{-\frac{2}{3}} - 2 \cdot x^{-3} + 7^{\frac{1}{2}} \cdot x \right)' = \left(x^{-\frac{2}{3}} \right)' - 2 \cdot (x^{-3})' + 7^{\frac{1}{2}} \cdot (x)' = \\ &= -\frac{2}{3} x^{-\frac{2}{3}-1} - 2 \cdot (-3) \cdot x^{-3-1} + 7^{\frac{1}{2}} \cdot 1 \cdot x^{1-1} = -\frac{2}{3} x^{-\frac{5}{3}} + 6 \cdot x^{-4} + 7^{\frac{1}{2}} = \frac{6}{x^4} - \frac{2}{3\sqrt[3]{x^5}} + \sqrt{7} \\ x' &= 1 \cdot x^{1-1} \end{aligned}$$

4) $y = \cos \frac{1-\sqrt{x}}{1+\sqrt{x}}$

$$\begin{aligned} y' &= \left(\cos \frac{1-\sqrt{x}}{1+\sqrt{x}} \right)' = -\sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \left(\frac{1-\sqrt{x}}{1+\sqrt{x}} \right)' = \\ &= -\sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{(1-\sqrt{x})' \cdot (1+\sqrt{x}) - (1-\sqrt{x}) \cdot (1+\sqrt{x})'}{(1+\sqrt{x})^2} = \\ &= -\sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{\left(0 - \frac{1}{2\sqrt{x}} \right) \cdot (1+\sqrt{x}) - (1-\sqrt{x}) \cdot \left(0 + \frac{1}{2\sqrt{x}} \right)}{1+2\sqrt{x}+x} = \\ &= -\sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{-\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} - \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{x}}}{1+2\sqrt{x}+x} = -\sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{-\sqrt{x}^{-1}}{1+2\sqrt{x}+x} \end{aligned}$$

5) $y = e^{sh^2 5x}$

$$y' = (e^{sh^2 5x})' = e^{sh^2 5x} \cdot (sh^2 5x)' = e^{sh^2 5x} \cdot 2sh5x \cdot ch5x \cdot 5 = 5 sh 10x e^{sh^2 5x}$$

6) $y = \ln \frac{(x+1)(x+3)^3}{(x+2)^3(x+4)}$

$$\begin{aligned} y &= \ln \frac{(x+1)(x+3)^3}{(x+2)^3(x+4)} = \ln(x+1) + \ln(x+3)^3 - \ln(x+2)^3 - \ln(x+4) = \\ &= \ln(x+1) + 3\ln(x+3) - 3\ln(x+2) - \ln(x+4) \end{aligned}$$

$$\begin{aligned} y' &= (\ln(x+1) + 3\ln(x+3) - 3\ln(x+2) - \ln(x+4))' = \frac{1}{x+1} - \frac{3}{x+2} + \frac{3}{x+3} - \frac{1}{x+4} = \\ &= \frac{(x+2)(x+3)(x+4) - 3(x+1)(x+3)(x+4) + 3(x+1)(x+2)(x+4) - (x+1)(x+2)(x+3)}{(x+1)(x+2)(x+3)(x+4)} = \\ &= \frac{x^3 + 9x^2 + 26x + 24 - 3x^3 - 24x^2 - 57x - 36 + 3x^3 + 21x^2 + 42x + 24 - x^3 - 6x^2 - 11x - 6}{(x+1)(x+2)(x+3)(x+4)} = \\ &= \frac{9x^2 - 24x^2 + 21x^2 - 6x^2 + 26x - 57x + 42x - 11x + 24 - 36 + 24 - 6}{(x+1)(x+2)(x+3)(x+4)} = \\ &= \frac{6}{(x+1)(x+2)(x+3)(x+4)} \end{aligned}$$

$$\begin{aligned}
7) \quad y &= \frac{\sin^2 x}{\operatorname{ctg} x + 1} + \frac{\cos^2 x}{\operatorname{tg} x + 1} \\
y &= \frac{\sin^2 x}{\operatorname{ctg} x + 1} + \frac{\cos^2 x}{\operatorname{tg} x + 1} = \frac{\sin^2 x}{\frac{\cos x}{\sin x} + 1} + \frac{\cos^2 x}{\frac{\sin x}{\cos x} + 1} = \\
y &= \frac{\sin^2 x}{\frac{1 - \sin x}{1 - \sin x}} + \frac{1 - \sin^2 x}{\frac{1}{\sin x}} = \sin^2 x(1 - \sin x) + \sin x(1 - \sin^2 x) = \sin^2 x - \sin^3 x + \sin x - \sin^3 x = \\
&= -2\sin^3 x + \sin^2 x + \sin x = -2 \cdot \frac{3 \sin x - \sin 3x}{4} + \frac{1 - \cos 2x}{2} + \sin x = \\
&= -\frac{3 \sin x - \sin 3x + 1 - 1 + 2\sin^2 x + 2 \sin x}{2} = -\frac{1}{2}(5 \sin x - \sin 3x + 2 \sin^2 x) \\
y' &= -\frac{1}{2}(5 \sin x - \sin 3x + 2 \sin^2 x)' = -\frac{1}{2}(5 \cos x + \cos 3x \cdot (3x)' + 2(\sin x' \cdot \sin x + \sin x \cdot \sin x')) = \\
&= -\frac{1}{2}(5 \cos x + 3 \cos 3x + 2(2 \cos x \cdot \sin x)) = -\frac{1}{2}(5 \cos x + 3 \cos 3x + 2 \sin x) \\
&= -\frac{1}{2}(5 \cos x + 3 \cos 3x + 2 \sin x) = \\
&= -\frac{1}{2}(5 \cos x + 3 \cos 3x) - \sin x - \text{первый вариант решения}
\end{aligned}$$

$$\begin{aligned}
y' &= \left(\frac{\sin^2 x}{\operatorname{ctg} x + 1} + \frac{\cos^2 x}{\operatorname{tg} x + 1} \right)' = \frac{2 \cdot \sin x \cdot \sin x \cdot \operatorname{ctg} x + 2 \cdot \sin x \cdot \sin x + \sin^2 x \cdot \left(-\frac{1}{\sin^2 x} + 0 \right)}{2 \cdot \left(-\frac{1}{\sin^2 x} + 0 \right)} + \\
&+ \frac{2 \cdot \cos x \cdot \cos x \cdot \operatorname{tg} x + 2 \cdot \cos x \cdot \cos x + \cos^2 x \cdot \frac{1}{\cos^2 x} + 0}{2 \cdot \left(\frac{1}{\cos^2 x} + 0 \right)} = \\
&= \frac{2 \cdot \sin^2 x \cdot \operatorname{ctg} x + 2 \cdot \sin^2 x - 1}{-\frac{2}{\sin^2 x}} + \frac{2 \cdot \cos^2 x \cdot \operatorname{tg} x + 2 \cdot \cos^2 x + 1}{\frac{2}{\cos^2 x}} = \\
&= -\sin^4 x(\operatorname{ctg} x + 1) + \cos^4 x(\operatorname{tg} x + 1) = -\frac{\sin^3 x}{\cos^3 x} - \frac{\sin^4 x}{\cos^4 x} + \frac{\sin x}{\cos x} + 1 - \text{второй вариант решения}
\end{aligned}$$

2. Найти производную данной функции в точке:

$$\begin{aligned}
1) \quad y &= \frac{\ln x}{x}, x_0 = e \\
y' &= \left(\frac{\ln x}{x} \right)' = \frac{\ln x' \cdot x - \ln x \cdot x'}{x^2} = \frac{1 - \ln x}{x^2} \\
\frac{1 - \ln e}{e^2} &= 0 - \text{второй вариант решения}
\end{aligned}$$

$$\begin{aligned}
2) \quad y &= \frac{\sqrt{x}}{\sqrt{x} + 1}, x_0 = 9 \\
y' &= \left(\frac{\sqrt{x}}{\sqrt{x} + 1} \right)' = \frac{\sqrt{x}' \cdot (\sqrt{x} + 1) - \sqrt{x} \cdot (\sqrt{x} + 1)'}{(\sqrt{x} + 1)^2} = \frac{\frac{1}{2\sqrt{x}} \cdot (\sqrt{x} + 1) - \sqrt{x} \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2} = \frac{\frac{1}{2\sqrt{x}} \cdot (\sqrt{x} + 1 - \sqrt{x})}{(\sqrt{x} + 1)^2} = \\
&= \frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2} = \frac{(2\sqrt{x})^{-1}}{(\sqrt{x} + 1)^2} = \frac{1}{2\sqrt{x}(\sqrt{x} + 1)^2} \\
\frac{1}{2\sqrt{9}(\sqrt{9} + 1)^2} &= \frac{1}{6 \cdot 16} = \frac{1}{96}
\end{aligned}$$

3. Используя логарифмическую производную, найти производные функций:

$$1) \quad y = x^{\ln x}$$

$$\ln y = \ln x^{\ln x} = \ln x \cdot \ln x$$

$$(\ln y)' = (\ln x \cdot \ln x)'$$

$$\frac{y'}{y} = \ln x' \cdot \ln x + \ln x \cdot \ln x' = 2 \frac{\ln x}{x}$$

$$y' = x^{\ln x} \cdot 2 \frac{\ln x}{x} = 2 \ln x \cdot x^{\ln x - 1}$$

$$2) \ y = \frac{(x^3 - 2) \cdot \sqrt[3]{(x-1)}}{(x+5)^4}$$

$$\ln y = \ln \frac{(x^3 - 2) \cdot \sqrt[3]{(x-1)}}{(x+5)^4} = \ln(x^3 - 2) + \frac{1}{3} \ln(x-1) - 4 \ln(x+5)$$

$$(\ln y)' = \left(\ln(x^3 - 2) + \frac{1}{3} \ln(x-1) - 4 \ln(x+5) \right)'$$

$$\frac{y'}{y} = \frac{3x^2}{x^3 - 2} + \frac{1}{3(x-1)} - \frac{4}{x+5}$$

$$y' = y \cdot \left(\frac{3x^2}{x^3 - 2} + \frac{1}{3(x-1)} - \frac{4}{x+5} \right) = \frac{(x^3 - 2) \cdot \sqrt[3]{(x-1)}}{(x+5)^4} \cdot \left(\frac{3x^2}{x^3 - 2} + \frac{1}{3(x-1)} - \frac{4}{x+5} \right)$$

$$3) \ y = (tg x)^{\cos x}$$

$$\ln y = \ln(tg x)^{\cos x} = \cos x \cdot \ln tg x$$

$$(\ln y)' = (\cos x \cdot \ln tg x)'$$

$$\frac{y'}{y} = \cos x' \cdot \ln tg x + \cos x \cdot \ln tg x' = -\sin x \cdot \ln tg x + \cos x \cdot \frac{1}{tg x} \cdot \frac{1}{\cos^2 x} = -\sin x \cdot \ln tg x + \frac{1}{\sin x}$$

$$y' = y \cdot \left(\frac{1}{\sin x} - \sin x \cdot \ln tg x \right) = (tg x)^{\cos x} \cdot \left(\frac{1}{\sin x} - \sin x \cdot \ln tg x \right)$$

4. Найти производную неявно заданной функции:

$$1) \ e^{xy} - \cos(x^2 + y^2) = 0$$

$$e^{xy} \cdot (xy)' + \sin(x^2 + y^2) \cdot (x^2 + y^2)' = 0$$

$$e^{xy} \cdot (y + xy') + \sin(x^2 + y^2) \cdot (2x + 2yy') = 0$$

$$ye^{xy} + xe^{xy}y' + 2x \sin(x^2 + y^2) + 2y \sin(x^2 + y^2) y' = 0$$

$$y'(xe^{xy} + 2y \sin(x^2 + y^2)) = -(ye^{xy} + 2x \sin(x^2 + y^2))$$

$$y' = -\frac{ye^{xy} + 2x \sin(x^2 + y^2)}{xe^{xy} + 2y \sin(x^2 + y^2)}$$

$$2) \ x \sin y + y \sin x = 0$$

$$\sin y + x \cos y \cdot y' + y' \sin x + y \cos x = 0$$

$$y'(\sin x + x \cos y) = -(\sin y + y \cos x)$$

$$y' = -\frac{(\sin y + y \cos x)}{(\sin x + x \cos y)}$$

5. Найти производную для заданных параметрически функций:

1) $x = t^3 + t, y = t^2 + t + 1$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{(t^2 + t + 1)'}{(t^3 + t)'} = \frac{2 \cdot t^{2-1} + 1 \cdot t^{1-1} + 0}{3 \cdot t^{3-1} + 1 \cdot t^{1-1}} = \frac{2t + 1}{3t^2 + 1}$$

2) $x = e^t \sin t, y = e^t \cos t$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{(e^t \cos t)'}{(e^t \sin t)'} = \frac{e^{t'} \cos t + e^t \cos t'}{e^{t'} \sin t + e^t \sin t'} = \frac{e^t \cos t + e^t(-\sin t)}{e^t \sin t + e^t \cos t} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

6. Найти уравнения касательной и нормали к данной кривой в точке x_0 :

$$y = e^x, x_0 = 0$$

$$y' = e^x$$

$$y'(x_0) = y'(e^x) = 1$$

$$y - 1 = x \Rightarrow y = x + 1 - \text{касательная}$$

$$y - 1 = -x \Rightarrow y = 1 - x - \text{нормаль}$$

7. Найти производные указанных порядков для следующих функций:

1) $y = -x \cdot \cos x, y'' = ?$

$$y' = (-x \cdot \cos x)' = -x' \cdot \cos x - x \cdot \cos x' = -\cos x + x \cdot \sin x$$

$$y'' = (-\cos x + x \cdot \sin x)' = -\cos x' + x' \cdot \sin x + x \cdot \sin x' = \\ = \sin x + \sin x + x \cdot \cos x = 2 \sin x + x \cos x$$

2) $y = e^{2x}, y^{(V)} = ?$

$$y^I = e^{2x'} = e^{2x} \cdot 2x' = e^{2x} \cdot (2'x + 2x') = 2e^{2x}$$

$$y^{II} = (2e^{2x})' = 2e^{2x} \cdot 2x' = 2e^{2x} \cdot (2'x + 2x') = 2^2 e^{2x}$$

$$y^{III} = (2^2 e^{2x})' = 2^2 e^{2x} \cdot 2x' = 2^2 e^{2x} \cdot (2'x + 2x') = 2^3 e^{2x}$$

$$y^{IV} = (2^3 e^{2x})' = 2^3 e^{2x} \cdot 2x' = 2^3 e^{2x} \cdot (2'x + 2x') = 2^4 e^{2x}$$

$$y^V = (2^4 e^{2x})' = 2^4 e^{2x} \cdot 2x' = 2^4 e^{2x} \cdot (2'x + 2x') = 2^5 e^{2x} = 32e^{2x}$$

3) $y = \ln(1 + x), y^{(n)} = ?$

$$y^I = (\ln(1 + x))' = \frac{1}{1 + x} \cdot (1 + x)' = \frac{1 \cdot (0 + 1)}{1 + x} = \frac{1}{1 + x}$$

$$y^{II} = \left(\frac{1 \cdot 1}{1 + x} \right)' = \frac{1' \cdot (1 + x) - 1 \cdot (1 + x)'}{(1 + x)^2} = \frac{0 \cdot (1 + x) - 1 \cdot (0 + 1)}{(1 + x)^2} = -\frac{1}{(1 + x)^2}$$

$$y^{III} = \left(-\frac{1}{(1 + x)^2} \right)' = -\frac{1' \cdot (1 + x)^2 - 1 \cdot (1 + x)^{2'}}{(1 + x)^3} = -\frac{0 \cdot (1 + x)^2 - 1 \cdot 2(0 + 1)^{2-1}}{(1 + x)^3} = \frac{2}{(1 + x)^3}$$

$$y^n = (-1)^{n+1} \frac{(n - 1)!}{(1 + x)^n}$$