15.077/IDS.147 Problem Set 2 (Python Version)
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Declaration: I pledge that the work submitted for this coursework is my own unassisted work unless stated otherwise.
Acknowledgement to: Harry Yu

In [2]:
```
import numpy as np
from scipy.special import xlogy
from scipy import stats
import pandas as pd
import matplotlib.pyplot as plt
from ipywidgets import interactive
import seaborn as sns
import statsmodels.api as sm
```

8.66 Posterior Distribution

Let the unknown probability that a basketball player makes a shot successfully be θ. Suppose your prior on θ is uniform on $[0, 1]$ and that she then makes two shots in a row. Assume that the outcomes of the two shots are independent.

Part (a): What is the posterior density of θ.

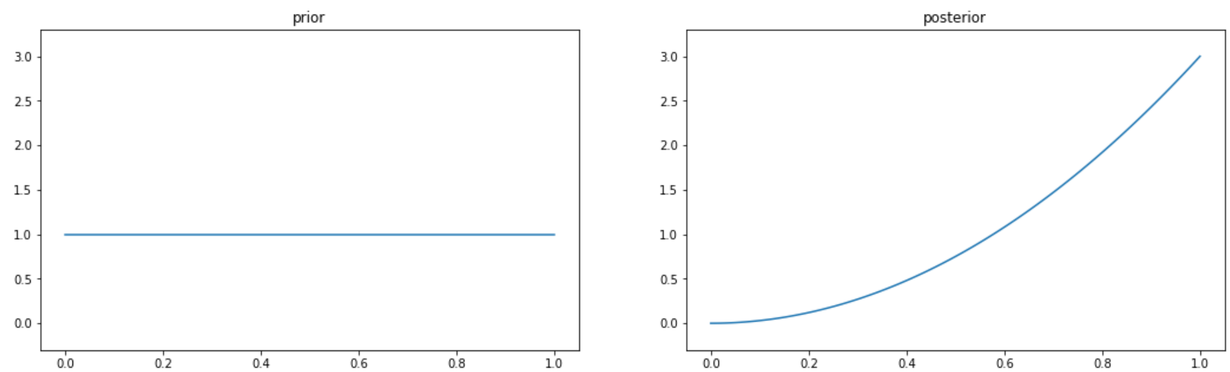
Solution: Notice the prior is given as $p_\theta(\theta) = 1$ for all $\theta \in [0, 1]$, and the likelihood is given as $p_{x|\theta}(x \mid \theta) = \theta^x(1-\theta)^{1-x}$ for all x \in {0,1}, we have the posterior density as

$$p_{\theta|x_1,x_2}(\theta|x_1 = 1,\ x_2 = 1) \propto p_{x|\theta}(1 \mid \theta)p_{x|\theta}(1 \mid \theta)p_\theta(\theta) = \theta^2$$

Normalising (by dividing it over integral over $[0, 1]$ yields the final density $p_{\theta|x_1,x_2}(\theta|x_1 = 1,\ x_2 = 1) = 3\theta^2$.

In [25]:
```
x1 = [0,1]; y1 = [1,1]
x2 = np.linspace(0,1,99); y2 = 3*(x2**2)
fig, (ax1,ax2) = plt.subplots(1, 2, figsize=(18,5))
ax1.plot(x1,y1)
ax1.set_ylim((-0.3,3.3))
ax1.set_title("prior")
ax2.plot(x2,y2)
ax2.set_ylim((-0.3,3.3))
ax2.set_title("posterior")
```

Out[25]:
```
Text(0.5, 1.0, 'posterior')
```



Part (b): What would you estimate the probability that she makes a third shot to be?

Solution: We would use the posterior mean, i.e.

$$\mathbb{E}(\theta \mid x_1 = 1,\ x_2 = 1) = \int_0^1 3\theta^3\, d\theta = \frac{3}{4}$$

9.24 Sign Test

Let X be a binomial random variable with n trials and probability p of success, i.e. $X \sim \mathrm{B}(n, p)$.

Part (a): What is the generalised likelihood ratio for testinig $H_0 : p = 0.5$ versus $H_1 : p \neq 0.5$.

Solution: take it for granted that the MLE of p is $\hat{p} = X/n$, then the generalised likelihood ratio is

$$\Lambda = \frac{\sup_{p=0.5} \mathbb{P}(X = x \mid p)}{\sup_{p\in[0,1]} \mathbb{P}(X = x \mid p)} = \frac{\mathbb{P}(X = x \mid p = 0.5)}{\mathbb{P}(X = x \mid p = x/n)} = \frac{C_x^n (1/2)^n}{C_x^n (x/n)^x (1 - x/n)^{n-x}} = \frac{1}{2^n (x/n)^x (1}$$

Part (b): Show that the test rejects for large values of $|X - n/2|$.

Solution: the likelihood ratio test rejects when $T(x) := -2 \ln \Lambda(x)$ is large. Notice that

$$T(x) = 2\left(n \ln 2 + x \ln\left(\frac{x}{n}\right) + (n - x) \ln\left(1 - \frac{x}{n}\right)\right), \quad x = 0, 1, \ldots, n$$

The following is a graph for $T(x)$ for varying n.

In [62]:
```
# set up plot

def T(x, n):
    return 2*(n*np.log(2)+xlogy(x,x/n)+xlogy(n-x,1-x/n))

def f(n):
    plt.figure(2)
    x = np.linspace(0,n,100)
    plt.plot(x, T(x,n))
    plt.scatter([n/2],[0],color="red")
    plt.ylim([0,7])
    plt.xlim([-0.3,5.3])
    plt.show

interactive_plot = interactive(f,n=(1,5))
output = interactive_plot.children[-1]
output.layout.height = "350px"
interactive_plot
```

```
interactive(children=(IntSlider(value=3, description='n', max=5, min=1), Output
(layout=Layout(height='350px'))…
```

As we can see, the function $T(x)$ is convex and have minimum at $x = n/2$ (if we extend T so that it is defined on $[0, n]$). One can formally show this by taking derivative of T:

$$T'(x) = \ln\left(\frac{x}{n}\right) + 1 + \ln\left(1 - \frac{x}{n}\right) - 1 = \ln\left(\frac{x}{n}\left(1 - \frac{x}{n}\right)\right)$$

and that $T'(x) = 0$ iff $x = n/2$. Therefore the value of $T(x)$ is large whenever $|x - n/2|$ is large, and hence be rejected.

Part (c): Using the null distribution of X, show how the significance level corresponding to a rejection region $|X - n/2| > k$ can be determined.

Solution: The significance level is

$$\mathbb{P}\left(\left|X - \frac{n}{2}\right| > k \;\middle|\; H_0\right) = \mathbb{P}\left(\left|X - \frac{n}{2}\right| > k \;\middle|\; p = 0.5\right)$$

Part (d): If $n = 10$ and $k = 2$, what is the significance level of the test?

Solution: Under H_0 we have $X \sim \mathrm{B}(10, 0.5)$ and hence
$\alpha = \mathbb{P}(|X - 5| > 2 \mid p = 0.5) = 2 \times \mathbb{P}(X \in \{0, 1, 2\} \mid p = 0.5)$

In [7]:
```
2*stats.binom.cdf(2,10,0.5)
```

Out[7]:
```
0.10937500000000003
```

Part (e): Use the normal approximation to the binomial distribution to find the significance level if $n = 100$ and $k = 10$.

Solution: Under H_0 we have $X \sim \mathrm{B}(100, 0.5)$, which is approximately distributed with

$Z \sim \mathrm{N}(50, 25)$. *Therefore* $\alpha = \mathbb{P}(|X - 50| > 10 \mid p = 0.5) = \mathbb{P}(|Z - 50| > 10.5) = 2 \times \mathbb{P}(Z < 39.5)$. *Note the continuity correction.*

In [8]:
```
2*stats.norm.cdf(39.5,50,5)
```

Out[8]:
```
0.035728841125633085
```

This analysis is the basis of the sign test, a typical application of which would be something like this: An experimental drug is to be evaluated on laboratory rats. Innpairs of litter mates, one animal is given the drug and the other is given a placebo. A physiological measure of benefit is made after some time has passed. Let X be the number of pairs for which the animal receiving the drug benefited more than its litter mate. A simple model for the distribution of X if there is no drug effect is binomial with $p = 0.5$. This is then the null hypothesis that must be made untenable by thedata before one could conclude that the drug had an effect.

9.38 Seasonality Variations in Suicide Rates

Yip et al. (2000) studied seasonal variations in suicide rates in England and Wales during 1982-1996, collecting counts shown in the following table:

In [3]:
```
suicide = pd.DataFrame({
    'Month' : ["Jan", "Feb", "Mar", "Apr", "May", "June", "July", "Aug", "Sept", 
    'Male' : [3755, 3251, 3777, 3706, 3717, 3660, 3669, 3626, 3481, 3590, 3605, 
    'Female' : [1362, 1244, 1496, 1452, 1448, 1376, 1370, 1301, 1337, 1351, 1416,
suicide.T
```

Out[3]:

	0	1	2	3	4	5	6	7	8	9	10	11
Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Male	3755	3251	3777	3706	3717	3660	3669	3626	3481	3590	3605	3392
Female	1362	1244	1496	1452	1448	1376	1370	1301	1337	1351	1416	1226

Do either the male or female data show seasonality?

Solution: We follow the procedure in 9.36. we first calculate the total counts:

In [13]:
```
male_total = sum(suicide["Male"])
female_total = sum(suicide["Female"])
print(f"Total count for male is {male_total}.")
print(f"Total count for female is {female_total}.")
```

```
Total count for male is 43229.
Total count for female is 16379.
```

We then perform a Goodness-of-fit test to test whether the data exhibits seasonality. Notice that the counts of male/female in each month is multinomially $\mathrm{Multinomial}(\vec{p} = (p_1, \ldots, p_{12}))$

*distributed , with $p_1 + ... + p_{12} = 1$. In particular, if no seasonality is present then $p_i \propto$ number of days present in the i-th month. We therefore set up the hypothesis test as following: $H_0 : \vec{p} = \vec{p}_0 := (31, 28, 31, \ldots, 31)/365$ and $H_1 : \vec{p} \neq \vec{p}_0$. **Remark**: One may also assume a simplified model with $\vec{p}_0 = (1, 1, \ldots, 1)/12$. We will discuss the effect of choosing this model later.*

In [17]:
```
p0 = np.array([31,28,31,30,31,30,31,31,30,31,30,31])/365
```

Under H_0, the expected count is as followed.

In [18]:
```
suicide_exp = pd.DataFrame({
    'Month' : ["Jan", "Feb", "Mar", "Apr", "May", "June", "July", "Aug", "Sept",
    'Male' : male_total*p0,
    'Female' : female_total*p0 })
suicide_exp.T
```

Out[18]:

	0	1	2	3	4	5	6	7	8	9
Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct
Male	3671.5	3316.2	3671.5	3553.07	3671.5	3553.07	3671.5	3671.5	3553.07	3671.5
Female	1391.09	1256.47	1391.09	1346.22	1391.09	1346.22	1391.09	1391.09	1346.22	1391.09

We may then obtain the χ^2 statistic. Notice that the χ^2 statistic follows a χ^2_{11} distribution. We may then look at the p-values. In python this could be done using the `chisquare` function in the `scipy.stats` library.

In [24]:
```
male_chisq, male_p = stats.chisquare(suicide["Male"], suicide_exp["Male"])
female_chisq, female_p = stats.chisquare(suicide["Female"], suicide_exp["Female"]
print(f"For Male: chisq statistic is {male_chisq}, and p-value is {male_p}")
print(f"For Female: chisq statistic is {female_chisq}, and p-value is {female_p}'
```

```
For Male: chisq statistic is 42.45023462019279, and p-value is 1.35334889595579
66e-05
For Female: chisq statistic is 50.525646061153765, and p-value is 5.03427231586
2472e-07
```

The p-value is very small, compared with typical significance level, say 5%. With this significance level, we reject H_0 for both male and female. The data provided is not compatible with the hypothesis that no seasonality is present, so there might be seasonality present in suicide rate which worth further investigation.

Remark: If we use a simpler model with $\vec{p}_0 = (1, 1, \ldots, 1)/12$ then we get the following result:

In [27]:
```
male_chisq_simpler, male_p_simpler = stats.chisquare(suicide["Male"])
female_chisq_simpler, female_p_simpler = stats.chisquare(suicide["Female"])
print(f"For Male: chisq statistic is {male_chisq_simpler}, and p-value is {male_p
print(f"For Female: chisq statistic is {female_chisq_simpler}, and p-value is {fe
```

```
For Male: chisq statistic is 74.56020264174514, and p-value is 1.64593642679719
53e-11
For Female: chisq statistic is 53.78564014897123, and p-value is 1.291533159564
3268e-07
```

As we might have have expected, we are rejecting H_0 with 5% (or even lower) significance level. However I prefer the previous model which takes number of dates into account since this is more accurate - this is indicated by an increase in the χ^2 statistics (especially for male).

9.47 Variance Stabilizing Transformation

Let X follow a Poisson distribution with mean λ. Show that the transformation $Y = \sqrt{X}$ is variance-stabilizing.

Solution: Notice that $\mathrm{Var}(X) = \sigma^2(\lambda) := \lambda$ *and* $g(\lambda) := \sqrt{\lambda}$ *has derivative* $g'(\lambda) = 1/(2\sqrt{\lambda})$. *By δ-method, we have*

$$\mathrm{Var}(Y) = \sigma^2(\lambda)(g'(\lambda))^2 = \lambda\left(\frac{1}{2\sqrt{\lambda}}\right)^2 = \frac{1}{4}$$

which is somewhat independent with the mean. Thus the transformation is variance stabilizing.

9.61 Volatility of Stocks

The files `haliburton` and `macdonalds` give the monthly returns on the stocks of these two companies from 1975 through 1999.

In [31]:
```
haliburton = pd.read_csv("./haliburton.txt", header=None)
macdonalds = pd.read_csv("./macdonalds.txt", header=None)
```

In [49]:
```
haliburton.columns
```

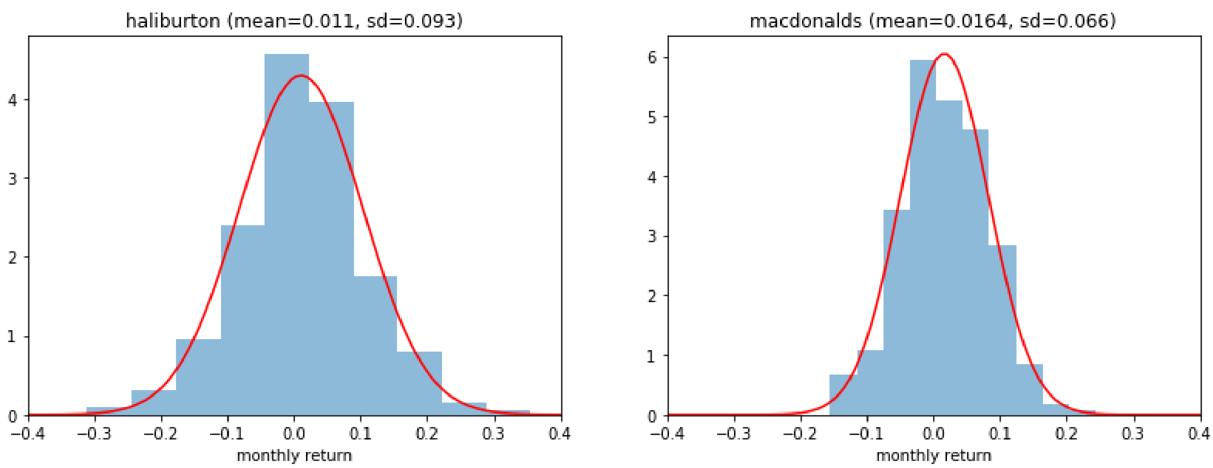
Out[49]:
```
Int64Index([0], dtype='int64')
```

Part (a): Make histograms of the returns and superimpose fitted normal densities. Comment on the quality of the fit. Which stock is more volatile?

Solution: see [https://stackoverflow.com/questions/17874063/is-there-a-parameter-in-matplotlib-pandas-to-have-the-y-axis-of-a-histogram-as-p (https://stackoverflow.com/questions/17874063/is-there-a-parameter-in-matplotlib-pandas-to-have-the-y-axis-of-a-histogram-as-p)](https://stackoverflow.com/questions/17874063/is-there-a-parameter-in-matplotlib-pandas-to-have-the-y-axis-of-a-histogram-as-p) for normalisation.

In [85]:
```
x = np.linspace(-0.4,0.4,100)

fig, ax = plt.subplots(nrows=1, ncols=2)

fig.tight_layout()
fig.subplots_adjust(wspace=0.2, hspace=0.2)
fig.set_figwidth(12)
fig.set_figheight(4)

ax[0].hist(haliburton[0], density=True, alpha=0.5)
ax[0].plot(x,stats.norm.pdf(x,haliburton.mean()[0],haliburton.std()[0]),color="re
ax[0].set_xlim([-0.4,0.4])
ax[0].set_xlabel("monthly return")
ax[0].set_title(f"haliburton (mean={np.round(haliburton.mean()[0], 4)}, sd={np.ro

ax[1].hist(macdonalds[0], density=True, alpha=0.5)
ax[1].plot(x,stats.norm.pdf(x,macdonalds.mean()[0],macdonalds.std()[0]),color="re
ax[1].set_xlim([-0.4,0.4])
ax[1].set_xlabel("monthly return")
ax[1].set_title(f"macdonalds (mean={np.round(macdonalds.mean()[0], 4)}, sd={np.ro
```

Out[85]:
```
Text(0.5, 1, 'macdonalds (mean=0.0164, sd=0.066)')
```



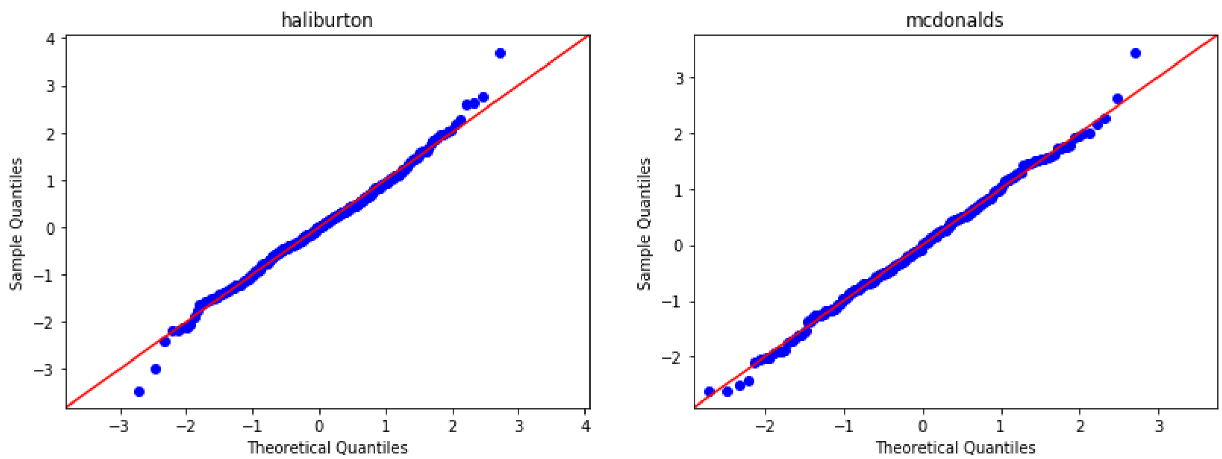
I would say the haliburton stock is more volatile since it has a higher fitted standard deviation. The fit is reasonably good, since the shape of density is generally captured, and is not shifted from the histogram.

Part (b): Make normal probability plots and again comment on the quality of the fit.

Solution: We use qqplot to look at the quality of fit.

In [93]:
```
fig, ax = plt.subplots(nrows=1, ncols=2)

fig.tight_layout()
fig.subplots_adjust(wspace=0.2, hspace=0.2)
fig.set_figwidth(12)
fig.set_figheight(4)

sm.qqplot(haliburton[0], stats.norm, fit=True, ax=ax[0], line="45")
ax[0].set_title("haliburton")
sm.qqplot(macdonalds[0], stats.norm, fit=True, ax=ax[1], line="45")
ax[1].set_title("mcdonalds")
```

Out[93]:
```
Text(0.5, 1, 'mcdonalds')
```



The fit is good since most of the points lie on the "$y = x$" line, indicates that the sample quantiles for each data points matches the theoretical quanties. One thing to note: there are some outliers at both ends of haliburton and upper end of mcdonalds. I would say the distribution of haliburton returns is slightly heavy-tailed, while the distribution of mcdonalds returns is slightly right-skewed.

14.7 Weighted Least Squares

Suppose that in the model $y_i = \beta_0 + \beta_1 x_i + e_i$, the errors have mean zero and are independent, but $\mathrm{Var}(e_i) = \rho_i^2 \sigma^2$, where the ρ_i are known constants, so the errors do not have equal variance. This situation arises when the y_i are averages of several observations at x_i; in this case, if y_i is an average of n_i independent observations, $\rho_i^2 = 1/n_i$. Because the variances are not equal, the theory developed in this chapter does not apply; intuitively, it seems that the observations with large variability should influence the estimates of β_0 and β_1 less than the observations with small variability. The problem may be transformed as follows:

$$\rho_i^{-1} y_i = \rho_i^{-1} \beta_0 + \rho_i^{-1} \beta_1 x_i + \rho_i^{-1} e_i$$

or

$$z_i = u_i\beta_0 + v_i\beta_1 + \delta_t, \quad u_i = \rho_i^{-1}, v_i = \rho_i^{-1}x_i \text{ and } \delta_i = \rho_i^{-1}e_i$$

Part (a): Show that the new model satisfies the assumptions of the standard statistical model.

Solution: Clearly the δ_i's are independent. Notice $\mathrm{Var}(\delta_i) = \mathrm{Var}(\rho_i^{-1}e_i) = \rho_i^{-2}\mathrm{Var}(e_i) = \rho_i^{-2}\rho_i^2\sigma^2 = \sigma^2$. *So the variances are constant, and the normal assumption holds.*

Part (b): Find the least squares estimates of β_0 and β_1.

Solution: It is much more convenient to express in vector notation. Define

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{pmatrix}, \quad \vec{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \quad V = \begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_N \end{pmatrix}$$

Then we have $\mathrm{cov}(\vec{Y}) = \sigma^2 V^2$. *The model is then transformed to*

$$V^{-1}\vec{Y} = V^{-1}X\vec{\beta} + V^{-1}\vec{e}$$

And the least square estimator for $\vec{\beta}$ is

$$\hat{\beta} = ((V^{-1}X)^T(V^{-1}X))^T(V^{-1}X)^T V^{-1}\vec{y} = (X^T V^{-2}X)^{-1}X^T V^{-2}\vec{y}$$

To obtain $\hat{\beta}_0$ and $\hat{\beta}_1$ in terms of x_i and y_i we consider the matrices:

$$X^T V^{-2} X = \begin{pmatrix} \sum \rho_i^{-2} & \sum \rho_i^{-2} x_i \\ \sum \rho_i^{-2} x_i & \sum \rho_i^{-2} x_i^2 \end{pmatrix}$$

with inverse

$$(X^T V^{-2} X)^{-1} = \frac{1}{\sum \rho_i^{-2} \sum \rho_i^{-2} x_i^2 - (\sum \rho_i^{-1} x_i)^2} \begin{pmatrix} \sum \rho_i^{-2} x_i^2 & -\sum \rho_i^{-2} x_i \\ -\sum \rho_i^{-2} x_i & \sum \rho_i^{-2} \end{pmatrix}$$

Moreover we have

$$X^T V^{-1}\vec{y} = \begin{pmatrix} \sum \rho_i^{-2} y_i \\ \sum \rho_i^{-2} x_i y_i \end{pmatrix}$$

and therefore we have

$$\hat{\beta}(\vec{y}) = \frac{1}{\sum \rho_i^{-2} \sum \rho_i^{-2} x_i^2 - (\sum \rho_i^{-1} x_i)^2} \begin{pmatrix} \sum \rho_i^{-2} x_i^2 \sum \rho_i^{-2} y_i - \sum \rho_i^{-2} x_i \sum \rho_i^{-2} x_i y_i \\ \sum \rho_i^{-2} \sum \rho_i^{-2} x_i y_i - \sum \rho_i^{-2} x_i \sum \rho_i^{-2} y_i \end{pmatrix}$$

Part (c): Show that performing a least squares analysis on the new model, as was done in part (b), is equivalent to minimizing

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \rho_i^{-2}$$

This is a weighted least squares criterion; the observations with large variances areweighted less.

Solution: This is almost trivial: just notice that for all i, we have

$$(z_i - u_i\beta_0 - v_i\beta_1)^2 = (\rho_i^{-1} y_i - \rho_i^{-1} u_i\beta_0 - \rho_i^{-1} v_i\beta_1)^2 = \rho_i^{-2}(y_i - \beta_0 - \beta_1 x_i)^2$$

Part (d): Find the variances of the estimates of part (b).

Solution: Just note that

$$\operatorname{cov}(\hat{\beta}(\vec{Y})) = ((X^T V^{-2} X)^{-1} X^T V^{-2}) \operatorname{cov}(\vec{Y}) \, ((X^T V^{-2} X)^{-1} X^T V^{-2})^T = \sigma^2 (X^T V^{-2} X)^{-1}$$

where $(X^T V^{-2} X)$ has been calculated above after some hard work. We therefore have

$$\operatorname{Var}(\hat{\beta}_0) = \frac{\sum \rho_i^{-2} x_i^2}{\sum \rho_i^{-2} \sum \rho_i^{-2} x_i^2 - (\sum \rho_i^{-1} x_i)^2}$$

$$\operatorname{Var}(\hat{\beta}_1) = \frac{\sum \rho_i^{-2}}{\sum \rho_i^{-2} \sum \rho_i^{-2} x_i^2 - (\sum \rho_i^{-1} x_i)^2}$$

14.55 Gasoline and Vapor

When gasoline is pumped into the tank of an automobile, hydrocarbon vapors in the tank are forced out and into the atmosphere, producing a significant amount of air pollution. For this reason, vapor-recovery devices are often installed on gasoline pumps. It is difficult to test a recovery device in actual operation, because all that can be measured is the amount of vapor actually recovered and, by means of a "sniffer", whether any vapor escaped into the atmosphere. To estimate the efficiency of the device, it is thus necessary to estimate the total amount of vapor in the tank by using its relation to the values of variables that can actually be measured. In this exercise, you will try to develop such a predictive relationship using data that were obtained in a laboratory experiment. The file `gasvapor` contains recordings of the following variables: initial tank temperature (F) temperature of the dispensed gasoline (F), initial vapor pressure in the tank (psi), vapor pressure of the dispensed gasoline (psi), and emitted hydrocarbons (g). **A prediction of emitted hydrocarbons is desired.**

First, randomly select 40 observations and set them aside. You will develop a predictive relationship based on the remaining observations and then test its strengthon the observations you have held out.

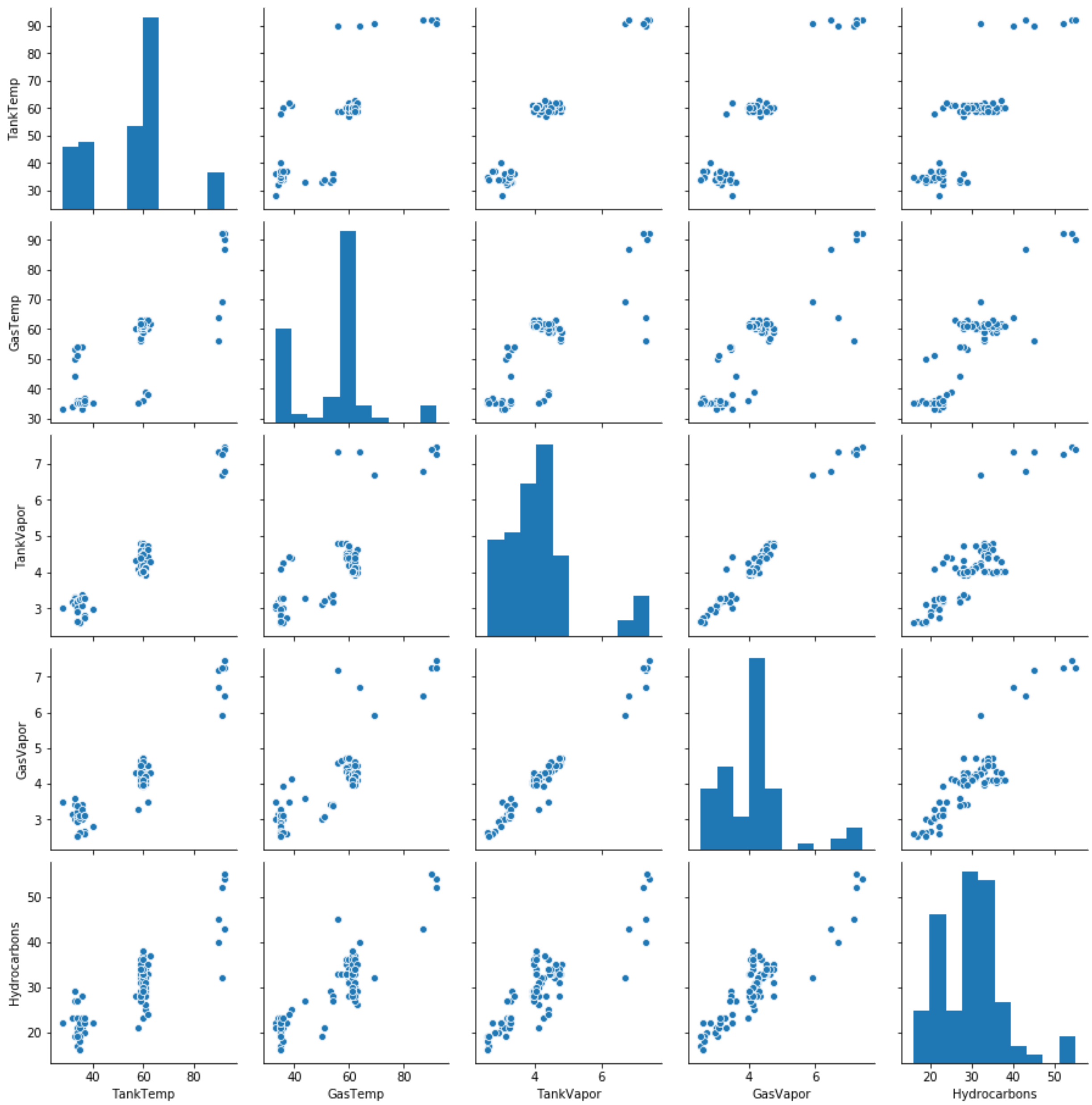
In [4]:
```
gas_train = pd.read_csv("./gasvapor_train.txt", quotechar="'")
gas_test = pd.read_csv("./gasvapor_test.txt", quotechar="'")   # the reserved 40
```

Part (a): Look at the relationships among the variables by scatterplots. Comment on which relationships look strong. Based on this information, what variables would you conjecture will be important in the model? Do the plots suggest that transformations will be helpful? Do there appear to be any outliers?

Solution: We first look at the pairplot.

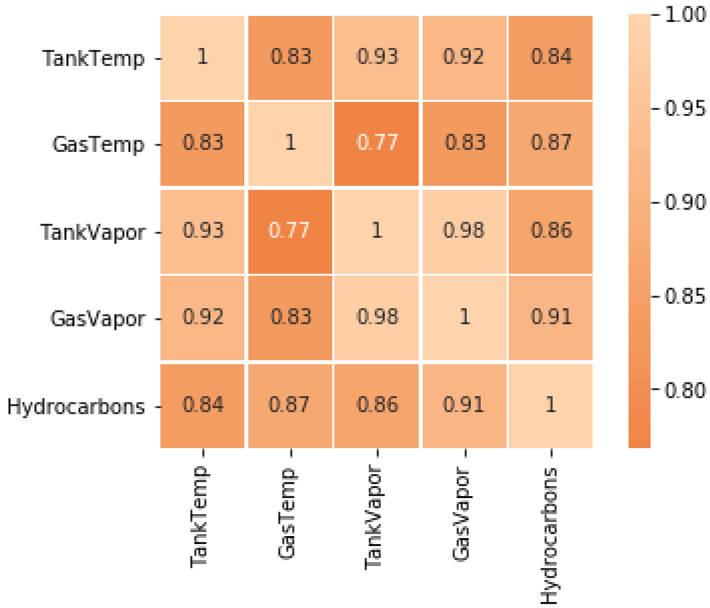
In [5]:
```
sns.pairplot(gas_train)
```

Out[5]: <seaborn.axisgrid.PairGrid at 0x1ae1c049c88>



One may also look at the correlation matrix.

In [12]:
```
p = sns.heatmap(gas_train.corr(), center=0, annot=True, square=True, linewidths=.
```



As we can see, most of the data are highly correlated. The correlation between `GasVapor` *and* `TankVapor` *is alarming, so we might possibly remove one of them. The correlation between* `TankTemp` *and* `TankVapor` *worth more careful treatment as well.*

For transformation, the plot above shows that the experiment was conducted with three choices of `TankTemp` *. One may consider using factor model (which is not covered in class until now!) Other than that, I do not think transformation is needed.*

Part (b): Try fitting a few different models and select two that you think are the best.

Solution: As we have discussed, no transformation is needed for the model. Therefore we use backward selection to select the two "best" models. (Clearly they are not the best objectively, it is just a systematic yet naive way to select variables). We begin with the full model containing all variables.

In [63]:
```
X = gas_train[[
    'TankTemp',
    'GasTemp',
    'TankVapor',
    'GasVapor']]
X = sm.add_constant(X)
model = sm.OLS(gas_train['Hydrocarbons'], X )
results = model.fit()
results.summary()
```

Out[63]:

OLS Regression Results

Dep. Variable:	Hydrocarbons	**R-squared:**	0.877
Model:	OLS	**Adj. R-squared:**	0.871
Method:	Least Squares	**F-statistic:**	142.7
Date:	Thu, 04 Mar 2021	**Prob (F-statistic):**	1.39e-35
Time:	23:57:54	**Log-Likelihood:**	-203.98
No. Observations:	85	**AIC:**	418.0
Df Residuals:	80	**BIC:**	430.2
Df Model:	4		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	0.5676	1.301	0.436	0.664	-2.021	3.156
TankTemp	-0.0264	0.065	-0.409	0.683	-0.155	0.102
GasTemp	0.1848	0.052	3.542	0.001	0.081	0.289
TankVapor	-2.4630	1.988	-1.239	0.219	-6.419	1.493
GasVapor	7.4800	2.003	3.735	0.000	3.494	11.466

Omnibus:	0.317	**Durbin-Watson:**	0.988
Prob(Omnibus):	0.853	**Jarque-Bera (JB):**	0.088
Skew:	0.070	**Prob(JB):**	0.957
Kurtosis:	3.072	**Cond. No.**	748.

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

We drop the variable having the highest p-value (excluding intercepts), thaht is `TankTemp` *in our case.*

In [14]:
```
X = gas_train[[
#    'TankTemp',
    'GasTemp',
    'TankVapor',
    'GasVapor']]
X = sm.add_constant(X)
model1 = sm.OLS(gas_train['Hydrocarbons'], X )
results1 = model1.fit()
results1.summary()
```

Out[14]:

OLS Regression Results

Dep. Variable:	Hydrocarbons	**R-squared:**	0.877
Model:	OLS	**Adj. R-squared:**	0.872
Method:	Least Squares	**F-statistic:**	192.2
Date:	Wed, 10 Mar 2021	**Prob (F-statistic):**	1.00e-36
Time:	00:33:30	**Log-Likelihood:**	-204.07
No. Observations:	85	**AIC:**	416.1
Df Residuals:	81	**BIC:**	425.9
Df Model:	3		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	0.6891	1.260	0.547	0.586	-1.817	3.196
GasTemp	0.1725	0.042	4.064	0.000	0.088	0.257
TankVapor	-2.9611	1.564	-1.894	0.062	-6.072	0.150
GasVapor	7.7619	1.871	4.148	0.000	4.039	11.485

Omnibus:	0.107	**Durbin-Watson:**	0.996
Prob(Omnibus):	0.948	**Jarque-Bera (JB):**	0.027
Skew:	0.041	**Prob(JB):**	0.987
Kurtosis:	2.971	**Cond. No.**	462.

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

We further drop the variable having the highest p-value (excluding intercepts), thaht is `TankVapor` in our case.

In [15]:
```
X = gas_train[[
#    'TankTemp',
    'GasTemp',
#    'TankVapor',
    'GasVapor']]
X = sm.add_constant(X)
model2 = sm.OLS(gas_train['Hydrocarbons'], X )
results2 = model2.fit()
results2.summary()
```

Out[15]:

OLS Regression Results

Dep. Variable:	Hydrocarbons	**R-squared:**	0.871
Model:	OLS	**Adj. R-squared:**	0.868
Method:	Least Squares	**F-statistic:**	277.7
Date:	Wed, 10 Mar 2021	**Prob (F-statistic):**	3.05e-37
Time:	00:33:33	**Log-Likelihood:**	-205.91
No. Observations:	85	**AIC:**	417.8
Df Residuals:	82	**BIC:**	425.1
Df Model:	2		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	0.5590	1.278	0.438	0.663	-1.983	3.101
GasTemp	0.2065	0.039	5.282	0.000	0.129	0.284
GasVapor	4.3497	0.512	8.492	0.000	3.331	5.369

Omnibus:	1.780	**Durbin-Watson:**	0.939
Prob(Omnibus):	0.411	**Jarque-Bera (JB):**	1.185
Skew:	-0.129	**Prob(JB):**	0.553
Kurtosis:	3.517	**Cond. No.**	241.

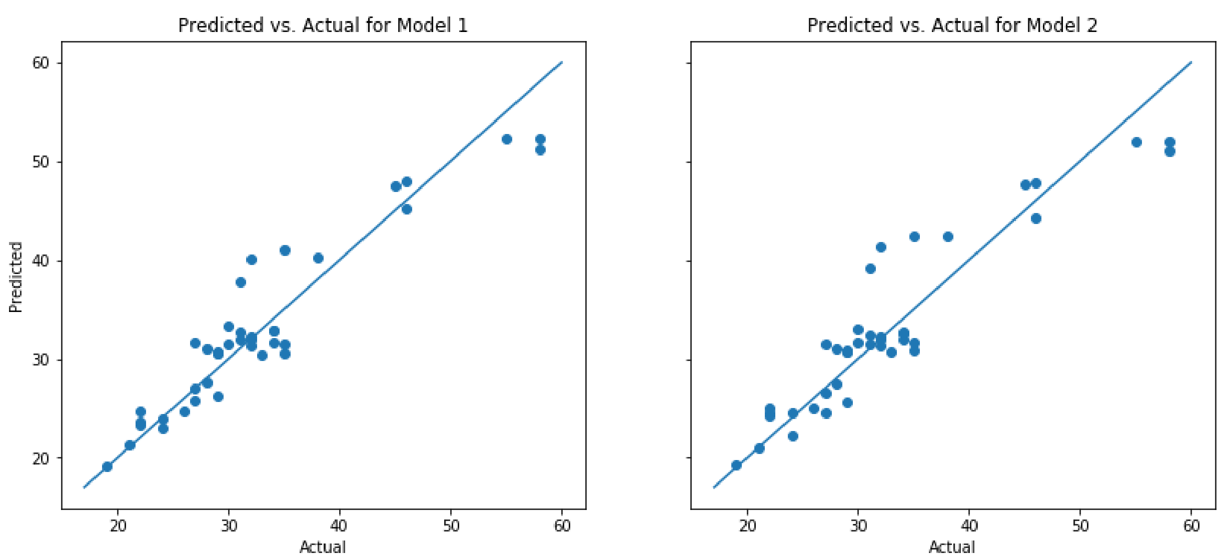
Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

The algorithm of backward selection shall terminate since the p-value for non-constant variables are very small (say, less than 5%). We have therefore obtain the two "best" models.

Part (c): Using these two models, predict the responses for the 40 observations you haveheld out.

In [20]:
```
predict_test1 = results1.predict(sm.add_constant(gas_test[['GasTemp','TankVapor',
predict_test2 = results2.predict(sm.add_constant(gas_test[['GasTemp','GasVapor']]
```

In [18]:
```
fig, ax = plt.subplots(nrows=1, ncols=2)

fig.tight_layout()
fig.subplots_adjust(wspace=0.2, hspace=0.2)
fig.set_figwidth(12)
fig.set_figheight(5)

xmin = gas_test["Hydrocarbons"].min() - 2
xmax = gas_test["Hydrocarbons"].max() + 2

ax[0].scatter(gas_test["Hydrocarbons"], predict_test1) # create scatter on each s
ax[0].plot([xmin, xmax], [xmin,xmax])
ax[0].set_title("Predicted vs. Actual for Model 1")  # give each subplot a title
ax[0].set(xlabel="Actual", ylabel="Predicted") # label each set of axes

ax[1].scatter(gas_test["Hydrocarbons"], predict_test2) # create scatter on each s
ax[1].plot([xmin, xmax], [xmin,xmax])
ax[1].set_title("Predicted vs. Actual for Model 2")  # give each subplot a title
ax[1].set(xlabel="Actual", ylabel="Predicted") # label each set of axes

for ax in ax.flat:
    ax.label_outer()
```

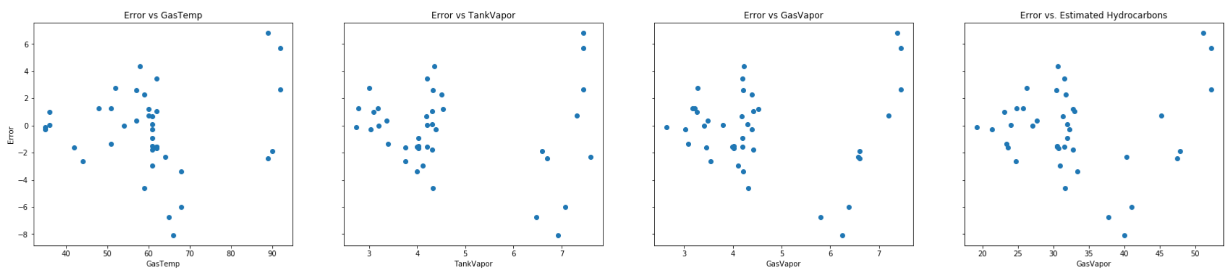


I think the plot looks almost the same, so there isn't a significant improvement when the variable `TankVapor` is added.

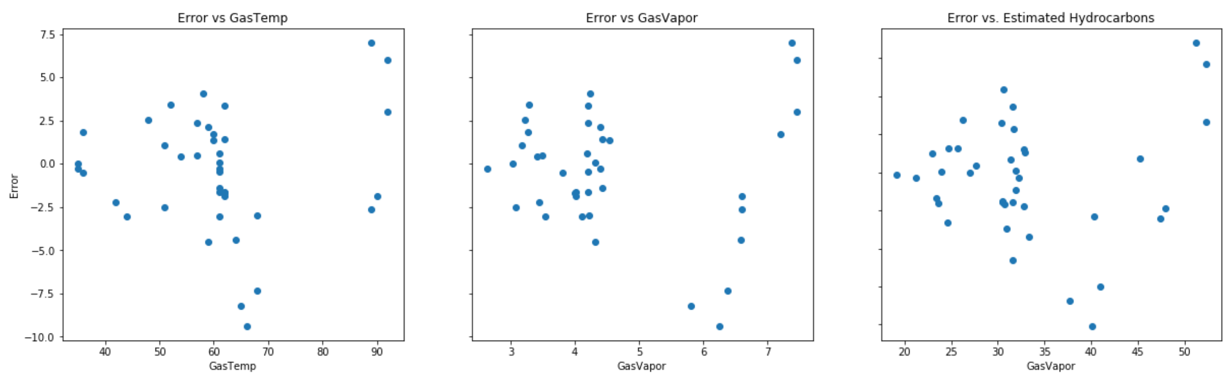
Plot prediction errors versus each of the indepen-dent variables.

In [21]:
```
err1 = gas_test["Hydrocarbons"] - predict_test1
err2 = gas_test["Hydrocarbons"] - predict_test2
```

In [38]:
```
namesArray = ['GasTemp', 'TankVapor', 'GasVapor']
dataArray = np.array(gas_test[namesArray])
fig, ax = plt.subplots(nrows=1, ncols=4)

fig.tight_layout()
fig.subplots_adjust(wspace=0.2, hspace=0.2)
fig.set_figwidth(26)
fig.set_figheight(5)

for i in range(3):
    ax[i].scatter(dataArray[:,i], err1) # create scatter on each subplot
    ax[i].set_title("Error" + ' vs ' + namesArray[i]) # give each subplot a title
    ax[i].set(xlabel=namesArray[i], ylabel="Error") # label each set of axes

ax[3].scatter(predict_test1, err1) # create scatter on each subplot
ax[3].set_title("Error vs. Estimated Hydrocarbons") # give each subplot a title
ax[3].set(xlabel=namesArray[i], ylabel="Error") # label each set of axes

for ax in ax.flat:
    ax.label_outer()
```



In [39]:
```
namesArray = ['GasTemp', 'GasVapor']
dataArray = np.array(gas_test[namesArray])
fig, ax = plt.subplots(nrows=1, ncols=3)

fig.tight_layout()
fig.subplots_adjust(wspace=0.2, hspace=0.2)
fig.set_figwidth(18)
fig.set_figheight(5)

for i in range(2):
    ax[i].scatter(dataArray[:,i], err2) # create scatter on each subplot
    ax[i].set_title("Error" + ' vs ' + namesArray[i]) # give each subplot a title
    ax[i].set(xlabel=namesArray[i], ylabel="Error") # label each set of axes

ax[2].scatter(predict_test1, err1) # create scatter on each subplot
ax[2].set_title("Error vs. Estimated Hydrocarbons") # give each subplot a title
ax[2].set(xlabel=namesArray[i], ylabel="Error") # label each set of axes

for ax in ax.flat:
    ax.label_outer()
```



Again the set of plots aren't significantly different as expected. I think the plots between error and independent variables are satisfactory. No further correlation is exhibited. However, the `error` vs. estimated hydrocarbons plots indicate that Heteroscedasticity is present in our model. It is now apparent that a log transform on `hydrocarbon` data should have been done, but note this is not apparent when doing initial diagonistics. Therefore I will not include the new model here.

Summarize the strength of the prediction by the root mean square prediction error:

$$\text{RMSPE} = \sqrt{\frac{1}{40}\sum_{i=1}^{40}(Y_i - \hat{Y}_i)^2}$$

where Y_i is the i-th observed value and $\hat{Y}_i$ is the predicted value.

In [41]:
```
rmspe1 = np.sqrt(np.sum(err1**2)/40)
rmspe2 = np.sqrt(np.sum(err2**2)/40)
print(f"The rmspe of first model is {rmspe1}.")
print(f"The rmspe of second model is {rmspe2}.")
```

```
The rmspe of first model is 3.0510681222820897.
The rmspe of second model is 3.446586830789645.
```

They have similar values.