5. (**Theoretical**) Show that the $D^4 f$ operator is given by:

$$D^4 f(x_j) \cong \frac{f(x_{j+2}) - 4f(x_{j+1}) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2})}{h^4} \tag{3.33}$$

For this operator, what is the order ($\mathcal{O}(h^k)$) of the approximation?

- Error de aproximación en términos de h

$$E(h) = f''''(x_j) - \Big(\big(f(x_j+2h) - 4f(x_j+h) + 6f(x_j) - 4f(x_j-h) + f(x_j-2h)\big)\Big) / h^4$$

- Expandimos en series de Taylor para hallar el $\mathcal{O}(h^k)$

$$f(x_j+2h) = f(x_j) + 2h f'(x_j) + \left(\frac{4h^2}{2!}\right) f''(x_j) + \left(\frac{8h^3}{3!}\right) \cdot f'''(x_j) + \frac{16h^4}{4!} f''''(x_j)$$

$$f(x_j+h) = f(x_j) + hf'(x_j) + \frac{h^2}{2!}f''(x_j) + \frac{h^3}{3!}f'''(x_j) + \frac{h^4}{4!}f''''(x_j)$$

$$f(x_j-h) = f(x_j) - hf'(x_j) + \frac{h^2}{2!}f''(x_j) - \frac{h^3}{3!}f'''(x_j) + \frac{h^4}{4!}f''''(x_j)$$

$$f(x_j-2h) = f(x_j) - 2hf'(x_j) + \left(\frac{4h^2}{2!}\right)f''(x_j) - \frac{8h^3}{3!}f'''(x_j) + \frac{16h^4}{4!}f''''(x_j)$$

Remplazamos en la fórmula del error de aproximación.

$$E(h) = f''''(x_j) - \Big(\Big(\Big(f(x_j) + 2h\,f'(x_j) + \left(\frac{4h^2}{2}\right) f''(x_j) + \left(\frac{8h^3}{6}\right) f'''(x_j) + \frac{16h^4}{24} f''''(x_j)\Big) - 4\Big(f(x_j) + h\,f'(x_j) + \frac{h^2}{2} f''(x_j) + \frac{h^3}{6} f'''(x_j) + \frac{h^4}{24} f''''(x_j)\Big) + 6f(x_j) - 4\Big(f(x_j) - h\,f'(x_j) + \frac{h^2}{2} f''(x_j) - \frac{h^3}{6} f''(x_j) + \frac{h^4}{24} f''''(x_j)\Big) + f(x_j) - 2h\,f'(x_j) + \left(\frac{4h^2}{2}\right) f''(x_j) - \frac{2h^3}{3} f'''(x_j)\Big)\Big) \Big/ h^4$$

$$E(h) = \Big(f''''(x_j) - \cancel{f(x_j)} - \cancel{2h\,f'(x_j)} - \cancel{2h^2 f''(x_j)} - \cancel{\frac{4h^3}{3} f'''(x_j)} - \cancel{\frac{2h^4}{3} f''''(x_j)} - \cancel{4f(x_j)} - \cancel{4h\,f'(x_j)} - \cancel{2h^2 f''(x_j)} - \cancel{\frac{2}{3}h^3 f'''(x_j)} - \cancel{\frac{h^4}{6} f''''(x_j)} + \cancel{6f(x_j)} - \cancel{4f(x_j)} + \cancel{4h\,f'(x_j)} - \cancel{2h^2 f''(x_j)} + \cancel{\frac{2}{3}h^3 f''(x_j)} - \cancel{\frac{h^4}{6} f''''(x_j)} + \cancel{f(x_j)} - \cancel{2h\,f(x_j)} + \cancel{2h^2 f''(x_j)} - \cancel{\frac{2}{3}h^3 f'''(x_j)}\Big) \Big/ h^4$$

$$E(h) = \frac{f''''(x_j) - 2f(x_j) - 4h\,f'(x_j) - 4h^2 f''(x_j) - 2h^3 f'''(x_j) - h^4 f''''(x_j)}{h^4}$$

$$E(h) = \overbrace{h^4 f''''(x_j)}^{O(h^K)} \,.\, \underbrace{-2f(x_j) - 4h\,f'(x_j) - 4h^2 f''(x_j) - 2h^3 f'''(x_j)}_{\text{Desarrollo de Taylor simplificado}}$$

- $f(x) = f(x_0) + f'(x_0)(x_j - x_0) + \frac{f''(x_0)(x_j - x_0)^2}{2!} +$

$$f'''(x_0)\frac{(x_j - x_0)^3}{3!} + \underbrace{f''''(x_0)\frac{(x_j - x_0)^4}{4!}}_{O(h)^4}$$