

5. (Theoretical) Show that the $D^4 f$ operator is given by:

$$D^4 f(x_j) \cong \frac{f(x_{j+2}) - 4f(x_{j+1}) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2}))}{h^4} \quad (3.33)$$

For this operator, what is the order ($\mathcal{O}(h^k)$) of the approximation?

● Error de aproximación en términos de h

$$E(h) = \frac{f^{(4)}(x_j) - \left(f(x_{j+2}) - 4f(x_{j+1}) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2}) \right)}{h^4}$$

● Expandimos en series de Taylor para hallar el $\mathcal{O}(h^k)$

$$f(x_j + 2h) = f(x_j) + 2h f'(x_j) + \left(\frac{4h^2}{2!} \right) f''(x_j) + \left(\frac{8h^3}{3!} \right) f'''(x_j) + \left(\frac{16h^4}{4!} \right) f^{(4)}(x_j)$$

$$f(x_j + h) = f(x_j) + h f'(x_j) + \frac{h^2}{2!} f''(x_j) + \frac{h^3}{3!} f'''(x_j) + \frac{h^4}{4!} f^{(4)}(x_j)$$

$$f(x_j - h) = f(x_j) - h f'(x_j) + \frac{h^2}{2!} f''(x_j) - \frac{h^3}{3!} f'''(x_j) + \frac{h^4}{4!} f^{(4)}(x_j)$$

$$f(x_j - 2h) = f(x_j) - 2h f'(x_j) + \frac{4h^2}{2!} f''(x_j) - \frac{8h^3}{3!} f'''(x_j) + \frac{16h^4}{4!} f^{(4)}(x_j)$$

● Reemplazamos en la fórmula del error de aproximación.

$$E(h) = f^{(4)}(x_j) - \left(\left(f(x_j) + 2h f'(x_j) + \left(\frac{4h^2}{2}\right) f''(x_j) + \left(\frac{8h^3}{6}\right) f'''(x_j) + \frac{16h^4}{24} f^{(4)}(x_j) \right) - 4 \left(f(x_j) + h f'(x_j) + \frac{h^2}{2} f''(x_j) + \frac{h^3}{6} f'''(x_j) + \frac{h^4}{24} f^{(4)}(x_j) \right) + f(x_j) - 2h f'(x_j) + \frac{4h^2}{2} f''(x_j) - \frac{2h^3}{3} f'''(x_j) \right) / h^4$$

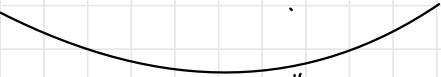
$$E(h) = \left(f^{(4)}(x_j) - \cancel{f(x_j)} - \cancel{2h f'(x_j)} - \cancel{2h^2 f''(x_j)} - \cancel{\frac{4h^3}{3} f'''(x_j)} - \cancel{\frac{2h^4}{3} f^{(4)}(x_j)} - \cancel{4 f(x_j)} - \cancel{4h f'(x_j)} - \cancel{2h^2 f''(x_j)} - \cancel{\frac{2h^3}{3} f'''(x_j)} - \cancel{\frac{h^4}{6} f^{(4)}(x_j)} + \cancel{6 f(x_j)} - \cancel{4 f(x_j)} + \cancel{4h f'(x_j)} - \cancel{2h^2 f''(x_j)} + \cancel{\frac{2h^3}{3} f'''(x_j)} - \cancel{\frac{h^4}{6} f^{(4)}(x_j)} + \cancel{f(x_j)} - \cancel{2h f'(x_j)} + \cancel{2h^2 f''(x_j)} - \cancel{\frac{2h^3}{3} f'''(x_j)} \right) / h^4$$

$$E(h) = \frac{f^{(4)}(x_j) - 2 f(x_j) - 4h f'(x_j) - 4h^2 f''(x_j) - 2h^3 f'''(x_j) - h^4 f^{(4)}(x_j)}{h^4}$$

$$E(h) = \overbrace{h^4 f^{(4)}(x_j)}^{O(h^4)} - \underbrace{2 f(x_j) - 4h f'(x_j) - 4h^2 f''(x_j) - 2h^3 f'''(x_j)}_{\text{Desarrollo de Taylor simplificado}}$$

Desarrollo de Taylor simplificado

$$\begin{aligned}
 \bullet \quad f(x) &= f(x_0) + f'(x_0)(x_j - x_0) + \frac{f''(x_0)(x_j - x_0)^2}{2!} + \\
 &\quad \frac{f'''(x_0)(x_j - x_0)^3}{3!} + \frac{f^{(4)}(x_0)(x_j - x_0)^4}{4!} + \dots
 \end{aligned}$$



$$O(h)^4$$