# A distance for evidential preferences with application to group decision making

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**Abstract**

In this paper, we focus on measuring the dissimilarity between preferences with uncertainty and imprecision, modelled by evidential preferences based on the theory of belief functions. Two issues are targeted: The first concerns the conflicting interpretations of *incomparability*, leading to a lack of consensus within the preference modelling community. This discord affects the value settings of dissimilarity measures between preference relations. After reviewing the state of the art, we propose to distinguish between two cases: *indecisive* and *undecided*, respectively modelled by a binary relation and union of all relations. The second concerns a flaw that becomes apparent when measuring the dissimilarity in the theory of belief functions. Existing dissimilarity functions in the theory of belief functions are not suitable for evidential preferences, because they measure the dissimilarity between preference relations as being identical. This is counter-intuitive and conflicting with almost all the related works. We propose a novel distance named Unequal Singleton Pair (USP) distance, able to discriminate specific singletons from others when measuring the dissimilarity. The advantages of USP distances are illustrated by the evidential preference aggregation and group decision-making applications. The experiments show that USP distance effectively improves the quality of decision results.

*Keywords:* Preference modelling; Theory of belief functions; Decision making; Distance measure

## 1. Introduction

Preference is a traditional topic in human history and its corresponding study is of great interest in various domains, such as sociology [1], economy [2], and more specifically group decision making [3]. As new applications emerge, preference modelling continues to attract the attention of recent research communities in computer science, such as social networks [4], and deep learning communities [5], *etc.*

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In reality, preferences are not always expressed in a firm or consistent manner, uncertainty and imprecision may exist in preference information, facing unknown situations, or multiple conflicting information sources. In this paper, we accept terms *uncertainty* and *imprecision* for two different concepts of imperfectness. Term *uncertainty* refers to a degree of conformity to reality, while term *imprecision* refers to quantitative default of knowledge on the information contained. A simple example is given here for a better understanding of the two terms, while other scenarios of imperfectness are also possible:

**Example 1.** In the US presidential election of 2020 between Joe Biden and Donald Trump, two pieces of information are given: the first is a statement, saying *Joe Biden has* 50% *chances to win.* The second is a data missing case, interpreted as *Biden or Trump* will win.

In this example, the first case expresses a piece of uncertain preference information, while the certainty is 50%. The second expresses a piece of preference information with imprecision. Indeed, the second is an extreme case of imprecision, also known as *total ignorance.*

Some previous works have already investigated imperfect preferences. For example, the authors of [6] proposed a fuzzy model to cope with imprecise preferences. Recently, *evidential preference* models based on the Theory of Belief Functions (TBF) have been proposed and attracted much attention in preference modelling venues [7–9]. TBF (also referred to as Dempster-Shafer Theory [10, 11] or Evidence Theory) is a powerful mathematical tool for the modelling of uncertainty and imprecision, widely applied in information fusion and decision making. In addition to the capability of characterizing uncertainty and imprecision, the evidential preference model handles as well the aforementioned ambiguity of the interpretation of "incomparability", in a unified framework.

Knowledge with uncertainty is depicted by degree values on the corresponding event, while knowledge with imprecision is depicted by the union set of possible singletons. Thus, the evidential preference model is able to express these two aspects of imperfectness in preference information faithfully. A more detailed introduction on the TBF and evidential preference model will be provided later in Section 2.

Dissimilarity measure between imperfect preference relations is another issue targeted in this work in addition to preference modelling. Dissimilarity plays a vital role in preference-aware systems concerning various applications. For example, distance-based consensus methods is an important category in preference aggregation. Indeed, some conventional voting rules are also considered distance-based consensus methods using different distance functions, as explained in [12]. In social network analysis, distances between preferences are applied for community detection and neighborhood-based recommendation systems [13, 14].

Dissimilarity functions depend predominantly on data representation. For example, for preferences represented as orders or list-wise, correlation coefficients are widely applied, such as Kendall's $\tau$, Spearman's $\rho$, Pearson's $r$ or their extended versions [15]. In the case of a pair-wise representation of the preferences, dissimilarities are measured between binary relations [16–19]. Bouyssou *et al.* [20] mentioned that *binary relation is the central tool in most models of preferences.* Thus, measuring the dissimilarity between pairwise preferences is a fundamental issue. Usually, between any two alternatives $a_i$ and $a_j$, four exclusive binary relations are possible: "strict preference ($a_i \succ a_j$)", "inverse strict preference ($a_i \prec a_j$)", "indifference ($a_i \approx a_j$)", and "incomparability ($a_i \sim a_j$)". The dissimilarity between preference relations are mainly measured through these four relations. However, the definition of "incomparability" may correspond to different interpretations, resulting in different dissimilarity value settings. Two major issues are encountered when measuring the dissimilarity between evidential preferences. The first is caused by the ambiguous interpretations of *incomparability* while the second exists in the context of the theory of belief functions.

*First issue: different interpretations of "incomparability"*

Relation *incomparability* between two alternatives is defined as *neither prefer, nor indifferent* (with the formal definition given in Section 2.1). However, this definition is ambiguous[1]. In the literature, we have identified two interpretations of *incomparability*, considered either as an *indecisive* or an *undecided* situation.

**Example 2.** *Indecisive case: Which job is better?* [21]

> Alice has two available job offers. The first job is poorly paid but corresponds well to Alice's interests while the second is well paid but does not match Alice's interests. As a consequence, Alice finds herself in a dilemma when she has to choose between the two jobs.

In Example 2, *incomparability* is caused by conflicting preferences associated with different criteria, making the decision indeterminable. We interpret this case as *indecisive.*

**Example 3.** *Undecided case: Missing opinions in Sushi survey.* [22]

> In the Sushi preference data set, the *respondent* (agent) is asked to give her/his preferences on 10 types of sushi randomly selected among 100 sushi types. The preferences are expressed in the forms of both score and order. However, the preferences on the other 90 sushi of each agent are missing, making them *incomparable*.

The second example shows a case where *incomparability* is caused by missing information, which relates to *the ignorance of knowledge.* We interpret this case as *undecided.*

To avoid misleading in engineering aspects, we need to mention that missing information is not always interpreted as *ignorance.* In some scenarios, *indifference* or *not preferred* could be better interpretations. For example, in democratic voting systems, blank votes are usually considered as *abstention*; In a top-k ranking system, alternatives without preference information are considered as *lower-ranked*.

*Incomparability* can be set without controversy when only one of the two situations mentioned before exist in the preference-aware systems. However, for the cases that merge with both situations, it is necessary to distinguish between the two interpretations. Dissimilarity measures between pair-wise preference relations may encounter such issues.

[1]The term *ambiguous* is applied to one identical definition with multiple possible interpretations.

Some works (*e.g.* [23, 24]) differentiate between the two relations as *indifference* and *incomparability*, some works interpret *incomparability* as an absence of knowledge (*e.g.* [18, 19, 25]), and some works interpret as a specific case (*e.g.* [17]). Indeed, this issue exists in different works but has rarely been systematically discussed in the research community.

The flaws of dissimilarity measure are not limited to preferences. In the Theory of Belief Functions, some problems in dissimilarity measures rise to be important in the evidential preference model.

*Second issue: a flaw in dissimilarity measures in the Theory of Belief Functions*

In TBF, each precise event is modelled by a *singleton*. Dissimilarity values between different singletons are always considered equal in existing measures. This is because two singletons are considered as either different or identical, respecting a binary relation. However, with this property applied by *evidential preferences*, dissimilarities between any two preference relations are equal. A straightforward example is that the dissimilarity between *strict preference* and *inverse strict preference* (denoted as $d(\succ, \prec)$) is identical to the dissimilarity between *strict preference* and *indifference* (denoted as $d(\succ, \approx)$), formally, $d(\succ, \prec) = d(\succ, \approx)$. This equality relation is counter-intuition and in conflict with all previous works in preference modelling [16–19, 25, 26].

For the smoothness of writing, related works, as well as comprehensive details about this flaw, are introduced in Section 2.3, after a preliminary introduction on the TBF.

*Contributions*

Facing the two issues above, three main contributions are made targeting the issues introduced above.

- The state of the art in dissimilarity measures over pairwise preferences is reviewed. Regarding the different interpretations of "incomparability", the disagreements are clarified and characterized through the evidential preference model.

- A novel metric function for dissimilarity measure between evidential preferences is proposed by extending the existing Jousselme distance [27], named Unequal Singleton Pair (USP) distance. USP distance takes into account the different dissimilarity values between singletons concerning the second issue and also handles well the case of imprecise information.

- The proposed USP distance is applied in group-decision making and is compared with other traditional and recent methods on both synthetic and real-world data sets. The results show that USP distance improves the quality of group decision making. Besides, a Condorcet's paradox elimination method for the evidential preference model is developed and applied as well.

The merits of the techniques proposed in this paper consist of three main parts, two of which are related to the evidential preference based on TBF.

- Evidential preferences are able to express imprecise cases, especially missing data cases faithfully;
- The ambiguous definition of *incomparability* is clarified by *undecided* and *indecisive* in a unified framework in evidential preference;
- USP distance is capable to discriminate singleton elements for decision making with the imprecise information taken into account. The advantage of this distance is demonstrated in group decision-making applications based on the evidential preference model.

This paper is organised as follows: preliminary notions on preference modelling, Theory of Belief Functions, and dissimilarity measures are introduced in Section 2. Afterwards, the first contribution is presented in Section 3, where dissimilarity measures over pairwise preferences are reviewed. In Section 4, the proposed novel distance for evidential objects is introduced, named Unequal Singleton Pair (USP) distance, along with a value-setting example and an application of group decision making. In Section 5, the advantages of USP distance over the other preference aggregation methods are illustrated with experiments on the sushi preference dataset in the real-world. Finally, conclusions, as well as important perspectives are, given in Section 6.

## 2. Preliminaries

In this section, preliminary knowledge on preferences and the theory of belief functions, as well as some properties of dissimilarity measures are introduced.

### *2.1. Preference modelling: background*

Preference modelling is usually based on order theory. In this paper, we use the widely accepted notions in studies of preferences from [28].

**Definition 1.** *(Binary Relation)* Given a finite set of $N$ alternatives $\mathcal{A} = \{a_1, a_2, a_3, \ldots, a_N\}$, a *binary relation* $R$ on the alternative set $\mathcal{A}$ is a subset of the Cartesian product $\mathcal{A} \times \mathcal{A}$, that is, a set of ordered pairs $(a_i, a_j)$ such that $a_i$ and $a_j$ are in $\mathcal{A} : R \subseteq \mathcal{A} \times \mathcal{A}$ [28].

**Definition 2.** *(Preference relation)* Between any two alternatives $a_i, a_j$, only four relations possibly exist $\{\succ, \succeq, \approx, \sim\}$, defined from binary relation $R$ as:

$$\begin{array}{cccl}
\text{Strict preference:} & a_i \succ a_j & \Leftrightarrow & a_i R a_j \text{ and } a_j \neg R a_i; \\
\text{Indifference:} & a_i \approx a_j & \Leftrightarrow & a_i R a_j \text{ and } a_j R a_i; \\
\text{Weak preference:} & a_i \succeq a_j & \Leftrightarrow & a_i \succ a_j \text{ or } a_i \approx a_j; \\
\text{Incomparability:} & a_i \sim a_j & \Leftrightarrow & a_i \neg \succ a_j \text{ and } a_i \neg \prec a_j \text{ and } a_i \neg \approx a_j.
\end{array}$$

Relation "weak preference" is the union of strict preference and indifference. Thus, between two alternatives $a_i$ and $a_j$, the four relations $\{\succ, \prec, \approx, \sim\}$ are exclusive and complete[2], illustrated in Figure 1.

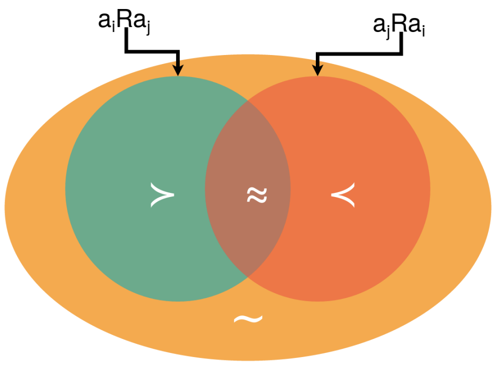



**Figure 1:** Venn diagram of preference relations.

Let $R$ be a binary relation on the alternative set $\mathcal{A}$, we have the definitions of different orders based on preference structure with relations "strict preference", "indifference" and "incomparability" :

**Definition 3.** *(Preference Structure)* A preference structure is a collection of binary relations defined on the set $\mathcal{A}$ and such that:

– for each couple $(a_i, a_j)$, $a_i, a_j \in \mathcal{A}$, at least one relation is satisfied;

– for each couple $(a_i, a_j)$, $a_i, a_j \in \mathcal{A}$, if one relation is satisfied, any other relation cannot be satisfied.

**Definition 4.** *(Total/Weak/Partial/Quasi (pre)order)*

| | | |
|---|---|---|
| $R$ is a *total order* | *iif.* | $R \in \{\succ, \prec\}$; |
| $R$ is a *weak order*[3] | *iif.* | $R \in \{\succ, \prec, \approx\}$; |
| $R$ is a *partial order* | *iif.* | $R \in \{\succ, \prec, \sim\}$; |
| $R$ is a *quasi-(pre)order* | *iif.* | $R \in \{\succ, \prec, \approx, \sim\}$. |

In this paper, orders without *incomparability* are called *complete orders*, *i.e.* total and weak orders are *complete orders.*

Dissimilarity measure for quasi (pre)orders is compatible with all the other three orders as quasi (pre)order is the most general concept. However, most of the dissimilarity measure methods are designed for complete orders. The difficulty falls on the dissimilarity measure on relation "incomparable" , with more details given in Section 3.

[2]As $a_i \succ a_j$ is equivalent to $a_j \prec a_i$, to avoid repetitive comparisons between two alternatives, we assume that $i < j$ in this article.

[3]In some work, weak order is also referred to as "partial ranking", while "partial order" is another type of preference structure.

*2.2. Theory of belief functions (TBF): notions*

Let $\Omega = \{\omega_1, \ldots, \omega_H\}$ be a finite set representing all possible status of a categorical attribute, the uncertainty and imprecision of this attribute is expressed by Basic Belief Assignment (BBA).

**Definition 5.** *(Basic Belief Assignment (BBA))* A *(normalized) Basic Belief Assignmen (BBA)* on $\Omega$ is a function $m : 2^\Omega \to [0, 1]$ such that:

$$m(\emptyset) = 0 \text{ and } \sum_{X \subseteq \Omega} m(X) = 1. \tag{1}$$

The subsets $X$ of $\Omega$ such that $m(X) > 0$ are called *focal elements*, while the finite set $\Omega$ is called *discernment framework*. $\Omega$ is also considered as *total ignorance* since it represents all the possibilities. A BBA representing *total ignorance* ($m(\Omega) = 1$) is also called a *vacuous* BBA. A BBA is *simple supported* if a non-zero value is assigned only on one singleton and $\Omega$. In this paper, we accept the term *evidential object* as an object described by imperfect information with both uncertainty and imprecision. Thus, an evidential object is represented by a BBA.

**Definition 6.** *(Categorical BBA) A categorical BBA* is a normalized BBA satisfying:

$$m(X) = 1 \ , \forall X \subset \Omega \text{ and } m(Y) = 0 \ , \forall Y \subseteq \Omega, Y \neq X. \tag{2}$$

A categorical BBA on element $X \in 2^\Omega$ is denoted as $X^0$. With the concept of simple support, we name a categorical BBA on one singleton as *categorically simple supported.*

In the process of information fusion, where multiple BBAs are combined into one, various combination rules are available. The selection of a pertinent combination rule depends on many criteria, in which an important one is *cognitively independent.*

**Definition 7.** *(Cognitively Independent Source)* Sources are considered as cognitively independent if the values of BBA on any source has no communication with the others.

The *conjunctive combination rule* proposed by [29] is applied for finding the consensus among multiple reliable and cognitively independent sources, defined as:

$$m_{conj}(X) = \sum_{Y_1 \cap \ldots \cap Y_s = X} \prod_{s=1}^{S} m_s(Y_s). \tag{3}$$

For cognitive dependent sources, the mean value rule is commonly applied. Given a discernment framework $\Omega$ and sources $S$, BBA on source $s \in S$ denoted as $m_s$, for $\forall X \in 2^\Omega$, mean value combination is defined as follows:

$$m_{mean}(X) = \frac{1}{S} \sum_{s=1}^{S} m_s(X). \tag{4}$$

These two combination rules are applied in the evidential preference aggregation process, with details given in Section 4.5.

*2.3. Properties of dissimilarity measures over evidential objects*

*Metrics*, or *distance functions* are usually applied to measure the dissimilarity between two objects.

**Definition 8.** *(Metric (distance))* A *metric (distance)* function $d$ on a set $\mathcal{T}$ is defined as:

$$d : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_{\geq 0},$$

where $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. Given any three elements $t_i, t_j, t_k \in \mathcal{T}$, the following conditions are satisfied:

$$\begin{array}{lr} 1.\ d(t_i, t_j) \geq 0 : & \text{non-negativity;} \\ 2.\ d(t_i, t_j) = 0 \Leftrightarrow t_i = t_j : & \text{identity of indiscernible;} \\ 3.\ d(t_i, t_j) = d(t_j, t_i) : & \text{symmetry;} \\ 4.\ d(t_i, t_j) \leq d(t_i, t_k) + d(t_k, t_j) : & \text{triangle inequality.} \end{array} \tag{5}$$

Usually, the term *dissimilarity* refers to *semi-metric*, in which the property of *triangle inequality* does not hold.

In TBF, different dissimilarity measures are applicable to BBAs. From geometrical and statistical views, these measures are categorized into three parts: distance [27, 30], conflicts[4] [31], and divergence [32]. Comprehensive surveys are respectively available in [32] and [33]. We only introduce the properties required in this paper.

As BBAs are defined in space of $2^\Omega$, in addition to the properties for metrics in Equation set (5), properties considering the structure of discernment are specifically required in TBF. The authors in [33] resumed three structural properties: Strong structural property (with consideration of the interaction between focal elements), weak structural property (with consideration of the cardinality of focal elements), and structural dissimilarity (with consideration of the interaction between sets of focal elements). We accentuate the *strong structural property* hereby, which is the strictest one.

**Property 1. Strong structural property**: A distance measure function $d$ between two BBAs $m_1$ and $m_2$ is *strongly structural* if its definition accounts for the interaction between the focal elements of $m_1$ and $m_2$.

According to [33], the structural property is usually considered as the axiomatic metric property for distance measure between evidence objects. A popular distance that is strongly structural is Jousselme distance [27], defined as:

$$d_{Jousselme}(m_1, m_2) = \sqrt{(\boldsymbol{m_1} - \boldsymbol{m_2})^T \boldsymbol{Jacc}(\boldsymbol{m_1} - \boldsymbol{m_2})}, \tag{6}$$

where $\boldsymbol{Jacc}$ is the matrix whose elements are Jaccard indices:

$$Jacc(X_1, X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|}, \text{for } X_1, X_2 \in 2^\Omega \setminus \emptyset. \tag{7}$$

Matrix $\boldsymbol{Jacc}$ is positive definite and the properties of metric (Equation set (5)) hold in Jousselme distance, proved in [34].

[4] Strictly, "conflict" is not a dissimilarity measure, or semi-metric, as the identity of in-discernment property does not hold.

## 3. Contribution I: A synthetic review of dissimilarity measures over pairwise preferences

Dissimilarity measures over preferences can be grouped into two categories: between pairwise preferences (binary relations) and between list-wise preferences (preference orders). In this section, the former one is focused, where axiomatic distances and Hamming distance are mostly applied. Among these distances, different value settings concerning *incomparability* are accepted, caused by the ambiguous interpretations aforementioned. In this section, starting with a synthetic review of dissimilarity over pair-wise preferences, we introduce how the evidential preference model distinguish these interpretations in a unified framework.

### *3.1. Axiomatic distances*

In axiomatic distances, axioms are used to help to define values of dissimilarity between preference relations. These axioms are extracted either from the definitions or intrinsic properties of preference relations. Axiomatic distances were originally proposed for preference aggregation in distance-based consensus models.

Early research was initiated by Kemeny and Snell [16], where a model for combining preferences (in weak orders) into a group consensus was proposed (a.k.a. Kemeny-Snell model, marked as KS model in this paper). However, the definition given in [16] to the partial order does not include relation *indifference* since it supposes that an alternative $a_i$ is either preferred or incomparable to another alternative $a_j$. J.M. Blin [25] adopted the KS model and proposed an aggregation method by distance minimization. Cook *et al.* [26] (marked as CS model) consider *incomparability* by introducing a *comparison matrix*. In the CS model, the *incomparability* is defined as *not compared* while the axioms respect definitions in Figure 1. Roy and Slowinski [17] (marked as RS model) proposed logical conditions to build a distance measure between pairs of binary relations. The authors in [18] (marked as KM model) respected the RS model and suggested a dissimilarity measure with supplementary conditions. Then, Jabeur *et al.* [19] proposed numeric values for dissimilarity measure between different relations based on [18] (marked as JMK model) by taking the centroid value in a restrained space created by the axioms. In this paper, we refer to the distances applied in different models by corresponding model names.

Several contradictions exist within these models, as depicted in Figure 2, where $\Omega$ denotes the *ignorance* of knowledge. In the RS model, *incomparability* is interpreted as:

> *Incomparability relation is the affirmation of the incapacity to establish the relation type: there is no indifference, no weak preference and no strict preference between the two alternatives* [17].

This interpretation matches the Venn diagram in Figure 2a. However, in other models, such as CS, KM, and JMK models, *incomparability* is interpreted as *the two alternatives are not compared* (*i.e. undecided*). This interpretation corresponds to the illustration in Figure 2b. Axioms accepted in each model are listed in Table 1. These axioms concern the preference relations $\succ, \prec, \approx, \sim, ?$ where the symbol ? represent ignorance induced from the *undecided* case, (*i.e.* $? = \{\succ \cup \prec \cup \approx \cup \sim\}$). Distance function between preference relations is


**(a)** "Incomparability" as indecisive. **(b)** "Incomparability" as undecided.

**Figure 2:** Venn diagrams of the preference relations.

denoted by $d_\Delta$. In order to distinguish the two interpretations of *incomparability*, we remark axioms on the *undecided* version as **BIS**.

**Table 1:** Axioms accepted in different preference models:

| Axiom | Content | Model reference |
|---|---|---|
| Axiom 1: | properties of metric (Equations (5)) | all models |
| Axiom 2: | $d_\Delta(\succ, \approx) = d_\Delta(\prec, \approx)$ and $d_\Delta(\succ, \sim) = d_\Delta(\prec, \sim)$ | RS model |
| **BIS**: | $d_\Delta(\succ, ?) = d_\Delta(\prec, ?)$ | CS and KM models |
| Axiom 3: | $d_\Delta(\succ, \approx) + d_\Delta(\approx, \prec) = d_\Delta(\succ, \prec)$ | all models |
| Axiom 4: | $d_\Delta(\succ, \sim) \leq d_\Delta(\approx, \sim)$ | RS model |
| **BIS** 1: | $d_\Delta(\succ, ?) \leq d_\Delta(\approx, ?)$ | CS model |
| **BIS** 2: | $d_\Delta(\succ, ?) = d_\Delta(\approx, ?)$ | KM model |
| Axiom 5: | $d_\Delta(\approx, \sim) \leq d_\Delta(\approx, \succ)$ | CS model |
| **BIS**: | $d_\Delta(\approx, ?) \leq d_\Delta(\approx, ?)$ | CS and KM models |
| Axiom 6: | $d_\Delta(\succ, \prec) = max(\{d_\Delta(R, R') : R, R' \in \{\succ, \prec, \approx, \sim\}\})$ | all models |

Axiom 2 implies that $\succ$ and $\prec$ are opposite relations, and symmetric to indifference $\approx$. Axiom 6 indicates that strict preference and inverse preference relations are most distinguished. CS and KM models accept the interpretation of *undecided* for *incomparability*, referring to the knowledge of ignorance. CS and KM models confuse the definition of "incomparability" with "ignorance", where "ignorance" is caused by having no knowledge of the preference relation.

Apart from axiomatic preference distances, Minkowski distances based on the encoding of different preference relations are often applied. The most popular one is the Hamming distance.

### *3.2. Hamming distance*

Hamming distance for preference relation is based on the encoding of preference. Given two alternatives $a_i$ and $a_j$, preference relations are encoded in binary by flattening preference matrix, with corresponding Hamming distance values resumed in Table 2.

**Table 2:** Encoding of preference relations and Hamming distance:

| $R$ | $a_i \succ a_j$ | | $a_i \prec a_j$ | | $a_i \approx a_j$ | | $a_i \sim a_j$ | |
|---|---|---|---|---|---|---|---|---|
| | $a_i$ | $a_j$ | $a_i$ | $a_j$ | $a_i$ | $a_j$ | $a_i$ | $a_j$ |
| $a_i$ | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| $a_j$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| Code | 0100 | | 0010 | | 0110 | | 0000 | |

| $d_{\Delta_H}$ | $\succ$ | $\prec$ | $\approx$ | $\sim$ |
|---|---|---|---|---|
| $\succ$ | 0 | 2 | 1 | 1 |
| $\prec$ | 2 | 0 | 1 | 1 |
| $\approx$ | 1 | 1 | 0 | 2 |
| $\sim$ | 1 | 1 | 2 | 0 |

### *3.3. Summary of distances between pair-wise relations*

The inequality relations applied in each distance are summarized in Figure 3. An impor-

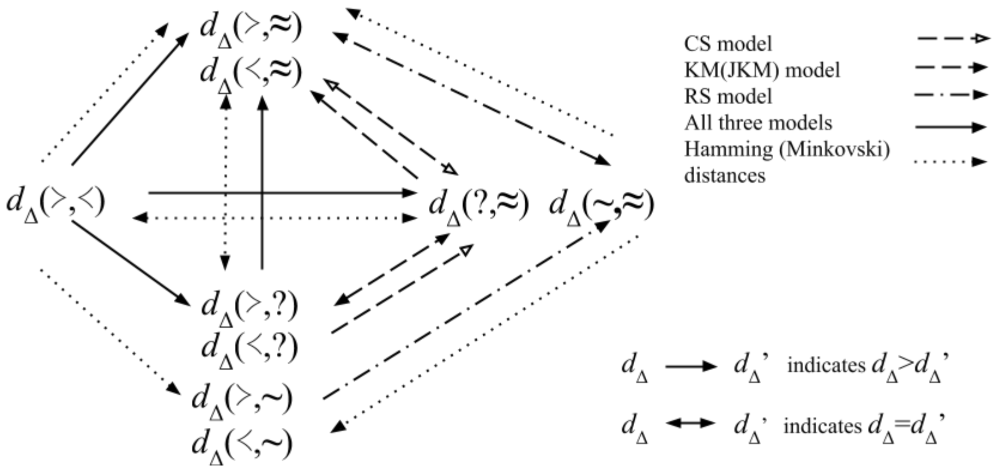



**Figure 3:** Inequality relation applied in axiomatic and Minkowski preference distances.
$d_\Delta$: distance between two preference relations;
?: incomparability as unobserved information (undecided);
$\sim$: incomparability as a specific preference relation (indecisive).

tant advantage of axiomatic methods is that these dissimilarity measures are metrics with the properties guaranteed. However, the above research works show that the dissimilarity measures vary with different axioms accepted, these methods have a strong dependency on the interpretation of different preference relations.

### *3.4. Clarification of "incomparability" in evidential preference model*

The different interpretations of *incomparability* can be simultaneously characterized by the evidential preference model in a unified framework.

The evidential preference model is originally proposed by [8] on weak orders (corresponding to Figure 2b) and extended to quasi orders (corresponding to Figure 2a) with the consideration of *incomparability* by [9]. Consider the case that a set of agents $\mathcal{U}$ expressing their preferences between every pair over alternative set $\mathcal{A}$. Each agent is a source, denoted as $u, u \in \mathcal{U}$. For any alternative pair $a_i, a_j \in \mathcal{A}$, four relations are possible. Therefore, the discernment framework $\Omega_{ij}^{pref}$ is defined as:

$$\Omega_{ij}^{pref} = \{\omega_{ij}^{R} | R \in \{\succ, \prec, \approx, \sim\}\}. \tag{8}$$

In evidential preference, the case of *undecided* is characterized by total ignorance $\Omega_{ij}^{pref}$, while the case of *indecisive* by a specific binary relation *incomparability* $\omega_{ij}^{\sim}$. Therefore, different interpretations are distinguished in a unified framework. The degree of uncertainty on preference relation is represented by values on singletons, which can be regarded as a fuzzy value[5]. The imprecision is characterized by values on union sets.

With the combination rules in the framework of the theory of belief functions, the evidential preference model is effective in group decision-making with imperfect preference information sources. A more in-depth discussion on this topic is given in Section 4.5.

## 4. Contribution II: USP distance - A novel measure suitable for evidential preferences

Despite that evidential preference can faithfully express all types of preference, none of the existing distances in the context of the theory of belief functions is able to properly measure the dissimilarity between evidential preferences, with aforementioned inequality relations respected. In this section, we analyze the properties that disable a relevant dissimilarity measure and propose a new distance adapted to evidential preferences, namely Unequal Singleton Pair (USP) distance. The analysis starts with a detailed introduction to this flaw.

### *4.1. A flaw in the dissimilarity measure over evidential preferences*

The fact that existing distances in the theory of belief functions are not relevant for evidential preference is caused by a contradiction between their required properties. In Section 3, it is concluded that Equation (9) is always applied in measuring the dissimilarity between preference types.

$$d_{\Delta}(\succ, \prec) > d_{\Delta}(\succ, \approx). \tag{9}$$

This inequality relation is rewritten as Equation (10) adapted to evidential preferences.

$$d(\{\omega^{\succ}\}^0, \{\omega^{\prec}\}^0) > d(\{\omega^{\succ}\}^0, \{\omega^{\approx}\}^0), \tag{10}$$

where $\{\omega^{R}\}^0$ represents the categorical BBA on singleton element $\omega^{R}$ (see Definition 6).

However, the existing dissimilarity measures do not distinguish values between singletons in TBF, thus, none of them satisfies Equation (10). In TBF, a discernment framework ($\Omega = \{\omega_1, \omega_2, \ldots, \omega_H\}$) requires that all the singleton elements are mutually exclusive (see

[5]This fuzzy value does not represent a fuzzy preference. Conventional fuzzy preference express the relation between two alternatives by only one membership with 1 representing strict preference, 0 the reverse strict preference, and 0.5 the indifference.

Section 2.2). With the property of exclusivity, any two elements $\omega_i, \omega_j \in \Omega$ are either identical or distinguished, with respect to a binary relation. Existing distances apply this binary relation by an assumption that dissimilarity values between singleton elements are equal, precisely:

**Assumption 1.** In a discernment framework $\Omega = \{\omega_1, \omega_2, \ldots, \omega_H\}$, the dissimilarity between any two different singletons is a constant (normalized as 1), formally, $\forall \omega_m, \omega_n \in \Omega, \omega_m \neq \omega_n$,

$$d(\{\omega_m\}^0, \{\omega_n\}^0) \equiv 1. \tag{11}$$

Obviously, Equation (10) and Equation (11) are contradictory. As a consequence, Assumption 1 is not suitable for measuring the dissimilarity between evidential preferences. To propose a theoretically correct solution, an analysis of necessary properties for evidential preferences is conducted.

*4.2. Dissimilarity measure properties for evidential preferences*

Three essential properties are required for dissimilarity measures on evidential preferences.

**Property 2.** The dissimilarity is measured by a metric, *i.e.* the properties of metric (Definition 8) hold.

**Property 3.** The dissimilarity measure takes the difference of dissimilarity between singletons into account.

Property 3 abandons Assumption 1. With the account that dissimilarities are measured between BBAs in the space of $2^\Omega$, the following property must hold as well, matching the property of *strongly structural* (see Property 1 in Section 2.3), Specifically described as:

**Property 4.** Dissimilarity function $d$ forms with such structure:

$$\forall X, Y \in 2^\Omega, d(X^0, Y^0) \leq d(\{X \setminus (X \cap Y)\}^0, Y^0). \tag{12}$$

By applying these properties on weak orders, represented by $\Omega = \{\omega^{\succ}, \omega^{\prec}, \omega^{\approx}\}$, the dissimilarity values are set as:

$$d(\{\omega^{\succ}\}^0, \{\omega^{\prec}\}^0) = 1; \tag{13a}$$

$$d(\{\omega^{\succ}\}^0, \{\omega^{\approx}\}^0) = d_\Delta(\prec, \approx) = p. \tag{13b}$$

where $p$ is a dissimilarity value between 0 and 1 (strictly smaller than 1).

With respect to Equation (12), the three relations below yield:

1. Between *strict preference* $\succ$ and union with *indifference* $\approx$:

$$d(\{\omega^{\succ}\}^0, \{\omega^{\succ}, \omega^{\approx}\}^0) < d(\{\omega^{\succ}\}^0, \{\omega^{\approx}\}^0)$$
$$\Rightarrow d(\{\omega^{\succ}\}^0, \{\omega^{\succ}, \omega^{\approx}\}^0) \in (0, p). \tag{14a}$$

Symmetrically,

$$d(\{\omega^{\prec}\}^0, \{\omega^{\prec}, \omega^{\approx}\}^0) < d(\{\omega^{\prec}\}^0, \{\omega^{\approx}\}^0)$$
$$\Rightarrow d(\{\omega^{\prec}\}^0, \{\omega^{\prec}, \omega^{\approx}\}^0) \in (0, p). \tag{14b}$$

2. Between strict preference $\succ$ and total ignorance (missing data):

$$d(\{\omega^{\succ}\}^0, \Omega^0) > d(\{\omega^{\succ}\}^0, \{\omega^{\succ}, \omega^{\approx}\}^0). \tag{15a}$$

Symmetrically,

$$d(\{\omega^{\prec}\}^0, \Omega^0) > d(\{\omega^{\prec}\}^0, \{\omega^{\prec}, \omega^{\approx}\}^0). \tag{15b}$$

3. Between strict preference $\succ$ and union of other two preference types ($\prec$ and $\approx$):

$$d(\{\omega^{\succ}\}^0, \{\omega^{\approx}\}^0) < d(\{\omega^{\succ}\}^0, \{\omega^{\prec}, \omega^{\approx}\}^0) < d(\{\omega^{\succ}\}^0, \{\omega^{\prec}\}^0). \tag{16a}$$

Symmetrically,

$$d(\{\omega^{\prec}\}^0, \{\omega^{\approx}\}^0) < d(\{\omega^{\prec}\}^0, \{\omega^{\succ}, \omega^{\approx}\}^0) < d(\{\omega^{\succ}\}^0, \{\omega^{\prec}\}^0). \tag{16b}$$

To apply Properties 2, 3, and 4, we propose Unequal Singleton Pair (USP) distance as a solution.

### *4.3. USP distance–a distance for unequal singleton pairs over BBAs*

In TBF, Jousselme distance[27] is widely applied because this distance is strongly structural by taking into account the imprecision in the structure with respect to Property 2 and Property 4. Inspired by the methodology in the conception of Jousselme distance, we propose a novel distance named Unequal Singleton Pair (USP) distance, with the Property 3 imposed.

Assumption 1 is guaranteed in by Jaccard index (Equation (6)), with Property 4, recalled here:

$$Jacc(X_1, X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|}, \text{for } X_1, X_2 \in 2^{\Omega} \setminus \emptyset.$$

To drop Assumption 3, we define USP distance by modifying Jaccard index matrix as follows.

$$d_{USP}(m_1, m_2) = \sqrt{(\boldsymbol{m_1} - \boldsymbol{m_2})^T \boldsymbol{Sim} (\boldsymbol{m_1} - \boldsymbol{m_2})}, \tag{17}$$

where $\boldsymbol{Sim}$ differs from $\boldsymbol{Jacc}$, also referring to the similarity matrix between different singleton elements. In $\boldsymbol{Sim}$, the similarity value between two elements $X_1$ and $X_2$, $(X_1, X_2 \in 2^{\Omega})$ is denoted as $sim(X_1, X_2)$. In the first place, with a similar form as Jaccard index, $sim(X_1, X_2)$ is defined as the division between two virtual functions: $resemb(X_1, X_2, \ldots, X_M)$ and $entire(X_1, X_2, \ldots, X_M)$:

$$sim(X_1, X_2) = \frac{resemb(X_1, X_2)}{entire(X_1, X_2)}, \tag{18}$$

where $resemb(X_1, X_2)$ describes the cause of the similarity between $X_1$ and $X_2$ (say *resemblance*), and $entire(X_1, X_2)$ the entire part concerned by $X_1$ and $X_2$. $resemb(\cdot)$ and $entire(\cdot)$ are defined as virtual functions. With Assumption 1, the functions $resemb_{ass1}(X_1, X_2, \ldots, X_M)$ and $entire_{ass1}(X_1, X_2, \ldots, X_M)$ are overrode as:

$$resemb_{ass1}(X_1, X_2, \ldots, X_M) = \mid \bigcap_{i=1}^{M} X_i \mid, \tag{19}$$

$$entire_{ass1}(X_1, X_2, \ldots, X_M) = \mid \bigcup_{i=1}^{M} X_i \mid. \tag{20}$$

Obviously, the similarity matrix $\boldsymbol{Sim}$ is identical to Jaccard matrix $\boldsymbol{Jacc}$.

Here, we propose a new method to drop Assumption 1. Define a set of elements in $2^\Omega$, $W = \{X_1, X_2, \ldots, X_M\}$, therefore $W \subseteq 2^\Omega$. Denote $resemb(W)$ for $resemb(X_1, X_2, \ldots, X_M)$ and $entire(W)$ for $entire(X_1, X_2, \ldots, X_M)$ to simplify the expression. The size of $W$ is defined by the number of elements $X \in 2^\Omega$, denoted by $|W|$. Singletons in $W$ is defined by the union of all elements in $W$, formally:

$$\cup W = \bigcup_{X_i \in W} X_i. \tag{21}$$

Denote the subset of $W$ by $W_{sub}$, by dropping Assumption 1, $entire(W)$ is defined as a generalized version of cardinal function on the union sets:

$$\begin{aligned} entire(W) &= \sum_{\omega \in \cup W} entire(\omega) - \sum_{W_{sub} \subseteq W, |W_{sub}|=2} resemb(W_{sub}) \\ &\quad + \sum_{W_{sub} \subseteq W, |W_{sub}|=3} resemb(W_{sub}) - \sum_{W_{sub} \subseteq W, |W_{sub}|=4} resemb(W_{sub}) \\ &\quad + \ldots + \sum_{W_{sub} \subseteq W, |W_{sub}|=|2^\Omega|} resemb(W_{sub}) \times (-1)^{|2^\Omega|} \\ &= \sum_{\omega \in \cup W} entire(\omega) + \sum_{t=1}^{|2^\Omega|} \sum_{W_{sub} \subseteq W, |W_{sub}|=t} resemb(W_{sub}) \times (-1)^t. \end{aligned} \tag{22}$$

In practice, with only the similarity between two singleton elements given, a unique solution for $entire(W)$ may not be reached by Equation (22). To enable and simplify the calculation, two more assumptions are imposed, defined as follows.

**Assumption 2.** In a set of exclusive elements $\Omega = \{\omega_1, \omega_2, \ldots, \omega_H\}$, any resemblance is only shared by maximal two elements, *formally*:

$$resemb(W) = 0, \quad \forall W \subseteq 2^{\Omega}, |W| \geq 3. \tag{23}$$

Assumption 2 guarantees that a unique solution exists for $resemb(\omega_m, \omega_n)$ with only pairwise similarity $sim(\omega_m, \omega_n)$ given.

**Assumption 3.** To normalize the element values in the calculation, *entire* part concerned by one singleton is assigned to 1, formally:

$$entire(\omega) = 1, \quad \forall \omega \in \Omega. \tag{24}$$

With Assumptions 2 and 3, given similarity $sim(\omega_m, \omega_n)$, the resemblance is calculable and a unique solution exists. Deduced from Equation (22), we have:

$$entire(X, Y) = \sum_{\omega \in X \cup Y} entire(\omega) - \sum_{\substack{\omega_m \in X \\ \omega_n \in Y \\ m \neq n}} resemb(\omega_m, \omega_n). \tag{25}$$

Hence, Equation (18) is overrode as:

$$sim(X_1, X_2) = \frac{\sum\limits_{\substack{\omega_m \in X_1 \\ \omega_n \in X_2 \\ m \neq n}} resemb(\omega_m, \omega_n)}{\sum\limits_{\omega \in X_1 \cup X_2} entire(\omega) - \sum\limits_{\substack{\omega_m \in X_1 \\ \omega_n \in X_2 \\ m \neq n}} resemb(\omega_m, \omega_n)}. \tag{26}$$

To guarantee the Assumption 2 given by Equation (23), the following condition holds:

$$\sum_{\substack{\omega_m, \omega_n \in \Omega \\ \omega_m \neq \omega_n}} sim(\omega_m, \omega_n) \leq 1. \tag{27}$$

Property 2 (property of the metric system) is guaranteed because the USP distance is in the format of the Mahalanobis distance. Moreover, extended from the Jousselme distance, Property 4 is kept by different values in the similarity matrix $\boldsymbol{Sim}$.

For a better understanding, we demonstrate the inference and calculation of the USP distance in the evidence preference by a graphical illustration, followed by an application of decision-making thereafter.

### *4.4. Illustrative example-USP for evidential preferences*

The calculation of the similarity matrix can be aided by graphics as depicted in Figure 4. Evidential preference is taken as an example, with four singletons elements: $\omega^{\succ}, \omega^{\prec}, \omega^{\approx}$ and $\omega^{\sim}$. In accordance with the exclusivity of the discernment framework, Figure 4a shows that singletons do not share any overlapped parts. Obviously, Figure 4a is isomorphic with Figure 1. In the conception of USP, we assume that the pairwise dissimilarity difference between singletons can be described by an overlapped zone in another space than the one in which they are defined. More precisely, outside the original $\mathcal{S}$ space, the singletons can have an overlapping part in a different $\mathcal{S}^*$ space, represented by the overlapping space between singletons in figure 4b. In this way, the overlapped part could be used to show the difference in dissimilarities between the singletons elements.


**(a)** Element in original space $\mathcal{S}$.

**(b)** Element in space $\mathcal{S}^*$ with different similarity values.

**Figure 4:** Graphical representation of similarity matrix calculation.

Thus, the overlap area in space $\mathcal{S}^*$ allows calculating the similarity between simple elements. In addition, the constraints set by the functions Assumption 2 and Assumption 3 are respectively interpreted as *the overlapped zones are shared by 2 elements*, and *the area of an element is valued as 1.*

Table 3 gives the similarity values between $\omega^R, R \in \{\succ, \prec, \approx, \sim\}$. The function $resemb(\cdot)$

**Table 3:** Similarity $sim_{\mathcal{S}^*}$ between singletons:

| $sim(\cdot)$ | $\omega^{\succ}$ | $\omega^{\prec}$ | $\omega^{\approx}$ | $\omega^{\sim}$ |
|---|---|---|---|---|
| $\omega^{\succ}$ | 1 | 0 | x | y |
| $\omega^{\prec}$ | 0 | 1 | x | y |
| $\omega^{\approx}$ | x | x | 1 | z |
| $\omega^{\sim}$ | y | y | z | 1 |

valuing the overlapped areas are assigned with $p_1, p_2, p_3$ corresponding to Figure 4 as follows:

$$\begin{aligned} resemb(\omega^{\succ}, \omega^{\approx}) = resemb(\omega^{\prec}, \omega^{\approx}) &= p_1; \\ resemb(\omega^{\succ}, \omega^{\sim}) = resemb(\omega^{\prec}, \omega^{\sim}) &= p_2; \\ resemb(\omega^{\approx}, \omega^{\sim}) &= p_3. \end{aligned} \tag{28}$$

Plugging Equation (28) in Equations (24) and (25), the following relations are established:

$$\begin{cases} x = \dfrac{p_1}{2-p_1} & \Rightarrow p_1 = \dfrac{2x}{1+x}; \\ y = \dfrac{p_2}{2-p_2} & \Rightarrow p_2 = \dfrac{2y}{1+y}; \\ z = \dfrac{p_3}{2-p_3} & \Rightarrow p_3 = \dfrac{2z}{1+z}. \end{cases} \tag{29}$$

Thus, similarities in $2^\Omega \times 2^\Omega$ are calculated by applying Equation (26), and Assumption 2 yields:

$$\begin{aligned} 2p_1 + p_3 \leq 1 \quad &\Rightarrow \frac{4x}{1+x} + \frac{2z}{1+z} \leq 1 \quad \Rightarrow \frac{2}{x+1} + \frac{1}{z+1} \leq \frac{5}{2}; \\ 2p_2 + p_3 \leq 1 \quad &\Rightarrow \frac{4y}{1+y} + \frac{2z}{1+z} \leq 1 \quad \Rightarrow \frac{2}{y+1} + \frac{1}{z+1} \leq \frac{5}{2}; \\ p_1 + p_2 \leq 1 \quad &\Rightarrow \frac{2x}{1+x} + \frac{2y}{1+y} \leq 1 \quad \Rightarrow \frac{1}{x+1} + \frac{1}{y+1} \leq \frac{3}{2}. \end{aligned} \tag{30}$$

Equation set (30) defines the limit of value setting, guarantee axiomatic properties, and metric properties. With this dissimilarity value calculation method, we apply USP distance in group decision-making applications.

### *4.5. Application: Group decision making*

The USP distance finds its application in group decision making between pairwise alternatives, represented by $a_i, a_j \in \mathcal{A}$. Respectively, the discernment framework is designated by $\Omega_{ij}$ (see Equation (8)) and the decided relation by $R_{ij}$. The procedure consists of two steps: combination and decision, as shown in Figure 5.

$$\left.\begin{matrix} m_1^{\Omega_{ij}} \\ \vdots \\ m_s^{\Omega_{ij}} \end{matrix}\right\} \xrightarrow{\boxed{\text{Conjunctive combination}}} m^{\Omega_{ij}} \xrightarrow{\boxed{\text{Decision}}} R_{ij} \in \{\succ, \prec, \approx, \sim\}$$

**Figure 5:** Group decision making procedure.

As mentioned above, in CS and KM models, *incomparability* interprets the absence of information, (referred to as *undecided*), however, the final consensus decision strategies are possible to converge to the absence of information (*undecided*) as a consequence, which is meaningless in decision making. To solve this problem, the conjunctive combination rule for BBAs is more adaptable. More specifically, LNS-CR (Conjunctive Rule for Large Number of Sources) [35] is applied as it is suitable for the aggregation of preferences without uncertainty, represented by categorically simple supported BBAs (see Definition 6).

In the decision step, a distance-based strategy is applied, which is originally introduced in [36], defined by:

$$R_{ij} = \underset{R \in \{\succ,\prec,\approx,\sim\}}{argmin} (d(m^{\Omega_{ij}}, \{\omega^R\}^0)), \tag{31}$$

where $m^{\Omega_{ij}}$ denotes a BBA for the evidential preference between the alternatives $a_i$ and $a_j$, and $\{\omega^R\}^0$ the categorical BBAs respectively representing the relations $\succ, \prec, \approx$, and $\sim$. With USP distance applied in the decision rule above, the dissimilarities in singleton pairs are significantly discriminated against. The advantages are further discussed with experiments in Section 5.

## 5. Illustrative examples and experiments

In this section, three experiments are conducted. In the first experiment, we compare decision-making methods by examples on synthetic data with traditional methods, the impact of missing data is particularly studied. In the second experiment, we compare the differences between USP distance and Jousselme distance in decision making, particularly confronting a Condorcet Paradox. Afterwards, in the third experiment, group decision making is evaluated based on real-world Sushi preference data.

### *5.1. On synthetic data*

In the first and second experiments, preferences are generated in quasi order with missing data. That is to say, between alternatives $a_i$ and $a_j$, in addition to the four preference relations $\succ$, $\prec$, $\approx$, and $\sim$ (*indecisive*), the case of missing information (*undecided*) is also considered, denoted as ?.

#### *5.1.1. Exp. 1: Impact of missing data in preference aggregation*

Firstly, preference aggregation results with different numbers of each preference relation are compared. Following aggregation rules compatible with quasi order are considered: Borda rule, distance-based consensus rule with KM distance and CS distance, evidential preference with Jousselme distance (denoted as $EP_J$), and with USP distance (denoted as $EP_{USP}$). In Section 3.1, we mentioned that KM and CS models are not applicable to measuring incomparability as *indecisive*. To make a fair comparison, preferences with incomparability interpreted as *undecided* (denoted with the symbol ?) rather than *indecisive* ($\sim$) are generated. Since traditional preference models without uncertainty are compared, BBAs are categorical in the evidential preference model.

To calculate USP distance, similarity values corresponding to Figure 6 are applied, where extreme values are taken, *i.e.* the equality conditions in Formulas (30) are accepted. Table 4 illustrates the agent number settings, with $\#_\succ$, $\#_\prec$, $\#_\approx$, $\#_\sim$. $\#_?$ denoting the number of agents giving each preference relation. These numbers are particularly set to highlight the differences among aggregation strategies.

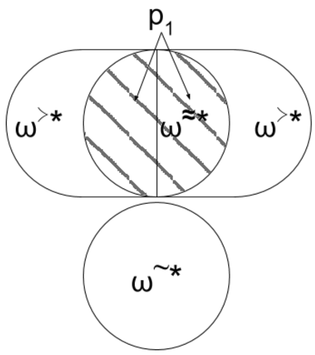
Aggregation results are illustrated in Table 5. $EP_XS$and $EP_X\Omega$ respectively represent the decision on a *singleton* (in space of $\Omega^{pref}$) and the power set (in space of $2^{\Omega^{pref}}$) with the evidential preference model. $X$ is written as $J$ and $USP$ to respectively denote Jousselme distance and USP distance. Only non-zero values in the combined BBAs are illustrated in

**Table 4:** Experiment settings of Exp. 1:

| $SettingN°$ | $\#_{\succ}$ | $\#_{\prec}$ | $\#_{\approx}$ | $\#_{\sim}$ | $\#_{?}$ |
|---|---|---|---|---|---|
| *1* | 5 | 4 | 4 | 0 | 7 |
| *2* | 5 | 3 | 3 | 0 | 20 |
| *3* | 5 | 5 | 4 | 0 | 7 |
| *4* | 5 | 5 | 5 | 0 | 0 |
| *5* | 5 | 5 | 5 | 0 | 20 |
| *6* | 5 | 5 | 0 | 0 | 0 |
| *7* | 5 | 0 | 5 | 0 | 0 |
| *8* | 5 | 0 | 5 | 0 | 20 |
| *9* | 5 | 0 | 4 | 0 | 20 |

**Table 5:** Results of Exp. 1:

| $N°$ | KM | CS | Borda | $EP_J$ S | $EP_J\Omega$ | $EP_{USP}$ S | $EP_{USP}$ $\Omega$ | combined BBA singletons $m$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | $\omega^{\succ}$ | $\omega^{\prec}$ | $\omega^{\approx}$ | $\Omega$ |
| *1* | ? | $\approx$ | $\succ$ | $\succ$ | $\Omega$ | $\approx$ | $\succ\cup\prec\cup\approx$ or $\Omega$ | 0.184 | 0.131 | 0.131 | 0.295 |
| *2* | ? | ? | $\succ$ | $\succ$ | $\Omega$ | $\succ$ | $\succ\cup\prec\cup\approx$ or $\Omega$ | 0.184 | 0.131 | 0.131 | 0.295 |
| *3* | ? | $\approx$ | $\approx$ | $\succ$ or $\prec$ | $\Omega$ | $\approx$ | $\succ\cup\prec\cup\approx$ or $\Omega$ | 0.164 | 0.164 | 0.118 | 0.295 |
| *4* | $\approx$ | $\approx$ | $\approx$ | $\succ$ or $\prec$ or $\approx$ | $\Omega$ | $\approx$ | $\succ\cup\prec\cup\approx$ or $\Omega$ | 0.148 | 0.148 | 0.148 | 0.296 |
| *5* | ? | ? | $\approx$ | $\succ$ or $\prec$ or $\approx$ | $\Omega$ | $\approx$ | $\succ\cup\prec\cup\approx$ or $\Omega$ | 0.148 | 0.148 | 0.148 | 0.296 |
| *6* | $\succ$ or $\prec$ | $\succ$ or $\prec$ or $\approx$ | $\approx$ | $\succ$ or $\prec$ | $\approx$ or $\Omega$ | $\succ$ or $\prec$ | $\succ\cup\prec$ or $\succ\cup\prec\cup\approx$ or $\succ\cup\prec\cup\sim$ or $\Omega$ | 0.25 | 0.25 | 0 | 0.25 |
| *7* | $\succ$ or $\approx$ | $\succ$ or $\approx$ | $\succ$ | $\succ$ or $\approx$ | $\succ\cup\approx$ or $\Omega$ | $\succ$ or $\approx$ | $\succ\cup\approx$ | 0.25 | 0 | 0.25 | 0.25 |
| *8* | ? | ? | $\succ$ | $\succ$ or $\approx$ | $\succ\cup\approx$ or $\Omega$ | $\succ$ or $\approx$ | $\succ\cup\approx$ | 0.25 | 0 | 0.25 | 0.25 |
| *9* | ? | ? | $\succ$ | $\succ$ | $\succ\cup\approx$ | $\succ$ | $\succ\cup\approx$ | 0.3 | 0 | 0.196 | 0.247 |




| $sim$ | $\succ$ | $\prec$ | $\approx$ | $\sim$ |
|---|---|---|---|---|
| $\succ$ | 1 | 0 | 1/3 | 0 |
| $\prec$ | 0 | 1 | 1/3 | 0 |
| $\approx$ | 1/3 | 1/3 | 1 | 0 |
| $\sim$ | 0 | 0 | 0 | 1 |

**Figure 6:** Similarity value settings and graphical illustration in Exp. 1.

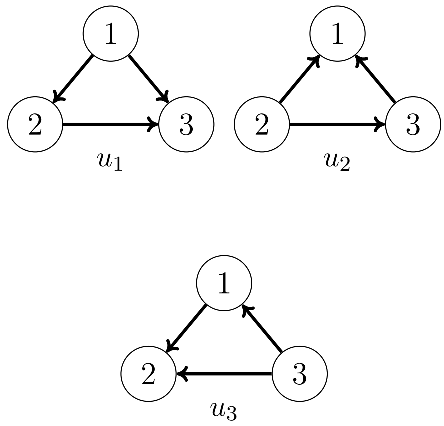
column "combined BBA" (without the value on $\emptyset$). From the results, we draw the following conclusions:

1. In $N°1, 2, 3, 5, 8, 9$, KM and CS models return *undecided* case (denoted by ?). This indicates that consensus rules based on axiomatic distances are not appropriate when a big part of data is missing.
2. From $N°1\tilde{}6$, we observe that $\Omega$ (equivalent to the *undecided* case ?) is one of the possible decision results in $EP_J\Omega$ and $EP_{USP}\Omega$. This issue is similar to the above one and reflects that decisions in space of $2^\Omega$ are easily influenced by missing data.
3. The decision result from the Borda rule is always precise. However, relations $\approx$, $\sim$ and ? are not distinguished. Indeed, the Borda rule merely compares the number of $\succ$ and $\prec$ with other types of preferences neglected.
4. In strategies for evidential preference model, between two distances in settings $N°$ 3 and 4, $EP_{USP}S$ gives a more precise result than $EP_JS$. This is because those decisions on singletons are discriminated against with USP distance.

*5.1.2. Exp. 2: Condorcet's paradox avoidance in preference aggregation*

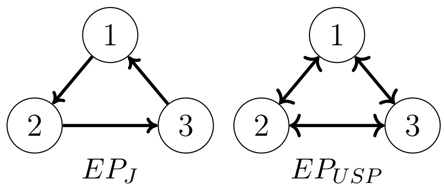
Condorcet's paradox is a traditional issue in preference aggregation problems, reflecting a phenomenon that collective preferences are cyclic while all agents' preferences are not. An initial discussion and with an avoidance solution evidential preference model is introduced in [9]. In this experiment, we demonstrate different results between $EP_{USP}$ and $EP_J$ strategies concerning Condorcet's paradox.

In Figure 7, we generate BBA of three agents $\mathcal{U} = \{u_1, u_2, u_3\}$ expressing their uncertain preferences on three alternatives $\mathcal{A} = \{a_1, a_2, a_3\}$. The aggregation results decided by rules $EP_J$ and $EP_{USP}$ are illustrated in Figure 8. It can be observed that $EP_J$ decision strategy returns a *cyclic* preference order, indicating Condorcet's paradox, while $EP_{USP}$ decision strategy returns *indifference* among the three alternatives. This shows the advantage of USP distance. Still, we need to point that $EP_{USP}$ does not definitively eliminate Condorcet's paradox. The example here demonstrates only a specific case. Condorcet's paradox is not definitively avoided by $EP_{USP}$. To avoid Condorcet's paradox, we develop an elimination method for the evidential preference model.




| | $m(\omega^{\succ})$ | $m(\omega^{\prec})$ | $m(\omega^{\approx})$ |
|---|---|---|---|
| BBA of agent $u_1$ | | | |
| $(a_1, a_2)$ | 0.7 | 0 | 0.3 |
| $(a_1, a_3)$ | 0.9 | 0 | 0.1 |
| $(a_2, a_3)$ | 0.7 | 0 | 0.3 |
| BBA of agent $u_2$ | | | |
| $(a_1, a_2)$ | 0 | 0.9 | 0.1 |
| $(a_1, a_3)$ | 0 | 0.7 | 0.3 |
| $(a_2, a_3)$ | 0.7 | 0 | 0.3 |
| BBA of agent $u_3$ | | | |
| $(a_1, a_2)$ | 0.7 | 0 | 0.3 |
| $(a_1, a_3)$ | 0 | 0.7 | 0.3 |
| $(a_2, a_3)$ | 0 | 0.9 | 0.1 |

**Figure 7:** BBA value settings of evidential preferences from 3 agents.




| | $m(\emptyset)$ | m($\omega^{\succ}$) | m($\omega^{\prec}$) | m($\omega^{\approx}$) | $m(\Omega)$ |
|---|---|---|---|---|---|
| $(a_1, a_2)$ | 0.147 | 0.186 | 0.075 | 0.166 | 0.426 |
| $(a_1, a_3)$ | 0.147 | 0.075 | 0.186 | 0.166 | 0.426 |
| $(a_2, a_3)$ | 0.147 | 0.186 | 0.075 | 0.166 | 0.426 |

**Figure 8:** Decision results from two BFpref consensus rules.

We denote the set of edges (preference relations) as:

$$\mathcal{R}_{Condorcet} = \{R_{ij}\}, R_{ij} \in \{\succ, \prec\},$$

where $i, j$ are the index of alternative pair $(a_i, a_j)$ in the cyclic graph representing Condorcet's paradox. Among all edges in the cyclic graph, the one whose corresponding BBA is closest to the categorical BBA of its opposite relation $\{\omega^{R_{ij}^{oppo}}\}^0$ is searched, where $R_{ij}^{oppo}$ represent the opposite relation to $R_{ij}$.Then, this edge is replaced by its opposite relation. Formally, the edge to be replaced, denoted by $R_{replace}$, is determined by:

$$R_{replace} = \underset{R_{ij} \in \mathcal{R}_{Condorcet}}{argmin} \; (d(m^{\Omega_{ij}}, \{\omega^{R_{ij}^{oppo}}\}^0)). \tag{32}$$

This method takes the idea from Dodgson's voting method [37], and it degrades to Dodgson's method in traditional cases when preferences are certain. This method for Condorcet's paradox elimination is applied in the following experiment on data from the real world.

### *5.2. Exp. 3: Aggregation of conflicting preferences over Sushi preference dataset*

In this part, we demonstrate the group decision-making with USP distance in the evidential preference model on the preference dataset [22] (referred to as SUSHI in the following text). In this dataset, only three preference relations exist between any two alternatives:

$\succ, \prec$ and $\approx$. Missing data are considered as total ignorance in the evidential preference model as $\succ \cup \prec \cup \approx$.

Several most recent Multi-Criteria Decision-Making (MCDM) methods are compared with our method. [38–41]. We have chosen PROMETHEE-EDAS [39], TOPSIS-WAA [40], and 3WD-MADM method [41]. These three methods are fuzzy extensions of traditional PROMETHEE [42], TOPSIS [43] and ELECTRE I [44] respectively, which are classic MCDM solutions. Besides, these methods are recent enough to demonstrate the effectiveness of the proposed method. It should be pointed out that the group decision-making method based on evidential preference is only adaptable for mono-criteria, where each alternative is represented by a single value. To adjust the MCDM method for group decision making, the score of each agent is considered as a criterion of alternatives. For example, the scores given by agents $u_1, u_2, u_3$ alternatives $a_1, a_2, a_3$ are represented by matrix $H$, as shown in Table 6. The weight of each criterion (agent) is identical and all agents play a beneficial role in MCDM.

**Table 6:** Example: preference data in score:

| Score | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|
| $a_1$ | 4 | 3 | 2 |
| $a_2$ | 1 | 3 | 2 |
| $a_3$ | 3 | 0 | 4 |

**Table 7:** Example: Scores converted from preference data in order:

| Order No. | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|
| $a_1$ | 3rd | 2nd | 1st |
| $a_2$ | 3rd | 1st | 2nd |
| $a_3$ | 2nd | 3rd | 1st |

$\Rightarrow$

| Converted score | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|
| $a_1$ | 0 | 1 | 2 |
| $a_2$ | 3 | 1 | 2 |
| $a_3$ | 2 | 3 | 1 |

The order data are converted into a score from 9 to 0, with the most preferred given 9 and least given 0. To apply the fuzzy MCDM method and normalize the score, both score and order data are fuzzified by:

$$Fuzzy(a_i, u_j) = \frac{Score(a_i, u_j)}{max(Score(a_i, u)|u \in \mathcal{U}))}, \tag{33}$$

where $H_{ij}$ represents score on alternative $a_i$ from agent $u_j$.

These methods do not consider incomplete cases. Therefore, they are not directly applicable for Sushi preference data. For comparison, missing data are completed by the average value of all existing data from agents in the corresponding region. In the 3WD-MADM method, only the average strategy is compared. Due to the consequence of the data completion strategy that alternatives are close, in parameter selection of ELECTRE I for outranking calculation, we have chosen a relatively small concordance threshold $p_{ELECTRE} = 0.55$ and large discordance threshold $q_{ELECTRE} = 0.45$. For a similar reason, the threshold in PROMETHEE-EDAS linear preference function $p_{PROMETHEE}$ should be small. We fix

$p_{PROMETHEE} = 0.05$ in the preference function integrated within PROMETHEE-EDAS. Without specific description, parameters are selected by default values as in [39–41].

*Introduction to Sushi preference dataset*

SUSHI gathers the preferences of voluntary agents located geographically in the *eastern* region (3257 agents) and *western* region (1742 agents) across Japan (respectively referred to as East and West Japan). Agents are required to express their preference over 10 sushis randomly selected from 100 sushis. The preferences are expressed in two forms: score and order. Scores range from 0 to 4 (5 levels in total). The sparsity of data is therefore 90%. In this case (preferences are given in two forms), the contradiction is inevitable. For example, between two sushis *magoru* and *ebi*, an agent may give *magoru* $\succ$ *ebi* in order but with identical scores. Details of the data set are depicted in Table 8.

**Table 8:** Sushi Preference Dataset:

| | # agents | # alternatives | sparsity | conflict between order and score | remark |
|---|---|---|---|---|---|
| East Region | 3257 | 100 | 90% | 3.45‰ | more oily sushi |
| West Region | 1742 | 100 | 90% | 3.44‰ | less oily sushi |

The study of Japanese food sociology tells some ground truth knowledge. People from East and West Japan have different taste preferences on Sushis. Generally, Japanese from East Japan prefer more oily and saltier food than from West Japan. The oily level denoted by *oil* $\in [0, 4]$ is also provided for each sushi type. Originally, level 0 refers to the oiliest one and 4 the least. For the consistency with literal significance, we inverse the order, *i.e.* 0 refers to the least oily and 4 the most.

*Aggregation and evaluation*

In SUSHI, the responses in score and order from an identical agent are cognitively *dependent*. Therefore, the mean rule (Equation (4)) is suitable for the aggregation of personal preference. The preference aggregation procedure is illustrated in Figure 9, as an extended version to Figure 5 in Section 4.5.

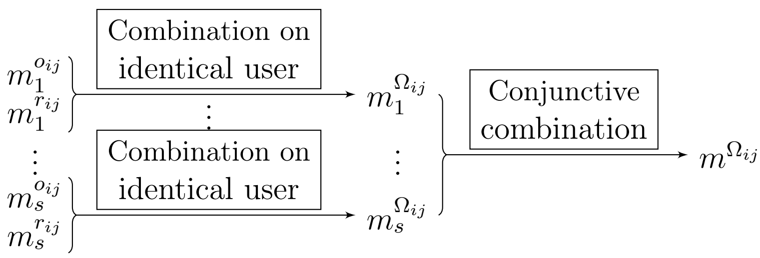



**Figure 9:** Combination procedure for sushi preferences.

The evaluation of preference aggregation results is tricky. If an evaluation criterion on aggregated preference is fixed, it is possible to find a method based on the optimization of this criterion. Therefore, the aggregation results are usually evaluated by the effects or consequences, such as feedback of services [45, 46] or knowledge of ground truths. These criteria require additional information.

In our experiments, we evaluate the preference aggregation results in two aspects. Firstly, we focus on the consensus degree between the final result and each expert. Secondly, we compare the result with the ground truth knowledge in terms of food habits in Japan. In the first evaluation, the Average Deviation Index (ADI) is applied, defined in Equation (34) [47]:

$$ADI = \frac{1}{|S|} \sum_{s \in S} \tau(\sigma_{agg}, \sigma_s), \tag{34}$$

where, $\sigma_{agg}$ refers to the aggregated preference order, $\sigma_s$ the preference order of agent $s$, and $\tau$ the normalized Fagin distance [15].

Table 9 summarizes the ADI of all methods in comparison, including four rules of evidential preferences on orders and scores. We observe that the decision rule of $EP_{USP}$ provides

**Table 9:** Aggregated conflicting Sushi preferences with ADI:

| | East Japan | | West Japan | |
|---|---|---|---|---|
| | Order | Score | Order | Score |
| $EP_J$ | 0.2685 | 0.2395 | 0.2661 | 0.2417 |
| $EP_{USP}$ | 0.1462 | **0.1172** | **0.1346** | **0.1103** |
| PROMETHEE-EDAS | 0.1821 | 0.1761 | 0.2603 | 0.2609 |
| TOPSIS-WAA | **0.1439** | 0.1960 | 0.1449 | 0.1666 |
| 3WD-MADM(ELECTRE I) | 0.4878 | 0.4638 | 0.6445 | 0.4356 |

aggregation result with more consensus than $EP_J$, on both scores and orders. Besides, $EP_{USP}$ returns a similar result as TOPSIS-WAA, with a little improvement. The results of 3WD-MADM and PROMETHEE-EDAS are relatively unsatisfactory. The reason is similar: 3WD-MADM is calculated based on pairwise rankings with integrated ELECTRE I. PROMETHEE-EDAS is calculated based on in/out flow on pairwise alternatives. Since the data set is highly sparse and missing values are filled in with the mean value of observed data, most criteria are identical between two alternatives, and the distances between alternatives are relatively small. In the result of ELECTRE I, almost half of the alternatives in our experiments are indifferently ranked as least preferred. In the result of PROMETHEE, most preferences are *weak*, which the ranking result unreliable.

However, ADI is not always a convincing and reliable homogeneous evaluation criterion. Lower ADI can be reached by applying a Kendall distance $\tau$ minimization strategy. This also explains the satisfactory performance of $EP_{USP}$. Indeed, in cases without *uncertainty* nor *imprecision*, the average value of USP distance between two preference orders is identical to Kendall distance with Fagin's extension [48].

To give a more concrete conclusion, in the second place, the preference aggregation results are evaluated with a piece of sociology knowledge in Japan about eating habits: people from east Japan usually prefer more oily sushi than people from west Japan. If the aggregated preferences can successfully discriminate the oiliness choice largely from the other region, the aggregation rule is believed to be appropriate. Thus, for verification, the *average oiliness index* (AOI) is implemented to the whole aggregated preference order. With the oiliness degree of sushi $a_i$ denoted by $oil(a_i)$, we define $AOI$ on the set of sushi $A_{sushi}$ as:

$$AOI(A_{sushi}) = \frac{1}{|A_{sushi}|} \sum_{a_i \in A_{sushi}} oil(a_i). \tag{35}$$

The comparison of $AOI$ on top-k sushis between East and West Japan is calculated respectively upon the difference:

$$AOI_{diff,k} = AOI(A_k^{East}) - AOI(A_k^{West}), \tag{36}$$

where, $AOI(A_k^{East})$ and $AOI(A_k^{West})$ refer respectively to the aggregation orders of top-$k$ sushi for East and West Japan.

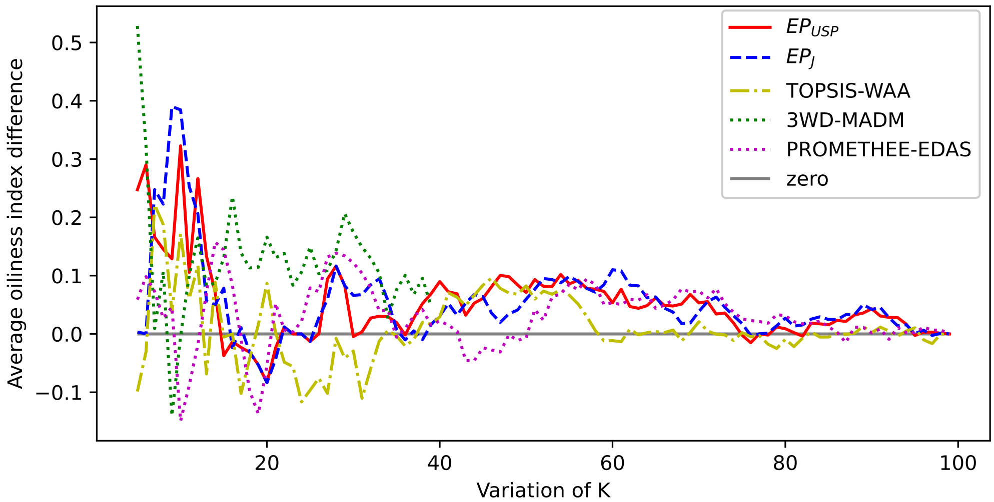



**Figure 10:** Sushi AOI difference between East Japan and West Japan.

Figure 10 depicts the $AOI_{diff}$ of top-$k$ preferred sushi decided by all methods in comparison, with $k$ varying from 5 to 100. For 3WD-MADM, $k$ varies from 5 to 40 as the last 60 sushis are all considered indifferently as not preferred. According to the definition of AOI in Equation (36), positive $AOI_{diff}$ means oily sushi is more preferred in east Japan than in west Japan, and vice versa for negative $AOI_{diff}$ values. It is obvious that $AOI_{diff}$ has positive values globally across different $k$ values for decision rules based on both Jousselme and USP distance, especially for top-10 sushis. The results of $EP_{USP}$ and $EP_J$ correspond generally the sociological knowledge mentioned before. Nevertheless, around $k = 20$, $AOI_{diff}$ values drop to negative, for the reason that, the 19th and 20th favorite sushis in the order are much less oily in the east than in the west Japan. The result of TOPSIS-WAA has a similar trend with *EP* methods but corresponds less with the sociology knowledge. 3WD-MADM and PROMETHEE-EDAS return rather random results. This does not imply that these two methods are ineffective in decision-making. Indeed, the unsatisfactory results are mainly caused by the data completion strategy. With most of the missing data completed by the mean value, the difference between alternatives is relatively little. Therefore, MCDM methods based on preference intensity encounter problems in distinguishing the alternatives.

Still, we need to note that *people in east Japan prefer more oily sushi* is a rough piece of knowledge rather than a strict rule. This statement does not imply that all oily sushi should be preferred in east Japan than west, and it is common that some oily sushis are preferred in both regions, as the existence of several oily sushi more welcomed in east Japan is always reasonable.

## 6. Conclusion and discussion

In this paper, we originally bring the discussions on the ambiguity in the interpretation of *incomparability*. We find that most of the previous studies do not distinguish between interpretations of incompatibility according to different situations. In some works, *incomparability* is interpreted as *undecided* because of certain criteria, while in others it is interpreted as *undecided* because of a lack of observed information. This ambiguity could lead to controversy in dissimilarity values, especially for pair-wise preference relations. Moreover, reasoning the *undecided* case as *indifference* is not a faithful way and may lead to unexpected consequences. This issue has rarely been discussed in the research community. Therefore, in order to clarify the different dissimilarity measures at the definition level, we identify and categorize the two interpretations of *incomparability* as *indecisive* and *undecided* respectively. With the help of the evidential preference model based on the theory of belief functions, we are able to distinguish *indecisive* and *undecided* interpretations and *indifference* relation faithfully in a unified framework: the *indifference* and *indecisive* cases are exclusive with other preference relations by being considered as specific binary relations, while the *undecided* case of *incomparability* is considered as *total ignorance* represented by the union of all possible relations.

In the second place, we reveal a flaw in existing dissimilarity measures in the theory of belief functions while applied on evidential preferences after a thorough study on existing distances between preference relations. Due to the property of belief functions, dissimilarities between exclusive singletons are considered equal, which raises a contradiction with all previous works in preference modelling. Obviously, this property of belief functions is not suitable for evidential preferences. To solve this issue, we propose a novel dissimilarity measure function, named USP distance, for BBAs (evidential objects with both uncertainty and imprecision). USP distance is an extension of Jousselme distance, which takes the difference of dissimilarity between singletons into consideration. In the case that dissimilarities between different singletons are equal, USP distance degrades to Jousselme distance. USP distance is applied in group decision making with evidential preference and shows a better performance than Jousselme distance. The advantages of USP distance are also justified by experimental comparison with several MCDM methods.

In this paper, the application of USP distance is limited in evidential preference aggregation. USP distance can have more potential utilities in decision making, not only limited in the circumstances of preference modelling. For example, in a decision system where some alternatives are riskier than others, it is unfavourable to choose the risky alternatives. Traditionally, such discrimination is realised by utility functions, as introduced in [49]. USP distance brings another possible solution. By assigning the pair-wise dissimilarity with different values, singletons are possible to be discriminated against by USP distance. A comparative study between the two solutions is in the scope of our future work.

Moreover, other distance-based applications, notably learning tasks on evidential objects with USP distance, are left to be justified. Here, we propose a potential issue in this task as a discussion topic without being solved, with the example of clustering over evidential preferences. Clustering preferences is an important technique, especially in recommendation systems and social networks where users are profiled by their preference information. For orders consisting of crisp preference relations, the USP distance degrades to Kendall's $\tau$ distance in total orders and Fagin's distance [48] in weak orders. Thus, it is appropriate to apply USP distance for the classification on complete preference orders. A similar application using Jousselme distance has been justified in [50]. Nevertheless, for incomplete and imprecise preferences, where the impact of ignorance is important, an issue becomes obvious: the missing preference information modelled by vacuous BBAs (see Definition 5) are identical and are classified into the same group based on the dissimilarity. With such results, imprecise information is brutally clustered into crisp clusters, losing the significance of clustering over preferences. A more rational classification result should be that these vacuous BBAs are not classifiable, *i.e.* the membership of these vacuous BBAs to any group is also a knowledge of total ignorance. Therefore, it is reasonable to doubt the appropriateness of existing distance functions in the theory of belief functions, including USP distance. Generally speaking, this issue is due to the failure of existing distances measuring over evidential objects with imprecision. To deal with this issue, a study on the properties of dissimilarity measure for learning tasks over evidential objects is necessary within the scope of our future work.

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