An Algebraic Introduction to Knots and Braids

Master of Mathematics

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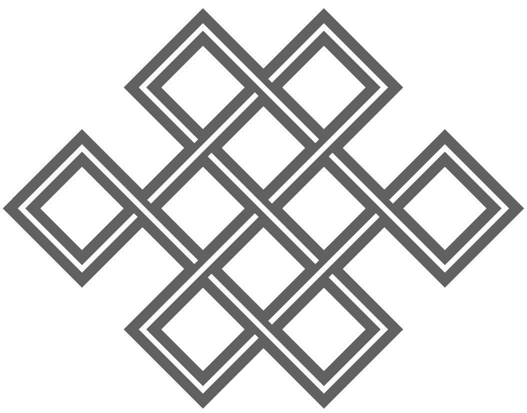

Contents

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Abstract

This dissertation investigates braids and knots, emphasizing how algebraic techniques provide insight into their topological structures. The first chapter focuses on the study of braids. We begin with an abstract characterization of the Artin braid group, then define topological braids and establish their equivalence, notably through an adaptation of Reidemeister's Theorem for braid diagrams. We demonstrate that the group structure arising from topological braids is isomorphic to the Artin braid group and leverage this isomorphism to explore structural properties. Key results include the homomorphism to the symmetric group and the computation of the braid group's centre.

The second chapter explores knot theory, again highlighting algebraic connections. After establishing foundational definitions and knot equivalence using Reidemeister's Theorem, we investigate knot invariants. This leads to the construction of the knot quandle, analysing its strengths (completeness) and weaknesses (isomorphism problem). We then derive more practical invariants from the quandle structure, including p-colorability and the Alexander polynomial.

The final chapter establishes the link between braids and knots. We present Alexander's Theorem, demonstrating that every knot or link can be represented as a closed braid. Subsequently, we explore Markov's Theorem, which provides the necessary moves to determine when different braids close to form equivalent links. Together, these theorems solidify the powerful connection enabling the application of braid theory to knot classification.

Introduction

The study of knots was considered by figures such as Gauss in the late 1700's however the rigorous study of these objects began with Tait in 1877 [Tai77]. After some development in the field, Artin published his seminal work on braids in 1925 [Art25] and the first notable connection between knots and braids was made by Alexander in 1923 [Ale23]. Knots and braids remained a primarily intellectual pursuit throughout the early and mid 1900s after some promising applications in physics proposed by Lord Kelvin were disproven at the turn of the century. In approximately the last 40 years however, knots and braids have experienced a renaissance with practical applications of knot theory being found in chemistry and biology and more recently braids have been used extensively in quantum computing and in statistical mechanics. This dissertation aims to give an introduction to how we can think about these structures mathematically and in particular, how leveraging algebra changes how we think about the core problems in the field. The focus of the final chapter will be to show how knots and braids, that at first appear to be completely different objects, are fundamentally connected to each other and how this connection lends itself to new perspectives on both objects.

We begin our exploration in the first chapter in Section 1.1 by defining the Artin braid group algebraically, in terms of a set of generators and relations. We will take a look at some key characteristics of this group such as the homomorphism onto the symmetric group in Theorem 1.1.9 and consider some examples that help illustrate its structure. We will then shift our focus to the topological view in Section 1.2 where we will define geometric, diagrammatic and polygonal braids. The goal of this section is to show that each of these topological structures exhibit a natural notion of equivalence and further, that the resulting equivalence classes can be resolved among all 3 of these structures. A key result in this section is Reidemeister's Theorem 1.2.10 in the context of braids. Reidemeister's Theorem is the classical theorem that allows us to draw equivalence between the natural equivalence classes exhibited in the 3 dimensional geometric braids and the 2 dimensional diagrammatic braids. This equivalence allows us to reduce the more complex notion of equivalence on the geometric braids, to the simpler notion on diagrammatic braids that consists of finite combinations of just 2 standard moves given in Definition 1.2.9.

Having set this scene, we will see how the group structure that naturally arises from the topological braids is isomorphic to the Artin braid group in Theorem 1.3.4. This is the first and one of the most fruitful instances that we will see of how topological and algebraic structures that represent the same underlying information give us different perspectives on this information. Indeed, the remainder of the first chapter provides a deeper exploration into this information that we call braids. We will look at a different representation of the Artin braid group in Section 1.3.3 and revisit some key homomorphisms that we had seen without the topological setting and use this new perspective to understand more deeply, the structure of the braid group in Section 1.4. In section 1.5, we will lean into the algebraic representation to help define an order on the braids. The chapter culminates in the computation of the centre of the braid group in Theorem 1.5.15. Throughout this final Section 1.5.3, we will discuss how the geometric intuition as to what the centre of the braid group should be, guides our rigorous algebraic process. This is a captivating section and it beautifully sums up what the power of the interplay between the topological and algebraic representations of our braids.

The second chapter shifts our focus to our second object of study, knots, starting with a translation of our intuitive notion of a knot as a closed loop of string into rigorously defined topological objects allowing us to conduct a more concrete discussion. We will examine how knot equivalence presents itself in this topological setting in Section 2.3 and we will discuss how we might present these 3 dimensional objects as 2 dimensional diagrams. This leads us, as with braids to Reidemeister's Theorem 2.3.4. Like with braids this allows us to reduce the complex notion of equivalence (isotopy) in the 3 dimensional setting down to the same set of moves that allow us to draw equivalence in the braid setting along with one extra move, this is some foreshadowing for the final chapter. We conclude our introduction to knots with a review of polygonal knots and how their own notion of equivalence may prove useful in some cases and we introduce prime knots as the primary objects that the field of knot theory aims to classify.

The central problem in knot theory is that of classification, that is how can we tell whether two knots are the same? This leads us to the study of knot invariants which we can define as properties of a knot that are preserved under the Reidemeister moves, as a direct consequence of Reidemeister's Theorem, this allows us to classify 3 dimensional knots. After introducing tricolourability in Definition 2.4.2 as a simple introduction to knot invariants, we conduct an intuitive construction of the knot quandle in Section 2.4.2. This is another major case where we use an algebraic object to aid our exploration of a topological structure, indeed the knot quandle is an algebraic structure that is designed to represent knots and be preserved up to isomorphism under the Reidemeister moves. We will discuss how the isomorphism problem prevents the knot quandle from being a useful invariant in its own right and we will see how we can make some considerations that allow us to construct new, more useful invariants from the knot quandle itself. The first of these will be an extension of tricolourability called p-colourability (Definition 2.4.21), indeed we will see how tricolourability is a way to visually represent a quandle homomorphism from the knot quandle to the dihedral quandle as demonstrated in Example 2.4.20. The second of these invariants will be the famous Alexander polynomial which we will investigate in Section 2.4.3, this is particularly interesting as this is not the typical route by which we arrive at this polynomial and it hints at the deeper structure of the objects that we are exploring.

As promised, the final chapter bridges the gap between knots and braids. We start by discussing the closure of a braid as given in Definition 3.1.1. Essentially we are saying that we can connect the top and bottom of a braid to get a knot or link, Alexander's Theorem 3.1.3 then tells us that strikingly, any knot can be represented by a braid up to equivalence. Working with the observation that different braids can close to form the same link, we set out to define a notion of equivalence on braids that relates to their closure, this culminates in Markov's Theorem 3.2.6. Markov's Theorem gives us a set of moves that can be performed in the braid setting that does not affect the closure of the braid in question. Together Alexander's Theorem and Markov's allow us to construct a bijective relationship between equivalence classes of knots and braids, this is showcased in Corollary 3.2.8. We complete the chapter with a short investigation as to what the closures of certain types of braid look like and how understanding these braids can give us a deeper understanding of the structure of their related knot.

1 Braid Groups

The Artin braid group on n strands is a finitely generated, infinite group which is non-commutative for $n \geq 3$. This group will be the primary object of study in this chapter and our first step will be to define it in terms of its generators and relations. We will then define geometric braids and examine how they act as a beautiful physical description of the Artin braid group. Throughout the chapter, we will explore different properties of braid groups in both the algebraic and geometric setting. In fact we will see it is often useful to hold these ideas simultaneously when investigating their characteristics as in, for example, Section 1.3.3. The chapter culminates in the proof of Theorem 1.5.15 that allows us to compute the centre of the braid groups.

1.1 The Artin Braid Group

We begin by defining the Artin braid group abstractly in terms of group generators and relations, followed by an examination of the three simplest cases of such groups. [KT08, Sec.1.1]

Definition 1.1.1 (The Artin Braid Group)
The Artin[1] *braid group B_n for any positive integer n is the group generated by $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the 'braid relations':*

1. $\sigma_i \sigma_j = \sigma_j \sigma_i \;\; \forall i, j = 1, \ldots, n-1 : |i-j| \geq 2$
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \;\; \forall i = 1, \ldots, n-2$

Example 1.1.2
By definition, B_1 has $n-1=0$ generators and is thus the trivial group $\{e\}$.

We consider B_2 which has one generating element, namely σ_1. Since both braid relations involve more than one generator, B_2 has no relations and is therefore isomorphic to the infinite cyclic group.

The group B_3 presents considerably more complexity than that of B_1 or B_2. It is the simplest non-commutative Artin braid group as we shall prove in the following proposition.

Proposition 1.1.3
The Artin braid group B_3 is non-commutative.

Proof.
Since B_3 is generated by σ_1 and σ_2 and since by the first braid relation as given in Definition 1.1.1, commutativity only exists between generators σ_i and σ_j if $|i-j| \geq 2$, we have:

$$\sigma_1 \sigma_2 \neq \sigma_2 \sigma_1$$

□

[1] Named after the prolific mathematician Emil Artin (also the namesake of the Artinian rings) who originally presented his theory of braids in his 1925 paper *Theorie der Zöpfe* [Art25] with an updated version in English published in 1947 [Art47].

Corollary 1.1.4
Every Artin braid group B_n with $n \geq 3$ is non-commutative.

Proof.
Proposition 1.1.3 deals with the case of B_3, and for $n > 3$, it is clear that B_n contains a copy of B_3 as a subgroup. $\square$

We further investigate the structure of B_3 in the following example.

Example 1.1.5
The group B_3 is generated by two generators σ_1 and σ_2 and we therefore must consider the second braid relation, that is, in this context:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \tag{1.1.1}$$

As noted in the proof of Proposition 1.1.3, the first braid relation does not apply in B_3. Now, if we define $x = \sigma_1\sigma_2\sigma_1$ and $y = \sigma_1\sigma_2$, we obtain an alternative set of generators for B_3. To see this, observe that $x^{-1}y = \sigma_1$ and after applying the second braid relation to x to get x', we observe $y^{-1}x' = \sigma_2$. These generators satisfy the unique relation $x^2 = y^3$. We can verify this easily:

$$y^3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 \tag{1.1.2}$$

Applying the second braid relation to the last 3 generators:

$$y^3 = \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 = x^2 \tag{1.1.3}$$

This relationship implies that x^2 lies in the centre of B_3 since it commutes with both x and y (alternatively $(\sigma_1\sigma_2\sigma_1)^2$ commutes with σ_1 and σ_2) and therefore every element in B_3, we will see more on the centre in Section 1.5.3.

1.1.1 Homomorphisms

We make the following observation regarding any homomorphism from the Artin braid group B_n.

Proposition 1.1.6
Given a group homomorphism $f : B_n \to G$ for some group G and the Artin braid group B_n, the elements $\{g_i = f(\sigma_i)\}_{i=1,\ldots,n-1}$ of G satisfy the braid relations:

1. $g_i g_j = g_j g_i \;\; \forall i,j = 1,\ldots,n-1 : |i-j| \geq 2$
2. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \;\; \forall i = 1,\ldots,n-2$

Proof.
For the first braid relation, let g_i and g_j be such that $|i-j| \geq 2$, we have:

$$g_i g_j = f(\sigma_i)f(\sigma_j) = f(\sigma_i\sigma_j) = f(\sigma_j\sigma_i) = f(\sigma_j)f(\sigma_i) = g_j g_i$$

For the second braid relation, consider g_i and g_{i+1} with $i = 1,\ldots,n-2$:

$$g_i g_{i+1} g_i = f(\sigma_i)f(\sigma_{i+1})f(\sigma_i) = f(\sigma_i\sigma_{i+1}\sigma_i) = f(\sigma_{i+1}\sigma_i\sigma_{i+1}) = f(\sigma_{i+1})f(\sigma_i)f(\sigma_{i+1}) = g_{i+1}g_i g_{i+1}$$

$\square$

Strikingly, the converse also holds as we can see in the lemma below.[2]

Lemma 1.1.7
If $g_1, \ldots, g_{n-1}$ are elements of a group G satisfying the braid relations, then there exists a unique group homomorphism $f : B_n \to G$ such that $f(\sigma_i) = g_i \; \forall i = 1, \ldots, n-1$.

Proof.
Let F_n be the free group generated by the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$. There is a unique group homomorphism $\bar{f} : F_n \to G$ such that $\bar{f}(\sigma_i) = g_i$ for all $i = 1, \ldots, n-1$. This homomorphism induces a group homomorphism $f : B_n \to G$ provided $\bar{f}(r^{-1}r') = 1$ or, equivalently, provided $\bar{f}(r) = \bar{f}(r')$ for all braid relations $r = r'$. For the first braid relation we have;

$$\bar{f}(\sigma_i\sigma_j) = \bar{f}(\sigma_i)\bar{f}(\sigma_j) = g_i g_j = g_j g_i = \bar{f}(\sigma_j)\bar{f}(\sigma_i) = \bar{f}(\sigma_j\sigma_i) \tag{1.1.4}$$

For the second braid relation we have;

$$\bar{f}(\sigma_i\sigma_{i+1}\sigma_i) = \bar{f}(\sigma_i)\bar{f}(\sigma_{i+1})\bar{f}(\sigma_i) = g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} = \bar{f}(\sigma_{i+1})\bar{f}(\sigma_i)\bar{f}(\sigma_{i+1}) = \bar{f}(\sigma_{i+1}\sigma_i\sigma_{i+1}) \tag{1.1.5}$$

□

Example 1.1.8
Consider the symmetric group S_5, recall that this group can be generated by the set of simple transpositions $\{(1,2),(2,3),(3,4),(4,5)\}$, let these be labelled as the set of generators s_1, s_2, s_3, s_4 respectively. It is easy to verify that these satisfy the braid relations. So by Lemma 1.1.7 there exists a unique homomorphism $\phi_5 : B_5 \to S_5$ such that $\phi_5(\sigma_i) = s_i$ for all $i = 1, 2, 3, 4$.

If we look at some of the properties of this map, it is clear that it must be surjective since each generator s_i is in the image of ϕ_5 since $s_i = \phi_5(\sigma_i)$ for each $i = 1, \ldots, 5$. Since ϕ_5 is a homomorphism, $im(\phi_5) = S_5$.

Consider the element $\sigma_1\sigma_1 \in B_5$, this would map to:

$$\phi_5(\sigma_1\sigma_1) = \phi_5(\sigma_1)\phi_5(\sigma_1) = s_1 s_1 = e_{S_5}$$

This tells us that the kernel of ϕ_5 is non trivial which implies that ϕ_5 is not injective.

Theorem 1.1.9
For the Artin braid group B_n with $n \geq 2$, there exists a unique surjective, non-injective homomorphism $\phi_n : B_n \to S_n$ such that $\phi_n(\sigma_i) = s_i$ for all $i = 1, \ldots, n-1$ where s_i are the generators of the symmetric group S_n.

Proof.
Follows the same pattern as Example 1.1.8. □

Remark 1.1.9.1. ϕ_n is sometimes called the natural projection.

The homomorphisms discussed in this section have deep implications that can be fully explored after outlining the physical interpretation of the Artin braid group. Indeed, we will see an application of Lemma 1.1.7 in Section 1.3.2 and use the consequences of this application to discuss Theorem 1.1.9 in greater detail in Section 1.4.

[2]An analogy of lemma 1.1.7 and its converse hold for any arbitrary group presentation.

1.2 Braids

The significant interest in the Artin braid group (and the origins of its name) arises from its natural representation by physical objects collectively referred to as braids. In this section we will discuss 3 distinct types of braid each with a corresponding notion of equivalence. As we work through these we will show that these there is a bijection between the respective equivalence classes of these three objects in Corollary 1.2.11 and through Proposition 1.2.14 and Theorem 1.2.16. Note that the term ‘braid’ is used informally here. We will clarify our language as we proceed. We will use some notions from topology as we proceed, for clarification on any topological points see [Mun00].

1.2.1 Geometric Braids

The first of these objects that we will look at are known as the *geometric braids*. Note that I is understood to be the interval $[0, 1] \subset \mathbb{R}$. By a *topological interval* we mean a topological space that is homeomorphic to I.

Definition 1.2.1 (Geometric Braid)
A geometric braid on $n \geq 1$ strings is a set $b \subset \mathbb{R}^2 \times I$ formed by n disjoint topological intervals called the strings of b such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1, 0, 0), (2, 0, 0), \ldots, (n, 0, 0)\}$$
$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(1, 0, 1), (2, 0, 1), \ldots, (n, 0, 1)\}$$

Note 1.2.1.1
Due to the compactness of I and the distinct start and end points of each string, we cannot have an instance where one of the strings ‘wraps around’ the other strings an infinite number of times. Other undesirable behaviours such as space filling curves are prevented by the fact that each string maps homeomorphically onto I.

It is clear that every string in some geometric braid b meets each plane $\mathbb{R}^2 \times t$ with $t \in I$ at exactly one point and each string connects a point $(i, 0, 0)$ to a point $(s(i), 0, 1)$ where $i, s(i) \in \{1, \ldots, n\}$ so the sequence $(s(1), \ldots, s(n))$ is a permutation of the set $\{1, \ldots, n\}$. This is called the *fundamental permutation* of b and is directly related to Theorem 1.1.9. We discuss this relationship in Section 1.4.

In order to rigorously define equality between geometric braids we will adapt the concept of homotopy to our specific use case.

Definition 1.2.2 (Homotopy)
Let X and Y be topological spaces and let $f : X \to Y$ and $g : X \to Y$ be continuous maps. We say that f is homotopic to g denoted $f \simeq g$, if there exists a continuous function $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ with $x \in X$. We call H a homotopy between f and g.

Example 1.2.3
We can now look at an explicit example of a homotopy, that is the classic example that there is a homotopy from a mug to a torus as we can see in Figure 1.2.1.
This is a nice visualization of our definition, we can see that through a continuous function we can move from our original embedding (the mug) to our final embedding (the torus).

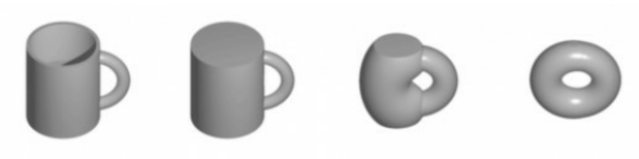

Figure 1.2.1: A homotopy from a mug to a torus.

Definition 1.2.4 (Braid Isotopy)
Two geometric braids b and b' on n strings are considered braid isotopic if there exists a continuous map

$$F : b \times I \to \mathbb{R}^2 \times I$$

such that for each $t \in I$ the map $F_t : b \to \mathbb{R}^2 \times I$ with $F_t(x) = F(x,t)$ is an injective continuous homeomorphism onto its own image where that image is a geometric braid on n strings, $F_0 = id_b : b \to b$ and $F_1(b) = b'$. We call F a braid isotopy between b and b'.

The intuition here is to keep the idea of small, continuous deformations that we saw in Example 1.2.3 and apply it to our geometric braids. The key difference between homotopy and braid isotopy is that in the latter, each stage of the deformation must itself be a geometric braid.

It is obvious that braid isotopy is an equivalence class on the class of geometric braids on n strings. In the following definition we call these equivalence classes, *braid isotopy equivalence classes.*

Definition 1.2.5 (Braids)
The braid isotopy equivalence classes of the geometric braids on n strings are called braids on n strings.

1.2.2 Braid Diagrams

Geometric braids are a subset of $\mathbb{R}^2 \times [0,1]$ and thus inherently three dimensional however since we are working on paper, we want to represent these structures as projections onto $\mathbb{R} \times \{0\} \times I$. Precisely, that is a projection along the second coordinate of a given geometric braid. For a particular geometric braid b this would look like Figure 1.2.2.

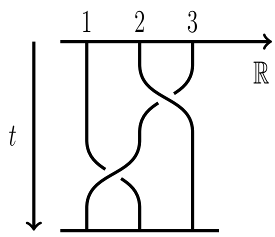

Figure 1.2.2: An example of a projection of a geometric braid with 3 strings along the second coordinate onto $\mathbb{R} \times \{0\} \times I$.

We can see that the projection in Figure 1.2.2 has points where the two projected strings meet. These points are called crossings. We call a projected string a strand. In order to determine which strand passes under the other at each crossing we must mark the under crossing strand. In the case of Figure 1.2.2 we break the undercrossing strand near the crossing. This preserves what will prove to be necessary information from the 3 dimensional structure (see the proof of Proposition 1.2.7).

Consider performing braid isotopies on b and how our projection might change as we perform such isotopies. It is clear that we might perform isotopies in such a way that the projection becomes ambiguous as in Figure 1.2.3. This projection is ambiguous because it is not clear at the bottom whether the first or third strand has emerged from behind the second.

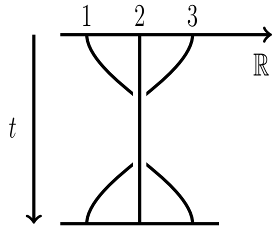

Figure 1.2.3: An example of an ambiguous projection of a geometric braid with 3 strings along the second coordinate onto $\mathbb{R} \times \{0\} \times I$.

The conclusion that we draw from Figure 1.2.3 is that when considering a rigorous definition for these projections in their own right, we should only consider the projections of those geometric braids that do not present this ambiguity. That is we want each crossing to be at a distinct coordinate and we want to avoid crossing over 'long distances'. These considerations lead us to the following definition.

Definition 1.2.6 (Braid Diagram)
A braid diagram on n strands is a set $D \subset \mathbb{R} \times I$ split as a union of n topological intervals called the strands of D such that the following conditions hold.

1. *The projection $\mathbb{R} \times I \to I$ maps each strand homeomorphically to I.*
2. *The unique start and endpoints of each strand are given by*

$$D \cap (\mathbb{R} \times \{0\}) = \{(1,0),(2,0),\dots,(n,0)\}$$
$$D \cap (\mathbb{R} \times \{1\}) = \{(1,1),(2,1),\dots,(n,1)\}$$

3. *Every point of $\mathbb{R} \times I$ belongs to at most 2 strands. At each intersection point (crossing), if two strands meet at $\mathbb{R} \times t_1 : t_1 \in I$, then there exists $\epsilon > 0$ such that the two strands do not share coordinates at any point in the space $\mathbb{R} \times (t_1, t_1 + \epsilon]$. The first coordinate of each strand must be strictly increasing or decreasing in the space $\mathbb{R} \times (t_1 - \epsilon, t_1 + \epsilon)$. One strand must be distinguished as an 'undercrossing' and one as an 'overcrossing' at each intersection point.*

Note 1.2.6.1
As with geometric braids, the compactness of I necessarily prevents an infinite number of crossings. Again, other undesirable behaviours such as space filling curves are prevented by the fact that each strand maps homeomorphically onto I.

We have that Figure 1.2.2 is a braid diagram on 3 strands. In fact, this represents the element $\sigma_2^{-1}\sigma_1$ in the Artin braid group (we will see why in Section 1.3). We also see that Figure 1.2.3 is not a braid diagram as it violates condition 3 of Definition 1.2.6. Note that we have adopted the convention, that to denote the undercrossing strand we have broken the strand near the crossing. We also adopt the convention that for a braid diagram on n strands we name each strand $1, 2, \dots, n$ according to its unique $\mathbb{R}$ coordinate when $t = 0$.

What we aim to obtain through Definition 1.2.6 is a projection that is able to represent every geometric braid up to braid isotopy. We then have a description of geometric braids that contains enough information to recover braid isotopy equivalence classes yet this information is conveyed in such a way that it is more practical in many instances. We set out to prove that our definition achieves this goal.

Proposition 1.2.7
Any braid diagram D represents a braid isotopy equivalence class of geometric braids.

Proof.
For a given braid diagram D, using the obvious identification $\mathbb{R} \times I = \mathbb{R} \times \{0\} \times I$, we can assume that D lies on $\mathbb{R} \times \{0\} \times I \subset \mathbb{R}^2 \times I$. In a small neighbourhood of every crossing of D, we slightly extend the strand marked as the undercrossing into $\mathbb{R} \times (0, +\infty) \times I$ by increasing the second coordinate while keeping the first and third coordinates. This transforms D into a geometric braid on n strings. We say that its equivalence class is the braid represented by D. $\square$

Note 1.2.7.1
It is clear that this proof can be adapted to changes in the direction of the second coordinate axis, here we take it to be 'into the page' when performing the described operation on a braid diagram such as that in Figure 1.2.2.

Theorem 1.2.8
Any braid b can be represented by a braid diagram.

Proof.
Select a geometric braid β that represents the braid b. It is clear that we should be able to perform braid isotopy on β to achieve the geometric braid β' such that the projection of β' onto $\mathbb{R} \times \{0\} \times I$ has crossings which satisfy condition 3 of Definition 1.2.6. It is trivial to note that conditions 1 and 2 of this definition are satisfied by such a projection of any geometric braid. $\square$

Note 1.2.8.1
We may denote the braid diagram of a given braid b by D_b.

Our goal now is to define equivalence classes on our braid diagrams and investigate how the geometric braids represented by these equivalence classes are related. In order to define equivalence on braid diagrams, we will follow the procedure due to Reidemeister. We start by defining the following moves.

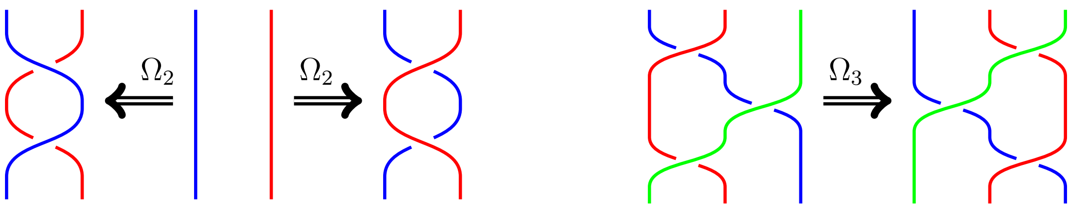

Figure 1.2.4: The two Reidemeister moves for braids, Ω_2 (left) and Ω_3 (right).

Notice that there are 2 moves that we refer to as Ω_2. We obtain the inverse of these moves, Ω_2^{-1} and Ω_3^{-1}, by reversing the directions of the arrows in Figure 1.2.4. It is with reference to these moves that we define equivalence on braid diagrams.

Definition 1.2.9 (Reidemeister Equivalence for Braids)
Two braid diagrams are Reidemeister equivalent if they are related by a finite sequence of Reidemeister moves Ω_2 and Ω_3.

Note 1.2.9.1
There is a fourth Reidemeister move denoted Ω_0 that allows for any strand of a braid diagram to be arbitrarily lengthened or shortened inside $\mathbb{R} \times I$. This move is generally considered trivial and thus not included in our full definition.

It is obvious that Reidemeister equivalence is an equivalence class on braid diagrams. We can relate this to braid isotopy equivalence classes due to the work of Reidemeister in [RBCS83, Chapter 3]. While the classical Theorem given in 2.3.4 was originally developed for knotted string as we shall see later, Reidemeister provided a version for braids as well.

Theorem 1.2.10 (Reidemeister's Theorem for Braids)
Two braid diagrams represent braid isotopic geometric braids if and only if these diagrams are Reidemeister equivalent.

Remark 1.2.10.1. Reidemeister's original proof included a third Reidemeister move Ω_1 that must be included to extend this notion to knotted string, we will see more on this in Section 2.3.

It is clear that if two braid diagrams are Reidemeister equivalent then there is a braid isotopy between the respective geometric braids they respectively represent. To see this, notice that the geometric braids represented in each diagram in Figure 1.2.4 are braid isotopic. When Ω_2 or Ω_3 (or their respective inverses) are applied to some given braid diagram, they only affect the position of the diagram inside a disk in $\mathbb{R} \times I$ leaving the rest of the diagram unchanged.

The converse, that any two geometric braids that are braid isotopic implies that their braid diagram representations are connected by a finite series of Reidemeister moves is very non trivial and a full proof of Theorem 1.2.10 can be found in [KT08, Theorem 1.6] or as mentioned, in Reidemeister's original book *Knotentheorie* translated into English at [RBCS83, Chapter 3].

We summarise what we have achieved in the following corollary.

Corollary 1.2.11
Let $\mathcal{D}_n$ be the set of Reidemeister equivalence classes of all braid diagrams on n strands. Let $\mathcal{B}_n$ be the set of braid isotopy equivalence classes of all geometric braids on n strings. The map $\psi : \mathcal{B}_n \to \mathcal{D}_n$ assigning to a Reidemeister equivalence class of a braid diagram the braid that it represents induces a bijection from the set $\mathcal{B}_n$ onto $\mathcal{D}_n$.

Proof.
We have a surjection by Proposition 1.2.7 and Reidemeister's Theorem 1.2.10 gives us the injection. □

1.2.3 Polygonal Braids

Braid diagrams carry less information than geometric braids while retaining the information necessary to characterise braid isotopy equivalence classes. An alternative way to do this is to restrict our view to a subset of geometric braids known as the *polygonal braids*.

Definition 1.2.12 (Polygonal Braids)
A geometric braid is polygonal if all its strands are formed by a finite number of consecutive linear segments.

Note 1.2.12.1
Since polygonal braids are a subset of geometric braids we retain our notion of a braid diagram. We are also able to use braid isotopy to define equivalence.

From this point on we may refer to braids (in the context of braid isotopy equivalence classes of geometric braids) and represent them in figures as braid diagrams without necessarily making the distinction between the two objects. We may also do the converse and refer to braid diagrams without necessarily drawing the equivalence in the braid setting. It is clear that we can do this by Corollary 1.2.11.

Example 1.2.13
We consider the braid in Figure 1.2.5.

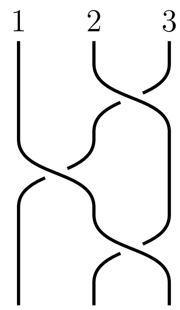

Figure 1.2.5: Some braid on 3 strings.

Now, if we try to approximate this braid using linear segments thus generating a polygonal braid, we could get the polygonal braid shown in Figure 1.2.6.

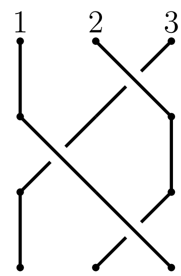

Figure 1.2.6: Braid diagram of a linear approximation of the braid in Figure 1.2.5. This is an example of a polygonal braid.

Of course we could obtain a different configuration of linear segments that would also be an approximation of Figure 1.2.5. The key takeaway from Example 1.2.13 that we set out to formalise, is that the polygonal braid in Figure 1.2.6 is in the same isotopy equivalence class as the geometric braid in Figure 1.2.5 (and so contains the relevant isotopy equivalence class information) however it is clear that the polygonal version may be fully described in a way that requires less information.

Proposition 1.2.14
Any geometric braid β can be approximated as a polygonal braid.

This is clear by the Linear Approximation Theorem, a full proof of which can be found in [Mag22, Pg.200].

As we have discussed, since polygonal braids are a subset of geometric braids we can pass to equivalence classes on the set of all polygonal braids based on braid isotopy. We can however develop a notion of equivalence on the lower information polygonal braids that does not apply to braids in general yet generates the same equivalence classes on polygonal braids as braid isotopy.

Definition 1.2.15 (Delta Move)
Let A, B and C be 3 distinct points in $\mathbb{R}^2 \times I$ such that the third coordinate of A is strictly less than the third coordinate of B that is strictly less than than the third coordinate of C (recall that the third coordinate increases as we move 'down' the braid). The Δ move $\Delta(ABC)$ applies to a polygonal braid $b \subset \mathbb{R}^2 \times I$ when b meets the triangle ABC precisely along the segment AC. Under this assumption the Δ move $\Delta(ABC)$ transforms the segment $AC \subseteq b$ into two segments formed by AB then BC. The inverse move $\Delta^{-1}(ABC)$ replaces the segments AB and BC with the segment AC in a similar manner.

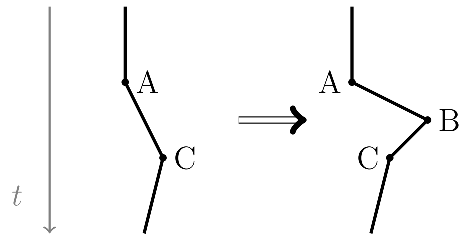

Figure 1.2.7: The Δ move $\Delta(ABC)$ on a strand of a polygonal braid. Δ^{-1} can be obtained by reversing the arrow.

Theorem 1.2.16
Two polygonal braids are braid isotopic if and only if one can be transformed into the other by a finite series of Δ moves.

It is obvious that polygonal braids related only by a finite sequence of Δ moves present a braid isotopy. The converse can be shown to be true and is done so in [KT08, Claim 1.7].

We have then another structure that can represent any geometric braid, that has its own notion of equivalence such that each equivalence class represents an equivalence class generated by braid isotopy on the geometric braids. Polygonal braids in particular are useful because they can be fully defined by the coordinates of the start and finish of each line segment, in particular this allows us to input braids into computers. For an application of Theorem 1.2.16 (although in a slightly different context) see Theorem 3.1.3.

At this point, we clarify that due to Corollary 1.2.11, Proposition 1.2.14 and Theorem 1.2.16 we can freely exchange between polygonal braids, geometric braids and braid diagrams and between delta moves, braid isotopy and Reidemeister moves. The equivalence classes on polygonal braids, geometric braids and braid diagrams generated by each respective notion of equivalence may be referred to as a braid. We also may freely change between using the word strand or string to refer to a topological interval defining a braid diagram or geometric braid.

1.3 Linking the Artin Braid Group to Braids

What we have discussed so far is ways to reduce the amount of information carried by geometric braids yet retain enough information to be able to recover the braid isotopy equivalence classes. In this section we will explore the greatest such reduction that is, to the Artin braid group. Our approach will be to prove that braids form a group in Section 1.3.1 and that this group is isomorphic to the Artin braid group in Section 1.3.2.

1.3.1 The Braid Group

We denote the set of all braids on n strings $\mathcal{B}_n$. We aim to define a group on $\mathcal{B}_n$ using a set of generators. We will first define these generating elements, then define operation and the identity element. We will demonstrate that the group axioms are then satisfied.

Generators
We define the generators on $\mathcal{B}_n$ to be the set of *elementary braids* on n strings, that is the braids with only one crossing denoted σ_i^+ and σ_i^- as shown in Figure 1.3.1 and 1.3.2 respectively.

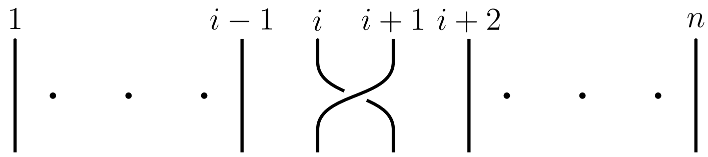

Figure 1.3.1: The elementary braid σ_i^+ on n strands.

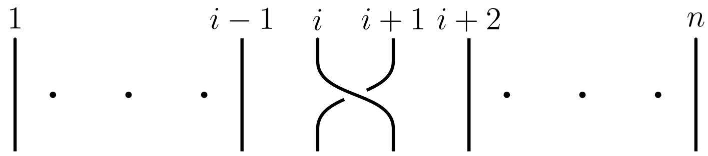

Figure 1.3.2: The elementary braid σ_i^- on n strands.

Note that by convention the $i+1$-th strand crossing over the i-th strand is considered the 'positive' crossing direction.

Operation
We define the product of two given braids b_1 and b_2, $b_1 b_2$ as placing b_1 on top of b_2, 'gluing' the ends of each strand together and shortening the strings as to fit in $\mathbb{R}^2 \times I$. We call this the braid group operation.

Example 1.3.1
Consider $\mathcal{B}_4$ the set of braids on 4 strands and the generators σ_2^+ and σ_3^+, we can see these represented as braid diagrams in Figure 1.3.3.

Now we can see the braid diagram of their product $\sigma_2^+\sigma_3^+$ under the braid group operation in Figure 1.3.4.

It is clear that the braid group operation is associative although we note that commutativity is restricted.

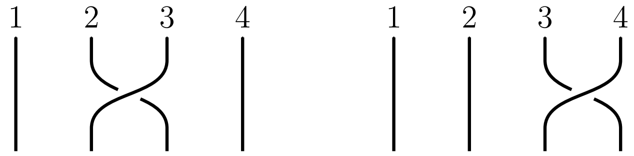

Figure 1.3.3: The braids σ_2^+ and σ_3^+ in $\mathcal{B}_4$.

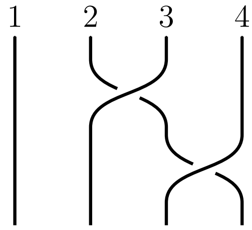

Figure 1.3.4: The braid diagram of $\sigma_2^+\sigma_3^+$ in $\mathcal{B}_4$.

Identity
We define the identity braid on $\mathcal{B}_n$ denoted $e_{\mathcal{B}_n}$,to be the braid with no crossings on n strings as depicted explicitly in Figure 1.3.5 on $\mathcal{B}_4$.

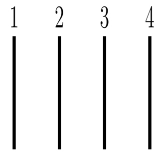

Figure 1.3.5: The identity braid e on $\mathcal{B}_4$.

It is clear that for all $b \in \mathcal{B}_n$, $be_{\mathcal{B}_n} = b = e_{\mathcal{B}_n}b$ by the Reidemeister move Ω_0 (recall that this is the move that allows for arbitrary lengthening and shortening of strands as given in Note 1.2.9.1).

It remains to show that every element in $\mathcal{B}_n$ has a two-sided inverse. Recall that a monoid is equivalent to a group with identity and no inverse element.

Lemma 1.3.2
Each b in $\mathcal{B}_n$ has a two-sided inverse.

Proof.
We claim that the braids $\{\sigma_i^+ : i = 1, \ldots, n-1\}$ and $\{\sigma_i^- : i = 1, \ldots, n-1\}$ generate $\mathcal{B}_n$ as a monoid. To see this, consider a braid b on n strings represented by a braid diagram D_b. Since our braid diagram has finitely many crossings, we may perform slight deformations of D_b in a neighbourhood of its crossing points to ensure that each crossing has a unique second coordinate in $D_b \subset \mathbb{R} \times I$. So we have the real numbers

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1$$

such that the intersection of D_b with each strip $\mathbb{R} \times [t_j, t_{j+1}]$ contains exactly one crossing. This intersection is then a diagram of either σ_i^+ or σ_i^- for some $i = 1, \ldots, n-1$. The resulting splitting of D_b as a product (in the braid group operation sense) of k braid diagrams shows that

$$b = D_b = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2} \ldots \sigma_{i_k}^{\epsilon_k}$$

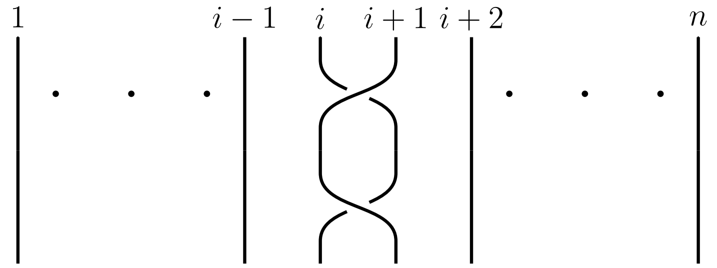

Figure 1.3.6: The braid diagram of $\sigma_i^+\sigma_i^-$.

where each ϵ_j is either $+$ or $-$ and $i_j = 1, \dots, n-1$ for all j.

It is clear from Figure 1.3.6, that $\sigma_i^+\sigma_i^- = \sigma_i^-\sigma_i^+ = e$ for all $i = 1, \dots, n-1$. In particular, we have that the corresponding braid diagrams are related by Ω_2. Therefore

$$b^{-1} = \sigma_{i_1}^{-\epsilon_1}\sigma_{i_2}^{-\epsilon_2}\dots\sigma_{i_k}^{-\epsilon_k}$$

is a two-sided inverse of b in $\mathcal{B}_n$. □

We conclude that $\mathcal{B}_n$ along with braid group operation defines a group known as *the braid group on n strings* that can be generated by $\{\sigma_i^+ : i = 1, \dots, n-1\}$.

1.3.2 The Braid Group is Isomorphic to the Artin Braid Group

In order to achieve the desired isotopy, we must first prove the following lemma.

Lemma 1.3.3
The elements in the set $\{\sigma_i^+ : i = 1, \dots, n-1\}$ satisfy the braid relations. That is:

1. $\sigma_i^+\sigma_j^+ = \sigma_j^+\sigma_i^+ \ \ \forall i, j = 1, \dots, n-1 : |i-j| \geq 2$
2. $\sigma_i^+\sigma_{i+1}^+\sigma_i^+ = \sigma_{i+1}^+\sigma_i^+\sigma_{i+1}^+ \ \ \forall i = 1, \dots, n-2$

Proof.
The first relation follows from the fact that each side of the relation is represented by isotopic braids as we see in Figure 1.3.7.

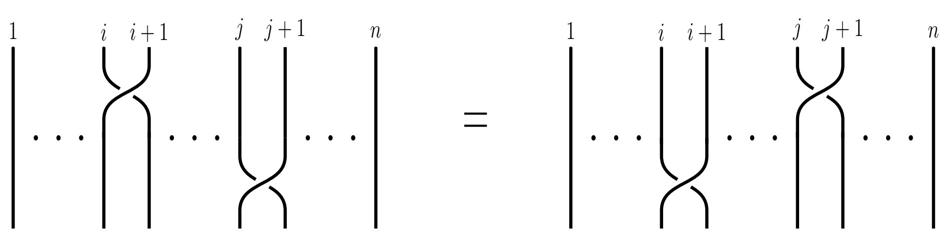

Figure 1.3.7: The braid diagram corresponding to the first braid relation in the braid group.

The second relation follows from the fact that each side of the relation is represented by braid diagrams that differ by only Ω_3.

□

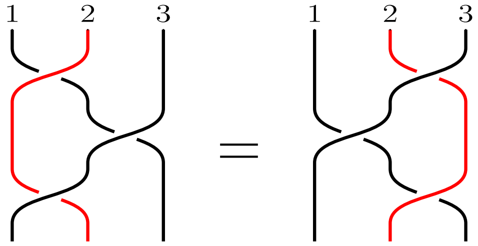

Figure 1.3.8: The braid diagram representing the second braid relation in the braid group. The second strand is highlighted in red to see Ω_3 clearly.

Theorem 1.3.4
For ϵ either $+$ or $-$, there is a unique homomorphism $\varphi_\epsilon : B_n \to \mathcal{B}_n$ such that $\varphi_\epsilon(\sigma_i) = \sigma_i^\epsilon$ for all $i = 1, \dots, n$. The homomorphism φ_ϵ is an isomorphism.

Proof.
Without loss of generality we prove the case where ϵ is taken to be $+$. The existence and uniqueness of φ_+ follows directly from Lemmas 1.1.7 and 1.3.3. We have from Lemma 1.3.2 that the set $\{\sigma_i^+ : i = 1, \dots, n-1\}$ generates $\mathcal{B}_n$ as a group. These generators belong to the image of φ_+ and so φ_+ is surjective.

For injectivity we construct a set theoretic map $\psi : \mathcal{B}_n \to B_n$ such that $\psi \circ \varphi_+ = id$. By the procedure described in the proof of Lemma 1.3.2, we represent some braid $b \in \mathcal{B}_n$ by a braid diagram D_b whose crossings have unique second coordinate in $D_b \subset \mathbb{R} \times I$. This leads to an expansion of the form

$$b = D_b = \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \dots \sigma_{i_k}^{\epsilon_k} \tag{1.3.1}$$

where each ϵ_j is either $+$ or $-$ and $i_j = 1, \dots, n-1$ for all j. Set

$$\psi(D_b) = (\sigma_{i_1})^{\epsilon_1} \dots (\sigma_{i_k})^{\epsilon_k}$$

where:

$$(\sigma_i)^+ = \sigma_i \quad \text{and} \quad (\sigma_i)^- = \sigma_i^{-1}$$

We claim that $\psi(D_b)$ only depends on b. By Theorem 1.2.10 we only need to show that $\psi(D_b)$ does not change under Reidemeister moves of D_b. We first verify Ω_0. Applications of Ω_0 that do not change the order of the second coordinates of each crossing preserve the expansion 1.3.1 and therefore $\psi(D_b)$. An application of Ω_0 that exchanges the order of the second coordinate of two crossings (as in Figure 1.3.7) replaces the term $\sigma_i^{\epsilon_i}\sigma_j^{\epsilon_j}$ with $\sigma_j^{\epsilon_j}\sigma_i^{\epsilon_i}$ for some $i, j = 1, \dots, n-1 : |i - j| \geq 2$. Under ψ, these expressions are sent to the same element by the first braid relation.

The move Ω_2 (respectively Ω_2^{-1}) on D_b inserts (removes) a term of the form $\sigma_i^+\sigma_i^-$ or $\sigma_i^-\sigma_i^+$ into the expansion 1.3.1 (this term looks like Figure 1.3.6). Clearly this preserves $\psi(D_b)$.

The move Ω_3 on D_b replaces a term of the form $\sigma_i^+\sigma_{i+1}^+\sigma_i^+$ with $\sigma_{i+1}^+\sigma_i^+\sigma_{i+1}^+$ in the expansion 1.3.1 (as in Figure 1.3.8). Under ψ these expressions are sent to the same element by the second braid relation. For the move Ω_3^{-1} we can conduct a similar procedure.

This shows that ψ is a well defined map from $\mathcal{B}_n$ to B_n. By construction $\psi \circ \varphi_+ = id$ so φ_+ is injective. □

This proof demonstrates that in the Artin braid group, the first braid relation 'accounts' for Ω_0, inversion 'accounts' for Ω_2 and the second braid relation 'accounts' for Ω_3, when we consider σ_i^+ to be the physical representation of the Artin generator σ_i (and with σ_i^- representing σ_i^{-1}).

From this point on we may leverage Theorem 1.3.4 to use the phrase *braids on n strands or strings* to refer to the elements of the Artin braid group B_n as well as in reference to the equivalence classes induced by braid isotopy on the geometric braids. We may switch between the identifiers B_n and $\mathcal{B}_n$ to refer to either of these groups and we shall write σ_i to denote σ_i^+ and $\sigma_i^- = (\sigma_i^+)^{-1} = \sigma_i^{-1}$. As we shall see the ability to switch between these representations is extremely useful.

1.3.3 The Birman Ko Lee Representation

We will now use the identifier φ_+ from Theorem 1.3.4 to help introduce a group that is an alternate presentation of the Artin braid group. As noted we will see how switching between using diagrams in the braid group to guide our intuition and using generators in the Artin braid group allow us to build a more complete picture of new concepts. [BKL98]

Definition 1.3.5 (Birman Ko Lee Braid Group)
The Birman Ko Lee braid group on n strands $\mathscr{B}_n$ is given by the generators

$$\{\sigma_{t,s} : t, s = 1, \cdots, n \text{ and } t \neq s\}$$

along with the following 'braid relations':

1. $\sigma_{t,s}\sigma_{r,q} = \sigma_{r,q}\sigma_{t,s}$ *if* $(t-s)(t-q)(s-r)(s-q) > 0$
2. $\sigma_{t,s}\sigma_{s,r} = \sigma_{t,r}\sigma_{t,s} = \sigma_{s,r}\sigma_{t,r}$ *if* $1 \leq r < s < t \leq n$

Note 1.3.5.1
Since there is no restriction on the ordering of s and t we can say that $\sigma_{t,s}^{-1} = \sigma_{s,t}$.

This representation is indeed similar to the Artin group except generators are not restricted to adjacent strands. To see this consider the braid diagram that corresponds to the generator $\sigma_{t,s}$ in $\mathscr{B}_n$ as in Figure 1.3.9.

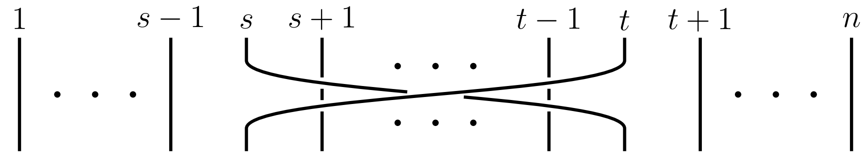

Figure 1.3.9: The Birman Ko Lee generater $\sigma_{t,s}$ braid diagram on n strands.

These generators are often referred to as band generators. We will now take a look at what the braid relations in Definition 1.3.5 are actually saying by considering their braid diagrams.

The first Birman Ko Lee braid relation refers to the idea of commutativity at a distance that we saw in Figure 1.3.7 for the Artin presentation with the added concept of nested commutativity whereby if a generator $\sigma_{r,q}$ is contained 'underneath' $\sigma_{t,s}$, that is $s < q < r < t$ then $\sigma_{r,q}\sigma_{t,s} = \sigma_{t,s}\sigma_{r,q}$ as we see in the braid diagram in Figure 1.3.10. Note that we leave out the braids 1 to $s-1$ and $t+1$ to n for simplicity.

The second Birman Ko Lee braid relation is referred to as a partial commutativity relation and once again we can make this clearer by looking at the braid diagram of a particular case as in Figure 1.3.11. Again note that all strands except s, r and t have been removed to simplify the diagram.

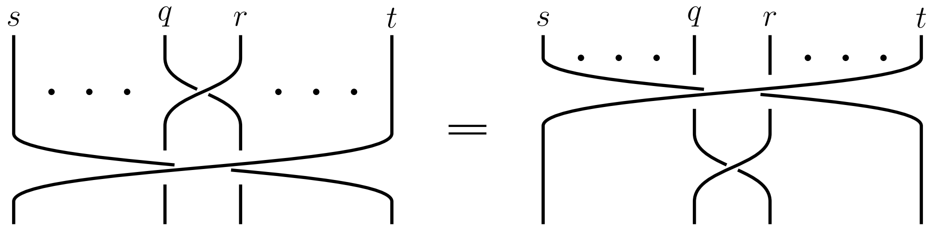

Figure 1.3.10: Braid diagram depicting nested commutativity from part of the first Birman Ko Lee braid relation, $\sigma_{r,q}\sigma_{t,s} = \sigma_{t,s}\sigma_{r,q}$ if $s < q < r < t$.

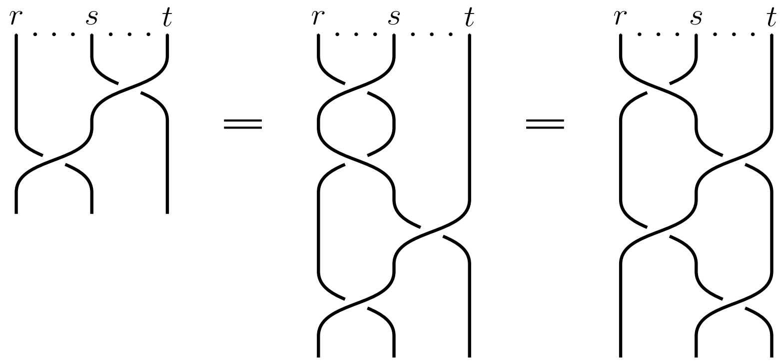

Figure 1.3.11: Braid diagram of the second Birman Ko Lee braid relation,$\sigma_{t,s}\sigma_{s,r} = \sigma_{t,r}\sigma_{t,s} = \sigma_{s,r}\sigma_{t,r}$ if $1 \leq r < s < t \leq n$.

We can verify that these diagrams are in fact equal by considering the Reidemeister moves. In the second diagram, we can enact the second Reidemeister move once on the strands r and s and we can then see that this must be equivalent to the first diagram. For the third diagram, we pull the $r-th$ strand through the crossing between s and t i.e. the third Reidemeister move, then we have precisely the second diagram and therefore our equivalence. So to summarise we have the observation that the braid relations in Definition 1.3.5 account for commutativity at a distance and the second and third Reidemeister moves with the extra condition of nested commutativity which is an obvious additional component.

Another way to see that these diagrams in Figure 1.3.11 are equivalent is to consider their Artin generator form on B_3 with strand r being strand 1 in B_3, strand s being strand 2 and strand t being strand 3. The first diagram is clearly just $\sigma_2\sigma_1$, and the third can be represented as $\sigma_1^{-1}\sigma_2\sigma_1\sigma_2$ which we can manipulate using the Artin braid relations as follows:

$$\sigma_1^{-1}\sigma_2\sigma_1\sigma_2 = \sigma_1^{-1}\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1$$

Here we use the second Artin braid relation (which corresponds to the third Reidemeister move) then the fact that inverses cancel (the second Reidemeister move). In a similar method, we can easily verify that the second diagram represented as $\sigma_1\sigma_1^{-1}\sigma_2\sigma_1$ is also equal to $\sigma_2\sigma_1$ just by using the fact that inverses cancel.

Here we have been able to use both the geometric braids and the Artin braid group to solidify our understanding of these new generators. It is indeed possible to write out the general Birman Ko Lee generator $\sigma_{t,s}$ in terms of Artin braid group generators which we see below.

$$\sigma_{t,s} = (\sigma_{t-1}\sigma_{t-2}\dots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\sigma_{s+2}^{-1}\dots\sigma_{t-1}^{-1})$$

This is however exceedingly long when t and s are far apart and not a good representation of what is a fairly simple braid. We can leverage this advantage in computational settings. Indeed, we see such an application in the original paper [BKL98].

1.4 Useful Homomorphisms

Using our identification between the Artin braid group B_n and the braid group $\mathcal{B}_n$ from Theorem 1.3.4, we will revisit the homomorphism $\phi_n : B_n \to S_n$ as it was defined in Example 1.1.8 and Theorem 1.1.9.

Recall Definition 1.2.1 and the discussion of the *fundamental permutation.* This fundamental permutation of a braid $b \in \mathcal{B}_n$ is precisely the element $\phi_n(b)$.

We can think of the homomorphism ϕ_n as forgetting about how the strands cross in the braid group $\mathcal{B}_n$ so then the group is just a set of permutations on n elements, exactly the description of the symmetric group. In particular we can see that ϕ_n loses structure since, for instance, the set of braids whose strings all start and end with the same first coordinate in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ respectively, is mapped to the identity element. In fact any arbitrary element in S_n is mapped to by an infinite number of braids that have fundamental permutations that agree with that element(provided $n \geq 2$). So we can think of the braid group as a generalization of the symmetric group, in particular we have endowed the symmetric group with a memory up to the braid relations and the group axioms of how the elements have been permuted to their current state.

We discussed braids that map to the identity permutation. By definition these braids make up precisely the kernel of ϕ_n.

Definition 1.4.1 (Pure Braids)
The kernel of the natural projection $\phi_n : B_n \to S_n$ is called the pure braid group on n strands denoted P_n with $n \geq 2$.

We can tell straight away from the first isomorphism theorem that P_n is a normal subgroup of B_n and so this definition makes sense. In fact, we can define generators for P_n. It is easiest to view these generators in terms of the Artin braid group generators along with their geometric interpretations. While it is possible to define P_n with generators and relations, it is seldom useful and somewhat messy so we will avoid it. A description of these relations can be found in [KT08, Corollary 1.19].

In terms of Artin braid group generators we have that the generators of the pure braid group denoted $A_{i,j}$, can be written as follows:

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \ : 1 \leq i < j \leq n$$

We illustrate this as a braid diagram in Figure 1.4.1. Dots represent strands that have been removed for simplicity.

With this in mind, we can now look at two homomorphisms that will prove to be extremely useful in Section 1.5. In the following we will make reference to the geometric and algebraic settings, that is using the identifier φ_+ as defined in Theorem 1.3.4 to switch between the Artin braid group and braid group settings. The first of these is known as the inclusion homomorphism and we will first define it on B_n.

Definition 1.4.2 (The Inclusion Homomorphism)
The inclusion homomorphism, sometimes called the natural inclusion is defined by:

$$\iota : B_n \to B_{n+1} \ : \ \iota(\sigma_i) = \sigma_i \quad \forall i = 1, \dots, n-1$$

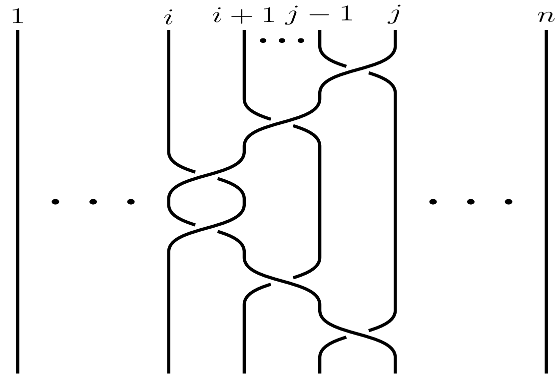

Figure 1.4.1: Pure braid group generator $A_{i,j}$ on n strands.

Geometrically this is equivalent to adding one strand with starting coordinate $(n+1,0,0)$ and ending coordinate $(n+1,0,1)$ to the braid $b \in B_n \subset \mathbb{R}^3$ such that the new strand does not interact with any of the original strands.

Proposition 1.4.3
The inclusion homomorphism $\iota : B_n \to B_{n+1}$ is injective for all n.

Proof.
Consider two braids $b_1, b_2 \in B_n$ such that $\iota(b_1)$ and $\iota(b_2)$ are equivalent up to braid isotopy. If we consider the geometric operation of ι, we can see that by restricting the isotopy between $\iota(b_1)$ and $\iota(b_2)$ to the left most n strands we achieve an braid isotopy between b_1 and b_2. □

It is clear that we can equivalently define the inclusion homomorphism on the pure braids P_n to P_{n+1} by $\iota(A_{i,j}) = A_{i,j} \forall i,j = 1,\dots,n$. Note that we use the same symbol to denote this inclusion map and that by the same argument as in the proof of Proposition 1.4.3, it is injective.

We will now define the forgetting homomorphism which is the one way inverse operation of the inclusion homomorphism on the pure braids.

Definition 1.4.4 (The Forgetting Homomorphism)
The forgetting homomorphism is defined by

$$f_n : P_n \to P_{n-1} \;:\; f_n(A_{i,j}) = A_{i,j} \;\forall i,j = 1,\dots,n-1 \quad \textit{and} \quad f_n(A_{i,n}) = e_{p_{n-1}} \;\forall i = 1,\dots,n-1$$

Geometrically, we can think of the forgetting homomorphism as removing the $n-th$ string from a braid $b \in P_n$ to obtain a new braid $f_n(b) \in P_{n-1}$. It is clear that for any braid $b' \in P_n$, if b' is isotopic to b then $f_n(b)$ is isotopic to $f_n(b')$. In the following example we use a geometric argument to verify that f_n is indeed a homomorphism on n strings.

Example 1.4.5
Considering f_n as defined in Definition 1.4.4 and two braids $b, d \in P_n$ where b' and d' represent b and d with the n-th string removed respectively. We have;

$$f_n(b)f_n(d) = b'd' = f_n(bd)$$

which is clear since under composition pure braids remain pure braids.

It is clear geometrically, that $f_n \circ \iota(b) = b$ for any $b \in P_n$ which yields another proof that ι is injective (in this case only on P_n) and implies that f_n is surjective.

We also ascertain the following set.

$$U_n = \ker(f_n : P_n \to P_{n-1})$$

Geometrically, this is all the pure braids in P_n where removing the n^{th} string yields the identity in P_n. We can also say that U_n is a normal subgroup of P_n by the first isomorphism theorem and the subject of the following theorem.

Theorem 1.4.6
For all $n \geq 2$, the group U_n is free on the $n-1$ generators $\{A_{i,n}\} \forall i = 1, \ldots, n-1$.

This theorem will be important in Section 1.5.3 however the proof goes beyond the scope of this paper and can be found in [KT08, Theorem 1.16].

1.5 Order, Fundamental Braids and the Centre

One example of the pure braids we learned about in the previous section are the square of the *fundamental braids.* The fundamental braid itself can be thought of geometrically as taking all the n strands in the identity position, laying them into a ribbon and then twisting the end of the ribbon 180 degrees, this is illustrated in Figure 1.5.1.

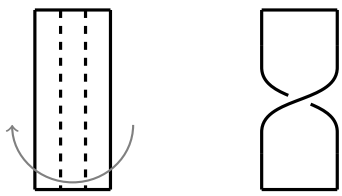

Figure 1.5.1: Visual intuition for the fundamental braids.

The fundamental braids are extremely interesting and we will take a closer look at them in Section 1.5.2. First of all we will work up to defining a partial order on B_n and our final Section 1.5.3 will focus on the centre and how it is generated.

1.5.1 Order

In order to build a means of testing the suitability of our partial order, first we will introduce the notion of a positive braid. [EM94]

Definition 1.5.1 (Positive Braid)
A braid b in B_n is considered a positive braid if it is an element of the positive submonoid B_n^+ generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

Note 1.5.1.1
Since elements of a submonoid do not necessarily have inverses, geometrically our definition makes sense, we are basically defining a positive braid to be a braid that only has crossings in the positive direction that is the same direction as in a standard generator.

This definition naturally extends to the idea of a negative braid.

Definition 1.5.2 (Negative Braid)
A braid b in B_n is considered a negative braid if it is an element of the negative submonoid B_n^- generated by $\sigma_1^{-1}, \sigma_2^{-1}, \cdots, \sigma_{n-1}^{-1}$.

Considering again the geometric interpretation of a negative braid, that would be a geometric braid made up of only negative crossings.

So we have an idea of what a positive braid is and what a negative braid is, we should take this into consideration when defining an order on B_n. We can define the partial order relation '$\leq$' as follows;

Definition 1.5.3 (The Partial Order Relation on B_n)
If for some pair $b_1, b_2 \in B_n$ there exists $p_l, p_r \in B_n^+$ such that $b_1 = p_l b_2 p_r$ then we say $b_2 \leq b_1$.

Note 1.5.3.1
It is a simple exercise to prove that this is indeed a partial order.

Example 1.5.4
Consider on B_3 the braids σ_1, which is positive and σ_2^{-1}, which is negative. We can say:

$$\sigma_1 = \sigma_2 \sigma_2^{-1} \sigma_1$$

Which satisfies our definition so we can say $\sigma_2^{-1} \leq \sigma_1$.

We will now take a look at some important properties of this partial ordering.

Proposition 1.5.5
For any positive braid $b \in B_n^+$ we can say $e \leq b$ where e is the identity braid on B_n.

Proof.
We can say that e satisfies $b = eeb$. □

Proposition 1.5.6
For any negative braid $b \in B_n^-$ we can say $b \leq e$ where e is the identity braid on B_n.

Proof.
We can use the fact that any negative braid $b \in B_n^-$ is made up of m generators of the form

$$b = \sigma_{b_1}^{-1} \sigma_{b_2}^{-1} \ldots \sigma_{b_m}^{-1}$$

so there exists a positive braid p of the form

$$p = \sigma_{b_m} \sigma_{b_{m-1}} \ldots \sigma_{b_2} \sigma_{b_1}$$

such that $pb = e$ and therefore $e = pbe$ which satisfies our definition and so $b \leq e$. □

Proposition 1.5.7
For all positive braids $b \in B_n^+$ and negative braids $d \in B_n^-$, $d \leq b$.

Proof.
By Proposition 1.5.6, we have that $d \leq e$ and by Proposition 1.5.5 we have that $e \leq b$. Since partial orders are transitive we have that $d \leq b$. □

Proposition 1.5.8
For braids $b_1, b_2 \in B_n$ we can say $b_1 \leq b_2 \iff b_2^{-1} \leq b_1^{-1}$

Proof.
$(\Rightarrow)$
If $b_1 \leq b_2$, then for some $a_l, a_r \in B_n^+$, we have $b_2 = a_l b_1 a_r$. Therefore,

$$b_2^{-1} = a_r^{-1} b_1^{-1} a_l^{-1} \implies b_1^{-1} = a_r b_2^{-1} a_l.$$

We have that $b_2^{-1} \leq b_1^{-1}$.

$(\Leftarrow)$
Holds by the same argument since $(b^{-1})^{-1} = b$ for all $b \in B_n$. □

So overall we can say our partial ordering defined in Definition 1.5.3 makes good sense when considering positive and negative braids. Remarkably, it is in fact possible to impose a left invariant total order on B_n known as the Dehornoy order and equally remarkably, P_n admits a biordering. While we have developed a lot of the tools that would be necessary to define the Dehornoy order, this goes beyond the scope of this paper, but more can be found in [Deh97].

1.5.2 The Fundamental Braids

We can now return to the fundamental braid, keep the visual intuition from Figure 1.5.1 in mind as we now define it abstractly in terms of Artin braid group generators.

Definition 1.5.9 (The Fundamental Braids)
The fundamental braid on B_n for $n \geq 2$ is defined as follows;

$$\Delta_n = b_1 b_2 \dots b_{n-1} : b_k = \sigma_k \sigma_{k-1} \dots \sigma_1$$

Example 1.5.10
We will write out the fundamental braids up to B_4 explicitly in terms of Artin braid group generators then we will look at the braid diagrams of these braids.

$$\begin{aligned}\Delta_2 &= (\sigma_1)\\ \Delta_3 &= (\sigma_1)(\sigma_2\sigma_1)\\ \Delta_4 &= (\sigma_1)(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\end{aligned}$$

The brackets that have been put in correspond to each b_k as in Definition 1.5.9 and a pattern that is clear in the braid diagrams in Figure 1.5.2.

1.5.3 The Centre

Recall that for the visual intuition, we thought of Δ_n as a 180 degree twist of a ribbon in which lay n strands in the identity position as in Figure 1.5.1 so clearly for Δ_n^2 we rotate the end of our ribbon 360 degrees as in Figure 1.5.3.

Recall that an element g of group G is in the centre of G or $Z(G)$ if g commutes with every element of G. It is fairly clear from Figure 1.5.3 that Δ_n^2 should be in the centre of B_n since by creating this twist we haven't really done anything except make a 360 degree twist in the ambient space, so if we pass a braid through this twist it should not change. It is not so clear that it should generate the entire centre of B_n as we shall see in Theorem 1.5.15. We will take a look at a concrete example before we take this notion any further.

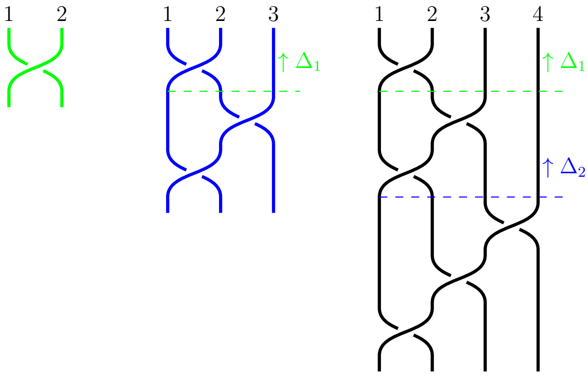

Figure 1.5.2: Braid diagrams corresponding to fundamental braids on B_2, B_3 and B_4 respectively. The dashed lines allow us to relate the braid diagrams to the abstract definition of a fundamental braid 1.5.9.

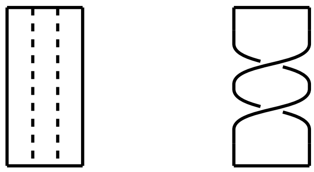

Figure 1.5.3: Visual intuition for Δ_n^2.

Example 1.5.11

Consider the braid group B_3 and the braid $b = \sigma_1^{-1}$ we want to see visually using braid diagrams that:

$$b\Delta_n^2 = \Delta_n^2 b$$

We will construct these diagrams below.

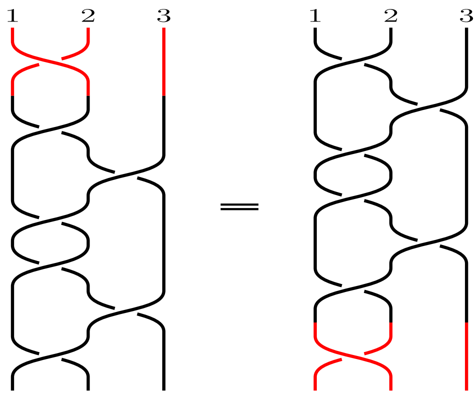

Figure 1.5.4: Braid diagrams showing $b\Delta_n^2 = \Delta_n^2 b$, b is highlighted in red.

We can see that to get the diagrams to look the same we simply use the second Reidemeister move where b connects to Δ_n^2 on each diagram then 'slide' the remaining generators to the right positions. It is interesting to note that on the square of the fundamental braid itself, any two strands interact exactly twice, allowing this 'sliding' to happen.

This should strengthen our geometric intuition that Δ_n^2 should be in the centre of B_n. We will provide an algebraic proof below adapted from [Val21, Appendix B].

Lemma 1.5.12
For the Artin braid group B_n and the fundamental braid Δ_n and $n \geq 2$, we have the following identities.

(i) $b_k\sigma_i = \sigma_{i-1}b_k \ \forall i = 2, \ldots, k$

(ii) $\Delta_n\sigma_1 = \sigma_{n-1}\Delta_n$

Where $b_k = \sigma_k\sigma_{k-1}\ldots\sigma_1$ *as in Definition 1.5.9.*

Proof.
First we prove part *(i)*.

$$\begin{aligned} b_k\sigma_i &= \sigma_k \ldots \sigma_{i+1}\sigma_i\sigma_{i-1} \ldots \sigma_1\sigma_i && \textit{First Braid Relation} \\ &= \sigma_k \ldots \sigma_{i+1}\sigma_i\sigma_{i-1}\sigma_i \ldots \sigma_1 && \textit{Second Braid Relation} \\ &= \sigma_k \ldots \sigma_{i+1}\sigma_{i-1}\sigma_i\sigma_{i-1} \ldots \sigma_1 && \textit{First Braid Relation} \\ &= \sigma_{i-1}b_k \end{aligned}$$

For part *(ii)* we claim that $b_n^2 = b_{n-1}b_n\sigma_1$ for some $n \geq 2$ We use the fact that σ_i commutes with b_j if and only if $j \leq i-2$ by the first braid relation.

$$\begin{aligned} b_n^2 = b_nb_n &= \sigma_n\sigma_{n-1}\sigma_nb_{n-2}b_{n-1} \\ &= \sigma_{n-1}\sigma_n\sigma_{n-1}b_{n-2}b_{n-1} \\ &= \sigma_{n-1}\sigma_nb_{n-1}^2 \end{aligned}$$

Using this identity we proceed as follows.

$$\begin{aligned} b_n^2 = \sigma_{n-1}\sigma_nb_{n-1}^2 &= (\sigma_{n-1}\sigma_n)(\sigma_{n-2}\sigma_{n-1})b_{n-2}^2 \\ &= (\sigma_{n-1}\sigma_n)(\sigma_{n-2}\sigma_{n-1})(\sigma_{n-3}\sigma_{n-2})b_{n-3}^2 \end{aligned}$$

Continuing in this way, we get:

$$b_n^2 = (\sigma_{n-1}\sigma_n)(\sigma_{n-2}\sigma_{n-1})(\sigma_{n-3}\sigma_{n-2})\ldots(\sigma_3\sigma_4)(\sigma_2\sigma_3)(\sigma_1\sigma_2)b_1^2$$

We can commute forward every other term in this chain so that all of these terms land between σ_1 and b_1^2 such that their order is maintained. Since $b_1^2 = \sigma_1\sigma_1$, we have

$$b_n^2 = b_{n-1}b_n\sigma_1$$

as desired. Now, we consider $\sigma_{n-1}\Delta_n$.

$$\begin{aligned} \sigma_{n-1}\Delta_n &= \sigma_{n-1}b_1 \ldots b_{n-2}b_{n-1} \\ &= b_1 \ldots \sigma_{n-1}b_{n-2}b_{n-1} \\ &= b_1 \ldots b_{n-1}^2 \\ &= b_1 \ldots b_{n-2}b_{n-1}\sigma_1 \\ &= \Delta_n\sigma_1 \end{aligned}$$

□

Lemma 1.5.13
For the Artin braid group B_n and the fundamental braid Δ_n and $n \geq 2$:

$$\Delta_n \sigma_i = \sigma_{n-i}\Delta_n \forall i = 1, \ldots, n-1$$

Proof.
We proceed by induction on n for all $i = 2, \ldots, n-1$. For $n = 2$, we have:

$$\Delta_2 \sigma_1 = \sigma_1 \sigma_1 = \sigma_1 \Delta_2$$

We can assume that our identity holds for some $n = k > 2$. For $n = k+1$ and $i = 2, \ldots, k$ we have:

$$\begin{aligned}
\Delta_{k+1}\sigma_i &= \Delta_k b_k \sigma_i \\
&= \Delta_k \sigma_{i-1} b_k && \textit{Lemma 1.5.12 part (i)} \\
&= \sigma_{k-(i-1)}\Delta_k b_k && \textit{Induction Hypothesis} \\
&= \sigma_{(k+1)-i}\Delta_{k+1}
\end{aligned}$$

We have the $i = 1$ case by Lemma 1.5.12 part *(ii)*. □

For the visual intuition on why Lemma 1.5.13 is the case, recall that we think of the fundamental braid as a 180 degree twist of the ambient space as in Figure 1.5.1. Passing a braid diagram through this structure is equivalent to 'travelling through the page' to view the braid diagram from the other side. We conclude our investigation with the following lemma.

Lemma 1.5.14
For the Artin braid group B_n, the fundamental braid Δ_n^2 is in the centre of B_n.

Proof.
Given some braid b in the Artin braid group B_n given by $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_m}$ for some $m \in \mathbb{N}$ and $i_j \in \{1, \ldots, n-1\}$ $\forall j = 1, \ldots, m$, let b' be the braid b where each element σ_{i_j} is replaced by σ_{n-i_j}. Note that $(b')' = b$. Applying Lemma 1.5.13, we have:

$$\Delta_n^2 b = \Delta_n b' \Delta_n = b'' \Delta_n^2 = b \Delta_n^2$$

□

It is certainly interesting to note the difficulty that we have in proving the statement of Lemma 1.5.14 in a purely Algebraic context when it is such a clear fact geometrically. Whenever we are working in the Artin braid group, we have removed all information about geometric braids that is not relevant to the isotopy classes. While working in this lower information environment is often informative, the extra information that exists in the geometric braids makes it much easier, in many instances, to intuitively see that results are in fact true. We use a mixture of geometric and algebraic arguments to prove the following Theorem.

Theorem 1.5.15
For the Artin braid group B_n and for $n \geq 2$, $Z(B_n) = Z(P_n)$ is the infinite cyclic group generated by Δ_n^2.

Proof.
We have that $\Delta_n^2 \in Z(B_n)$ by Lemma 1.5.14. To show that all elements in $Z(P_n)$ are powers of Δ_n^2 we proceed by induction. For $n = 2$ it is clear that P_2 is generated by $\Delta_2^2 = \sigma_1^2$ and thus

$Z(P_2) = P_2$. For the inductive step we first pick $\beta \in Z(P_n)$, where $n \geq 3$. Recall the forgetting homomorphism $f_n : P_n \to P_{n-1}$ which removes the n-th strand, and its kernel $U_n = \ker(f_n)$. Since f_n is surjective and U_n is its kernel, we have the short exact sequence:

$$1 \longrightarrow U_n \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow 1.$$

Furthermore, the inclusion map ι acts as a section for f_n, meaning $f_n \circ \iota$ is the identity map on P_{n-1}. This property means the short exact sequence is split. A split short exact sequence gives rise to a semidirect product structure. Therefore, P_n is isomorphic to the semidirect product $U_n \rtimes P_{n-1}$. So we have that for any pure braid β, we can uniquely write $\beta = \iota(\beta')\beta_n$ where $\beta' = f_n(\beta) \in P_{n-1}$ and $\beta_n \in U_n$. Recalling the generators of the pure braid group $A_{i,j}$ as in Figure 1.4.1, define $\gamma_n = A_{1,n}A_{2,n} \dots A_{n-1,n}$, γ_5 is shown in Figure 1.5.5 as an example.

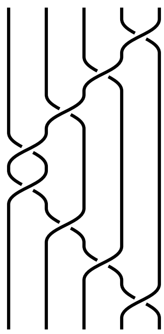

Figure 1.5.5: Braid diagram of γ_5.

A simple geometric argument allows us to state that γ_n commutes with any element of $\iota(P_{n-1}) \subseteq P_n$ and therefore with $\iota(\beta')$. Since β was chosen to be in the centre of P_n, we must have that it commutes with γ_n. So γ_n must commute with $\beta_n = \iota(\beta')^{-1}\beta$ so the group G generated by β_n and γ_n is abelian. It is clear that G is a subgroup of U_n and by Theorem 1.4.6 we have that U_n is free and therefore G is free by the Nielsen-Schreier Theorem [Sti93, Pgs.103-104]. This implies that $G \cong \mathbb{Z}$ since $\mathbb{Z}$ is the only abelian free group. We define the homomorphism $l_{i,j} : P_n \to \mathbb{Z}\ \forall\ 1 \leq i < j \leq n$ as

$$l_{i,j}(A_{r,s}) = \begin{cases} 1 & i,j = r,s \\ 0 & i,j \neq r,s \end{cases}$$

for all pairs (r,s) distinct from (i,j). It is easy to check that this a homomorphism. Clearly, we have that $l_{1,n}(\gamma_n) = 1$ so γ_n must generate $G \cong \mathbb{Z}$. In particular $\exists k \in \mathbb{N} : \gamma_n^k = \beta_n$. Since the forgetting homomorphism is surjective and surjective homomorphisms map centres into centres, $\beta' = f_n(\beta) \in Z(P_{n-1})$. We use our induction assumption, $\beta' = (\Delta_{n-1}^2)^m$ for some $m \in \mathbb{Z}$. We now show that $m = k$.

As before we expand β as $\beta = \iota(\beta')\beta_n = \iota((\Delta_{n-1}^2)^m)\gamma_n^k$, noting that $l_{i,n}(\gamma_n^k) = k$ and $l_{i,n}(\iota(\Delta_{n-1}^2)) = 0$ for all $i = 1, \dots, n-1$ it follows that $l_{i,n}(\beta) = l_{i,n}(\iota((\Delta_{n-1}^2)^m)\gamma_n^k) = k$ for all $i = 1, \dots, n-1$. In particular, $l_{i,n}(\beta)$ does not depend on i. Since $\beta \in Z(P_n)$, we have $\sigma_{n-1}\beta\sigma_{n-1}^{-1} \in Z(P_n)$ and we have that $l_{i,n}(\sigma_{n-1}\beta\sigma_{n-1}^{-1})$ does not depend on $i = 1, \dots, n-1$. Using our expansion of β and from the definitions we obtain

$$l_{1,n}(\sigma_{n-1}\beta\sigma_{n-1}^{-1}) = l_{1,n-1}(\beta) = m$$

and

$$l_{n-1,n}(\sigma_{n-1}\beta\sigma_{n-1}^{-1}) = l_{n-1,n}(\beta) = k$$

Thus $m = k$. So finally, we have:

$$\beta = \iota((\Delta_{n-1}^2)^m)\gamma_n^k = \iota((\Delta_{n-1}^2)^k)\gamma_n^k = (\iota(\Delta_{n-1}^2)\gamma_n)^k = (\Delta_n^2)^k$$

That is, $Z(P_n) = (\Delta_n^2)$.

To show that $Z(B_n) = (\Delta_n^2)$ we use again that surjective homomorphisms map centres into centres and that there exists a surjective homomorphism defined in Theorem 1.1.9 that maps B_n onto S_n. Since $Z(S_n) = \{e_{S_n}\}$ we have that:

$$Z(B_n) \subseteq Z(P_n) \subseteq (\Delta_n^2) \subseteq Z(B_n)$$

and therefore

$$Z(B_n) = Z(P_n) = (\Delta_n^2)$$

It is easy to see geometrically that $(\Delta_n^2) \cong \mathbb{Z}$. □

2 Knots and Links

We now want to take a look at braid theory's famous older brother, knot theory. In this chapter we will explore how we adapt our understanding of what a 'knot' is so that it can be a useful object in a mathematical setting in Section 2.1. Note that this first section is purely informal and aims to provide the correct motivation and intuitions for the rest of our discussions and the rigorous definitions given in Section 2.2. We will then motivate the most prominent open question in the field; how can we tell different knots apart? We will then investigate some methods that have so far been found to help answer this question and discuss the success of each with a particular focus on how can leverage algebra to help us understand these inherently topological objects.

2.1 Introduction to Knots and Links

Knots are not likely to be something that is completely new to any reader, whenever we hear the word knot we might think of a length of string that has been looped, passed through itself then pulled tight.

Figure 2.1.1: How we normally think of a 'knot'.

The information about the knot that we are interested in is how the string passes through itself. To see this we want to 'tease out' the 'knot' as in Figure 2.1.1. This is our first consideration to make this 'knot' mathematically interesting.

The second thing we need to consider to make this 'knot' mathematically useful is that we want to use something similar to what we used for braids to define equivalence, that was braid isotopy 1.2.4. We will formally define what these objects are in Section 2.2, for now think of them as pieces of string in $\mathbb{R}^3$ with the standard topology. As it stands, our 'knot' is currently homotopic to a point, to see this, recall the idea of a continuous deformation from Example 1.2.3 and imagine shortening the 'loose' ends of our 'knot' incrementally until they meet. Indeed no matter how we looped our string through itself, it would still be homotopic to just a single point. A possible solution to this is to consider the fact that the circle is not contractible, that

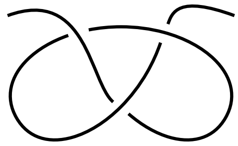

Figure 2.1.2: Our 'knot' after we have 'teased it out'.

is it is not homotopic to a single point (for incontractibility see [Hat00, Chapter 0]). So in order to stop our knots being from being contractible we join up the two loose ends of our 'knot' to create a closed loop.

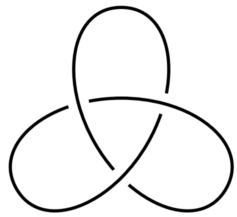

Figure 2.1.3: A mathematical knot, the trefoil.

Now we have a mathematical knot, in particular the knot in Figure 2.1.3 is called a trefoil and is referred to as the simplest non trivial knot. The trivial knot is called the unknot which is just a closed loop of string as illustrated in Figure 2.1.4.

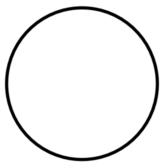

Figure 2.1.4: The unknot

At this point it will also be useful to mention the concept of links, links are just multiple knots that may be looped together, for example the trivial link with n components is just n copies of the unknot that do not interact with each other. The simplest non trivial version of a link is called the Hopf link which is illustrated in Figure 2.1.5. Note we abuse the language here slightly, as we will see in Section 2.2, knots themselves are considered links with 1 component.

It should be noted that the diagrams in Figures 2.1.3, 2.1.4 and 2.1.5 are all projections of a 3 dimensional knot or link onto 2 dimensions (recall the notion of braid diagrams in Definition 1.2.6), we will formalise this notion in Section 2.3.1. For now we want to take the objects in Figures 2.1.3, 2.1.4 and 2.1.5 and define them away from the concept of pieces of 'string'. Instead we must define them in a topological setting to give us some really solid foundations on which to build our discussion.

2.2 Topological Definition

In order to proceed, as within Section 1.2, we assume a very small amount of topology on the readers part, all the relevant terminology can be found in [Mun00] and while we do need the subject to define our knots sufficiently rigorously, it is not necessary to have a full understanding

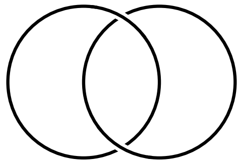

Figure 2.1.5: The Hopf link.

to proceed with further concepts. We proceed with our construction of a definition for knots and links.

Definition 2.2.1 (Hausdorff Space)
A topological space X is called a Hausdorff space if for each pair of distinct points $x_1, x_2 \in X$ there exist disjoint open neighbourhoods $U_1, U_2 \in X$ of x_1 and x_2 respectively.

Definition 2.2.2 (Second Countable Space)
A topological space X is called second countable if there exists some countable collection

$$\mathcal{U} = \{U_i\}_{i=1}^{\infty}$$

of open subsets of X such that any open subset of X can be written as a union of elements of some subfamily of $\mathcal{U}$. That is, X has countable basis.

Note that in the following we are referring to a *topological manifold* however we omit the adjective for brevity. [Hat00, Pg.231]

Definition 2.2.3 (Manifold and Submanifold)
An n-dimensional manifold M is a second countable Hausdorff space such that every point $x \in M$ has an open neighbourhood that is homeomorphic to $\mathbb{R}^n$. A manifold S is a submanifold of M if S is a subspace of M.

Note 2.2.3.1
It is clear that S necessarily has dimension less than or equal to n and that S inherits its topology from M.

In the following definition we say that a manifold M is *closed* if it is compact and every point $m \in M$ has an open neighbourhood $U \subseteq M$ such that U is homeomorphic to $\mathbb{R}^3$. We also say that a 1 dimensional closed submanifold L of M is *locally flat* if every point of L has a neighbourhood $U \subseteq M$ such that the pair $(U, U \cap L)$ is homeomorphic to the pair $(\mathbb{R}^3, \mathbb{R} \times \{0\} \times \{0\})$.

Definition 2.2.4 (Geometric Link)
Let M be a 3 dimensional manifold. A geometric link in M is a locally flat, closed, 1 dimensional submanifold of M.

Note 2.2.4.1
We adopt the convention that unless otherwise specified, M should be taken to be $\mathbb{R}^3$.

Including the locally flat criteria in this definition implies that our geometric link is a subset of M° (the union of all open subsets contained in M) and excludes any kind of *locally wild* behaviour inside M°. We define a point l in our geometric link to be locally wild if there does not exist a neighbourhood of l that is *ambient isotopic* to a straight line in M.

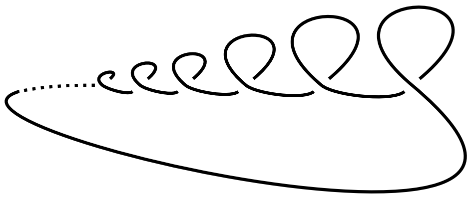

Figure 2.2.1: A geometric link that exhibits locally wild behaviour.

Note 2.2.4.2
We define link isotopy in Definition 2.3.2. In this context, we can say that u, a neighbourhood of l and i a straight line in M are ambient isotopic if there exists a self isotopy (Definition 2.3.1) $\{F_s\}$ from M to M such that $F_1(u) = i$.

Examples of geometric links that exhibit locally wild behaviour are loops that repeat infinitely while getting incrementally smaller as in Figure 2.2.1 or fractals. Since it is a closed 1 dimensional manifold, a geometric link must consist of a finite number of components that are homeomorphic to the standard unit circle:

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

A space homeomorphic to S^1 is called a topological circle. A geometric link consisting of $n \geq 1$ topological circles is called an n component link. A trivial example of a 3 component link is the boundary of 3 disjoint embedded 2-disks in $\mathbb{R}^3$ that we see in Figure 2.2.2.

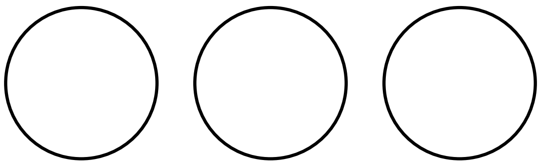

Figure 2.2.2: A trivial example of a 3 component geometric link in $\mathbb{R}^3$.

Clearly, this is as we stated earlier, that a trivial link that has in this case 3 components is just 3 copies of the unknot. Our last step is to define a geometric knot.

Definition 2.2.5 (Geometric Knot)
A geometric knot is a 1 component geometric link.

We may refer to a section of a geometric braid as a strand or string. The specific section we are referring to will be made clear by the context.

2.3 Equivalence

With geometric braids we defined equivalence in a way that captures the information of how the strings cross over each other. With knots we aim to be able to say that two geometric knots are equivalent if the way in which the string passes through itself in each is the same. We proceed as follows.

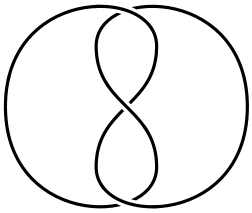

Figure 2.3.1: A different representation of the trefoil from Figure 2.1.3.

Definition 2.3.1 (Self Isotopy)
A self isotopy of a manifold M onto itself is a continuous family of homeomorphisms

$$\{F_s : M \to M\}_{s \in I}$$

such that $F_0 = id_M$.

The fact that the family is continuous means that the mapping

$$F : M \times I \to M \quad , \quad (x, s) \mapsto F_s(x)$$

with $x \in M, s \in I$ is continuous. For a proof of this see [KT08, Sec.1.7.1].

Definition 2.3.2 (Link Isotopy)
For two geometric links L and L' in some 3 dimensional manifold M, there exists a link isotopy between L and L' if there exists a self isotopy $\{F_s\}$ from M onto itself such that $F_1(L) = L'$.

Note 2.3.2.1
This definition is often generalised to ambient isotopy as it describes a continuous deformation of the ambient space. For more detail on this see [BZH13, Chapter 1].

The intuition here, is the idea of continuously deforming the manifold that the geometric links are a submanifold of and the geometric links 'come along for the ride' in some sense.

It is clear that link isotopy forms an equivalence class on all geometric links. The corresponding equivalence classes are called *links* (or *knots* with respect to geometric knots). For instance the geometric knot depicted in Figure 2.3.1 is link isotopic to the trefoil in 2.1.3 so we also call the geometric knot in Figure 2.3.1 a trefoil as they are in the same link isotopy equivalence class.

Note here that it is common to abuse language and refer to knots when we actually mean links (noting that all knots are links). We may follow this convention and it will be made clear if a distinction is necessary.

2.3.1 Link Diagrams

As mentioned in Section 2.1, all of our geometric links are being represented as a 2 dimensional projection of a 3 dimensional object, these projections are called *link diagrams* (or knot diagrams for 1 component links). This is similar to the notion of braid diagrams(Definition 1.2.6) in that link diagrams serve as a projection of our links onto 2 dimensions. Unfortunately we will not be able to take the approach of defining link diagrams independently then proving their association to geometric links as it goes beyond the scope of this paper. This approach can however be found in [KT08, Sec.2.1.2]. We will proceed with the idea that link diagrams are a projection of geometric links onto a plane as in Figure 2.3.2.

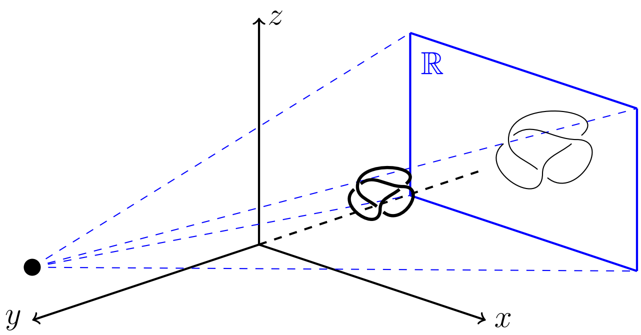

Figure 2.3.2: Intuition behind knot diagrams. Think about the shadow that the geometric knot casts on the plane.

One thing to note about these projections is the crossing points, that is the point on the projection where the strand passes over itself. Like braids we denote the *undercrossing* strand by breaking the strand near the crossing.

It can be shown that all geometric links are link isotopic to a geometric link that has a 'nice' link diagram in that each crossing can be clearly seen, crossings are well defined and crossings are the only points the diagram crosses over itself. To see this imagine performing link isotopy on the geometric link in Figure 2.3.2 and think about how this affects the projection.

This is analogous to Theorem 1.2.8 however defining a 'nice' link diagram is as noted, beyond the scope of this paper and a full discussion can be found in [KT08, Sec.2.1.2].

It is possible to define a notion of equivalence on knot diagrams in their own right. This construction involves Ω_2, Ω_3 (and Ω_0) from our construction of Reidemeister equivalence on braids 1.2.10. As mentioned in Remark 1.2.10.1, we must include another move in this context, outlined in Figure 2.3.3.

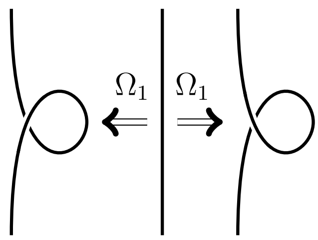

Figure 2.3.3: The Reidemeister move Ω_1.

We obtain the inverse Ω_1^{-1} by reversing the arrows in Figure 2.3.3. Note that, like Ω_2 there are two moves that we refer to as Ω_1. Putting these moves together gives us the following definition of Reidemeister equivalence for links.

Definition 2.3.3 (Reidemeister Equivalence for Links)
Two link diagrams L and L' are Reidemeister Equivalent if L can be transformed into L' by a finite sequence of the moves Ω_1, Ω_2 and Ω_3 and their inverses.

Note 2.3.3.1
As with braids Ω_0, which allows for arbitrary lengthening and shortening of strands, is assumed. We are using the same word to describe equivalence among links diagrams as with braid diagrams however this we will make clarifications where appropriate.

We can now state Reidemeister's Theorem for links however we just refer to this as Reidemeister's Theorem since this is the purpose for which it originally came about.

Theorem 2.3.4 (Reidemeister's Theorem)
Two link diagrams represent link isotopic geometric links if and only if these diagrams are Reidemeister equivalent.

As noted earlier, the proof of this Theorem can be found in Artin's original book *Knotentheorie* translated into English at [RBCS83].

As with braids, we will use Reidemeister's Theorem and the fact that any knot can be represented by a knot diagram, to switch freely between the geometric and diagram settings and their corresponding notions of equivalence. We may refer to the Reidemeister equivalence classes of link diagrams as links.

2.3.2 Prime Knots

Throughout this paper we will primarily be interested in *prime knots*. In order to get a sense of these objects we must define knot composition. This requires the following.

Definition 2.3.5 (Arc)
For a given link diagram D we call an arc of D any subset of D that starts as the undercrossing strand at some crossing and ends as the undercrossing strand at the next crossing.

The phrase 'next crossing' here is taken to mean the first crossing we encounter as we travel along the knot from the first undercrossing. So for example the unknot in Figure 2.1.4 has 1 arc while the trefoil in Figure 2.4.2 has 3 where each arc is a different colour.

We define *knot composition* to be the process by which we take two knots and isolate an arc on each, at this arc we make a cut and then glue the loose ends from one knot onto the other as in Figure 2.3.4.

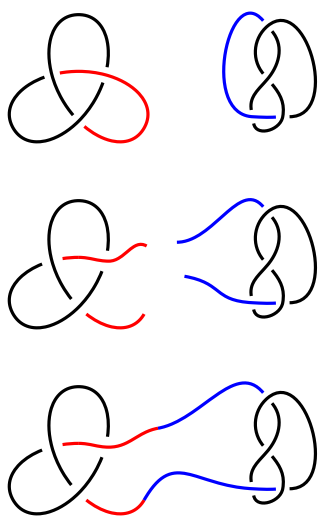

Figure 2.3.4: The knot composition procedure.

Definition 2.3.6 (Prime Knots)
A knot is called a prime knot if it is not a composition of two non trivial (not the unknot) knots.

It has been tabulated that there are 7 (single component) prime knots with 7 or fewer crossings (in their respective knot diagrams), 165 prime knots with 10 or fewer crossings and 1,701,936 prime knots with 16 or fewer crossings as uncovered in [HTW98]. More prime knots have been calculated however making so called *prime knot tables* for knots with a larger and larger number of crossings is exceedingly difficult due to the exponential increase in complexity.

We may adopt the notation used in the table in Appendix A where a knot is named C_n where C is the number of crossings (on the standard projection) and n is an arbitrary identifier linked to the order in which these knots where originally discovered.

2.3.3 Polygonal Knots

As was the case with geometric braids, we aim to find ways that allow us to reduce the vast amounts of information carried by a geometric knot yet retain the information that is necessary to recover isotopy equivalence classes. Analogously to braids, we can look at a subset of geometric links known as polygonal links.

Definition 2.3.7 (Polygonal Link)
A link is called polygonal if each of its components are formed by a finite number of consecutive linear segments.

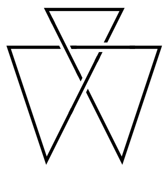

Figure 2.3.5: A polygonal trefoil.

Note 2.3.7.1
Recall Definition 1.2.12 of polygonal braids, Definition 2.3.7 is the same principal for links. Figure 2.3.5 represents a polygonal trefoil. We refer to any given linear segment as an edge and any point where two linear segments meet as a vertex.

Proposition 2.3.8
Any geometric link can be approximated by a polygonal link.

Proof.
The proof follows from the Linear Approximation Theorem [Mag22, Pg.200]. □

We extend our notion of Δ moves 1.2.15 to the knot setting.

Definition 2.3.9 (Delta Move)
Let L be a polygonal link and D be a triangle in $\mathbb{R}^3$. If we denote the edges of D by u, v and w then without loss of generality, if $L \cap D = u$, then $L' = (L \setminus u) \cup (v \cup w)$ defines a new polygonal link in $\mathbb{R}^3$. We say that L' results from a Δ move. The inverse process is given by Δ^{-1}.

Note 2.3.9.1
It will be clear in which context we are applying a delta move and therefore whether the knot or braid definition applies. We define orientation in Definition 2.4.7, any Δ move performed on an oriented polygonal knot must retain orientation.

Since polygonal links are a subset of geometric links, we can pass to link isotopy classes. An equivalent set of equivalence classes on the polygonal links only can be drawn from the theorem below.

Theorem 2.3.10
Two polygonal links are link isotopic if and only if one can be transformed into the other by a finite series of Δ moves.

It is clear that links connected by a finite sequence of Δ moves are link isotopic. Proof of the converse can be found in [Kaw96, Appendix A.4]. Like polygonal braids, polygonal links allow for knots to be accurately input into a computer and also allow for some algorithmic proofs. For an application of polygonal links and Δ moves see the proof of Theorem 3.1.3.

2.4 Invariants and Quandles

We have been able to pass to isotopy equivalence classes. We have stated that the geometric knot in Figure 2.3.1 is equivalent to that in Figure 2.1.3 however this is not immediately obvious. Indeed how do we even know that the trefoil is not equivalent to the unknot? How do we know there isn't some complicated way of transforming the trefoil depicted in Figure 2.3.1 into the unknot?

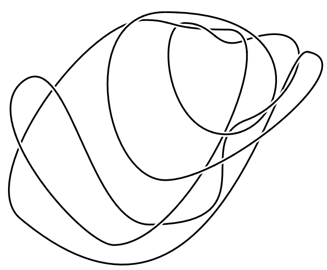

Figure 2.4.1: Some knot.

It is possible to show through Reidemeister moves that the knot in Figure 2.4.1 is in fact the unknot but this is far from obvious. Therefore it is natural to ask for a method for easily being able to tell whether two knots are equivalent. Finding such a method has been the pursuit of knot theorists for over 100 years since Tait began his intensive study in the 1870s and this remains the most important open question in the field.[3] The search for this method is not without progress and in the rest of this chapter, we will discuss some methods that have been developed in an attempt to do so. The overarching goal will be to arrive at the Alexander polynomial through the atypical algebraic route.

2.4.1 Invariants

Whenever we are considering a method to distinguish between different links we want to be able to calculate some object such as a number, polynomial or group from a link. Furthermore, such an object may be evaluated at link diagrams and it should remain the same under the

[3]Knot theory first became a popular field of study in the late 1800s as Sir William Thompson (later Lord Kelvin) developed his theory of vortex atoms which modelled atoms as knots. Peter Guthrie Tait is often referred to as the father of the field as he was the first to attempt to classify and tabulate knots. It appears however, (as seems to be the case with most mathematical fields) that Gauss was the first to discuss the field in his diary when he was 17.

Reidemeister moves. Of course, this means that one of these objects can represent multiple distinct links but it does give us a way to categorise links as equivalent 'in the eyes of' certain objects. These objects are called invariants.

Definition 2.4.1 (Invariant)
An invariant is any property of a link diagram that is preserved under Reidemeister moves.

That is, an invariant is a property of a link that is the same for every geometric link in that isotopy class, for instance an invariant would be a property shared by the unknot as represented in Figure 2.1.4 and Figure 2.4.1. The perfect invariant would be an algebraic object that would be easy to calculate from any given link diagram and would be unique for each distinct link, possibly even under some strengthening of our definition of equivalence such as passing to oriented isotopy equivalence classes (orientation is defined in Definition 2.4.7). Many strong invariants have been found however it is yet to be proven that a perfect invariant as described above does exist.

In this section we will present two known invariants and discuss their strengths and weaknesses. We will primarily follow and expand upon, the work of Kauffman in [GM16, Pgs.7-19] and we will take the end of this section to its logical conclusion in Section 2.4.3. We will draw upon several other sources that shall be specified throughout.

A simple example of an invariant is p-colourability attributed to Ralph Fox who first developed the concept as a way to help students understand homomorphisms from the algebraic object of the knot quandle onto a dihedral quandle. These terms will be properly defined in due course and we can see this homomorphism for ourselves in Example 2.4.20. See [Azr12] for a full historical account. In order to introduce this we will first look at a simple, intuitive case known as tricolourability.

Definition 2.4.2 (Tricolourability)
In order for a link diagram to be tricolourable each arc must be coloured by up to 3 different colours satisfying the following conditions:

1. *At least 2 colours are used.*
2. *At any crossing, if 2 colours are used all 3 must be used.*

By colouring here, we literally mean 'colouring in' an arc. Consider the following example on the trefoil.

Example 2.4.3
One way (and indeed the only possible way) to colour the trefoil would be to colour each arc a different colour as in Figure 2.4.2. Here we have satisfied the conditions of Definition 2.4.2,

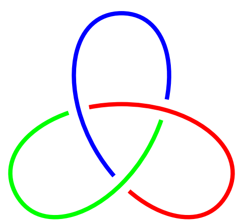

Figure 2.4.2: The trefoil tricoloured.

indeed we have used 2 colours and at each crossing, all arcs are of a different colour so we say that the trefoil is tricolourable.

Now that we have a suitable understanding of what tricolourability is we can prove that it is an invariant.

Proposition 2.4.4
Tricolourability is an invariant.

Proof.
Our strategy will be to illustrate that tricolourability is preserved by the Reidemeister moves. In particular, we will assume that we have a tricolourable link L and perform each of the three Reidemeister moves on L and show that in each case tricolourability is preserved. Note that while there are different configurations of the third Reidemeister move, it will suffice to show one.

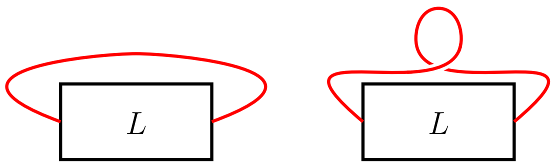

Figure 2.4.3: The first Reidemeister move on L tricoloured.

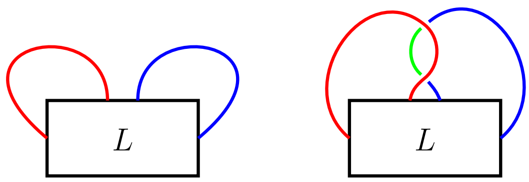

Figure 2.4.4: The second Reidemeister move on L tricoloured.

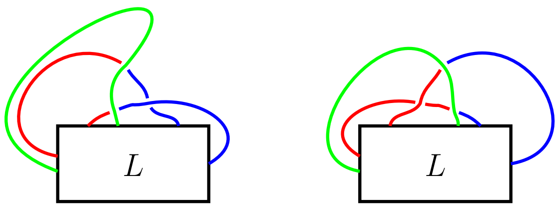

Figure 2.4.5: The third Reidemeister move on L tricoloured.

We can see that in Figures 2.4.3, 2.4.4 and 2.4.5, it is clear that whether we choose to do a Reidemeister move or its inverse, we can always tricolour the affected area in a way that does not change how L is tricoloured. Recall there were two moves that we refer to as Ω_1 and Ω_2, to see the analogous moves simply invert the crossings in Figures 2.4.3 and 2.4.4. □

It is worth noting here, that the strategy of proving that some property is an invariant by checking if it varies under each of the Reidemeister moves is an extremely common application of Reidemeister's Theorem 2.3.4. From Proposition 2.4.4 we know that tricolourability is an invariant and from Example 2.4.3 we know that the trefoil is tricolourable so indeed every possible projection of the trefoil is tricolourable. We can leverage this to prove the following proposition.

Proposition 2.4.5
The trefoil and the unknot are not equivalent.

Proof.
We have that the trefoil is tricolourable. If we consider the standard projection of the unknot as in Figure 2.1.4, it is clear that it only has 1 arc and hence can only be coloured in 1 way with 1 colour, this violates the first condition of Definition 2.4.2 and therefore every possible projection of the unknot is not tricolourable. □

We have that tricolourability splits the class of all links into two subclasses, those that are tricolourable and those that are not. This means that if we establish that two knot diagrams are not in the same subclass, we have that they cannot be equivalent but it also means that there are many examples of links that are not equivalent yet fit into the same subclass. See Example 2.4.33 for an example of a knot other than the trefoil that is tricolourable.

2.4.2 Constructing the Quandle

There exists a generalisation of tricolourability to what we call p-colourability. In order to derive this extension we must replace our colours with arbitrary labels and instead of the criteria in Definition 2.4.2 being met at the crossings, we want some method of constructing equations at the crossings, and some restrictions these equations must follow in order to remain invariant under the Reidemeister moves.

In order to make our construction be an invariant we must consider the property of orientation (see Remark 2.4.8.1) which we will begin a development of in the following example.

Example 2.4.6
We know that a circle can be thought of as a 1 dimensional manifold, we define orientation on such a manifold as a choice of either forwards or backwards direction at every point such that given any neighbourhood of some point all directions in that neighbourhood agree. This can be thought of as 'direction of travel' around the 1 dimensional manifold.

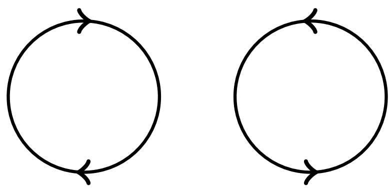

Figure 2.4.6: The two possible orientations of the circle.

We see in Figure 2.4.6 that a circle can be oriented in 2 distinct ways.

In the following definition we recall the fact that all components of an n component link are homeomorphic to a circle. (See Definition 2.2.4).

Definition 2.4.7 (Oriented Link)
We say that a component of a link is oriented if it is homeomorphic to an oriented circle.

Note 2.4.7.1
Since a circle has 2 possible orientations as in Example 2.4.6, an n component link has 2^n possible orientations. Indeed, any single component knot has 2 possible orientations.

Note 2.4.7.2
We extend our notion of link isotopy in Definition 2.3.2 to link isotopy among oriented links to be an orientation preserving link isotopy.

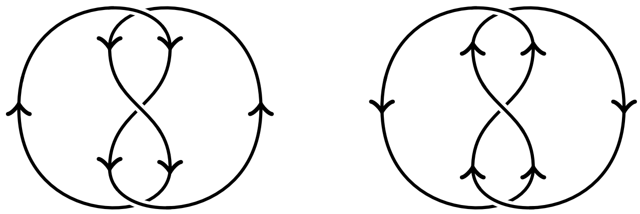

Figure 2.4.7: The two possible orientations of the trefoil.

Example 2.4.8
We consider the possible orientations of the trefoil in Figure 2.4.7.

It is interesting to note that if we strengthen our notion of equivalence to distinguish between different orientations of links then we uncover an extended version of Reidemeister equivalence for knots 2.3.3 with 20 moves, more can be found in [KT08, Sec.2.1.3].

Remark 2.4.8.1. In this section we go on to use orientation to construct an invariant known as the *non involutary knot quandle* in Definition 2.4.14. It is possible to construct a similar object known as the *involutary knot quandle* without the use of orientation that, while carrying some information about the associated knot, is not an invariant. See [Joy82, Sec.2] and [Sta15] for more.

Orientation implies that we have two types of crossing in our link diagrams, a left handed crossing and a right handed crossing.

Definition 2.4.9 (Left/Right Handed Crossing)
On an oriented component of a link, a crossing is left (respectively right) handed if from the perspective of the incoming undercrossing arc, the overcrossing arc is directed left (respectively right).

Note 2.4.9.1
If you make a thumbs up sign with your right hand and point it along the direction of the overcrossing and the undercrossing is travelling in the same direction that your fingers are curling then it is a right handed crossing, if not it is left handed.

Now we can talk about how we will formulate our equations at each type of crossing. A useful way to do this is to make the product of the label on the overcrossing arc with the label on the incoming undercrossing arc be equal to the label on the outgoing undercrossing arc. We need a different binary operation for our labels on right and left handed crossings (see Remark 2.4.8.1). We will call the right handed operation $*$ and the left handed operation $\star$. We illustrate this method for constructing crossing equations below.

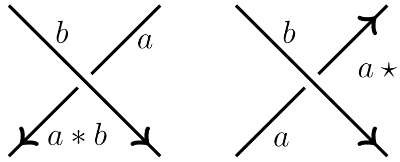

Figure 2.4.8: A left handed (left) and right handed (right) crossing annotated with arbitrary labels and a method for constructing crossing equations using the binary operations $*$ and $\star$.

If we construct equations in this way at each crossing of a given knot we get a set of equations. We want to impose some set of restrictions on this set of equations to ensure that they will be invariant under the Reidemeister moves. We can ensure this by enforcing that our binary operations satisfy a number of rules corresponding to the Reidemeister moves. We can deduce these rules by labelling the Reidemeister moves in the same manner as outlined in Figure 2.4.8, that is by following our method for constructing equations at the crossings.

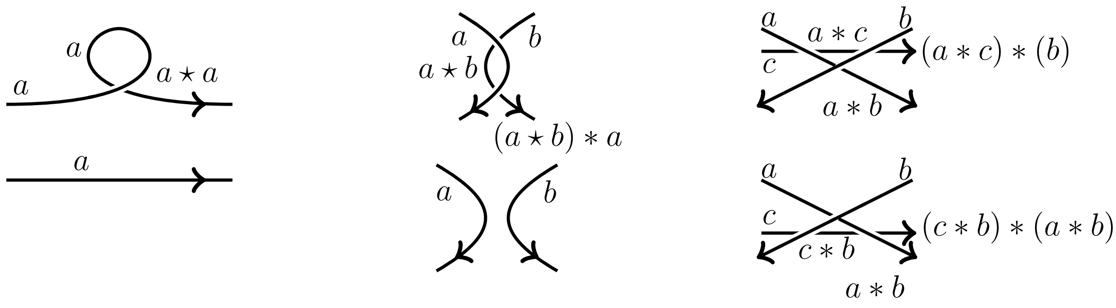

Figure 2.4.9: The method for constructing crossing equations applied to diagrams of the Reidemeister moves.

We deduce a set of rules based on Figure 2.4.9 that ensures our set of equations will be an invariant. Note that it would be a simple exercise to swap all crossing directions in Figure 2.4.9 to obtain an analogous set of rules with flipped binary operations which we will include in our set below.

$$a * a = a \quad \text{and} \quad a \star a = a \quad \text{for all labels } a, \tag{2.4.1}$$

$$(a \star b) * a = b \quad \text{and} \quad (a * b) \star a = b \quad \text{for all labels } a, b, \tag{2.4.2}$$

$$(a * c) * b = (a * b) * (c * b) \quad \text{and} \quad (a \star c) \star b = (a \star b) \star (c \star b) \quad \text{for all labels } a, b, c. \tag{2.4.3}$$

We see that these rules correspond to Reidemeister moves 1, 2 and 3 respectively as we expect. We define the algebraic structure that these rules gives rise to in the definition below.

Definition 2.4.10 (Quandle)
A set Q with binary operations $*$ and $\star$ such that $(Q, *, \star)$ satisfies conditions 2.4.1,2.4.2 and 2.4.3 is called a quandle.

Example 2.4.11
Any group $(G, \cdot)$ can be transformed into the quandle $(G, \triangleleft, \triangleright)$ by the conjugation operations as given below:

$$a \triangleleft b = a \cdot b \cdot a^{-1}$$
$$b \triangleright a = a^{-1} \cdot b \cdot a$$

Noting that $b \triangleright a = a^{-1} \triangleleft b$, it is a nice exercise to verify that these satisfy the quandle axioms.

Remark 2.4.11.1. The interplay between quandles and groups is very interesting, in fact, by the example above we can think of a quandle as what is left in a group if we ignore multiplication and only consider the operation of conjugation. More can be found in [BN20].

We use Example 2.4.11 to prove the following proposition.

Proposition 2.4.12
Quandles are not necessarily associative.

Proof.
Using the construction of a quandle on a non-commutative group $(G, \cdot)$ as in Example 2.4.11, we can see;

$$(a \triangleleft b) \triangleleft c = a \cdot b \cdot a^{-1} \triangleleft c = a \cdot b \cdot a^{-1} \cdot c \cdot a^{-1} \cdot b^{-1} \cdot a$$
$$a \triangleleft (b \triangleleft c) = a \triangleleft b \cdot c \cdot b^{-1} = a \cdot b \cdot c \cdot b^{-1} \cdot a^{-1}$$

Which are not necessarily equal since G is non-commutative. □

We will now introduce two more examples of quandles that will be very important for us later.

Example 2.4.13
It is a useful exercise to verify that the following examples do indeed form quandles.

1. *The dihedral quandle of order n is the quandle with set $\mathbb{Z}/n\mathbb{Z}$ and binary operations defined by*
$$a * b = a \star b = 2b - a$$
2. *The Alexander quandle is the quandle with set $\mathbb{Z}[t, t^{-1}]$ (the Laurent polynomials) and binary operations defined by*
$$x * y = tx + (1 - t)y$$
$$x \star y = t^{-1}x + (1 - t^{-1})y$$

Quandles where introduced independently by Joyce [Joy82] and Matveev [Mat82] in 1982 who both discussed the notion of a *knot quandle*.

Definition 2.4.14 (Knot Quandle)
For a knot k and an oriented diagram of k with labelled arcs, the set of generators $\{Q\}$, defined as the set of all labels on the arcs of k, along with the binary operations $*$ and $\star$ and the set of relations given by the set of equations constructed at each crossing of k as in Figure 2.4.10, form the knot quandle of k, $\mathcal{Q}(k)$.

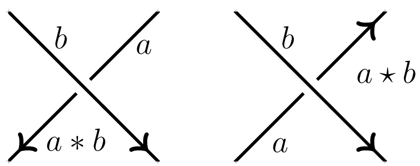

Figure 2.4.10: Method for construction of crossing equations for left (left) and right (right) handed crossings.

Remark 2.4.14.1. We define the knot quandle using generators and relations as it is generally infinite. We expect this if we consider an alternate presentation of the knot quandle, that is as the the fundamental group of the knot complement. We will not get into this interesting description in this paper but more can be found in [Rol76, Chapter 3].

Now we are in a position to consider an oriented knot, label its arcs and construct a set of crossing equations thereby gaining the generators and relations for the relevant knot quandle.

Example 2.4.15
We consider an oriented trefoil with arbitrary labels.

From Figure 2.4.11 we can gain a set of equations at each crossing using the method outlined in Figure 2.4.10 as follows:

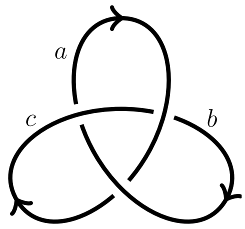

Figure 2.4.11: An oriented trefoil with labelled arcs.

$$a * b = c \tag{2.4.4}$$
$$b * c = a \tag{2.4.5}$$
$$c * a = b \tag{2.4.6}$$

These equations are the relations of the knot quandle of the oriented trefoil in Figure 2.4.11. We denote the trefoil 3_1 *and say:*

$$\mathcal{Q}(3_1) = \{a, b, c : a * b = c, b * c = a, c * a = b\}$$

is the knot quandle of the trefoil.

Theorem 2.4.16
The knot quandle is an invariant up to isomorphism.

Proof.
Both Joyce and Matveev prove this fact in their original papers on the quandle. We will not replicate these proofs here. [Joy82][Mat82] □

From our construction of the quandle, we should expect Theorem 2.4.16 to be the case. We illustrate this in the following example.

Example 2.4.17
In order to give an example of how the knot quandle is an invariant under the Reidemeister moves, consider the knot quandle of the trefoil as obtained in 2.4.15. If we performed the first Reidemeister move on the arc labelled a in Figure 2.4.11, we would gain the crossing equation and therefore relation in the knot quandle $a * a = a$ *(assuming the loop produced a right handed crossing) which, using our quandle axioms gives*

$$a * a = a \tag{2.4.7}$$
$$a = a \tag{2.4.8}$$

as a new relation. The quandle given by $\{a, b, c : a * b = c, b * c = a, c * a = b\}$ *is clearly isomorphic to the new quandle given by* $\{a, b, c : a * b = c, b * c = a, c * a = b, a = a\}$.

Problems arise using the knot quandle directly as an invariant when the knots and links become more complex. For a given knot we may derive a huge set of crossing equations and we lack a definitive way to tell if these are different than the given equations for any other knot. This is analogous to deciding if different presentations of finitely generated groups are isomorphic known as the isomorphism problem proposed by Dehn in 1911 [Deh11] which is known to be undecidable for non-abelian groups.

What we need to do then is take a look at properties of the knot quandle that do not change over isomorphism classes of the knot quandles. We can take a look at an example of such a property in the following.

Definition 2.4.18 (Quandle Homomorphism)
Given quandles $(Q, *, \star)$ and $(Q', *', \star')$, a quandle homomorphism is a map $h : Q \to Q'$ such that

$$h(a * b) = h(a) *' h(b) \text{ and } h(a \star b) = h(a) \star' h(b)\ \forall a, b \in Q$$

Corollary 2.4.19
Let k be some knot and X be some quandle. The number of homomorphisms from the knot quandle $\mathcal{Q}(k)$ to X is a knot invariant.

Proof.
Suppose this where not the case, then there would exist two isomorphic presentations of the knot quandle $\mathcal{Q}(k)$ that have a different number of homomorphisms to X which is absurd. □

Example 2.4.20
Consider the knot quandle for the trefoil as derived in Example 2.4.15 as follows,

$$\mathcal{Q}(3_1) = \{a, b, c : a * b = c, b * c = a, c * a = b\}$$

Now consider the dihedral quandle of order 3 as was defined in part 1 of Example 2.4.13 with underlying set $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ and binary operations given by $a * b = a \star b = 2b - a$. We will denote this quandle $\mathcal{Z}_3$. Now consider the map given by

$$h : \mathcal{Q}(3_1) \to \mathcal{Z}_3 : a \mapsto 0, b \mapsto 1, c \mapsto 2$$

It is easy to check that this is a quandle homomorphism. The labels on the arcs of the trefoil are as in Figure 2.4.12.

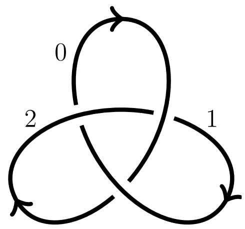

Figure 2.4.12: The labels on the trefoil after h has been applied.

The crossing equations map to the following.

$$\begin{aligned} 2 &\equiv_3 2 \\ 4 - 1 &\equiv_3 0 \\ -2 &\equiv_3 1 \end{aligned}$$

In this example, the homomorphism we have constructed is exactly equivalent to the condition for tricolourability in Definition 2.4.2, indeed as we noted earlier, tricolourability was developed by Fox as a visual representation of this very homomorphism.

Note also that there are clearly 9 possible ways that we could tricolour the trefoil, this corresponds to the 9 homomorphisms that could be constructed in a similar way to h in the example above. We derive the following generalisation.

Definition 2.4.21 (p-Colourability)
Given a prime number p, we say that a link diagram L is p-colourable if every arc in the diagram can be labelled using the numbers in the set $\{0, 1, \ldots, p-1\}$ with at least 2 of the labels distinct such that at each crossing we have

$$2x - y - z \equiv_p 0$$

where x is the value assigned to the overstrand and y and z are the values assigned to the understrands of the crossing.

This definition requires a brief explanation on why we choose p to be prime. While we could consider all integers greater than 1, we choose to exclude non-primes since their p-colourability is determined by their prime divisors. For example, if a knot is 15 colourable, it is both 5-and 3-colourable. It is also worth noting that there does not exist a 2-colourable knot however every link is clearly 2 colourable, indeed an n component link is at least n colourable. We will now take a look at a simple example to further illustrate p-colourability.

Example 2.4.22
Consider the cinquefoil, a knot with 5 crossings, we can label the arcs as in Figure 2.4.13.

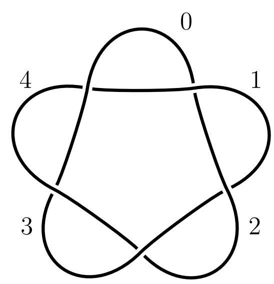

Figure 2.4.13: Cinquefoil with arcs labelled to show 5-colourability.

It can be easily verified that each crossing satisfies our criteria. For example the upper right most crossing has overstrand 1 and understrands 0 and 2, so our formula from Definition 2.4.21 would read $2(1) - 0 - 2 \equiv_5 0$.

Proposition 2.4.23
A knot k is p-colourable if and only if the knot quandle $\mathcal{Q}(k)$ admits at least one homomorphism to the dihedral quandle $\mathcal{Z}_p$.

Proof.
Using the construction of such a homomorphism as in Example 2.4.20, the data of this homomorphism is exactly equivalent to an assignation of a colour (label) to each arc. The homomorphism condition is precisely the condition that this is a valid colouring by the Definition of p-colourability 2.4.21. $\square$

Theorem 2.4.24
p-colourability is an invariant.

Proof.
If a knot k is p-colourable, then the knot quandle $\mathcal{Q}(k)$ must admit at least one homomorphism into $\mathcal{Z}_p$ by Proposition 2.4.23. By Corollary 2.4.19 this is invariant. $\square$

We now see that p-colourability is a stronger invariant than tricolourability. While tricolourability divides the class of all possible knot diagrams into two subclasses, p-colourability divides this class into uncountably many subclasses since there are countably many primes and for two distinct primes p_1 and p_2, if a knot is p_1-colourable that says nothing about whether it is p_2-colourable.

Yet still we face the same problem as we did with tricolourability, for example the trefoil is still not the only 3-colourable knot (see Example 2.4.33). In general, we could have multiple distinct knots in any given subclass, for example the cinquefoil and the knot 10_{132} in Figure 2.4.14 are both 5-colourable.

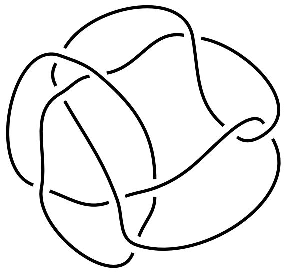

Figure 2.4.14: The knot 10_{132}.

2.4.3 The Alexander Polynomial

Our construction of the knot quandle permits a further generalisation to an even stronger invariant. We have seen how the existence of a quandle homomorphism from a knot quandle into $\mathcal{Z}_P$ is an invariant with an easy method to determine the existence of such a homomorphism. Now we will investigate the nature of the homomorphisms from some knot quandle into the Alexander quandle as defined in Example 2.4.13. Recall that this was as the quandle with underlying set $\mathbb{Z}[t, t^{-1}]$ and binary operations defined by

$$x * y = tx + (1-t)y \tag{2.4.9}$$
$$x \star y = t^{-1}x + (1-t^{-1})y \tag{2.4.10}$$

where t is understood to be an indeterminate. We denote this quandle Λ. Notice that if we let $t = -1$ and work over $\mathbb{Z}/p\mathbb{Z}$, we obtain precisely the quandle operations that define p-colourability. We construct a homomorphism in the following example.

Example 2.4.25

Consider again, the knot quandle for the trefoil.

$$\mathcal{Q}(3_1) = \{a, b, c : a * b = c, b * c = a, c * a = b\}$$

We define the homomorphism

$$A : \mathcal{Q}(3_1) \to \Lambda : a \mapsto a, b \mapsto b, c \mapsto c$$

where $a, b, c \in \mathbb{Z}$. We will see that what integers these map to is not the primary point of interest. The crossing equations (relations) map to the following.

$$ta + (1-t)b - c = 0 \tag{2.4.11}$$
$$tb + (1-t)c - a = 0 \tag{2.4.12}$$
$$tc + (1-t)a - b = 0 \tag{2.4.13}$$

We know that because Λ is a quandle, this system of equations must be an invariant. We know from the discussion following Example 2.4.17, that using this system as our invariant directly is not a useful strategy. We need a property of this system of equations that does not change as we perform Reidemeister moves and therefore change this set of equations. To find this we must first put our equations into matrix form.

Definition 2.4.26 (Alexander Matrix)
The Alexander Matrix of a knot k is the relations of the knot quandle $\mathcal{Q}(k)$ mapped by the homomorphism

$$A : \mathcal{Q}(k) \to \Lambda$$

(where A is as constructed in Example 2.4.25) in matrix form.

To continue Example 2.4.25 on the trefoil.

Example 2.4.27
We want to put our system of equations 2.4.11 to 2.4.13 in matrix form, this achieves the following matrix.

$$\begin{bmatrix} t & (1-t) & -1 \\ -1 & t & (1-t) \\ (1-t) & -1 & t \end{bmatrix}$$

This then is an Alexander matrix of the trefoil.

To explain the next step we need to make reference to the *Alexander module*, which is a $\mathbb{Z}[t, t^{-1}]$-module over the fundamental group of the knot complement, this is difficult to define but what is important for us is that it can be presented as a square matrix. For more on this and a deeper look at what we are constructing, [Rol76, Chapter 7] is a great starting point and [HLW17] gives a much more modern deep dive.

The Alexander matrix encodes a particular presentation of the Alexander module. In particular there are redundant relations in the module represented by the matrix calculated directly from the relation in the knot quandle. We remove these redundancies by deleting any one row and any one column, we call this the reduced Alexander matrix.

Definition 2.4.28 (Reduced Alexander Matrix)
The reduced Alexander matrix is the Alexander matrix with one row and one column removed.

Example 2.4.29
We remove one row and one column from the matrix in Example 2.4.27

$$\begin{bmatrix} t & (1-t) \\ -1 & t \end{bmatrix}$$

This is a reduced Alexander matrix of the trefoil. Now if we calculate the determinant of this matrix we achieve;

$$t^2 - t + 1$$

This determinant is what we are looking for, if our system of equations are equivalent to some other system, then the matrix that they represent must be performing the same linear transformation and therefore must have the same determinant (subject to Note 2.4.30.1).

Definition 2.4.30 (Alexander Polynomial)
The Alexander polynomial is the determinant of the reduced Alexander matrix. We denote the Alexander polynomial of a link L by $\Delta_L(t)$. The normal form of the Alexander polynomial is as a symmetric Laurent polynomial.

Note 2.4.30.1
Normalising the Alexander polynomial involves multiplying through by $t^{\pm a} : a \in \mathbb{Z}$*. We can do this because the Alexander polynomial is only an invariant up to a factor of* $t^{\pm a}$ *for some* $a \in \mathbb{Z}$*. The reason for this is that* t *and* t^{-1} *are the two units of the ring* $\mathbb{Z}[t, t^{-1}]$*.*

Example 2.4.31
If we normalise the determinant gathered in Example 2.4.29 we get:

$$\Delta_{3_1}(t) = t - 1 + t^{-1}$$

the Alexander polynomial of the trefoil.

Theorem 2.4.32
The Alexander polynomial is an invariant up to a factor of $t^{\pm a} : a \in \mathbb{Z}$*.*

For a proof see Alexander's original paper [Ale28].

This is the first example we have seen of a successful class of invariant known as *knot polynomials*. A list of Alexander polynomials for knots up to 9 crossings can be found in Appendix B using a slightly different normal form than the one we have adopted here. As an invariant, the Alexander polynomial performs much better than p-colourability as is illustrated using 3-colourability in the following example.

Example 2.4.33
Consider the knot 6_1*, we can see in Figure 2.4.15 that it is tricolourable.*

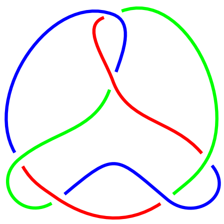

Figure 2.4.15: The knot 6_1 tricoloured.

We know from Figure 2.4.2 that the trefoil is also tricolourable so we cannot use tricolourability to prove that these knots are not equivalent. Let's calculate the Alexander polynomial of 6_1*. First we must compute the corresponding knot quandle. We label the arcs as in Figure 2.4.16 and choose and orientation.*

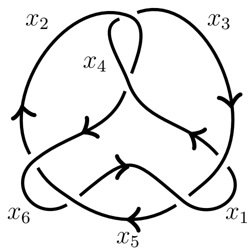

Figure 2.4.16: The knot 6_1 oriented with labelled arcs.

Generating relations at each crossing, we find the following to be the knot quandle of 6_1*:*

$$\begin{aligned}\mathcal{Q}(6_1) = \{x_1, x_2, x_3, x_4, x_5, x_6 : & x_6 \star x_5 = x_1, x_5 \star x_6 = x_2,\\ & x_4 * x_2 = x_3, x_2 * x_4 = x_6,\\ & x_1 \star x_3 = x_4, x_3 \star x_1 = x_5\}\end{aligned}$$

From Definition 2.4.26, we can represent the Alexander matrix as follows.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1-t^{-1} & t^{-1} \\ 0 & -1 & 0 & 0 & t^{-1} & 1-t^{-1} \\ 0 & 1-t & -1 & t & 0 & 0 \\ 0 & t & 0 & 1-t & 0 & -1 \\ t^{-1} & 0 & 1-t^{-1} & -1 & 0 & 0 \\ 1-t^{-1} & 0 & t^{-1} & 0 & -1 & 0 \end{bmatrix}$$

Removing the 6th row and 5th column, we get the reduced Alexander matrix as in Definition 2.4.28.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & t^{-1} \\ 0 & -1 & 0 & 0 & 1-t^{-1} \\ 0 & 1-t & -1 & t & 0 \\ 0 & t & 0 & 1-t & -1 \\ t^{-1} & 0 & 1-t^{-1} & -1 & 0 \end{bmatrix}$$

After some calculation that we shall omit, we find the determinant of this matrix to be

$$-2 + 5t^{-1} - 2t^{-2}$$

which we can normalise to find the Alexander polynomial:

$$\Delta_{6_1}(t) = -2t + 5 - 2t^{-1}$$

Since this is different than that which we calculated for the trefoil, we have that by the fact that the Alexander polynomial is an invariant, the trefoil is not equivalent to 6_1.

Notice an application to p-colourability here, if we evaluate the Alexander polynomial for the trefoil at $t = -1$ we get 3. Indeed if we do this for any knot we can say that a knot is p-colourable if and only if $p|\Delta_k(-1)$ where Δ_k is the Alexander polynomial of some knot k. We expect this due to the fact that, as was noted below equation 2.4.10, if we plug -1 into our operations 2.4.9 and 2.4.10 we recover the quandle operations that define p-colourability, for more detail see [KL18]. The value of the Alexander polynomial evaluated at $t = -1$ is often referred to as the *determinant* of a knot and a list of such values for all knots up to 9 crossings can be found in Appendix B.

Despite the Alexander Polynomial's outperformance of p-colourability it still has some significant drawbacks, we still have knots that are different but have the same Alexander polynomial. Indeed, the Alexander polynomial is the same for a trefoil and its mirror image and it cannot detect orientation. It is in fact well known that the quandle cannot produce an invariant that detects orientation [Joy79, Pg.53] and we should expect this if we consider the form of the knot quandle described in Remark 2.4.14.1. We can also calculate, that the Alexander polynomial for 6_1 as found in Example 2.4.33 is the same as that of the knot 9_{46} as in Figure 2.4.17.

To summarise, we have constructed the knot quandle which is an invariant however the isomorphism problem prevents it from being directly useful. In order to get around this, we find properties of the quandle that do not vary across isomorphism classes.

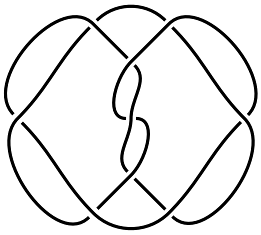

Figure 2.4.17: The knot 9_{46}.

3 Linking Links to Braids

We now want to turn our attention back to braid groups or rather show how they can aid our exploration of knots. To make the connection between braids and knots we can consider what would happen if we took a braid and connected the final position of each strand to the starting position of each strand as in the diagram below.

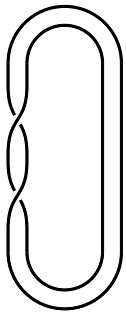

Figure 3.0.1: The closure of the braid $\sigma_1\sigma_1$.

It is immediately clear that we can close any braid and indeed whenever we do so we create some link, for instance the diagram in Figure 3.0.1 is that of the Hopf link. The question is can we relate knots back to closed braids? If this were possible we would be able to associate knots with an Artin braid group element. This will be the subject of this chapter.

3.1 Alexander's Theorem

In this section we aim prove Alexander's theorem that states that any link has a closed braid representation. In order to do this we need to give a concrete definition to the concept of closed braids that we saw in Figure 3.0.1.

Definition 3.1.1 (Closed Braid)
In $\mathbb{R}^3$, we consider the coordinate axis $l = \{0,0\} \times \mathbb{R} \subseteq \mathbb{R}^3$ that meets the plane $\mathbb{R}^2 \times \{0\}$ at the origin $O = \{0,0,0\}$. The counter-clockwise rotation about O in the plane $\mathbb{R}^2 \times \{0\}$ determines a positive direction of rotation about l. An oriented geometric link $L \subseteq \mathbb{R}^3 \setminus l$ is a closed n braid if the vector from O to any point $x \in L$ rotates in the positive direction about l when x moves along L in the direction determined by the orientation of L.

Note 3.1.1.1
We adopt the convention that all closed braids are oriented anticlockwise relative to l as in the definition above.

In order to see that any braid b can be closed, consider the Euclidean 2-disk D given with radius $\max\{x, y : (x, y, z) \in b\}$ centred such that $b \subset D \times I$. Using the identifier $(x, 0) = (x, 1)\ \forall x \in D$ we then have the solid torus $V = D \times S^1 \subseteq \mathbb{R}^3$. It is clear that we can centre this torus around the coordinate axis l and take the canonical orientation to be the direction that b travels from the point where each strand has third coordinate 0 to the point where each strand has third coordinate 1. It is clear that this process results in a closed braid. It is clear that if two braids are isotopic they have the same closure. It follows that similar operations can be applied in the 2 dimensional setting and that the 2 and 3 dimensional cases can be resolved. We call the link that results from a braid being closed in this way the closure of a braid. For more details see [KT08, Sec.2.2.3].

Example 3.1.2
The link in Figure 3.0.1 is the Hopf link as in Figure 2.1.5. The closure of the braid σ_1 is the unlink with 2 components. The closure of the braid $\sigma_1^{\pm 3}$ is the trefoil.

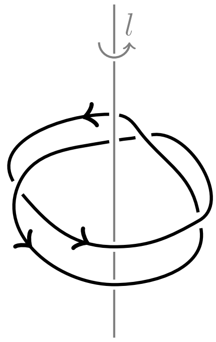

Figure 3.1.1: The trefoil as a closed braid.

In general, the braid closure of $\sigma_1^{\pm q}$ is known as the $(2, q)$ torus link. Closed braids can be equivalently defined as links in the torus (for details on this definition see [KT08, Sec.2.2.5]) indeed we can see that if we pick a euclidean 2-disk D in $\mathbb{R}^3$ that lies in an open half plane bounded by l with it's centre in $\mathbb{R}^2 \times \{0\}$ and we rotate D about l, we sweep a solid torus $V = D \times S^1 \subseteq \mathbb{R}^3$. It is clear that if we pick D sufficiently large then $L \subseteq V$.

Theorem 3.1.3 (Alexander's Theorem)
Any oriented link in $\mathbb{R}^3$ is isotopic to a closed braid.

Proof.
It will suffice to show that any oriented polygonal link $L \subset \mathbb{R}^3$ is isotopic to a closed braid due to Theorem 2.3.10. Moving slightly the vertices of L in $\mathbb{R}^3$ we obtain a link that is isotopic to L. We use such small deformations to ensure that $L \subset \mathbb{R}^3 \setminus l$ and the edges of L do not lie in planes containing the axis l.
Let $AC \subset L \subset \mathbb{R}^3 \setminus l$ be an edge of L where L is oriented from A to C. The edge AC is positive (respectively negative) if the vector from the origin $O \in l$ to a point $x \in AC$ rotates in the positive (respectively negative) direction about l when x moves from A to C. The assumption that AC does not lie in a plane containing l implies that L is strictly positive or negative. We say that AC is *accessible* if there exists a point $B \in l$ such that the triangle ABC meets L only along AC.
If all edges of L are positive then L is a closed braid and there is nothing to prove. Consider a negative edge AC of L. We replace AC with a sequence of positive edges as follows. If AC is accessible then we pick $B \in l$ such that the triangle ABC only meets L along AC. In the plane defined by ABC we take a slightly bigger triangle $AB'C$ containing B in it's interior meeting l only at B and meeting L only along AC as we see below. We apply to L the Δ move $\Delta(AB'C)$

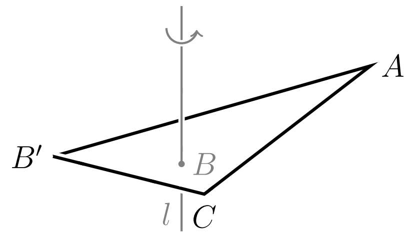

Figure 3.1.2: The triangle $AB'C$.

replacing AC with two positive edges AB' and $B'C$. The resulting polygonal link is isotopic to L and has one negative edge less than L.
Now suppose that AC is not accessible. Note that every point $P \in AC$ is contained in an accessible subsegment of AC. To see this we choose $B \in l$ such that the segment PB meets L only at P and then slightly 'thicken' this segment inside the triangle ABC to obtain a new triangle P^-BP^+ meeting L along $P^-P^+ \subset AC$ containing P. Then P^-P^+ is an accessible subsegment of AC. Since AC is compact, we can split it into a finite number of consecutive accessible subsegments. We apply to each of them the Δ move as above choosing the corresponding points $B \in l$ distinct and choosing B' close enough to B to stay away from the other edges of L. Since AC does not lie in a plane containing l, the triangles determining these Δ moves meet only at the common vertices of consecutive subsegments of AC therefore these Δ moves do not hinder each other and may be performed in an arbitrary order. They replace $AC \subset L$ with a sequence of positive edges beginning at A and ending at C. The resulting polygonal link is isotopic to $L \subset \mathbb{R}^3$. Applying this procedure inductively to all negative edges of L we obtain a closed braid that is isotopic to L. $\square$

Now that we have shown that any given link can be represented by a closed braid, we can associate a braid word with any given link. This can be computationally very difficult and in the paper [Jon86] that earned him the fields medal, Jones computed closed braid representatives for the 249 knots of crossing number less than or equal to 10, constructing the first table of closed braid representatives of knots. The version of the proof that we have followed for Theorem 3.1.3 is an adaptation of Alexander's original method [Ale23] and although it is clearly an algorithmic proof, there have been no successful computer algorithms made based on this proof and Jones constructed his table by hand. Since then via different approaches to the proof of Theorem 3.1.3, algorithms have been developed that can be input into a computer such as that due to Yamada [Yam87] later adapted by Vogel [Vog90] that can efficiently calculate the braid group representation of a given link. A list of appropriate braid representations for each single component prime knot up to 9 crossings can be found in Appendix B.

3.2 Markov's Theorem

We now have that any link has an Artin braid group representation however this representation is not unique.

Proposition 3.2.1
The braid word representation of a link is not an invariant.

Proof.
We can see in Figure 3.2.1 two closed braids that represent the 2 component unlink that have braid word representations $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3$. Clearly these braids are not equivalent by the fact that they are on a different number of strands. We can also see from Figure 3.2.1

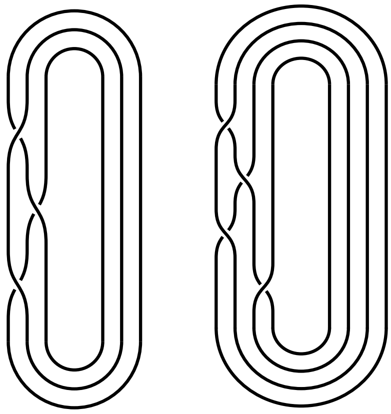

Figure 3.2.1: The closure of the braids $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3$.

that their closed braids are only a Reidemeister move different, so braid word representations are not invariant under Reidemeister moves. □

So in order to make our braid word representations useful, we want to separate the class of all braids into equivalence classes where a braid β is equivalent to a braid β' if and only if their closures represent the same closed link.

Definition 3.2.2 (Markov Equivalence)
We call two braids β and β' Markov equivalent if their closure represents the same link. We denote this relationship $\beta \sim_M \beta'$.

Similarly to how we define braid and link equivalence with the Reidemeister moves 2.3.4, we would like a set of moves that we can perform on braids that define Markov equivalence. In A.A. Markov's 1935 paper [Mar35], he outlines a set of moves known as *Markov moves.*

In order to understand where these moves come from, we need to ask what equivalence in braids is missing when compared to equivalence in links. Indeed, we know that braids and links share the second and third Reidemeister moves so the obvious thing that we need to account for is the lack of the first Reidemeister move in the braid setting. More precisely, we want to determine a move that we can do on a braid that corresponds to the first Reidemeister move (and its inverse) whenever we close the braid. This is achieved through the following.

Definition 3.2.3 (Stabilisation/Destabilisation)
Stabilisation is the transformation of the braid β on n strands into the braid $\sigma_n^{\pm 1}\iota(\beta)$ where ι is the inclusion map. The inverse of this move, where we take the a braid on $n+1$ strands of the form $\sigma_n^{\pm 1}\beta'$ such that β' is on n strands, and reduce it to $\beta' \in B_n$ is called destabilisation.

To see that this is indeed equivalent to the first Reidemeister move whenever we close the braids consider the following example.

Example 3.2.4
Consider again, the closed braids from the proof of Proposition 3.2.1. If we take the braid $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ and transform it into the braid $\iota(\sigma_1\sigma_2^{-1}\sigma_1^{-1})\sigma_3$, we are clearly performing a stabilisation move.

We can see in Figure 3.2.2 that adding the generator σ_3 and closing the braid corresponds to the first Reidemeister move on the closed braid. This is highlighted in blue.

The second thing that we need to consider is that if when we close a braid, a Reidemeister move may become available from the top and bottom generators of the braid.

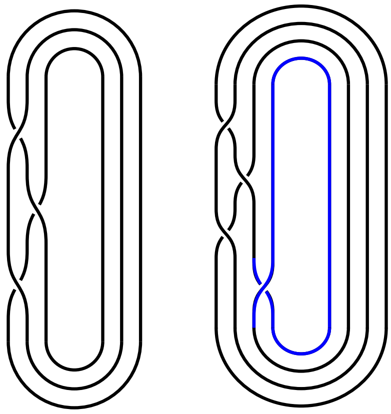

Figure 3.2.2: A visualisation of how we can account for the first Reidemeister move for closed braids in the braid setting.

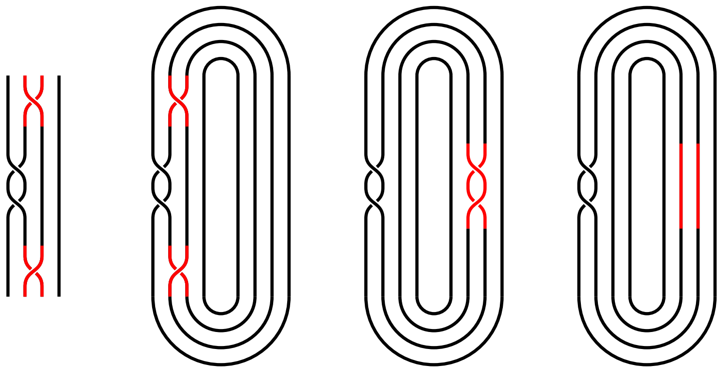

Figure 3.2.3: An illustration of how closing a braid can give access to Reidemeister moves that are unavailable in the braid setting.

We can see from Figure 3.2.3 that if we have a braid on n strands of the form $\gamma\beta\gamma^{-1}$ such that γ and β are braids on n strands and we then close the braid, we can perform Reidemeister moves on the resulting link so that we get a link precisely equivalent to the closure of β. We formalise this move in the braid setting in the following.

Definition 3.2.5 (Conjugation)
Conjugation is the transformation of a braid $\beta \in B_n$ to the braid $\gamma\beta\gamma^{-1}$ where γ is some other braid in B_n.

Combining Definitions 3.2.3 and 3.2.5 with braid isotopy we get the following key theorem.

Theorem 3.2.6 (Markov's Theorem)
Two braids are Markov equivalent if and only if they are connected by a finite sequence of braid isotopy 1.2.4, conjugation, stabilisation and destabilisation.

The first proof of Markov's theorem came almost 40 years after its initial announcement in 1974 due to Birman in [Bir74]. Today there are at least 5 known different proofs, the other 4 can be found in [Mor86],[Tra98],[BM02] and [LR97]. Unfortunately, despite great selection, all the proofs listed are sufficiently complicated to go beyond the scope of this paper and we will not state them here. This is interesting because geometrically, this is so obviously the case from the way we formulated Definitions 3.2.3 and 3.2.5, however a formal proof requires a higher level of rigour.

From the above theorem, it is clear that $\sim_M$ as defined in Definition 3.2.2 is an equivalence relation on the disjoint union $\bigsqcup_{n\geq 1} B_n$ of all braid groups. We illustrate this in the following example.

Example 3.2.7
From our counterexample in the proof of Proposition 3.2.1, it is clear that

$$\sigma_1\sigma_2^{-1}\sigma_1^{-1} \sim_M \sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3$$

by stabilisation/destabilisation. For a further example it is clear that σ_1 *and* σ_1^{-1} *have the same closure yet are not equivalent in the Artin braid group. Note that we can see that a destabilisation to the identity element in* B_1 *shows that they have the same closure however to demonstrate an algebraic approach to the Markov moves we will take a different approach. We have*

$$\begin{aligned}
\sigma_1 \sim_M \sigma_2^{-1}\sigma_1 \sim_M (\sigma_1\sigma_2)^{-1}\sigma_2^{-1}\sigma_1(\sigma_1\sigma_2) &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2 &\quad (3.2.1)\\
&= \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^2\sigma_2 &\quad (3.2.2)\\
&= \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2 &\quad (3.2.3)\\
&= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_2 &\quad (3.2.4)\\
&= \sigma_2\sigma_1^{-1} \sim_M \sigma_1^{-1} &\quad (3.2.5)
\end{aligned}$$

so σ_1 *and* σ_1^{-1} *are indeed Markov equivalent.*

Recall that by Definition 3.1.1, all closed braids inherently have an orientation thus, $\sim_M$ gives us a description of the set of isotopy classes of oriented links in $\mathbb{R}^3$ euclidean space. This is given in the following corollary that nicely sums up what we have achieved in this section and the previous.

Corollary 3.2.8
Let $\mathcal{L}$ *be the set of equivalence classes of all oriented links in* $\mathbb{R}^3$ *euclidean space. The map* $\psi : \bigsqcup_{n\geq 1} B_n/\sim_M \to \mathcal{L}$ *assigning to a braid the isotopy class of its closure induces a bijection from the quotient set* $\bigsqcup_{n\geq 1} B_n/\sim_M$ *onto* $\mathcal{L}$.

Proof.
We have a surjection by Alexander's Theorem 3.1.3 and Markov's Theorem 3.2.6 gives us the injection. □

3.3 Further Examples

In the following, recall that the fundamental permutation of a braid $\beta \in B_n$ is given by the image of β in the usual homomorphism onto the symmetric group S_n as defined in Theorem 1.1.9.

Proposition 3.3.1
For any $\beta \in B_n$*, the number of components of the closure of* β *is equal to the number of cycles in the decomposition of the fundamental permutation of* β *as a product of commuting cycles.*

Proof.
Consider the cycle $\sigma = (a_1, \dots, a_k)$ in the symmetric group S_n and suppose that it is a part of the decomposition of the fundamental permutation of β as a product of commuting cycles. Arguing geometrically, it is clear that if we close the braid β and we travel around $\hat{\beta}$ (where we use $\hat{\beta}$ to denote the closure of β) starting at the top of the strand a_k and travelling in the direction of the canonical orientation, we see that we travel from the strand a_k to the strand a_{k-1} and so on until we reach the strand a_1 at which point we return to the start point at the strand a_k. It follows that the cycle σ defines a component of $\hat{\beta}$. □

Corollary 3.3.2
The closure of the pure braid $\beta \in P_n$ is n copies of the unknot.

Proof.
It is clear that the decomposition of the fundamental permutation of β as a product of commuting cycles yields $(1)(2)\dots(n)$ and therefore has n components by Proposition 3.3.1. It is also clear geometrically that a 1-cycle defines a copy of the unknot. □

Example 3.3.3
As was discussed in Example 1.1.2, B_2 is isomorphic to the infinite cyclic group. We define this isomorphism by:

$$wr_2 : B_2 \to \mathbb{Z} \quad , \quad \sigma_1 \mapsto 1$$

We consider the pure braids P_2 and the homomorphism from B_2 to the symmetric group S_2 as defined in Theorem 1.1.9 as ϕ_2. Recall that the image of a braid in ϕ_2 is precisely its fundamental permutation. It is hence, clear that any pure braid $p \in P_2$ we must have $wr_2(p) = m$ where m is an even integer. If we consider the pure braid $wr_2^{-1}(0) = e_{B_2}$, we have that its closure is two copies of the unknot. The closure of the pure braid $wr_2^{-1}(2) = \sigma_1\sigma_1$ is the Hopf link as we seen in Figure 3.0.1.

If we consider instead the odd integers we see that $wr_2^{-1}(1) = \sigma_1$ closes to the unknot, $wr_2^{-1}(3) = \sigma_1\sigma_1\sigma_1$ closes to the trefoil and $wr_2^{-1}(5) = \sigma_1\sigma_1\sigma_1\sigma_1\sigma_1$ closes to the cinquefoil. This pattern continues for the positive odd integers and the negatives just correspond to the mirror image of their positive counterparts. We expect that all of these should close to knots by Proposition 3.3.1.

Recall Definition 1.5.1 of a positive braid as an element of the positive submonoid B_n^+ generated by $\sigma_1, \sigma_2, \dots, \sigma_n$.

Definition 3.3.4 (Positive Permutation Braid)
A braid $\beta \in B_n^+$ is called a positive permutation braid if no pair of its strings cross more than once.

We define the following homomorphism.

$$wr : B_n \to \mathbb{Z} \quad , \quad \sigma_i \mapsto 1$$

We seen this given as an isomorphism from B_2 in Example 3.3.3 however for $n \geq 3$ it is an homomorphism. This is often referred to as the *writhe* of a braid $\beta \in B_n$ and it determines the *algebraic crossing number* of a braid. For example, the braid in Figure 1.5.5 has algebraic crossing number 0 and the braid in Figure 1.2.5 has algebraic crossing number 3. In the following we say that a braid β is conjugate to a braid β' if there exists a braid α such that $\beta = \alpha\beta'\alpha^{-1}$.

Lemma 3.3.5
If $\beta, \beta' \in B_n^+$ are conjugate and have isotopic closure, then $wr(\beta) = wr(\beta')$.

The proof of this result involves a tool called *Seifert circles* and is therefore omitted however a full proof can be found in [MH05, Lemma 1], we use it to prove the following. For more on Seifert circles see [Rol76, Chapter 5].

Theorem 3.3.6
Any positive permutation braid β which closes to the unknot is conjugate to $\sigma_1\sigma_2\dots\sigma_{n-1}$.

Proof.
Since β closes to a single component knot, all generators must appear at least once otherwise the fundamental permutation will not be a single n-cycle. It is clear that $\sigma_1\sigma_2\ldots\sigma_{n-1}$ closes to the unknot, to see this, consider the closure of σ_1, clearly this is just the unknot. Assuming that the closure of of $\sigma_1\ldots\sigma_k$ is the unknot for some k, it is clear that $\sigma_1\ldots\sigma_k\sigma_{k+1}$ has the same closure as $\sigma_1\ldots\sigma_k$ by a single destabilisation. So $\sigma_1\sigma_2\ldots\sigma_{n-1}$ is the unknot for all $n \geq 1$ by induction. Using Lemma 3.3.5, we have that $wr(\beta) = wr(\sigma_1\sigma_2\ldots\sigma_{n-1}) = n-1$ so we conclude that each generator must appear in β exactly once.

We can write $\beta = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{n-1}}$ where $i_j = 1,\ldots,n-1 \forall j = 1,\ldots,n-1$. Notice that if we perform the operation

$$\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{n-1}} \to \sigma_{i_{n-1}}\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{n-2}}$$

we have precisely performed a conjugation. We can notice here that we can represent β up to conjugation by placing the generators $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{n-1}}$ on a circle. We proceed by showing that we can reorder these generators on the circle to be in the desired order.

For the first step of our induction, notice that we can just define σ_1 to be the first generator on the circle. Assume that $\sigma_1,\ldots,\sigma_k$ lie in order for some $k = 2,\ldots,n-1$. We know that any generators that lie between σ_k and σ_{k+1} must be of the form $\sigma_j : j > k+1$ and so they can commute backwards beyond σ_1, so $\sigma_1,\ldots,\sigma_k,\sigma_{k+1}$ is in order.

$\square$

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Appendices

A Knot Table

The following knot table shows all single component knots up to 8 crossings and the first 36 9 crossing knots named using C_n notation where C is the number of crossings and n is an identifier based on the order each knot was tabulated in historically. The table also shows all 2 and 3 component links up to 8 crossings labelled with the notation C_n^m where m denotes the number of components. This was obtained from [MJ02].

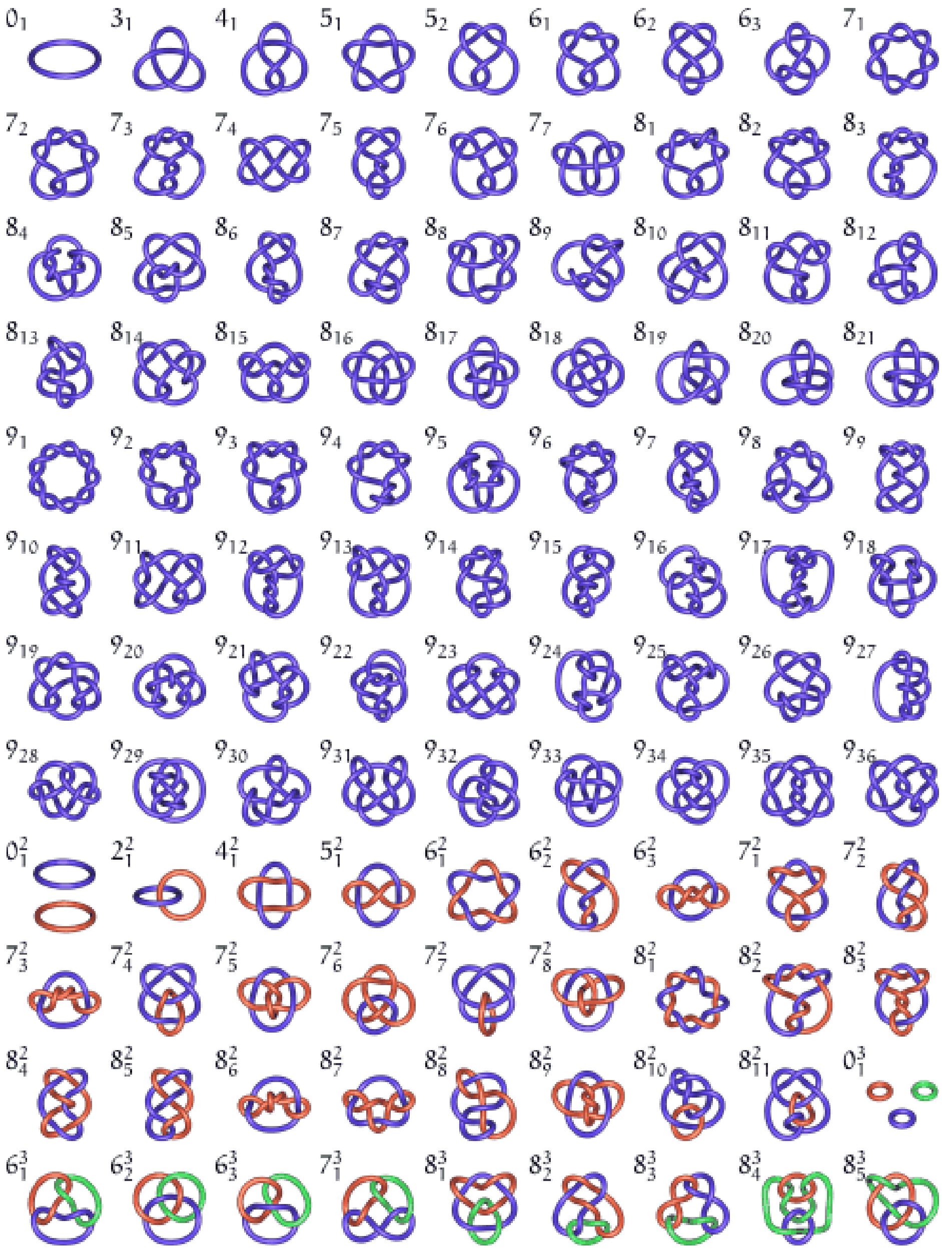

Figure A.0.1: A knot table showcasing all single component prime knots up to the first 36 9 crossing knots and all 2 and 3 component links each up to 8 crossings.

B Table of Key Values

The following table contains data for each single component prime knot up to and including those with 9 crossings, all knots up to 8 crossings and the first 36 9 crossing knots can be seen in Appendix A.

The *Braid Notation* column denotes a braid that can be closed for each knot. These braid words are *minimal* in that they are as simple as possible, for a rigorous approach to defining this see [Git04]. For ease of reading, the column entry $i; j; -k$ refers to the braid $\sigma_i \sigma_j \sigma_k^{-1}$. The *Braid Index* column refers to the number of strands that this braid occurs on.

The *Alexander Polynomial* is as in Definition 2.4.30. Note that the normal form used in this table is that all t have positive degree.

The *Determinant* column refers to the value of the Alexander polynomial evaluated at $t = -1$. Recall that this determines for which p a knot is p-colourable.

This table was obtained from [LM25], which should be noted is a well documented source for data on a wide range of invariants.

Name	Braid Index	Braid Notation	Alexander Polynomial	Determinant
3_1	2	1;1;1	$1 - t + t^2$	3
4_1	3	1;-2;1;-2	$1 - 3t + t^2$	5
5_1	2	1;1;1;1;1	$1 - t + t^2 - t^3 + t^4$	5
5_2	3	1;1;1;2;-1;2	$2 - 3t + 2t^2$	7
6_1	4	1;1;2;-1;-3;2;-3	$2 - 5t + 2t^2$	9
6_2	3	1;1;1;-2;1;-2	$1 - 3t + 3t^2 - 3t^3 + t^4$	11
6_3	3	1;1;-2;1;-2;-2	$1 - 3t + 5t^2 - 3t^3 + t^4$	13
7_1	2	1;1;1;1;1;1;1	$1 - t + t^2 - t^3 + t^4 - t^5 + t^6$	7
7_2	4	1;1;1;2;-1;2;3;-2;3	$3 - 5t + 3t^2$	11
7_3	3	1;1;1;1;1;2;-1;2	$2 - 3t + 3t^2 - 3t^3 + 2t^4$	13
7_4	4	1;1;2;-1;2;2;3;-2;3	$4 - 7t + 4t^2$	15
7_5	3	1;1;1;1;2;-1;2;2	$2 - 4t + 5t^2 - 4t^3 + 2t^4$	17
7_6	4	1;1;-2;1;3;-2;3	$1 - 5t + 7t^2 - 5t^3 + t^4$	19
7_7	4	-1;2;-1;2;-3;2;-3	$1 - 5t + 9t^2 - 5t^3 + t^4$	21
8_1	5	1;1;2;-1;2;3;-2;-4;3;-4	$3 - 7t + 3t^2$	13
8_2	3	1;1;1;1;1;-2;1;-2	$1 - 3t + 3t^2 - 3t^3 + 3t^4 - 3t^5 + t^6$	17
8_3	5	1;1;2;-1;-3;2;-3;-4;3;-4	$4 - 9t + 4t^2$	17
8_4	4	-1;-1;-1;2;-1;2;3;-2;3	$2 - 5t + 5t^2 - 5t^3 + 2t^4$	19
8_5	3	1;1;1;-2;1;1;1;-2	$1 - 3t + 4t^2 - 5t^3 + 4t^4 - 3t^5 + t^6$	21
8_6	4	1;1;1;1;2;-1;-3;2;-3	$2 - 6t + 7t^2 - 6t^3 + 2t^4$	23
8_7	3	-1;-1;-1;-1;2;-1;2;2	$1 - 3t + 5t^2 - 5t^3 + 5t^4 - 3t^5 + t^6$	23
8_8	4	-1;-1;-1;-2;1;3;-2;3;3	$2 - 6t + 9t^2 - 6t^3 + 2t^4$	25
8_9	3	1;1;1;-2;1;-2;-2;-2	$1 - 3t + 5t^2 - 7t^3 + 5t^4 - 3t^5 + t^6$	25
8_10	3	-1;-1;-1;2;-1;-1;2;2	$1 - 3t + 6t^2 - 7t^3 + 6t^4 - 3t^5 + t^6$	27
8_11	4	1;1;2;-1;2;2;-3;2;-3	$2 - 7t + 9t^2 - 7t^3 + 2t^4$	27
8_12	5	1;-2;1;3;-2;-4;3;-4	$1 - 7t + 13t^2 - 7t^3 + t^4$	29
8_13	4	1;1;-2;1;-2;-2;-3;2;-3	$2 - 7t + 11t^2 - 7t^3 + 2t^4$	29
8_14	4	1;1;1;2;-1;2;-3;2;-3	$2 - 8t + 11t^2 - 8t^3 + 2t^4$	31
8_15	4	1;1;-2;1;3;2;2;2;3	$3 - 8t + 11t^2 - 8t^3 + 3t^4$	33
8_16	3	-1;-1;2;-1;-1;2;-1;2	$1 - 4t + 8t^2 - 9t^3 + 8t^4 - 4t^5 + t^6$	35
8_17	3	1;1;-2;1;-2;1;-2;-2	$1 - 4t + 8t^2 - 11t^3 + 8t^4 - 4t^5 + t^6$	37
8_18	3	1;-2;1;-2;1;-2;1;-2	$1 - 5t + 10t^2 - 13t^3 + 10t^4 - 5t^5 + t^6$	45
8_19	3	1;1;1;2;1;1;1;2	$1 - t + t^3 - t^5 + t^6$	3
8_20	3	1;1;1;-2;-1;-1;-1;-2	$1 - 2t + 3t^2 - 2t^3 + t^4$	9
8_21	3	1;1;1;2;-1;-1;2;2	$1 - 4t + 5t^2 - 4t^3 + t^4$	15
9_1	2	1;1;1;1;1;1;1;1;1	$1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8$	9

Name	Braid Index	Braid Notation	Alexander Polynomial	Determinant
9_2	5	1;1;1;2;-1;2;3;-2;3;4;-3;4	$4 - 7t + 4t^2$	15
9_3	3	1;1;1;1;1;1;1;2;-1;2	$2 - 3t + 3t^2 - 3t^3 + 3t^4 - 3t^5 + 2t^6$	19
9_4	4	1;1;1;1;1;2;-1;2;3;-2;3	$3 - 5t + 5t^2 - 5t^3 + 3t^4$	21
9_5	5	1;1;2;-1;2;2;3;-2;3;4;-3;4	$6 - 11t + 6t^2$	23
9_6	3	1;1;1;1;1;1;2;-1;2;2	$2 - 4t + 5t^2 - 5t^3 + 5t^4 - 4t^5 + 2t^6$	27
9_7	4	1;1;1;1;2;-1;2;3;-2;3;3	$3 - 7t + 9t^2 - 7t^3 + 3t^4$	29
9_8	5	1;1;-2;1;-2;-3;2;4;-3;4	$2 - 8t + 11t^2 - 8t^3 + 2t^4$	31
9_9	3	1;1;1;1;1;2;-1;2;2;2	$2 - 4t + 6t^2 - 7t^3 + 6t^4 - 4t^5 + 2t^6$	31
9_10	4	1;1;2;-1;2;2;2;2;3;-2;3	$4 - 8t + 9t^2 - 8t^3 + 4t^4$	33
9_11	4	-1;-1;-1;-1;2;-1;-3;2;-3	$1 - 5t + 7t^2 - 7t^3 + 7t^4 - 5t^5 + t^6$	33
9_12	5	1;1;-2;1;3;-2;3;4;-3;4	$2 - 9t + 13t^2 - 9t^3 + 2t^4$	35
9_13	4	1;1;1;1;2;-1;2;2;3;-2;3	$4 - 9t + 11t^2 - 9t^3 + 4t^4$	37
9_14	5	-1;-1;-2;1;3;-2;3;-4;3;-4	$2 - 9t + 15t^2 - 9t^3 + 2t^4$	37
9_15	5	-1;-1;-1;-2;1;3;-2;-4;3;-4	$2 - 10t + 15t^2 - 10t^3 + 2t^4$	39
9_16	3	1;1;1;1;2;2;-1;2;2;2	$2 - 5t + 8t^2 - 9t^3 + 8t^4 - 5t^5 + 2t^6$	39
9_17	4	-1;2;-1;2;2;2;-3;2;-3	$1 - 5t + 9t^2 - 9t^3 + 9t^4 - 5t^5 + t^6$	39
9_18	4	1;1;1;2;-1;2;2;2;3;-2;3	$4 - 10t + 13t^2 - 10t^3 + 4t^4$	41
9_19	5	-1;2;-1;2;2;3;-2;-4;3;-4	$2 - 10t + 17t^2 - 10t^3 + 2t^4$	41
9_20	4	1;1;1;-2;1;3;-2;3;3	$1 - 5t + 9t^2 - 11t^3 + 9t^4 - 5t^5 + t^6$	41
9_21	5	-1;-1;-2;1;-2;3;-2;-4;3;-4	$2 - 11t + 17t^2 - 11t^3 + 2t^4$	43
9_22	4	-1;2;-1;2;-3;2;2;2;-3	$1 - 5t + 10t^2 - 11t^3 + 10t^4 - 5t^5 + t^6$	43
9_23	4	1;1;1;2;-1;2;2;3;-2;3;3	$4 - 11t + 15t^2 - 11t^3 + 4t^4$	45
9_24	4	1;1;-2;1;3;-2;-2;-2;3	$1 - 5t + 10t^2 - 13t^3 + 10t^4 - 5t^5 + t^6$	45
9_25	5	1;1;-2;1;3;2;2;-4;3;-4	$3 - 12t + 17t^2 - 12t^3 + 3t^4$	47
9_26	4	-1;-1;-1;2;-1;2;-3;2;-3	$1 - 5t + 11t^2 - 13t^3 + 11t^4 - 5t^5 + t^6$	47
9_27	4	1;1;-2;1;-2;-2;3;-2;3	$1 - 5t + 11t^2 - 15t^3 + 11t^4 - 5t^5 + t^6$	49
9_28	4	1;1;-2;1;3;-2;-2;3;3	$1 - 5t + 12t^2 - 15t^3 + 12t^4 - 5t^5 + t^6$	51
9_29	4	1;-2;-2;3;-2;1;-2;3;-2	$1 - 5t + 12t^2 - 15t^3 + 12t^4 - 5t^5 + t^6$	51
9_30	4	-1;-1;2;2;-1;2;-3;2;-3	$1 - 5t + 12t^2 - 17t^3 + 12t^4 - 5t^5 + t^6$	53
9_31	4	1;1;-2;1;-2;3;-2;3;3	$1 - 5t + 13t^2 - 17t^3 + 13t^4 - 5t^5 + t^6$	55
9_32	4	-1;-1;-3;2;-3;-1;2;-1;2	$1 - 6t + 14t^2 - 17t^3 + 14t^4 - 6t^5 + t^6$	59
9_33	4	-1;-3;2;-3;-1;2;2;-1;2	$1 - 6t + 14t^2 - 19t^3 + 14t^4 - 6t^5 + t^6$	61
9_34	4	-1;2;-1;2;-3;2;-1;2;-3	$1 - 6t + 16t^2 - 23t^3 + 16t^4 - 6t^5 + t^6$	69
9_35	5	1;1;2;-1;2;2;3;-2;-2;4;-3;2;4;3	$7 - 13t + 7t^2$	27
9_36	4	-1;-1;-1;2;-1;-1;-3;2;-3	$1 - 5t + 8t^2 - 9t^3 + 8t^4 - 5t^5 + t^6$	37
9_37	5	1;1;-2;1;3;-2;-1;-4;3;-2;3;-4	$2 - 11t + 19t^2 - 11t^3 + 2t^4$	45
9_38	4	1;1;2;2;-3;2;-1;2;3;3;2	$5 - 14t + 19t^2 - 14t^3 + 5t^4$	57
9_39	5	-1;-1;-2;1;3;2;-1;-4;-3;2;-3;-4	$3 - 14t + 21t^2 - 14t^3 + 3t^4$	55
9_40	4	1;-2;1;3;-2;1;3;-2;3	$1 - 7t + 18t^2 - 23t^3 + 18t^4 - 7t^5 + t^6$	75
9_41	5	-1;-1;-2;1;3;2;2;-4;-3;2;-3;-4	$3 - 12t + 19t^2 - 12t^3 + 3t^4$	49
9_42	4	-1;-1;-1;2;1;1;-3;2;-3	$1 - 2t + t^2 - 2t^3 + t^4$	7
9_43	4	1;1;1;2;1;1;-3;2;-3	$1 - 3t + 2t^2 - t^3 + 2t^4 - 3t^5 + t^6$	13
9_44	4	1;1;1;2;-1;-1;-3;2;-3	$1 - 4t + 7t^2 - 4t^3 + t^4$	17
9_45	4	-1;-1;-2;1;-2;-1;-3;2;-3	$1 - 6t + 9t^2 - 6t^3 + t^4$	23
9_46	4	-1;2;-1;2;-3;-2;1;-2;-3	$2 - 5t + 2t^2$	9
9_47	4	-1;2;-1;2;3;2;-1;2;3	$1 - 4t + 6t^2 - 5t^3 + 6t^4 - 4t^5 + t^6$	27
9_48	4	-1;-1;-2;1;-2;-1;3;-2;1;-2;3	$1 - 7t + 11t^2 - 7t^3 + t^4$	27
9_49	4	1;1;2;1;1;-3;2;-1;2;3;3	$3 - 6t + 7t^2 - 6t^3 + 3t^4$	25

Index of Definitions