Question 1: Residential Bid Price Function and Location Choice (40 P.)

Consider the following decision problem of an individual household who decides to buy some land to live near the central business district (CBD) of the city. All employment and goods are localised in the CBD city center. The price of land is decreasing with increasing distance to the CBD, however, commuting costs are increasing with distance to the CBD.

The household faces the budget constraint $y = p_z z + P(t)q + k(t)$, with income y, price of the composite good p_z, quantity z of the composite good, price of land $P(t)$ as a function of distance t from the CBD, quantity of land q, and commuting cost $k(t)$ as a function of distance t from the CBD. The household has to optimise the parameters z, q, and t.

(a) Intuitively, explain the compensation principle between land prices and commuting costs in the spatial equilibrium.

Households are compensated with lower land prices for the increased commuting costs they occur by living further away from the CBD. The land price premium should correspond to the savings in commuting costs obtained from living closer to the CBD in equilibrium. Thus, from an economic stand point, the distance to the CBD does not matter in spatial equilibrium, as the combined costs of commuting costs and land price premium amount to the same at every distance t.

Summarized, in equilibrium the marginal benefit of an increase in distance equals its marginal cost.

(b) Assume a fixed distance, $t = t_0$ such that the household can only optimize between the quantities of land and the composite good. Specify the equation of all possible combinations of quantities that can be consumed. Determine the slope coefficient.

The function becomes $y = p_z z + P(t_0)q + k(t_0)$, where $k(t_0)$, p_z and $P(t_0)$ are constants. The quantity of the composite good will be defined by

$$z = y - \frac{P(t_0)q + k(t_0)}{p_z}, \tag{1}$$

where q is the input variable. The slope of the coefficient is the derivative of z with respect to q, which is

$$\frac{\partial z}{\partial q} = -\frac{P(t_0)}{p_z}. \tag{2}$$

As prices are always positive, the slope coefficient is negative, meaning that an increase in the composite good consumption results in a decrease of the quantity of land consumed, and vice versa.

(c) Assume a fixed amount $= z_0$ of the composite good, such that the household can only vary parameters q and t. Specify all possible combinations in a tq-diagram. Consider a marginal change in the distance from the CBD. Through which channels does this change affect the optimal amount of q? Show that for the condition $\frac{\partial k}{\partial t} = -\frac{\partial P}{\partial t} q$, the household can achieve the optimal quantity of land q for distance t.

In a tq-diagram all possible combinations of t and q are specified by $q = \frac{y - p_z z_0 - k(t)}{P(t)}$, where y, p_z, and z_0 are constants and $k(t)$ and $P(t)$ are functions of the input variable t. A marginal change in the distance t is specified by the derivative:

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial t}(\frac{y - p_z z_0 - k(t)}{P(t)}) = \frac{\frac{\partial P}{\partial t}(p_z z_0 + k(t) - y) - P(t) \times \frac{\partial k}{\partial t}}{P(t)^2} \tag{3}$$

Therefore, a marginal change in distance t affects the optimal amount of q through the land price channel (the price level $P(t)$ as well as its change with respect to distance $\frac{\partial P}{\partial t}$) and the commuting cost channel (the cost level $k(t)$ as well as the changes in commuting costs with respect to distance $\frac{\partial k}{\partial t}$). The income y as well as the price of the composite good p_z and the amount z_0 also determine the optimal quantity of land q, but are assumed to remain constant in this model, and thus don't vary with a change of distance t.

Inserting $\frac{\partial k}{\partial t} = \frac{\partial P}{\partial t} \times q$ into equation 3, we receive:

$$= \frac{\frac{\partial P}{\partial t}(p_z z_0 + k(t) - y) + P(t) \times \frac{\partial P}{\partial t} \times q}{P(t)^2} \tag{4}$$

$$= \frac{\frac{\partial P}{\partial t}(p_z z_0 + k(t) - y + P(t) \times q)}{P(t)^2} \tag{5}$$

where $p_z z_0 + k(t) - y + P(t) \times q$ represents the budget constraint (under the assumption that the entire budget is used, this term equals zero). Therefore, the quantity of land q is maximised with respect to the distance under the condition $\frac{\partial k}{\partial t} = \frac{\partial P}{\partial t} \times q$, as $\frac{\partial q}{\partial t} = 0$ indicates a global maximum ($q(t)$ is a concave function and thus the First Order Condition holds).

When we maximise the budget function y with respect to the distance t, we receive $\frac{\partial y}{\partial t} = 0 = \frac{\partial P}{\partial t} \times q + \frac{\partial k}{\partial t}$ which corresponds to the condition $\frac{\partial k}{\partial t} = -\frac{\partial P}{\partial t} q$. Thus, we can set the amount of land q at any given distannce t, such that the change y w.r.t. t is zero. Therefore, we are in a spatial equilibrium, as described in question a. The optimal choice of q at a given t is such that a marginal benefit of an increase in distance equals its marginal cost.

Consider now an individual household i located at a distance t from the central business district (CBD). The budget constraint of the individual household is given as $y_i - k(t) = p_z z + p(t)q$, with individual income y_i. For simplicity, we assume that $k(t) = k_i t$ for household i. A continuum of households lives in the economy, such that the quantity of land q is infinitively small. Hence, the land cost can be specified as $R(t) = p(t)q$ for each household located at distance t to the CBD. Each household belongs either to type $i = 1$ or $i = 2$. It holds, that $k_1 > k_2$. All households consume the same amount of the composite good.

(d) Calculate the location point at which land costs R for both types of households must be identical. Which condition must hold to maintain a spatial equilibrium? Explain.

Land costs R are the same for both household types at the intersection of the two functions $y_1 - k_1 t = p_z z + p(t)q$ and $y_2 - k_2 t = p_z z + p(t)q$. As both households consume the same amount of the composite good z and its price p_z can be assumed to be the same for both households $p_z z$ can be ignored, as it just represents an upward shift for both household groups. Therefore, the target function can be simplified to $y_1 - k_1 t = p(t)q = y_2 - k_2 t$. Solving for t, we obtain the location point

$$t = \frac{y_1 - y_2}{k_1 - k_2}. \tag{6}$$

As $y_1 > y_2$ and $k_1 > k_2$, the location point at which land costs R are identical for both types of households will be a positive distance.

In equilibrium, household 1 will rent houses closer to the city center, as they have higher commuting costs $k_1 > k_2$ and are thus willing to pay more for shorter commuting times, while households 2 will rent beyond the intersection point t. Thus, in equilibrium all households rent one house and are spatially segregated. Furthermore, to maintain a spatial equilibrium the houshold type with the larger commuting costs k_i must also earn a higher income y_i. This stylized model, however, neglects the fact that higher income households usually demand higher lot sizes q, which could result in a flatter rent gradient for k_1 and consequently lead to a cessation of perfect segregation between household types.

Because of the pollution in the city center, the quality of the environment is assumed to improve with each additional distance unit from the CBD. This effect is represented by parameter E. The environmental quality is a location-specific public good. The budget constraint of the individual household i can be rewritten as given as $y_i - k_i t + Et = p_z z + R(t)$.

(e) Determine the new residential bid price function of the household. Calculate and discuss the implications of the rent gradient.

The residential bid price function indicates how much households are willing to pay for an acre of land at various different locations in the city (i.e. distance to the CBD). Should both households value the improvement of the environment the same amount, the rent gradient of household 1 will remain steeper as that of household 2. Thus, the residential bid price function is determined as

$$p(t) = \frac{y_1 - k_1 t + Et - p_z z}{q} \tag{7}$$

up until the intersection point $t = \frac{y_1 - y_2}{k_1 - k_2}$, and then

$$p(t) = \frac{y_2 - k_2 t + Et - p_z z}{q}. \tag{8}$$

The introduction of E does not change the distance to the intersection point, given the same valuation of improvements by both households. However, the slope of the rent gradient for both households increases (becomes less negative), whereby the new rent gradient is

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial t} \frac{y_i - k_i t + Et - p_z z}{q} = \frac{-k_i + E}{q}, \tag{9}$$

increasing the overall price level (i.e. the total amount of rent obtained by the landlords, defined as the area beneath the bid price function, increases).

Furthermore, the city will expand as a consequence of the flatter rent gradient. The new border is at $0 = \frac{y_2 - k_2 t - p_z z}{q}$, which can be solved for t to obtain $t = \frac{y_2 - p_z z}{k_2 - E}$, meaning that an increase E will result in a smaller denominator and hence a border more distant from the CBD. The difference to the old border, $t = \frac{y_2 - p_z z}{k_2}$, is equivalent to the expansion of the border: $\Delta t = \frac{y_2 - p_z z}{k_2 - E} - \frac{y_2 - p_z z}{k_2}$.

In the case that the benefits of the environmental quality outweigh the increased commuting costs with an increase in distance t, the rent gradient could even become positive, whereby the prices of housing increase with the distance to the CBD. This phenomenon has frequently been observed in many cities throughout the world, especially in the industrial age, where the wealthy moved to the suburbs and country side.

Question 2: Bid Price Function and Land Use of Firms (20 P.

Consider the locational choice of a firm under perfect competition in a spatial equilibrium model. We have a centralized city, such that the firm has to take into account the distance t from the local market place in the center. The firm accessibility of potential customers therefore decreases with the distance to the local market. Land prices $P(t)$ decrease with the distance. Furthermore, the firm has to choose the optimal amount q of occupied land units. The firm's profit is defined as the residual after operating costs and land costs are deducted from the revenues. The variables are defined as follows:

$V = V(t, q)$ revenues
$C = C(V, t, q)$ operating costs
$R = P(t)q$ land costs

(a) Specify the profit function and calculate its total differential.

The profit function π is defined as

$$\pi = V(t, q) - C(V, t, q) - P(t)q. \tag{10}$$

As the revenues function is dependent on the operating costs function, which are both simulateously dependent on t and q, as well as the price of land $P(t)$ depending on t, I use the chain rule (and a dependency graph as an aide) to derive to total differential of the profit function using only partial differentials. Thus, I obtain the total differential as

$$d\pi = \frac{\partial \pi}{\partial V}\frac{\partial V}{\partial q} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial V}\frac{\partial V}{\partial q} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial q} - \frac{\partial \pi}{\partial q} + \frac{\partial \pi}{\partial V}\frac{\partial V}{\partial t} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial V}\frac{\partial V}{\partial t} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial t} - \frac{\partial \pi}{\partial P}\frac{\partial P}{\partial t} \times q \tag{11}$$

(b) Based on your result in a), hold q and t constant, respectively, to specify the optimality condition for the firm's locational choice as well as the optimal size of occupied space demanded by the firm. Derive the slope of the bid price function of the firm.

Holding t constant, all changes with respect to t become 0, leaving us with the optimality condition for the size of occupied space demanded by the firm as

$$\frac{\partial \pi}{\partial q} = P(t) = \frac{\partial \pi}{\partial V}\frac{\partial V}{\partial q} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial V}\frac{\partial V}{\partial q} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial q}. \tag{12}$$

Holding q constant, all changes with respect to q become 0. Thus, at a given level of profits (for example, $\pi = 0$), the optimality condition for the firm's location is given by

$$\frac{\partial \pi}{\partial P}\frac{\partial P}{\partial t} \times q = \frac{\partial \pi}{\partial V}\frac{\partial V}{\partial t} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial V}\frac{\partial V}{\partial t} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial t}. \tag{13}$$

The first term on the right hand side V_t represents the loss in total revenue that results from moving away from the cente rof the city, the next two terms $C_V \cdot V_t$ are the change in operating costs, and the last term C_t is the familiar change in the amount spent on land.

The slope of the bid price function for the firm is

$$\frac{\partial \pi}{\partial P}\frac{\partial P}{\partial t} = \frac{\frac{\partial \pi}{\partial V}\frac{\partial V}{\partial t} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial V}\frac{\partial V}{\partial t} - \frac{\partial \pi}{\partial C}\frac{\partial C}{\partial t}}{q}. \tag{14}$$