1. Leontief Expansion

$$(I - \rho W)^{-1} = \sum_{i=0}^{\infty} (\rho W)^i \tag{1}$$

multiplying both sides with $(I - \rho W)$, we can prove the Leontief expansion

$$(I - \rho W)^{-1}(I - \rho W) = I = \sum_{i=0}^{\infty} (\rho W)^i (I - \rho W) \tag{2}$$

given

$$\begin{aligned}
\sum_{i=0}^{\infty} (\rho W)^i (I - \rho W) &= ((\rho W)^0 + (\rho W)^1 + (\rho W)^2 + ... + (\rho W)^i)(I - \rho W) \\
&= (I + (\rho W)^1 + (\rho W)^2 + ... + (\rho W)^i)(I - \rho W) \\
&= I - (\rho W) + (\rho W) - (\rho W)^2 + (\rho W)^2 - (\rho W)^3 + ... - (\rho W)^{i+1} \\
&= I - (\rho W)^{i+1}
\end{aligned} \tag{3}$$

where $(\rho W)^{i+1}$ converges to 0 with $\mid \rho \mid < 1$ and W being row standardised.

2. OLS Bias

$$\begin{aligned}
y &= \rho W y + \epsilon \\
&= (I - \rho W)^{-1} \epsilon
\end{aligned} \tag{4}$$

Substituting $X = Wy$, the OLS estimator becomes

$$\hat{\beta}_{OLS} = ((Wy)'(Wy))^{-1}(Wy)'y \tag{5}$$

Using 4, we obtain

$$\hat{\beta}_{OLS} = ((Wy)'(Wy))^{-1}(Wy)'(\rho W y + \epsilon) \tag{6}$$

$$\mathbb{E}[\hat{\beta}_{OLS}] = \rho + \mathbb{E}[(Wy)'(Wy)]^{-1}\mathbb{E}[(Wy)'\epsilon] \tag{7}$$

Using 4, substituting $C = W(I - \rho W)^{-1}$ and using trace's cyclic property, among others, we can rearrange the second term to

$$\begin{aligned}
\mathbb{E}[(Wy)'\epsilon] &= \mathbb{E}[(W(I - \rho W)^{-1}\epsilon)'\epsilon] \\
&= \mathbb{E}[(C\epsilon)'\epsilon] \\
&= \mathbb{E}[\epsilon' C' \epsilon] \\
&= \mathbb{E}[tr(\epsilon' C' \epsilon)] \\
&= tr(C)\mathbb{E}[\epsilon\epsilon']
\end{aligned} \tag{8}$$

where $tr(C)$ can be rearranged to the following using the Leontief expansion as follows

$$\begin{aligned}
tr(C) &= tr(W(I - \rho W)^{-1}) \\
&= tr(W\sum_{i=0}^{\infty}(\rho W)^i) \\
&= tr(W(I + (\rho W)^1 + (\rho W)^2 + ... + (\rho W)^i)) \\
&= tr(W + \rho W^2 + \rho^2 W^3 + ... + \rho^i W^{i+1}) \\
&= tr(W) + \rho tr(W^2) + \rho^2 tr(W^3) + ... + \rho^i tr(W^{i+1})) \\
&= \frac{1}{\rho}\sum_{i=1}^{\infty}\rho^i tr(W^i))
\end{aligned} \tag{9}$$