Math Basics

Session 2: (Linear Algebra Fundamentals)

Agenda:

1	Vectors and Matrices
2	Matrix Operations
3	Linear Transformations and Eigenvalues

Vectors and Matrices

Vector

A vector is an ordered list of numbers (scalars) that can represent a point in space. Vectors have both **magnitude** (length) and **direction**. In the context of AI, vectors are used to **represent data points**, **features**, or **weights**.

Vectors are typically represented as **columns** (column vectors) or **rows** (row vectors). For example, a 3-dimensional vector can be written as:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Vectors

Examples

- **Position Vector**: In a 2D space, a vector $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ can represent a point 3 units to the right and 4 units up from the origin.

- **Feature Vector**: In machine learning, a vector can represent features of a data point, such as $\mathbf{x} = \begin{bmatrix} \text{age} \\ \text{income} \end{bmatrix} = \begin{bmatrix} 30 \\ 50000 \end{bmatrix}$.

Operations on Vectors

Addition

Vectors of the same dimension can be added by adding corresponding elements:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Operations on Vectors

Subtraction

Similar to addition, but subtracting corresponding elements.

Scalar Multiplication

Multiplying a vector by a scalar (number) scales each element of the vector:

$$c \cdot \mathbf{v} = c \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \end{bmatrix}$$

Matrices

A matrix is a rectangular array of numbers arranged in rows and columns. Matrices can represent multiple vectors, data tables, or linear transformations.

A matrix A with m rows and n columns is denoted as $A_{m \times n}$. For example, a 2x3 matrix looks like this:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Matrices

Examples

- **Data Matrix:** In a dataset, each row could represent a different data point (case), and each column could represent a different feature (variable).

- **Transformation Matrix:** Matrices can be used to perform transformations like rotations, scaling, or translations in space

Why Vectors and Matrices are Important in AI and Machine Learning ?

- **Data Representation**: Vectors and matrices are used to represent datasets, where each row is a data point and each column is a feature.
- **Linear Transformations**: Matrices can represent transformations applied to data, such as scaling, rotating, and translating data points in space.
- **Matrix Operations:** Essential for performing calculations in machine learning algorithms, such as linear regression, neural networks, and principal component analysis (PCA).

Summary

- Vectors represent points or directions in space and can be used to model features of data points.

- Matrices represent more complex data structures and transformations and are fundamental tools for organizing and manipulating data in AI.

- Understanding vector and matrix operations is crucial for implementing and optimizing machine learning algorithms.

Matrix Operations

Matrix Operations

This segment covers essential matrix operations that are fundamental for understanding how data is manipulated and transformed in AI and machine learning. Matrix operations enable the efficient computation of complex transformations, solving systems of linear equations, and performing various data manipulations.

Matrix Multiplication

Matrix multiplication is a way of combining two matrices to produce a new matrix. The number of columns in the first matrix must match the number of rows in the second matrix for multiplication to be possible.

Notation:

If A is an m×n matrix and B is an n×p matrix, the product AB is an m×p matrix.

Formula:

Each element of the resulting matrix C=AB is calculated as the dot product of a row in A and a column in B:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Matrix Multiplication

Example

- Let A be a 2×3 matrix and B be a 3×2 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

- The resulting matrix $C = AB$ is:

$$C = \begin{bmatrix} (1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11) & (1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12) \\ (4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11) & (4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Determinant of a Matrix

The determinant is a scalar value that can be computed from the elements of a square matrix. It provides important properties of the matrix, such as whether it is invertible (non-singular).

Notation:

For a square matrix A, the determinant is denoted as det(A) or |A|.

Formula for a 2x2 Matrix

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Determinant of a Matrix

Example

For a matrix $A = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$, the determinant is:

$$\det(A) = (3 \cdot 6) - (8 \cdot 4) = 18 - 32 = -14$$

Inverse of a Matrix

The inverse of a matrix A is a matrix A^{-1} such that when it is multiplied by A, it results in the identity matrix I: $A \times A^{-1} = I$

Conditions:

A matrix must be square (same number of rows and columns) and have a non-zero determinant to have an inverse.

Formula for a 2x2 Matrix

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Inverse of a Matrix

Example

For $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$:

$$\det(A) = (4 \cdot 6) - (7 \cdot 2) = 24 - 14 = 10$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Transpose of a Matrix

The transpose of a matrix is obtained by flipping the matrix over its diagonal, converting rows into columns.

Notation:

If A is a matrix, the transpose is denoted by A^T

Example

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Summary

- **Matrix multiplication** is crucial for data transformations and neural network operations.
- **Determinants** help determine the properties of matrices, such as invertibility.
- **Matrix inverses** are used to solve linear equations and optimize models.
- **Matrix transposition** is essential for aligning data dimensions and preparing matrices for multiplication.

Types of Matrices

Square Matrix

A matrix with the same number of rows and columns (e.g., n×n). It's used often in linear transformations and finding determinants.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Types of Matrices

Diagonal Matrix

A square matrix where all non-diagonal elements are zero. Only the diagonal elements may be non-zero.

Example: $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Types of Matrices

Identity Matrix

A special type of diagonal matrix where all the diagonal elements are 1. It acts as the multiplicative identity in matrix operations.

Example: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Types of Matrices

Zero Matrix

A matrix where all elements are zero. It's the additive identity in matrix operations.

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Types of Matrices

Row Matrix

A matrix with only one row.

Example: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Types of Matrices

Symmetric Matrix

A matrix with only one row.

A square matrix that is equal to its transpose $(A = A^T)$

Example: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Linear Transformations and Eigenvalues

Linear Transformations and Eigenvalues

This segment covers the concepts of linear transformations and eigenvalues, which are fundamental in understanding how data can be manipulated and analyzed in various dimensions. These concepts play a crucial role in areas such as dimensionality reduction, stability analysis, and optimization in AI and machine learning.

Linear Transformations

A linear transformation is a function that maps vectors from one space to another, preserving vector addition and scalar multiplication. Essentially, it transforms vectors while maintaining the structure of the space.

Matrix Representation

Matrix Representation Linear transformations can be represented by matrices.
If x is a vector and A is a matrix, then Ax is a linear transformation of x.

Example: If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then:

$$A\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Common Types of Linear Transformations

Scaling

Changes the size of vectors. Represented by a diagonal matrix where each diagonal element scales the corresponding dimension.

Example: Scaling by a factor of 2, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, scales vectors by 2.

Common Types of Linear Transformations

Rotation

Rotates vectors around the origin. Represented by a rotation matrix.

Example in 2D: $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ rotates vectors by an angle θ.

Common Types of Linear Transformations

Reflection

Flips vectors over a specified line (in 2D) or plane (in 3D).

Example in 2D: Reflection over the x-axis, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

Common Types of Linear Transformations

Shearing

Distorts the shape of vectors in one direction while keeping the other direction unchanged.

Example in 2D: Horizontal shear in 2D, $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, where k is the shear factor.

Eigenvalues and Eigenvectors

In the context of a linear transformation, an eigenvector of a matrix A is a non-zero vector that only changes by a scalar factor when A is applied to it. The scalar is known as the eigenvalue.

Eigenvalue (λ): A scalar indicating how much the eigenvector is stretched or compressed.

Eigenvector (v): A vector that does not change its direction during the transformation.

Eigenvalues and Eigenvectors

Mathematical Formulation

$$A\mathbf{v} = \lambda\mathbf{v}$$

Where:

- **A** is the transformation matrix.
- **v** is the eigenvector.
- **λ** is the eigenvalue.

Eigenvalues and Eigenvectors

Finding Eigenvalues and Eigenvectors

- To find eigenvalues, solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

- Once eigenvalues (λ) are known, solve $(A-\lambda I)v=0$ to find the corresponding eigenvectors.

Eigenvalues and Eigenvectors

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$:

1. Calculate the characteristic equation:

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 \cdot 1 = 0$$

2. Solve for λ\lambdaλ:

$$\lambda^2 - 7\lambda + 10 = 0 \Rightarrow (\lambda - 5)(\lambda - 2) = 0$$

So, the eigenvalues are λ=5 and λ=2.

3. Substitute each λ back into (A−λI)v=0 to find the eigenvectors.