ST2131
AY21/22 SEM 2
github/jovyntls

01. COMBINATORIAL ANALYSIS

The Basic Principle of Counting

- **basic principle of counting** $\rightarrow$ Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- **generalized basic principle of counting** $\rightarrow$ If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, there are n_2 possible outcomes of the 2nd exp, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1!\,n_2!\dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \le r \le n$$

N5 - The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Multinomial Coefficients

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!\,n_2!\dots n_r!}$$

N6 - represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \dots, n_3$, where $n_1 + n_2 + \dots + n_r = n$

N7 - The Multinomial Theorem: $(x_1 + x_2 + \dots + x_r)^n$

$$= \sum_{(n_1,\dots,n_r):n_1+n_2+\dots+n_r=n} \frac{n!}{n_1!\,n_2!\dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n, \quad x_i > 0, \quad i = 1, 2, \dots, r$

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n$

Proof. let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- **sample space** $\rightarrow$ The *set* of all outcomes of an experiment
- **event** $\rightarrow$ Any *subset* of the sample space
- **complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E.
- **subset** $\rightarrow E \subset F$ is all of the outcomes in E that are also in F.
  - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws: $\quad (\bigcup_{i=1}^{n} \mathbf{E_i})^c = \bigcap_{i=1}^{n} \mathbf{E_i^c} \quad$ and $\quad (\bigcap_{i=1}^{n} \mathbf{E_i})^c = \bigcup_{i=1}^{n} \mathbf{E_i^c}$

Axioms of Probability

definition 1: relative frequency

$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$. problems: (1) $\frac{n(E)}{n}$ may not converge when $n \to \infty$. (2) $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated.

Axioms (definition 2)

Consider an experiment with sample space S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:

1. $0 \le P(E) \le 1$ 2. $P(S) = 1$

3. For mutually exclusive events, $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$. same for finite case

mutually exclusive $\rightarrow$ events for which $E_iE_j = \emptyset$ when $i \neq j$

Simple Propositions

N1 - $P(\emptyset) = 0$

N6 - **probability function** $\iff$ it satisfies the 3 axioms.

N8 - if $E \subset F$, then $P(E) \le P(F)$

N10 - Inclusion-Exclusion identity where $n = 3$

$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$

N11 - **Inclusion-Exclusion identity** -

$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1}E_{i_2}) + \dots$

$+ (-1)^{r+1} \sum_{i_1 < i_2 < \dots < i_r} P(E_{i_1}E_{i_2}\dots E_{i_r}) + \dots + (-1)^{n+1} P(E_1E_2\dots E_n)$

(i) $P(\bigcup_{i=1}^{n} E_i) \le \sum_{i=1}^{n} P(E_i)$ (based on Inclusion-Exclusion identity)

(ii) $P(\bigcup_{i=1}^{n} E_i) \ge \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_iE_j)$

(iii) $P(\bigcup_{i=1}^{n} E_i) \le \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_iE_j) + \sum_{k<j<i} P(E_iE_jE_k)$

(iv) and so on.

Sample Space having Equally Likely Outcomes

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \dots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.

N1 - for any event E, $P(E) = \frac{\text{\# of outcomes in } E}{\text{\# of outcomes in } S} = \frac{\text{\# of outcomes in } E}{n}$

increasing sequence of events $\{E_n, n \ge 1\} \rightarrow E_1 \subset E_2 \subset \dots \subset E_n \subset \dots$

decreasing sequence of events $\{E_n, n \ge 1\} \rightarrow E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$

increasing: $\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$ decreasing: $\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$

N2 - for both *increasing* and *decreasing* sequence, $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$

03. CONDITIONAL PROBABILITY AND INDEPENDENCE

Conditional Probability

if $P(F) > 0$, then $P(E|F) = \frac{P(E \cap F)}{P(F)}$

multiplication rule:

$P(E_1 \dots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\dots P(E_n|E_1E_2\dots E_{n-1})$

N3 - **axioms of probability** apply to conditional probability

1. $0 \le P(E|F) \le 1$ 2. $P(S|F) = 1$ where S is the sample space

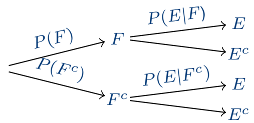
3. If E_i $(i \in \mathbb{Z}_{\ge 1})$ are mutually exclusive, then $P(\bigcup_{1}^{\infty} E_i|F) = \sum_{1}^{\infty} P(E_i|F)$

N4 - If we define $Q(E) = P(E|F)$, then all previously proven results apply.

- $P(E_1 \cup E_2|F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$

Total Probability & Bayes' Theorem

conditioning formula - $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$



$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(F)\cdot P(E|F)}{P(E)}$$

$$P(F^c|E) = \frac{P(EF^c)}{P(E)} = \frac{P(F^c)\cdot P(E|F^c)}{P(E)}$$

Total Probability

theorem of total probability - Suppose $F_1, F_2, \dots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$, then $P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(F_i)P(E|F_i)$

Bayes Theorem

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(F_j)P(E|F_j)}{\sum_{i=1}^{n} P(F_i)P(E|F_i)}$$

application of bayes' theorem

$$P(B_1 \mid A) = \frac{P(A|B_1)\cdot P(B_1)}{P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2)}$$

Let A be the event that the person test positive for a disease.

B_1: the person has the disease. B_2: the person does not have the disease.

true positives: $P(B_1 \mid A)$	false negatives: $P(\bar{A} \mid B_1)$
false positives: $P(A \mid B_2)$	true negatives: $P(\bar{A} \mid B_2)$

Independent Events

N1 - E and F are independent $\iff P(EF) = P(E)\cdot P(F)$

N2 - E and F are independent $\iff P(E|F) = P(E)$

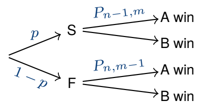
N3 - E and F are independent $\iff E$ and F^c are independent.

N4 - if E, F, G are independent, then E will be independent of any event formed from F and G. (e.g. $F \cup G$)

N6 - (E and F are indep) $\wedge$ (E and G are indep) $\not\Rightarrow E$ and FG are independent

N7 - For independent trials with probability p of success, probability of m successes before n failures, for $m, n \ge 1$,

method 1



method 2

$$P_{n,m} = \sum_{k=m}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$

$= P(\ge n$ successes in $m+n-1$ trials)

04. RANDOM VARIABLES

- **random variable** $\rightarrow$ a real-valued function defined on the sample space

Types of Random Variables

r.v.	-	$E(X)$
binomial	X = # of successes in n trials w/ replacement	np
negative binomial	X = # of trials until k successes	k/p
geometric	X = # of trials until a success	$1/p$
hypergeometric	X = # of successes in n trials, no replacement	rn/N

- X is a **Bernoulli r.v.** with parameter p if $\rightarrow$

$$p(x) = \begin{cases} p, & x = 1, \text{ ('success')} \\ 1-p, & x = 0 \quad \text{('failure')} \end{cases}$$

- Y is a **Binomial r.v.** with parameters n and $p \rightarrow \quad Y = X_1 + X_2 + \dots + X_n$ where $X_1, X_2, \dots, X_n$ are independent Bernoulli r.v.'s with parameter p.
  - $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $P(k$ successes from n independent trials each with probability p of success)

$$E(Y) = np, \quad Var(Y) = np(1-p)$$

- **Negative Binomial** $\rightarrow X$ = number of trials until k successes are obtained
  - e.g. number of balls drawn (with replacement) until k red balls are obtained
- **Geometric** $\rightarrow X$ = number of trials until a success is obtained

- $P(X=k)=(1-p)^{k-1}\cdot p$ where k is the number of trials needed
- e.g. number of balls drawn (with replacement) until 1 red ball is obtained

- **Hypergeometric** $\rightarrow X =$ number of trials until success, *without replacement*
  - $P(X=k)=\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$, $k=0,1,\dots,n$ (for m red balls of N balls)
  - e.g. number of red balls out of n balls drawn without replacement

Properties

N1 - if $X\sim$ Binomial(n,p), and $Y\sim$ Binomial$(n-1,p)$, then

$$E(X^k)=np\cdot E[(Y+1)^{k-1}]$$

N2 - if $X\sim$ Binomial(n,p), then for $k\in\mathbb{Z}^+$,

$$P(X=k)=\frac{(n-k+1)p}{k(1-p)}\cdot P(X=k-1)$$

Coupon Collector Problem

Q. Suppose there are N distinct types of coupons. If T denotes the number of coupons needed to be collected for a complete set, what is $P(T=n)$?

A. $P(T\geq n-1)=P(T\geq n)=P(T=n)+P(T>n)$

$\Rightarrow P(T=n)=P(T>n-1)-P(T>n)$

Let $A_j=\{$no type j coupon is contained among the first $n\}$

$P(T>n)=P(\bigcup_{j=1}^N A_j)$

$P(T>n)=\sum_j P(A_j)$ - coupon j is not among the first n collected

$-\sum\sum_{j_1<j_2}P(A_{j_1}A_{j_2})$ - coupon j_1 and j_2 are not the first n

$+\cdots+(-1)^{N+1}P(A_1A_2\cdots A_N)$ by *inclusion-exclusion identity*

$P(A_{j_1}A_{j_2}\cdots A_{j_k})=(\frac{N-k}{N})^n$

Hence $P(T>n)=\sum_{i=1}^{N-1}\binom{N}{i}(\frac{N-i}{N})^n(-1)^{i+1}$

Probability Mass Function

probability mass function , pmf of X (*discrete*) $\rightarrow$ $p(a)=P(X=a)$

- if X assumes one of the values $x_1,x_2,\dots$, then $\sum_{i=1}^\infty p(x_i)=1$

Cumulative Distribution Function

- **cumulative distribution function (cdf)** of a r.v. $X\rightarrow$ the function F defined by $F(x)=P(X\leq x),\quad -\infty<x<\infty$
  - $F(x)$ is defined on the entire real line. (aka *distribution function*)

pmf,

a	1	2	4
$p(a)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$F(a)=\sum p(x)$ for all $x\leq a$

cdf, $F(a)=\begin{cases}0, & a<1\\ 1/2, & 1\leq a<2\\ 3/4, & 2\leq a<4\\ 1, & a\geq 4\end{cases}$

Expected Value

- aka population mean/sample mean, μ

discrete: $E(X)=\sum_x x\cdot p(x)$

continuous: $E(X)=\int_{-\infty}^{\infty}x\cdot f(x)\,dx$

$$E[g(x)]=\int_{-\infty}^{\infty}g(x)f(x)\,dx$$

N1 - if a and b are constants, then $E(aX+b)=aE(X)+b$

N3 - for a non-negative r.v. Y, $E(Y)=\int_0^\infty P(Y>y)\,dy$

Proof. $\int_0^\infty P(Y>y)\,dy=\int_0^\infty\int_y^\infty f_Y(x)\,dx\,dy$ (because $f(x)=\frac{d}{dx}F(x)$)

$=\int_0^\infty xf_Y(x)\,dx\quad=E(Y)$

- I is an indicator variable for event A if $I=\begin{cases}1,\text{ if } A \text{ occurs}\\ 0,\text{ if } A^c \text{ occurs}\end{cases}$. then

$E(I)=P(A)$.

finding expectation of f(x)

- method 1, using pmf of Y: let $Y=f(X)$. Find corresponding X for each Y.
- method 2, using pmf of X: $E[f(x)]=\sum_x f(x)p(x)$

Variance

If X is a r.v. with mean $\mu=E[X]$, then the **variance** of X is defined by

$$\begin{aligned}Var(X)&=E[(X-\mu)^2]\\&=E(x^2)-[E(x)]^2\end{aligned}$$

- $Var(aX+b)=a^2Var(x)$
- $Var(X)=\sum_{x_i}(x_i-\mu)^2\cdot p(x_i)$ (deviation $\cdot$ weight)

Poisson Random Variable

a r.v. X is said to be a **Poisson r.v.** with parameter λ if for some $\lambda>0$,

$$P(X=i)=e^{-\lambda}\cdot\frac{\lambda^i}{i!}$$
$$E(X)=\lambda,\quad Var(X)=\lambda$$

- $\sum_{i=0}^\infty P(X=i)=1$
- **Poisson Approximation of Binomial** - if $X\sim$ Binomial(n,p), where n is large and p is small, then $X\dot\sim$Poisson(λ) where $\lambda=np$.
  - ✓ *weak* dependence is ok
- **2 ways** to look at the Poisson distribution
  1. an approximation to the binomial distribution with large n and small p
  2. counting the number of events that occur at *random* at certain points in time

Poisson distribution as random events

Let $N(t)$ be the number of events that occur in time interval $[0,t]$.

N1 - If the 3 assumptions are true, then $N(t)\sim$ Poisson(λt).

N2 - If λ is the *rate of occurrences* of events per unit time, then the number of occurrences in an interval of length t has a Poisson distribution with mean λt.

$$P(N(t)=k)=\frac{e^{-\lambda t}(\lambda t)^k}{k!}\text{, for } k\in\mathbb{Z}_{\geq 0}$$

o(h) notation

$o(h)$ stands for any function $f(h)$ such that $\lim_{h\to 0}\frac{f(h)}{h}=0$

- $o(h)+o(h)=o(h)$
- $\frac{\lambda t}{n}+o(\frac{t}{n})\doteq\frac{\lambda t}{n}$ for large n

Expected Value of sum of r.v.

For a r.v. X, let $X(s)$ denote the value of X when $s\in\mathcal{S}$

N1 - $E(x)=\sum_i x_iP(X=x_i)=\sum_{s\in\mathcal{S}}X(s)p(s)$ where $\mathcal{S}_i=\{s:X(s)=x_i\}$

N2 - $E(\sum_{i-1}^n)=\sum_{i=1}^n E(X_i)$ for r.v. $X_1,X_2,\dots,X_n$

e.g. distribution of time to next event

Q. suppose an accident happens at a rate of 5 per day. Find the distribution of time, starting from now, until the next accident.

A. Let $X=$ time (in days) until the next accident.

Let $V=$ be the number of accidents during time period $[0,t]$.

$V\sim$ Poisson$(5t)$ $\quad\Rightarrow P(V=k)=\frac{e^{-5t}\cdot(5t)^k}{k!}$

$P(X>t)=P($no accidents happen during $[0,t])=P(V=0)=e^{-5t}$

$P(X\leq t)=1-e^{-5t}$

example: finding pdf

Q - Find the pdf of $(b-a)X+a$ where a,b are constants, $b>a$. The pdf of X is given by $f(x)=\begin{cases}1, & 0\leq X\leq 1\\ 0, & \text{otherwise}\end{cases}$.

A. Let $Y=(b-a)X+a$.

cdf, $F_Y(y)=P(Y\leq y)=P((b-a)X+a\leq y)=P(X\leq\frac{y-a}{b-a})$

$F_Y(y)=\int_0^{\frac{y-a}{b-a}}1\,dx=\frac{y-a}{b-a},\quad a<y<b$

$f_Y(y)=\frac{d}{dy}F_Y(y)=\begin{cases}\frac{1}{b-a}, & a<y<b\\ 0, & \text{otherwise}\end{cases}$

05. CONTINUOUS RANDOM VARIABLES

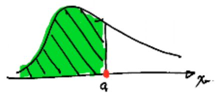
X is a **continuous r.v.** $\rightarrow$ if there exists a nonnegative function f defined for all real $x\in(-\infty,\infty)$, such that $P(X\in B)=\int_B f(x)\,dx$

N1 - $P(X\in(-\infty,\infty))=\int_{-\infty}^\infty f(x)\,dx=1$

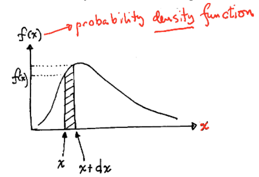
N2 - $P(a\leq X\leq b)=\int_a^b f(x)\,dx$

N3 - $P(X=a)=\int_a^a f(x)\,dx=0$

N4 - $P(X<a)=P(X\leq a)=\int_{-\infty}^a f(x)\,dx$



N5 - interpretation of **probability density function**



$$P(x<X<x+dx)=\int_x^{x+dx}f(y)\,dy$$
$$\approx f(x)\cdot dx$$

pdf at x, $f(x)\approx\frac{P(x<X<x+dx)}{dx}$

N6 - if X is a continuous r.v. with pdf $f(x)$ and cdf $F(x)$, then $f(x)=\frac{d}{dx}F(x)$. (*Fundamental Theorem of Calculus*)

N7 - median of X, x occurs where $F(x)=\frac{1}{2}$

Generating a Uniform r.v.

if X is a continuous r.v. with cdf $F(x)$, then

- **N8** - $F(X)=U\sim uniform(0,1)$.

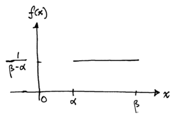
Proof. let $Y=F(X)$. then cdf of Y, $F_Y(y)$ =

$P(Y\leq y)=P(F(X)\leq y)=P(X\leq F^{-1}(y))=F(F^{-1}(y))=y$.

- **N9** - $X=F^{-1}(U)\sim$ cdf $F(x)$.
  - generating a r.v. from a uniform(0, 1) r.v. and a r.v. with cdf $F(x)$.

Uniform Random Variable

X is a **uniform r.v.** on the interval (α,β), $X\sim Uniform(\alpha,\beta)$ if its pdf is given by



$$f(x)=\begin{cases}\frac{1}{\beta-\alpha}, & \alpha<x<\beta\\ 0, & \text{otherwise}\end{cases}$$

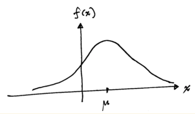
$$E(X)=\frac{\alpha+\beta}{2},\quad Var(X)=\frac{(\beta-\alpha)^2}{12}$$

if $X\sim Uniform(\alpha,\beta)$, then $\frac{x-\alpha}{\beta-\alpha}\sim Uniform(0,1)$

Normal Random Variable

X is a **normal r.v.** with parameters μ and σ^2, $X\sim N(\mu,\sigma^2)$ if the pdf of X is given by

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\quad -\infty<x<\infty$$
$$E(x)=\mu,\quad Var(X)=\sigma^2$$



if $X\sim N(\mu,\sigma^2)$, then $\frac{X-\mu}{\sigma}\sim N(0,1)$

if $Y\sim N(\mu,\sigma^2)$ and a is a constant, $F_y(a)=\Phi(\frac{a-\mu}{\sigma})$

standard normal distribution $\rightarrow X\sim N(0,1)$

- $F(x)=P(X\leq x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{1}{2}y^2}\,dy=\Phi(x)$
- $\Phi(x)=1-\Phi(-x)$

Normal Approximation to the Binomial Distribution

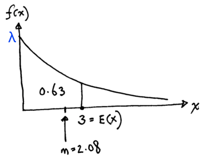
if $S_n \sim Binomial(n,p)$, then $\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0,1)$ for large n.

$\mu = np, \quad \sigma^2 = np(1-p)$

Exponential Random Variable

a *continuous* r.v. X is a **exponential r.v.**, $\quad X \sim Exponential(\lambda)$ or $Exp(\lambda)$

if for some $\lambda > 0$, its pdf is given by



$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

$$P(X < a) = \int_0^a \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda a}$$

- an exponential r.v. is *memoryless*.
  - a non-negative r.v. is **memoryless** $\rightarrow$ if $P(X > s + t \mid X > t) = P(X > s)$ for all $s, t > 0$.

Gamma Distribution

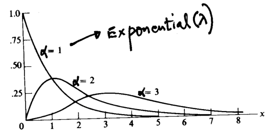
a r.v. X has a **gamma distribution**, $\quad X \sim Gamma(\alpha, \lambda)$

with parameters (α, λ), $\lambda > 0$ and $\alpha > 0$ if its pdf is given by

$$f(x) \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E(X) = \frac{\alpha}{\lambda} \quad Var(X) = \frac{\alpha}{\lambda^2}$$

where the gamma function $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha - 1}\, dy$.



Gamma densities ($\lambda = 1$).

N1 - $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

Proof. using integration by parts of LHS to RHS

N2 - if α is an integer n, then $\Gamma(n) = (n-1)!$

N3 - if $X \sim Gamma(\alpha, \lambda)$ and $\alpha = 1$, then $X \sim Exp(\lambda)$.

N4 - for events occurring randomly in time following the 3 assumptions of poisson distribution, the **amount of time elapsed** until a total of n events has occurred is a gamma r.v. with parameters (n, λ).

- time at which the n-th event occurs, $T_n \sim Gamma(n, \lambda)$
- number of events in time period $[0, t]$, $N(t) \sim Poisson(\lambda t)$

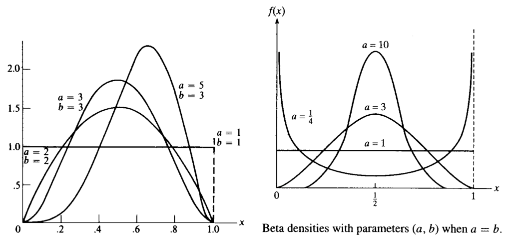
N5 - $Gamma(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}) = \chi^2_n$ (chi-square distribution to n degrees of freedom)

Beta Distribution

a r.v. X is said to have a **beta distribution**, $X \sim Beta(a, b)$

$$f(x) = \begin{cases} \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{a}{a+b} \quad Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$$



Beta densities with parameters (a, b) when $a = b$.

N1 - $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$

N2 - $\beta(a = 1, b = 1) = Uniform(0, 1)$

N3 - $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Cauchy Distribution

a r.v. X has a cauchy distribution, $\quad X \sim Cauchy(\theta)$

with parameter θ, $\infty < \theta < \infty$ if its density is given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2}, \; -\infty < x < \infty$$

06. JOINTLY DISTRIBUTED RANDOM VARIABLES

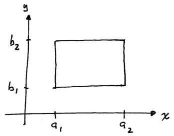
Joint Distribution Function

the **joint cumulative distribution function** of the pair of r.v. X and Y is $\rightarrow$

$F(x, y) = P(X \leq x, Y \leq y)$, $-\infty < x < \infty$, $\; -\infty < y < \infty$

N1 - **marginal cdf of** X, $F_X(x) = \lim_{y \to \infty} F(x, y)$.

N2 - **marginal cdf of** Y, $F_Y(y) = \lim_{x \to \infty} F(x, y)$.



N3 - $P(X > a, Y > b) = 1 - F_X(a) - F_Y(b) + F(a, b)$

N4 - $P(a_1 < X \leq a_2, b_1 < Y \leq b_2)$

$= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1)$

Joint Probability Mass Function

if X and Y are both discrete r.v., then their **joint pmf** is defined by

$p(i, j) = P(X = i, Y = j)$

N1 - **marginal pmf of** X, $P(X = i) = \sum_j P(X = i, Y = j)$

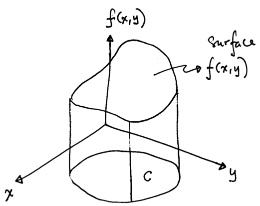
N2 - **marginal pmf of** Y, $P(Y = i) = \sum_i P(X = i, Y = j)$

Joint Probability Density Function

the r.v. X and Y are said to be *jointly continuous* if there is a function $f(x, y)$ called the **joint pdf**, such that for any two-dimensional set C,

$$P[(X, Y) \in C] = \iint_C f(x, y)\, dx\, dy$$

= volume under the surface over the region C.



N1 - if $C = \{(x, y) : x \in A, y \in B\}$, then $P(X \in A, Y \in B) = \int_B \int_A f(x, y)\, dx\, dy$

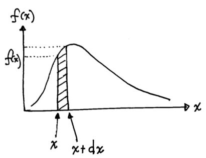
N2 - $F(a, b)$

$= P\Big(X \in (-\infty, a], Y \in (-\infty, b]\Big)$

$= \int_{-\infty}^b \int_{-\infty}^a f(x, y)\, dx\, dy$

N3 - $f(a, b) = \frac{\delta^2}{\delta a \delta b} F(a, b)$

interpretation of pdf

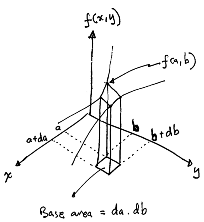


$$P(x < X < x + dx) = \int_x^{x+dx} f(y)\, dy$$

$$\approx f(x)\, dx$$

pdf at x, $f(x) \approx \frac{P(x < X < x + dx)}{dx}$

interpretation of joint pdf



$P(a < X < a + da, b < Y < b + db)$

$= \int_b^{b+db} \int_a^{a+da} f(x, y)\, dx\, dy$

$\approx f(a, b)\, da\, db \quad$ (density of probability)

marginal pdf of X, $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

marginal pdf of Y, $f_Y(x) = \int_{-\infty}^{\infty} f(x, y)\, dx$

Independent Random Variables

N1 - X, Y are **independent** $\rightarrow P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

N2 - X and Y are **independent** $\rightarrow$

$P(X \leq a, Y \leq b) = P(X \leq a) \cdot P(Y \leq b)$

or $F(a, b) = F_X(a) \cdot F_Y(b) \quad \Rightarrow$ joint cdf is the product of the marginal cdfs

N3 - *discrete case*: discrete r.v. X and Y are **independent** $\iff$

$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all x, y.

N4 - *continuous case*: jointly continuous r.v. X and Y are **independent** $\iff$

$f(x, y) = f_X(x) \cdot f_Y(y)$ for all x, y.

$-\infty < x < \infty, \quad -\infty < y < \infty$

N5 - independence is a **symmetric** relation $\rightarrow X$ indep of $Y \iff Y$ indep of X

if X and Y are independent, then for any functions h and g,

$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

Sum of Independent Random Variables

N1 - for independent, continuous r.v. X and Y having pdf f_X and f_Y,

$$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\, dy$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a - y) f_Y(y)\, dy$$

Distribution of Sums of Independent r.v.

for $i = 1, 2, \ldots, n$,

1. $X_i \sim Gamma(t_i, \lambda) \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim Gamma(\sum_{i=1}^n t_i, \lambda)$
2. $X_i \sim Exp(\lambda) \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim Gamma(n, \lambda)$
3. $Z_i \sim N(0, 1) \quad \Rightarrow \quad \sum_{i=1}^n z_i^2 \sim \chi^2_n = Gamma(\frac{n}{2}, \frac{1}{2})$
4. $X_i \sim N(\mu_i, \sigma_i^2) \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
5. $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2) \Rightarrow X + Y \sim Poisson(\lambda_1 + \lambda_2)$
6. $X \sim Binom(n, p), Y \sim Binom(m, p) \Rightarrow X + Y \sim Binom(n + m, p)$

Conditional Distribution (discrete)

for discrete r.v. X and Y, the **conditional pmf** of X given that $Y = y$ is

$$P_{X|Y}(x|y) = P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)}$$

for discrete r.v. X and Y, the **conditional pdf** of X given that $Y = y$ is

$$F_{X|Y}(x|y) = P(X \leq x | Y = y) = \sum_{a \leq x} P(X = a, Y = y) = \sum_{a \leq x} P_{X|Y}(a|y)$$

N0 - equivalent notation:

- $P_{X|Y}(x|y) = P(X = x | Y = y)$
- $P_X(x) = P(X = x)$

N1 - if X is independent of Y, then $P_{X|Y}(x|y) = P_X(x)$

Conditional Distribution (continuous)

for X and Y with joint pdf $f(x, y)$, the **conditional pdf** of X given that $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad \text{for all } y \text{ s.t. } f_Y(y) > 0$$

$$F_{X|Y}(a|y) = P(X \leq a | Y = y) = \int_{-\infty}^{a} f_{X|Y}(x|y)\, dx$$

N1 - for any set A, $P(X \in A | Y = y) = \int_A f_{X|Y}(x|y)\, dy$

N2 - if X is independent of Y, then $f_{X|Y}(x|y) = f_X(x)$.

! "find the marginal/conditional pdf of Y" $\Rightarrow$ must include the **range** too!!

Joint Probability Distribution of Functions of r.v.

Let X_1 and X_2 be jointly continuous r.v. with joint pdf $f_{x_1, x_2}(x_1, x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ satisfy

1. the equations $y_1 = g_1(X_1, X_2)$ and $y_2 = g_2(X_1, X_2)$ can be *uniquely* solved for x_1, x_2 in terms of y_1 and y_2

2. $g_1(x_1,x_2)$ and $g_2(x_1,x_2)$ have continuous partial derivatives at all points (x_1,x_2) such that $J(x_1,x_2) = \begin{vmatrix} \frac{\delta g_1}{\delta x_1} & \frac{\delta g_1}{\delta x_2} \\ \frac{\delta g_2}{\delta x_1} & \frac{\delta g_2}{\delta x_2} \end{vmatrix} = \frac{\delta g_1}{\delta x_1} \cdot \frac{\delta g_2}{\delta x_2} - \frac{\delta g_2}{\delta x_1} \cdot \frac{\delta g_1}{\delta x_2} \neq 0$

then

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\frac{1}{|J(x_1,x_2)|}$$

where $x_1 = h_1(y_1,y_2), x_2 = h_2(y_1,y_2)$

07. PROPERTIES OF EXPECTATION

- for a **discrete** r.v. X, $E(X) = \sum_x x \cdot p(x) = \sum_x \cdot P(X=x)$
- for a **continuous** r.v. X, $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$
- for a **non-negative integer-valued** r.v. Y, $E(Y) = \sum_{i=1}^{\infty} P(Y \geq i)$
- for a **non-negative** r.v. Y, $E(Y) = \int_{-\infty}^{\infty} P(Y > y)\,dy$

Expectations of Sums of Random Variables

for X and Y with joint pmf $p(x,y)$ and joint pdf $f(x,y)$,

$$E[g(x,y)] = \sum_y \sum_x g(x,y)p(x,y)$$

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy$$

N2 - if $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$

N4 - for r.v.s X and Y, if $X \geq Y$, then $E(X) \geq E(Y)$

N5 - let $X_1,\ldots,X_n$ be independent and identically distributed r.v.s having distribution $P(X_i \leq x) = F(x)$ and expected value $E(X_i) = \mu$.

if $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$, then $E(\bar{X}) = \mu$

N6 - $\bar{X}$ is the **sample mean**. $\Rightarrow$ sample mean = population mean

! trick: express a r.v. as a sum of r.v. with easier to find expectation

examples

- hypergeometric with r red balls out of N balls with n trials
  - indicator r.v. $=1$ if the ith ball selected is red
  - $P(Y_i = 1) = \frac{r}{N} \Rightarrow E(Y_i) = \frac{r}{N} \Rightarrow E(X) = \sum_{i=1}^{n} Y_i = n\frac{r}{N}$
- coupon collector problem:
  - let X = number of coupons collected for a complete set
  - let X_i = additional number to be collected to obtain distinct type after i distinct types have been collected. $X_i \sim Geometric(p = \frac{N-i}{N})$
  - $E(X) = \sum_{i=1}^{N-1} E(X_i) = 1 + \frac{1}{\frac{N-1}{N}} + \frac{1}{\frac{N-2}{N}} + \cdots + \frac{1}{\frac{1}{N}}$

    $= N(\frac{1}{N} + \frac{1}{N-1} + \cdots + 1)$

Covariance, Variance of Sums and Correlations

covariance $\rightarrow$ measure of *linear relationship*

$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])]$$
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

N1 - X and Y are independent $\Rightarrow Cov(X,Y) = 0$

N2 - $Cov(X,Y) = 0 \not\Rightarrow X$ and Y are independent. *Proof.* let $E(X) = 0$, $E(XY) = 0 \Rightarrow Cov(X,Y) = 0$, but not independent e.g. non-linear relationship

Covariance properties

1. $Cov(X,Y) = Cov(Y,X)$
2. $Cov(X,X) = Var(X)$
3. $Cov(aX,Y) = aCov(X,Y)$
4. $Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n}\sum_{j-1}^{m} Cov(X_i,Y_j)$
5. $Cov(I_A,I_B) = P(B)[P(A|B) - P(A)]$

for variance:

N1 - $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i,X_j)$

N2 - if $X_1,\ldots,X_n$ are *pairwise independent*, $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$

N3 - for n independent and identically distributed r.v. with variance σ^2,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$Var(\bar{X}) = \frac{\sigma^2}{n} \qquad E(S^2) = \sigma^2$$

$\Rightarrow S^2$ is an *unbiased estimator* for σ^2.

Correlation

correlation of two r.v. X and Y, $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$

N1 - $-1 \leq \rho(X,Y) \leq 1$ where -1 and 1 denote a perfect negative and positive linear relationship respectively.

N2 - $\rho(X,Y) = 0 \Rightarrow$ no *linear* relationship - uncorrelated

N3 - $\rho(X,Y) = 1 \Rightarrow Y = aX + b, a = \frac{\delta y}{\delta x} > 0$

N4 - for independent events A, B with indicator r.v. I_A, I_B: $Cov(I_A,I_B) = 0$.

N5 - deviation is not correlated with the sample mean. For independent & identically distributed r.v. $X_1,X_2,\ldots,X_n$ with variance σ^2, then $Cov(X_i - \bar{X}, \bar{X}) = 0$.

Conditional Expectation

the **conditional expectation** of X given that $Y = y$, $\forall y$ s.t. $P_Y(y) > 0$, is:

$$E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y) = \sum_x x \cdot p_{X|Y}(x|y)$$

$$E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\,dx$$

! note the range for $f_{X|Y}(x|y)$

N1 - If $X,Y \sim Geometric(p)$, then $P(X=i|X+Y=n) = \frac{1}{n-1}$, a uniform distribution.

N2 - $E(X|X+Y=n) = \sum_{i=1}^{n-1} i \cdot P(X=i|X+Y=n) = \frac{n}{2}$

discrete case: $E[g(x)|Y=y] = \sum_x g(x)P_{X|Y}(x|y)$

continuous case: $E[g(x)|Y=y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)\,dx$

then $E(X) = E_{\text{w.r.t. } y}(E_{\text{w.r.t. } X|Y=y}(X|Y))$

Deriving Expectation

$E(X) = E_Y(E_X(X|Y))$

discrete case: $E(X) = \sum_y E(X|Y=y)P(Y=y)$

continuous case: $E(X) = \int_{-\infty}^{\infty} E(X|Y=y)f_Y(y)\,dy$

N3 - 3 methods for finding $E(X)$ given $f(x,y)$

1. using $E(g(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy$ $\Rightarrow$ let $g(x,y) = x$
2. using $E(X) = \int_{-\infty}^{\infty} xf_X(x)\,dx$
3. using $E(X) = \int_{-\infty}^{\infty} E(X|Y=y)f_Y(y)\,dy$

N4 - $E(\sum_{i=1}^{N} X_i) = E_N(E(\sum_{i=1}^{N} X_i|N)) = \sum_{n=0}^{\infty} E(\sum_{i=1}^{N} X_i|N=n) \cdot P(N=n)$

Computing Probabilities by Conditioning

discrete: $P(E) = \sum_y P(E|Y=y)P(Y=y)$

continuous: $P(E) = \int_{-\infty}^{\infty} P(E|Y=y)f_Y(y)\,dy$

Proof. X is an indicator r.v.; $E(X|Y=y) = P(X=1|Y=y) = P(E|Y=y)$

N5 - $P(X<Y) = \int P(X<Y|Y=y) \cdot f_Y(y)$

Conditional Variance

$$Var(X|Y) = E[(X - E(X|Y))^2 \mid Y]$$
$$= E(X^2|Y) - [E(X|Y)]^2$$

N6 - $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

N7 - $E(f(Y)) = E(f(Y)|Y=t) = E(f(t)|Y=t)$

$= E(f(t))$ if $N(t)$ and Y are independent

Moment Generating Functions

moment generating function $M(t)$ of the r.v. $X \rightarrow$

$M(t) = E(e^{tX})$ for all real values of t

- if X is *discrete* with pmf $p(x)$, $M(t) = \sum_x e^{tx} \cdot p(x)$
- if X is *continuous* with pdf $f(x)$, $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$M(t)$ is called the **mgf** because *all moments of* X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t=0$.

- the n^{th} moment of of X is given as $E(X^n) = \sum_x x^n \cdot p(x)$
  - $M'(0) = E(X)$, $M''(0) = E(X^2)$, $M^n(0) = E(X^n)$, $n \geq 1$
  - $M'(t) = E(X^n e^{tX})$, $n \geq 1$

if X and Y are independent and have mgf's $M_X(t)$ and $M_Y(t)$ respectively,

N10 - the mgf of $X+Y$ is $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

N11 - if $M_X(t)$ exists and is finite in some region about $t=0$, then the distribution of X is **uniquely** determined. $M_X(t) = M_Y(t) \iff X = Y$

joint mgf: $E[e^{tX+sY}] = \int\int e^{tx+sy}f(x,y)\,dy\,dx$

Common mgf's

- $X \sim Normal(\mu,\sigma^2)$, $M(t) = e^{\frac{\sigma^2 t^2}{2} + \mu t}$
- $X \sim Binomial(n,p)$, $M(t) = (pe^t + (1-p))^n$
- $X \sim Poisson(\lambda)$, $M(t) = \exp[\lambda(e^t - 1)]$
- $X \sim Exp(\lambda)$, $M(t) = \frac{\lambda}{\lambda - t}$

08. LIMIT THEOREMS

Markov's Inequality $\rightarrow$ if X is a non-negative r.v., $\forall a > 0$, $P(X \geq a) \leq \frac{E(x)}{a}$.

Chebyshev's inequality $\rightarrow$ if X is an r.v. with finite mean μ and variance σ^2, then for any value of $k > 0$, $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$.

N1 - if $Var(X) = 0$, then $P(X = E[X]) = 1$

weak law of large numbers $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s, each with finite mean $E[X_i] = \mu$. Then, for any $\epsilon > 0$, $P\{|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \epsilon\} \rightarrow 0$ as $n \rightarrow \infty$

central limit theorem $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s each having mean μ and variance σ^2. Then the distribution of $\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n \rightarrow \infty$.

- aka: $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \rightarrow z \sim N(0,1)$
- for $-\infty < a < \infty$, as $n \rightarrow \infty$,

  $P(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a) \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx = F(a)$ - cdf of $N(0,1)$

N2 - Let $Z_1, Z_2, \ldots$ be a sequence of r.v.s with distribution functions F_{Z_n} and moment generating functions $M_{Z_n}, n \geq 1$. Let Z be a r.v. with distribution function F_Z and mgf M_Z.

If $M_{Z_n}(t) \rightarrow M_Z(t)$ for all t, then $F_{Z_n}(t) \rightarrow F_Z(t)$ for all t at which $F_Z(t)$ is continuous.

strong law of large numbers $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distribution r.v.s, each having finite mean $\mu = E[X_i]$. Then, with probability 1, $\frac{X_1 + \cdots + X_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$

Chernoff bounds $\rightarrow$ Suppose X is a r.v. and a constant.

$P(X \geq a) \leq e^{-ta}M_X(t)$, for all $t > 0$

$P(X \leq a) \leq e^{-ta}M_X(t)$, for all $t < 0$

Jensen's Inequality $\rightarrow$ If $f(x)$ is a convex function ($f''(x) \geq 0$), and the expectation exists and is finite, $E(f(X)) \geq f(E(X))$

approximations - $\lim_{n\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}$