MA2101

Isomorphisms

Definitions Let F be a mapping $F : S \to T$.

- Surjection: $\forall t \in T,\ \exists s \in S \quad F(s) = t$
- Injection: $\forall s_1, s_2 \in S$
  $F(s_1) = F(s_2) \Rightarrow s_1 = s_2$
- Bijection: Surjection and Injection

Homomorphism Let $\phi : U \to V$ be a mapping. It is a homomorphism if

$$\phi(u + v) = \phi(u) + \phi(v)$$
$$\phi(au) = a\phi(u)$$

If this homomorphism is <u>also</u> a bijection, then this is an isomorphism.

Linear mappings

Definition A mapping $T : V \to W$ is called a linear mapping if $\forall a \in \mathcal{F}, \forall u, v \in V$:

$$T(u + v) = T(u) + T(v)$$
$$T(au) = aT(u)$$

Set of linear mappings The set of all linear mappings from V to W is denoted $\mathcal{L}(V, W)$. It is a vector space.

Basis as a linear mapping Let B be a basis of V, where $\dim(V) = n$.

- Then $B \in \mathcal{L}(\mathcal{F}^n, V)$, i.e. it is a linear mapping from $\mathcal{F}^n$ to V.
- However, some $x \in \mathcal{L}(\mathcal{F}^n, V)$ is not necessarily a basis, as it is not necessarily a bijection.

Change of basis Let y and z be basis of an n-dimensional vector space V.

- z has inverse z^{-1}
- Define $P = z^{-1} \circ y$, then $P : \mathcal{F}^n \to \mathcal{F}^n$, and since composition of bijections is a bijection, P is a bijection, so P is an isomorphism of F^n with itself.
- Hence, $y = z \circ P$, where P is an isomorphism of F^n with itself.

Kernel and range

Let $T \in \mathcal{L}(V, W)$.

Kernel

- The kernel of T, denoted $\mathrm{Ker}(T)$, is the set of elements in V that are mapped to the zero vector in W.
- The dimension of $\mathrm{Ker}(T)$ is called the nullity of T, denoted as $\mathrm{Null}(T)$.
- $\mathrm{Ker}(T)$ is a subspace of V.
- T is injective $\Leftrightarrow \mathrm{Ker}(T) = \{\vec{0}\}$

Range

- The range of T, denoted $\mathrm{Rang}(T)$, is the set of vectors in W that can be expressed as Tv for some $v \in V$.
- The dimension of $\mathrm{Rang}(T)$ is called the rank of T, denoted as $\mathrm{Rank}(T)$.
- $\mathrm{Rang}(T)$ is a subspace of W.
- T is surjective $\Leftrightarrow \mathrm{Rang}(T) = W$

Fundamental theorem

$$\dim(V) = \mathrm{Null}(T) + \mathrm{Rank}(T)$$

- Let $T \in \mathcal{L}(V, W)$ where $\dim(V) > \dim(W)$. Then T cannot be injective.
- Let $T \in \mathcal{L}(V, W)$ where $\dim(V) < \dim(W)$. Then T cannot be surjective.
- For linear maps from V to itself,
  bijectivity $\Leftrightarrow$ injectivity $\Leftrightarrow$ surjectivity

Duality

Dual space

Definition The dual space of a vector space V, is the set of all linear mappings from V to $\mathcal{F}$, $\mathcal{L}(V, \mathcal{F})$.

- We denote the dual space as $\hat{V}$
- If $\alpha \in \hat{V}$, then $\forall u \in V,\ \alpha(u) \in \mathcal{F}$

Example Let $\mathcal{F}^n$ be the vector space of column vectors.

- The dual space of $\mathcal{F}^n$ is the space of $1 \times n$ row vectors.
- Its canonical dual basis ϵ^i are the row vectors $\begin{pmatrix}1 & 0 & 0 & \cdots\end{pmatrix}, \quad \begin{pmatrix}0 & 1 & 0 & \cdots\end{pmatrix}, \quad \cdots$

Dual basis

Definition Let z_i be a basis for V.

- For any $v \in V$, write $v = \sum a^i z_i$.
- Denote the dual vectors as ζ^i. Define them as follows:
$$\zeta^i(v) = a^i$$
- Letting the ζ act on z, we have
$$\zeta^i(z_j) = I^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$
- The set of all $\zeta^i \in \hat{V}$ that satisfies $\zeta^i(z_j) = I^i_j$ is called the dual basis of z_i.

Show basis

1. (Show that they belong to $\hat{V}$) Show that each ζ^i is a linear mapping from V to $\mathcal{F}$.
2. (Show they are linearly independent) Consider $\sum p_i \zeta^i = \vec{0}$, and let it act on z_j.
$$0 = p_i \zeta^i(z_j) = p_i I^i_j = p_j$$
3. (Show that they span) We do not know the dimension of $\hat{V}$. Thus, we show that every element of $\hat{V}$ can be expressed as a linear combination of ζ^i.
   - Let $\beta = (\alpha z_i)\zeta^i$ for some dual vector α.
   $$\beta(z_j) = \alpha(z_i)\zeta^i(z_j) = \alpha(z_i)I^i_j = \alpha(z_j)$$
   - Hence, any dual vector β can be expressed as a linear combination of ζ^i, where the components are $\alpha(z_i)$.
4. This implies $\dim(\hat{V}) = \dim(V)$.
5. Since in our definition, a^i is unique to v, so the dual basis is also unique.

Alternative definition

- Let z be a basis for V, so $z \in \mathcal{L}(\mathcal{F}^n, V)$.
- Let $\alpha \in \mathcal{F}^n$. Let $p \in \hat{\mathcal{F}}^n$.
- Define $\zeta \in \mathcal{L}(\hat{\mathcal{F}}^n, \hat{V})$ as follows:
$$\zeta p(z\alpha) = pa$$
- Here, $\zeta p \in \hat{V}$, and $z\alpha \in V$.

Why duality?

- $v \in V$ is linear when acting on $\alpha \in \hat{V}$.
- We claim that α is the dual vector of v, but we can also regard v as the dual vector of α.

Transpose mapping

Let $\alpha \in \hat{V}$ and let $v \in V$. For each LT $T : V \to V$, define $\hat{T} : \hat{V} \to \hat{V}$ as follows:

$$\hat{T}(\alpha)(v) = \alpha(Tv)$$

Then $\hat{T}$ is the transpose of T. Since $\hat{T}$ is linear, so $\hat{T} \in \mathcal{L}(\hat{V}, \hat{V})$.

Dual of the dual space

Dual of dual space of V is isomorphic to V.

Tensor products

Definition

- Let V be a vector space. Let $\hat{V}$ be its dual.
- For each $v \in V$ and $\alpha \in \hat{V}$, define their tensor product $v \otimes \alpha$ to be the element of $\mathcal{L}(V, V)$ as follows:
$$(v \otimes \alpha)(w) = \alpha(w)v$$
  where $w \in V$.
- The tensor product is linear in both ways.
$$(u + v) \otimes \alpha = u \otimes \alpha + v \otimes \alpha$$
$$v \otimes (\alpha + \beta) = v \otimes \alpha + v \otimes \beta$$
- Let β be a dual vector.
$$\beta \circ (v \otimes \alpha) = \beta(v)\alpha$$

Basis

Let z_i be a basis of V and let ζ^j be the dual basis.

- For the following, consider $V = \mathcal{F}^n$ and $z_i = e_i$ for a concrete example.
- For each pair of (i, j), $z_i \otimes \zeta^j \in \mathcal{L}(V, V)$.
- The full set of $z_i \otimes \zeta^j$ is a basis for $\mathcal{L}(V, V)$.
- $\dim(\mathcal{L}(V, V)) = \dim(V)^2$
- Define T^i_j as follows:
$$T(z_j) = T^i_j(z_i)$$

Show span

- Let $T \in \mathcal{L}(V,V)$.
- Since $T(z_i) \in V$, then express it as a linear combination as follows, where := removes the summations
$$T(z_i) = \sum_j T_i^j(z_j) := T_i^j(z_j)$$
- Next, define S in the following way, where := removes the summations.
$$S = \sum_i \left(\left(\sum_j T_i^j z_j \right) \otimes \zeta^i \right) := T_i^j z_j \otimes \zeta^i$$
S is a linear combination of things in $\mathcal{L}(V,V)$, so $S \in \mathcal{L}(V,V)$.
- Let S act on z_k. Summations are not present here, but use the above definitions to confirm that they work:
$$\begin{aligned} S(z_k) &= T_i^j z_j \otimes \zeta^i(z_k) = \zeta^i(z_k) T_i^j z_j \\ &= I_k^i T_i^j z_j = T_k^j z_j = T(z_k) \end{aligned}$$
By linearity, we have $S(v) = T(v)$ for all $v \in V$. Hence
$$T = S = T_i^j z_j \otimes \zeta^i = T_i^j \left(z_j \otimes \zeta^i \right)$$
and any arbitrary T can be expressed as a linear combination of $z_j \otimes \zeta^i$. Hence, $z_i \otimes \zeta^j$ forms a basis for $\mathcal{L}(V,V)$.

Extracting components Let ζ^i be the dual basis of z_j.
$$\zeta^i \left(T(z_j) \right) = \zeta^i \left(T_j^k(z_k) \right) = T_j^k I_k^i = T_j^i$$
In particular,
$$\epsilon^i \left(T(e_j) \right) = \epsilon^i \left(T_j^k(e_k) \right) = T_j^k I_k^i = T_j^i$$
where e_j are the canonical basis vectors of V and ϵ^i are the canonical basis vectors of $\hat{V}$.

Transpose of tensor product The transpose of $v \otimes \alpha$ is $\alpha \otimes v$.

Trace of tensor product The trace of $v \otimes \alpha$ is $\alpha(v)$. That is, the trace of the outer product is the inner product.

Matrices

Definition A $n \times m$ matrix is just a list of m vectors belonging to $\mathcal{F}^n$.

Convention We may refer to the elements of a matrix in the following way:
$$\begin{pmatrix} M_1^1 & M_2^1 & M_3^1 \\ M_1^2 & M_2^2 & M_3^2 \\ M_1^3 & M_2^3 & M_3^3 \end{pmatrix}$$

- Subscripts: which column vector from the list you are referring to
- Superscripts: which component of a given vector you are referring to
- M_j^i is the (i,j)-entry in a given matrix

Vector space The set $\mathcal{M}_{n\times m}$ of $n \times m$ matrices is a vector space, with usual matrix addition, and scalar multiplication.

Matrix of a linear transformation From the tensor products section,
$$T = T_i^j \left(z_j \otimes \zeta^i \right)$$
where $z_j \otimes \zeta^i$ is a basis for $\mathcal{L}(V,V)$. Then T_i^j is the matrix of T relative to the basis z_i.

Transpose of $T : V \to V$ From the transpose mapping section, we have define the transpose mapping as follows:
$$\hat{T}(\alpha)(v) = \alpha(Tv)$$
where $\alpha \in \hat{V}$ and $v \in V$. Let $\alpha = \zeta^i$ and $v = z_j$. Then
$$\begin{aligned} \zeta^i \left(T z_j \right) &= z_j \left(\hat{T} \zeta^i \right) \\ \zeta^i \left(T_j^i z_i \right) &= z_j \left(\hat{T}_i^j \zeta^j \right) \\ T_j^i &= \hat{T}_i^j \end{aligned}$$
Hence the transpose matrix is obtained by flipping along the diagonal.

Transpose of $T : V \to W$ The transpose of $T : V \to W$ is $\hat{T} : \hat{W} \to \hat{V}$.

Trace is zero on commutators
$$\operatorname{tr}(MN) = \sum_i M_k^i N_i^k = N_i^k M_k^i = \operatorname{tr}(NM)$$

$\mathcal{L}(V,W)$ for $V \neq W$

Let $\dim(V) = n$ and $\dim(W) = m$.

Tensor product For each $x \in W$ and $\alpha \in \hat{V}$, define their tensor product $x \otimes \alpha$ to be the element of $\mathcal{L}(V,W)$ as follows, where $v \in V$:
$$(x \otimes \alpha)(v) = \alpha(v)x$$

Basis

- Let x_i be a basis of W, and let ζ^j be a dual basis for V.
- Then the full set $x_i \otimes \zeta^j$ is a set of $n \times m$ elements of $\mathcal{L}(V,W)$ which forms a basis.

LT Any $T \in \mathcal{L}(V,W)$ can be expressed as $T_j^i \left(x_i \otimes \zeta^j \right)$, where the T_j^i can be assembled into a $m \times n$ matrix.

Note

- Given bases for V and W, we can map any LT in $\mathcal{L}(V,W)$ to a matrix.
- This mapping is a linear mapping from $\mathcal{L}(V,W)$ to $\mathcal{M}_{m\times n}$.

Matrix of a product

Matrix multiplication Define matrix multiplication as
$$M_j^i N_k^j = M_1^i N_k^1 + M_2^i N_k^2 + \cdots$$

Multiplying transformations

Definition

- Let U, V, W be vector spaces such that $S \in \mathcal{L}(U,V)$ and $T \in \mathcal{L}(V,W)$.
- Then $T \circ S \in \mathcal{L}(U,W)$ and
- $\mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S)$

Example Let $v, w \in V$ and $\alpha, \beta \in \hat{V}$. Then
$$(v \otimes \alpha) \circ (w \otimes \beta) = \alpha(w) v \otimes \beta$$

Matrix of the tensor product Since
$$v \otimes \alpha = (v^i z_i) \otimes (p_j \zeta^j) = v^i p_j \left(z_i \otimes \zeta^j \right)$$
So the matrix is just $v^i p_j$. For example, when $\dim(V) = \dim(\hat{V}) = 3$,
$$\begin{pmatrix} v^1 p_1 & v^1 p_2 & v^1 p_3 \\ v^2 p_1 & v^2 p_2 & v^2 p_3 \\ v^3 p_1 & v^3 p_2 & v^3 p_3 \end{pmatrix} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$$
Sometimes, the tensor product is also called the outer product of a column vector with a row vector.

Change of basis

Dual basis ζ Let ζ^j be the dual basis to z_i. Let ϵ^j be the dual canonical basis for $\mathcal{F}^n$. Then we have
$$\zeta^j = \epsilon^j \circ z^{-1}$$

Linear map T Let $T : V \to V$. Then $\mathcal{M}(T)$ relative to basis z is the same as the matrix of
$$z^{-1} \circ T \circ z$$
relative to the canonical basis of $\mathcal{F}^n$, because
$$\epsilon^j z^{-1} T z e_i = \zeta^j z_i = T_i^j$$
Then, consider $y = z \circ P$. Note that $y^{-1} = P^{-1} \circ z^{-1}$. Then $\mathcal{M}(T)$ relative to basis y is
$$\begin{aligned} y^{-1} \circ T \circ y &= (P^{-1} \circ z^{-1}) \circ T \circ (z \circ P) \\ &= P^{-1}(z^{-1} T z) P \end{aligned}$$
Viewing this as a product of transformations, then the matrix is also the product of

- Matrix of P^{-1} rel. to canonical basis of $\mathcal{F}^n$
- Matrix of T rel. to the basis z of V
- Matrix of P rel. to the canonical basis of $\mathcal{F}^n$

Components Let z be a basis of V, let $v \in V$, let $a = z^{-1} v$, so a is a column vector of the components of v. The components relative to another basis $y = z \circ P$ is:
$$y^{-1} v = (P^{-1} z^{-1}) v = P^{-1} a$$

Alternative dual basis definition We may define a dual basis in the following way:
$$(\zeta p)(za) = pa$$
where $p \in \hat{\mathcal{F}^n}$. Since pa is a scalar, its value should not depend on the choice of basis. Then obviously $pa = pPP^{-1}a$. From the previous section, then the components of a dual vector will change from p to pP.