MA2101

Matrices

Describing change Let T be a 2D LT.

$$\mathcal{M}(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This means that $T(\hat{i}) = \begin{pmatrix} a \\ c \end{pmatrix}$ and $T(\hat{j}) = \begin{pmatrix} b \\ d \end{pmatrix}$, i.e. the columns tell us how the basic unit vectors change under the transformation.

Rotation matrix (2D) An anticlockwise rotation by θ is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Shear matrix (2D) A shear parallel to the x-axis by θ is given by

$$\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$$

Tut1 Q1 A matrix can be decomposed as a sum:

$$A = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$$

The first term is symmetric, and the second term is antisymmetric.

Also, the second term is traceless in the following:

$$B = \frac{\operatorname{tr}(B)}{n} I + \left(B - \frac{\operatorname{tr}(B)}{n} I \right)$$

Applying this to the first term of the previous equation, we CAN decompose a matrix into: symmetric traceless + multiple of identity + antisymmetric.

Tut1 Q4 The exponential of a matrix is defined as:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

Tut2 Q1 To check that a vector lies on a plane, the dot product of the vector, and the normal vector, should be 0.

Tut2 Q5 For a matrix A, $\det e^A = e^{\operatorname{tr}(A)}$.

Determinants

The determinant of a matrix tells us how the area (2D) / volume (3D) of the basic box changes with the transformation associated with the matrix.

2D Let T be a 2D LT.

$$|\det T| = \left| T\hat{i} \times T\hat{j} \right| = \left| T\hat{i} \right| \left| T\hat{j} \right| \sin\theta$$

We use this to obtain the formula for the determinant of a 2×2 matrix.

3D Let T be a 3D LT.

$$\begin{aligned} |\det T| &= \left| (T\hat{i} \times T\hat{j}) \cdot T\hat{k} \right| \\ &= \left| T\hat{i} \times T\hat{j} \right| \left| T\hat{k} \right| \cos\theta \end{aligned}$$

This is also known as the triple product.

- Same under cyclic perm of vars
- Same under swapping $\times$ and $\cdot$
- Negates under swapping a pair of vars

LA1 Defn 2.5.2 The determinant of a $n \times n$ square matrix $\boldsymbol{A}$ is defined as:

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots & \\ +a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{ij} = (-1)^{i+j} \det(\boldsymbol{M_{ij}})$$

where $\boldsymbol{M_{ij}}$ is a $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the ith row and jth column. The scalar value A_{ij} is called the (i,j)-cofactor of $\boldsymbol{A}$.

LA1 Defn 2.5.24 The adjoint of a square matrix $\boldsymbol{A}$ is defined as:

$$\operatorname{adj}(\boldsymbol{A}) = (A_{ij})^{\mathrm{T}}_{n \times n}$$

where A_{ij} is the (i,j)-cofactor of $\boldsymbol{A}$.

Properties Let A, B be square matrices of order n, and c a scalar. Then

1. $\det cA = c^n \det A$
2. $\det AB = \det BA = \det A \times \det B$
3. $\det A = \det A^{\mathrm{T}}$
4. If A invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

Inverse property Let A, B, C be square matrices of the same order. Then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Eigenvectors

Let T be a LT, let u be a vector. If $Tu = \lambda u$ for some scalar λ, then u is an eigenvector of T corresponding to eigenvalue λ.

Find eigenvalues Solve $\det(\lambda I - A) = 0$.

Find eigenvectors Solve $\det(\lambda I - A)x = 0$, substituting the specific eigenvalue λ.

Product of eigenvalues The product of eigenvalues is the determinant. It tells us how much the volume of the basic box changes.

Sum of eigenvalues The trace of a matrix, denoted $\operatorname{tr}(A)$, is defined to be the sum of diagonal entries. Note that $\operatorname{tr}(A) = \sum \lambda$.

Properties of trace The trace of only makes sense for square matrices. Let A, B, C, P be order n matrices.

1. Same under cyclic permutations
$$\operatorname{tr}(ABC) = \operatorname{tr}(BCA) = \operatorname{tr}(CAB)$$
2. Same under matrix change of basis
$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(APP^{-1}) = \operatorname{tr}(AI) = \operatorname{tr}(A)$$
3. $\operatorname{tr}(A) = \operatorname{tr}(A^{\mathrm{T}})$
4. Is a LT, $T : \mathcal{M}_{n \times n} \to \mathbb{R}$
5. Is surjective (consider $\frac{k}{n}\operatorname{tr}(I)$)
6. Is NOT injective (consider change of basis with different P)

Diagonal form of LT

Column vector relative to new basis Consider $\mathbb{R}^2$. Let $\{\hat{i}, \hat{j}\}$ be the standard basis. Let $\{u, v\}$ be another basis. Define

$$u = p_1^1\hat{i} + p_1^2 = \begin{pmatrix} p_1^1 \\ p_1^2 \end{pmatrix}$$
$$v = p_2^1\hat{i} + p_2^2 = \begin{pmatrix} p_2^1 \\ p_2^2 \end{pmatrix}$$

Note that p_b^a means the scalar that belongs to row a and column b. Then $P = \begin{pmatrix} p_1^1 & p_2^1 \\ p_1^2 & p_2^2 \end{pmatrix}$ takes $(\hat{i}, \hat{j})$ to (u, v). Since both are bases, then $\det P \neq 0$.

We want to express a vector x using the new basis. We want to find α, β such that

$$x = \begin{pmatrix} a \\ b \end{pmatrix}_{(\hat{i},\hat{j})} = a\hat{i} + b\hat{j} = \alpha u + \beta v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(u,v)}$$

It is a fact that

$$\begin{pmatrix} a \\ b \end{pmatrix}_{(\hat{i},\hat{j})} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(u,v)}$$

and since P is invertible,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(u,v)} = P^{-1} \begin{pmatrix} a \\ b \end{pmatrix}_{(\hat{\imath},\hat{\jmath})}$$

so we have a way of expressing the vector relative to the new basis.

Matrix relative to new basis Let T be a 2D LT, and let x, y be 2D vectors. Declare

$$T_{(\hat{\imath},\hat{\jmath})} x_{(\hat{\imath},\hat{\jmath})} = y_{(\hat{\imath},\hat{\jmath})}$$

and by algebra:

$$P^{-1} T_{(\hat{\imath},\hat{\jmath})} P P^{-1} x_{(\hat{\imath},\hat{\jmath})} = P^{-1} y_{(\hat{\imath},\hat{\jmath})}$$
$$\left(P^{-1} T_{(\hat{\imath},\hat{\jmath})} P\right) P^{-1} x_{(\hat{\imath},\hat{\jmath})} = P^{-1} y_{(\hat{\imath},\hat{\jmath})}$$
$$\left(P^{-1} T_{(\hat{\imath},\hat{\jmath})} P\right) x_{(u,v)} = y_{(u,v)}$$

so the matrix relative to the new basis is

$$\left(P^{-1} T_{(\hat{\imath},\hat{\jmath})} P\right)$$

Row vector relative to new basis (Tut2 Q2) Let $\boldsymbol{c}$ be a column vector, and $\boldsymbol{r}$ be a row vector. Under a change of basis,

$$\boldsymbol{c} \to P^{-1}\boldsymbol{c}$$

Since $\boldsymbol{rc}$ is a number, it has to stay unchanged under a change of basis:

$$\boldsymbol{rc} \to (\boldsymbol{r}P)(P^{-1}\boldsymbol{c}) = \boldsymbol{rc}$$

So we hypothesize (and it works), that

$$\boldsymbol{r} \to \boldsymbol{r}P$$

In a similar fashion, notice $\boldsymbol{r}M\boldsymbol{c}$ is a number.

$$\boldsymbol{r}M\boldsymbol{c} \to (\boldsymbol{r}P)(P^{-1}MP)(P^{-1}\boldsymbol{c}) = \boldsymbol{r}M\boldsymbol{c}$$

This is another way to make sense of the change of basis formula.

Diagonalization The matrix of a transformation relative to its own eigenvectors (assuming they form a basis) is diagonal, i.e.

$$P^{-1}TP = D$$
$$T = PDP^{-1}$$

and we use this to calculate powers of matrices.

Vector Spaces

Addition Addition is a mapping $f : V \times V \to V$.

Scalar multiplication Scalar multiplication is a mapping $\mathcal{F} \times V \to V$.

Axioms A vector space is a set V with an addition and scalar multiplication such that

- Addition is commutative:
$$u + v = v + u \quad \forall u, v \in V$$
- Addition is associative:
$$(u + v) + w = u + (v + w) \quad \forall u, v, w \in V$$
- There is an additive identity:
$$\exists 0 \in V \quad v + 0 = v \quad \forall v \in V$$
- Every $v \in V$ has an additive inverse:
$$\forall v \in V,\ \exists w \in V \quad v + w = 0$$
- There is a multiplicative identity:
$$\exists 1 \in \mathcal{F},\ \forall v \in V \quad 1v = v$$
- Multiplication is distributive both ways:
$$\forall a, b \in \mathcal{F},\ \forall u, v \in V \quad a(u + v) = au + av$$
$$\forall a, b \in \mathcal{F},\ \forall u, v \in V \quad (a + b)u = au + bu$$

Subspaces

Definition A subset U of a vector space V is a subspace if U is a vector space, with the same scalar multiplication and addition as in V.

Verification Three things to verify:

- Existence of additive identity (zero)
- Closed under addition
- Closed under scalar multiplication

The rest of the vector space axioms will follow.

Sum of subspaces Let U_1, U_2 be subspaces wrt V. Then

$$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1,\ u_2 \in U_2\}$$

Direct sum If U_1 and U_2 above were disjoint, then $U_1 + U_2$ is the direct sum of U_1 and U_2, and is denoted by $U_1 \oplus U_2$.

Isomorphisms

Definitions Let F be a mapping $F : S \to T$.

- Surjection
$$\forall t \in T,\ \exists s \in S \quad F(s) = t$$
- Injection
$$\forall s_1, s_2 \in S \quad F(s_1) = F(s_2) \Rightarrow s_1 = s_2$$
- Bijection: Surjection and Injection

Homomorphism Let $\phi : U \to V$ be a mapping. It is a homomorphism if

$$\phi(u + v) = \phi(u) + \phi(v)$$
$$\phi(au) = a\phi(u)$$

If this homomorphism is also a bijection, then this is an isomorphism.

Infinite isomorphisms Let V be a vector space. The mapping $v \to cv\ \forall v \in V$ is an isomorphism, and there are infinitely many different c.

Finite dimensional A vector space is finite dimensional over $\mathcal{F}$ if it is isomorphic to $\mathcal{F}^n$ for some finite integer n.

Span, LI, Basis

Linear combination A linear combination of vectors v_i is

$$a^1 v_1 + a^2 v_2 + \cdots + a^n v_n = \sum a^i v_i$$

Note the use of superscripts instead of subscripts, for the scalars.

Span The span of a list of vectors is the set of all linear combinations of the vectors.

Linearly independent A list of vectors v_i is linearly independent if

$$\sum a^i v_i = 0 \quad \Rightarrow \quad \forall i\ (a^i = 0)$$

Basis A basis is a span that is linearly independent.

- Every finite-dimensional vector space has a basis.
- Every vector can be expressed uniquely as a linear combination of the vectors in the basis.

Decompose into direct sum Let U be a subspace of a finite-dimensional vector space W.

Then there exists V, a subspace of W, such that $W = U \oplus V$, and $\dim(W) = \dim(U \oplus V) = \dim(U) + \dim(V)$.

Basis as a mapping

A basis can be thought of as a mapping $\phi : \mathcal{F}^n \to V$, i.e. it turns a list of numbers (components) into a vector associated with the basis. This mapping is a vector space isomorphism.

Using the definition of ϕ above, then $z_i = \phi(e_i)$, where e_i are the canonical basis vectors, forms a basis for V. Thus, a basis is just a specific example of the infinitely many vector space isomorphism between $\mathcal{F}^n$ and V.

LaTeX stuff

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	`\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}`
$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$	vmatrix
$\mathcal{F}$ ζ $\hat{i}$ A^T	`\mathcal{F} \zeta \hat{i} A^T`