1-var Calculus

Integration by parts

$$\int u\ dv = uv - \int v\ du$$

Quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Fundamental theorem

Suppose f is continuous on $[a,b]$.

Part 1 $F(x) = \int_a^x f(t)\ dt$ is continuous on $[a,b]$ and differentiable on (a,b), and its derivative is $f(x)$.

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\ dt = f(x)$$

Part 2 Let F be an antiderivatve of f on $[a,b]$. Then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Mean value theorem

Suppose f is continuous on $[a,b]$, and f is differentiable on (a,b). Then there exists a c such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Limit definition of derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

and letting $z = x+h$,

$$f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

2-var Calculus

Local extrema

Critical point is an interior point of the domain of a function $f(x,y)$ where $f_x = f_y = 0$ or where one or both of f_x and f_y do not exist.

Solve for local extrema Solve $f_x = 0$ and $f_y = 0$ simultaneously.

Second derivative test Suppose f and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then,

1. Local max at (a,b) if $f_{xx} < 0$ and discriminant > 0.
2. Local min at (a,b) if $f_{xx} > 0$ and discriminant > 0.
3. Saddle point at (a,b) if discriminant < 0.
4. Inconclusive at (a,b) if discriminant $= 0$.

Discriminant is defined as $f_{xx}f_{yy} - f_{xy}^2$.

Fubini's theorem Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y)\ dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\ dxdy$$

Series

Useful inequalities

$$n! < n^n \quad \sqrt{n} < n \quad \ln n < n^c$$

Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$$

Convergence of some series

1. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$ diverges using comparison test with harmonic series.
2. $\displaystyle\sum_{n=0}^{\infty}\frac{\ln n}{n^k}$, where $k > 2$, converges using comparison test with p-series. Start with $\ln n < n$.

Absolute and conditional convergence

Absolutely convergent $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty}|a_n|$ converges.

Conditionally convergent $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Convergence tests

Geometric series $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.

p-series $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.

n-th term test If $\lim\limits_{n\to\infty} a_n$ does not exist or $\neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Alternating Series test If $a_{n+1} \le a_n$ and $\lim\limits_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Absolute convergence If a series is absolutely convergent, then it is convergent.

Convergence tests for series with positive terms

a_n and b_n are all positive.

Integral test Suppose f is continuous, positive, decreasing on $[1, \infty)$ such that $a_n = f(n)$ for all n. Then

$$\sum_{n=1}^{\infty} a_n \text{ convergent} \Leftrightarrow \int_1^{\infty} f(x)\, dx \text{ convergent}$$

and the biconditional holds for divergence as well.

Comparison test Suppose $a_n \leq b_n$ for all n.

(i) $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent

(ii) $\sum a_n$ divergent $\Rightarrow \sum b_n$ divergent

Limit Comparison test If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$, then both series converge, or both series diverge.

Ratio test Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

(i) $0 \leq L < 1 \Rightarrow \sum a_n$ is absolutely convergent (and convergent).

(ii) $L > 1 \Rightarrow \sum a_n$ is divergent.

Root test Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \mathrm{L}$.

(i) $0 \leq L < 1 \Rightarrow \sum a_n$ is absolutely convergent (and convergent).

(ii) $L > 1 \Rightarrow \sum a_n$ is divergent.

Power series

Radius of convergence Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, exactly one of the following possibilities holds:

(a) The series converges at $x = a$ only ($R = 0$)

(b) The series converges for all x ($R = \infty$)

(c) There is a positive R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

R is the radius of convergence. We can compute R by the following methods:

(i) If $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$, then $R = 1/L$.

(ii) If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$, then $R = 1/L$.

Interval of convergence The interval of convergence is the interval that consists of all values of x for which the series converges. Usually, the interval of convergence is

$$(a-R, a+R)$$

but the series might converge at endpoints, which need to be tested separately.

Differentiation/Integration Let

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

such that the power series has radius of convergence $R > 0$. Then f is differentiable on the interval $|x-a| < R$ and

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

Taylor series The Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Maclaurin series The Maclaurin series generated by f is the Taylor series at $x = 0$.

Standard Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad -1 < x \leq 1$$

$$\begin{aligned} \ln \frac{1+x}{1-x} &= 2 \tanh^{-1} x \\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad |x| < 1 \end{aligned}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad |x| \leq 1$$

Limits relating to e^x

$$\begin{aligned} L &= \lim_{n\to\infty} \left(\frac{n+k}{n}\right)^n \\ &= \lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n \\ \ln L &= \lim_{n\to\infty} \left(n \ln\left(1 + \frac{k}{n}\right)\right) \\ &= \lim_{n\to\infty} \left(\ln\left(1 + \frac{k}{n}\right) \div \frac{1}{n}\right) \qquad \frac{0}{0} \\ &= \lim_{n\to\infty} \left(\frac{-\frac{k}{n^2}}{1 + \frac{k}{n}} \div \frac{-1}{n^2}\right) \\ &= \lim_{n\to\infty} \left(\frac{k}{1 + \frac{k}{n}}\right) \\ &= k \\ L &= e^k \end{aligned}$$