

# MA2101

## Matrices

**Describing change** Let  $T$  be a 2D LT.

$$\mathcal{M}(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This means that  $T(\hat{i}) = \begin{pmatrix} a \\ c \end{pmatrix}$  and  $T(\hat{j}) = \begin{pmatrix} b \\ d \end{pmatrix}$ , i.e. the columns tell us how the basic unit vectors change under the transformation.

**Rotation matrix (2D)** An anticlockwise rotation by  $\theta$  is given by

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

**Shear matrix (2D)** A shear parallel to the  $x$ -axis by  $\theta$  is given by

$$\begin{pmatrix} 1 & \tan \theta \\ 0 & 1 \end{pmatrix}$$

**Tut1 Q1** A matrix can be decomposed as a sum:

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

The first term is symmetric, and the second term is antisymmetric.

Also, the second term is traceless in the following:

$$B = \frac{\text{tr}(B)}{n}I + \left(B - \frac{\text{tr}(B)}{n}I\right)$$

Applying this to the first term of the previous equation, we CAN decompose a matrix into: symmetric traceless + multiple of identity + antisymmetric.

**Tut1 Q4** The exponential of a matrix is defined as:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

**Tut2 Q1** To check that a vector lies on a plane, the dot product of the vector, and the normal vector, should be 0.

**Tut2 Q5** For a matrix  $A$ ,  $\det e^A = e^{\text{tr}(A)}$ .

## Determinants

The determinant of a matrix tells us how the area (2D) / volume (3D) of the basic box changes with the transformation associated with the matrix.

**2D** Let  $T$  be a 2D LT.

$$|\det T| = \left|T\hat{i} \times T\hat{j}\right| = \left|T\hat{i}\right| \left|T\hat{j}\right| \sin \theta$$

We use this to obtain the formula for the determinant of a  $2 \times 2$  matrix.

**3D** Let  $T$  be a 3D LT.

$$\begin{aligned} |\det T| &= \left|(T\hat{i} \times T\hat{j}) \cdot T\hat{k}\right| \\ &= \left|T\hat{i} \times T\hat{j}\right| \left|T\hat{k}\right| \cos \theta \end{aligned}$$

This is also known as the triple product.

- Same under cyclic perm of vars
- Same under swapping  $\times$  and  $\cdot$
- Negates under swapping a pair of vars

**LA1 Defn 2.5.2** The determinant of a  $n \times n$  square matrix  $A$  is defined as:

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots & \text{if } n > 1 \end{cases}$$

where

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

where  $M_{ij}$  is a  $(n-1) \times (n-1)$  matrix obtained from  $A$  by deleting the  $i$ th row and  $j$ th column. The scalar value  $A_{ij}$  is called the  $(i, j)$ -cofactor of  $A$ .

**LA1 Defn 2.5.24** The adjoint of a square matrix  $A$  is defined as:

$$\text{adj}(A) = (A_{ij})_{n \times n}^T$$

where  $A_{ij}$  is the  $(i, j)$ -cofactor of  $A$ .

**Properties** Let  $A, B$  be square matrices of order  $n$ , and  $c$  a scalar. Then

1.  $\det cA = c^n \det A$
2.  $\det AB = \det BA = \det A \times \det B$
3.  $\det A = \det A^T$
4. If  $A$  invertible, then  $\det A^{-1} = \frac{1}{\det A}$ .

**Inverse property** Let  $A, B, C$  be square matrices of the same order. Then  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ .

## Eigenvectors

Let  $T$  be a LT, let  $u$  be a vector. If  $Tu = \lambda u$  for some scalar  $\lambda$ , then  $u$  is an eigenvector of  $T$  corresponding to eigenvalue  $\lambda$ .

**Find eigenvalues** Solve  $\det(\lambda I - A) = 0$ .

**Find eigenvectors** Solve  $\det(\lambda I - A)x = 0$ , substituting the specific eigenvalue  $\lambda$ .

**Product of eigenvalues** The product of eigenvalues is the determinant. It tells us how much the volume of the basic box changes.

**Sum of eigenvalues** The trace of a matrix, denoted  $\text{tr}(A)$ , is defined to be the sum of diagonal entries. Note that  $\text{tr}(A) = \sum \lambda$ .

**Properties of trace** The trace of only makes sense for square matrices. Let  $A, B, C, P$  be order  $n$  matrices.

1. Same under cyclic permutations

$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$$

2. Same under matrix change of basis

$$\text{tr}(P^{-1}AP) = \text{tr}(APP^{-1}) = \text{tr}(AI) = \text{tr}(A)$$

3.  $\text{tr}(A) = \text{tr}(A^T)$
4. Is a LT,  $T: \mathcal{M}_{n \times n} \rightarrow \mathbb{R}$
5. Is surjective (consider  $\frac{k}{n}\text{tr}(I)$ )
6. Is NOT injective (consider change of basis with different  $P$ )

## Diagonal form of LT

**Column vector relative to new basis** Consider  $\mathbb{R}^2$ . Let  $\{\hat{i}, \hat{j}\}$  be the standard basis. Let  $\{u, v\}$  be another basis. Define

$$\begin{aligned} u &= p_1^1 \hat{i} + p_1^2 \hat{j} = \begin{pmatrix} p_1^1 \\ p_1^2 \end{pmatrix} \\ v &= p_2^1 \hat{i} + p_2^2 \hat{j} = \begin{pmatrix} p_2^1 \\ p_2^2 \end{pmatrix} \end{aligned}$$

Note that  $p_b^a$  means the scalar that belongs to row  $a$  and column  $b$ . Then  $P = \begin{pmatrix} p_1^1 & p_2^1 \\ p_1^2 & p_2^2 \end{pmatrix}$  takes  $(\hat{i}, \hat{j})$  to  $(u, v)$ . Since both are bases, then  $\det P \neq 0$ .

We want to express a vector  $x$  using the new basis. We want to find  $\alpha, \beta$  such that

$$x = \begin{pmatrix} a \\ b \end{pmatrix}_{(\hat{i}, \hat{j})} = a\hat{i} + b\hat{j} = \alpha u + \beta v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(u, v)}$$

It is a fact that

$$\begin{pmatrix} a \\ b \end{pmatrix}_{(\hat{i}, \hat{j})} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(u, v)}$$

and since  $P$  is invertible,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(u,v)} = P^{-1} \begin{pmatrix} a \\ b \end{pmatrix}_{(\hat{i},\hat{j})}$$

so we have a way of expressing the vector relative to the new basis.

**Matrix relative to new basis** Let  $T$  be a 2D LT, and let  $x, y$  be 2D vectors. Declare

$$T_{(\hat{i},\hat{j})} x_{(\hat{i},\hat{j})} = y_{(\hat{i},\hat{j})}$$

and by algebra:

$$\begin{aligned} P^{-1} T_{(\hat{i},\hat{j})} P P^{-1} x_{(\hat{i},\hat{j})} &= P^{-1} y_{(\hat{i},\hat{j})} \\ (P^{-1} T_{(\hat{i},\hat{j})} P) P^{-1} x_{(\hat{i},\hat{j})} &= P^{-1} y_{(\hat{i},\hat{j})} \\ (P^{-1} T_{(\hat{i},\hat{j})} P) x_{(u,v)} &= y_{(u,v)} \end{aligned}$$

so the matrix relative to the new basis is

$$(P^{-1} T_{(\hat{i},\hat{j})} P)$$

**Row vector relative to new basis (Tut2 Q2)** Let  $\mathbf{c}$  be a column vector, and  $\mathbf{r}$  be a row vector. Under a change of basis,

$$\mathbf{c} \rightarrow P^{-1} \mathbf{c}$$

Since  $\mathbf{r}\mathbf{c}$  is a number, it has to stay unchanged under a change of basis:

$$\mathbf{r}\mathbf{c} \rightarrow (\mathbf{r}P)(P^{-1}\mathbf{c}) = \mathbf{r}\mathbf{c}$$

So we hypothesize (and it works), that

$$\mathbf{r} \rightarrow \mathbf{r}P$$

In a similar fashion, notice  $\mathbf{r}M\mathbf{c}$  is a number.

$$\mathbf{r}M\mathbf{c} \rightarrow (\mathbf{r}P)(P^{-1}MP)(P^{-1}\mathbf{c}) = \mathbf{r}M\mathbf{c}$$

This is another way to make sense of the change of basis formula.

**Diagonalization** The matrix of a transformation relative to its own eigenvectors (assuming they form a basis) is diagonal, i.e.

$$\begin{aligned} P^{-1}TP &= D \\ T &= PDP^{-1} \end{aligned}$$

and we use this to calculate powers of matrices.

## Vector Spaces

**Addition** Addition is a mapping  $f : V \times V \rightarrow V$ .

**Scalar multiplication** Scalar multiplication is a mapping  $\mathcal{F} \times V \rightarrow V$ .

**Axioms** A vector space is a set  $V$  with an addition and scalar multiplication such that

- Addition is commutative:

$$u + v = v + u \quad \forall u, v \in V$$

- Addition is associative:

$$(u + v) + w = u + (v + w) \quad \forall u, v, w \in V$$

- There is an additive identity:

$$\exists 0 \in V \quad v + 0 = v \quad \forall v \in V$$

- Every  $v \in V$  has an additive inverse:

$$\forall v \in V, \exists w \in V \quad v + w = 0$$

- There is a multiplicative identity:

$$\exists 1 \in \mathcal{F}, \forall v \in V \quad 1v = v$$

- Multiplication is distributive both ways:

$$\forall a, b \in \mathcal{F}, \forall u, v \in V \quad a(u + v) = au + av$$

$$\forall a, b \in \mathcal{F}, \forall u, v \in V \quad (a + b)u = au + bu$$

## Subspaces

**Definition** A subset  $U$  of a vector space  $V$  is a subspace if  $U$  is a vector space, with the same scalar multiplication and addition as in  $V$ .

**Verification** Three things to verify:

- Existence of additive identity (zero)
- Closed under addition
- Closed under scalar multiplication

The rest of the vector space axioms will follow.

**Sum of subspaces** Let  $U_1, U_2$  be subspaces wrt  $V$ . Then

$$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$$

**Direct sum** If  $U_1$  and  $U_2$  above were disjoint, then  $U_1 + U_2$  is the direct sum of  $U_1$  and  $U_2$ , and is denoted by  $U_1 \oplus U_2$ .

## Isomorphisms

**Definitions** Let  $F$  be a mapping  $F : S \rightarrow T$ .

- Surjection

$$\forall t \in T, \exists s \in S \quad F(s) = t$$

- Injection

$$\forall s_1, s_2 \in S \quad F(s_1) = F(s_2) \Rightarrow s_1 = s_2$$

- Bijection: Surjection and Injection

**Homomorphism** Let  $\phi : U \rightarrow V$  be a mapping. It is a homomorphism if

$$\begin{aligned} \phi(u + v) &= \phi(u) + \phi(v) \\ \phi(au) &= a\phi(u) \end{aligned}$$

If this homomorphism is also a bijection, then this is an isomorphism.

**Infinite isomorphisms** Let  $V$  be a vector space. The mapping  $v \rightarrow cv \quad \forall v \in V$  is an isomorphism, and there are infinitely many different  $c$ .

**Finite dimensional** A vector space is finite dimensional over  $\mathcal{F}$  if it is isomorphic to  $\mathcal{F}^n$  for some finite integer  $n$ .

## Span, LI, Basis

**Linear combination** A linear combination of vectors  $v_i$  is

$$a^1 v_1 + a^2 v_2 + \dots + a^n v_n = \sum a^i v_i$$

Note the use of superscripts instead of subscripts, for the scalars.

**Span** The span of a list of vectors is the set of all linear combinations of the vectors.

**Linearly independent** A list of vectors  $v_i$  is linearly independent if

$$\sum a^i v_i = 0 \quad \Rightarrow \quad \forall i \quad (a^i = 0)$$

**Basis** A basis is a span that is linearly independent.

- Every finite-dimensional vector space has a basis.
- Every vector can be expressed uniquely as a linear combination of the vectors in the basis.

**Decompose into direct sum** Let  $U$  be a subspace of a finite-dimensional vector space  $W$ .

Then there exists  $V$ , a subspace of  $W$ , such that  $W = U \oplus V$ , and  $\dim(W) = \dim(U \oplus V) = \dim(U) + \dim(V)$ .

## Basis as a mapping

A basis can be thought of as a mapping  $\phi : \mathcal{F}^n \rightarrow V$ , i.e. it turns a list of numbers (components) into a vector associated with the basis. This mapping is a vector space isomorphism.

Using the definition of  $\phi$  above, then  $z_i = \phi(e_i)$ , where  $e_i$  are the canonical basis vectors, forms a basis for  $V$ . Thus, a basis is just a specific example of the infinitely many vector space isomorphism between  $\mathcal{F}^n$  and  $V$ .

## LaTeX stuff

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	<code>\begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</code>
$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$	<code>\begin{vmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{vmatrix}</code>
$\mathcal{F} \quad \zeta \quad \hat{i} \quad A^T$	<code>\mathcal{F} \quad \zeta \quad \hat{i} \quad A^T</code>