Computation of Cournot-Nash equilibria by entropic regularization

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Final Project Optimal Transport

January 2020

Plan of the presentation

Presentation of the problem

Cournot Nash Equilibria

- The objective of the article is to study a class of games with continuum of players where equilibria can be obtained by the minimization of a certain functional related to optimal transport.
- The problem are approximate by entropic regularization technique.
- Our work will be to adapt this problem to a concrete problem from finance : the delta hedging of a portfolio in a competitive model with market impact.

Theoretical analysis

Definition of the cost function

The games we are studying may be rewritte as the optimization of this cost function :

$$\Psi_{i,j}(\mu) = c_{i,j} + f_j(\mu_j) + \sum_{k \in J} \phi_{k,j}\mu_k \tag{1}$$

This formulation can be interpreted as follow :

1. Classic Transport Cost $c_{i,j}$: what is the penalisation for the actor x_i to choose y_j
2. Congestion cost $f_j(\nu_j)$: depending on how many people choose j
3. Iteration cost : $\sum_{k \in J} \phi_{kj}\mu_k$ impact on others decision on our choice.

Theoretical analysis

Entropic penalization

The objective of this article is to solve this problem by adding an entropic penalization term and use the Dykstras' algorithm.

$$MK_\epsilon(\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \{ \sum_{i,j \in I \cdot J} c_{ij} \gamma ij + \epsilon \sum_{i,j \in I \cdot J} \gamma_{ij} (ln(\gamma_{ij}) - 1) \} \quad (2)$$

In order to not compute $prox_G^{KL}()$ directly, we will used the extended algorithm : Dykstra proximal splitting algorithm. It consists by splitting G as a sum of more elementary functionals and calculate the prox on each elementary function.

Theoretical analysis

Presentation of the algorithm

Data: nbr Itter,G,
Result: $\gamma^{(nbrItter)}$
initialization :
$\gamma^{(0)} = \bar{\gamma}, z^{(0)} = z^{(1)} = ... = e, e_{ij} = 1, (i,j) \in I \cdot J$
while $n \leq nbrItter$ **do**
$\gamma^{(n)} = prox^{KL}_{G_n}(\gamma^{(n-1)} \odot z^{(n-L)})$
$z^{(n)} = z^{(n-1)} \odot (\gamma^{(n-1)} \oslash \gamma^{(n)})$
end

Algorithm 1: Dykstras' algorithm for KL projections on the intersection of convex sets

Related financial problem

Market Impact

- Dynamic hedging problem is one of the most important concepts of quantitative finance.
- These actors sometimes hold the same positions, and they can decide whether they want to hedge or not.
- Therefore, hedging these large positions may have an impact on the markets.
- The most common strategy to be hedge against the risk of variation of the price of an action if the "delta hedge" its portfolio by having a certains quantity δ of action $\forall t \in [0, T]$

Related financial problem

Market Impact

In order to modelize the interaction of the players, we will introduce two notions of market impact.

- The permanent impact on the market due to the action of a specific trader on the market.It can be interpreted as the modification of the strategy of the other actors because of our action on the market.
- Temporary Market Impact : In order to modelize short variation of the price due to a strong action on the market which will revert to the previous value, we can add a temporary market impact.

Related financial problem

Fundamental equation for delta hedging with market impact

Classical Black & Scholes market :

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_t &= S_0 e^{(\mu + \frac{\sigma^2}{2})t + \sigma\sqrt{(t)}Z} \ where\ Z \sim Normal(0,1) \end{aligned} \tag{3}$$

Black & Scholes market with market impact $\widetilde{S}_t = \hat{S}_t \cdot (1 + \alpha \nu_t)$

The associated cost function is :

$$\Psi_{i,j} = pX_iY_j + \delta_{t_j}\hat{S}_T - C_T + (C_0 e^{rt_j} - \delta_{t_j}\widetilde{S}_{t_j})e^{(T-t_j)r} \tag{4}$$

Numerical findings

Definition of G

Let's G defined by :

$$G(\nu) = \chi\{\Lambda_1(\gamma) = \mu\} + E_1(\nu) + E_2(\nu) \tag{5}$$

$$E_1(\nu) = \sum_{j \in J} F_j(\nu_j) \ where \quad F_j(\nu_{t_j}) = -\alpha\delta_{t_j} S_{t_j} \frac{\nu_{t_j}^2}{2} e^{(T-t_j)r} := \nu_j^2 \zeta_j \tag{6}$$

$$E_2(\nu) = \frac{1}{2} \sum_{k,j \in J \cdot J} \phi_{kj} \nu_k \nu_j \ where \ \phi_{k,j} = S_{t_j} \cdot (t_j - t_k) \tag{7}$$

Numerical findings

Computation of the proximal operators

We will split G into 2 functions : G$_1$ *and* G_2

$$G_1 = \chi\{\Lambda_1(\gamma) = \mu\} \; and \; (prox^{KL}_{G_1}(\theta))_{ij} \quad = \frac{\mu_i \theta_{ij}}{\sum_{k \in J} \theta_{ik}} \tag{8}$$

$$G_2 = \frac{1}{\epsilon} E \circ \Lambda_2 \tag{9}$$

$$prox^{K}_{G_2} L(\theta)_{ij} = \theta_{ij} exp(-\frac{\nu_j \zeta_j + \sum_{k \in J} \phi_{kj} \nu_k}{\epsilon}) \tag{10}$$

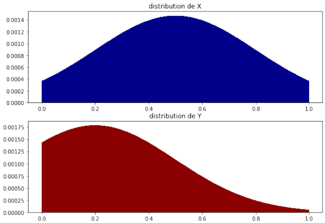
Where ν is the solution of the system :

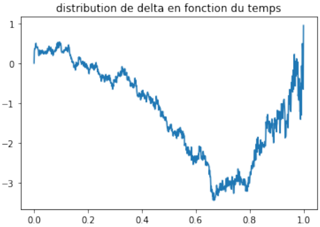
$$\nu_j = (\sum_{i \in I} \theta_{ij}) exp(-\frac{\zeta_j \nu_j + \sum_{k \in K} \psi_{kj} \nu_k}{\epsilon}) \tag{11}$$

Numerical findings

Presentation of δ and the distribution of X and Y

The figure 12 presents the distribution of X and Y taken to simulate our data. We choose normalized Gaussian variables. The figure 12 presents the delta quantity traders need to hold at every time to have a perfectly hedge portfolio.

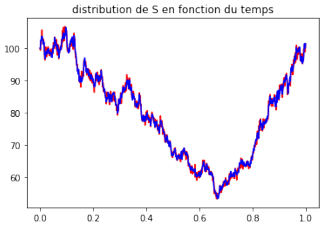


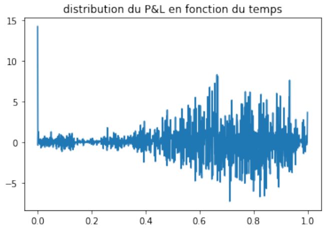


Numerical findings

Presentation of S_t and the PL Computation

The figure 13 presents the variation of the price of the stock. We plot in blue the variation without market impact and in red with a small market impact.
The figure 13 presents the shape of the variation of the profit and loss of a portfolio hedge with a delta hedging strategy.

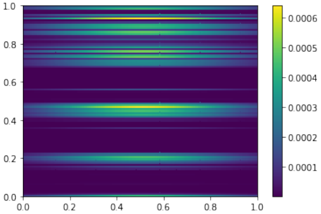




Numerical findings

Resulting matrix γ for the simplified problem

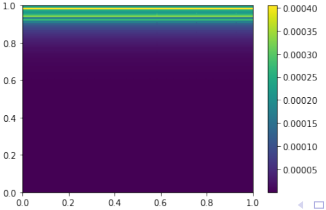
- In order to solve the non linear problem, I use the Newton Krylov algorithm which does not converge all the time and is low to compute. I define a maximum number of itteration in order to accelerate the convergence.
- The figure 14 presents γ in the case we consider G $= G_1$.



Numerical findings

Resulting matrix γ for the global problem

The figure 15 presents the results in case case of $G = G_1 + G_2$. The best strategy seems to hedge the position at the last minute. Indeed as the second derivative the price of the financial product by the stock price (called Gamma $:= \frac{\partial C^2}{\partial S^2}$) is always negative, buying stocks will make the price rise. Thus, the Cournot Nash equilibria is when every one hedgeits position just before the end.



Critics

- In practice, knowing μ and σ is a real challenge as the Black & Scholes model is too simple to model the dynamic of a stock.
- One possible further step of this research project is to study others derivative products such as the collar which has its gamma ($\frac{\partial C^2}{\partial S^2}$) not always negative.

Conclusion

- Through this study, we achieve to find the Cournot Nash Equilibria of a game based on the delta hedging of a portfolio in a competitive model with market impact.
- We used the Dykstras' algorithm for KL projections on the intersection of convex sets in order to easily compute the gradient proximal operator.
- This results might be usefull in order to study the behaviour of financial competitor and develop new strategies based on these knowledges.