OPTIMAL TRANSPORT

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ABSTRACT

The objective of the article [1] is to study a class of games with continuum of players where equilibria can be obtained by the minimization of a certain functional related to optimal transport. The problem are approximate by entropic regularization technique . Our work will be to adapt this problem to a concrete problem from finance : the delta hedging of a portfolio in a competitive model with market impact.

***K*eywords** Game Theory · Optimal Transport · Portfolio Hedging

1 Introduction

Dynamic hedging problems is one of the most important concepts of quantitative finance. The optimal hedge strategy is not always clear. Large positions on the market may be held by a unique actor, or a few traders or even the entire sell-side community.

When an actor need to hedge a large position, it has an impact on the markets.

Moreover, many actors sometimes hold the same positions, and they can decide when they want to hedge or not. The question this article is trying to answer is whether or not a basic model of market impact with several actors may have a Cournot Nash equilibria.

Cournot-Nash equilibria can be obtained by the minimization of a certain functional on the set of measures on the space of strategies. This functional typically involves two terms: an optimal transport cost and a more standard integral functional which may capture both congestion and attractive eects.

The paper [1] presents a powerful way to compute a Cournot-Nash equilibria by approximate optimal transport by adding an entropic penalization term.

Doing so, the problem becomes projecting for the Kullback-Leibler divergence a given joint measure on the set of measures with xed marginals. This powerful method is intimately related to Sinkhorn algorithm [2].

In order to take advantage of the power of entropic regularization on Wasserstein gradient ows, Peyre [3] introduced an extension of Dykstra's algorithm which he called Dykstra proximal splitting. Then, this algorithms was extended by Chizat [4] and now perfectly well-suited to the computation of Cournot-Nash equilibria.

In Section 2, we will recall the Cournot-Nash equilibria problem as presented in the paper [1].

In Section 3, we will present the hedging problem and formulate it as a Cournot-Nash equilibria problem. We support his section on the work of Almgrem [5] and Guéant [6] on portfolio hedging and market impact.

In Section 4, we will present the regularization methods and the Dikstras' algorithm used to solve this problem. We can reffer to the two article : [3] and [4] for the proof of convergence of this algorithm.

In Section 5, we will present our numerical results.

2 Presentation of the problem

In order to understand the Cournot-Nash equilibrium problem, we will first introduced some notations.

- X = $\{x_i\}$: kind of different actors
- $\mu = \{\mu_i\}$: proportion of the actors x_i
- $Y = \{y_j\}$: the different choices of strategy
- $\nu = \{\nu_j\}$: proportion of actors who choose the strategy y_j.
- $\gamma = \{\gamma_{i,j}\}$: the probability matrix that the player x_i choose the strategy y_j
- $\Psi = \{\Psi_{i,j}\}$: cost for i to choose j depending on ν

As γ is a probability matrix and X, Y are two distributions. We have the following constraints :

$$\begin{aligned} \sum_i \mu_i &= 1 \\ \sum_j \nu_j &= 1 \\ \forall i \in I \sum_j \gamma_{i,j} &= \mu_i \\ \forall j \in J \sum_i \gamma_{i,j} &= \nu_j \end{aligned} \tag{1}$$

We can now defined the cost function of the problem :

$$\Psi_{i,j}(\mu) = c_{i,j} + f_j(\mu_j) + \sum_{k \in J} \phi_{k,j}\mu_k \tag{2}$$

This formulation can be interpreted as follow :

1. Classic Transport Cost $c_{i,j}$: what is the penalisation for the actor x_i to choose y_j
2. Congestion cost $f_j(\nu_j)$: depending on how many people choose j
3. Iteration cost : $\sum_{k \in J} \phi_{kj}\mu_k$ impact on others decision on our choice.

As this problem is a competitive problem, we are trying to find a solution where every actor is making the best possible choice regarding others ones.

More precisely, the definition of the Cournot Nash equilibrium a matrix γ which satisfies the feasibility constraints and such that :

$$if \ \gamma_{ij} > 0 \ then \ \Psi_{ij}[\mu] = \min_{k \in J} \Psi_{ik}[\mu] \tag{3}$$

We search a probability matrix μ which is consistent with cost minimizing behaviours of players.

We now introduce some notations in order to simplify the readability of the optimization problem :

$$\begin{aligned} C_\mu &= \{\gamma = (\gamma_{i,j}) \, \Gamma_1(\gamma) = \mu\} \\ C^\nu &= \{\gamma = (\gamma_{i,j}) \, \Gamma_2(\gamma) = \nu\} \end{aligned} \tag{4}$$

$$\begin{aligned} P(Y) &= \{\mu \in \mathbb{R}^J_+ : \sum_{i,j \in I \cdot J} \mu_j = 1\} \\ MK(\mu) &= \inf_{\gamma \in \Pi(\mu,\nu)} \{c \cdot \gamma = \sum_{i,j \in IxJ} c_{ij}\gamma_{ij}\} \end{aligned} \tag{5}$$

$$E(\mu) = \sum_{j \in J} F_j(\nu_j) + \frac{1}{2} \sum_{k,j \in JxJ} \phi_{kj} \nu_k \nu_j$$
$$where\ F_j(t) = \int_0^t f_j(s) ds \tag{6}$$

The Optimization problem can now be written as :

$$\inf_{\nu \in P(Y)} MK(\nu) + E(\nu) \tag{7}$$

3 Related financial problem

3.1 Introduction to portfolio hedging

Dynamic hedging problems is one of the most important concepts of quantitative finance. The optimal hedge strategy is not always clear. Large positions on the market may be held by a unique actor, or a few traders or even the entire sell-side community. These actors sometimes hold the same positions, and they can decide whether they want to hedge or not. Therefore, hedging these large positions may have an impact on the markets. This paper aims to study and develop Trading and Hedging strategies using Game Theory. We will consider each bank involved in a hedging activity as a “player” in a given game that we will describe hereunder. Note that the timing of these hedging strategies is crucial for the “players”. This will help us find the optimum timing.

First, we will introduce the fundamental equations we will need to modelize our system.

3.2 Introduction to the Delta Hedging of a portfolio

3.2.1 Modelisation of the market and presentation of the derivative product

The derivative product we will consider is a call.

A call option is a financial contract between two parties, the buyer and the seller of this type of option.

- The buyer of the call option has the right, but not the obligation, to buy an agreed quantity of stocks (the underlying S) from the seller of the option at a certain time (the expiration date T) for a certain price (the strike price K).
- The seller is obligated to sell the commodity or financial instrument to the buyer if the buyer so decides. The buyer pays a fee (called a premium C_0) for this right.

Thus the price of the call à t = 0 and t = T is :

$$C_0 = E[(S_T - K)^+]$$
$$C_T = (S_T - K)^+ \tag{8}$$

This financial product depends on the price of an underlying assets S_T. in order to avoid loosing money, the seller needs to hedge his position against the risk of a large variation of the price of S_T

3.2.2 Delta Hedging Portfolio Strategy in a Black Scholes model

We will study the optimal behaviour to hedge a portfolio when there is only two possible product :

- a riskless product : B_t
- a risky product S_t which is the underlying of our call option. It's followed the BlackScholes equation.

In the Black Scholes model, the price of a stock has a drift $\mu \in \mathbb{R}$ and a volatility $\sigma \in \mathbb{R}$ which follow a Brownian Motion W_t. The related Stochastic differential equation is :

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$
$$S_t = S_0 e^{(\mu + \frac{\sigma^2}{2})t + \sigma\sqrt{(t)}Z} \tag{9}$$
$$where\ Z \sim Normal(0, 1)$$

The riskless product follows the equation :

$$\begin{aligned} dB_t &= rB_t dt \\ B_t &= B_0 e^{rt} \end{aligned} \tag{10}$$

If we call $C(t, St)$ the value of the call at time t and our portfolio V_t with a self finance strategy. As we want a risk neutral portfolio, we want

$$\forall t \in \mathbb{R} \frac{d(C_t - V_t)}{dW_t} = 0 \tag{11}$$

As we have a self finance strategy, all the money not invested in the risky asset is invested in the riskless asset. We have then, α_t in stocks and $v_t - \alpha_t X_t$ in risk-less product.

Moreover, we have : by applying the Ito formula to $C(t, S_t)$

$$dC(t, St) = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial^2 S} d < S >_t \tag{12}$$

and :

$$dV_t = \alpha_t dX_t + (V_t - \alpha X_t) dB_t \tag{13}$$

Thus: we need to have :

$$\alpha_t = \frac{\partial C}{\partial S} \ \forall t \in [0, T] \tag{14}$$

As we have :

$$\forall t \in [0, T] : C_{t,S_t} = \phi(d_1(t)) S_t - K e^{-rt} \phi(d_2(t)) \tag{15}$$

$$\alpha_t = \frac{\partial C}{\partial S} = \phi(d_1(t)) \tag{16}$$

Where :

$$\begin{aligned} d_1(t) &= \frac{log(\frac{S_t}{K}) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \\ &and \\ \phi(u) &= \frac{1}{2\pi} \int_{x=0}^{u} e^{\frac{-x^2}{2}} \end{aligned} \tag{17}$$

In practice, the traders cannot being hedge $\forall t$ because of the transaction fees, they hedge their position in average one time per week with a stop loss in order to avoid loosing too much money.

3.3 Modelization of the market impact into a Black Scholes model

An important phenomena the Black Scholes model doesn't take into account is the variation of the price due to our own action (sell or buy) on the market. In this section we will explain two kinds of market impact : temporary and permanent, in order to add interaction between actors in our model. One can find more detail about this model in [5] and [6].

3.3.1 Permanent Market Impact

The permanent impact on the market due to the action of a specific trader on the market can be modelize as :

$$d\hat{S}_t = (\mu + l(\nu, t)) \hat{S}_t dt + \sigma \hat{S}_t dWt \tag{18}$$

where :

$$l(\nu, t) = \int_{u=0}^{t} \nu_u^{\beta} du \tag{19}$$

It can be interpreted as the modification of the strategy of the other actors because of our action on the market.

In order to simplify our model, we will assumed that the permanent market is at the first order :

$$l(\nu, t) = \int_{u=0}^{t} \nu_u \cdot \beta du \tag{20}$$

We can compute the integral of $l(\nu, t)$:

$$\begin{aligned} \int_{t=0}^{T} l(\nu, t)dt &= \int_{t=0}^{T}\int_{u=0}^{t} \nu_u \cdot \beta dudt \\ \int_{t=0}^{T} l(\nu, t)dt &= \beta \cdot \int_{t=0}^{T} \nu_t (T-t)dt \end{aligned} \tag{21}$$

Thus the real price of the risky asset is $\hat{S_T}$:

$$\begin{aligned} &\hat{S}_T = S_0 \cdot e^{(\mu T + \int_{t=0}^{T} l(\nu,t)dt) + \sigma\sqrt{T} Z(0,1)} \\ &\hat{S}_T = S_T \cdot e^{\int_{t=0}^{T} l(\nu,t)dt)} \end{aligned} \tag{22}$$

Which is approximately equal to :

$$\begin{aligned} &\hat{S}_T = S_T \cdot (1 + \int_{t=0}^{T} l(\nu, t)dt)) \\ &\hat{S}_T = S_T \cdot (1 + \beta \cdot \int_{t=0}^{T} \nu_t (T-t)dt) \end{aligned} \tag{23}$$

3.3.2 Temporary Market Impact

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In order to modelize short variation of the price due to a strong action on the market which will revert to the previous value, we can add a temporary market impact. Instead of paying S_t in order to buy the stock, we will pay $\widetilde{S}_t$

$$\begin{gathered} \widetilde{S}_t = \hat{S}_t \cdot r(t) \\ where\ r(t) = (1 + \nu_t)^\alpha \end{gathered} \tag{24}$$

In order to simplify our model, we will assumed a first order approximation of $r(t)$:

$$r(t) = (1 + \nu_t \cdot \alpha) \tag{25}$$

3.4 Modelization as an Optimal Transport Problem

In order to modelize this problem with Game theory, we will make the following assumptions :

- Each actor sold a known quantity of Call to the client.
- Each actor is looking to delta-hedge its books.

In order to match the definition of the article, we will define :

- X = $\{x_i\}$ A trader which need to hedge its Portfolio
- $\mu = \{\mu_i\}$ proportion sell by x_i to the client
- $Y = \{Y_j\}$ the time where the trader decide to hedge their Portfolio. $Y \in [0, 1]^J$
- $\nu = \{\nu_j\}$ proportion of actors who choose to hedge at time Y_j
- $\gamma = \{\gamma_{i,j}\}$ probability matrix that the player x_i choose the strategy y_j
- $\Psi = \{\Psi_{i,j}\}$ cost for i to choose j depending on ν

Before defining the cost function, we can enumerate the different steps of the hedging of the portfolio by the actor x_i

- We receive C_0 \$ at t=0 and we used this money to buy risk less product B_t
- just before t_j we have : $C_0 e^{rt_j}$ \$
- we buy δ_{i,t_j} Stocks at time t_j
- we pay fees $p \cdot X_i \cdot Y_j$ which depends on the moment and the banks who process to the transaction. (the bi-linear model is for simplification)
- at the end, we need to pay $C_T = (S_T - K)^+$
- we can sell our stocks and get : $\delta_{t1} S_T$
- the rest of the money get us : $(C_0 e^{rt_1} - \delta_{t1}\widetilde{S_{t1}})e^{(T-t1)r}$

The cost function is thus :

$$\Psi_{i,j} = p \cdot X_i \cdot Y_j + \delta_{t_j}\hat{S}_T - C_T + (C_0 e^{rt_j} - \delta_{t_j}\widetilde{S}_{t_j})e^{(T-t_j)r} \tag{26}$$

As we have :

$$\begin{aligned} \widetilde{S}_t &= \hat{S}_t \cdot (1 + \alpha\nu_t) \\ \widetilde{S}_t &= S_t \cdot (1 + \beta \cdot \int_{u=0}^{t} \nu_u (t-u) du) \cdot (1 + \alpha\nu_t) \end{aligned} \tag{27}$$

Which is at first order equal to :

$$\widetilde{S}_t = S_t \cdot (1 + \beta \cdot \int_{u=0}^{t} \nu_u (t-u) du + \alpha\nu_t) \tag{28}$$

We can now decompose the cost into 3 parts :

$$c_{i,j} = p \cdot X_i \cdot Y_j + \delta_{t_j}(S_T - S_{t_j} e^{rT - t_j}) - (S_T - K)^+ + C_0 \cdot e^{rT} \tag{29}$$

$$f_j(\nu_j) = -\alpha\delta_{t_j} S_{t_j} \nu_{t_j} e^{(T-t_j)r} \tag{30}$$

$$\sum_{k \in J} \phi_{k,j}\mu_k = \sum_{k<j} \beta S_{t_j} \nu_k \cdot (t_j - t_k) \tag{31}$$

thus :

$$\phi_{k,j} = S_{t_j} \cdot (t_j - t_k) \tag{32}$$

$$\mu_k = \beta\nu_k \tag{33}$$

We have find a formulation of our problem with the form :

$$\inf_{\nu \in P(Y)} MK(\nu) + E(\nu) \tag{34}$$

In the following section, we will explain the algorithm used to solved this problem and its assumption on $MK(\nu)$ and $E(\nu)$.

4 Resolution of the problem

4.1 Entropic regularization formula of the problem

Solving (34) in practice (even if E is convex) might be dicult because of the transport cost term MK for which it is expensive to compute a sub-gradient. There is however a simple regularization of MK which is much more convenient to handle: the entropic regularization. Given a regularization parameter $\epsilon \geq 0$, let us dene for every $\nu \in P(Y)$:

With the previous notation, we can write :

$$MK_\epsilon(\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \{ \sum_{i,j \in I \cdot J} c_{ij} \gamma ij + \epsilon \sum_{i,j \in I \cdot J} \gamma_{ij} (ln(\gamma_{ij}) - 1) \} \tag{35}$$

The first order optimality conditions give the following Gibbs form for γ

$$\gamma_{ij} = a_i exp(-\frac{1}{\epsilon}(c_{ij} + f_j(\nu_j) + \sum_{k \in J} \phi_{kj} \nu_k)) \tag{36}$$

And then use the Dykstra proximal splitting algorithm

for $a_i \geq 0$ such that

$$a_i = \frac{\mu_i}{\sum_{j \in J} exp(-\frac{1}{\epsilon}(c_{ij} + f_j(\nu_j) + \sum_{k \in J} \phi_{kj} \nu_k))} \tag{37}$$

4.2 Proximal Splitting Scheme

We can rewrite (35) with the Kullback Leiber divergence defined by :

$$KL(\gamma|\theta) = \sum_{i,j \in I \cdot J} \gamma_{ij} (ln(\frac{\gamma_{ij}}{\theta_{ij}} - 1)) \tag{38}$$

Then, we have :

$$\begin{gathered} \sum_{i,j \in I \cdot J} c_{ij} \gamma ij + \epsilon \sum_{i,j \in I \cdot J} \gamma_{ij} (ln(\gamma_{ij}) - 1) = \epsilon KL(\gamma|\bar{\gamma}) \\ where\ \bar{\gamma}_{ij} = e^{-\frac{c_{ij}}{\epsilon}} \end{gathered} \tag{39}$$

We now can reformulate the problem as a proximal problem :

$$\begin{gathered} prox_G^{KL}(\bar{\gamma}) = \min_{\gamma \in (R)_+^{I \cdot J}} \{ KL(\gamma|\bar{\gamma}) + G(\gamma) \} \\ with\ G(\gamma) = \chi\{\Lambda_1(\gamma) = \mu\} + \frac{1}{\epsilon} E(\Lambda_2(\gamma)) \end{gathered} \tag{40}$$

and $\Lambda_1(\gamma)$ is the first marginal of γ
and $\Lambda_2(\gamma)$ is the second marginal of γ

We split G as the sum of elementary function :

$$\begin{gathered} G = \sum_{l=1}^{L} G_l \\ G_n = G_{n\ modulo(L+1)+1} \end{gathered} \tag{41}$$

Finally, the algorithm used to solve the problem is :

Data: nbr Itter,G,
Result: $\gamma^{(nbrItter)}$
initialization : $\gamma^{(0)} = \bar{\gamma}, z^{(0)} = z^{(1)} = ... = e, e_{ij} = 1, (i,j) \in I \cdot J$
while $n \leq nbrItter$ **do**
  $\gamma^{(n)} = prox_{G_n}^{KL}(\gamma^{(n-1)} \odot z^{(n-L)})$
  $z^{(n)} = z^{(n-1)} \odot (\gamma^{(n-1)} \oslash \gamma^{(n)})$
end

Algorithm 1: Dykstras' algorithm for KL projections on the intersection of convex sets

Where :

$$\begin{aligned}(\gamma \odot \theta)_{ij} &= \gamma_{ij} \cdot \theta_{ij} \\ (\gamma \oslash \theta)_{ij} &= \frac{\gamma_{ij}}{\theta_{ij}}\end{aligned} \tag{42}$$

4.3 Computation of the proximal operators

To solve our problem, we will need to split G into two parts. We can then compute the prox of each part.

4.3.1 Computation of $prox_{G_1}^{KL}$

$$\begin{aligned}G_1 &= \chi\{\Lambda_1(\gamma) = \mu\} \\ (prox_{G_1}^{KL}(\theta))_{ij} &= \frac{\mu_i \theta_{ij}}{\sum_{k \in J} \theta_{ik}}\end{aligned} \tag{43}$$

4.3.2 Computation of $prox_{G_2}^{KL}$

$$\begin{aligned}E(\nu) &= E_1(\nu) + E_2(\nu) \\ E_1(\nu) &= \sum_{j \in J} F_j(\nu_j) \\ where\ F_j(\nu_{t_j}) &= \int_0^{\nu_{t_j}} f_j(s)ds = -\alpha\delta_{t_j} S_{t_j} \frac{\nu_{t_j}^2}{2} e^{(T-t_j)r} := \zeta_j \nu_{t_j}^2\end{aligned} \tag{44}$$

$$E_2(\nu) = \frac{1}{2} \sum_{k,j \in J \cdot J} \phi_{kj} \nu_k \nu_j$$

$$where\ \phi_{k,j} = S_{t_j} \cdot (t_j - t_k)$$

$$G_2 = \frac{1}{\epsilon} E \circ \Lambda_2 \tag{45}$$

$$prox_{G_2}^{K} L(\theta)_{ij} = \theta_{ij} exp(-\frac{\nu_j \zeta_j + \sum_{k \in J} \phi_{kj} \nu_k}{\epsilon}) \tag{46}$$

where ν is the solution of the system :

$$\nu_j = (\sum_{iI} \theta_{ij}) exp(-\frac{\nu_j \zeta_j + \sum_{kJ} \phi_{kj} \nu_k}{\epsilon}) \tag{47}$$

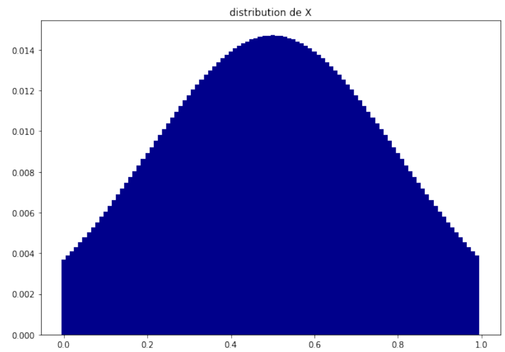
Which can be solved by Newton's step.

Now, we have all we need to solve the problem. In the next part we will present the numerical results.

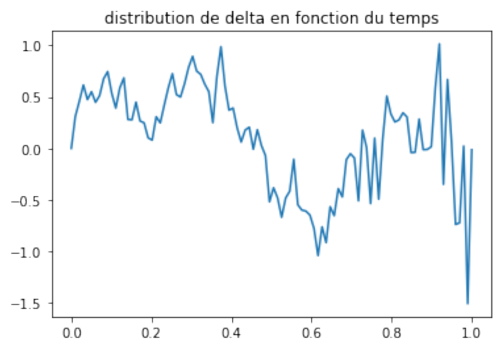
5 Numerical Results

5.1 Set up of the environment

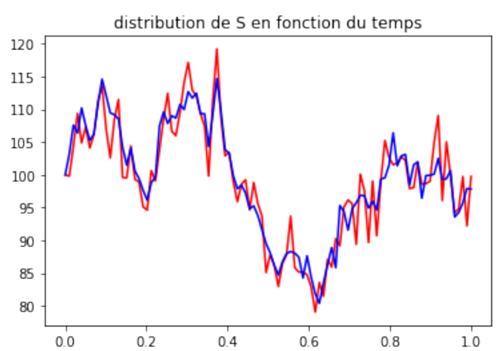
The figure 5.1 presents the distribution of X taken to simulate our data. We choose a normalized Gaussian variable.



The figure 5.1 presents the delta quantity traders need to hold at every time to have a perfectly hedge portfolio.



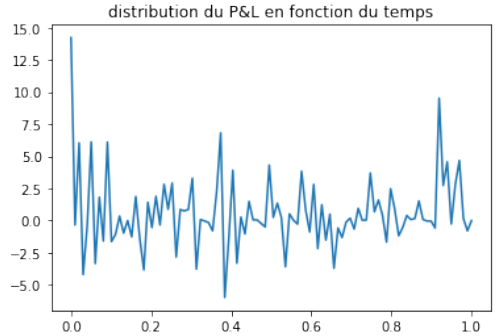
The figure 5.1 presents the variation of the price of the stock. We plot in blue the variation without market impact and in red with a small market impact.



The figure 5.1 presents the shape of the variation of the profit and loss of a portfolio hedge with a delta hedging strategy.

5.2 Resulting matrix γ

One of the difficulty of this method is the divergence of $\bar{\gamma}$ if epsilon is too small or c_{ij} too big. Moreover, solving a non linear problem at each step of the algorithm is heavy to compute.



I use the Newton Krylov algorithm from scipy to solve the non linear problem as it is a method design to solve large scale non linear problem. The idea of this method is to compute the inverse of the Jacobian with an iterative Krylov method istead of computing the inverse directly.

However, this method does not converge all the time and is very low to compute. The convergence also depends on the initialization of the Newton Krylov algorithm which is not obvious to decide. In order to accelerate the algorithm and obtain numerical results, I set a maximal iteration at 1000 for the linear problem.

The figure 5.2 presents γ in the case we consider G = G_1.

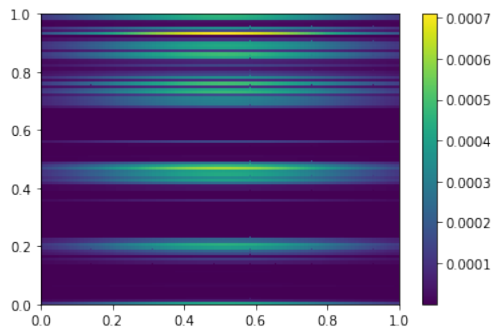


Figure 1: γ as the solution of the simplified problem G = G_1

The figure 5.2 presents the results in case case of $G = G_1 + G_2$. As in this case, there is a long and short term market impact, we see that the best strategy seems to hedge the position at the last minute. Indeed as the second derivative the price of the financial product by the stock price (called Gamma := $\frac{\partial C^2}{\partial S^2}$) is always negative, buying stocks will make the price rise. Thus, the Cournot Nash equilibria is when every one hedge at the last moment.

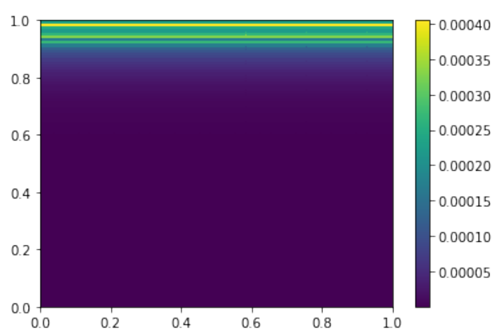


Figure 2: γ as the solution to our problem

6 Conclusion

Through this study, we achieve to find the Cournot Nash Equilibria of a game based on the delta hedging of a portfolio in a competitive model with market impact.

We used the Dykstras' algorithm for KL projections on the intersection of convex sets in order to easily compute the gradient proximal operator.

This study might be seen as a first step in the study of the market impact of delta hedging strategy. This results might be usefull in order to study the behaviour of financial competitor and develop new strategies based on these knowledges.

In practice, knowing μ and σ is a real challenge as the Black & Scholes model is too simple to modelize the dynamic of a stock.

One possible further step of this research project is to study others derivative products such as the collar which has its gamma ($\frac{\partial C^2}{\partial S^2}$) not always negative.

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