Calculus Notes

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Contents

Notes based off of Stewart's Calculus Textbook + Calculus I and II courses taught by professor Stangeby and Davis.

1 Epsilon Delta Limit

1.1 Definition

$$\lim_{x \to c} f(x) = L \tag{1}$$

if and only if, given any number $\varepsilon > 0$, we can find a number $\delta > 0$ which will depend on ε, for which

$$|f(x) - L| < \varepsilon \tag{2}$$

whenever

$$0 < |x - c| < \delta. \tag{3}$$

1.2 Concise Definition

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : |x - c| < \delta \implies |f(x) - L| < \varepsilon \tag{4}$$

1.3 Variations

1.3.1 Left Handed Limit

$$\forall \varepsilon > 0, \quad \exists \delta : -\delta < x - c < 0 \implies |f(x) - L| < \varepsilon \tag{5}$$

1.3.2 Right Handed Limit

$$\forall \varepsilon > 0, \quad \exists \delta : 0 < x - c < \delta \implies |f(x) - L| < \varepsilon \tag{6}$$

1.3.3 Limit Approaches Positive Infinity

$$\forall M > 0, \quad \exists \delta > 0 : |x - c| < \delta \implies f(x) > M \tag{7}$$

1.3.4 Limit Approaches Negative Infinity

$$\forall N < 0, \quad \exists \delta > 0 : |x - c| < \delta \implies f(x) < N \tag{8}$$

1.3.5 x Approaches Positive Infinity

$$\forall \varepsilon > 0, \quad \exists \delta : x > \delta \implies |f(x) - L| < \varepsilon \tag{9}$$

1.3.6 x Approaches Negative Infinity

$$\forall \varepsilon > 0, \quad \exists \delta : x < \delta \implies |f(x) - L| < \varepsilon \tag{10}$$

Example

Prove that

$$\lim_{x \to 2} x^3 = 8 \tag{11}$$

If

$$|\,\mathrm{f(x)} - L| < \varepsilon \tag{12}$$
$$|x^3 - 8 - L| < \varepsilon \tag{13}$$

when

$$|x - 2| < \delta \tag{14}$$

$$|x - 2||x^2 + 2x + 4| < \varepsilon \tag{15}$$
$$\delta|x^2 + 2x + 4| > |x - 2||x^2 + 2x + 4| \tag{16}$$

Suppose $\delta < 1$, then

$$-1 < x - 2 < 1 \tag{17}$$
$$1 < x < 3 \tag{18}$$
$$7 < x^2 + 2x + 4 < 19. \tag{19}$$

Tip: $\delta < 1$ can be chosen because the function is continuous along the closed interval. If it is not continuous, a smaller value of delta would need to be chosen.

$$|x^2 + 2x + 4| < 19 \tag{20}$$
$$|x - 2||x^2 + 2x + 4| < 19\delta \tag{21}$$
$$\varepsilon = 19\delta \tag{22}$$

If $\varepsilon > 19$, let $\delta = 1$. Then,

$$|x - 2| < 1 \tag{23}$$
$$7 < x^2 + 2x + 4 < 19 \tag{24}$$
$$|x - 2||x^2 + 2x + 4| < 19 < \varepsilon \tag{25}$$

$\therefore$ given $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{19}\}$. Then, if $|x - 2| < \delta$, $|x^3 - 8 - L| < \varepsilon$ where $L = 0$, proving that $\lim_{x\to 2} x^3 = 8$, as required.

2 Limit Laws

1. **Sum and Difference Law**

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \tag{26}$$

2. **Constant Multiple Law**

$$\lim_{x\to c} cf(x) = c\lim_{x\to c} f(x) \tag{27}$$

3. **Product Law**

$$\lim_{x\to c}\,[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) \tag{28}$$

4. **Quotient Law**

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \text{where} \lim_{x\to c} g(x) \neq 0 \tag{29}$$

5. **Power Law**

$$\lim_{x\to c}\,[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n \tag{30}$$

6. **Substitution Law**

$$\lim_{x\to c} f(x) = f(c) \tag{31}$$

7. **Common Factor Cancelation Law**

$$\lim_{x \to c} \frac{f(x)(x-c)}{g(x)(x-c)} = \lim_{x \to c} \frac{f(x)}{g(x)} \tag{32}$$

8. **Sin Trig Limit**

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{33}$$

9. **Cos Trig Limit**

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \tag{34}$$

10. **Squeeze Theorem**
Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq c$ in an open interval containing c such that:

$$f(x) \leq g(x) \leq h(x). \tag{35}$$

If

$$\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x), \tag{36}$$

Then,

$$\lim_{x \to c} g(x) = L. \tag{37}$$

3 Continuity

$q(x)$ is continuous at $x = c$ if

$$\lim_{x \to c} q(x) = q(c). \tag{38}$$

A function is continuous if it is differentiable everywhere.

$$\begin{aligned} \lim_{x \to c} f(x) - f(c) &= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) \cdot (x - c) \qquad (39) \\ &= f'(c) \lim_{x \to c} (x - c) \qquad (40) \\ &= 0 \qquad (41) \end{aligned}$$

Thus,

$$\lim_{x \to c} f(x) = f(c). \tag{42}$$

3.1 Intermediate Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$, and L is a number that lies in between $f(a)$ and $f(b)$, the there exists a number c such that $a < c < b$ and $f(c) = L$.

3.2 Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$, then $f(x)$ has both a maximum and minimum value on $[a, b]$.

3.3 Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and $f(x)$ is differentiable on the open interval (a, b), there exists a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{43}$$

4 Differentiation Laws

1. **Definition**
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{44}$$
2. **Constant Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} c = 0 \tag{45}$$
3. **Constant Multiple Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} (cf(x)) = c \frac{\mathrm{d}}{\mathrm{d}x} f(x) \tag{46}$$
4. **Power Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} (x^n) = nx^{n-1} \tag{47}$$
5. **Sum and Difference Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} (f(x) + g(x)) = f'(x) + g'(x) \tag{48}$$
6. **Product Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} (f(x)g(x)) = f'(x)g(x) + g'(x)f(x) \tag{49}$$
7. **Quotient Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \tag{50}$$
8. **Chain Rule**
$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x) \tag{51}$$
9. **Basic Trig Rules**
$$\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x) \tag{52}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x) \tag{53}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \tan(x) = \sec^2(x) \tag{54}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sec(x) = \sec(x)\tan(x) \tag{55}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \csc(x) = -\csc(x)\cot(x) \tag{56}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cot(x) = -\csc^2(x) \tag{57}$$

Curve Sketching

1. Find the **domain** and **endpoints** of the function.
2. Find x and y **intercepts**.
3. Determine if there is any **symmetry**.
4. Find all horizontal, vertical or oblique **asymptotes**.
5. Determine where the function is **increasing or decreasing**.
6. Find **local maximums and minimums**.
7. Determine **concavity** and points of inflection.

8. Find **absolute maximum and minimum**.
9. Figure out **range** of the function.
10. Do sketch.

5 Integrals

The left-hand Riemann Sum (endpoints on the left) is defined as:

$$A = \sum_{i=1}^{n} f(x_{i-1})\Delta x. \tag{58}$$

The right-hand Riemann Sum (endpoints on the right) is defined as:

$$A = \sum_{i=1}^{n} f(x_i)\Delta x. \tag{59}$$

With the right-hand Riemann Sum, the integral is defined as:

$$\int f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x. \tag{60}$$

5.1 Delta Epsilon Definition

$$\forall \varepsilon > 0, \quad \exists \delta : n > \delta \implies |\sum_{i=1}^{n} f(x_{i-1})\Delta x - \sum_{i=1}^{n} f(x_i)\Delta x| < \varepsilon \tag{61}$$

6 Fundamental Theorem of Calculus

6.1 Part I

Let f be continuous on $[a, b]$. The function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)dt \tag{62}$$

is continuous on $[a, b]$, differentiable on (a, b), and has derivative $F'(x) = f(x)$.

Proof

For all x and $x + h \in (a, b)$,

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt && (63)\\ &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt && (64)\\ &= \int_x^{x+h} f(t)dt && (65) \end{aligned}$$

Thus,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt \tag{66}$$

We will now show that this equals $f(x)$. By the extreme value theorem, f must take on a maximum value $f(M)$, and a minimum value $f(m)$ on the continuous interval $[x, x+h]$.

$$\int_x^{x+h} f(m)dt \leq \int_x^{x+h} f(t)dt \leq \int_x^{x+h} f(M)dt \tag{67}$$

$$f(m) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(M) \tag{68}$$

$$f(m) \leq F'(x) \leq f(M) \tag{69}$$

As $h \to 0$, $f(m)$ and $f(M)$ both approach $f(x)$. Therefore, $F'(x) = f(x)$.

Tip: Remember to apply chain rule/u-substitution if the upper or lower bound is not a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x) \tag{70}$$

6.2 Part II

If $F'(x)$ is continuous on $[a, b]$, then

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{71}$$

Proof

Let

$$G(x) = \int_a^x F'(t)dt \tag{72}$$

Then, $G(x) = F(x) + C$. However,

$$G(a) = \int_a^a F'(t)dt = 0. \tag{73}$$

Thus,

$$C = -F(a) \tag{74}$$

Therefore, for all $x \in [a, b]$,

$$\int_a^b F'(t)dt = F(b) - F(a). \tag{75}$$

7 U-Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{76}$$

If $u = g(x)$ and $du = g'(x)dx$.

Tip: Remember to adjust bounds if integral is definite.

8 Volume

8.1 Disk or Washer Method

For a solid with a known continuous function for cross-sectional area, $A(x)$, the volume can be calculated as

$$V = \int_a^b A(x)dx. \tag{77}$$

Example

Let's calculate the volume of a sphere with radius r. We know that

$$A(x) = \pi y^2 = \pi(r^2 - x^2). \tag{78}$$

Thus,

$$V = \int_{-r}^{r} \pi(r^2 - x^2) \tag{79}$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \tag{80}$$

$$= \frac{4}{3}\pi r^3. \tag{81}$$

8.2 Cylindrical Shells Method

For shapes rotated about the y axis, the volume can be calculated as

$$V = \int_a^b 2\pi x f(x)dx \tag{82}$$

Example

Let's calculate the volume of the solid obtained by rotating the region bounded by $y = 2x^3 - x^3$ and $y = 0$ about the y-axis.
The circumference of the cylinder is $2\pi x$. The height of the cylinder is $2x^3 - x^3$. The thickness is dx. Thus,

$$V = \int_0^2 (2\pi x)(2x^3 - x^3)dx \tag{83}$$

$$= 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 \right]_0^2 \tag{84}$$

$$= \frac{16}{5}\pi. \tag{85}$$

9 The Average Value of a Function

The average value of a function f on the interval $[a, b]$ is defined as

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)dx \tag{86}$$

9.1 Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx \tag{87}$$

9.2 Second Mean Value Theorem for Integrals

If f and g are continuous on $[a, b]$ and g is non-negative, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx \tag{88}$$

9.3 Inverse Functions

If f has an inverse function, and the function is differentiable at a, then

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}. \tag{89}$$

Tip: Remember to adjust domain and range when deriving the inverse function.

10 Natural Logarithmic Function

A logarithm function is a non-constant differentiable function f, defined for $x \in \{\mathbb{R}, (0, \infty)\}$ such that for all $a > 0$ and $b > 0$:

$$f(a \cdot b) = f(a) + f(b) \tag{90}$$

10.1 Logarithmic Function Derivatives

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (91)\\
&= \lim_{h\to 0}\frac{f(\frac{x+h}{x})}{h} && (92)\\
&= \lim_{h\to 0}\frac{f(1+\frac{h}{x})}{h} && (93)\\
&= \lim_{h\to 0}\frac{f(1+\frac{h}{x})}{\frac{h}{x}}\frac{1}{x} && (94)\\
&= \lim_{k\to 0}\frac{f(1+k)-f(1)}{k}\frac{1}{x}, \quad k = \frac{h}{x} && (95)\\
&= f'(1)\frac{1}{x} && (96)
\end{aligned}$$

For simplicity, let's let $f'(1) = 1$, and we define this function as the natural logarithm.

$$\ln x = \int_1^x \frac{dt}{t},\ x > 0 \tag{97}$$

We can also define e^x to be the inverse of the natural logarithm.

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \tag{98}$$

Using the definition of $f'(1) = 1$ from Equation 95, we can also write e^x as a limit.

$$1 = f'(1) = \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \to 0} \ln(1+x)^{1/x} \tag{99}$$

$$e = e^1 = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x} \tag{100}$$

Proof: $e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ when $x \geq 0$

$e^x \geq 1$ when $x >= 0$ since $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x > 0$, and $e^0 = 1$.
Now, let's integrate both sides of this equation.

$$\int_0^x e^t dt \geq \int_0^x 1 dt \tag{101}$$

$$e^x \geq x + 1 \tag{102}$$

$$\tag{103}$$

Now, by mathematical induction, for $k > 0$,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} \tag{104}$$

$$\int_0^x e^t dt \geq \int_0^x 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^k}{k!} \tag{105}$$

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \tag{106}$$

$$\tag{107}$$

11 Inverse Trigonometric Functions

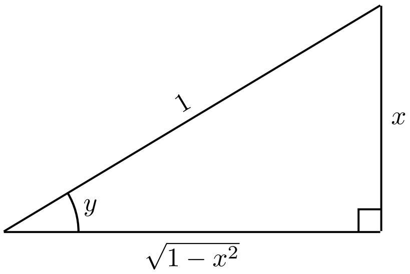


Figure 1: A geometric depiction of $y = \sin^{-1} x$.

Table 1: Domain and Range of Inverse Trigonometric Functions.

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \tan^{-1} x$	$x \in \mathbf{R}$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

11.1 Derivatives of Inverse Trigonometric Functions

$$y = \sin^{-1} x \Rightarrow \sin y = x \Leftrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} \cos y = 1 \Leftrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < -1 \tag{108}$$

Using similar logic as shown above, we can find the derivatives of the other inverse trigonometric functions.

1. $$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \quad (-1,1) \tag{109}$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \quad (-1,1) \tag{110}$$

3. $$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2} \tag{111}$$

4. $$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = -\frac{1}{x\sqrt{1-x^2}}, \quad (-\infty,-1)\cup(1,\infty) \tag{112}$$

5. $$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{x\sqrt{1-x^2}}, \quad (-\infty,-1)\cup(1,\infty) \tag{113}$$

6. $$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = -\frac{1}{1+x^2} \tag{114}$$

12 Differential Equations

12.1 Separable Differential Equations

A separable differential equation is a first-order differential equation in which the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be separated into a function of x and a function of y.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(x)} \tag{115}$$

The solutions to this equation can be found by integrating both functions.

$$\int h(y)dy = \int g(x)dx \tag{116}$$

Proof

We can use the chain rule to solve Equation 115.

$$h(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \tag{117}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\int h(y)dy\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\int g(x)dx \tag{118}$$

$$\int h(y)dy = \int g(x)dx \tag{119}$$

13 First-Order Linear Differential Equations

First-order linear differential equations are of the form

$$y' + P(x)y = Q(x). \tag{120}$$

To solve these equations, we must try to find an integration factor $I(x)$ such that

$$I(x)\left(y' + P(x)y\right) = (I(x)y)'. \tag{121}$$

Substituting this into Equation 120 gives us

$$(I(x)y)' = I(x)Q(x) \tag{122}$$

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right]. \tag{123}$$

Finding the Integration Factor

Expanding Equation 121,

$$I(x)y' + I(x)P(x)y = I'(x)y + I(x)y' \tag{124}$$

$$I(x)P(x) = I'(x). \tag{125}$$

This is now a separable differential equation we can solve.

$$\int \frac{dI}{I} = \int P(x)dx \tag{126}$$

$$\ln|I| = \int P(x)dx \tag{127}$$

$$I = Ae^{\int P(x)dx} \tag{128}$$

14 Second-Order Linear Differential Equations

A second-order linear differential equation has the form

$$P(x)y'' + Q(x)y' + R(x)y = G(x) \tag{129}$$

where P, Q, R, and G are continuous functions. However, we will only be going over the equations where P, Q, and R are constants.

14.1 Homogeneous Equations

Equations with $G(x) = 0$ are homogeneous equations.

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{130}$$

The general solution to this type of differential equation is a linear combination of two linearly independent solutions y_1 and y_2. In other words, the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{131}$$

where c_1 and c_2 are constants. To begin, let's assume the solution to the differential equation is of the form

$$y = e^{rx} \tag{132}$$

where r is a constant. Plugging this in to Equation 130 would produce

$$(ar^2 + br + c)e^{rx} = 0. \tag{133}$$

This expression would only be true if r is a root of

$$ar^2 + br + c = 0, \tag{134}$$

which is also known as the auxiliary equation. Now, we can separate the solutions into 3 different cases based on the discriminant of the auxiliary equation.

14.1.1 Case 1: $b^2 - 4ac > 0$

If the roots r_1 and r_2 of the auxiliary equation are real and unequal, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{135}$$

14.1.2 Case 2: $b^2 - 4ac = 0$

If there is only one real root of the auxiliary equation, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}. \tag{136}$$

14.1.3 Case 3: $b^2 - 4ac < 0$

If the roots the auxiliary equation are complex, $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \tag{137}$$

14.2 Nonhomogeneous Equations

A second-order nonhomogeneous linear differential equation with constant coefficients is of the form

$$ay'' + by' + cy = G(x). \tag{138}$$

The general solution to this equation can be written as

$$y(x) = y_p(x) + y_c(x) \tag{139}$$

where y_p is a particular solution of the nonhomogeneous equation, and y_c is the general solution of the complementary homogeneous equation,

$$ay'' + by' + cy = 0. \tag{140}$$

There are two primary methods of finding the particular solution for a nonhomogeneous equation: the method of undetermined coefficients and the method of variation of parameters.

14.3 The Method of Undetermined Coefficients

We can guess the form of the particular solution with undetermined constants, and substitute it into the differential equation. If there is solution to each of the constants, then that function is a particular solution to the differential equation.

Table 2: Common Guesses for $y_p(x)$.

$G(x)$	$y_p(x)$
$P(x)$ (a polynomial of degree n)	$Q(x)$ (a polynomial of degree n)
$P(x)e^{sx}$	$Q(x)e^{sx}$
$P(x)\cos sx$ or $P(x)\sin sx$	$Q_1(x)\cos sx + Q_2(x)\sin sx$
$P(x)e^{sx}\cos sx$ or $P(x)e^{sx}\sin sx$	$Q_1(x)e^{sx}\cos sx + Q_2(x)e^{sx}\sin sx$

If $y_p(x)$ is a solution to the complementary homogeneous equation, multiply it by x.

14.4 The Method of Variation of Parameters

Suppose we have solved the complementary homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$y(x) = c_1 y_(x) + c_2 y_2(x). \tag{141}$$

Then, the particular solution of the nonhomogeneous equation is of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{142}$$

Let's find the first derivative of this particular solution.

$$y_p' = u_1'y_1 + u_2'y_2' + u_1y_1' + u_2y_2' \tag{143}$$

Because u_1 and u_2 are arbitrary function, we can impose a condition on them to simplify our calculations. Let

$$u_1'y_1 + u_2'y_2 = 0. \tag{144}$$

Then,

$$y_p' = u_1y_1' + u_2y_2' \tag{145}$$
$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' \tag{146}$$

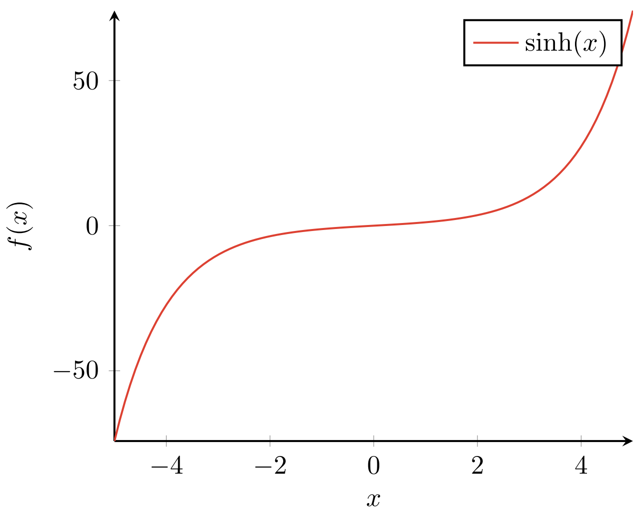
Substituting this into the differential results in

$$a(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = G \tag{147}$$
$$u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = G \tag{148}$$
$$a(u_1'y_1' + u_2'y_2') = G \tag{149}$$

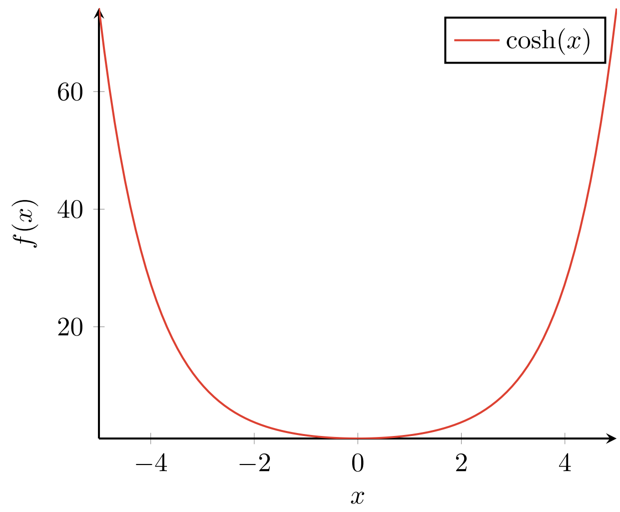
Lastly, Equations 144 and 149, can be used to solve for u_1 and u_2.

15 Hyperbolic Functions

1.
$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{150}$$

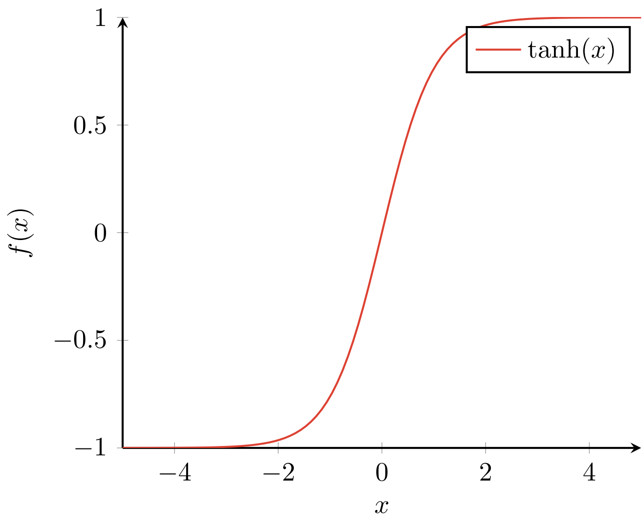


2.
$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{151}$$



3.

$$\tanh x = \frac{\sinh x}{\cosh x} \tag{152}$$



4.

$$\operatorname{csch} x = \frac{1}{\sinh x} \tag{153}$$

5.

$$\operatorname{sech} x = \frac{1}{\cosh x} \tag{154}$$

6.

$$\coth x = \frac{\cosh x}{\sinh x} \tag{155}$$

7.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad x \in \mathbf{R} \tag{156}$$

8.

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geq 1 \tag{157}$$

9.

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1 \tag{158}$$

15.1 Derivatives of Hyperbolic Functions

1. $$\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x \tag{159}$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sin x \tag{160}$$

3. $$\frac{\mathrm{d}}{\mathrm{d}x} \tanh x = \operatorname{sech}^2 x \tag{161}$$

4. $$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{csch} x = -\operatorname{csch} x \coth x \tag{162}$$

5. $$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sech} x = \operatorname{sech} x \tanh x \tag{163}$$

6. $$\frac{\mathrm{d}}{\mathrm{d}x} \coth x = -\operatorname{csch}^2 x \tag{164}$$

7. $$\frac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} \tag{165}$$

8. $$\frac{\mathrm{d}}{\mathrm{d}x} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}} \tag{166}$$

9. $$\frac{\mathrm{d}}{\mathrm{d}x} \tanh^{-1} x = \frac{1}{1-x^2} \tag{167}$$

10. $$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}} \tag{168}$$

11. $$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}} \tag{169}$$

12. $$\frac{\mathrm{d}}{\mathrm{d}x} \coth^{-1} x = \frac{1}{1-x^2} \tag{170}$$

16 L'Hopital's Rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{171}$$

Proof

We will prove L'Hopital's Rule by splitting the problem into 3 cases and proving only the right-hand limit (left-hand limit proof is similar).

Case 1:

If $f(c) = g(c) = 0$, there exists an interval (c, b) such that $g(x)$ and either strictly increasing or decreasing for $x \in (c, b)$. $g(x)$ is non-zero since $g(c) = 0$. Thus, by Cauchy's Mean Value

Theorem, there exists $a \in (c, x)$ such that

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(a)}{g'(a)} \tag{172}$$

$$\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \tag{173}$$

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(a)}{g'(a)} = \lim_{a \to c^+} \frac{f'(a)}{g'(a)} = L \tag{174}$$

Case 2:

If

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c^+} g(x) = \pm\infty \tag{175}$$

By the delta-epsilon definition of the limit, for every $\varepsilon > 0$, there exists, $\delta > 0$ such that

$$c < x < c + \delta \quad \text{and} \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon. \tag{176}$$

Again, by Cauchy's Mean Value Theorem,

$$\frac{f'(a)}{g'(a)} = \frac{f(b) - f(x)}{g(b) - g(x)} = \frac{\frac{f(x)}{g(x)} - \frac{f(b)}{g(x)}}{1 - \frac{g(b)}{g(x)}} \tag{177}$$

$$\frac{f'(a)}{g'(a)} \left(1 - \frac{g(b)}{g(x)}\right) = \frac{f(x)}{g(x)} - \frac{f(b)}{g(x)} \tag{178}$$

$$\frac{f'(a)}{g'(a)} = \frac{f(x)}{g(x)} - \left(\frac{f(b)}{g(x)} - \frac{f'(a)g(b)}{g'(a)g(x)}\right) \tag{179}$$

$$\frac{f'(a)}{g'(a)} = \frac{f(x)}{g(x)} - r(x) \tag{180}$$

Since $r(x)$ tends to 0 as $x \to c^+$, we may choose $\delta > 0$ such that $|r(x)| < \varepsilon$ for all $x \in (c, c + \delta)$ and as a result,

$$L - 2\varepsilon < \frac{f(x)}{g(x)} < L + 2\varepsilon. \tag{181}$$

Case 3:

For limits, where $x \to \infty$, we can use the clever substitution, $t = x^{-1}$.

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{f(t^{-1})}{g(t^{-1})} = \lim_{x \to 0^+} \frac{f'(t^{-1})}{g'(t^{-1})} = L \tag{182}$$

17 Integration By Parts

Integration by parts is just another way of writing the product rule for derivatives.

$$\int u dv = uv - \int v du \tag{183}$$

18 Trigonometric Integrals and Substitutions

18.1 Integrating Expressions with Sine or Cosine

The following methods can be used to solve integrals of the form

$$\int \sin^m x \cos^n x dx. \tag{184}$$

1. If the power of cosine is odd, use the identity $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Then, substitute $u = \sin x$.
2. If the power of sine is odd, use the identity $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine. Then, substitute $u = \cos x$.
3. If the powers of both sine and cosine are even, use the identity

   $$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{185}$$

   to express the remaining factors in terms of cosine.

The product to sum identities can be used to solve integrals of the form

$$\int \sin mx \cos nx dx. \tag{186}$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right] \tag{187}$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right] \tag{188}$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A - B) + \sin(A + B)\right] \tag{189}$$

The reduction formulas for

$$\int \sin^n x dx \quad \text{and} \quad \int \cos^n x dx \tag{190}$$

are

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx \tag{191}$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx \tag{192}$$

18.2 Integrating Expressions with Tangent or Secant

The process is quite similar to sine and cosine where we can use identities to simplify the expression.

$$\tan^2 x = \sec^2 x - 1 \tag{193}$$

$$\tag{194}$$

Additionally it would be useful to know the integral of tangent and secant.

$$\int \tan x dx = \ln|\sec x| + C \tag{195}$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C \tag{196}$$

Secant Integral Proof

Let $u = \sec x + \tan x$. Then,

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx \tag{197}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \tag{198}$$

$$= \int \frac{du}{u} \tag{199}$$

$$= \ln|\sec x + \tan x| + C \tag{200}$$

18.3 Trigonometric Substitution

Trigonometric expressions can be sometimes substituted in to simplify the integral. The following substitutions are useful.

1. $$\int \sqrt{a^2 - x^2} dx \tag{201}$$

   Set $x = a \sin \theta$.

2. $$\int \sqrt{a^2 + x^2} dx \tag{202}$$

   Set $x = a \tan \theta$.

3. $$\int \sqrt{x^2 - a^2} dx \tag{203}$$

   Set $x = a \sec \theta$.

19 Partial Fractions

The method of partial fractions splits a rational function into a sum of simpler rational functions which can be then integrated. For the following fraction, we can solve for values of A, B, C, and D, to simplify the expression.

$$\frac{x^2 + 3x + 5}{(x+1)^2(x^2 + x + 1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C + Dx}{x^2 + x + 1} \tag{204}$$

20 Improper Integrals

1. $$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx \tag{205}$$

2. $$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx \tag{206}$$

3. If f is continuous on $[a, b)$, and discontinuous at b, then

   $$\int_a^b f(x)dx = \lim_{x \to b^-} \int_a^x f(x)dx. \tag{207}$$

4. If f is continuous on $(a, b]$, and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{x \to a^+} \int_x^b f(x)dx. \tag{208}$$

20.1 Comparison Theorem

Suppose that $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.
- If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

21 Additional Integration Applications

21.1 Arc Length

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx. \tag{209}$$

21.2 Surface Area

The surface area for rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx. \tag{210}$$

21.3 Moments and Centers of Mass

To calculate the moment of a region of a function about the y-axis, we can use the following formula.

$$M_y = \rho \int_a^b x f(x)dx. \tag{211}$$

Likewise, for the x-axis, we have

$$M_x = \rho \int_a^b \frac{1}{2} [f(x)]^2 \, dx. \tag{212}$$

The center of mass can be found by dividing moment of area by area.

22 Parametric Equations

22.1 Derivatives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \tag{213}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} \tag{214}$$

22.2 Area

$$A = \int_{t_1}^{t_2} y(t)x'(t)dt \tag{215}$$

22.3 Arc Length and Surface Area

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} dt \tag{216}$$

$$S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} dt \tag{217}$$

23 Polar Coordinates

Polar equations are of the form

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{218}$$

r and θ are can be found by

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}. \tag{219}$$

23.1 Derivatives

By using parametric equation derivative formula,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}. \tag{220}$$

23.2 Area

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \tag{221}$$

23.3 Arc Length

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} d\theta \tag{222}$$

24 Sequences

A sequence is a list of numbers written in a definite order.

The sequence $\{a_1, a_2, a_3, \ldots\}$ can be denoted by

$$\{a_n\}_{n=1}^{\infty} \quad \text{or} \quad \{a_n\}. \tag{223}$$

1. A sequence is **monotonic** if it is either increasing or decreasing.
2. A sequence is bounded above if there exists a number M such that $a_n \leq M$ for all n.
3. A sequence is bounded below if there exists a number m such that $a_n \geq m$ for all n.

4. If it is both bounded above and below, it is a **bounded sequence**.
5. Every bounded, monotonic sequence is convergent.

25 Series

A series is the sum of the terms of a sequence.

If $\lim_{x\to c} a_n \neq 0$, then the series $\sum a_n$ is divergent.

The nth term of a geometric series of the form

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \dots \tag{224}$$

is given by

$$s_n = \frac{a(1-r^n)}{1-r}. \tag{225}$$

A geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$.

The power series

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \tag{226}$$

converges if $p > 1$ and diverges if $p \leq 1$.

25.1 Integral Test

If $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, then the series

$$\sum_{n=1}^{\infty} f(n) \tag{227}$$

is convergent if and only if the improper integral

$$\int_1^{\infty} f(x)dx \tag{228}$$

is convergent.

25.2 Remainder Estimation

Suppose $f(k) = a_k$ where f is a continuous, positive, and decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Let s be the sum of the infinite series, and s_n be the sum of the first n terms. If the remainder, $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx. \tag{229}$$

25.3 Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

25.4 Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \tag{230}$$

where c is a finite positive number, then either both series converge or both series diverge.

25.5 Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \quad b_n > 0 \tag{231}$$

satisfies

1. $b_{n+1} \leq b_n$ for all n
2. $\lim_{n\to\infty} b_n = 0$

then the series converges.

Proof

Considering the even partial sums, $\{s_{2n}\}$, we find that

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}. \tag{232}$$

Thus,

$$0 \geq s_2 \geq s_4 \geq s_6 \geq \dots \geq s_{2n} \geq \dots. \tag{233}$$

However, we can also write the general term as

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots - (b_{2n-2} - b_{2n-1}) - b_{2n}. \tag{234}$$

All the terms in parentheses are positive, and $b_{2n} \geq 0$. Therefore, by the monotonic sequence theorem, because $\{s_{2n}\}$ is an increasing sequence which is bounded above, it converges. Now, we need to prove that the odd terms, $\{s_{2n+1}\}$, converge as well. Suppose

$$\lim_{n\to\infty} s_{2n} = s. \tag{235}$$

Then,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s. \tag{236}$$

25.6 Alternating Series Remainder Estimation

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

1. $b_{n+1} \leq b_n$ for all n
2. $\lim_{n\to\infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}. \tag{237}$$

25.7 Absolute Convergence

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent.

A series is **conditionally convergent** if it is convergent but not absolutely convergent.

25.8 Ratio Test

1. If
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \tag{238}$$
then the series $\sum a_n$ is absolutely convergent.

2. If
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \tag{239}$$
then the series $\sum a_n$ is divergent.

3. If
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L = 1 \tag{240}$$
then the test is inconclusive.

25.9 Root Test

1. If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \tag{241}$$
then the series $\sum a_n$ is absolutely convergent.

2. If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \tag{242}$$
then the series $\sum a_n$ is divergent.

3. If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1 \tag{243}$$
then the test is inconclusive.

25.10 Power Series

For a given power series $\sum_{n=0}^{\infty} c_N (x-a)^n$, there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There exists a number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

For a function represented by a power series, is differentiable within the interval of convergence.

25.11 Taylor Series

The Taylor Series can be defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{244}$$

The Maclaurin Series is a special case of the Taylor Series where $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{245}$$

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq R$, then the remainder $R_n(x)$ of the Taylor Series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}. \tag{246}$$

Below are some common Maclaurin Series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad R = 1 \tag{247}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad R = \infty \tag{248}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad R = \infty \tag{249}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad R = \infty \tag{250}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad R = 1 \tag{251}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad R = 1 \tag{252}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots \qquad R = 1 \tag{253}$$

25.12 Fourier Series

The Fourier Series of a function $f(x)$ is of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{254}$$

To solve for the coefficients, we can start by integrating both sides of the equation.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)dx \tag{255}$$

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right) \tag{256}$$

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 + 0 \tag{257}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \tag{258}$$

To solve for a_n for $n \geq 1$, we can multiply both sides of the equation by $\cos mx$ and integrate.

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} \left(a_0 \cos mx + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos mx \right) dx \tag{259}$$

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx \right) \tag{260}$$

$$\tag{261}$$

It is not hard to show that

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases} \tag{262}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0. \tag{263}$$

Thus,

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = 0 + a_m \pi \tag{264}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \tag{265}$$

Similarly, we can solve for b_n by multiplying both sides of the equation by $\sin mx$ and integrating.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \tag{266}$$

The Fourier series can apply to a wider class of functions: a piecewise continuous function with a finite number of discontinuities. The Fourier series of a square-wave function can be represented as

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{2n-1}. \tag{267}$$

If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series is convergent. At the points where f is discontinuous, the Fourier series converges to the average of the left-hand and right-hand limits of f.

If f has a period other than 2π, then an u-substitution can be applied: $x = \frac{Lt}{\pi}$, where $2L$ is the period of f.

The Fourier series of a function $f(x)$ can be represented as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{268}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \tag{269}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \tag{270}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx. \tag{271}$$

26 Vectors and Geometry

The distance between a point and a plane is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{272}$$

A cylinder is a surface that consists of all lines called rulings that are parallel to a given line and pass through a given plane curve.

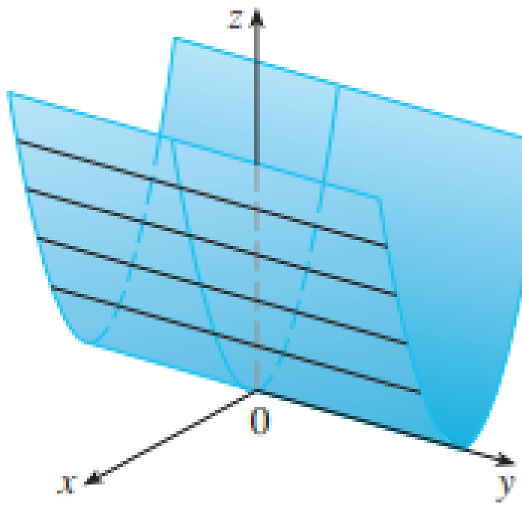


Figure 2: The surface $z = x^2$ is a cylinder (Image Source: Stewart's Textbook).

A quadric surface is the graph of a second-degree equation in three variables. The general form of a quadric surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0. \tag{273}$$

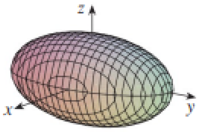
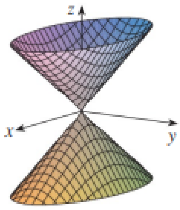
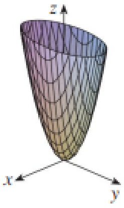
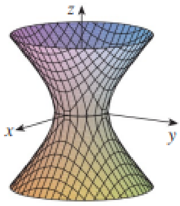
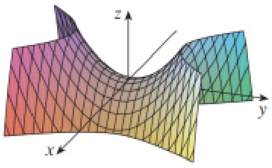
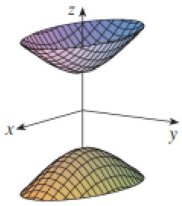
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Figure 3: Graphs of quadric surfaces (Image Source: Stewart's Textbook).

27 Vector Functions

A vector function is a function that takes a real number as input and gives a vector as output. A vector function can be written as

$$\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle. \tag{274}$$

The same limits and continuity properties also apply to vector functions.

$$\lim_{t\to c}\mathbf{r}(t)=\langle\lim_{t\to c}f(t),\lim_{t\to c}g(t),\lim_{t\to c}h(t)\rangle. \tag{275}$$

The set of all points in space where $x=f(t)$, $y=g(t)$, and $z=h(t)$ is called the space curve.

27.1 Derivatives and Integrals

$$\mathbf{r}'(t)=\langle f'(t),g'(t),h'(t)\rangle \tag{276}$$

$$\int\mathbf{r}(t)dt=\left\langle\int f(t)dt,\int g(t)dt,\int h(t)dt\right\rangle \tag{277}$$

The same integration and derivative rules apply to vector functions as well.

27.2 Arc Length

The arc length of a vector function $\mathbf{r}(t)$ is given by

$$L = \int_a^b |\mathbf{r}'(t)|\, dt. \tag{278}$$

A parameterization $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t) \neq \mathbf{0}$ and $\mathbf{r}'(t)$ is continuous.

If $\mathbf{r}(t)$ is smooth, the unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}. \tag{279}$$

Curvature is now defined as

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{280}$$

$$= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}. \tag{281}$$

For a special case of $y = f(x)$, the curvature is given by

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}. \tag{282}$$

The normal vector is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}. \tag{283}$$

Since $T \cdot T = 1$, after taking the derivative of both sides, we get $T' \cdot T + T \cdot T' = 0$. Thus, $T' \cdot T = 0$.

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the binormal vector.

The normal plane is normal to $\mathbf{T}(t)$, and the osculating plane is normal to $\mathbf{B}(t)$.

27.3 Velocity and Acceleration

$$\mathbf{v}(t) = \mathbf{r}'(t) \tag{284}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \tag{285}$$

However, it is often useful to represent acceleration in terms of the unit tangent and normal vectors.

$$\mathbf{a}(t) = v'\mathbf{T} + \kappa v^2 \mathbf{N} \tag{286}$$

$$v' = \frac{\mathbf{a} \cdot \mathbf{v}}{v} \tag{287}$$

28 Multivariable Functions

28.1 Limits

If f is defined on a subset D of $\mathbf{R}^n$, then $\lim_{x\to a} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon. \tag{288}$$

28.2 Partial Derivatives

The partial derivative of f with respect to x is

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}. \tag{289}$$

Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a,b). If the partial derivatives f_x and f_y are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b). \tag{290}$$

28.3 Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0). \tag{291}$$

28.4 Linear Approximation

If $z = f(x,y)$, the f is differentiable at (a,b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{292}$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

The increment Δz represents the change in value of f when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.

28.5 Continuity

If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

28.6 Differentials

The total differential is defined by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \tag{293}$$

28.7 Chain Rules

Suppose that $z = f(x,y)$, $x = g(t)$, and $y = h(t)$ are differentiable functions. Then,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{294}$$

Suppose that $z = f(x,y)$, $x = g(s,t)$, and $y = h(s,t)$ are differentiable functions. Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}. \tag{295}$$

28.8 Implicit Differentiation

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \tag{296}$$

28.9 Directional Derivatives

The directional derivative of f at (x, y) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h} \tag{297}$$

$$= f_x(x, y)a + f_y(x, y)b. \tag{298}$$

28.10 Gradient Vector

The gradient of f is the vector function

$$\nabla f = \langle f_x, f_y \rangle. \tag{299}$$

The directional derivative can be expressed as the dot product of the gradient vector and the direction vector.

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \tag{300}$$

The directional derivative is maximized when $\mathbf{u}$ is in the direction of the gradient vector.

Proof of the Existence of the Gradient Vector

$o(\vec{h})$ is a function such that

$$\lim_{\vec{h} \to \vec{0}} \frac{o(\vec{h})}{|\vec{h}|} = 0. \tag{301}$$

f is differentiable $\vec{x}$ if and only if there exists a gradient vector $\nabla f(\vec{x})$ such that

$$f(\vec{x} + \vec{h}) - f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{h} + o(\vec{h}). \tag{302}$$

Clairaut's Theorem can be used to determine if a vector function is a gradient. $\nabla = \langle P, Q \rangle$ is a gradient if and only if $P_y = Q_x$.

28.11 Tangential Planes to Level Surfaces

At a point (x_0, y_0, z_0) on the surface of $F(x, y, z) = k$, the tangential plane is defined as,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{303}$$

28.12 Maximum and Minimum Values

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

28.13 Second Derivatives Test

Suppose that the second partial derivatives of f are continuous on a disk D that contains the point (a, b) and that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2. \tag{304}$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
3. If $D < 0$, then f has a saddle point at (a, b) which is not a local minimum or maximum.

28.14 Extreme Values

If f is continuous on a closed, bounded set D in $\mathbf{R}^2$, then f has both a maximum and a minimum value on D at some points in D.

28.15 Lagrange Multipliers

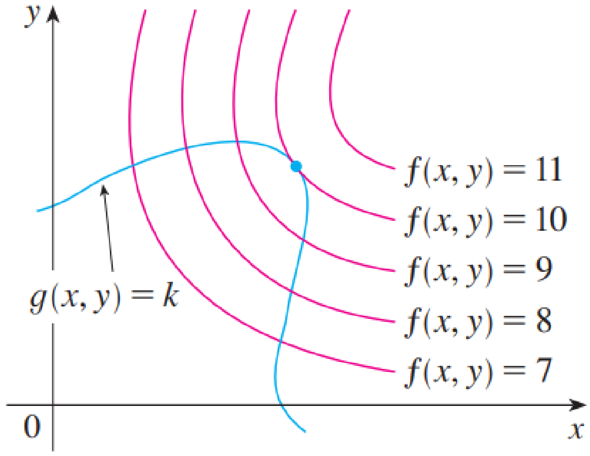


Figure 4: A visualization of the level curves of $f(x, y)$ and the constraint function $g(x, y)$ (Image Source: Stewart's Textbook).

The intuitive idea behind Lagrange multipliers is that the maximum or minimum of a function subject to a constraint occurs when the gradient of the function is parallel to the gradient of the constraint function.

$$\nabla f(x, y) = \lambda \nabla g(x, y) \tag{305}$$

Using this formula and the constraint function, we can solve for the critical points where there might be a maximum or minimum.

28.16 Differentiation with Respect to a Parameter

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(x, t) dt = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt \tag{306}$$

If $a(x)$ and $b(x)$ are constants, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a}^{b} f(x, t) dt = \int_{a}^{b} \frac{\partial}{\partial x} f(x, t) dt. \tag{307}$$