Modern Physics Notes

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1 Quantization of Light

Something is **quantized** if it can only occur in discrete amounts.

Max Planck was the first person to propose that electromagnetic radiation is quantized with his studies of blackbody radiation.

He assumed that radiation of frequency f can be emitted only in integral multiples of basic quantum hf where $h = 6.626 \times 10^{-34}$ Js.

$$E = 0, hf, 2hf, 3hf, \ldots \tag{1}$$

Later on Einstein showed that the photoelectric effect can only be explained if light is quantized.

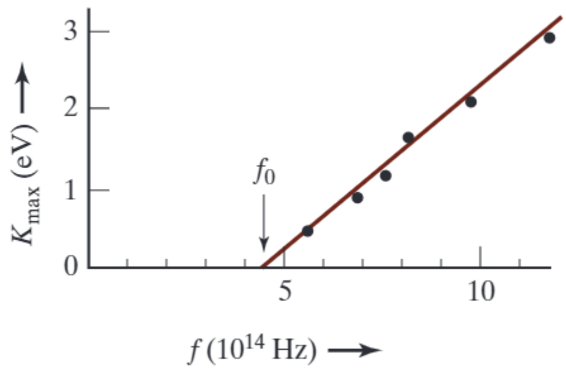

Figure 1: A plot of energy of the ejected electron vs the frequency of light.

When a photon of light hits a metal surface, it only ejects an electron if the energy of the photon is greater than the work function of the metal defined by

$$\phi = hf_0. \tag{2}$$

Therefore, the energy of the ejected electron is given by

$$K_{\max} = hf - \phi. \tag{3}$$

The current depends linearly on the intensity of the light.

The final piece of evidence for quantization of light was the Compton effect. Compton assumed that the photons are particles with energy and momentum and during collisions, energy and momentum are conserved.

He assumed that the photon is a particle with energy $E = hf$ and momentum $p = \frac{hf}{c}$.

Compton related the change in wavelength with angle of scattering using the equation,

$$\Delta\lambda = \lambda - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta), \tag{4}$$

and he found the equation to be consistent with his experimental results.

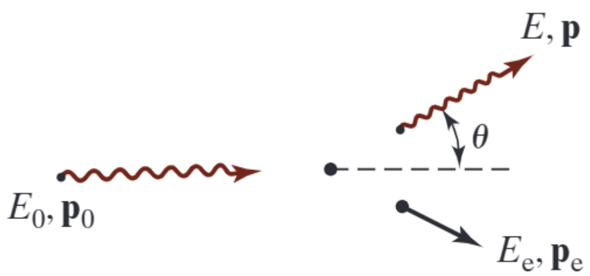

Figure 2: Compton Scattering.

2 Quantization of Atomic Energy Levels

By the 19th century, it was known that the atoms and molecules of any one chemical species emit and absorb light a wavelengths characteristic of that species.

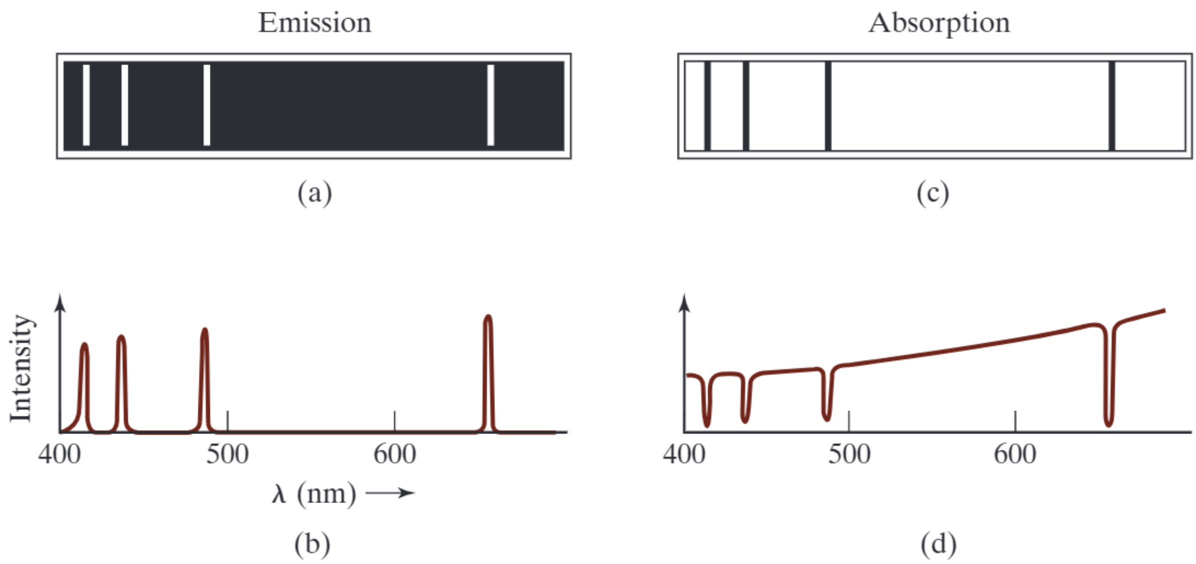

Figure 3: Absorption and emission spectra of hydrogen.

Johann Balmer found that the wavelengths of hydrogen fitted the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{n^2}\right). \tag{5}$$

Rydberg generalized this formula to

$$\frac{1}{\lambda} = R\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right), \tag{6}$$

where R is the Rydberg constant and n are integers.

Bohr tried to explain this phenomenon of energy levels by assuming that electrons orbit around the nucleus based on electrostatic forces.

$$mv^2 = \frac{ke^2}{r^2} \tag{7}$$

Furthermore, he assumed that the angular momentum of the electron is quantized,

$$mvr = n\hbar, \tag{8}$$

where $\hbar = \frac{h}{2\pi}$. Thus,

$$r = \frac{n^2\hbar^2}{ke^2m} = n^2 a_B \tag{9}$$

where a_B is the Bohr radius.

Substituting this into the equation for energy, Bohr found that

$$E_n = -\frac{ke^2}{2r} = -\frac{ke^2}{2a_B n^2} = -\frac{hcR}{n^2} = -\frac{E_R}{n^2}, \tag{10}$$

which has precisely the form of the Rydberg formula. E_R is called the Rydberg energy.

$$E = \frac{ke^2}{2a_B}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) = E_R\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right). \tag{11}$$

Furthermore, Morsely confirmed this by measuring frequencies of X-rays emitted by atoms.

He found that $\sqrt{f} \propto (Z-1)$, so $E \propto (Z-1)^2$.

3 Matter Waves

De Broglie proposed that material particles, such as electrons, should show a particle wave duality just like photons.

$$E = hf \quad \text{and} \quad p = \frac{h}{\lambda} \tag{12}$$

λ is called the de Broglie wavelength and p is the momentum of the particle.

This was verified by Davisson and Germer who showed that electrons can be diffracted by a crystal lattice.

3.1 The Quantum Wave Function

Born found that the light energy E in any small volume $\mathrm{d}V$ at $\mathbf{r}$ is proportional to the electromagnetic wave function squared times change in volume.

$$E \propto [\varepsilon(\mathbf{r}, t)]^2 \, \mathrm{d}V \tag{13}$$

In other words,

$$\text{probable number of photons in } \mathrm{d}V \text{ at } \mathbf{r} \propto [\varepsilon(\mathbf{r}, t)]^2 \, \mathrm{d}V. \tag{14}$$

Born proposed that a similar relation should apply to electron waves and other matter waves:

$$\text{probable number of particles in } \mathrm{d}V \text{ at } \mathbf{r} \propto [\Psi(\mathbf{r}, t)]^2 \, \mathrm{d}V. \tag{15}$$

The wave function is generally complex. Furthermore, Ψ is continuous and single valued. The intensity of the wave is defined as $|\Psi|^2$ which gives the probability of finding a particle at any particular position.

3.2 The Uncertainty Relation for Position and Momentum

From the de Broglie relation,

$$p = \frac{h}{\lambda} = \hbar k \tag{16}$$

$$\Delta p = \hbar \Delta k. \tag{17}$$

Furthermore, mathematically it can be proven that the following relation holds true,

$$\Delta x \Delta k \geq \frac{1}{2}. \tag{18}$$

Combining these two equations, we get the Heisenberg uncertainty principle,

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{19}$$

Similarly, the time-energy uncertainty relation is

$$\Delta E \Delta t \geq \frac{\hbar}{2}. \tag{20}$$

As photon wavelength decreases, the uncertainty in position decreases.

As photon wavelength decreases, the uncertainty in momentum increases.

4 The One Dimensional Schrödinger Equation

The general standing wave of a quantum system can be represented in the form

$$\Psi(x,t) = \psi(x)e^{-i\omega t} \tag{21}$$

where $\psi(x)$ is the spatial part of the wave function and ω is the angular frequency. The state of a quantum system is completely described by its wave function.

$$\int |\Psi(x,t)|^2 \, \mathrm{d}x = 1. \tag{22}$$

The differential equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \frac{2m}{\hbar^2}\left[U(x) - E\right]\psi \tag{23}$$

is known as the time independent Schrödinger equation where ψ is the wavefunction only dependent on x, $U(x)$ is the potential energy, and E is the total energy of the system. This equation is basically the $F = ma$ equation of the quantum world.

4.1 Quantum Operators

For every classical observable, there is a quantum mechanical counterpart that is an operator.

Table 1: Quantum Operators.

Classical Observable	Operator	1D Representation
Position	$\hat{x}$	x
Momentum	$\hat{p}$	$-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$
Kinetic Energy	$\hat{T}$	$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
Potential Energy	$\hat{U}$	$U(x)$
Total Energy	$\hat{H}$	$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+U(x)$

For a system described by a wavefunction, the average value of an observable $\hat{Q}$ is

$$\langle\hat{Q}\rangle = \int \Psi^*\hat{Q}\Psi \, \mathrm{d}x. \tag{24}$$

Using this expected value notation, we can also write standard deviation as

$$\sigma_Q = \sqrt{\langle\hat{Q}^2\rangle - \langle\hat{Q}\rangle^2}. \tag{25}$$

4.2 The Particle within a Rigid Box

Let us determine the allowed energies of a particle in a rigid box of length a using Schrödinger's equation. The potential energy of a rigid box is defined to be zero inside the box and infinite outside the box.

$$U(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & x < 0 \text{ or } x > a \end{cases} \tag{26}$$

Schrödinger's equation reduces to

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{2mE}{\hbar^2}\psi. \tag{27}$$

We will consider two cases: $E \geq 0$ and $E < 0$. The continuity of ψ requires that $\psi(0) = \psi(a) = 0$.

If $E < 0$, then we can define $\alpha = \sqrt{-2mE}/\hbar$, so

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \alpha^2\psi. \tag{28}$$

The general solution to this equation is

$$\psi(x) = Ae^{\alpha x} + Be^{-\alpha x}. \tag{29}$$

Using the boundary conditions, we find that $A = B = 0$. This contradicts the assumption that $E < 0$, meaning that $E < 0$ is not possible.

For the second case, $E \geq 0$, we can define $k = \sqrt{2mE}/\hbar$, so the general solution to this equation is

$$\psi(x) = A\sin kx + B\cos kx. \tag{30}$$

We see that based on boundary conditions, k must be multiple of $\frac{\pi}{a}$.

Thus,

$$E = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2n^2\pi^2}{2ma^2}. \tag{31}$$

4.3 The Free Particle

The second application of the Schrödinger equation is to investigate the possible energies of a free particle. The potential energy of a free particle is constant and can be chosen to be zero. This simplifies the Schrödinger equation to

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{2mE}{\hbar^2}\psi. \tag{32}$$

We will look for solutions for all x in the range $(-\infty, \infty)$. Similar to the particle in a rigid box, we will consider two cases: $E \geq 0$ and $E < 0$.

If $E < 0$, then we can let $\alpha = \sqrt{-2mE}/\hbar$, so the general solution to this equation is

$$\psi(x) = Ae^{\alpha x} + Be^{-\alpha x}. \tag{33}$$

When $x \to \pm\infty$, $\psi(x)$ blows up, so the only possible value for the constants is $A = B = 0$. This contradicts the assumption that $E < 0$.

If $E \geq 0$, we can let $k = \sqrt{2mE}/\hbar$, so the general solution this time will be of the form

$$\begin{align} \psi(x) &= A\sin kx + B\cos kx \tag{34} \\ &= Ce^{ikx} + De^{-ikx}. \tag{35} \end{align}$$

The key thing to note here is taht neither term blows up as $x \to \pm\infty$. Furthermore, rather than having discrete energy levels, like the particle in a rigid box, the free particle has a continuous spectrum of energies since there are no boundary conditions.

4.4 The Particle in a Non-Rigid Box

For a non-rigid box, the potential energy can be expressed

$$U(x) = \begin{cases} 0 & 0 \leq x \leq a \\ U_0 & x < 0 \text{ or } x > a \end{cases} \tag{36}$$

where the potential energy U_0 outside the box is finite. However, this model can be improved by accounting for the fact that the potential energy transition between the box and the outside is smooth like a rounded well and not sudden as in the definition above.

Furthermore, to simplify the rounded well definition, we shall assume that the potential is constant $U(x) = U_0$ for $x < 0$ and $x > a$.

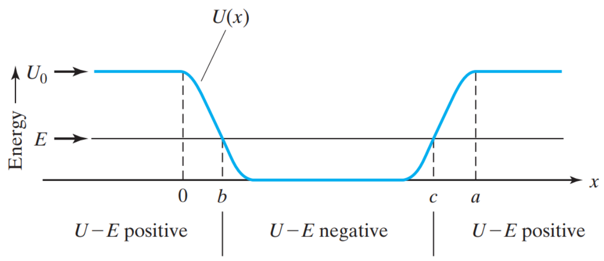

Figure 4: Realistic rounded well potential energy function for a non-rigid box.

The turning points $x = b$ and $x = c$ are known as the turning points, and the regions $x < b$ and $x > c$ are known as the classically forbidden regions since classical particles with energy E cannot penetrate there.

Using the methods above, we find that the wave function of the free particle can be expressed in the following form:

$$\psi(x) = \begin{cases} Ae^{\alpha x} + Be^{-\alpha x}, & E < U_0, \quad \alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} \\ Ce^{ikx} + De^{-ikx}, & E > U_0, \quad k = \sqrt{\frac{2m(E - U_0)}{\hbar^2}} \end{cases} \tag{37}$$

4.5 Quantum Tunneling

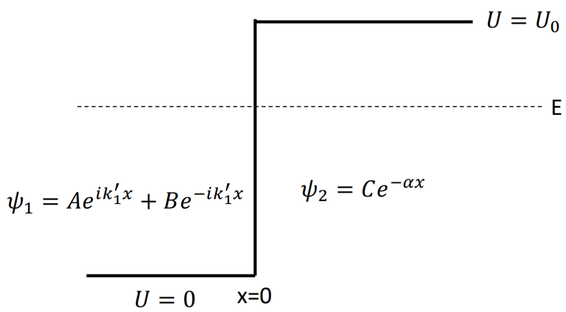

Figure 5: A potential step function.

Consider the above potential step function, to determine the coefficients A, B, and C, we would need three equations. Firstly, the energy reflection coefficient $R_e = |B|^2/|A|^2$ and $R_e + T_e = 1$ can be used to determine A and B. Secondly, the wave function must be continuous at the boundary, so $\psi_1(0) = \psi_2(0)$ and $\psi_1'(0) = \psi_2'(0)$. This is known as an infintely long barrier. However, if the barrier is finite, then there is a probability that the particle can tunnel through the barrier.

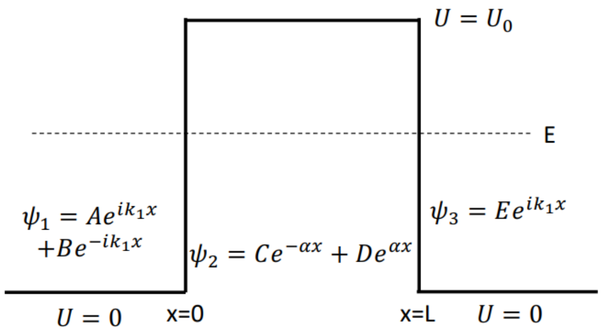

Figure 6: A finite barrier.

The probability of tunneling is

$$T \approx 16 \left(\frac{E}{U_0}\right)\left(1 - \frac{E}{U_0}\right) e^{-2\alpha L} \tag{38}$$

where

$$\alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}. \tag{39}$$

5 Special Relativity

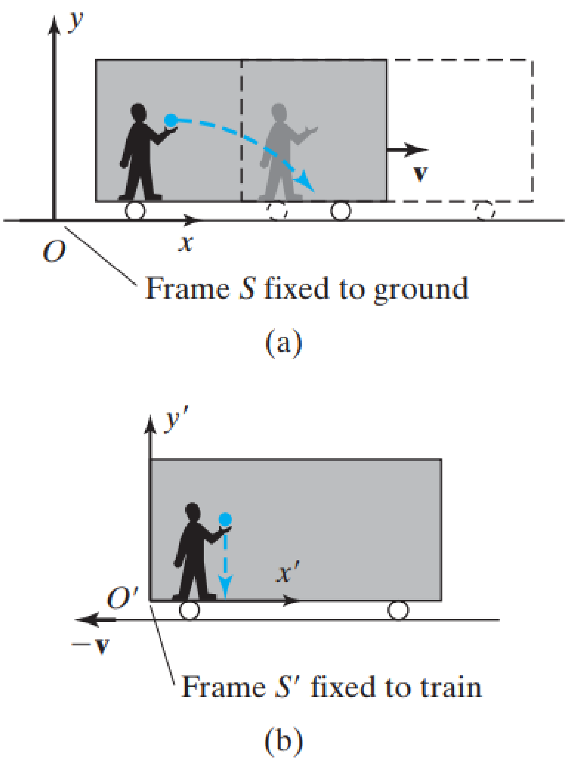

Figure 7: Motion of the ball as seen by an observer on the ground (a) and an observer on the train (b).

In classical relativity, the velocity of the ball relative to the person on the ground can be calculated by the equation

$$v_{BP} = v_{BT} + v_{TP}. \tag{40}$$

However, this fails to explain why explains why the speed of light $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is constant in all inertial frames of reference as found in the Michealson-Morley experiment.

5.1 The Postulates of Special Relativitiy

1. The laws of Newtonian mechanics hold in an entire family of reference frames, any one of which moves uniformly relative to any other.
2. There can be only one reference frame in which light travels at the same speed c in all directions (and, more generally, in which all laws of electromagnetism are valid).

5.2 Time Dialation, Length Contraction

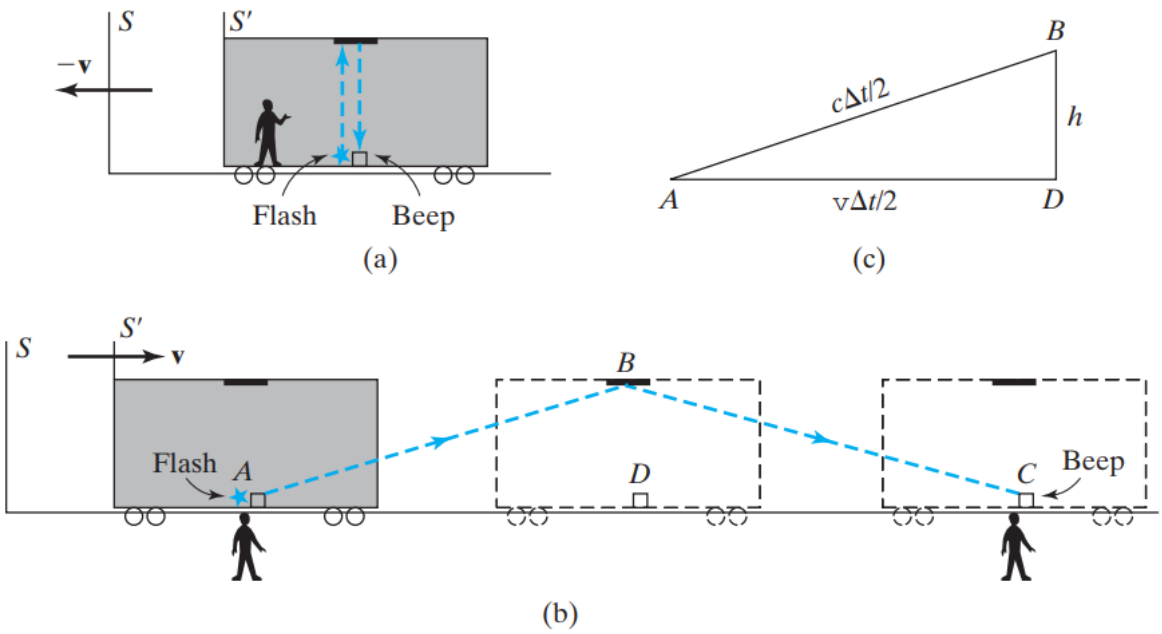

Figure 8: A person inside the train (a) and a person on the ground (b) both measures the time it takes for light to reach a sensor in a train.

In the figure above, the person in the train would measure that the time taken would be $\Delta t' = \frac{2h}{c}$. $\Delta t'$ os often known as proper time. However, this is not the same as the time measured by the person on the ground. By the pythagorean theorem,

$$\left(\frac{c\Delta t}{2}\right)^2 = h^2 + \left(\frac{v\Delta t}{2}\right)^2 \tag{41}$$

$$\Delta t = \frac{2h}{\sqrt{c^2 - v^2}}. \tag{42}$$

This can be expressed as

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} = \gamma \Delta t' \tag{43}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

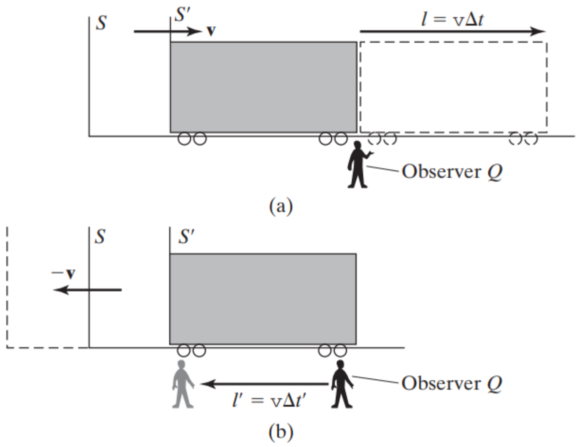

Figure 9: A train moving a distance of $v\Delta t$ to the right as observed by a person at rest (a). A person appearing to move left taking the frame of reference of the train (b).

A similar expression can be found for length contraction. For an observer on the train, the distance would be measured to be $l' = v\Delta t'$. For the observer on the ground, the distance would be measured to be $l = v\Delta t$. Therefore, these two equations imply that

$$l = \frac{l'}{\gamma}. \tag{44}$$

5.3 The Lorentz Transformation

The Lorentz transformation calculates the coordinates x', y', z', and t' of an event in a frame S' using coordinates x, y, z, and t measured in another frame S.

$$x' = \gamma(x - vt) \tag{45}$$
$$y' = y \tag{46}$$
$$z' = z \tag{47}$$
$$t' = \gamma(t - \frac{vx}{c^2}) \tag{48}$$

Using this transformation, we can determine the relative velocity between two frames of reference.

$$u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \tag{49}$$
$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \tag{50}$$
$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \tag{51}$$

5.4 The Doppler Effect

The Doppler effect is the change in frequency of a wave in relation to an observer moving relative to the source of the wave. When the source is approaching the observer, the observed frequency is

$$f_{\text{obs}} = f_{\text{source}}\sqrt{\frac{1+\beta}{1-\beta}}. \tag{52}$$

When the source is moving away from the observer, the observed frequency is

$$f_{\text{obs}} = f_{\text{source}}\sqrt{\frac{1-\beta}{1+\beta}}. \tag{53}$$

5.5 Relativistic Momentum, Mass, and Energy

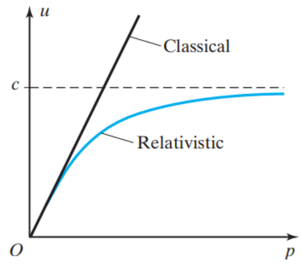

Figure 10: In classical mechanics, the velocity exceeds the speed of light as momentum grows. However, in relativity, the velocity cannot exceed the speed of light.

The relativistic mass is defined as

$$m_{\text{var}} = \frac{m_0}{\sqrt{1-\beta^2}} \tag{54}$$

the relativistic momentum is

$$p = m_{\text{var}} v, \tag{55}$$

and the relative energy is

$$E = \gamma m_0 c^2. \tag{56}$$

Since all bodies must have energy even at rest $E = mc^2$, the relativistic Kinetic energy is

$$K = (\gamma - 1) m_0 c^2. \tag{57}$$

The relation between relativistic momentum and energy is

$$E = \sqrt{(pc)^2 + (mc^2)^2}. \tag{58}$$