Fluid Mechanics Notes

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1 Dimensional Analysis

Dimensional analysis is a method of reducing the number of variables in a problem by considering the dimensions of the variables.

Primary dimensions are dimensions that cannot be formed by other dimension. In fluid mechanics, typically the primary dimensions we will use is mass (M), length (L), and time (T).

Secondary dimensions are dimensions that can be formed by the primary dimensions. For example density (ρ) has dimensions of $\frac{M}{L^3}$, and viscosity (μ) has dimensions of $\frac{M}{LT}$.

The Buckingham Pi theorem states that if an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k - r$ independent dimensionless products (π terms), where r is the minimum number of reference dimensions required to describe the variables involved in the problem.

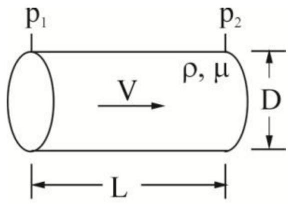

Figure 1: Steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal pipe.

Suppose we want to determine the pressure drop per unit length along the pipe. The following are steps to solve the problem using dimensional analysis:

1. List out important variables in the problem: pressure drop per unit length $\Delta p/L$, average flow velocity $\mathbf{V}$, pipe diameter D, fluid density ρ, and fluid viscosity μ.

2. Express the variables in terms of primary dimensions:

$$[\Delta p/L] = ML^{-2}T^{-2} \tag{1}$$
$$[\mathbf{V}] = LT^{-1} \tag{2}$$
$$[D] = L \tag{3}$$
$$[\rho] = ML^{-3} \tag{4}$$
$$[\mu] = ML^{-1}T^{-1} \tag{5}$$

3. Determine the number of π terms: $k = 5$ variables, $r = 3$ primary dimensions, so $k - r = 2$ π terms.

4. Select three repeating variables: ρ, $\mathbf{V}$, and D. The three repeating variables must have independent dimensions, and the dependent variable $\Delta p/L$ cannot be a repeating variable.

5. Form the π terms:

$$\pi_1 = \left(\frac{\Delta p}{L}\right) \rho^a \mathbf{V}^b D^c = ML^{-2}T^{-2}(ML^{-3})^a(LT^{-1})^b L^c = M^{1+a}L^{-2-3a+b+c}T^{-2-b} = 1 \tag{6}$$

Solving the above equation, we get $a = -1$, $b = -2$, and $c = 1$. Therefore,

$$\pi_1 = \frac{(\Delta p/L)\, D}{\rho \mathbf{V}^2}. \tag{7}$$

The same process can be done for π_2.

$$\pi_2 = \mu \rho^a \mathbf{V}^b D^c = ML^{-1}T^{-1}(ML^{-3})^a(LT^{-1})^b L^c = M^{1+a}L^{-1-3a+b+c}T^{-1-b} = 1 \tag{8}$$
$$\pi_2 = \frac{\mu}{\rho \mathbf{V} D} \tag{9}$$

6. Therefore, the pressure drop per unit length can be determined by the relationship between π_1 and π_2.

$$\pi_1 = f(\pi_2) \tag{10}$$

$$\frac{(\Delta p/L)\, D}{\rho \mathbf{V}^2} = f\left(\frac{\mu}{\rho \mathbf{V} D}\right) \tag{11}$$

2 Introduction

A fluid is a substance that deforms continuously under the application of a shear stress.

2.1 Fluid Analysis Methods

There are two approaches to analyzing fluid flow: statistical and continuum. The statistical approach treats the fluid as a collection of molecules, and the continuum approximation treats the fluid as a continuous substance.

The Knudsen number, Kn, is a dimensionless parameter which indicates the validity of the continuum approximation.

$$Kn = \frac{\text{microscopic length scale}}{\text{macroscopic length scale}} \tag{12}$$

When $Kn \ll 1$, the continuum approximation is valid.

2.2 Forces

There are two types of forces acting on a fluid element: body forces and surface forces. Body forces are forces that act on the fluid element without physical contact, such as gravity. Surface forces are forces that act on the fluid element through physical contact, such as pressure.

Surface forces consist of normal forces, which act perpendicular to the surface, and shear forces, which act parallel to the surface.

2.3 Viscosity

In Newtonian fluids, the shear stress is proportional to the rate of deformation.

$$\tau = \mu \frac{du}{dy} \tag{13}$$

where τ is the shear stress, μ is the dynamic viscosity, u is the velocity of the fluid, and y is the distance from the surface.

Non-Newtonian fluids can be further classified by how the shear stress varies with deformation rate.

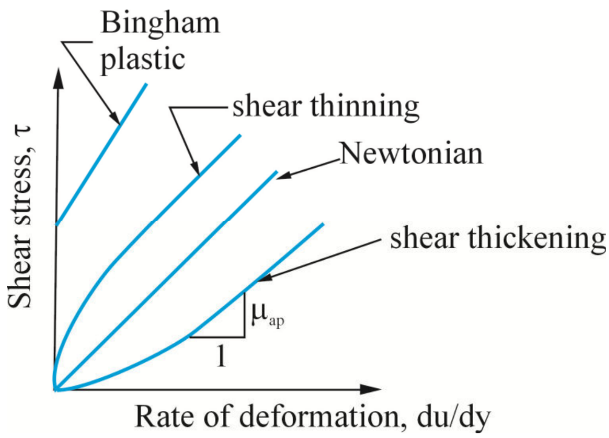

Figure 2: Variation of shear stress and strain in different fluids.

In liquids, an increase in temperature results in a decrease in viscosity because at higher temperature, the molecules have more kinetic energy to overcome the intermolecular forces. In gases, an increase in temperature results in an increase in viscosity because at higher temperature, the molecules collide more frequently.

Kinematic viscosity is the ratio of dynamic viscosity to density.

$$\nu = \frac{\mu}{\rho} \tag{14}$$

2.4 Compressibility

The compressibility of fluids can be characterized by the bulk modulus E_V.

$$E_V = -\frac{dp}{dV/V} \tag{15}$$

dp is the change in pressure and dV/V is the fractional change in volume.

The larger the bulk modulus, the less compressible the fluid.

3 Hydrostatics

The pressure field equation for a fluid in which there are no shear stresses is

$$-\nabla p - \rho g\hat{k} = \rho\vec{a} \tag{16}$$

where $\vec{a}$ is the acceleration of the fluid element, ρ is the density of the fluid, g is the acceleration due to gravity, and p is the pressure. In other words, the equation states that

$$-\text{Pressure Force} - \text{Gravity Force} = \text{Net Force}. \tag{17}$$

When the fluid is at rest, $\vec{a} = 0$, so the equation simplifies to

$$\frac{dp}{dz} = -\rho g \tag{18}$$

If this is an incompressible fluid, where ρ and g are constant, then or

$$p_1 - p_2 = \rho g(z_2 - z_1) \tag{19}$$

where p_1 and p_2 are the pressures at heights z_1 and z_2 respectively. For compressible fluids, density can be substituted using the ideal gas equation $\rho = \frac{p}{RT}$, so

$$p_2 = p_1 e^{-\frac{g(z_2 - z_1)}{RT_0}} \tag{20}$$

3.1 Hydrostatic Forces on a Gate

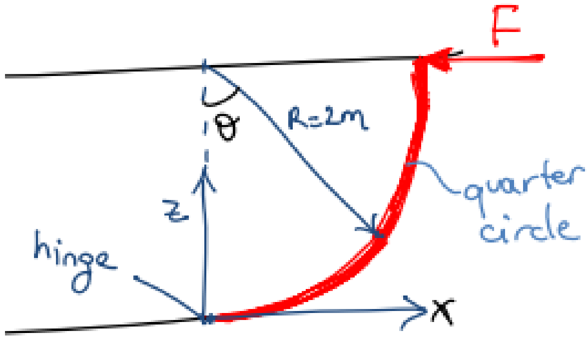

Figure 3: A circular gate holding up water with radius $R = 2\,\text{m}$ and a width $w = 4\,\text{m}$.

Suppose we want to determine the force required to hold the gate in place as shown in the figure above. The first method is to use the pressure prism.

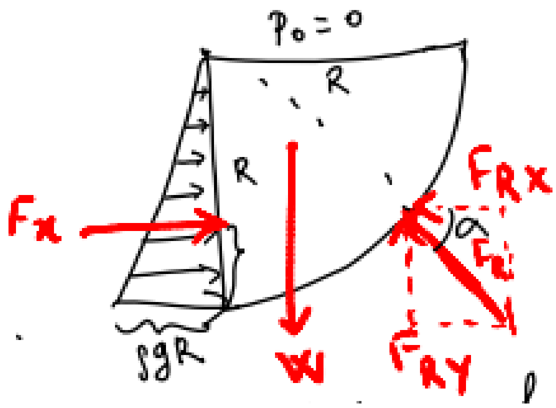

Figure 4: Free body diagram of the liquid above the gate.

$$F_{RX} = \rho g R^2 w/2 = 120000\,\text{N} \tag{21}$$

$$F_{RY} = \rho g \frac{\pi R^2}{4} w = 60000\pi\,\text{N} \tag{22}$$

$$F_R = \sqrt{F_{RX}^2 + F_{RY}^2} = 223451.5\,\text{N} \tag{23}$$

$$\alpha = \tan^{-1}\left(\frac{F_{RY}}{F_{RX}}\right) = 57.5^\circ \tag{24}$$

$$M_{\text{hinge}} = F_R R \cos\alpha \tag{25}$$

$$F = \frac{M_{\text{hinge}}}{R} = 120000\,\text{N} \tag{26}$$

The second method is to parameterize the surface and integrate. The vector equation for the surface is

$$\vec{r}(\theta, y) = R\sin\theta\hat{i} + y\hat{j} + (R - R\cos\theta)\hat{k}. \tag{27}$$

The partial derivatives are

$$\vec{r}_\theta = R\cos\theta\hat{i} + R\sin\theta\hat{k} \tag{28}$$

$$\vec{r}_y = \hat{j} \tag{29}$$

Their cross product is

$$\vec{r}_\theta \times \vec{r}_y = -R\sin\theta\hat{i} + R\cos\theta\hat{k}. \tag{30}$$

Therefore, the moment can be calculated as

$$\begin{aligned}
M &= \iint_S (\vec{r} \times \vec{n}) \cdot \hat{j}(-p\,\mathrm{d}S) &(31)\\
&= \iint_S \left(\vec{r} \times \frac{\vec{r}_\theta \times \vec{r}_y}{|\vec{r}_\theta \times \vec{r}_y|}\right) \cdot \hat{j}(-p\,|\vec{r}_\theta \times \vec{r}_y|\,\mathrm{d}\theta\,\mathrm{d}y) &(32)\\
&= \iint_S (\vec{r} \times (\vec{r}_\theta \times \vec{r}_y)) \cdot \hat{j}(-p\,\mathrm{d}\theta\,\mathrm{d}y) &(33)\\
&= \int_0^{\pi/2}\int_0^w \rho g R^3 \sin\theta\cos\theta\,\mathrm{d}y\,\mathrm{d}\theta. &(34)
\end{aligned}$$

And the force is calculated to be 120000 N, the same as before.

3.2 Buoyancy

Archimedes' principle states that the buoyant force on an object is equal to the weight of the fluid displaced by the object.

$$F_{\text{buoyant}} = \rho g V_{\text{displaced}} \tag{35}$$

Stability of a body can be determined what happens when it is displaced from its equilibrium position.

A fully submerged body is stable if the center of gravity is below the center of buoyancy.

The stability of a partially submerged body depends on its shape.

3.3 Linear Motion

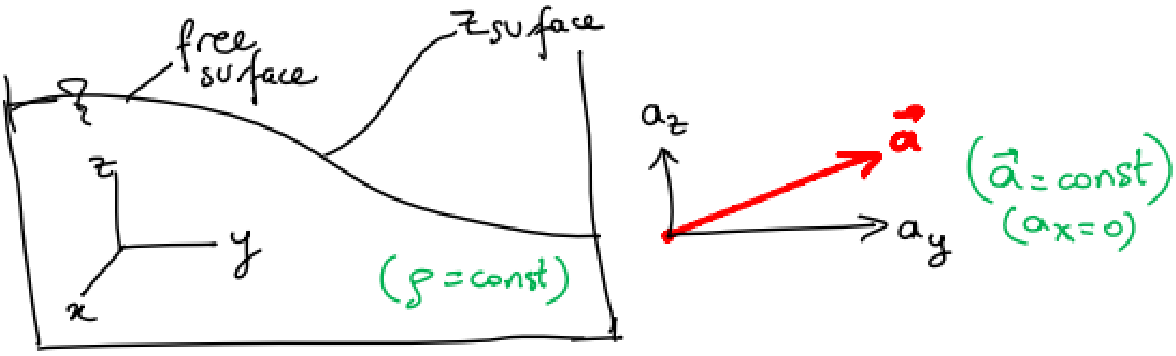

Figure 5: A container with liquid is accelerated in the y and z direction.

To determine the shape of the free surface, we can apply Equation 16.

$$-\nabla p - \rho g\hat{k} = \rho\vec{a} \tag{36}$$

$$-\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right) - \rho g\hat{k} = \rho a_y\hat{j} + \rho a_z\hat{k} \tag{37}$$

(38)

The equation of motion in component form can be expressed as

$$\frac{\partial p}{\partial x} = 0 \tag{39}$$

$$\frac{\partial p}{\partial y} = -\rho a_y \tag{40}$$

$$\frac{\partial p}{\partial z} = -\rho g - \rho a_z \tag{41}$$

Thus, the total differential of p is

$$\mathrm{d}p = \frac{\partial p}{\partial y}\,\mathrm{d}y + \frac{\partial p}{\partial z}\,\mathrm{d}z = -\rho a_y\,\mathrm{d}y - \rho(g + a_z)\,\mathrm{d}z \tag{42}$$

Integrating gives the following:

$$p = -\rho a_y y - \rho(g + a_z)z + C \tag{43}$$

The boundary condition at the free surface is that the pressure is equal to the atmospheric pressure. Therefore, substituting in $p = p_{\mathrm{atm}}$, and solving for $z = z_s$ gives

$$z_s = \frac{C - p_{\mathrm{atm}}}{\rho(g + a_z)} - \frac{a_y}{g + a_z}y \tag{44}$$

$$= C_1 - \frac{a_y}{g + a_z}y \tag{45}$$

Now suppose the container has length L and is filled with water to a height H.

The volume at rest must be the same as the volume when the container is accelerating.

$$\int_0^L Z_s\,\mathrm{d}y \cdot w = LwH \tag{46}$$

$$C_1 L - \frac{a_y}{g + a_z}\frac{L^2}{2} = LH \tag{47}$$

$$\tag{48}$$

Substituting C_1 into the equation gives

$$z_s = H - \frac{a_y}{g + a_z}\left(\frac{L}{2} - y\right). \tag{49}$$

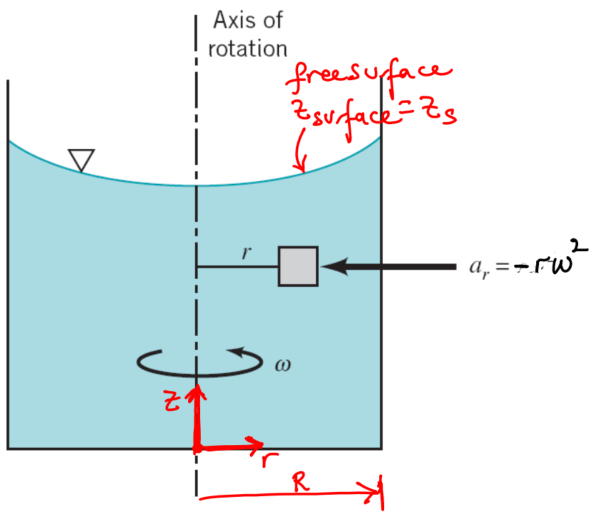

Figure 6: A cylindrical container with liquid in rotation.

Now let us consider fluid in rotation motion. This time, the pressure gradient in cylindrical coordinates can be expressed as

$$\nabla p = \frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{k}. \tag{50}$$

Again using Equation 16, we find that

$$\frac{\partial p}{\partial r} = \rho r \omega^2, \quad \frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} = -\rho g. \tag{51}$$

Solving for p gives

$$p = \rho\omega^2\frac{r^2}{2} - \rho g z + C. \tag{52}$$

Applying the boundary condition at the free surface gives

$$z_s = \frac{\omega^2}{2g}r^2 + C_1. \tag{53}$$

This shows that the free surface is parabolic.

4 Flowing Fluids

There are two ways to describe fluid motion: Lagrangian and Eulerian. In the Lagrangian approach, a small mass of fluid in a flow field is followed as it moves. In the Eulerian approach, the flow field is divided into small windows, and the properties of the fluid in each window are observed.

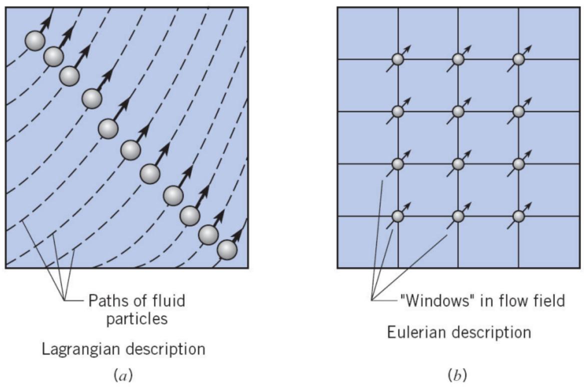

Figure 7: The two methods of flow visualization.

A streamline is a line that is tangent to the local velocity vector at every point along the line at a given instant.

A streamtube is an arbitrary closed curve in 3 dimensional space that is formed by streamlines.

A stream filament is a streamtube in which the cross-sectional area of the filament is small enough to have a constant velocity over the cross-sectional area of the filament.

A pathline is the line traced out by a given fluid particle as it moves from one point to another (the actual path a particle takes).

A streak line consists of a line that connects all fluid particles that have passed through the same point in space at a previous time.

In steady flows, all fluid flow properties at any given point are time independent. In unsteady flows, the properties of the fluid at a given point change with time.

Reynold's number is a dimensionless quantity that characterizes the turbulence of a flow.

$$Re = \frac{\rho \mathbf{V} L}{\mu} \tag{54}$$

where ρ is the density of the fluid, $\mathbf{V}$ is the velocity of the fluid, L is the characteristic length, and μ is the dynamic viscosity of the fluid. The higher the Reynold's number, the more turbulent the flow.

Suppose we have a velocity vector field $\mathbf{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$. The acceleration of the fluid element can be calculated by taking the substantial derivative of the velocity field. The substantial derivative is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{55}$$

In the following subsections, we will discuss the three key equations of fluid mechanics: the continuity equation, the momentum equation, and the energy equation.

4.1 Conservation of Mass

Reynold's transport theorem states that the time rate of change of B of the control mass system is equal to the time rate of change of B inside the control volume plus the time rate of net flow of B from the control volume through the control surface where $B = mb$ is some mass dependent property.

$$\frac{\mathrm{d}B_{\mathrm{sys}}}{\mathrm{d}t} = \frac{\mathrm{d}B_{\mathrm{cv}}}{\mathrm{d}t} + \iint_{\mathrm{cs}} \rho b \mathbf{V} \cdot \mathrm{d}A \tag{56}$$

Therefore, since $\dfrac{\mathrm{d}m_{\mathrm{sys}}}{\mathrm{d}t} = 0$, the continuity equation can be expressed as

$$\frac{\mathrm{d}m_{\mathrm{cv}}}{\mathrm{d}t} + \dot{m}_{\mathrm{out}} - \dot{m}_{\mathrm{in}} = 0 \tag{57}$$

or in integral form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V} \rho\,\mathrm{d}V + \iint_{\mathrm{cs}} \rho\mathbf{V} \cdot \mathrm{d}A = 0. \tag{58}$$

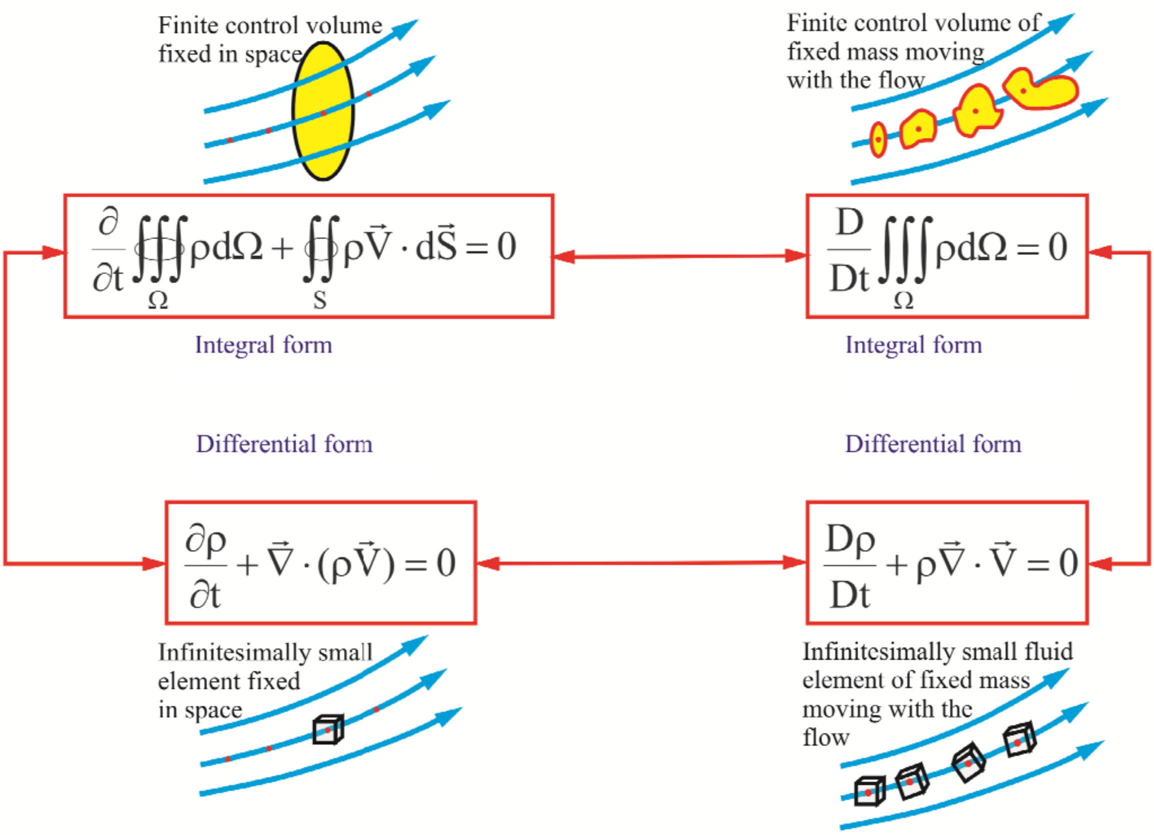

Figure 8: Four different forms of the continuity equation.

The equation can also further be simplified in special cases.

1. For steady flow,

$$\iint_{A_1} \rho_1 \mathbf{V}_1 \cdot \mathrm{d}A_1 = \iint_{A_2} \rho_2 \mathbf{V}_2 \cdot \mathrm{d}A_2 \tag{59}$$

2. For steady, incompressible flow,

$$\iint_{A_1} \mathbf{V}_1 \cdot \mathrm{d}A_1 = \iint_{A_2} \mathbf{V}_2 \cdot \mathrm{d}A_2 \tag{60}$$

3. For steady, incompressible, one dimensional flow,

$$V_1 A_1 = V_2 A_2 \tag{61}$$

   or

$$\nabla \cdot \mathbf{V} = 0 \tag{62}$$

4. For steady, compressible, one dimensional flow,

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \tag{63}$$

4.2 Conservation of Energy

The Bernoulli equation for conservation of energy applies only to steady, incompressible, irrotational flow where frictional forces are negligible, and there is no shaft work or heat transfer. It states that the sum of the pressure, kinetic, and potential energies per unit mass is constant along a streamline.

$$p + \frac{1}{2}\rho \mathbf{V}^2 + \rho g z = \text{constant} \tag{64}$$

p is termed the static pressure, $\frac{1}{2}\rho\mathbf{V}^2$ is the dynamic pressure, and $\rho g z$ is the hydrostatic pressure.

The stagnation pressure is the pressure when all the dynamic pressure is converted into static pressure:

$$p_2 = p_1 + \frac{1}{2}\rho \mathbf{V}_1^2. \tag{65}$$

4.3 Conservation of Momentum

Since momentum is mass dependent, Reynold's transport theorem can be applied to the conservation of momentum.

The most general form of the momentum equation is

$$\sum F_{\mathrm{cv}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{cv} \mathbf{V} \, \mathrm{d}V + \iint_{\mathrm{cs}} \rho \mathbf{V} (\rho \mathbf{V} \cdot \mathrm{d}A). \tag{66}$$

For steady flow, the equation simplifies to

$$\sum F_{\mathrm{cv}} = \iint_{\mathrm{cs}} \mathbf{V} (\rho \mathbf{V} \cdot \mathrm{d}A). \tag{67}$$

For steady, one dimensional flow, the equation further simplifies to

$$\sum F_{\mathrm{cv}} = \sum \left(\rho_{\mathrm{out}} V_{\mathrm{out}} A_{\mathrm{out}}\right) \mathbf{V}_{\mathrm{out}} - \sum \left(\rho_{\mathrm{in}} V_{\mathrm{in}} A_{\mathrm{in}}\right) \mathbf{V}_{\mathrm{in}}. \tag{68}$$

Finally, if we add the incompressibility condition, the equation becomes

$$\sum F_{\mathrm{cv}} = \sum \dot{m}_{\mathrm{out}} \mathbf{V}_{\mathrm{out}} - \sum \dot{m}_{\mathrm{in}} \mathbf{V}_{\mathrm{in}}. \tag{69}$$

5 Open Channel Flows

Open channel flow refers to the flow of liquids in channels open to the atmosphere and is characterized by the presence of a liquid-gas interface called the free surface.

Open channel flows can be classified as tranquil, critical, or supercritical depending on the Froude number.

$$Fr = \frac{\mathbf{V}}{\sqrt{gL}} \tag{70}$$

where g is the acceleration due to gravity, L is the characteristic length, and $\mathbf{V}$ is the velocity of the fluid.

When $Fr < 1$, the flow is tranquil. When $Fr = 1$, the flow is critical. When $Fr > 1$, the flow is supercritical.

The Froude number is also the ratio of the flow speed to the wave speed $Fr = \frac{\mathbf{V}}{c_0}$.

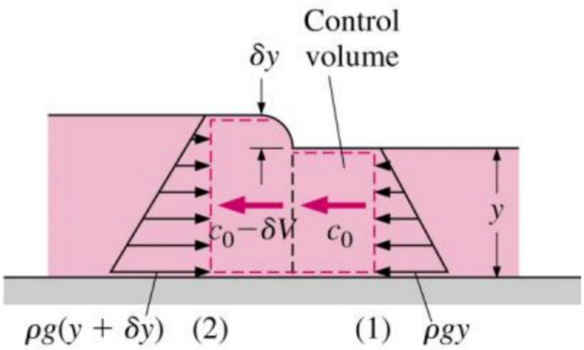

Figure 9: Control volume relative to an observer travelling with the wave.

The wave speed can be calculated using momentum and continuity equations. TODO: show derivation

$$c_0 = \sqrt{gy} \tag{71}$$

6 Compressible Flows

6.1 The Speed of Sound

Sound is caused by pressure disturbances that propagate through a material. The speed of sound is calculated to be

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{72}$$

where s denotes entropy since the process is isentropic.

Using isentropic relations, the equation becomes

$$c = \sqrt{\gamma RT} \tag{73}$$

More generally, the speed of sound can be expressed in terms of the bulk modulus E_V

$$c = \sqrt{\frac{E_V}{\rho}}. \tag{74}$$

6.2 The Mach Number

The Mach number is a dimensionless number which is defined as the ratio of the velocity of the fluid to the speed of sound in the fluid.

$$M = \frac{\mathbf{V}}{c} \tag{75}$$

If $M \leq 0.3$, the flow is incompressible. If $M > 0.3$, the flow is compressible.

If $M < 1$, the flow is subsonic. If $M = 1$, the flow is sonic. If $M > 1$, the flow is supersonic.

6.3 Compressible Bernoulli Equation

Before we already discussed the compressible momentum and mass equations, but now we will discuss the compressible energy equation.

The compressible Bernoulli equation states that

$$e + \frac{p}{\rho} + \frac{1}{2}\mathbf{V}^2 + gz = \text{constant} \tag{76}$$

where e is the internal energy per unit mass. $e + \frac{p}{\rho}$ is the enthalpy per unit mass.

6.4 Stagnation Temperature

The stagnation temperature for compressible flows is defined as the temperature of the fluid when all the kinetic energy is converted into enthalpy.

$$T_0 = T + \frac{\mathbf{V}^2}{2c_p} \tag{77}$$