Heat Transfer Notes

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1 Mechanisms of Heat Transfer

1.1 Conduction

Conduction is the heat transfer through a stationary medium as a result of a temperature difference.

In gases and liquids, conduction occurs through collisions and diffusion of molecules. In solids, conduction occurs through lattice vibrations.

Fourier's law of heat conduction states that the heat flux is proportional to the temperature gradient:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} = kA\frac{T_1 - T_2}{\Delta x} \tag{1}$$

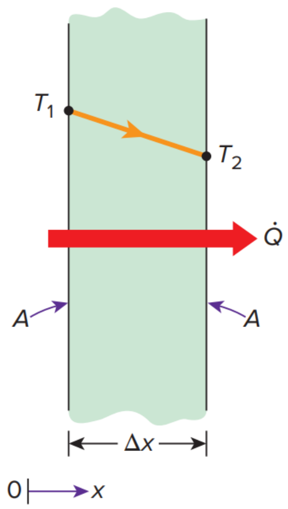

Figure 1: Heat conduction through a large plane wall.

Materials with closely packed atoms have high thermal conductivity.

Table 1: Thermal Conductivity of Various Materials

Material	k (W/m · K)
Diamond	2300
Copper	400
Gold	317
Aluminum	200
Iron	80
Glass	0.78
Liquid water	0.6
Air	0.026

1.2 Convection

Convection is the heat transfer between a solid surface and a moving fluid. It is a combination of conduction and advection which is energy transfer because of bulk motion of the fluid.

$$\dot{Q}_{\text{conv}} = hA(T_s - T_\infty) \tag{2}$$

where h is the convective heat transfer coefficient, A is the surface area, T_s is the surface temperature, and T_∞ is the fluid temperature far from the surface.

1.3 Radiation

Radiation is energy emitted by all matter in the form of electromagnetic waves. Thermal radiation is emitted by all bodies at finite temperatures. Radiation is typically volumetric, but opaque objects emit radiation from their surfaces.

The amount of radiation depends on surface temperature, surface emissivity, and surface area. A surface emissivity of 1 is a black body, while a surface emissivity of 0 is a perfect reflector.

$$\dot{Q}_{\text{rad}} = \varepsilon \sigma A T_s^4 \tag{3}$$

$\sigma = 5.670 \times 10^{-8}\,\text{W/m}^2{\cdot}\text{K}^4$ is the Stefan-Boltzmann constant.

The rate at which radiation is absorbed by a surface is given by

$$\dot{Q}_{\text{abs}} = \alpha \dot{Q}_{\text{rad}} \tag{4}$$

where α is the surface absorptivity.

When a small surface is surrounded by a large surface, the radiation heat transfer is given by

$$\dot{Q}_{\text{rad}} = \varepsilon \sigma A (T_s^4 - T_{\text{surr}}^4) \tag{5}$$

2 Steady-State Heat Conduction

2.1 Heat Conduction Equation

The 1D heat conduction equation in Cartesian coordinates is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{6}$$

$\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density, and c_p is the specific heat (Read More).

At steady-state, $\dfrac{\partial T}{\partial t} = 0$, so the heat conduction equation simplifies to

$$\frac{\partial^2 T}{\partial x^2} = 0. \tag{7}$$

For cylindrical coordinates, the heat conduction equation is

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right). \tag{8}$$

Similarly, for spherical coordinates, the heat conduction equation is

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right). \tag{9}$$

2.2 Thermal Resistance

Steady heat conduction systems can be analyzed using the concept of thermal resistance which is analogous to electrical resistance.

If we define thermal resistance for conduction to be

$$R_{\text{cond}} = \frac{L}{kA} \tag{10}$$

in units of °C/W, then by Equation 1, we have

$$\dot{Q}_{\text{cond}} = \frac{T_1 - T_2}{R_{\text{cond}}}. \tag{11}$$

With similar reasoning, the thermal resistance for convection is given by

$$R_{\text{conv}} = \frac{1}{hA}, \tag{12}$$

and the thermal resistance for radiation is given by

$$R_{\text{rad}} = \frac{1}{h_{\text{rad}} A} \tag{13}$$

where $h_{\text{rad}} = \varepsilon\sigma(T_s^2 + T_{\text{surr}}^2)(T_s + T_{\text{surr}})$.

For a system with both radiation and convection in parallel, if $T_{\text{surr}} \approx T_\infty$, then the radiation effect can be accounted for by replacing h in the convection resistance relation by

$$h_{\text{comb}} = h + h_{\text{rad}}. \tag{14}$$

Example

Consider a 0.8 m high and 1.5 m wide glass window with a thickness of 8 mm with a thermal conductivity of $k = 0.78\,\text{W/m}\cdot\text{K}$. Using additional properties given in the figure below, calculate the steady rate of heat transfer, and the temperature of the inner surface of the window glass.

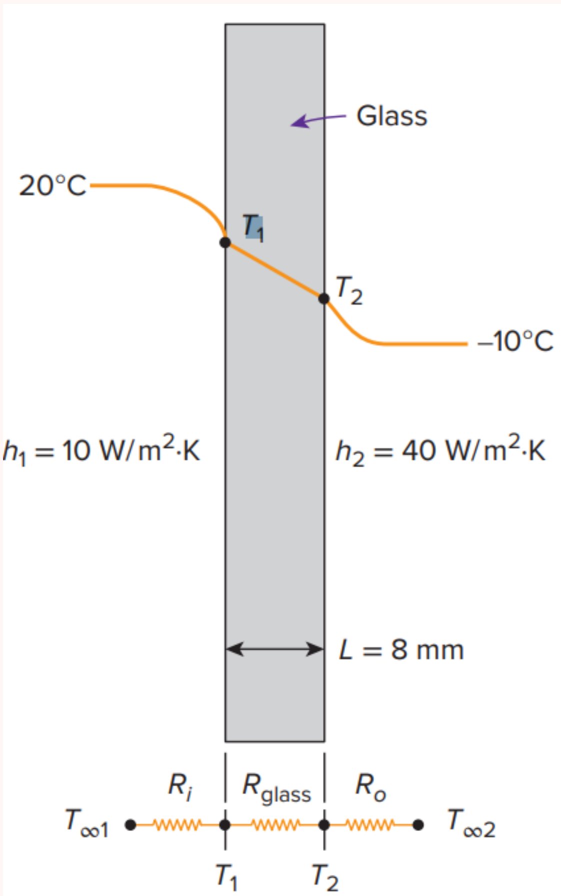

Figure 2: Example heat resistance problem.

The total area of the window is $A = 0.8 \cdot 1.5 = 1.2\,\text{m}$. The total thermal resistance is

$$R_{\text{total}} = R_i + R_{\text{glass}} + R_o = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} = 0.1127\,°\text{C/W}. \tag{15}$$

The steady rate of heat transfer is

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} = 266\,\text{W}. \tag{16}$$

The temperature of the inner surface of the window glass is

$$T_1 = T_{\infty 1} - \dot{Q} R_i = -2.2\,°\text{C}. \tag{17}$$

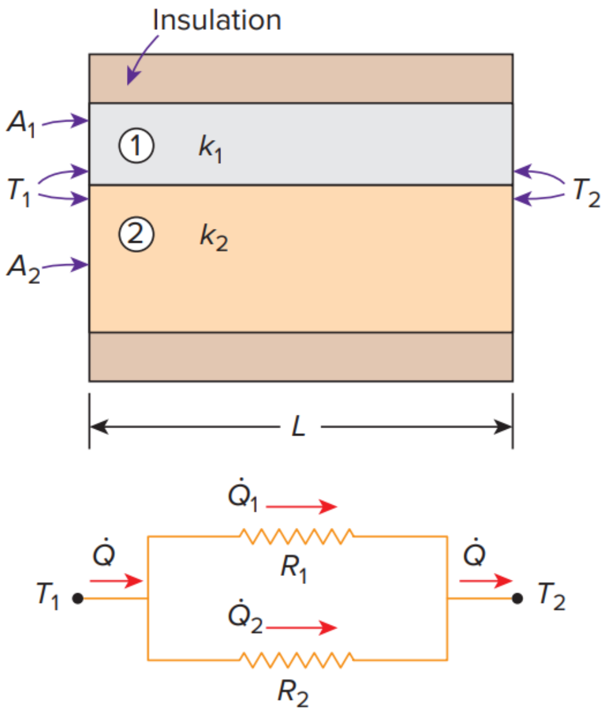

Figure 3: When two resistances are in parallel, the total resistance is $R_{\text{total}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$.

2.3 Contact Resistance

In the previous analysis of heat conduction through multilayer solids, we assumed that the layers were in perfect contact. In reality, the surface is microscopically rough, and there is a temperature drop at the interface.

The temperature drop can be expressed using Newton's law of cooling:

$$\dot{Q} = h_c A \Delta T \tag{18}$$

The thermal contact conductance is defined as

$$h_c = \frac{\dot{Q}}{A \Delta T} \tag{19}$$

and the thermal contact resistance is defined as

$$R_c = \frac{1}{h_c}. \tag{20}$$

Note that the thermal contact resistance is in units of $\mathrm{m}^2{\cdot}\mathrm{K/W}$, meaning that to get total resistance, we must divide by the area of the contact $R_{\text{total}} = \frac{1}{h_c A}$.

The value of thermal contact resistance depends on the surface roughness, material properties, as well as the temperature and pressure at the interface, and the type of fluid trapped in the interface.

2.4 Heat Conduction In Cylinders and Spheres

The heat transfer through a cylinder is given by

$$\dot{Q} = \frac{2\pi k L (T_1 - T_2)}{\ln(r_2/r_1)} \tag{21}$$

where r_1 and r_2 are the inner and outer radii of the cylinder respectively, and L is the length of the cylinder.

Therefore, the thermal resistance for a cylinder is given by

$$R_{\text{cyl}} = \frac{\ln(r_2/r_1)}{2\pi k L}. \tag{22}$$

Similarly, the thermal resistance for a sphere is given by

$$R_{\text{sph}} = \frac{r_2 - r_1}{4\pi k r_1 r_2}. \tag{23}$$

Consider an insulated pipe as shown in the diagram below.

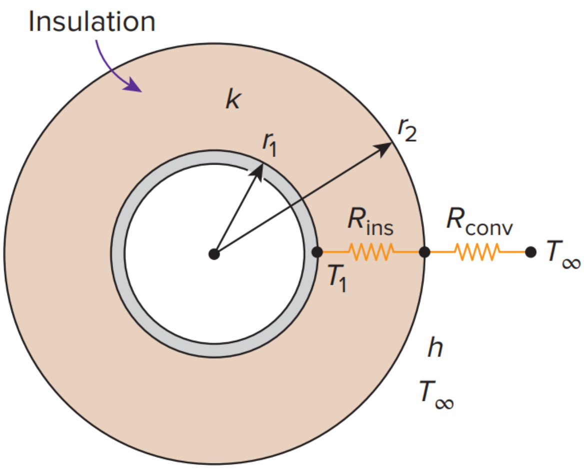

Figure 4: An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

The rate of heat transfer is given by

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{ins}} + R_{\text{conv}}} = \frac{T_1 - T_\infty}{\frac{\ln(r_2/r_1)}{2\pi k L} + \frac{1}{h_{\text{out}} 2\pi r_2 L}} \tag{24}$$

Adding more insulation material to the pipe does not always reduce the heat transfer rate. This is because it would decrease the convection resistance. Equation 24 can be differentiated to find the **critical radius of insulation** at which the heat transfer rate is minimized.

$$r_{\text{crit},cyl} = \frac{k}{h} \tag{25}$$

The critical radius of insulation for a sphere is given by

$$r_{\text{crit},sph} = \frac{2k}{h}. \tag{26}$$

2.5 Finned Surfaces

There are two ways to increase the rate of heat transfer from a surface to a surrounding medium: increase the convection heat transfer coefficient or increase the surface area. To increase surface area, high-conductivity materials known as fins are attached to the surface.

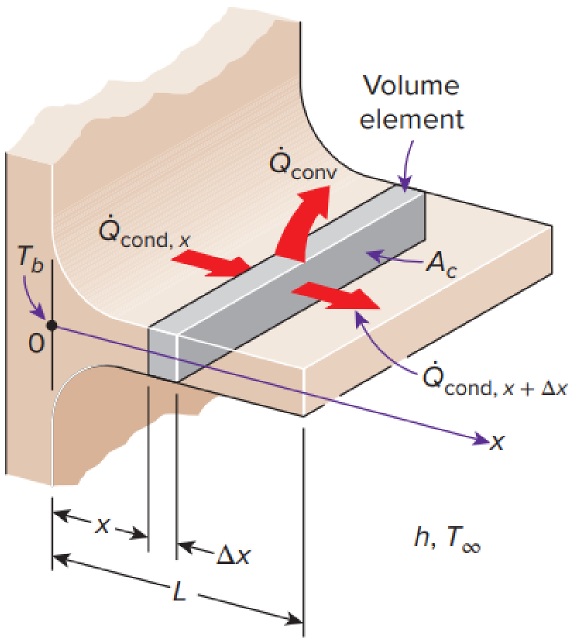

Figure 5: Volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and perimeter of p.

Essentially, three equations are needed to analyze the heat transfer through a fin: the energy balance equation, the heat convection equation, and Fourier's law of heat conduction.

$$\dot{Q}_{\text{cond},x} = \dot{Q}_{\text{cond},x+\Delta x} + \dot{Q}_{\text{conv}} \tag{27}$$

$$\dot{Q}_{\text{conv}} = hp\Delta x(T - T_\infty) \tag{28}$$

$$\dot{Q}_{\text{cond}} = -kA_c\frac{\partial T}{\partial x} \tag{29}$$

For the case of constant cross-section, and constant thermal conductivity, this simplifies to

$$\frac{\partial^2\theta}{\partial x^2} - m^2\theta = 0 \tag{30}$$

where $\theta = T - T_\infty$ is the temperature excess and $m^2 = \frac{hP}{kA_c}$. The general solution to this differential equation is

$$\theta = C_1e^{mx} + C_2e^{-mx}. \tag{31}$$

Infinitely Long Fin

For an infinitely long fin, the boundary conditions are $\theta(0) = \theta_b$ and $\theta(\infty) = 0$. Thus,

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} \tag{32}$$

$$\dot{Q}_{\text{long fin}} = hA_c\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0} = \sqrt{hpkA_c}(T_b - T_\infty) \tag{33}$$

Finite Length Fin

The heat transfer equation for a finite-length fin can be approximated using two approaches: either by assuming the fin tip is insulated or by applying a corrected fin length.

For the case of an insulated tip, the boundary condition applied at $x = L$ is $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$. This results in

$$\dot{Q}_{\text{ins fin}} = \sqrt{hpkA_c}(T_b - T_\infty)\tanh(mL). \tag{34}$$

If the fin tip is not insulated, the length of the fin can be corrected to account for the heat loss at the tip.

$$L_c = L + \frac{A_c}{p} \tag{35}$$

Using L_c in Equation 34 will yield a more accurate result for a non-insulated tip.

Fin Efficiency

Fin efficiency η is defined as the ratio of the actual heat transfer from the fin to the heat transfer if the entire fin were at base temperature.

For an infinitely long fin, the fin efficiency is

$$\eta_{\text{long fin}} = \frac{\sqrt{hpkA_c}(T_b - T_\infty)}{hpL(T_b - T_\infty)} = \frac{1}{mL}. \tag{36}$$

For a finite-length fin with an insulated tip, the fin efficiency is b

$$\eta_{\text{ins fin}} = \frac{\sqrt{hpkA_c}(T_b - T_\infty)\tanh(mL)}{hpL(T_b - T_\infty)} = \frac{\tanh(mL)}{mL}. \tag{37}$$

Fin Effectiveness

Fin effectiveness ε is defined as the ratio of the actual heat transfer from the fin to the heat transfer if there was no fin.

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\eta_{\text{fin}} \tag{38}$$

$$\varepsilon_{\text{long fin}} = \frac{\sqrt{hpkA_c}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}} \tag{39}$$

Thermal Resistance

The thermal resistance of a heat sink is given by

$$R = \frac{T_b - T_\infty}{\dot{Q}_{\text{total}}} = \frac{1}{h\left(A_{\text{no fin}} + \eta_{\text{fin}}A_{\text{fin}}\right)}. \tag{40}$$

3 Transient Heat Conduction

3.1 Lumped System Analysis

Some bodies appear to behave like a lump whose temperature remains uniform at any time during the heat transfer process.

The temperature of the lumped system can be found using the energy balance equation.

$$hA_s(T_\infty - T) = mc_p\frac{\mathrm{d}T}{\mathrm{d}t} \tag{41}$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \tag{42}$$

where $b = \frac{hA_s}{\rho V c_p}$ is the time constant.

Lumped system analysis is only valid when the Biot number is less than 0.1.

$$Bi = \frac{hL_c}{k} < 0.1 \tag{43}$$

$L_c = V/A_s$ is the characteristic length.

3.2 One Dimensional Transient Heat Conduction

When Lumped System Analysis is not applicable, the heat conduction equation must be solved directly.

$$\text{Differential Equation:} \quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{44}$$

$$\text{Boundary Conditions:} \quad \frac{\partial T(0,t)}{\partial x} = 0 \quad \text{and} \quad -k\frac{\partial T(L,t)}{\partial x} = h(T(L,t) - T_\infty) \tag{45}$$

$$\text{Initial Condition:} \quad T(x,0) = T_i \tag{46}$$

They can be simplified by non-dimensionalizing the equation using the Fourier number $Fo = \frac{\alpha t}{L^2}$ for dimensionless time, the Biot number $Bi = \frac{hL}{k}$ for dimensionless heat transfer coefficient, and $X = \frac{x}{L}$ for dimensionless distance from the center.

$$\text{Differential Equation:} \quad \frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau} \tag{47}$$

$$\text{Boundary Conditions:} \quad \frac{\partial \theta(0,\tau)}{\partial X} = 0 \quad \text{and} \quad \frac{\partial \theta(1,\tau)}{\partial X} = -Bi(\theta(1,\tau)) \tag{48}$$

$$\text{Initial Condition:} \quad \theta(X,0) = 1 \tag{49}$$

Table 2: Solutions to the One-Dimensional Transient Heat Conduction Equation for a plane of wall thickness $2L$, a cylinder of radius r_0 and a sphere of radius r_0.

Geometry	Solution	λ_n's are the roots of
Plane wall	$\theta = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n x/L)$	$\lambda_n \tan \lambda_n = \text{Bi}$
Cylinder	$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \tau} J_0(\lambda_n r/r_o)$	$\lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} = \text{Bi}$
Sphere	$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x/L)}{\lambda_n x/L}$	$1 - \lambda_n \cot \lambda_n = \text{Bi}$

3.3 Semi-infinite Solids

A semi-infinite solid is an idealized body that has a single plane surface and extends to infinity in all directions.

Again we will try to solve the heat conduction equation, except that we cannot use the separation of variables method. Instead, we will substitute a new variable $\eta = \frac{x}{\sqrt{4\alpha t}}$ to simplify the equation.

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\eta^2} = -2\eta \frac{\mathrm{d}T}{\mathrm{d}\eta} \tag{50}$$

The boundary conditions are $T(0) = T_s$ and $T(\eta \to \infty) = T_i$.

The solution to this equation is

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du = erf(\eta). \tag{51}$$

The heat flux at the surface can be determined using Fourier's law.

$$\dot{q}_s = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} \tag{52}$$

4 Forced Convection

4.1 Physical Mechanism of Convection

Heat transfer through a fluid involves both convection and conduction. The Nusselt number represents the enhancement of heat transfer through a fluid layer as a result of convection relative to conduction across the same fluid layer.

$$Nu = \frac{\dot{q}_{\text{conv}}}{\dot{q}_{\text{cond}}} = \frac{hL}{k} \tag{53}$$

4.2 Thermal Boundary Layer

Similar to a velocity boundary layer, a thermal boundary layer develops when a fluid a specified temperature flows over a surface at a different temperature. The thickness of the thermal boundary layer is defined as the distance from the surface at which the temperature difference $T - T_s$ equals $0.99(T_\infty - T_s)$.

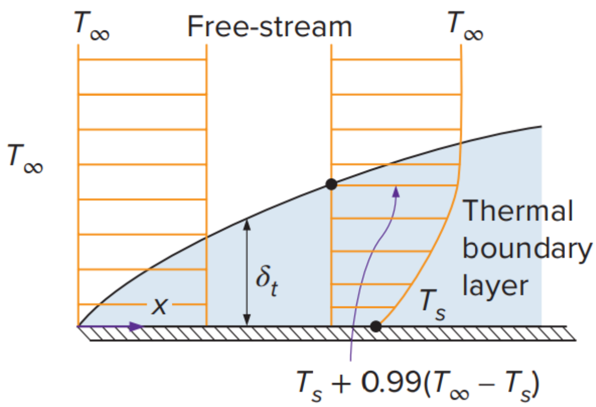

Figure 6: Thermal boundary layer on a flat plate.

The Prandtl number describes the relative thickness of the velocity and thermal boundary layers.

$$Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{k} \tag{54}$$

4.3 Parallel Flow Over Flat Plates

The Reynolds number determines whether a flow is laminar or turbulent. The Reynolds number at a distance x from the leading edge of a flat plate is given by

$$Re_x = \frac{\rho \mathbf{V} x}{\mu} = \frac{\mathbf{V} x}{\nu} \tag{55}$$

Laminar flow occurs when $Re_x < 5 \times 10^5$ and $Pr > 0.6$. Turbulent flow occurs when $5 \times 10^5 \leq Re_x \leq 10^7$ and $0.6 \leq Pr \leq 60$.

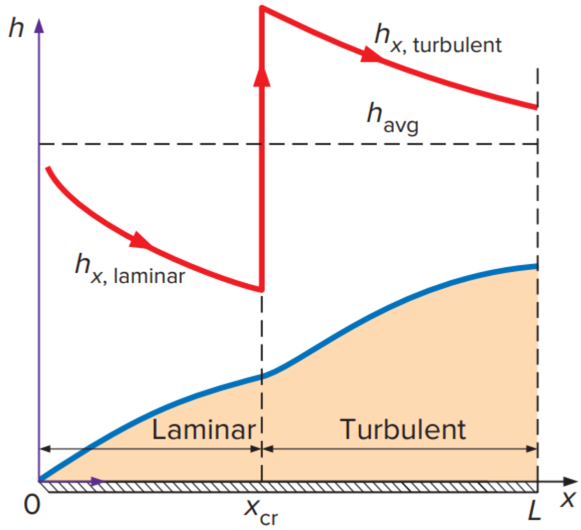

Figure 7: The average heat transfer coefficient for a flat plate with combined laminar and turbulent flow.

The average convective heat transfer coefficient can be determined by the relations:

$$\text{Laminar:} \quad Nu = \frac{hL}{k} = 0.664 Re_L^{0.5} Pr^{1/3} \tag{56}$$

$$\text{Turbulent:} \quad Nu = \frac{hL}{k} = 0.037 Re_L^{0.8} Pr^{1/3} \tag{57}$$

5 Radiation Heat Transfer

When radiation strikes a surface, it is either absorbed, reflected, or transmitted. Radiation flux incident on a surface is called irradiation and denoted by G.

$$\text{Absorptivity:} \quad \alpha = \frac{\dot{Q}_{\text{abs}}}{G} \tag{58}$$

$$\text{Reflectivity:} \quad \rho = \frac{\dot{Q}_{\text{ref}}}{G} \tag{59}$$

$$\text{Transmissivity:} \quad \tau = \frac{\dot{Q}_{\text{tr}}}{G} \tag{60}$$

$$\tau + \rho + \alpha = 1 \tag{61}$$

We will make two assumptions: the surface is diffuse and gray. Gray surfaces have the same emissivity at all wavelengths, while diffuse surfaces have the same emissivity in all directions.

5.1 Kirchhoff's Law

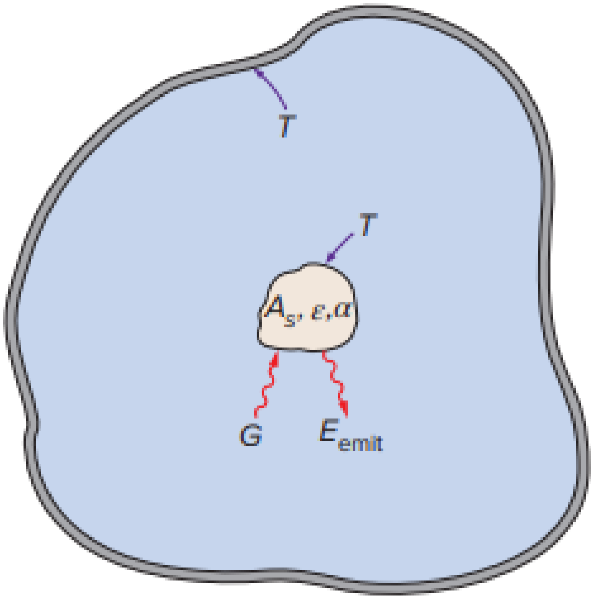

Figure 8: A small body contained in a large isothermal enclosure.

Kirchhoff's law states the total hemispherical emissivity of a surface at temperature T is equal to its total hemispherical absorptivity for radiation coming from a blackbody at the same temperature.

$$\varepsilon(T) = \alpha(T) \tag{62}$$

5.2 The View Factor

Radiation heat transfer between surfaces depends on the orientation of the surfaces relative to each other as well as their radiation properties. The view factor from a surface i to a surface j is denoted by F_{ij} which is the fraction of radiation leaving surface i that strikes surface j.

The reciprocity relation determines the counterpart view factor.

$$A_i F_{ij} = A_j F_{ji} \tag{63}$$

Additionally, the sum of the view factors to surface i to all surfaces of the enclosure is equal to 1.

$$\sum_{j=1}^{n} F_{ij} = 1 \tag{64}$$

5.3 Radiation Heat Transfer in Two-Surface Enclosures

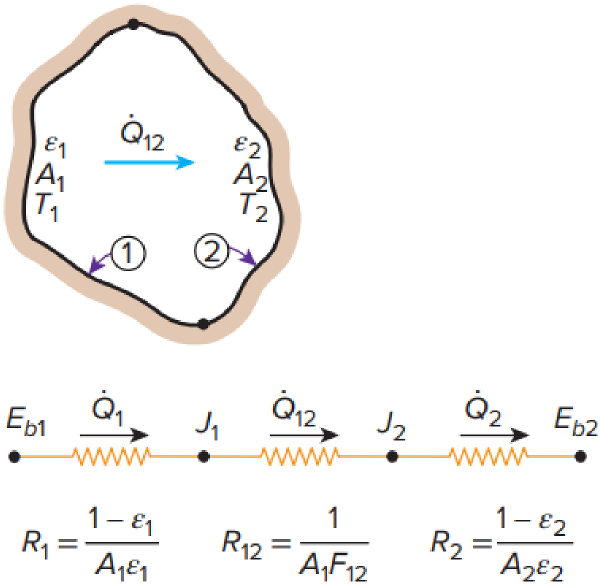

Figure 9: Schematic of a two-surface enclosure and the radiation network associated with it.

The net radiative heat transfer between two surfaces is given by

$$\dot{Q}_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{12} + R_2} = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1-\varepsilon}{A_1\varepsilon_1} + \frac{1}{A_1F_{12}} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}. \tag{65}$$