Differential Equation Notes

Last Updated: December 15, 2024

Contents

1 Introduction

An equation that contains derivatives of one or more unknown functions with respect to one or more independent variables is said to be a **differential equation**.

If an unknown function depend on a single independent variable, the differential equation is said to be an **ordinary differential equation** (ODE). Otherwise, it is called a **partial differential equation** (PDE). For example, all ODEs are of the form

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}), \tag{1}$$

where t is the independent variable and y is the dependent variable.

The **order** of a differential equation is the highest derivative that appears in the equation.

Linear Differential Equations

An nth order ODE $F(t, y, y', y'', \dots, y^{(n)}) = 0$ is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t). \tag{2}$$

The functions $a_0(t), a_1(t), \dots, a_n(t)$ are called the **coefficients** of the equation. Equation 2 is said to be **homogeneous** if $g(t) = 0$ for all t. Otherwise, it is said to be **nonhomogeneous**.

Solutions for a Differential Equation

A **solution** of the ODE (1) on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \dots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)} = f\left(t, \phi, \phi', \phi'', \dots, \phi^{(n-1)}\right) \tag{3}$$

for every t in $\alpha < t < \beta$.

Initial Value Problems

An **initial value problem** for an nth order ODE on an interval I consists of Equation 1 together with n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots, \quad y^{(n-1)}(t_0) = y_{n-1} \tag{4}$$

prescribed at a point $t_0 \in I$, where $y_0, y_1, \dots, y_{n-1}$ are given constants.

2 First Order Differential Equations

2.1 Separable Equations

A first order ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ is **separable** if the right side can be written in the form $f(x, y) = p(x)q(y)$. p must only depend on x and q must only depend on y.

Separable differential equations can be solved by using the differentials $\mathrm{d}x$ and $\mathrm{d}y = y'(x)\,\mathrm{d}x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x), q(y) \tag{5}$$

$$\mathrm{d}y = p(x)q(y)\,\mathrm{d}x \tag{6}$$

$$q(y)^{-1}\,\mathrm{d}y = p(x)\,\mathrm{d}x \tag{7}$$

$$\int q(y)^{-1}\,\mathrm{d}y = \int p(x)\,\mathrm{d}x \tag{8}$$

Exampe: Newton's Law of Cooling

Newton's Law of Cooling states that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_0) \tag{9}$$

where T is the temperature of the object at time t, T_0 is the temperature of the surrounding medium, and k is a positive constant.

Below are a few common mathematical terminology for the quantities that appear in the equation:

- t is an **independent variable**.
- T is a **dependent variable**.
- k and T_0 are the **parameters**.

The equation is separable.

$$\frac{\mathrm{d}T}{u - T_0} = k\,\mathrm{d}t \tag{10}$$

$$\ln|T - T_0| = kt + C \tag{11}$$

$$T = T_0 + Ce^{kt} \tag{12}$$

2.2 Linear Equations

A differential equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t) \tag{13}$$

is said to be a **linear first order ODE**.

First order linear ODEs can be solved by using the method of integrating factors.

The integrating factor is

$$I(t) = e^{\int p(t)\,\mathrm{d}t} \tag{14}$$

Multiplying the integrating factor by Equation 13 gives

$$I(x)\left(\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y\right) = g(t)I(t) \tag{15}$$

$$[I(t)y]' = g(t)I(t) \tag{16}$$

$$y = \frac{1}{I(t)}\left[\int I(t)g(t)\,\mathrm{d}t + C\right] \tag{17}$$

2.3 Difference Between Linear and Nonlinear Equations

Linear Theorem

If the functions p and g are continuous on an open interval $I = (\alpha, \beta)$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(y)y = g(t) \tag{18}$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0 \tag{19}$$

where y_0 is an arbitrary prescribed initial value.

Nonlinear Theorem

Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{20}$$

Example

Consider the differential equation

$$y' = y^2 \tag{21}$$

with the initial condition $y(0) = 1$. Since $f(t, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous for all t and y, the theorem guarantees that there is a unique solution to the initial value problem.

2.4 Autonomous Equations

A differential equation that can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \tag{22}$$

is said to be **autonomous** (the right side is not dependent on t).

The **equilibrium solutions** of an autonomous equation can be found by locating the roots of $f(y) = 0$ (also known as critical points).

Equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\left(1 - \frac{y}{K}\right)y \tag{23}$$

are known as **logistic** equations. The constant r is known as the **intrinsic growth rate**. Logistic equations implement the concept of **carrying capacity** K also known as **saturation level**.

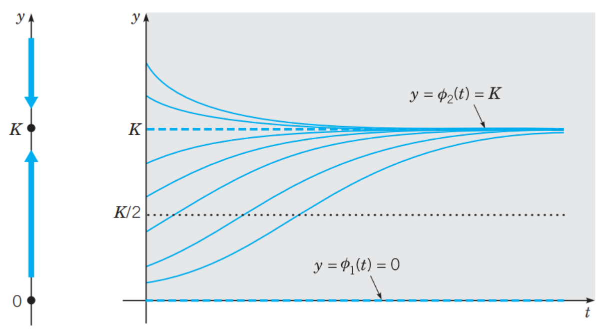


Figure 1: Phase line (left) and integral curve (right) of logistic equation. The equilibrium solution $y = K$ is asymptotically stable since solutions tend to approach the line $y = K$ as $t \to \infty$.

The general solutions to the logistic equation can be found using separation of variables.

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \tag{24}$$

We can also introduce a threshold T to the logistic equation, so that the population does not grow unless a certain threshold is reached.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\left(1 - \frac{y}{K}\right) y \left(\frac{y}{T} - 1\right) \tag{25}$$

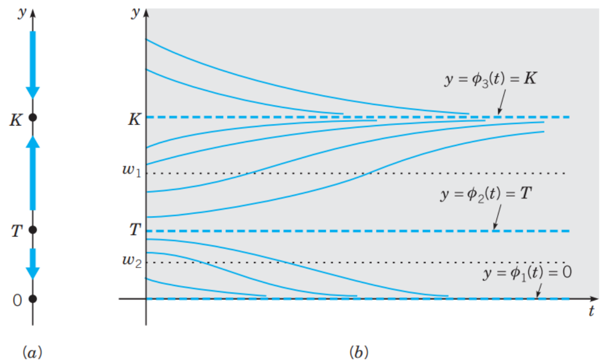


Figure 2: Logistic growth with a threshold. The equilibrium solution $y = T$ is asymptotically unstable.

3 Systems of First Order Equations

This section combines chapter 3 and 6 of the textbook.

3.1 Systems of Two First Order Linear Equations

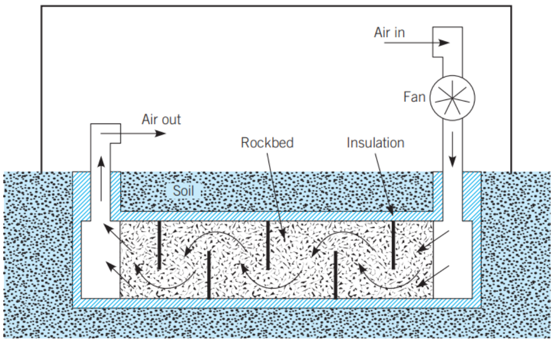


Figure 3: Simple greenhouse system.

Consider the following greenhouse system. Hot air during the day is pumped underground, transferring heat to the rocks below during the day when a certain temperature threshold is reached. At night, when the temperature above is cooled down below a certain threshold, the hot air from the rocks is pumped back up into the greenhouse.

Let us define the following variables:

- $u_1(t)$, $u_2(t)$ - Air temperature of the greenhouse and air temperature of the rockbed.
- m_1, m_2 - Total masses of air and rock.
- C_1, C_2 - Specific heats of air and rock.
- A_1, A_2 - Area of above ground greenhouse enclosure and area of the air-rock interface.

- h_1,h_2 - Heat transfer coefficients across interface areas A_1 and A_2.
- T_a - Temperature of air external to greenhouse.

Using the law of conservation of energy, we get the differential equations

$$m_1 C_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} = -h_1 A_1 (u_1 - T_a) - h_2 A_2 (u_1 - u_2) \tag{26}$$

$$m_2 C_2 \frac{\mathrm{d}u_2}{\mathrm{d}t} = -h_2 A_2 (u_2 - u_1) \tag{27}$$

Substituting the variables

$$k_1 = \frac{h_1 A_1}{m_1 C_1}, \quad k_2 = \frac{h_2 A_2}{m_1 C_1}, \quad \epsilon = \frac{m_1 C_1}{m_2 C_2} \tag{28}$$

into the system of equations, we get

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = -(k_1 + k_2)u_1 + k_2 u_2 + k_1 T_a \tag{29}$$

$$\epsilon \frac{\mathrm{d}u_2}{\mathrm{d}t} = \epsilon k_2 u_1 - \epsilon k_2 u_2 \tag{30}$$

Let us provide values for the constants.

$$k_1 = \frac{7}{8}, \quad k_2 = \frac{3}{4}, \quad \epsilon = \frac{1}{3}, \quad T_a = 16°C \tag{31}$$

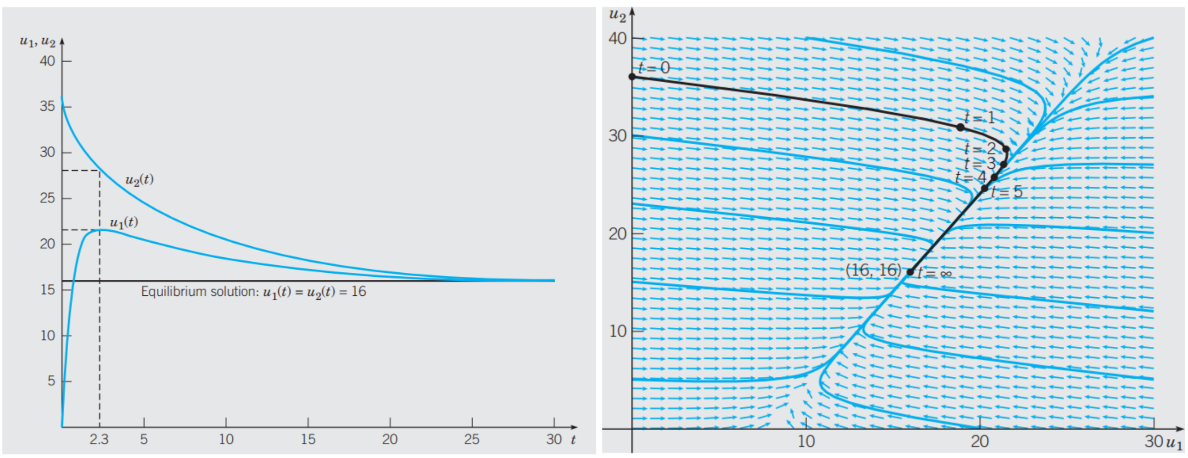
We can also rewrite the equation in matrix form.

$$\begin{bmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}t} \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix} \tag{32}$$

The method for solving equations $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{K}\mathbf{u} + \mathbf{b}$ will be discussed later, but as for now the solution will be simply presented.

$$\mathbf{u} = \begin{bmatrix} 8e^{-t/8} - 24e^{-7t/4} + 16 \\ 16e^{-t/8} + 4e^{-7t/4} + 16 \end{bmatrix} \tag{33}$$

The graphs for u_1 and u_2 vs t are called **component plots**. u_1 and u_2 are called the **state variables** of the system, and their plane is called the **state plane** or **phase plane**. A plot of sample trajectories on the phase plane is called a **phase portrait**.



(a) (b)

Figure 4: (a) Component plots of the greenhouse system. (b) Phase portrait of the greenhouse system.

Existence and Uniqueness of Solutions

Let each of the functions $p_{11}, \ldots, p_{22}, g_1$, and g_2 be continuous on an open interval $I = \alpha < t < \beta$, let t_0 be any point in I, and let x_0 and y_0 be any given numbers. Then, there exists a unique solution of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} p_{11}(t)x + p_{12}(t)y + g_1(t) \\ p_{21}(t)x + p_{22}(t)y + g_2(t) \end{pmatrix} \tag{34}$$

that also satisfies the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$.

Transformation of a Second Order Equation to a System of First Order Equations

One or more higher-order equations can always be transformed into a system of first order equations.

Consider the second order equation

$$y'' + p(t)y' + q(t)y = g(t). \tag{35}$$

We can introduce new variables $x_1 = y$ and $x_2 = y'$ to transform the equation into a system of first order equations.

$$x_1' = x_2 \tag{36}$$

$$x_2' = -q(t)x_1 - p(t)x_2 + g(t). \tag{37}$$

3.2 Homogeneous Linear Systems with Constant Coefficients

Converting $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$

If $\mathbf{A}$ has an inverse, then the only critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ is $\mathbf{x}_{\text{eq}} = -\mathbf{A}^{-1}\mathbf{b}$. If we let $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{x}_{\text{eq}}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}_{\text{eq}} + \tilde{\mathbf{x}}\right) = \mathbf{A}(\mathbf{x}_{\text{eq}} + \tilde{\mathbf{x}}) + \mathbf{b} \tag{38}$$

$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \mathbf{A}\tilde{\mathbf{x}} \tag{39}$$

since $\dfrac{\mathrm{d}\mathbf{x}_{\text{eq}}}{\mathrm{d}t} = 0$ and $\mathbf{A}\mathbf{x}_{\text{eq}} + \mathbf{b} = 0$. Thus, if $\tilde{\mathbf{x}}$ is a solution of the homogeneous system $\tilde{\mathbf{x}}' = \mathbf{A}\tilde{\mathbf{x}}$, then $\mathbf{x} = \mathbf{x}_{\text{eq}} + \tilde{\mathbf{x}}$ is a solution of the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

Superposition and Linear Independence

Superposition Principle

Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{40}$$

Then, the expression

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \tag{41}$$

is also a solution of the system for any constants c_1 and c_2.

If there is a prescribed initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, then it is possible to choose c_1 and c_2 so that the solution satisfies the initial condition.

$$c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) = \mathbf{x}_0 \tag{42}$$

$$\begin{pmatrix} x_{11}(t_0) & x_{12}(t_0) \\ x_{21}(t_0) & x_{22}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \tag{43}$$

The determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} \tag{44}$$

is called the **Wronskian determinant**. The initial value problem has a unique solution if and only if $W[\mathbf{x}_1, \mathbf{x}_2](t_0) \neq 0$. $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be **linearly independent** if $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ for all t.

Solution to a General System

Let us consider a general system of two first order linear homogeneous differential equations with constant coefficients.

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}, \tag{45}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{46}$$

If we substitute into the equation

$$\mathbf{x} = e^{\lambda t}\mathbf{v}, \tag{47}$$

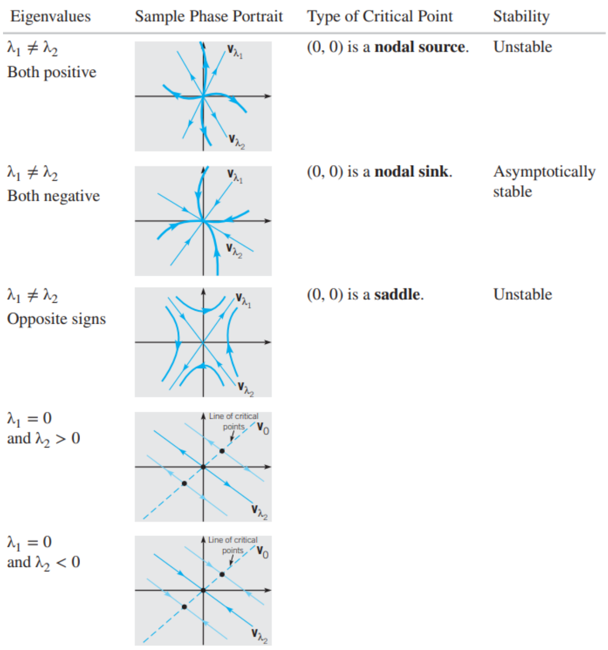
we get

$$\lambda e^{\lambda t}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v} \tag{48}$$

$$\lambda \mathbf{v} = \mathbf{A}\mathbf{v} \tag{49}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0. \tag{50}$$

Thus, λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is an eigenvector of $\mathbf{A}$.



Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\lambda_1 \neq \lambda_2$ Both positive		(0, 0) is a **nodal source**.	Unstable
$\lambda_1 \neq \lambda_2$ Both negative		(0, 0) is a **nodal sink**.	Asymptotically stable
$\lambda_1 \neq \lambda_2$ Opposite signs		(0, 0) is a **saddle**.	Unstable
$\lambda_1 = 0$ and $\lambda_2 > 0$			
$\lambda_1 = 0$ and $\lambda_2 < 0$			

Figure 5: Phase portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with positive eigenvalues.

3.3 Complex Eigenvalues

The following is a procedure for finding the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has complex eigenvalues.

1. Identify the complex conjugate eigenvalues $\lambda = \mu \pm i\nu$.

2. Determine the eigenvector $\mathbf{v}$ corresponding to each eigenvalue λ by solving the equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0. \tag{51}$$

3. Express the eigenvectors in the form $\mathbf{v} = \mathbf{a} + i\mathbf{b}$.
4. Write the solution $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda = \mu + i\nu$ as

$$\mathbf{x}_1 = e^{\mu t}(\mathbf{a}\cos\nu t - \mathbf{b}\sin\nu t) + ie^{\mu t}(\mathbf{a}\sin\nu t + \mathbf{b}\cos\nu t). \tag{52}$$

5. The general solution is the linear combination of the solution for each eigenvector.

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots \tag{53}$$

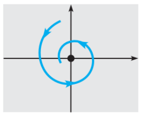
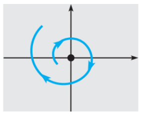
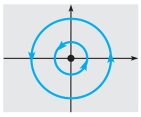
Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\lambda = \mu \pm i\nu$ $\mu < 0$		$(0, 0)$ is a **spiral sink**.	Asymptotically stable
$\lambda = \mu \pm i\nu$ $\mu > 0$		$(0, 0)$ is a **spiral source**.	Unstable
$\lambda = \mu \pm i\nu$ $\mu = 0$		$(0, 0)$ is a **center**.	Stable

Figure 6: Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has complex eigenvalues.

3.4 Repeated Eigenvalues

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ has a repeated eigenvalue λ with multiplicity 2. One solution is given by

$$c_1\mathbf{x}_1 = e^{\lambda t}\mathbf{v}. \tag{54}$$

For the second solution, we can guess that it is of the form

$$\mathbf{x}_2 = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}. \tag{55}$$

If we plug this into the original equation, we see that

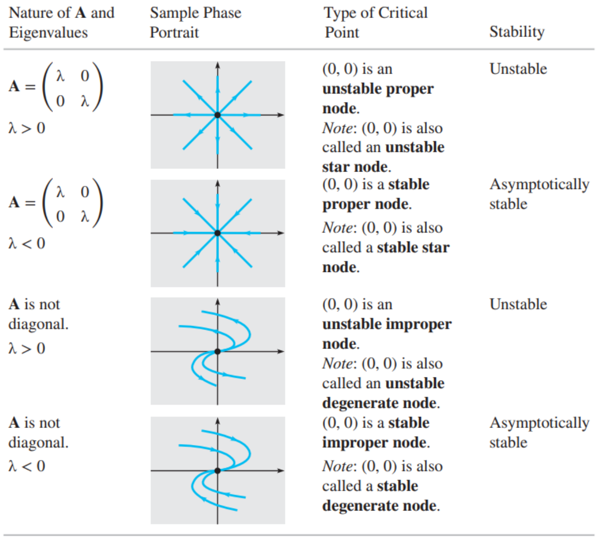
$$(\mathbf{v} + \lambda\mathbf{w})e^{\lambda t} + \lambda\mathbf{v}te^{\lambda t} = \mathbf{A}te^{\lambda t}\mathbf{v} + \mathbf{A}\mathbf{w}e^{\lambda t}. \tag{56}$$

For the equation to hold, we must have

$$(A - \lambda I)\mathbf{v} = 0 \tag{57}$$
$$(A - \lambda I)\mathbf{w} = \mathbf{v}. \tag{58}$$

Thus, the general solution is

$$\mathbf{x} = c_1 e^{\lambda t}\mathbf{v} + c_2\left(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}\right). \tag{59}$$



Nature of **A** and Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda > 0$		(0, 0) is an **unstable proper node**. *Note*: (0, 0) is also called an **unstable star node**.	Unstable
$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda < 0$		(0, 0) is a **stable proper node**. *Note*: (0, 0) is also called a **stable star node**.	Asymptotically stable
A is not diagonal. $\lambda > 0$		(0, 0) is an **unstable improper node**. *Note*: (0, 0) is also called an **unstable degenerate node**.	Unstable
A is not diagonal. $\lambda < 0$		(0, 0) is a **stable improper node**. *Note*: (0, 0) is also called a **stable degenerate node**.	Asymptotically stable

Figure 7: Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has repeated eigenvalues.

3.5 General First Order Linear Systems

First, we will introduce/review a few linear algebra concepts and notation before we show the steps to solving first order linear systems.

Existence and Uniqueness Theorem

For the linear system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{60}$$

if $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval $t_0 \in (\alpha, \beta)$, then there exists a unique solution in the interval $t_0 \in (\alpha, \beta)$.

Wronskian

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be solutions of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and the let $\mathbf{X}$ be the $n \times n$ matrix whose columns are $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then, the **Wronskian** is defined as

$$W = W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det \mathbf{X}(t). \tag{61}$$

$W = 0$ if and only if the solutions are linearly dependent.

Matrix Exponential

The Taylor series definition of matrix exponential is

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}. \tag{62}$$

Properties of Matrix Exponential

$$\begin{aligned} e^{\mathbf{0}t} &= \mathbf{I} && (63)\\ \frac{\mathrm{d}}{\mathrm{d}t}e^{\mathbf{A}t} &= \mathbf{A}e^{\mathbf{A}t} && (64)\\ e^{\mathbf{A}(t+\tau)} &= e^{\mathbf{A}t}e^{\mathbf{A}\tau} && (65)\\ \mathbf{A}e^{\mathbf{A}t} &= e^{\mathbf{A}t}\mathbf{A} && (66)\\ \left(e^{\mathbf{A}t}\right)^{-1} &= e^{-\mathbf{A}t} && (67)\\ e^{(\mathbf{A}+\mathbf{B})t} &= e^{\mathbf{A}t}e^{\mathbf{B}t} \text{ if } \mathbf{AB}=\mathbf{BA}. && (68) \end{aligned}$$

Solution to a General System using Matrices

In general, the solution to the initial value problem of a homogeneous linear ode is given by

$$\mathbf{x}(t) = e^{\mathbf{A}(t)}\mathbf{c}, \tag{69}$$

where $\mathbf{c}$ is a constant vector.

Usually it is quite difficult to determine the entries of $e^{\mathbf{A}(t)}$ in terms of elementary functions using the Taylor series definition. However, for the class of matrices,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \tag{70}$$

it follows that

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}. \tag{71}$$

Therefore, when $\mathbf{A}$ has n linearly independent eigenvectors, using eigendecomposition, the solution to the system is

$$\mathbf{x}(t) = e^{\mathbf{A}t}x_0 = \mathbf{X}e^{\mathbf{D}t}\mathbf{X}^{-1}\mathbf{x}_0, \tag{72}$$

where $\mathbf{X}$ is the matrix whose columns are the eigenvectors of $\mathbf{A}$.

3.6 Defective Matrices

Suppose $\mathbf{A}$ is a real $n \times n$ matrix and λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity m. Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be linearly independent solutions of $(\mathbf{A} - \lambda\mathbf{I}_n)^{m-1}\mathbf{v}_k = \mathbf{0}$. Then,

$$\mathbf{x}_k = e^{\lambda t}\left[\mathbf{v}_k + t(\mathbf{A} - \lambda\mathbf{I}_n)\mathbf{v}_k \cdots + \frac{t^{m-1}}{(m-1)!}(\mathbf{A} - \lambda\mathbf{I}_n)^{m-1}\mathbf{v}_k\right] = e^{\lambda t}e^{(\mathbf{A}-\lambda\mathbf{I}_n)t}\mathbf{v}_k \tag{73}$$

where $k = 1, \dots, m$.

4 Numerical Approximations

4.1 Euler's Method

The solution $y = \phi(t)$ to the initial value problem $\dfrac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$, $y(t_0) = y_0$ can be approximated using Euler's method. Suppose we have a several time steps $t_0 < t_1 < \cdots < t_n$ for $n = 1, 2, \dots$. The approximation of $y = \phi(t)$ at t_n is given by

$$y_n = y_{n-1} + f(t_{n-1}, y_{n-1})(t_n - t_{n-1}). \tag{74}$$

4.2 Accuracy of Numerical Methods

There are two fundamental sources of error in numerical methods: **truncation error** and **round-off error**.

Global truncation error is defined as the difference between the solution of the initial value problem $y = \phi(t)$ and its numerical approximation y_n.

$$E_n = \phi(t_n) - y_n \tag{75}$$

If we know $y_{n-1} = \phi(t_{n-1})$, then this error is known as local truncation error.

Round-off error are errors that arise from the finite number of decimal digits used in the calculation by the computer. The round off error is defined by

$$R_n = y_n - Y_n, \tag{76}$$

where Y_n is the exact value of y_n.

Local truncation error can be calculated using the Taylor series.

$$\phi(t_n + h) = \phi(t_n) + h\phi'(t_n) + \frac{h^2}{2}\phi''(t_n) \tag{77}$$

Therefore, the local truncation error is

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1} = \frac{1}{2}\phi''(t_n)h^2. \tag{78}$$

4.3 Improved Euler and Runge-Kutta Methods

Instead of approximating the integral in the equation

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f(t, \phi(t))\,\mathrm{d}t \tag{79}$$

using a rectangle like in the standard Euler method, we can use a trapezoid. This is known as the **improved Euler method**. The improved Euler method is given by

$$\begin{align} y_{n+1} &= y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right) \tag{80}\\ &= y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right). \tag{81} \end{align}$$

The Runge-Kutta method has a local truncation error proportional to h^5 instead of h^3 like the improved Euler method or h^2 for the standard Euler method.

The Runge-Kutta method is defined as

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{82}$$

where

$$\begin{align} k_1 &= f(t_n, y_n) \tag{83}\\ k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \tag{84}\\ k_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \tag{85}\\ k_4 &= f(t_n + h, y_n + hk_3). \tag{86} \end{align}$$

5 Second Order Differential Equations

Second order ODEs can be written in the form

$$y'' + p(t)y' + q(t)y = g(t). \tag{87}$$

Again, they satisfy the same existence and uniqueness theorem as first order ODEs, and the principle of superposition applies. They can also be converted to a system of two first order equations using the trick as described in Section 3.1.

5.1 Linear Homogeneous Equations with Constant Coefficients

We will find the general solution to the homogeneous equation $ay'' + by' + cy = 0$ in this section. Making the substitutions $x_1 = y$ and $x_2 = y'$, we get the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \mathbf{x}. \tag{88}$$

The eigenvalues are the roots of the characteristic polynomial

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \frac{1}{a}\left(a\lambda^2 + b\lambda + c\right). \tag{89}$$

If λ is an eigenvalue of $\mathbf{A}$, then the corresponding eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}. \tag{90}$$

Below are the different types of solutions depending on the roots of the characteristic polynomial.

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \lambda_1 \neq \lambda_2 \tag{91}$$

$$y = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \quad \text{if } \lambda_1 = \lambda_2 = \lambda \tag{92}$$

$$y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t) \quad \text{if } \lambda = \alpha \pm i\beta \tag{93}$$

5.2 Method of Undetermined Coefficients

For the equation $ay'' + by' + cy = g(t)$, particular solutions for $y(t)$ can be guessed as listed in Table 1. Then, the coefficients can be solved for, and the general solution is the superposition of the homogeneous solution $ay'' + by' + cy = 0$ and the particular solution.

Table 1: Common Guesses for $y(t)$.

$g(t)$	$y(t)$
$P(t)$ (a polynomial of degree n)	$Q(t)$ (a polynomial of degree n)
$P(t)e^{st}$	$Q(t)e^{st}$
$P(t)\cos st$ or $P(t)\sin st$	$Q_1(t)\cos st + Q_2(t)\sin st$
$P(t)e^{st}\cos st$ or $P(t)e^{st}\sin st$	$Q_1(t)e^{st}\cos st + Q_2(t)e^{st}\sin st$

Multiply guess by t if the guess is a solution to the homogeneous equation.

5.3 Variation of Parameters

The variation of parameters method can be applied to any linear nonhomogeneous equation or system.

Variation of Parameters for Two First Order Equations

Consider the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{94}$$

First we must find a fundamental set of solutions to the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. This may be challenging if $\mathbf{P}(t)$ is not constant. The **fundamental matrix** $\mathbf{X}(t)$ is the matrix whose columns are the fundamental solutions to the homogeneous system.

The idea for variation of parameters is that the general solution is of the form

$$\mathbf{x} = u_1(t)\mathbf{x}_1(t) + u_2(t)\mathbf{x}_2(t) = \mathbf{X}(t)\mathbf{u}(t), \tag{95}$$

where $u_1(t)$ and $u_2(t)$ are unknown functions to be determined.

Substituting Equation 95 into Equation 94, we get

$$\mathbf{X}'(t)\mathbf{u}(t) + \mathbf{X}(t)\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{X}(t)\mathbf{u}(t) + \mathbf{g}(t) \tag{96}$$

which simplifies to

$$\mathbf{u}'(t) = \mathbf{X}^{-1}(t)\mathbf{g}(t) \tag{97}$$

since $\mathbf{X}'(t) = \mathbf{P}(t)\mathbf{X}(t)$.

Integrating gives us

$$\mathbf{u}(t) = \int \mathbf{X}^{-1}(t)\mathbf{g}(t)\,\mathrm{d}t + \mathbf{c}. \tag{98}$$

$\mathbf{X}^{-1}(t)$ is

$$\mathbf{X}^{-1}(t) = \frac{1}{W[\mathbf{x}_1, \mathbf{x}_2](t)} \begin{pmatrix} x_{22}(t) & -x_{12}(t) \\ -x_{21}(t) & x_{11}(t) \end{pmatrix}. \tag{99}$$

Thus, the general solution to the nonhomogeneous system is

$$\mathbf{x}(t) = \mathbf{X}(t)\left(\int \mathbf{X}^{-1}(t)\mathbf{g}(t)\,\mathrm{d}t + \mathbf{c}\right). \tag{100}$$

Method of Variation of Parameters for Second Order Equations

If y_1 and y_2 form a complete set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ corresponding to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{101}$$

then a particular solution of the nonhomogeneous equation is given by

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\,\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\,\mathrm{d}t. \tag{102}$$

The general solution is then $y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$.

6 The Laplace Transform

Let f be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$F(s) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t. \tag{103}$$

The Laplace transform of f is defined by both F and $\mathcal{L}\{f\}$.

Linearity

The Laplace transform is a linear operator:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \tag{104}$$

Existence

Suppose

1. f is piecewise continuous on $[0, \infty)$, and
2. f is of exponential order, meaning there exists constants $M \geq 0$, $K > 0$, and a such that $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

The Laplace transform $F(s)$ exists for all $s > a$.

6.1 Properties of the Laplace Transform

1. $$\mathcal{L}\{e^{ct} f(t)\} = F(s - c), \quad s > a + c \tag{105}$$

2. $$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \text{if } f'(t) \text{ is piecewise continuous on } [0, \infty) \tag{106}$$

3. $$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \tag{107}$$

4. $$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \tag{108}$$

6.2 The Inverse Laplace Transform

If $f(t)$ is a piecewise continuous function on $[0, \infty)$ and $\mathcal{L}\{f(t)\} = F(s)$, then f is the inverse Laplace transform of F, denoted by $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Linearity

$$\mathcal{L}^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 \mathcal{L}^{-1}\{F_1(s)\} + c_2 \mathcal{L}^{-1}\{F_2(s)\}. \tag{109}$$

Existence

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, and $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$, then $f(t) = g(t)$ at all points where both f and g are continuous. In other words, both f and g are uniquely determined by their Laplace transforms.

Table 2: Elementary Laplace Transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
16.	$\delta(t-c)$	e^{-cs}
17.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
18.	$t^nf(t)$	$(-1)^nF^{(n)}(s)$

6.3 Solving Differential Equations using the Laplace Transform

1. Transform the differential equation into an algebraic equation in the s-domain using the linearity of the Laplace transform and its operational properties.
2. Solve the algebraic equation for $Y(s)$.
3. Find the inverse Laplace transform of $Y(s)$ to get the solution $y(t)$.

6.4 Discontinuous and Periodic Functions

The Unit Step Function

The unit step function $u_c(t)$ is defined as

$$u_c(t) = \begin{cases} 0 & \text{if}\, t < c \\ 1 & \text{if}\, t \geq c \end{cases} \tag{110}$$

The Laplace transform of the unit step function is

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}. \tag{111}$$

The unit step function is important since it is used to calculate the laplace transform of time shifted functions.

$$u_c(t)f(t-c) = e^{-cs}F(s) \tag{112}$$

Periodic Functions

A function f is periodic with period T if $f(t) = f(t+T)$ for all t.

The Laplace transform of a periodic function is

$$\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st} f(t)\,\mathrm{d}t}{1-e^{-sT}}. \tag{113}$$

Impulse Functions

The impulse function $\delta(t)$ has two properties:

1.
$$\delta(t-t_0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-t_0) = 0 \tag{114}$$

2. For any function continuous on an interval $a \le t_0 < b$ containing t_0,
$$\int_a^b f(t)\delta(t-t_0)\,\mathrm{d}t = f(t_0). \tag{115}$$

The Laplace transform of the impulse function is

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}. \tag{116}$$

6.5 Convolutions

Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0,\infty)$. The convolution of f and g is defined as

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau. \tag{117}$$

Convolutions are commutative, distributive, and associative.

The convolution theorem states that if $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \ge 0$, then

$$F(s)G(s) = \mathcal{L}\{f*g\}. \tag{118}$$

7 Orthogonal Functions and Fourier Series

7.1 Orthogonal Functions

The inner product of two piecewise continuous functions ψ and ϕ on $[a,b]$ is defined as

$$\langle\psi,\phi\rangle = \int_a^b \psi(t)\phi(t)\,\mathrm{d}t. \tag{119}$$

Below are some properties of the inner product.

$$\langle\psi,\phi\rangle = \langle\phi,\psi\rangle \tag{120}$$

$$\langle c_1\psi_1 + c_2\psi_2,\phi\rangle = c_1\langle\psi_1,\phi\rangle + c_2\langle\psi_2,\phi\rangle \tag{121}$$

$$\langle\psi,\psi\rangle = 0 \,\text{if and only if}\, \psi = 0. \tag{122}$$

Two functions ψ and ϕ are orthogonal on $[a,b]$ if $\langle\psi,\phi\rangle = 0$.

The Norm of a function is defined as

$$||\psi|| = \sqrt{\langle\psi,\psi\rangle}. \tag{123}$$

A set $S = \{\psi_1, \psi_2, \dots\}$ of functions is orthogonal on $[a,b]$ if $\langle\psi_i,\psi_j\rangle = 0$ for all $i \neq j$.

Additionally, S is orthonormal if $||\psi_i|| = 1$ for all i.

7.2 Fourier Series

The set of functions

$$\left\{\frac{1}{2}, \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) : \quad m = 1, 2, \ldots\right\} \tag{124}$$

is orthogonal on $[-L, L]$.

Using this fact, we can represent any function as an infinite series of sines and cosines. The Fourier series of a function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \tag{125}$$

The coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x \tag{126}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \tag{127}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{128}$$

8 Partial Differential Equations

8.1 Separation of Variables

In this section, we will go over the steps of solving the simplest type of partial differential equation using the method of separation of variables.

The heat conduction equation is given by

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}. \tag{129}$$

To find solutions for this equation, we will make the assumption that $u(x,t) = X(x)T(t)$ is a product of functions of x and t. Substituting this into the heat conduction equation, we get

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda. \tag{130}$$

It is crucial to realize that the only way this equation can hold is if both sides are equal to a constant. This is because the left side is a function of x only and the right side is a function of t only.

This gives us two different ordinary differential equations to solve,

$$X'' + \lambda X = 0 \tag{131}$$

$$T' + \alpha^2 \lambda T = 0. \tag{132}$$

For the first equation, the general solution is of the form

$$X(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right). \tag{133}$$

Applying the boundary value conditions $X(0) = X(L) = 0$, we find that

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots \tag{134}$$

with $\lambda = \left(\frac{n\pi}{L}\right)^2$.

Thus, the solution to the second equation is proportional to

$$T(t) = e^{-n^2\pi^2\alpha^2 t/L^2}, \tag{135}$$

and we conclude that the functions

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\alpha^2 t/L^2} \tag{136}$$

are solutions to the partial differential equation.

To satisfy the initial conditions $u(x,0) = f(x)$, we can make use of the fact that the solution is in the form of a Fourier series. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\alpha^2 t/L^2}. \tag{137}$$

For $t = 0$, we have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \tag{138}$$

so

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{139}$$