Vibrations and Waves Notes

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Contents

Notes based on the textbook, Vibrations and waves King, George C, and the PHY293 course.

1 Simple Harmonic Oscillators

The chapter will introduce simple harmonic motion (SHM), describing its characteristics and the mathematical equations governing the displacement, velocity, acceleration, and energy of simple harmonic oscillators using a simple mass-spring system.

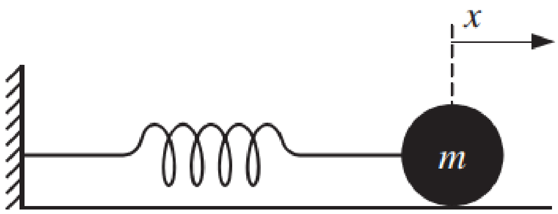


Figure 1: A simple harmonic oscillator consisting of a mass attached to a spring.

1.1 Motion of a Simple Harmonic Oscillator

Hook's law states that the force exerted by a spring is proportional to the displacement of the spring from its equilibrium position. The force exerted by the spring is given by

$$\boxed{F = -kx = ma} \tag{1}$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{2}$$

$$\tag{3}$$

If we let $\boxed{\omega = \sqrt{\frac{k}{m}}}$ be the angular frequency ($\omega = 2\pi/T$), then the equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x. \tag{4}$$

The general solution to this differential equation is

$$\boxed{x(t) = A\cos(\omega t + \phi),} \tag{5}$$

where A is the amplitude of the oscillation and ϕ is the phase angle.

1.2 Energy of a Simple Harmonic Oscillator

Now, let's calculate the energy of the system using Equation 1.

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -kx \tag{6}$$

Multiplying both sides by $v\,\mathrm{d}t = \mathrm{d}x$, we get

$$mv\,\mathrm{d}v = -kx\,\mathrm{d}x \tag{7}$$

$$\int mv\,\mathrm{d}v = -\int kx\,\mathrm{d}x \tag{8}$$

$$\frac{1}{2}mv^2 = -\frac{1}{2}kx^2 + C. \tag{9}$$

This is the same as the law of conservation of energy:

$$\boxed{E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.} \tag{10}$$

2 Damped Harmonic Oscillators

This chapter will discuss damped oscillations, where real oscillating systems, like an apple on a string, lose energy due to dissipative forces such as air resistance.

2.1 Motion of a Damped Harmonic Oscillator

The damping force F_d is proportional to its velocity v as long as v is not too large.

$$F_d = -bv. \tag{11}$$

Including the damping force in the equation of motion, we get

$$ma = -kx - bv \tag{12}$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{13}$$

If we let $\omega_0 = \sqrt{\frac{k}{m}}$ and $\gamma = \frac{b}{m}$, then the equation becomes

$$\boxed{\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0.} \tag{14}$$

ω_0 is known as the resonant frequency, the original frequency of the oscillator without damping or any applied force.

Example: Pendulum

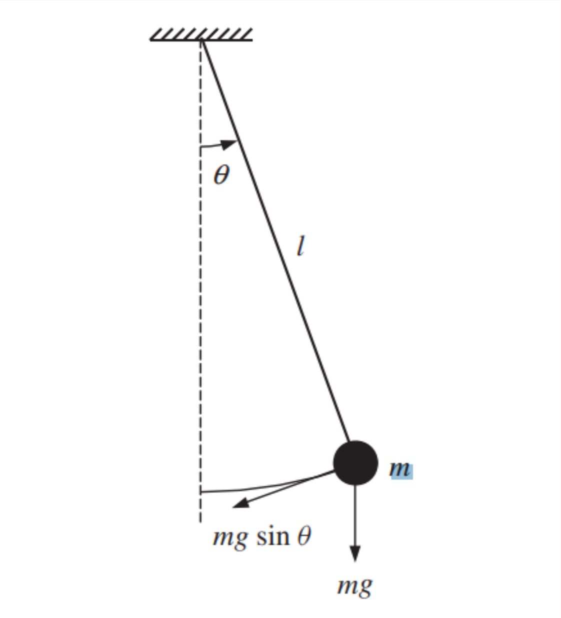


Figure 2: A simple pendulum of mass m and length l.

By Newton's second law, we obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta. \tag{15}$$

For small angles, $\sin\theta \approx \theta$, and the equation becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta \tag{16}$$

which is simple harmonic motion. Thus,

$$\omega = \sqrt{\frac{g}{l}} \tag{17}$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mg}{l}x^2, \tag{18}$$

where $x = l\sin\theta \approx l\theta$.

2.2 Motion of a Lightly Damped Harmonic Oscillator

If we assume that the degree of damping is small, then we will suppose that the equation of motion is of the form

$$\boxed{x = A_0 e^{-\beta t}\cos\omega t.} \tag{19}$$

Taking the derivatives of x with respect to t, we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -A_0 e^{-\beta t}\left(\omega\sin\omega t + \beta\cos\omega t\right) \tag{20}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = A_0 e^{-\beta t}\left[2\beta\omega\sin\omega t + \left(\beta^2 - \omega^2\right)\cos\omega t\right]. \tag{21}$$

$$\tag{22}$$

Substituting this into Equation 14, we get

$$A_0 e^{-\beta t}\left[(2\beta\omega - \gamma\omega)\sin\omega t + (\beta^2 - \omega^2 + \gamma\beta + \omega_0^2)\cos\omega t\right] = 0. \tag{23}$$

Since the above equation must hold for all t, we must have

$$2\beta\omega - \gamma\omega = 0 \tag{24}$$

$$\beta^2 - \omega^2 + \gamma\beta + \omega_0^2 = 0. \tag{25}$$

This gives us the following solutions for β and ω:

$$\boxed{\beta = \frac{\gamma}{2}} \tag{26}$$

$$\boxed{\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}.} \tag{27}$$

Each successive maxima A_n decreases by the same fractional amount.

$$\ln\frac{A_n}{A_{n+1}} = \frac{\gamma T}{2} \tag{28}$$

This is known as the logarithmic decrement.

2.3 Motion of a Heavily Damped Harmonic Oscillator

If we assume that the degree of damping is sufficiently large such that there are no oscillations, then we will suppose that the equation of motion is of the form

$$\boxed{x = e^{-\beta t}f(t).} \tag{29}$$

Taking the derivatives, and substituting into Equation 14, we get

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \left(\omega_0^2 - \gamma^2/4\right) f = 0 \tag{30}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \alpha^2 f. \tag{31}$$

When α is negative, this leads to an imaginary oscillatory solution corresponding with light damping. When α is positive, this corresponds to the desired solution

$$\boxed{f(t) = Ae^{\alpha t} + Be^{-\alpha t}.} \tag{32}$$

2.4 Critical Damping

When $\alpha = 0$, the solution to Equation 31 is

$$\boxed{f(t) = A + Bt.} \tag{33}$$

This is known as critical damping, where the system returns to equilibrium in the shortest time without oscillation.

To summarize, the three cases of damping are as follows:

1. $\gamma^2/4 < \omega_0^2$: Light damping; damped oscillations
2. $\gamma^2/4 = \omega_0^2$: Critical damping; quickest return to equilibrium without oscillation
3. $\gamma^2/4 > \omega_0^2$: Heavy damping; exponential decay of displacement

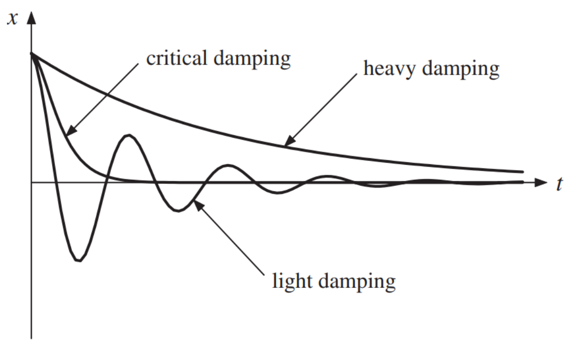


Figure 3: The three cases of damping.

2.5 Rate of Energy Loss in a Damped Harmonic Oscillator

The total energy in the pendulum is still given by Equation 10. We can calculate the energy of a very lightly damped oscillator with the assumption that $\gamma^2/4 \ll \omega_0^2 \implies \omega = \omega_o$. Equation 19 simplifies to

$$x = A_0 e^{-\gamma t/2} \cos \omega_0 t. \tag{34}$$

Taking the derivative, we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -A_0 \omega_0 e^{-\gamma t/2} \left(\sin \omega_0 t + (\gamma/2\omega_0) \cos \omega_0 t\right) \tag{35}$$

$$\approx -A_0 \omega_0 e^{-\gamma t/2} \sin \omega_0 t. \tag{36}$$

Substituting this into Equation 10, we get

$$E = \frac{1}{2}A_0^2 e^{-\gamma t}\left(m\omega_0^2 \sin^2 \omega_0 t + k\cos^2 \omega_0 t\right) \tag{37}$$
$$= \frac{1}{2}kA_0^2 e^{-\gamma t} \tag{38}$$
$$= E_0 e^{-\gamma t}. \tag{39}$$

Note that compared with the amplitude, instead of having a $\gamma/2$ term in the exponent, it now only has γ. Defining $\tau = 1/\gamma$ as the decay time or time constant, we get

$$\boxed{E = E_0 e^{-t/\tau}.} \tag{40}$$

The energy of an oscillator is dissipated because it does work against the damping at the rate of damping force times velocity.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (-bv)v \tag{41}$$

2.6 Quality Factor

The quality factor Q is defined as

$$\boxed{Q = \frac{\omega_0}{\gamma}.} \tag{42}$$

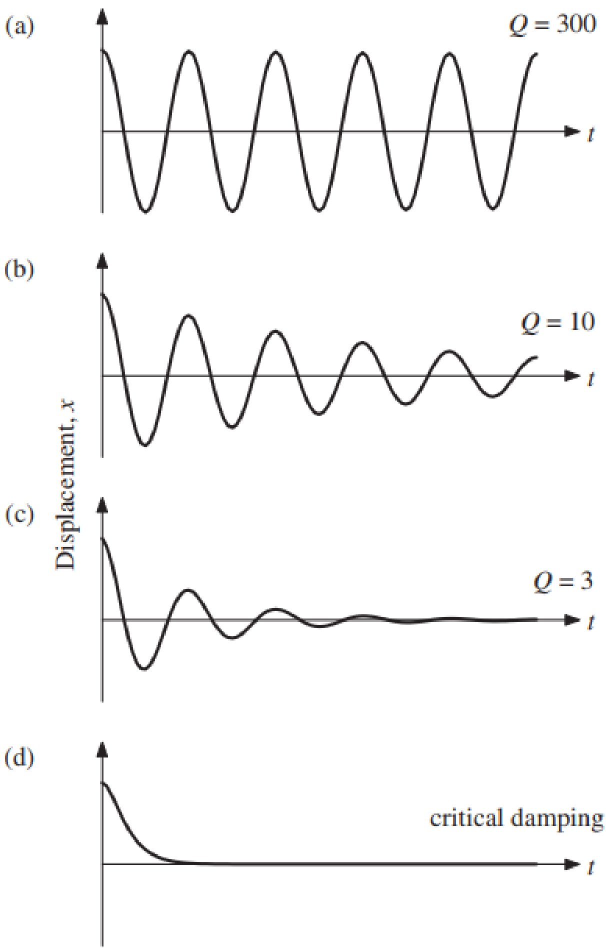


Figure 4: With a higher quality factor, the amplitude of the oscillator decreases more slowly.

Example: Electron Q Factor

An electron in an excited atom behaves like a damped harmonic oscillator when the atom radiates light. The lifetime τ is 10^{-8} s and the wavelength λ is 500 nm. Deduce the Q factor.

Since $\gamma = 1/\tau$ and $\omega_0 = 2\pi f = 2\pi c/\lambda$, we get

$$Q = \frac{\omega_0}{\gamma} = \frac{2\pi c/\lambda}{1/\tau} = \frac{2\pi c\tau}{\lambda} = \frac{2\pi \cdot 3 \times 10^8 \cdot 10^{-8}}{500 \times 10^{-9}} \approx 4 \times 10^7 \tag{43}$$

3 Forced Oscillations

This section will cover forced oscillations, where a periodic driving force is applied to the system.

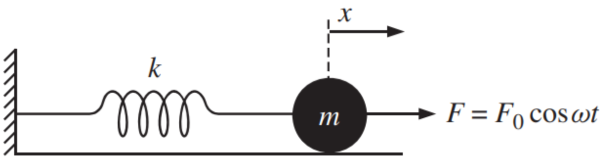


Figure 5: Applying a periodic driving force to a harmonic oscillator.

3.1 Motion of a Forced Harmonic Oscillator

Let us consider an ideal system with a driving force $F_d = F_0 \cos \omega t = ka \cos \omega t$. We will assume that there is no damping, and that the angular frequency of the oscillator is the same as the driving force.

$$\boxed{m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = ka \cos \omega t} \tag{44}$$

Because at all frequencies, the mass moves periodically at the same frequency ω as the driving force, this suggests that the displacement x can be written as

$$x = A(\omega) \cos(\omega t - \delta) \tag{45}$$

$A(\omega)$ is the amplitude of the oscillator, and δ is the phase angle **between the driving force and the displacement**.

Substituting x and its second derivative into Equation 44, and using $\omega_0 = \sqrt{k/m}$, we get

$$-\omega^2 A(\omega) \cos(\omega t - \delta) + \omega_0^2 A(\omega) \cos(\omega t - \delta) = \omega_0^2 a \cos \omega t \tag{46}$$

$$-\omega_0^2 A(\omega)(\cos \omega t \cos \delta + \sin \omega t \sin \delta) + \omega_0^2 A(\omega)(\cos \omega t \cos \delta + \sin \omega t \sin \delta) = \omega_0^2 a \cos \omega t \tag{47}$$

After equating the coefficients of $\cos \omega t$ and $\sin \omega t$, we get

$$A(\omega)(1 - \omega^2/\omega_0^2) \cos \delta = a \tag{48}$$

and

$$A(\omega)(1 - \omega^2/\omega_0^2) \sin \delta = 0. \tag{49}$$

Dividing the two equations, we get $\tan \delta = 0$, which implies that $\delta = 0$ or π. Substituting this in gives us

$$\boxed{A(\omega) = \frac{a}{1 - \omega^2/\omega_0^2}.} \tag{50}$$

Because the amplitude was defined as positive, $\delta = 0$ when $\omega < \omega_0$ and $\delta = \pi$ when $\omega > \omega_0$.

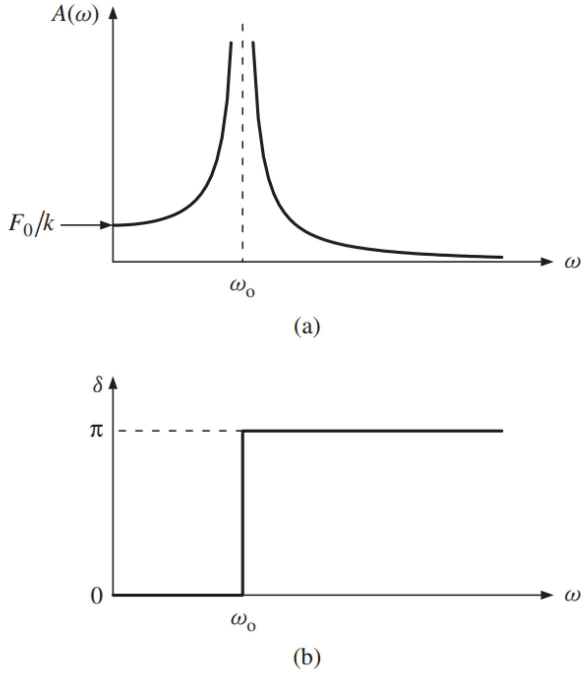


Figure 6: Plots of $A(\omega)$ and δ as a function of ω.

3.2 Motion of a Forced Damped Harmonic Oscillator

Adding the damping term $b\dfrac{\mathrm{d}x}{\mathrm{d}t}$ to Equation 44, dividing by m, and making the substitutions $\gamma = b/m$ and $\omega_0 = \sqrt{k/m}$, we get

$$\boxed{\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \omega_0^2 a \cos\omega t.} \tag{51}$$

Again, substituting $x = A(\omega)\cos(\omega t - \delta)$ into the equation, we get the two equations

$$A(\omega)\left[\left(\omega_0^2 - \omega^2\right)\cos\delta + \omega\gamma\sin\delta\right] = \omega_0^2 a \tag{52}$$
$$\left(\omega_0^2 - \omega^2\right)\sin\delta = \omega\gamma\cos\delta. \tag{53}$$

Therefore, we have

$$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}. \tag{54}$$

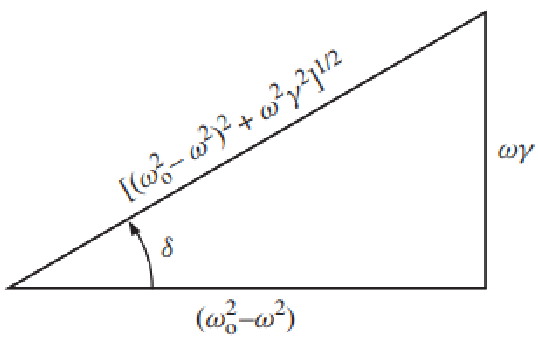


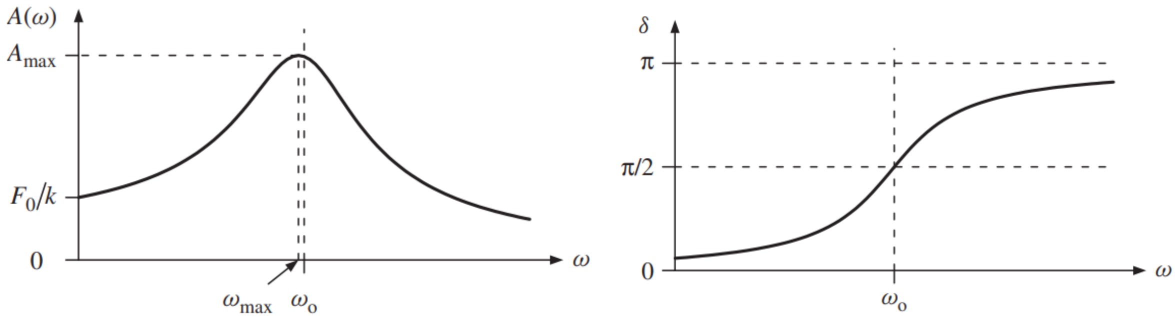
Figure 7: Geometrical construction for the phase angle δ.

If we substitute the values of $\sin\delta$ and $\cos\delta$ derived by the above phase angle triangle into the amplitude equation, we get

$$\boxed{A(\omega) = \frac{a\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}.} \tag{55}$$

The frequency for maximum amplitude does not occur at $\omega=\omega_0$ anymore, but at $\boxed{\omega = \omega_0\sqrt{1-\gamma^2/2\omega_0^2}.}$ Thus, the max amplitude will be

$$\boxed{A_{\text{max}} = \frac{a\omega_0/\gamma}{\sqrt{1-\gamma^2/4\omega_0^2}}.} \tag{56}$$



(a) (b)

Figure 8: (a) A plot of $A(\omega)$ as a function of ω for a forced damped oscillator. (b) A plot of δ as a function of ω for a forced damped oscillator.

Example: Electrical Circuits

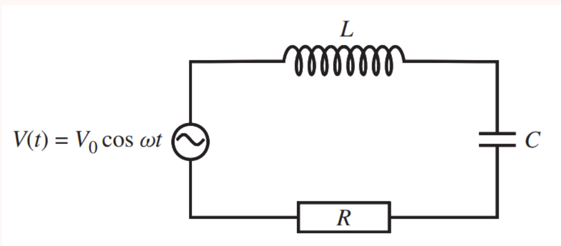


Figure 9: An LCR resonance circuit that is driven by an alternating voltage $V_0\cos\omega t$.

The differential equation for this circuit is

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = V_0\cos\omega t. \tag{57}$$

Comparing this with Equation 51, we see that

$$\omega_0^2 = \frac{1}{LC}, \quad \gamma = \frac{R}{L}, \quad a = V_0C. \tag{58}$$

3.3 Power Absorbed During Forced Oscillations

Differentiating the position of the oscillator with respect to time, we get

$$v = -v_0(\omega)\sin(\omega t - \delta) \tag{59}$$

where $v_0(\omega) = \omega A(\omega)$ is the velocity amplitude. The rate of energy loss due to damping is equal to the damping force times the velocity.

$$P(t) = b[v(t)]^2 = b\left[v_0(\omega)\right]^2 \sin^2(\omega t - \delta) \tag{60}$$

The average power absorbed over a complete cycle of oscillation is given by

$$\bar{P}(\omega) = \frac{1}{T}\int_{t_0}^{t_0+T} P(t)\,\mathrm{d}t = \frac{b\left[v_0(\omega)\right]^2}{T}\int_{t_0}^{t_0+T} \sin^2(\omega t - \delta)\,\mathrm{d}t. \tag{61}$$

The integral of $\sin^2(\omega t - \delta)$ over a complete cycle is $T/2$, so the average power absorbed is

$$\bar{P}(\omega) = \frac{b\left[v_0(\omega)\right]^2}{2} \tag{62}$$

$$= \frac{\omega^2 F_0 \gamma}{2m\left[\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2\right]} \tag{63}$$

$$\boxed{\approx \frac{F_0^2}{2m\gamma\left(4\Delta\omega^2/\gamma^2 + 1\right)}.} \tag{64}$$

The approximation, $\omega \approx \omega_0$ was made where $\Delta\omega = \omega - \omega_0$. The maximum value of $\bar{P}(\omega)$ occurs at $\omega = \omega_0$ and is given by

$$\boxed{\bar{P}_{\max} = \frac{F_0^2}{2m\gamma}.} \tag{65}$$

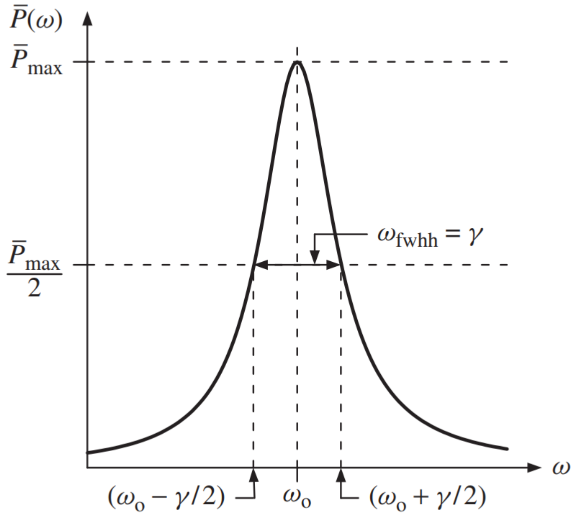


Figure 10: The power resonance curve of a forced oscillator. The full width at half height ω_{fwhh} is equal to γ.

3.4 Transient Response

The initial behavior of the oscillator before it settles down is its transient response.

Suppose that a driving force is applied to a damped oscillator and x_1 is a solution to the equation.

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x_1}{\mathrm{d}t} + \omega_0^2 x_1 = \frac{F_0}{m}\cos\omega t. \tag{66}$$

Also, suppose that x_2 is a solution to the damped oscillator without the driving force.

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x_2}{\mathrm{d}t} + \omega_0^2 x_2 = 0. \tag{67}$$

Then, adding these two equations gives us

$$\frac{\mathrm{d}^2 x_1 + x_2}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x_1 + x_2}{\mathrm{d}t} + \omega_0^2 (x_1 + x_2) = \frac{F_0}{m} \cos \omega t. \tag{68}$$

We see that $x_1 + x_2$ is also a solution to the forced oscillator.

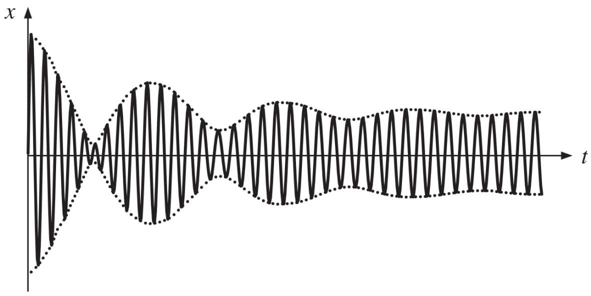


Figure 11: An example of the transient response of a forced oscillator.

3.5 The Use of Complex Numbers

Complex numbers can be used to greatly simplify the mathematics used to solve differential equations for oscillators. Let

$$z = Ae^{i(\omega t + \phi)} \tag{69}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \omega A e^{i(\omega t + \phi + \pi/2)} \tag{70}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \omega^2 A e^{i(\omega t + \phi + \pi)}. \tag{71}$$

Notice that $\mathrm{Re}(z) = A \cos \omega t + \phi = x$. Instead of substituting x into the differential equation, we can substitute z and its derivatives into the differential equation.

4 Coupled Oscillators

Coupling oscillators can lead to multiple ways which the system can oscillate leading to multiple frequencies. These different ways are known as **normal modes**, and the frequencies are known as **normal frequencies**.

4.1 Superposition of Normal Modes

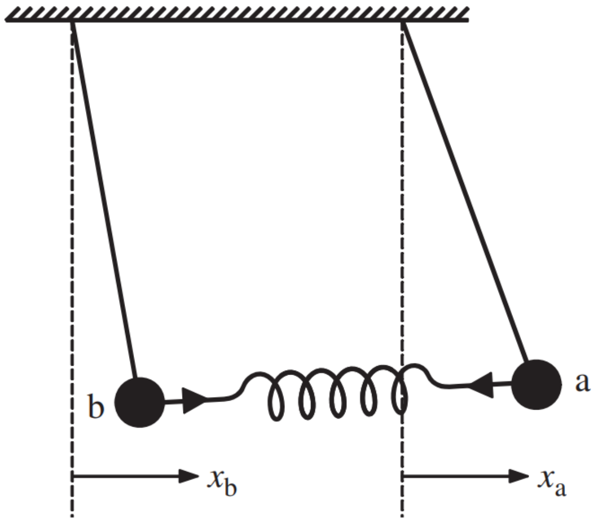


Figure 12: Two oscillators are coupled together by a spring.

The forces acting on the masses are

$$F_a = -\frac{mgx_a}{l} - k(x_a - x_b) \tag{72}$$

$$F_b = -\frac{mgx_b}{l} + k(x_a - x_b). \tag{73}$$

The resultant equations of motion are

$$m\frac{\mathrm{d}^2 x_a}{\mathrm{d}t^2} + \frac{mgx_a}{l} + k(x_a - x_b) = 0 \tag{74}$$

$$m\frac{\mathrm{d}^2 x_b}{\mathrm{d}t^2} + \frac{mgx_b}{l} - k(x_a - x_b) = 0. \tag{75}$$

Adding the two equations of motion gives us the equation for the first normal mode.

$$\frac{\mathrm{d}^2(x_a + x_b)}{\mathrm{d}t^2} + \frac{g}{l}(x_a + x_b) = 0 \tag{76}$$

Subtracting the two equations of motion gives us the equation for the second normal mode.

$$\frac{\mathrm{d}^2(x_a - x_b)}{\mathrm{d}t^2} + \left(\frac{g}{l} + \frac{2k}{m}\right)(x_a - x_b) = 0. \tag{77}$$

The two normal frequencies are

$$\omega_1 = \sqrt{\frac{g}{l}} \tag{78}$$

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}. \tag{79}$$

If we introduce the new variables $q_1 = x_a + x_b$ and $q_2 = x_a - x_b$, we can rewrite the equations of motion as

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} + \omega_1^2 q_1 = 0 \tag{80}$$

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} + \omega_2^2 q_2 = 0. \tag{81}$$

The solutions to the equations of motion are

$$q_1 = C_1 \cos(\omega_1 t + \phi_1) \tag{82}$$
$$q_2 = C_2 \cos(\omega_2 t + \phi_2). \tag{83}$$

We can rewrite x_a and x_b in terms of q_1 and q_2 as

$$x_a = \frac{1}{2}(q_1 + q_2) \tag{84}$$
$$x_b = \frac{1}{2}(q_1 - q_2). \tag{85}$$

Any solution of two masses can be written as a superposition of the two normal modes.

4.2 Matrix Representation of Coupled Oscillators

For this example, we will solve the equations of motion for two coupled masses using matrices.

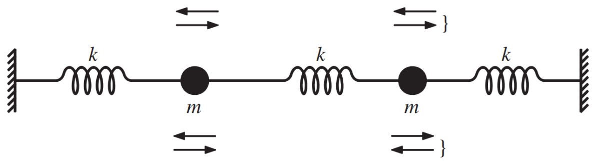


Figure 13: Two mass spring oscillators coupled together by a third spring.

The equations of motion for this mass spring system are

$$m\frac{\mathrm{d}^2 x_a}{\mathrm{d}t^2} = kx_b - 2kx_a \tag{86}$$
$$m\frac{\mathrm{d}^2 x_b}{\mathrm{d}t^2} = kx_a - 2kx_b. \tag{87}$$

If we substitute $x_a = A\cos\omega t$ and $x_b = B\cos\omega t$ into the equations of motion, we get

$$-Am\omega^2 \cos\omega t = kB\cos\omega t - 2kA\cos\omega t \tag{88}$$
$$-Bm\omega^2 \cos\omega t = kA\cos\omega t - 2kB\cos\omega t. \tag{89}$$

Our goal is to look for normal mode of solutions where both masses oscillate at the same frequency ω. In other words, this can be transposed into an eigenvalue problem where ω^2 is the eigenvalue. Representing the equation in matrix form,

$$\begin{bmatrix} \frac{2k}{m}, -\frac{k}{m} \\ -\frac{k}{m}, \frac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix} \tag{90}$$
$$\begin{bmatrix} \frac{2k}{m} - \omega^2, -\frac{k}{m} \\ -\frac{k}{m}, \frac{2k}{m} - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{91}$$

The equation has non-zero solutions if and only if the determinant of the matrix is zero. Thus, we have

$$\left(\frac{2k}{m} - \omega^2\right)^2 - \left(-\frac{k}{m}\right)^2 = 0. \tag{92}$$

The roots of this quadratic equation are $\omega_1^2 = \frac{k}{m}$ and $\omega_2^2 = \frac{3k}{m}$. Substituting these values back into the equation give $A = B$ and $A = -B$ respectively.

4.3 Forced Coupled Oscillators

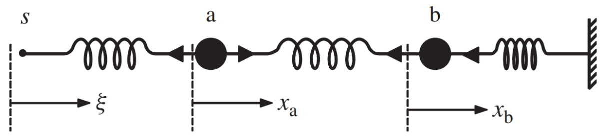


Figure 14: The end s of the spring is moved harmonically as $\xi = a\cos\omega t$

The equations of motion are

$$\frac{\mathrm{d}^2 x_a}{\mathrm{d}t^2} + \frac{2k}{m}x_a - \frac{k}{m}x_b = \frac{F}{m}\cos\omega t \tag{93}$$

$$\frac{\mathrm{d}^2 x_b}{\mathrm{d}t^2} - \frac{k}{m}x_a + \frac{2k}{m}x_b = 0. \tag{94}$$

As always, we can add and subtract them.

$$\frac{\mathrm{d}^2(x_a + x_b)}{\mathrm{d}t^2} + \frac{k}{m}(x_a + x_b) = \frac{F}{m}\cos\omega t \tag{95}$$

$$\frac{\mathrm{d}^2(x_a - x_b)}{\mathrm{d}t^2} + \frac{3k}{m}(x_a - x_b) = \frac{F}{m}\cos\omega t. \tag{96}$$

If we let $q_1 = x_a + x_b$ and $q_2 = x_a - x_b$, we can rewrite the equations of motion as

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} + \frac{k}{m}q_1 = \frac{F}{m}\cos\omega t \tag{97}$$

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} + \frac{3k}{m}q_2 = \frac{F}{m}\cos\omega t. \tag{98}$$

These equations for the different modes of oscillation can be then solved using methods as described in the previous sections.

5 Travelling Waves

Travelling waves, which manifest in various physical phenomena from ripples on a pond to seismic activity and electromagnetic waves, are central to many scientific fields.

5.1 Travelling Wave Examples

Waves must retain their shape as they travel.

Rope

A travelling wave on a rope can be described by a Gaussian wave function

$$y(x,t) = Ae^{-(x^2/a^2)}. \tag{99}$$

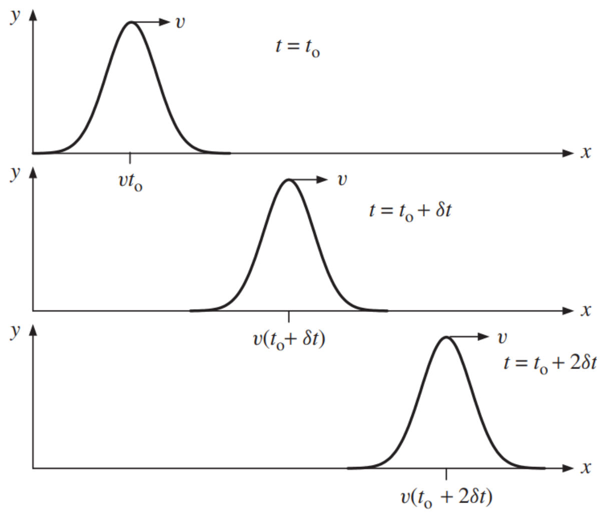


Figure 15: The shape of the rope will have a general form $f(x - vt)$.

Sinusoidal Waves

A travelling sinusoidal wave can be described by

$$y(x,t) = A \sin \frac{2\pi}{\lambda}(x - vt). \tag{100}$$

The wave number is $k = 2\pi/\lambda$. The angular frequency is $\omega = 2\pi v/\lambda$.

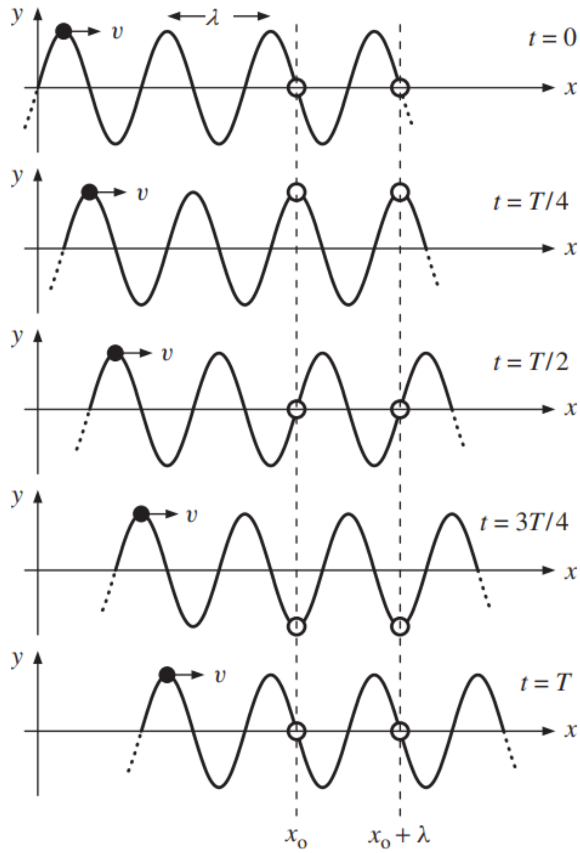


Figure 16: Schematic representation of a travelling sinusoidal wave of wavelength λ and period T.

5.2 The Wave Equation

The general form of any wave motion is given by

$$y(x,t) = f(x - vt) + g(x + vt). \tag{101}$$

We start with the function $f(x - vt)$ and change variables to $u = x - vt$ to obtain $f(u)$. Then,

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} \tag{102}$$

and

$$\frac{\partial^2 f}{\partial x^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial^2 u}{\partial x^2}. \tag{103}$$

Since $\dfrac{\partial u}{\partial x} = 1$ and $\dfrac{\partial^2 u}{\partial x^2} = 0$, we get

$$\frac{\partial^2 f}{\partial x^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}u^2} \tag{104}$$

Differentiation with respect to t now gives

$$\frac{\partial^2 f}{\partial t^2} = v^2\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}. \tag{105}$$

Combining the previous two equations gives us the wave equation

$$\boxed{\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x^2}.} \tag{106}$$

5.3 The Equation of a Vibrating String

We now derive the equation of motion for transverse vibrations on a taut string.

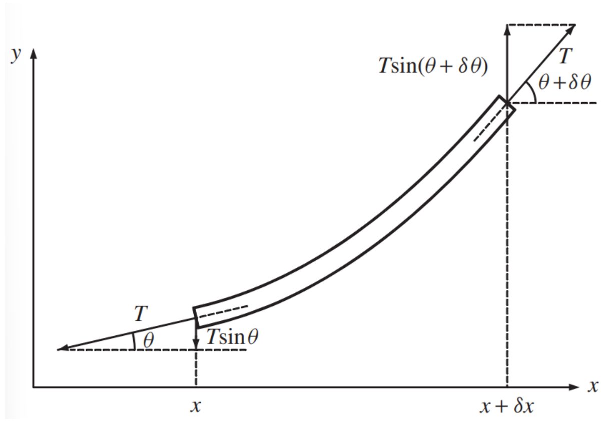


Figure 17: Segment of a taut string carrying a wave.

From the diagram above, it can be seen that the transverse force at x is equal to

$$F_y = T\left(\frac{\partial y}{\partial x}\right)_x. \tag{107}$$

Similarly, the transverse force at $x + \partial x$ is equal to

$$T\left[\left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\partial x\right)\right] \tag{108}$$

since $\left(\frac{\partial y}{\partial x}\right)_{x+\partial x} = \left(\frac{\partial y}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\partial x\right)$. Therefore, the resultant transverse force on the string is

$$T\left[\left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)\partial x - \left(\frac{\partial y}{\partial x}\right)_x\right] = T\left(\frac{\partial^2 y}{\partial x^2}\right)\partial x. \tag{109}$$

We also know that the mass of the string segment is $\mu\partial x$ where μ is the mass per unit length. By Newton's second law, we have

$$\mu\partial x\frac{\partial^2 y}{\partial t^2} = T\left(\frac{\partial^2 y}{\partial x^2}\right)\partial x \tag{110}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}. \tag{111}$$

By comparing this with the wave equation, we see that the speed of the wave is given by

$$\boxed{v = \sqrt{\frac{T}{\mu}}.} \tag{112}$$

5.4 The Energy In A Wave

As the wave moves along the string, the segments will have kinetic energy

$$K = \frac{1}{2}\mu\partial x\left(\frac{\partial y}{\partial t}\right)^2. \tag{113}$$

Furthermore, the stretched string will have a potential energy U. To calculate potential energy, we will assume that the extended length ∂s can be approximated by

$$\partial s = \frac{\partial x}{\cos\theta} = \frac{\partial x}{\left(1-\sin^2\theta\right)^{1/2}} \approx \partial x\left(1+\frac{1}{2}\sin^2\theta\right) \approx \partial x\left(1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right). \tag{114}$$

Thus,

$$U = T\left(\partial s - \partial x\right) = \frac{1}{2}T\partial x\left(\frac{\partial y}{\partial x}\right)^2. \tag{115}$$

The total energy of the wave can be calculated by summing up the kinetic and potential energies.

$$E = \frac{1}{2}\int\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x \tag{116}$$

For example, if we want to calculate the energy in one wavelength of the wave $y = A\sin kx - \omega t$, we can substitute the function into equation 116 to get

$$E = \frac{1}{2}\mu\omega^2 A^2\lambda. \tag{117}$$

The average power of the wave would then be

$$P = E\frac{v}{\lambda} = \frac{1}{2}\mu\omega^2 A^2 v. \tag{118}$$

Power can be also calculated using the formula

$$P = Fv = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \tag{119}$$

where T is the tension in the string.

5.5 Waves At Discontinuities

When a wave encounters a boundary between two different media, there will be an incident wave (the wave before hitting the boundary), a transmitted wave (the wave after the boundary), and a reflected wave (the wave which travels backwards).

Let us consider the case where two strings with different values of μ are joined together at a boundary $x = 0$. Two conditions must be satisfied at the discontinuity:

1. The displacements of the strings must be the same at $x = 0$ for all times.
2. There must be a continuity in the transverse restoring force at the boundary.

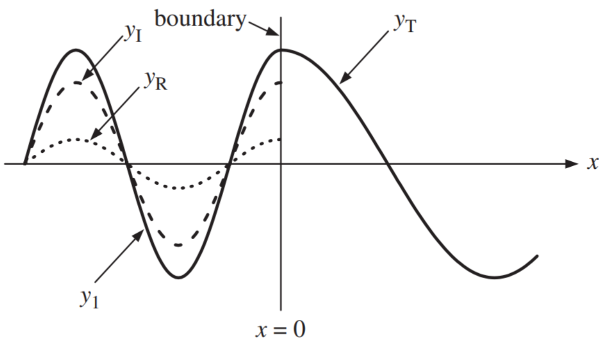


Figure 18: The incident (y_I), reflected (y_R), and transmitted (y_T) waves at a boundary.

Suppose we defined the incident wave, transmitted wave, and reflected wave as

$$y_I = A_1 \cos(\omega t - k_1 x) \tag{120}$$
$$y_T = A_2 \cos(\omega t - k_2 x) \tag{121}$$
$$y_R = B_1 \cos(\omega t + k_1 x) \tag{122}$$

where $k = \omega/v$ is the wave number.

The left-hand side of the string is given by $y_1 = y_I + y_R$ and the right-hand side is given by $y_2 = y_T$.

By condition 1, we have

$$y_1 = y_2 \tag{123}$$
$$A_1 + B_1 = A_2. \tag{124}$$

Condition 2 gives

$$\partial y_1/\partial x = \partial y_2/\partial x \tag{125}$$
$$k_1 A_1 - k_1 B_1 = k_2 A_2. \tag{126}$$

Since $v = \omega/k = \sqrt{T/\mu}$, the transmission coefficient of amplitude is given by

$$T = \frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}. \tag{127}$$

The reflection coefficient of amplitude is given by

$$R = \frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}. \tag{128}$$

$\sqrt{\mu}$ is known as the impedance of the string. When the second string has a higher mass per unit length than the first string, the reflected wave will be inverted.

Impedance is something that hinders progression.

$$Z_{\text{electrical}} = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \tag{129}$$
$$Z_{\text{acoustic}} = \sqrt{B\rho} \tag{130}$$
$$Z_{\text{mechanical}} = v\mu = \sqrt{F_T \mu}. \tag{131}$$

A useful equality is

$$1 + R = T. \tag{132}$$

Similarly, we also have the reflection and transmission coefficients of power which can be found in terms of R and T as

$$R_e = R^2 \tag{133}$$
$$T_e = \frac{k_2}{k_1} T^2 \tag{134}$$
$$R_e + T_e = 1. \tag{135}$$

Example: Transmission Coefficient of Amplitude

A wave reaches boundary at which 24% of the power gets transmitted and the incident and reflected waves are out of phase. What is the value of the transmission coefficient for the wave's amplitude?

We know that $R_e = 1 - T_e = 0.76$ and $R = \sqrt{R_e} = \pm 0.87$. Since the reflected wave is out of phase, $R = -0.87$. Thus, $T = 1 + R = 0.13$.

5.6 Waves in Higher Dimensions

The wave equation in two dimensions is given by

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2} \tag{136}$$

where $v^2 = S/\Omega$, S is the tension in surface, and Ω is the mass per unit area.

For radial two dimensional waves, the wave equation is given by

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}. \tag{137}$$

The three-dimensional wave equation is given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}. \tag{138}$$

Example: Sound

The speed of sound is different in various media. For example, in gases, the speed of sound is given by

$$v = \sqrt{\frac{\gamma R T}{M}} \tag{139}$$

where γ is the adiabatic index (1.67 for monoatomic, 1.4 for diatomic), R is the gas constant, T is the temperature, and M is the molar mass of the gas.

The speed of sound in a liquid is given by

$$v = \sqrt{\frac{B}{\rho}} \tag{140}$$

where B is the bulk modulus and ρ is the density of the liquid.

The speed of sound in a solid is given by

$$v = \sqrt{\frac{Y}{\rho}} \tag{141}$$

where Y is the Young's modulus and ρ is the density of the solid.

The intensity of a sound wave is in terms of average power per unit area.

$$I = \frac{\sqrt{\rho B} A^2 \omega^2}{2} = \frac{\Delta p_{max}^2}{2\rho v} \tag{142}$$

where ρ is the density of the medium, B is the bulk modulus, A is the amplitude of the wave, Δp_{max}^2 is the pressure amplitude, and ω is the angular frequency.

Sound intensity with known distances can be calculated using the formula

$$I_2 = I_1 e^{-\alpha(r_2 - r_1)} \left(\frac{r_1}{r_2}\right)^{N-1} \tag{143}$$

where α is the attenuation coefficient, r_1 and r_2 are the distances, and N is the number of dimensions.

Sound intensity level in decibels is given by

$$\beta = 10 \log_{10}\left(\frac{I}{I_0}\right) \tag{144}$$

where $I_0 = 10^{-12}$ W/m^2 is the threshold of hearing.

5.7 Interference

Two waves are **coherent** if they are monochromatic (same frequency and wavelength), and there is a constant phase relation between them (in phase, out of phase, or something in between).

For two coherent waves, constructive and destructive interference is defined by their phase difference s.

$$s = n\lambda \iff \phi = 2n\pi \quad \text{Constructive Interference} \tag{145}$$

$$s = (n + 1/2)\lambda \iff \phi = (2n+1)\pi \quad \text{Destructive Interference} \tag{146}$$

Huygen postulated that every point on a wavefront acts as a source of secondary wavelets.

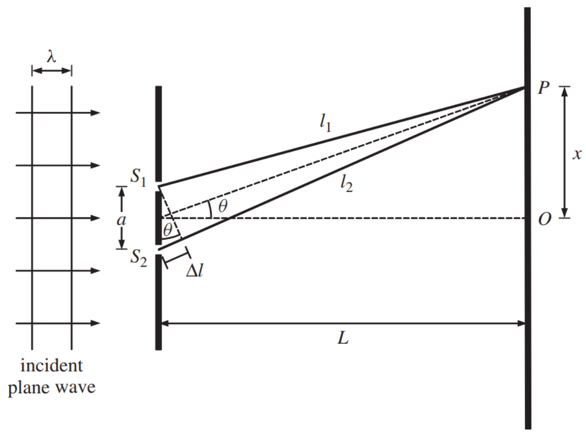


Figure 19: Young's double slit experiment.

In Young's double slit experiment, light diffracts through two slits and creates an interference pattern on a screen. When L is much larger than A and when θ is small, the intensity of the interference pattern is given by

$$I = I_0 \cos^2\left(\frac{\pi a \sin\theta}{\lambda}\right). \tag{147}$$

mth order constructive interference occurs when

$$a\sin\theta = m\lambda. \tag{148}$$

nth order destructive interference occurs when

$$a\sin\theta = (n + 1/2)\lambda. \tag{149}$$

6 Standing Waves

The general equation to represent the displacement of a standing wave is

$$y(x,t) = (A\sin kx + B\cos kx)(\cos\omega t + \sin\omega t). \tag{150}$$

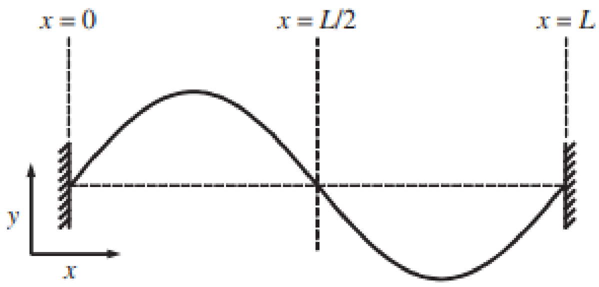


Figure 20: Standing wave on a taut string.

For a system fixed at both ends like a string, the angular frequency is given by

$$\omega_n = \frac{n\pi v}{L}, \tag{151}$$

and the displacement is given by the equation

$$y(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t). \tag{152}$$

$n = 1$ is known as the fundamental frequency $f_1 = \frac{v}{2L}$, $n = 2$ is the second harmonic $f_2 = \frac{v}{L}$, $n = 3$ is the third harmonic $f_3 = \frac{3v}{2L}$, and so on. The difference between two consecutive harmonic is the fundamental frequency f_1.

A system open at both ends have the same frequencies as a system fixed at both ends, but the nodes and antinodes are swapped.

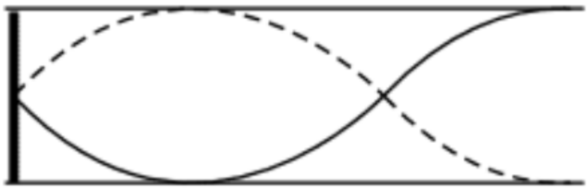


Figure 21: A standing wave in a pipe with an open and closed end.

When one end of a pipe is closed, only odd harmonics are present, and the difference between two consecutive harmonics is $2f_1$.

$$f_1 = \frac{v}{4L}, f_3 = \frac{3v}{4L}, f_5 = \frac{5v}{4L}, \ldots \tag{153}$$

6.1 The Energy of a Standing Wave

Applying Equation 116 to a standing wave gives

$$K_n = \int_0^L \frac{1}{2}\mu A_n^2 \omega_n^2 \sin^2\left(\frac{n\pi}{L}x\right)\sin^2(\omega_n t)\,\mathrm{d}x \tag{154}$$

$$U_n = \int_0^L \frac{1}{2}\mu v^2 A_n^2 \left(\frac{n\pi}{L}\right)^2 \cos^2\left(\frac{n\pi}{L}x\right)\cos^2(\omega_n t)\,\mathrm{d}x \tag{155}$$

$$E_n = K_n + U_n = \frac{1}{4}\mu\omega_n^2 A_n^2 L. \tag{156}$$

The average power of a standing wave can be calculated by replacing L with v in the equation above.

$$P = \frac{1}{4}\mu\omega_n^2 A_n^2 v. \tag{157}$$

6.2 Superposition of Standing Waves

Any shape $f(x)$ of a string with fixed end points ($f(0) = f(L) = 0$) can be written as a superposition of sine functions which have frequencies of $n\pi/L$ with appropriate values of coefficient of A_n.

$$f(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right). \tag{158}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x. \tag{159}$$

7 Dispersion of Waves

7.1 Superposition of Waves in Non-Dispersive Media

Let us consider the superposition of two waves of the same amplitude with different frequencies ω_1 and ω_2.

$$\psi = A\cos(k_1 x - \omega_1 t), \quad \psi_2 = A\cos(k_2 x - \omega_2 t) \tag{160}$$

In a non-dispersive medium, these two waves travel at the same velocity.

$$v = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} \tag{161}$$

The superposition of these two waves is given by

$$\psi = \psi_1 + \psi_2 = 2A\cos\left[\frac{k_2 - k_1}{2}x - \frac{\omega_2 - \omega_1}{2}t\right]\cos\left[\frac{k_2 + k_1}{2}x - \frac{\omega_1 + \omega_2}{2}t\right]. \tag{162}$$

This results in two frequencies being created: $\frac{\omega_1+\omega_2}{2}$ and $\frac{\omega_2-\omega_1}{2}$. The latter is known as the **beat** frequency.

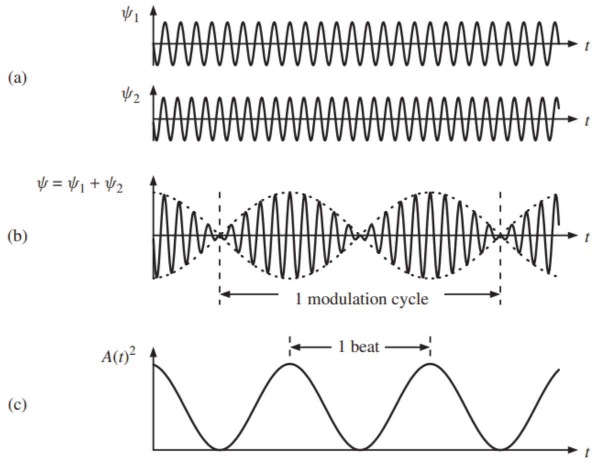


Figure 22: The superposition of two monochromatic waves with the same amplitude but slightly different frequencies. The resultant wave is contained within an **envelope** as shown by the dotted lines.

7.2 Phase and Group Velocities

In dispersive medium, the velocity and frequency of the wave are functions of the wavenumber k.

Again we turn back to the superposition of two waves with different frequencies ω_1 and ω_2 except this time in a dispersive medium.

$$\psi = \psi_1 + \psi_2 = 2A\cos\left[\frac{k_2 - k_1}{2}x - \frac{\omega_2 - \omega_1}{2}t\right]\cos\left[\frac{k_2 + k_1}{2}x - \frac{\omega_1 + \omega_2}{2}t\right]. \tag{163}$$

Let

$$\omega_0 = \frac{\omega_1 + \omega_2}{2}, \quad k_0 = \frac{k_1 + k_2}{2}, \quad \Delta\omega = \frac{\omega_2 - \omega_1}{2}, \quad \Delta k = \frac{k_2 - k_1}{2}. \tag{164}$$

Then,

$$\psi = A(x,t)\cos(k_0 x - \omega_0 t) \tag{165}$$

where

$$A(x,t) = 2A\cos(\Delta k x - \Delta\omega t). \tag{166}$$

The phase velocity is given by

$$v = \frac{\omega_0}{k_0}. \tag{167}$$

The amplitude of the crest must remain constant as the envelope travels, so

$$\Delta k x - \Delta \omega t = \text{constant}. \tag{168}$$

Differentiating this equation with respect to t, we can solve for the group velocity:

$$v_g = \left(\frac{\mathrm{d}\omega}{\mathrm{d}k} \right)_{k=k_0}. \tag{169}$$

Group velocity can be related with phase velocity by the equation

$$v_g = \frac{\mathrm{d}(kv)}{\mathrm{d}k} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = v + \lambda \frac{\mathrm{d}v}{\mathrm{d}\lambda}. \tag{170}$$

Normal dispersion occurs when $v_g < v$ and anomalous dispersion occurs when $v_g > v$.

7.3 The Dispersion Relation

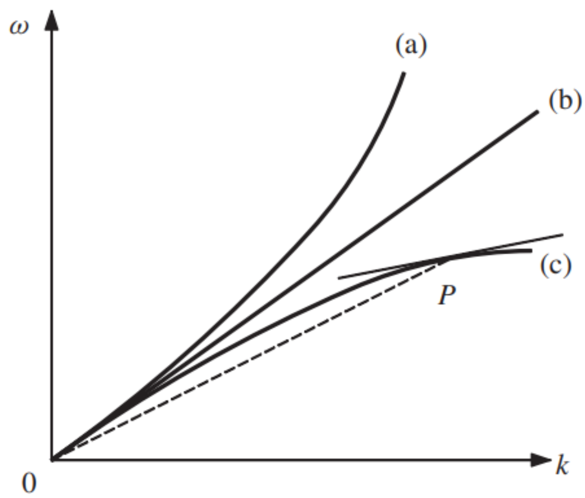


Figure 23: Plots of frequency ω against wavenumber k for various dispersion relations. (a) corresponds to anomalous dispersion, (b) corresponds to no dispersion, and (c) corresponds to normal dispersion.