Racetrack Problem

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Question

Along a 1000 mile race track are placed 50 gallons of gas, split between any number of gas cans, placed anywhere. The racecar run on 20 miles per gallon and starts on an empty tank that can hold up to 50 gallons.

Given any configuration of the gas cans, is there always a starting position that the racer can select to complete the entire track?

Notation

A configuration $C = \{d_i, r_i\}_{i=1}^n$ describes n gas cans with a pair of sequences

d_i : distance, in miles, to the $(i+1)$th gas can

r_i : additional range, in miles, provided the ith gas can

(equal to 20 mpg * gallons of gas)

subject to the constraints

$$d_i, r_i > 0$$

$$\sum_{i=1}^n d_i, r_i = 1000$$

Definition: For a configuration to be *completable*, we must have

$$\forall 1 \leq j \leq n, \quad \sum_{i=1}^j r_i \geq \sum_{i=1}^j d_i$$

(At all times, the range of the racecar is enough to reach the next gas can.)

Proof

Let $C = \{d_i, r_i\}_{i=1}^n$ be any configuration. Set the first gas can as the 0 position on the track. Assume the racer cannot complete the entire track starting from there. Then $\exists j \geq 1 :$

$$\sum_{i=1}^{j-1} r_i \geq \sum_{i=1}^{j-1} d_i \qquad \text{The racecar is able to reach the } j\text{th can,} \tag{1}$$

$$\sum_{i=1}^{j} r_i < \sum_{i=1}^{j} d_i \qquad \text{but runs out of gas before the } (j+1)\text{th can.} \tag{2}$$

Combining (2) with our constraints for $\{d_i\}$ and $\{r_i\}$ we get

$$\sum_{i=1}^{j-1} r_i + r_j + \sum_{i=j+1}^{n} r_i = \sum_{i=1}^{j-1} d_i + d_j + \sum_{i=j+1}^{n} d_i \tag{3}$$

$$\implies \sum_{i=j+1}^{n} r_i > \sum_{i=j+1}^{n} d_i \tag{4}$$

Define S as the excess gas accrued from the 1st to the jth can
      T as the excess gas accrued from the $(j+1)$th to the nth can.

$$S := \sum_{i=1}^{j-1} r_i - \sum_{i=1}^{j-1} d_i \geq 0$$

$$T := \sum_{i=j+1}^{n} r_i - \sum_{i=j+1}^{n} d_i > 0$$

Now we have a starting position to complete the track.

Step 1: Start at $(j+1)$th can and continue past nth can to the 0 position. (4) guarantees that $\nexists j' > j : $ (1) and (2) hold, i.e., the racecar won't get stuck along this section.

Step 2: Continue from 1st can to jth can. (1) guarantees enough range.

Step 3: From jth can to $(j+1)$th, use (3) to verify that the racecar has enough range complete the final leg: $d_j \leq T + S + r_j$

By definition, the configuration is *completable* by relabeling the gas cans

$$r : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$
$$i \mapsto ((i - j - 1) \mod n) + 1$$

and starting at the position of the new first gas can.

$\square$