Estimating e on a checkerboard

Samuel Hunter

8 September 2019

Method

Adapted from `https://twitter.com/JohnAllenPaulos/status/743085219511554049`
Take an n x n checkerboard and label its squares 1 to $m = n^2$. Randomly generate m whole numbers between 1 and m. For each of the numbers generated, place a penny on the square with the same number. The ratio of total squares, m, to squares with no pennies, U, is approximately e.

Solution

For the ith square ($1 \leq i \leq m$), let X_i be a random variable denoting the number of pennies on the ith square after all pennies have been placed.

$$X_i \sim Binomial(m, \frac{1}{m}) \tag{1}$$

Define a new indicator random variable

$$Y_i = \begin{cases} 0, & \text{if } X_i = 0 \qquad \text{square is empty} \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

Then Y_i is a Bernoulli trial with expected value

$$\mathbb{E}[Y_i] = 1 * \mathbb{P}(X_i = 0) + 0 * \mathbb{P}(X_i > 0) = (1 - \frac{1}{m})^m \tag{3}$$

Now over all of the squares, the number of empty squares is given by

$$U = \sum_{i=1}^{m} Y_i \tag{4}$$

Take its expected value:

$$\begin{aligned}
\mathbb{E}[U] &= \mathbb{E}[\sum_{i=1}^{m} Y_i] \\
&= \sum_{i=1}^{m} \mathbb{E}[Y_i] && \text{by linearity of expectation} \\
&= \sum_{i=1}^{m} (1 - \frac{1}{m})^m && \text{by (3)} \\
&= m(1 - \frac{1}{m})^m && (5)
\end{aligned}$$

Remember, we are interested in the ratio

$$\frac{\text{total squares}}{\text{empty squares}} = \frac{m}{U}$$

which on average is

$$\frac{m}{\mathbb{E}[U]} = \frac{\cancel{m}}{\cancel{m}(1 - \frac{1}{m})^m} = (1 - \frac{1}{m})^{-m}$$

Taking the limit as total squares increases without bound,

$$\lim_{m\to\infty} (1 - \frac{1}{m})^{-m} = (\lim_{m\to\infty} (1 - \frac{1}{m})^m)^{-1} = (e^{-1})^{-1} = e \qquad (6)$$

□