Remembering a sequence of shapes

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Question

A sequence of m unique shapes is shown to a participant. After given time to memorize the it, the participant is given n unique shapes to recreate the sequence, which contain the original m. 1 point is awarded for each shape placed in its original position. What is the expected number of points the participant would earn if he/she guessed randomly?

Solution

Define the indicator random variable

$$X_k = \begin{cases} 1, & \text{if the } k\text{th shape is correctly placed} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $1 \leq k \leq m \leq n$

Proposition
X_k is a Bernoulli trial with expected value $1/n$.

Proof
For the first shape the participant has n choices, so $\mathbb{E}[X_1] = 1/n$. For a subsequent shape ($k \geq 2$) to be correct ($X_k = 1$), the (kth) shape cannot have been placed in the previous $(k-1)$ positions and it must be selected from the remaining $(n-k)$ shapes.

$$
\begin{aligned}
\mathbb{E}[X_k] &= 0 \cdot \mathbb{P}(X_k = 0) + 1 \cdot \mathbb{P}(X_k = 1) && (2)\\
&= \mathbb{P}(X_k = 1) && (3)\\
&= \frac{P(n-1, k-1) \cdot \frac{1}{n-k}}{P(n,k)} && (4)\\
&= \frac{\cancel{n-1}}{n} \cdot \frac{\cancel{n-2}}{\cancel{n-1}} \cdot \ldots \cdot \frac{\cancel{n-k}}{\cancel{n-(k-1)}} \cdot \frac{1}{\cancel{n-k}} = \frac{1}{n} && (5)
\end{aligned}
$$

Note: $P(n,k)$ represents the k-permutations of n and is equal to $\frac{n!}{(n-k)!}$.

In other words, the probability of a shape being correctly placed does not depend on its position in the sequence.

Considering all k shapes, the score is given by $Y = \sum_{i=1}^{m} X_k$.
Since the X_k's are independent and identically distributed, $Y \sim Binomial(m, \frac{1}{n})$.

$$
\begin{aligned}
\mathbb{E}[Y] &= m \cdot \frac{1}{n} = \frac{m}{n} && (6)\\
\mathrm{Var}(Y) &= m \cdot \frac{1}{n} \cdot (1 - \frac{1}{n}) = \frac{m(n-1)}{n^2} && (7)
\end{aligned}
$$