Take an $n \times n$ checkerboard and label its squares 1 to $m = n^2$. Randomly generate m whole numbers between 1 and m. For each of the numbers generated, place a penny on the square with the same number. The ratio of total squares (m) to squares with no pennies is approximately e.

For the ith square, let

X_i be a random variable denoting the number of pennies on the ith square after all pennies have been placed.

Then X_i follows a binomial distribution with m trials and probability of success $\frac{1}{m}$.

Define a new indicator random variable

$$Y_i = \begin{cases} 1 & \text{if } X_i = 0 \text{ (square is empty)} \\ 0 & \text{otherwise} \end{cases}$$

Then Y_i is a Bernoulli trial with expected value

$$\mathbb{E}[Y_i] = 1 \cdot \mathbb{P}(X_i = 0) + 0 \cdot \mathbb{P}(X_i > 0) = \left(1 - \frac{1}{m}\right)^m$$

Now over all of the squares, define the number of empty squares as

$$U = \sum_{i=1}^{m} Y_i$$

Taking expected value is straight-forward:

$$\begin{aligned} \mathbb{E}[U] &= \mathbb{E}\left[\sum_{i=1}^{m} Y_i\right] \\ &= \sum_{i=1}^{m} \mathbb{E}[Y_i] \quad \text{by linearity of expectation} \\ &= \sum_{i=1}^{m} \left(1 - \frac{1}{m}\right)^m \\ &= m\left(1 - \frac{1}{m}\right)^m \end{aligned}$$

Remember, we are interested in the ratio

$$\frac{\text{total squares}}{\text{empty squares}} = \frac{m}{u},$$

which on average is

$$\frac{m}{\mathbb{E}[u]} = \frac{\cancel{m}}{\cancel{m}(1-\frac{1}{m})^m} = \left(1-\frac{1}{m}\right)^{-m}$$

Taking the limit as total squares increases without bound,

$$\lim_{m\to\infty}\left(1-\frac{1}{m}\right)^{-m} = \left(\lim_{m\to\infty}\left(1-\frac{1}{m}\right)^m\right)^{-1} = (e^{-1})^{-1} = e \quad \square$$