Dirac comb and exponential frequency spectra in chaotic dynamics

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An exponential frequency power spectral density is a well known property of many continuous time chaotic systems and has been attributed to the presence of Lorentzian-shaped pulses. Here a stochastic modelling of such fluctuations are presented, describing these as a super-position of pulses with fixed shape and constant duration. Closed form expressions are derived for the lowest order moments, auto-correlation function and frequency power spectral density in the case of periodic pulse arrivals and a random distribution of pulse amplitudes. In general, the spectrum is a Dirac comb located at multiples of the periodicity time and modulated by the pulse spectrum. Randomness in the pulse arrival times is investigated by numerical realizations of the process and the results are discussed in the context of some well-known chaos models.

I. INTRODUCTION

An intrinsic property of deterministic chaos in continuous time systems is an exponential frequency power spectral density (PSD) for the fluctuations.[1–10] This has been observed in numerous experiments and model simulations of fluids and magnetized plasmas. Recently, the exponential spectrum has been attributed to the presence of Lorentzian pulses in the temporal dynamics.[11–21] Weakly non-linear systems are often characterized by a quasi-periodic oscillations, resulting in a frequency power spectral density resembling a Dirac comb.[19–27] Far from the linear instability threshold the spectral peaks broaden and in many cases an exponential spectrum results.[1–27]

Many chaotic systems, including the Lorenz and the Rössler models, display quasi-periodic orbits with Lorentzian-shaped pulses close to the primary instability threshold. The associated PSD has sharp peaks at frequencies corresponding to the periodicity of the oscillations, resembling a Dirac comb. The Lorentizan-shaped pulses lead to an exponential modulation of the amplitude of the spectral peaks. With period-doubling the density of spectral peaks increases and in the chaotic state the spectral peaks broadens and the PSD is eventually an exponential function of frequency.

In this contribution, we present a stochastic model that describes a super-position of Lorentzian pulses and the resulting frequency spectra.[28–30] The model is based on the process known as shot noise or filtered Poisson process.[31–41] This model has recently been used to describe intermittent fluctuations in turbulent fluids and plasmas.[17,18]

For a super-position of pulses with fixed shape and constant duration closed form expressions are here derived for the lowest order moments, auto-correlation function and frequency power spectral density in the case of periodic pulse arrivals and a random distribution of pulse amplitudes. In general, the spectrum is a Dirac comb located at multiples of the periodicity time and modulated by the pulse spectrum. Randomness in the pulse arrival times is investigated by numerical realizations of the process and the results are discussed in the context of some well-known chaos models.

The contribution is structured as follows. In Sec. II, a motivating example for studying periodic pulse trains in connection to chaotic motion is presented. In Sec. III the stochastic model for a super-position of pulses is presented and its PSD for general arrival times is derived. In Sec. IV the case of periodic pulse arrivals is analyzed in detail with a particular

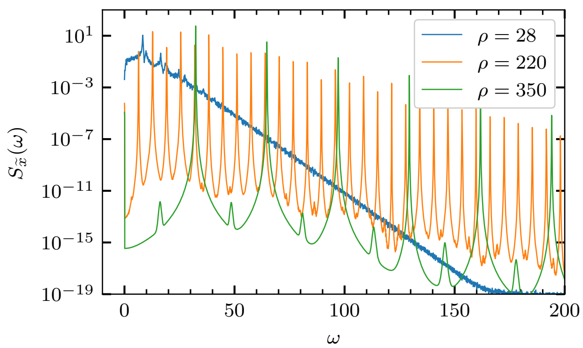

FIG. 1. Power spectral density of the x-variable in the Lorenz system for $\sigma = 10$, $\beta = 8/3$ and various values of ρ.

focus on Lorentzian pulses. Finally, in Sec. VI it is demonstrated that the stochastic model describes the chaotic dynamics of the Lorenz system.

II. THE LORENZ SYSTEM

A canonical chaos system is given by the Lorenz equations describing weakly non-linear thermal convection in an inversely stratified fluid

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \tag{1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \tag{2}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z. \tag{3}$$

Here x, y and z are the variables and σ, ρ and β are the model parameters. Time series of the x-variable and the associated frequency PSD are presented in Figs. 2 and 1 for $\sigma = 10$, $\beta = 8/3$ and three different values of the model parameter ρ.

For $\rho = 350$ the solutions consists of periodic oscillations and the frequency PSD resemble a Dirac comb with an exponential modulation of the peak amplitudes. As shown in Fig. 2, the oscillations are well described by Lorentzian-shaped pulses. Following a period doubling bifurcation, the solution for $\rho = 220$ is still regular and the PSD is again dominated by a Dirac-like comb. For $\rho = 28$ the solution is chaotic and the PSD has an exponential shape for high frequencies and some narrow peaks for low frequencies. In the following, these features of the chaotic dynamics will be analyzed by describing the fluctuations as a super-position

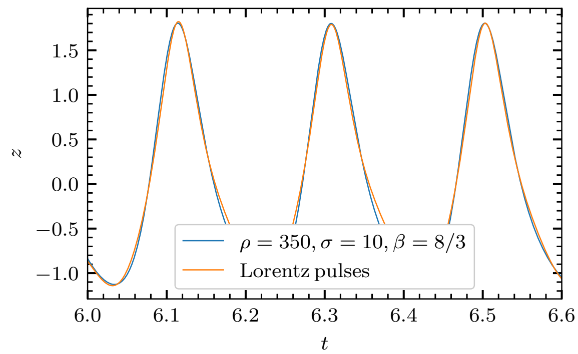

FIG. 2. Excerpt of the time series of the z-variable in the Lorenz system for $\sigma = 10$, $\beta = 8/3$ and $\rho = 350$ compared to a superposition of Lorentz pulses.

of Lorentzian-shaped pulses.

III. THE POWER SPECTRAL DENSITY OF A SUM OF PULSES

In this section, we develop an expression for the power spectral density of the a shot noise process for general arrival times. This is based on the formalism developed for filtered Poisson processes, also called shot noise processes.

We consider a train of $K(T)$ pulses arriving in the interval $[0, T]$ with randomly distributed arrival times $\{t_k\}_{k=1}^{K(T)}$ and randomly distributed amplitudes $\{A_k\}_{k=1}^{K(T)}$. The pulses have a characteristic shape φ and a characteristic duration time τ_d.

Following these remarks, the process is written as a convolution between a pulse train f_K and a pulse shape φ:

$$\Phi_K(t) = \int_{-\infty}^{\infty} \mathrm{d}s\, \varphi\left(\frac{t}{\tau_\mathrm{d}} - s\right) f_K(s), \tag{4}$$

where

$$f_K(s) = \sum_{k=1}^{K(T)} A_k \delta\left(s - \frac{t_k}{\tau_\mathrm{d}}\right) \tag{5}$$

and δ denotes the Dirac delta function. This can be viewed as a point process f_K passed through a filter with response function ϕ, hence the name. Note that for i.i.d. uniform pulse arrivals, $K(T)$ is a Poisson process.

We normalize the pulse shape such that

$$\int_{-\infty}^{\infty} |\varphi(s)| \mathrm{d}s = 1. \tag{6}$$

We also introduce the notation

$$\rho_\varphi(s) = \frac{1}{I_2} \int\limits_{-\infty}^{\infty} \mathrm{d}u \, \varphi(u)\varphi(u+s) \tag{7}$$

and

$$\varrho_\varphi(\theta) = \frac{1}{I_2} |\mathcal{F}[\varphi](\theta)|^2 \tag{8}$$

for the auto-correlation and the power spectral density of the pulse shape, respectively. Here,

$$I_n = \int_{-\infty}^{\infty} \varphi(s)^n \mathrm{d}s. \tag{9}$$

Note that the functions ρ_φ and ϱ_φ form a Fourier transform pair, where the definition of the Fourier transform is given in Appendix A. Throughout this contribution, we will use Lorentzian pulses, which are detailed in Appendix D.

To find the PSD, we start from Eq. (4), and take the Fourier transform as defined in Appendix A:

$$\mathcal{F}_T[\Phi_K](\omega) = \int\limits_0^T \mathrm{d}t \, \exp(-i\omega t)\Phi_K(t) = \int\limits_0^T \mathrm{d}t \, \exp(-i\omega t) \int\limits_{-\infty}^{\infty} \mathrm{d}s \, \varphi(s) f_K\left(\frac{t}{\tau_\mathrm{d}} - s\right) \tag{10}$$

where we have exchanged the functions in the convolution given by Eq. (4). A change of variables $u(t) = t - \tau_\mathrm{d} s$ gives

$$\mathcal{F}_T[\Phi_K](\omega) = \int\limits_{-\infty}^{\infty} \mathrm{d}s \, \varphi(s) \exp(-i\tau_\mathrm{d}\omega s) \int\limits_{-\tau_\mathrm{d} s}^{T-\tau_\mathrm{d} s} \mathrm{d}u \, f_K\left(\frac{u}{\tau_\mathrm{d}}\right) \exp(-i\omega u). \tag{11}$$

We assume that $\varphi(s)$ is negligible after a few τ_d. Moreover, since no pulses arrive for negative times, $f_K(u) = 0$ for $u < 0$. Assuming $T/\tau_\mathrm{d} \gg 1$, we can therefore approximate the limits of the second integral in Eq. (11) as $u \in [0, T]$, and the two integrals become independent. This gives

$$\mathcal{F}_T[\Phi_K](\omega) = \mathcal{F}[\varphi](\tau_\mathrm{d}\omega)\mathcal{F}_T[f_K](\omega). \tag{12}$$

The power spectral density (PSD) of the stationary process Φ is therefore

$$\mathcal{S}_\Phi(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle |\mathcal{F}_T[\Phi_K](\omega)|^2 \right\rangle = |\mathcal{F}[\varphi](\tau_\mathrm{d}\omega)|^2 \lim_{T\to\infty} \frac{1}{T} \left\langle |\mathcal{F}_T[f_K](\omega)|^2 \right\rangle, \tag{13}$$

where $\mathcal{S}_\Phi(\omega)$ is independent of K, since the average is over all random variables. The power spectrum is thus the product of the power spectrum of the pulse shape and the power spectrum of the point process. Non-uniform arrivals only affect the point process, so this will be isolated in the analysis in Sec. III A.

Using Eq. (8), the full power spectral density of Φ can be written as

$$\mathcal{S}_\Phi(\omega) = I_2 \varrho_\varphi(\tau_\mathrm{d}\omega) \lim_{T\to\infty} \frac{1}{T}\big\langle |\mathcal{F}_T[f_K](\omega)^2| \big\rangle. \tag{14}$$

A. The power spectral density for general arrival times

The Fourier transform of the point process is

$$\mathcal{F}_T[f_K](\omega) = \tau_\mathrm{d} \sum_{k=1}^{K} A_k \exp(-i\omega t_k). \tag{15}$$

Multiplying this expression with its complex conjugate and averaging over all random variables gives (for a general distribution of arrivals $P_{t_1,t_2,\dots,t_K}(t_1, t_2, \dots, t_K)$, assuming amplitudes are i.i.d. and independent of the arrival times):

$$\begin{aligned}
&\frac{1}{T}\big\langle |\mathcal{F}_T[f_K](\omega)^2| \big\rangle \\
&\qquad = \sum_{K=0}^{\infty} P_K(K;T,\tau_\mathrm{w}) \frac{\tau_\mathrm{d}^2}{T} \sum_{k=1}^{K}\sum_{l=1}^{K} \int_{-\infty}^{\infty} \mathrm{d}t_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}t_K P_{t_1,\dots,t_K}(t_1,\dots,t_K) \\
&\qquad\quad \times \int_0^\infty \mathrm{d}A_1 P_A(A_1) \cdots \int_0^\infty \mathrm{d}A_K P_A(A_K) A_k A_l \exp(i\omega(t_l - t_k)). \\
&\qquad = \sum_{K=0}^{\infty} P_K(K;T,\tau_\mathrm{w}) \frac{\tau_\mathrm{d}^2}{T} \sum_{k,l=1}^{K} \langle A_k A_l\rangle \langle \exp(i\omega(t_l - t_k))\rangle.
\end{aligned} \tag{16}$$

In this equation, there are K terms where $k = l$ and $K(K-1)$ terms where $k \neq l$. Summing over all these terms, we have

$$\begin{aligned}
\frac{1}{T}\big\langle |\mathcal{F}_T[f_K](\omega)^2| \big\rangle = \tau_\mathrm{d}^2 \sum_{K=0}^{\infty} P_K(K;T,\tau_\mathrm{w}) \\
\left\{ \frac{K}{T}\langle A^2\rangle + \frac{1}{T}\langle A\rangle^2 \sum_{k=1}^{K}\sum_{l\neq k} \langle \exp(i\omega(t_l - t_k))\rangle \right\}.
\end{aligned} \tag{17}$$

The average inside the exponential sum is the joint characteristic function of t_l and t_k. Exchanging the order of k and l in the double sum is the same as taking the complex conjugate of this characteristic function, so we get

$$\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \frac{\tau_\mathrm{d}^2\langle K\rangle}{T}\big\langle A^2\big\rangle + \frac{\tau_\mathrm{d}^2}{T}\langle A\rangle^2\sum_{K=0}^{\infty}P_K(K;T,\tau_\mathrm{w})\sum_{k=2}^{K}\sum_{l=1}^{k-1}2\mathrm{Re}[\langle\exp(i\omega(t_l-t_k))\rangle]. \quad (18)$$

1. Uniformly distributed i.i.d arrivals

As an example, we show that the expression in Eq. (18) is consistent with the established result for $K(T)$ a pure Poisson point process.

Now t_l, t_k are i.i.d. uniformly distributed arrivals on $[0,T]$. We therefore have that

$$\langle\exp(i\omega(t_l-t_k))\rangle = \langle\exp(i\omega t_l)\rangle\langle\exp(-i\omega t_k)\rangle = 2\frac{1-\cos(\omega T)}{\omega^2T^2}. \quad (19)$$

All terms in the double sum are equal, and we get

$$\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \frac{\tau_\mathrm{d}^2\langle K\rangle}{T}\big\langle A^2\big\rangle + 2\frac{\tau_\mathrm{d}^2}{T}\langle A\rangle^2\langle K(K-1)\rangle\frac{1-\cos(\omega T)}{\omega^2T^2}. \quad (20)$$

In this case, K is Poisson distributed with mean and variance equal to T/τ_w and we get

$$\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \frac{\tau_\mathrm{d}^2}{\tau_\mathrm{w}}\big\langle A^2\big\rangle + \frac{2\tau_\mathrm{d}^2}{\tau_\mathrm{w}^2}\langle A\rangle^2\frac{1-\cos(\omega T)}{\omega^2T^2}, \quad (21)$$

which gives, using $\gamma = \tau_\mathrm{d}/\tau_\mathrm{w}$,

$$\lim_{T\to\infty}\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \tau_\mathrm{d}\gamma\big\langle A^2\big\rangle + 2\pi\tau_\mathrm{d}\gamma^2\langle A\rangle^2\delta(\tau_\mathrm{d}\omega), \quad (22)$$

which is the standard expression for the Poison process.[39] Identifying the last term as containing a Dirac delta in the limit $T\to\infty$ makes sense in the theory of distributions[46]. A Poisson process gives a flat spectrum, so the only frequency variation in the full spectrum will be due to the pulse function.

2. Periodic arrival times

We consider the situation where the periodicity is known, but the exact arrivals are not. This corresponds to uncertainty in where the measurement starts in relation to the first arrival time. If the arrivals are periodic, the marginal PDF of arrival k given that the starting time is s, is

$$P_{t_k|s}(t_k|s) = \delta(t_k-\tau_\mathrm{p}k-s). \quad (23)$$

Since each arrival is deterministic, the joint PDF with known starting point is is the product of the marginal PDFs, and we have

$$\langle \exp(i\omega(t_l - t_k))\rangle = \exp(i\omega\tau_\mathrm{p}(l-k)). \tag{24}$$

Note that this is independent of s for all starting points, so for now we need not consider s further. We have from Eq. (18)

$$\begin{aligned}\sum_{k=2}^{K}\sum_{l=1}^{k-1} 2\mathrm{Re}[\langle \exp(i\omega(t_l - t_k))\rangle] &= \sum_{k=2}^{K}\sum_{l=1}^{k-1} 2\cos(\tau_\mathrm{p}\omega(l-k)) \\ &= \frac{K-1+\cos(\tau_\mathrm{p}\omega K) - K\cos(\tau_\mathrm{p}\omega)}{\cos(\tau_\mathrm{p}\omega)-1} = \frac{\cos(\tau_\mathrm{p}\omega K)-1}{\cos(\tau_\mathrm{p}\omega)-1} - K. \end{aligned}\tag{25}$$

Due to the periodicity, there are $\lfloor T/\tau_\mathrm{p}\rfloor$ events in a time series of length T. We use $P_K(K;T,\tau_\mathrm{p}) = \delta(K - \lfloor T/\tau_\mathrm{p}\rfloor)$. Inserting this and Eq. (25) into Eq. (18) gives

$$\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \frac{\tau_\mathrm{d}^2}{T}\lfloor T/\tau_\mathrm{p}\rfloor\big\langle A^2\big\rangle + \frac{\tau_\mathrm{d}^2}{T}\lfloor T/\tau_\mathrm{p}\rfloor\langle A\rangle^2\left[\lfloor T/\tau_\mathrm{p}\rfloor^{-1}\frac{\cos(\tau_\mathrm{p}\omega\lfloor T/\tau_\mathrm{p}\rfloor)-1}{\cos(\tau_\mathrm{p}\omega)-1} - 1\right]. \tag{26}$$

For $T/\tau_\mathrm{p} \gg 1$, $\lfloor T/\tau_\mathrm{p}\rfloor/T \approx 1/\tau_\mathrm{p}$, and we have (writing $K = \lfloor T/\tau_\mathrm{p}\rfloor$)

$$\lim_{K\to\infty}\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \frac{\tau_\mathrm{d}^2}{\tau_\mathrm{p}}\big\langle A^2\big\rangle - \frac{\tau_\mathrm{d}^2}{\tau_\mathrm{p}}\langle A\rangle^2 + \frac{\tau_\mathrm{d}^2}{\tau_\mathrm{p}}\langle A\rangle^2\lim_{K\to\infty}\frac{1}{K}\frac{\cos(\tau_\mathrm{p}\omega K)-1}{\cos(\tau_\mathrm{p}\omega)-1}. \tag{27}$$

Let us consider the last part of the last term,

$$\lim_{K\to\infty}\frac{1}{K}\frac{\cos(\tau_\mathrm{p}\omega K)-1}{\cos(\tau_\mathrm{p}\omega)-1}. \tag{28}$$

For integer n and $\tau_\mathrm{p}\omega \neq 2\pi n$, this limit is zero. For $\tau_\mathrm{p}\omega \to 2\pi n$, this limit tends to ∞. We might therefore consider Eq. (28) proportional to a train of δ-pulses located at $\tau_\mathrm{p}\omega = 2\pi n$.

Setting $\tau_\mathrm{p}\omega = 2\pi n + \epsilon$ where $\epsilon \ll 1$ and expanding the cosine in the denominator, we have

$$\lim_{K\to\infty}\frac{1}{K}\frac{\cos(\tau_\mathrm{p}\omega K)-1}{\cos(\tau_\mathrm{p}\omega)-1} \approx \lim_{K\to\infty}\frac{2}{K}\frac{1-\cos(\epsilon K)}{\epsilon^2}. \tag{29}$$

This is on the same form as we had when deriving Eq. (22), so we conclude that

$$\lim_{K\to\infty}\frac{1}{K}\frac{\cos(\tau_\mathrm{p}\omega K)-1}{\cos(\tau_\mathrm{p}\omega)-1} \sim 2\pi\sum_{n=-\infty}^{\infty}\delta(\tau_\mathrm{p}\omega - 2\pi n). \tag{30}$$

Inserting this into Eq. (26) gives the full expression for the PSD of a train of delta pulses with randomly distributed amplitudes:

$$\lim_{T\to\infty}\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle = \tau_\mathrm{d}\gamma A_\mathrm{rms}^2 + 2\pi\tau_\mathrm{d}\gamma^2\langle A\rangle^2\sum_{n=-\infty}^{\infty}\delta(\tau_\mathrm{d}\omega - 2\pi n\gamma). \tag{31}$$

This equation will be discussed in the following section.

IV. THE SHOT NOISE PROCESS WITH PERIODIC ARRIVALS

The full power spectral density of Φ is given by multiplying Eq. (31) by the power spectrum of the pulse functions, Eq. (8), as given by Eq. (14):

$$\mathcal{S}_\Phi(\omega) = \tau_\mathrm{d}\gamma A_\mathrm{rms}^2 I_2 \varrho_\varphi(\tau_\mathrm{d}\omega) + 2\pi\tau_\mathrm{d}\gamma^2 \langle A\rangle^2 I_2 \varrho_\varphi(\tau_\mathrm{d}\omega) \sum_{n=-\infty}^{\infty} \delta(\tau_\mathrm{d}\omega - 2\pi n\gamma). \tag{32}$$

There are two main differences from the uniformly distributed pulses, given by Eq. (22): A_rms enters into the first term instead of $\langle A^2\rangle$, and there is a contribution of delta spikes at integer multiples of $2\pi/\tau_\mathrm{p}$, with an envelope given by the pulse shape. We may view the first term as the average spectrum, due to the randomness of the amplitude distribution, while the second term containing the sum of delta pulses is due to the periodicity of the pulse arrivals. Accordingly, the first term vanishes for degenerately distributed amplitudes, $p_A(A) = \delta(A - \langle A\rangle)$. For a symmetric amplitude distribution around 0, $\langle A\rangle = 0$ and $A_\mathrm{rms}^2 = \langle A^2\rangle$. The periodicity is canceled out and only the first term remains.

In Fig. 3, the power spectral density of a synthetically generated shot noise is presented for exponentially distributed amplitudes (blue line) and symmetrically Laplace distributed amplitudes (orange line). The arrivals are periodic and the pulses have a Lorentzian shape. The analytic expression Eq. (32) for both cases is given by the black and green dashed lines respectively. The Dirac comb with decaying amplitudes is easily seen in the case with exponential amplitudes. We emphasize that the main effect of the periodicity, the Dirac comb, is completely cancelled out by the symmetrically distributed pulse amplitudes.

A. The correlation function

By the Wiener-Khinchin theorem,

$$R_\Phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathcal{S}_\Phi(\omega)\exp(i\omega t) \tag{33}$$

$$= \gamma A_\mathrm{rms}^2 I_2 \rho_\varphi(t/\tau_\mathrm{d}) + \gamma^2\langle A\rangle^2 I_2 \sum_{n=-\infty}^{\infty} \varrho_\varphi(2\pi n\gamma)\exp(i2\pi n\gamma t/\tau_\mathrm{d}). \tag{34}$$

By using the Poisson summation formula and properties of the Fourier transform as detailed in Appendix C, we can write R_Φ as

$$R_\Phi(t) = \gamma A_\mathrm{rms}^2 I_2 \rho_\varphi\left(\frac{t}{\tau_\mathrm{d}}\right) + \gamma\langle A\rangle^2 I_2 \sum_{m=-\infty}^{\infty} \rho_\varphi\left(\frac{m}{\gamma} + \frac{t}{\tau_\mathrm{d}}\right). \tag{35}$$

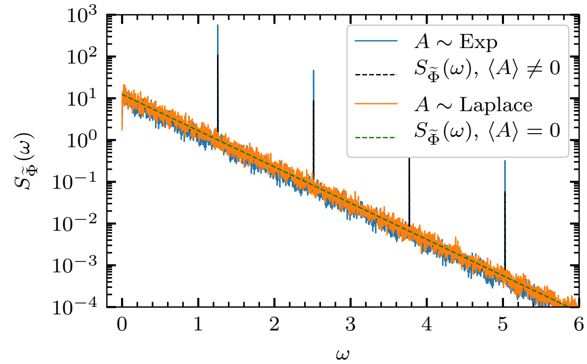

FIG. 3. The power spectral density of a shot noise process with periodic arrival times for exponentially (blue) and Laplace (orange) distributed amplitudes. The analytic expression is given by the black and green dashed lines, respectively.

Writing $m/\gamma + t/\tau_\mathrm{d} = (m + t/\tau_\mathrm{p})/\gamma$, we see that the correlation function consists of a central peak with followed by periodic modulations at integer multiples of τ_p. Again, for degenerate amplitudes the correlation function only consists of the periodic train: there is no randomness left in the signal and so the correlation function does not decay for large times. For symmetric amplitudes, only the central peak remains.

In Fig. 4, the auto-correlation function of a synthetically generated shot noise is presented for exponentially distributed amplitudes (blue line) and symmetrically Laplace distributed amplitudes (orange line). The arrivals are periodic and the pulses have a Lorentzian shape. The analytic expression Eq. (32) for both cases is given by the black and green dashed lines respectively. For exponentially distributed amplitudes, the periodicity is clearly seen. This effect is again completely cancelled out by symmetrically distributed amplitudes.

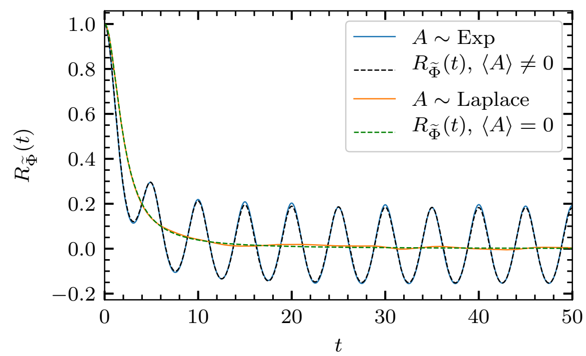

FIG. 4. The auto-correlation function of a shot noise process with periodic arrival times for exponentially (blue) and Laplace (orange) distributed amplitudes. The analytic expression is given by the black and green dashed lines, respectively.

B. The mean value and standard deviation

The mean value of the shot noise process with periodic pulses, assuming a uniform starting time distribution in $[0, \tau_\mathrm{p})$, is given by

$$
\begin{aligned}
\langle \Phi_K \rangle &= \sum_{k=1}^{K} \int \mathrm{d}A\, P_A(A) \int \mathrm{d}s\, P_s(s) A \varphi\left(\frac{t-\tau_\mathrm{p}k-s}{\tau_\mathrm{d}}\right) \\
&= \frac{\langle A\rangle}{\tau_\mathrm{p}} \sum_{k=1}^{K} \int_0^{\tau_\mathrm{p}} \mathrm{d}s\, \varphi\left(\frac{t-\tau_\mathrm{p}k-s}{\tau_\mathrm{d}}\right) \\
&= \frac{\langle A\rangle}{\tau_\mathrm{p}} \sum_{k=1}^{K} \int_{(t-\tau_\mathrm{p}(k+1))/\tau_\mathrm{d}}^{(t-\tau_\mathrm{p}k)/\tau_\mathrm{d}} \tau_\mathrm{d} \mathrm{d}u\, \varphi(u) = \gamma \langle A\rangle \int_{(t-(K+1)\tau_\mathrm{p})/\tau_\mathrm{d}}^{(t-\tau_\mathrm{p})/\tau_\mathrm{d}} \mathrm{d}u\, \varphi(u) \\
\langle \Phi \rangle &= \lim_{T\to\infty} \sum_{K=1}^{\infty} p_K(K; \tau_\mathrm{w}, T) \langle \Phi_K \rangle = \gamma \langle A\rangle I_1.
\end{aligned} \tag{36}
$$

In the last step, we let $T \to \infty$ giving $K \to \infty$ and set the upper integration limit to ∞ to avoid the effect due to the signal starting at $t = 0$. This is the expected result from Campbell's theorem. This is also consistent with the fact that the square mean value is given by the zero-frequency delta function in the power spectrum, $\mathcal{S}_\Phi(\omega) = 2\pi\langle\Phi\rangle^2\delta(\omega) + \ldots$.

The second moment is most conveniently found by noting that

$$
\langle \Phi \rangle^2 = \langle \Phi(t)\Phi(t) \rangle = R_\Phi(0) = \gamma A_\mathrm{rms}^2 I_2 + \gamma \langle A\rangle^2 I_2 \sum_{m=-\infty}^{\infty} \rho_\varphi\left(\frac{m}{\gamma}\right), \tag{37}
$$

where we have used that $\rho_\varphi(0) = 1$. This can be verified by calculating the second moment directly as was done for the first. In Appendix B, it is shown that this is also equivalent to an extention of Campbell's theorem. We get the variance

$$\Phi_{\mathrm{rms}}^2 = \gamma A_{\mathrm{rms}}^2 I_2 + \gamma \langle A \rangle^2 I_2 \left(\sum_{m=-\infty}^{\infty} \rho_\varphi(m/\gamma) - \gamma \frac{I_1^2}{I_2} \right). \tag{38}$$

In the case $\gamma \ll 1$, only the $m = 0$ term in the sum gives a contribution, $\rho_\varphi(0) = 1$, giving

$$\lim_{\gamma \to 0} \Phi_{\mathrm{rms}}^2 = \gamma \langle A^2 \rangle I_2, \tag{39}$$

where we neglect the γ^2-contribution of the last term in the bracket. Thus, in the limit of no pulse overlap, the variance for the case of periodic pulses is equivalent to the case of Poisson distributed pulses.

In the case $\gamma \gg 1$, we can write $m/\gamma = m\triangle_t \to t$ and treat the sum as an integral, $\gamma \sum_m \rho(m/\gamma)(1/\gamma) \approx \gamma \int \rho(t) \triangle_{\mathrm{t}} = \gamma I_1^2/I_2$, where the sum is over all integers and the integral is over all reals. The terms inside the bracket cancel, and we get

$$\lim_{\gamma \to \infty} \Phi_{\mathrm{rms}}^2 = \gamma A_{\mathrm{rms}}^2 I_2. \tag{40}$$

Since $A_{\mathrm{rms}}^2 = \langle A^2 \rangle - \langle A \rangle^2 \leq \langle A^2 \rangle$, the periodic pulse overlap gives lower variance than the Poisson distributed pulses as there is less randomness in the signal. For exponential amplitudes, the variance in the periodic case is a factor 2 smaller. For amplitudes with zero mean value, it is equal to the Poisson case while for fixed amplitudes, the signal has no variance as pulses will accumulate until the rate of accumulation exactly matches the rate of decay, after which the signal will remain constant.

C. Quasi-periodic pulses

In this section, we present the effect of quasi-periodicity in the arrival time distribution on the second-order statistics of the shot noise process. Here, we model quasi-periodicity using a uniform distribution for each arrival around the periodic arrival time, so that the distribution of the k'th arrival time given the starting time s is

$$P_{t_k}(t_k|s) = \begin{cases} \frac{1}{2\tau_{\mathrm{p}}\kappa}, & -\tau_{\mathrm{p}}\kappa \leq t_k - \tau_{\mathrm{p}}k - s \leq \tau_{\mathrm{p}}\kappa \\ 0, & \text{else} \end{cases}. \tag{41}$$

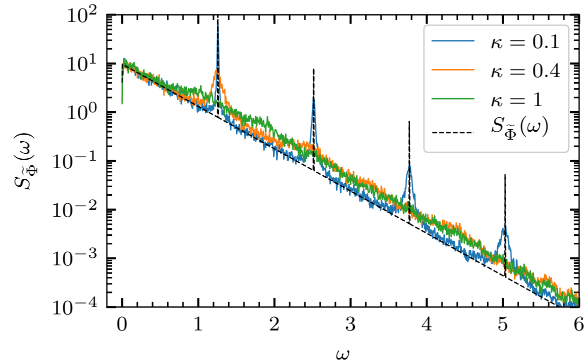

FIG. 5. The power spectral density of a shot noise process with quasi-periodic arrival times for different values of the κ-parameter. The analytic expression for purely periodic pulses is given by the black dashed line.

In the limit $\kappa \to 0$, we recover the periodic arrivals, while for $\kappa > 1$, the probability distributions of adjacent arrivals overlap. We emphasize that this is still a very restrictive formulation: even for $\kappa > 1$, each arrival is guaranteed to be centered on the time corresponding to the periodic arrival time, and the number of arrivals in a given interval is fixed up to end effects.

In Figs. 5 and 6, the effect of this quasi-periodicity is presented. The full lines give the power spectral densities and the auto-correlation functions of the shot noise process with quasi-periodic arrival times for different values of the κ-parameter, Lorentzian pulses and exponentially distributed amplitudes. The black dashed line gives the analytic prediction for purely periodic pulses. Even moderate deviations from pure periodicity quickly destroy the Dirac comb. For $\kappa = 1$, the spectrum and correlation function are already difficult to distinguish from the case of Poisson distributed arrivals. Thus, quasi-periodic phenomena in for example turbulent fluids cannot be expected to produce more than the first peak of the Dirac comb.

V. MULTIPLE PERIODICITIES: ROUTE TO CHAOS

We now consider a situation where we have multiple periodicities, each with their own amplitudes and possible offsets, such that we can write the Fourier transform of the point

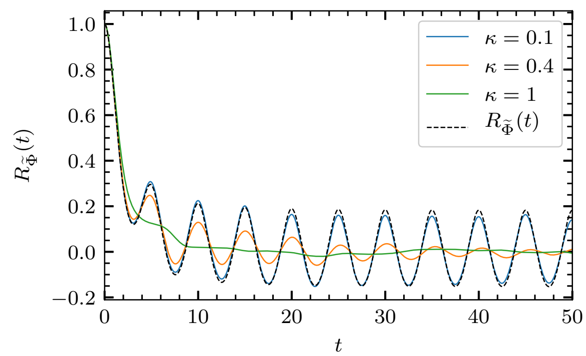

FIG. 6. The auto-correlation function of a shot noise process with quasi-periodic arrival times for different values of the κ-parameter. The analytic expression for purely periodic pulses is given by the black dashed line.

process as

$$\mathcal{F}_T[f_K](\omega) = \tau_\mathrm{d} \sum_{p=1}^{P} \sum_{k=1}^{K^p} A_k^p \exp(-i\omega[\tau^p k + \alpha^p]), \tag{42}$$

where τ^p are the periods, $\{A_k^p\}_{k=1}^{K_p}$ are the arrivals connected to the p'th periodicity, α^p are constant offsets for the first arrivals and $K^p = \lfloor (T - \alpha^p)/\tau^p \rfloor$. We assume that arrivals for different periodicities are independent. Further, we arange the periods in decreasing order, $\tau^1 \geq \tau^2 \geq \tau^3 \geq \ldots$. For large enough T that the offsets can be neglected, this leads to an increasing order in the number of events, $K^1 \leq K^2 \leq K^3 \leq \ldots$. We get

$$\begin{aligned}
&|\mathcal{F}_T[f_K](\omega)^2| \\
&\quad= \tau_\mathrm{d}^2 \sum_{p,q=1}^{P} \sum_{k=1}^{K^p} \sum_{l=1}^{K^q} A_k^p A_l^q \exp(-i\omega[\tau^p k - \tau^q l + \alpha^p - \alpha^q]) \\
&\quad= \tau_\mathrm{d}^2 \sum_{p=1}^{P} \sum_{k=1}^{K^p} (A_k^p)^2 \\
&\quad+ \tau_\mathrm{d}^2 \sum_{p=1}^{P} \sum_{\substack{k,l=1 \\ k\neq l}}^{K^p} A_k^p A_l^p \exp(-i\omega\tau^p[k-l]) \\
&\quad+ \tau_\mathrm{d}^2 \sum_{\substack{p,q=1 \\ p\neq q}}^{P} \sum_{k=1}^{K^p} \sum_{l=1}^{K^q} A_k^p A_l^q \exp(-i\omega[\tau^p k - \tau^q l + \alpha^p - \alpha^q]).
\end{aligned}$$

Taking the average over all amplitudes and gathering terms $(k,l)+(l,k)$ in the second double sum as well as terms $(p,q)+(q,p)$ in the triple sum, we get

$$
\begin{aligned}
&\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle \\
&\qquad= \frac{\tau_\mathrm{d}^2}{T}\sum_{p=1}^{P}K^p\big\langle(A^p)^2\big\rangle + \frac{\tau_\mathrm{d}^2}{T}\sum_{p=1}^{P}\sum_{k=2}^{K^p}\sum_{l=1}^{k-1}\langle A^p\rangle^2 2\cos(\omega\tau^p[k-l]) \\
&\qquad\quad+ \frac{\tau_\mathrm{d}^2}{T}\sum_{p=2}^{P}\sum_{q=1}^{p-1}\sum_{k=1}^{K^p}\sum_{l=1}^{K^q}2\langle A^p\rangle\langle A^q\rangle \\
&\qquad\quad\exp\left(i\omega\frac{(k-l)(\tau^p+\tau^q)}{2}\right)\cos\left(\omega\frac{(k+l)(\tau^p-\tau^q)}{2}+\omega(\alpha^p-\alpha^q)\right).
\end{aligned}
$$

The first two terms just gives a sum of the result in (31) over all periods. To investigate the last term, we consider the special case where $\tau^p = \tau\,\forall p$ so $K^p = K\,\forall p$. Then we have that the last term is

$$
\begin{aligned}
&\frac{\tau_\mathrm{d}^2}{T}\sum_{p=2}^{P}\sum_{q=1}^{p-1}\langle A^p\rangle\langle A^q\rangle 2\cos(\omega(\alpha^p-\alpha^q))\sum_{k=1}^{K}\sum_{l=1}^{K}\exp(i\omega\tau(k-l)) \\
&\qquad= \frac{\tau_\mathrm{d}^2}{T}\sum_{p=2}^{P}\sum_{q=1}^{p-1}\langle A^p\rangle\langle A^q\rangle 2\cos(\omega(\alpha^p-\alpha^q))\frac{\cos(K\omega\tau)-1}{\cos(\omega\tau)-1}.
\end{aligned}
\tag{43}
$$

This contains exactly the expression found in Eq. (29), so we have that

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\big\langle|\mathcal{F}_T[f_K](\omega)^2|\big\rangle \\
&\qquad= \sum_{p=1}^{P}\left\{\tau_\mathrm{d}\gamma(A_\mathrm{rms}^p)^2 + 2\pi\tau_\mathrm{d}\gamma^2\langle A^p\rangle^2\sum_{n=-\infty}^{\infty}\delta(\tau_\mathrm{d}\omega-2\pi n\gamma)\right\} \\
&\qquad+\sum_{p=2}^{P}\sum_{q=1}^{p-1}\langle A^p\rangle\langle A^q\rangle 2\cos(\omega(\alpha^p-\alpha^q))2\pi\tau_\mathrm{d}\gamma\sum_{n=-\infty}^{\infty}\delta(\tau\omega-2\pi n).
\end{aligned}
\tag{44}
$$

Here, $\gamma = \tau_\mathrm{d}/\tau$. Seeing that we can exchange $2\sum_{p=2}^{P}\sum_{q=1}^{p-1}$ with $\sum_{p=1}^{P}\sum_{q=1,\,q\neq p}^{P}$, we can write the full expression as

$$\lim_{T\to\infty}\frac{1}{T}\left\langle|\mathcal{F}_T[f_K](\omega)^2|\right\rangle = \sum_{p=1}^{P}\tau_\mathrm{d}\gamma(A_\mathrm{rms}^p)^2 + \\ \sum_{p=1}^{P}2\pi\tau_\mathrm{d}\gamma^2\langle A^p\rangle\left[\langle A^p\rangle + \sum_{q\neq p}\langle A^q\rangle\cos(\omega(\alpha^p-\alpha^q))\right]\sum_{n=-\infty}^{\infty}\delta(\tau_\mathrm{d}\omega - 2\pi n\gamma) \\ = \sum_{p=1}^{P}\tau_\mathrm{d}\gamma(A_\mathrm{rms}^p)^2 + 2\pi\tau_\mathrm{d}\gamma^2\sum_{p=1}^{P}\sum_{q=1}^{P}\langle A^p\rangle\langle A^q\rangle\cos(\omega(\alpha^p-\alpha^q))\sum_{n=-\infty}^{\infty}\delta(\tau_\mathrm{d}\omega - 2\pi n\gamma). \quad (45)$$

Thus, we get the same expression as for only one periodicity, except that we make the replacements

$$A_\mathrm{rms}^2 \to \sum_{p=1}^{P}(A_\mathrm{rms}^p)^2,$$
$$\langle A\rangle^2 \to \sum_{p=1}^{P}\sum_{q=1}^{P}\langle A^p\rangle\langle A^q\rangle\cos(\omega(\alpha^p-\alpha^q)).$$

In this case, the correction to the second term in the expression for the PSD depends on the offset between the different pulse trains. In particular, if there is no offset, $\alpha^p - \alpha^q = 0$, we just get the double sum over all mean values of the amplitudes. In this model, this means that adding further pulses with the same periodicity does not affect the density of the spikes in the Dirac comb. As period doubling can be seen as both decreasing τ_p and adding more pulses, this result shows that only decreasing τ_p affects the density of the Dirac comb.

VI. APPLICATION TO THE LORENZ ATTRACTOR

The predictions for the PSD of the stochastic model can be compared to that from numerical simulations of Lorenz system. In Fig. 7 the low-frequency part of the spectrum is presented for $\rho = 350$ as well as the predicted Dirac comb for a super-position of Lorentzian pulses with duration $\tau_\mathrm{d} = 0.039$ and periodicity $\tau_\mathrm{p} = 0.194$. This is clearly a good description of the oscillations in the Lorenz system.

In the chaotic state for $\rho = 28$ the PSD presented in Fig. 8 has some low-frequency peaks with higher harmonics on top of a exponential spectrum. This spectrum can be reproduced by a super-position of quasi-periodic Lorentzian with duation $\tau_\mathrm{d} = 0.135$, periodicity $\tau_\mathrm{p} =$

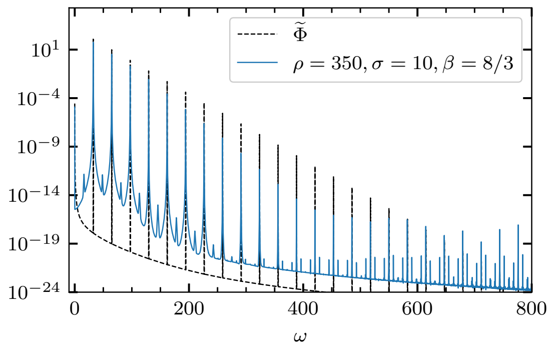

FIG. 7. The power spectral density of a Lorenz system with $\rho = 350$, $\sigma = 10$ and $\beta = 8/3$ compared to the frequency power spectral density of a synthetic shot noise process with periodic arrival times and exponentially distributed amplitudes.

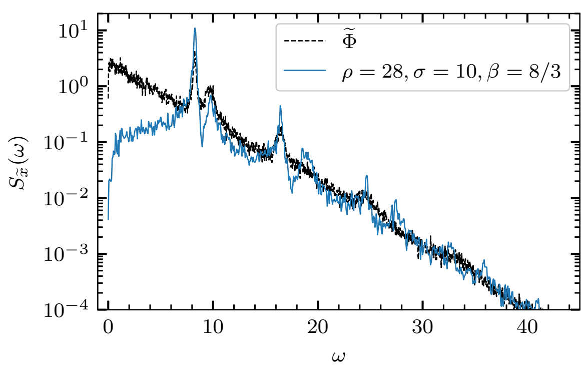

FIG. 8. The power spectral density of a Lorenz system with $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$ compared to the power spectral density of a synthetic shot noise process with quasi-periodic arrival times, exponentially distributed amplitudes and $\kappa = 0.1$.

0.643 and $\kappa = 0.1$ for the distribution of pulse arrivals. This is an excellent description of the PSD for the Lorenz system except for the very lowest frequencies which is likely due to the chaotic nature of the fluctuations.

VII. ACKNOWLEDGEMENTS

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Appendix A: Definitions of the Fourier transform and the Power spectral density

The PSD of a random process $\Phi(t)$ is defined as

$$\mathcal{S}_\Phi(\omega) = \lim_{T\to\infty} \frac{1}{T}\langle|\mathcal{F}_T[\Phi](\omega)|^2\rangle, \tag{A1}$$

where

$$\mathcal{F}_T[\Phi_K](\omega) = \int\limits_0^T \mathrm{d}t\, \exp(-i\omega t)\Phi(t) \tag{A2}$$

is the Fourier transform of the random variable over the domain $[0, T]$.

Analytical functions which fall rapidly enough to zero (such as the pulse function) have the Fourier transform

$$\mathcal{F}[\varphi](\theta) = \int\limits_{-\infty}^{\infty} \mathrm{d}s\, \varphi(s) \exp(-i\theta s) \tag{A3}$$

and the inverse transform

$$\varphi(s) = \mathcal{F}^{-1}[\mathcal{F}[\varphi](\theta)](s) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{d}\theta\, \exp(i\theta s)\mathcal{F}[\varphi](\theta). \tag{A4}$$

Note that here, θ and s are non-dimensional variables, as opposed to t and ω.

Appendix B: The extended Campbell's theorem

For a full discussion of Campbell's theorem for the mean value of a shot noise process as well as various extentions, we refer to [Rice, Campbell, Pecseli]. It can be shown that for i.i.d. waiting times W with distribution p_W and mean value τ_w, we have in our notation

$$\begin{aligned}&\left\langle\Phi^2\right\rangle = \gamma\left\langle A^2\right\rangle I_2 + \\ &2\gamma\langle A\rangle^2 I_2 \sum_{k=1}^{\infty} \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \cdots \int_0^\infty \mathrm{d}s_k p_W(s_1)p_W(s_2)\cdots p_W(s_k)\rho_\varphi\left(\frac{1}{\tau_\mathrm{d}}\sum_{n=1}^{k} s_n\right).\end{aligned} \tag{B1}$$

The k'th order integral can be compactly written as $\langle\rho_\phi(S_k/\tau_\mathrm{d})\rangle$, where $S_k = \sum_{n=1}^k s_n$. All s_n are i.i.d., with distribution p_W, and we denote the corresponding characteristic function as C_W. We get

$$\left\langle\rho_\varphi\left(\frac{1}{\tau_\mathrm{d}}S_k\right)\right\rangle = \int\limits_{-\infty}^{\infty} \mathrm{d}S\, p_S(S;k)\rho_\varphi\left(\frac{1}{\tau_\mathrm{d}}S\right) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{d}u \int\limits_{-\infty}^{\infty} \mathrm{d}S\, C_S(u;k)\exp(-iSu)\rho_\varphi\left(\frac{1}{\tau_\mathrm{d}}S\right). \tag{B2}$$

As C_S is the characteristic function of the sum of k i.i.d. random variables, we get $C_S(u;k) = C_W(u)^k$. Further, we see that this equation contains the Fourier transform of ρ_φ, so we have

$$\left\langle \rho_\varphi\left(\frac{1}{\tau_\mathrm{d}} S_k\right)\right\rangle = \frac{\tau_\mathrm{d}}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{d}u\, C_W(u)^k \varrho_\varphi(\tau_\mathrm{d} u), \tag{B3}$$

which gives

$$\langle \Phi^2\rangle = \gamma\langle A^2\rangle I_2 + 2\gamma\langle A\rangle^2 I_2 \sum_{k=1}^{\infty} \frac{\tau_\mathrm{d}}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{d}u\, C_W(u)^k \varrho_\varphi(\tau_\mathrm{d} u). \tag{B4}$$

Note that in this equation, only C_W^k depends on k, so we may take the sum over k into the integral and investigate $\sum_k$ For periodic arrivals, $p_W(w) = \delta(w - \tau_w)$, giving $C_W(u) = \exp(iu\tau_w)$, and we get

$$\langle \Phi^2\rangle = \gamma\langle A^2\rangle I_2 + 2\gamma\langle A\rangle^2 I_2 \sum_{k=1}^{\infty} \rho_\varphi\left(\frac{1}{\tau_\mathrm{d}} k\tau_\mathrm{w}\right). \tag{B5}$$

As $A_\mathrm{rms}^2 I_2 = \langle A^2\rangle I_2 - \langle A\rangle^2 I_2 = \langle A^2\rangle I_2 - \langle A\rangle^2 I_2 \rho_\varphi(0)$ and $\rho_\varphi(s) = \rho_\varphi(-s)$, this is equivalent to Eq. (37).

Appendix C: The Poisson summation formula

Here, we briefly present the well-known Poisson summation formula, which is treated in a number of textbooks[42–45]. For our purposes, the formulation used in Corollary VII.2.6 in[45] is the most useful. The statement in the book is for functions on general Euclidian spaces, but we repeat it here only for our special case (the real line):

The Poisson summation formula Suppose the Fourier transform of f and its inverse are defined as in Eq. (A3) and Eq. (A4) respectively. Further suppose that $|f(s)| \leq A(1 + |s|)^{-1-\delta}$ and $|\mathcal{F}[f](\theta)| \leq A(1 + |\theta/2\pi|)^{-1-\delta}$ with $A > 0$ and $\delta > 0$. Then

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \mathcal{F}[f](2\pi n), \tag{C1}$$

where both series converge absolutely.

- Note that the inequality conditions guarantee that both $|f(s)|$ and $|\mathcal{F}[f](\theta)|$ are integrable, which again guarantees that both f and its Fourier transform are continuous and vanish at ∞ (Theorem I.1.2 in[45])).

- Using properties of the Fourier transform, the summation formula can be cast to a number of different forms:

$$\sum_{n=-\infty}^{\infty} \mathcal{F}[f](2\pi n) = \sum_{m=-\infty}^{\infty} f(m) \tag{C2}$$

$$\sum_{n=-\infty}^{\infty} \gamma\mathcal{F}[f](2\pi n\gamma) = \sum_{m=-\infty}^{\infty} f(m/\gamma) \tag{C3}$$

$$\sum_{n=-\infty}^{\infty} \gamma\mathcal{F}[f](2\pi n\gamma)\exp(i2\pi n\gamma t/\tau_\mathrm{d}) = \sum_{m=-\infty}^{\infty} f(m/\gamma + t/\tau_\mathrm{d}). \tag{C4}$$

- By using the definitions of ρ and ϱ given in Eq. (7) and Eq. (8) respectively, as well as the Fourier transform, we have that if Eq. (C1) holds for φ, then by Eq. (C4) we have

$$\sum_{m=-\infty}^{\infty} \varphi(m+s) = \sum_{n=-\infty}^{\infty} \mathcal{F}[\varphi](2\pi n)\exp(i2\pi ns),$$

$$\int_{-\infty}^{\infty} \mathrm{d}u\,\varphi(u) \sum_{m=-\infty}^{\infty} \varphi(s+m) = \int_{-\infty}^{\infty} \mathrm{d}u\,\varphi(u) \sum_{m=-\infty}^{\infty} \mathcal{F}[\varphi](2\pi n)\exp(i2\pi ns),$$

$$\sum_{m=-\infty}^{\infty} \rho_\varphi(m) = \sum_{n=-\infty}^{\infty} \varrho_\varphi(2\pi n),$$

which means that the summation formula holds for the correlation function and power spectrum of the pulse as well. This does not necessarily work in reverse - if one of the sums over φ or its Fourier transform diverges, we cannot exchange the summation and the integral in the second step. Consider as an example the one-sided exponential pulse (detailed in Appendix E). Here, the pulse function does not fulfill the Poisson summation formula as the Fourier transform goes as θ^{-1}, and so the sum diverges. Its correlation function and power spectrum do, however, fulfill the conditions and therefore the formula.

Appendix D: The Lorentz pulse

The Lorentz pulse is given by

$$\varphi(s) = (1+s^2)^{-1}/\pi. \tag{D1}$$

Its Fourier transform is

$$\mathcal{F}[\varphi](\theta) = \exp(-|\theta|), \tag{D2}$$

the integrals are $I_n =$, and we have the correlation function

$$\rho_\varphi(s) = 4\,(4 + s^2)^{-1}, \tag{D3}$$

and spectrum

$$\varrho_\varphi(\theta) = 2\pi \exp(-2|\theta|). \tag{D4}$$

In general, the full sum of the correlation function is given by

$$\sum_{m=-\infty}^{\infty} \rho_\varphi\left(\frac{m}{\gamma} + \frac{t}{\tau_\mathrm{d}}\right) = \gamma\pi[\coth(2\gamma\pi - i\gamma\pi t/\tau_\mathrm{d}) + \coth(2\gamma\pi + i\gamma\pi t/\tau_\mathrm{d})]. \tag{D5}$$

Two special cases of this are of interest in the current contribution. For $t = 0$, we get

$$\sum_{m=-\infty}^{\infty} \rho_\varphi\left(\frac{m}{\gamma}\right) = 2\gamma\pi \coth(2\gamma\pi), \tag{D6}$$

while in the limit $\gamma \to 0$ we get the expected result

$$\lim_{\gamma\to 0} \sum_{m=-\infty}^{\infty} \rho_\varphi\left(\frac{m}{\gamma} + \frac{t}{\tau_\mathrm{d}}\right) = \rho_\varphi(t/\tau_\mathrm{d}). \tag{D7}$$

Appendix E: Table of pulses

For reference, we here present some relations for other pulse functions.

Name	$\varphi(s)$	$\mathcal{F}[\varphi](\theta)$	$\rho_\varphi(s)$	$\varrho_\varphi(\theta)$
One-sided exponential	$\begin{cases} 0, & s < 0 \\ \exp(-s), & s \geq 0 \end{cases}$	$(1 + \mathrm{i}\theta)^{-1}$	$\exp(-\|s\|)$	$2\,(1+\theta^2)^{-1}$
Symmetric exponential	$\exp(-\|s\|)$	$2\,(1+\theta^2)^{-1}$	$2\exp(-\|s\|)[1+\|s\|]$	$8(1+\theta^2)^{-2}$
Sech	$\mathrm{sech}(s)/\pi$	$\mathrm{sech}[\pi\theta/2]$	$s\,\mathrm{csch}(s)$	$\pi^2\mathrm{sech}[\pi\theta/2]^2/2$
Gauss	$\exp(-s^2/2)/\sqrt{2\pi}$	$\exp(-\theta^2/2)$	$\exp(-s^2/4)$	$2\sqrt{\pi}\exp(-\theta^2)$

A few reasonable results for the infinite sums can be obtained:

Name	$\sum_{m=-\infty}^{\infty} \rho_\varphi(m/\gamma)$
One-sided exponential	$\coth\left(\frac{1}{2\gamma}\right)$
Symmetric exponential	$2\coth\left(\frac{1}{2\gamma}\right) + \frac{1}{\gamma}\mathrm{csch}\left(\frac{1}{2\gamma}\right)^2$

Appendix F: Representation of delta functions under finite sampling

In this contribution, we frequently plot delta functions superposed on a waveform. A true representation of a continuous-time delta function would be a line extending out of the plot domain. Alternatively, we could indicate the delta spikes by arrows or stars on the ends. The first solution does not give an indication of the amplitude of the delta, while the second makes for very busy figures.

We have instead elected to represent the Dirac delta by its discrete analog, the Kronecker delta. For $t \to \triangle_{\mathrm{t}} n$, we have $\omega = 2\pi f \to 2\pi m/\triangle_{\mathrm{t}}$. A Dirac delta at a given angular frequency ω_* is then given by $\delta(\omega - \omega_*) = \delta(2\pi(m-k)/\triangle_{\mathrm{t}}) = \frac{\triangle_{\mathrm{t}}}{2\pi}\delta_{m-k}$ where k is the nearest integer to $\triangle_{\mathrm{t}}\omega_*/2\pi$. That is, the Dirac delta is approximated as a boxcar of width equal to the sampling step and a height equal to the inverse of the sampling step. This indicates the amplitude of the delta spikes, separates them from any superposed functions and tends to better approximations for finer sampling.

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