Mathematical Economics
Supervision 4

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Question 1

(a)

i.

$$\begin{aligned}
&x'(t) + x(t) - t = 0 \\
&x'(t)e^t + x(t)e^t = te^t \\
&\frac{\mathrm{d}}{\mathrm{d}t} x(t)e^t = te^t \\
&x(t)e^t = \int te^t \mathrm{d}t = te^t - \int e^t \mathrm{d}t = te^t - e^t + C \\
&x(t) = t - 1 + Ce^{-t} \\
&x'(t) = 1 - Ce^{-t}
\end{aligned}$$

ii.

$$\begin{aligned}
&x'(t) + t^2 x(t) = 5t^2 \\
&\frac{\mathrm{d}}{\mathrm{d}t} x(t)e^{\frac{t^3}{3}} = 5t^2 e^{\frac{t^3}{3}} \\
&x(t)e^{\frac{t^3}{3}} = \int 5t^2 e^{\frac{t^3}{3}} \mathrm{d}t = 5e^{\frac{t^3}{3}} + C \\
&x(t) = 5 + Ce^{-\frac{t^3}{3}}
\end{aligned}$$

Since $x(0) = 5 + C = 6$, $C = 1$ and $x(t) = 5 + e^{-\frac{t^3}{3}}$.

(b)

i.

$$\begin{aligned}
&\frac{y'(t)}{y(t)} = -\frac{1}{2t} \\
&\frac{\mathrm{d}}{\mathrm{d}t} \ln y(t) = -\frac{1}{2} t^{-1} \\
&\ln y(t) = -\frac{1}{2} \ln t + C \\
&y(t) = e^C t^{-\frac{1}{2}} = At^{-\frac{1}{2}},\ A > 0
\end{aligned}$$

ii.

$$\begin{aligned}
y'(t) &= -\frac{t}{y(t)} \\
y'(t)y(t) &= -t \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}y(t)^2 &= -t \\
\frac{y(t)^2}{2} &= -\frac{t^2}{2} + C \\
y(t) &= \pm\sqrt{C - t^2},\ C > 0
\end{aligned}$$

Question 2

(a)

The problem we face is

$$\max_{c(t)} \int_0^T \ln c(t) \mathrm{d}t$$

subject to

$$\begin{aligned}
s'(t) &= -c(t) \\
s(0) &= s_0, s(T) = s_T
\end{aligned}$$

where $s_T < s_0$. This means we are finding the optimal consumption path given a predetermined initial stock and target stock at time T, where consumption directly reduces the total stock one-for-one. Since the utility function is increasing but concave in $c(t)$, we should expect the optimal consumption path to spread $s_0 - s_T$ of consumption equally over time T, with consumption at any single period being $\frac{s_0 - s_T}{T}$. To verify this, the Hamiltonian for this problem is

$$H(c(t), \pi(t)) = \ln c(t) - \pi(t)c(t)$$

and the Pontryagin principle gives the necessary conditions for a maximum as

$$\frac{\partial H}{\partial c(t)} = \frac{1}{c(t)} - \pi(t) = 0 \tag{1}$$

$$s'(t) = \frac{\partial H}{\partial \pi(t)} = -c(t) \tag{2}$$

$$\pi'(t) = -\frac{\partial H}{\partial s(t)} = 0 \tag{3}$$

From (3) and (1),

$$\begin{aligned}
\pi(t) &= \bar{\pi} \\
c(t) &= \frac{1}{\bar{\pi}}
\end{aligned}$$

which shows that the optimal consumption path is constant over time. With (2),

$$s'(t) = -\frac{1}{\bar{\pi}}$$
$$s(t) = -\frac{t}{\bar{\pi}} + C$$

which shows that the state variable (total stock) is a linear function of time. If we want to, we can also solve for $\bar{\pi}$ and C using the boundary conditions:

$$s(0) = C = s_0$$
$$s(T) = s_0 - \frac{T}{\bar{\pi}} = s_T$$
$$\bar{\pi} = \frac{T}{s_0 - s_T}$$

and this confirms the earlier intuition that consumption is spread equally over time with consumption at each period being $\frac{s_0 - s_T}{T} = \frac{1}{\bar{\pi}}$.

(b)

The utility function is again increasing but concave. We should expect the same optimal consumption path as in (a) in that case. We can confirm this again if we want to:

$$H(c(t), \pi(t)) = c(t)^\alpha - \pi(t)c(t)$$

and for a maximum

$$\frac{\partial H}{\partial c(t)} = \alpha c(t)^{\alpha-1} - \pi(t) = 0$$
$$s'(t) = \frac{\partial H}{\partial \pi(t)} = -c(t)$$
$$\pi'(t) = -\frac{\partial H}{\partial s(t)} = 0$$

and again,

$$\pi(t) = \bar{\pi}$$
$$c(t) = \left(\frac{\alpha}{\bar{\pi}}\right)^{\frac{1}{1-\alpha}}$$
$$s(t) = -\left(\frac{\alpha}{\bar{\pi}}\right)^{\frac{1}{1-\alpha}} \cdot t + C$$
$$s(0) = C = s_0$$
$$s(T) = -\left(\frac{\alpha}{\bar{\pi}}\right)^{\frac{1}{1-\alpha}} \cdot T + s_0 = s_T$$
$$\frac{\alpha}{\bar{\pi}} = \left(\frac{s_0 - s_T}{T}\right)^{1-\alpha}$$
$$\bar{\pi} = \alpha\left(\frac{T}{s_0 - s_T}\right)^{1-\alpha}$$
$$c(t) = \left(\frac{\alpha}{\bar{\pi}}\right)^{\frac{1}{1-\alpha}} = \frac{s_0 - s_T}{T}$$
$$s(t) = s_0 - \frac{s_0 - s_T}{T} t$$

which is the same result as before.

(c)

U is strictly increasing and concave. This means $U'(\cdot) > 0, U''(\cdot) < 0$. The Hamiltonian for a generic utility function with these properties is

$$H(U(t), c(t), \pi(t)) = U(c(t)) - \pi(t)c(t)$$

and the solution must satisfy

$$\begin{aligned} \frac{\partial H}{\partial c(t)} &= U'(c(t)) - \pi(t) = 0 \\ s'(t) &= \frac{\partial H}{\partial \pi(t)} = -c(t) \\ \pi'(t) &= -\frac{\partial H}{\partial s(t)} = 0 \end{aligned}$$

As always, since $s(t)$ never appears in the Hamiltonian,

$$\begin{aligned} \pi(t) &= \bar{\pi} \\ U'(c(t)) &= \bar{\pi} \ \forall t \in [0, T] \end{aligned}$$

If $U'(c(t))$ is strictly decreasing in $c(t)$, (as implied by $U''(\cdot) < 0$), there is at most one value of $c(t)$ that can satisfy the above condition. Furthermore the condition does not change as t changes. Thus $c(t)$ must be constant over t if there is a non-negative $c(t)$ that satisfies $U'(c(t)) = \bar{\pi}$. We can implicitly define some $c^*(\bar{\pi})$ such that $U'(c^*(\bar{\pi})) = \bar{\pi}$. We know that if $c^*(\bar{\pi})$ exists it must be unique and invariant to t, so

$$\begin{aligned} s'(t) &= -c^*(\bar{\pi}) \\ s(t) &= s_0 - c^*(\bar{\pi})t \\ s(T) &= s_0 - c^*(\bar{\pi})T = s_T \\ c^*(\bar{\pi}) &= \frac{s_0 - s_T}{T} \end{aligned}$$

Which is the result we got. We haven't used the fact that $U'(\cdot) > 0$, but it stands to reason that if the utility function is strictly increasing the solution is optimal. If for example the utility function is concave but decreasing, the solution is suboptimal and the best thing to do is to consume $s_0 - s_T$ in any one period and nothing in all other periods. And if the utility function is concave but increasing in some areas and decreasing in others (for example if the utility function is quadratic), the solution may be optimal for smaller values of $s_0 - s_T$ but suboptimal for larger values of $s_0 - s_T$ where evenly spreading out consumption brings us to the decreasing portion of U in all periods.

Question 3

(a)

There is now some form of depreciation of the stock variable directly proportional to the total stock, and the Hamiltonian is now

$$H(c(t), s(t), \pi(t)) = \ln c(t) - \pi(t)[s(t) + c(t)]$$

and the conditions for a maximum are

$$\frac{\partial H}{\partial c(t)} = \frac{1}{c(t)} - \pi(t) = 0 \tag{1}$$

$$s'(t) = \frac{\partial H}{\partial \pi(t)} = -s(t) - c(t) \tag{2}$$

$$\pi'(t) = -\frac{\partial H}{\partial s(t)} = \pi(t) \tag{3}$$

From (3),

$$\pi(t) = \pi_0 e^t$$

Substituting into (1),

$$\begin{aligned}\frac{1}{c(t)} &= \pi_0 e^t \\ c(t) &= \frac{1}{\pi_0} e^{-t}\end{aligned}$$

and solving the differential equation for (2),

$$\begin{aligned}s'(t) + s(t) &= -\frac{1}{\pi_0} e^{-t} \\ s(t)e^t &= C - \frac{t}{\pi_0} \\ s(t) &= \left(C - \frac{t}{\pi_0}\right) e^{-t} \\ s(0) &= C = s_0 \\ s(T) &= \left(s_0 - \frac{T}{\pi_0}\right) e^{-T} = s_T \\ s_0 - \frac{T}{\pi_0} &= s_T e^T \\ \pi_0 &= \frac{T}{s_0 - s_T e^T}\end{aligned}$$

This gives us

$$\begin{aligned}c(t) &= \frac{s_0 - s_T e^T}{T} e^{-t} \\ s(t) &= \left(s_0 - \frac{(s_0 - s_T e^T)t}{T}\right) e^{-t}\end{aligned}$$

(b)

For $c(t)$ to be positive, $s_0 - s_T e^T$ must be positive. Intuitively, if $s_T e^T > s_0$, this means that even when we consume nothing and leave the stock alone, the stock variable at time T will go below s_T just from depreciation. This is perhaps seen more easily if we express the violation as $s_0 e^{-T} < s_T$. Thus there is no positive level of consumption that allows for the boundary conditions to be met.

Question 4

(a)

The party seeks to maximize

$$\int_0^T (-U^2 - hp)e^{rt}\mathrm{d}t$$

subject to

$$\begin{aligned} p &= \phi(U) - a\varepsilon \\ \phi(U) &= j - kU \\ \dot{\varepsilon} &= b(p - \varepsilon) \\ \varepsilon(0) &= \varepsilon_0 \end{aligned}$$

This is an optimal control problem with ε, the inflation expectations of the electorate, as the state variable. Changes in the state variable are governed by the differential equation, where the control variables are unemployment and inflation, although this can be thought of as a problem with either one of them as the control variable. The Phillips relation constrains us to just control unemployment and allow inflation to vary or vice versa; the party cannot freely control both at once.

Taking a closer look at the different conditions, we can see that inflation (p) is linearly decreasing in unemployment (U) and in inflation expectations (ε), holding everything else constant. This negative relationship between unemployment and inflation is standard. It is the negative relationship between inflation and inflation expectations which looks somewhat uncommon, but it does suggest that inflation expectations cannot systematically diverge from realized inflation. Inflation expectations get revised whenever the realized rate of inflation undershoots or overshoots expectations, with the revision being linearly proportional to the degree of over or undershooting. If inflation overshoots expectations, inflation expectations get revised upwards and the increase in ε is associated with a lower realized inflation rate, reducing the gap between p and ε.

Theoretically there are no restrictions on p, but U should never be negative.

(b)

The three conditions taken together imply

$$\begin{aligned} p &= j - kU - a\varepsilon \\ \dot{\varepsilon} &= b(j - kU - a\varepsilon - \varepsilon) \\ &= b[j - kU - (1 + a)\varepsilon] \end{aligned}$$

and the party is constrained to maximize

$$\int_0^T [-U^2 - h(j - kU - a\varepsilon)]e^{rt}\mathrm{d}t$$

From this expression we can see that the Phillips relation constrains the economy such that the reelection probability has a quadratic relationship with unemployment, and $V(U, p(U))$ admits a maximum at $U = \frac{hk}{2}$. Given that voters weigh recent experiences more heavily when it's voting season, it is likely that the optimal policy will set $U(T) = \frac{hk}{2}$ to capitalize on the political gains from the most politically popular point on the Phillips curve. In any case, the Hamiltonian becomes

$$H(U(t), \varepsilon(t), \pi(t), t) = (-U^2 - hj + hkU + ha\varepsilon)e^{rt} + \pi(t)b[j - kU - (1+a)\varepsilon]$$

and the necessary conditions for an optimal solution are

$$\frac{\partial H}{\partial U} = (hk - 2U)e^{rt} - k\pi(t) = 0 \tag{1}$$

$$\dot{\varepsilon} = \frac{\partial H}{\partial \pi(t)} = b[j - kU - (1+a)\varepsilon] \tag{2}$$

$$\dot{\pi}(t) = -\frac{\partial H}{\partial \varepsilon} = b(1+a)\pi(t) - hae^{rt} \tag{3}$$

(c)

From (3),

$$\begin{aligned}
\dot{\pi}(t) - b(1+a)\pi(t) &= -hae^{rt} \\
\dot{\pi}(t)e^{-b(1+a)t} - b(1+a)\pi(t)e^{-b(1+a)t} &= -hae^{[r-b(1+a)]t} \\
\pi(t)e^{-b(1+a)t} &= \frac{ha}{b(1+a) - r}e^{[r-b(1+a)]t} + C \\
\pi(t) &= \frac{ha}{b(1+a) - r}e^{rt} + Ce^{b(1+a)t}
\end{aligned}$$

Because there is no fixed endpoint for the state variable, the transversality condition $\pi(T) = 0$ applies. Therefore,

$$\begin{aligned}
\pi(T) &= \frac{ha}{b(1+a) - r}e^{rT} + Ce^{b(1+a)T} = 0 \\
Ce^{b(1+a)T} &= \frac{ha}{r - b(1+a)}e^{rT} \\
C &= \frac{ha}{r - b(1+a)}e^{[r-b(1+a)]T}
\end{aligned}$$

and the optimal co-state path works out to be

$$\begin{aligned}
\pi(t) &= \frac{ha}{b(1+a) - r}e^{rt} + \frac{ha}{r - b(1+a)}e^{[r-b(1+a)]T} \cdot e^{b(1+a)t} \\
&= \frac{ha}{b(1+a) - r}\left(e^{rt} - e^{rT + b(1+a)(t-T)}\right)
\end{aligned}$$

(d)

From (1),

$$\begin{aligned}
(hk - 2U)e^{rt} &= k\pi(t) \\
hk - 2U &= k\pi(t)e^{-rt} \\
U &= \frac{k(h - \pi(t)e^{-rt})}{2} \\
&= \frac{k\left[h - \frac{ha}{b(1+a)-r}\left(e^{rt} - e^{rT+b(1+a)(t-T)}\right)e^{-rt}\right]}{2} \\
&= \frac{k\left[h - \frac{ha}{b(1+a)-r}(1 - e^{[b(1+a)-r](t-T)})\right]}{2} \\
&= \frac{hk}{2}\left[1 + \frac{a}{b(1+a)-r}\left(e^{[b(1+a)-r](t-T)} - 1\right)\right]
\end{aligned}$$

This seems to reduce to a simple exponential function increasing in t (albeit with very messy coeffcients and intercepts). As previously guessed, U reduces to $\frac{hk}{2}$ when $t = T$ for the reasons discussed. For U to always be non-negative, we require that $U(0) \geq 0$, such that

$$\begin{aligned}
1 + \frac{a}{b(1+a)-r}\left(e^{-[b(1+a)-r]T} - 1\right) &\geq 0 \\
b(1+a) - r + a\left(e^{-[b(1+a)-r]T} - 1\right) &\geq 0
\end{aligned}$$

There is no neat expression for this condition; it could be satisfied or violated by many combinations of valid values for the different parameters inside. As an aside, if the Phillips relation was in the more conventional form, where $a < 0$ and inflation is positively associated with inflation expectations, then the condition will always be satisfied and U will always be non-negative.