Part 2B 2019 Macro Growth 1

1.

Consider a continuous time Solow growth model. There is a large set of identical firms indexed by "i". The production technology of firm i is represented by:

$$Y^i(t) = A^i(t)K^i(t)^\alpha L^i(t)^{1-\alpha}, \ \alpha \in (0, 1),$$

where $Y^i(t)$ corresponds to output of firm i, $K^i(t)$ is the capital stock used by firm i, and $L^i(t)$ is labour employed by firm i. A Productivity factor is given by $A^i(t) = Y(t)^\phi$, where $Y(t)$ is the aggregate output and $\phi \in (0, 1)$. Moreover, $\phi + \alpha < 1$. The labour force grows at a constant rate n and households save a fraction $s \in (0, 1)$ of income.

The economy is closed which implies that investment equals savings. Aggregate capital stock evolves according to the following equation of motion:

$$\dot{K}(t) = I(t) - \delta K(t), \ \delta > 0,$$

where $I(t)$ denotes investment and δ is the depreciation rate of the capital stock.

(a) What is the intuition behind $A(t) = Y(t)^\phi$?

(b) Show that the economy exhibits a balanced growth path with a positive long-run growth rate in output per worker.

(c) Does the economy converge to this balanced growth path equilibrium? Explain.

(d) Suppose that the economy is initially on a balanced growth path equilibrium. Consider a change in immigration laws such that it is harder for immigrants to move from another country, so the economy's underlying labour force growth rate decreases ($n' < n$). Describe the effects of such a policy on the dynamics of output per worker of this economy. Be sure to distinguish between short-run and long-run effects.

(a) The intuition behind $A(t) = Y(t)^\phi$ with $0 < \phi < 1$ and $Y(t) = \sum Y^i(t)$ is that aggregate total factor productivity across firms, $A(t)$ (as opposed to within firms $A^i(t)$), arises from externalities to individual output;

these externalities may be attributable to producer agglomeration (shared infrastructure, shared skills pools); the result may be aggregate increasing returns. Ciccone and Hall (1996) find more or less plausibly that density of producers explains differences in productivity across US state locations.

(b) Along a balanced growth path (BGP) the rate of growth of the capital stock g_K is constant. We also expect $g_K = g_Y$ and, in terms that are per unit of labour input, $g_k = g_y$ as well as $g_k = g_K - n$. The capital accumulation process is written as $\dot{K} = sY - \delta K$, so that, substituting from the production function we have $\dot{K} = sK^{\frac{\alpha}{1-\phi}}L(t)^{\frac{1-\alpha}{1-\phi}} - \delta K$, from which it follows that $\frac{\dot{K}}{K} + \delta = sK^{\frac{\alpha}{1-\phi}-1}L(t)^{\frac{1-\alpha}{1-\phi}}$, that is, $g_K + \delta = sK^{\frac{\alpha}{1-\phi}-1}L(t)^{\frac{1-\alpha}{1-\phi}}$. We now take logarithms and differentiate with regard to time to obtain, after some manipulation, $g_K = \frac{1-\alpha}{1-(\alpha+\phi)}\frac{\dot{L}}{L} = \boxed{\frac{1-\alpha}{1-(\alpha+\phi)}}n$, which is indeed <u>constant</u>. With $k = \frac{K}{L}$ we have $g_k = g_K - n = \frac{\phi}{1-(\alpha+\phi)}n$, which we can call [**1**]

From the aggregate version of the production function it then follows that indeed $g_Y = g_K$, $g_y = g_k$. Output per worker thus grows at the constant rate $\frac{\phi}{1-(\alpha+\phi)}n$ along the BGP, this rate depending on the share of capital α, the rate of growth of labour input n and on the externality-effects parameter ϕ. [Note that the rate of growth of labour input is often confused, in simple Solow, with the rate of growth of population, quite a different matter historically and socially.]

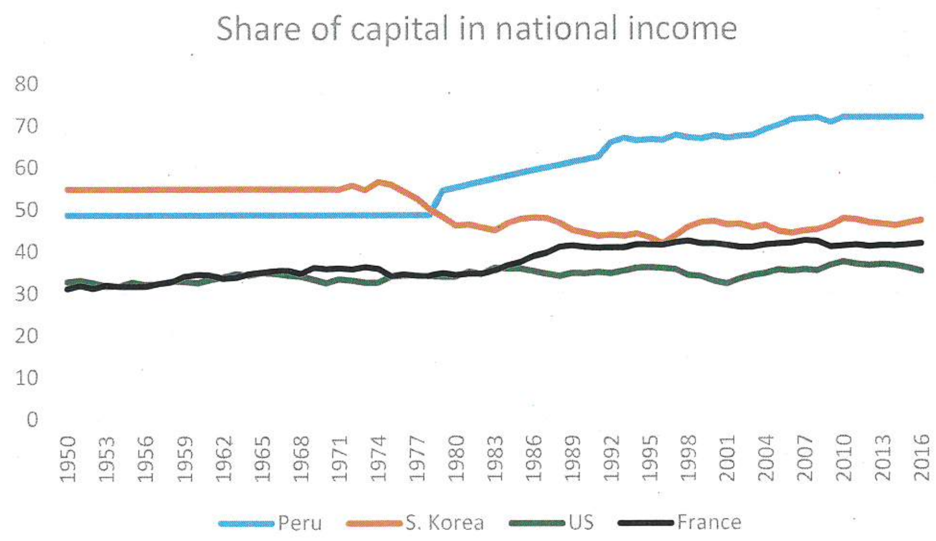
(c) To examine properties of convergence to the BGP it is convenient to examine the growth of k relative to n in [**1**] and use this to convert the system to <u>stationarity</u> by defining: $\tilde{k} = \frac{K}{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}}$, a variable which, because of the relationship between the two growth rates, is stationary (i.e., not subject to growth) along the BGP. From $\dot{K} = sK^{\frac{\alpha}{1-\phi}}L(t)^{\frac{1-\alpha}{1-\phi}} - \delta K$ it follows that $\frac{\dot{K}}{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}} = s\frac{K^{\frac{\alpha}{1-\phi}}L(t)^{\frac{1-\alpha}{1-\phi}}}{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}} - \frac{\delta K}{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}}$. But it can be seen from the construction of $\tilde{k}$ that $\frac{\dot{K}}{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}} = \dot{\tilde{k}} + \frac{1-\alpha}{1-(\alpha+\phi)}n\tilde{k}$. Hence by further substitution we arrive at $\dot{\tilde{k}} = s\tilde{k}^{\frac{\alpha}{1-\phi}} - \left(\delta + \frac{1-\alpha}{1-(\alpha+\phi)}n\right)\tilde{k}$. The growth rate of the stationary variable $g_{\tilde{k}}$ is identified by dividing this expression through by $\tilde{k}$: since $\frac{\alpha}{1-\phi} < 1$ this is decreasing in $\tilde{k}$. Moreover this growth rate goes to infinity as the stationary variable tends to zero while its limit is negative if the stationary variable tends to infinity. We can thus infer that there is a (unique) $\tilde{k}^*$ such that $g_{\tilde{k}}(\tilde{k}^*) = 0$, and that for any initial $\tilde{k}$ the economy converges to a long run equilibrium along the BGP such that $g_{\tilde{k}}(\tilde{k}^*) = 0$

(d) Because the long-run growth of the labour force decreases, so does the long run growth of output per unit of labour input. If we define, accordingly, $\tilde{y} = \frac{Y}{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}}$, we see that $y = \frac{Y}{L} = \tilde{y}\frac{L^{\frac{1-\alpha}{1-(\alpha+\phi)}}}{L}$. Now $\tilde{y}$ has the same dynamics as $\tilde{k}$, so, as n decreases $\tilde{y}$ converges (asymptotically) towards the new steady state; output per unit of labour input starts by increasing; but then its growth rate drops towards the new lower long run growth rate; in the long run both productivity growth and output per unit of labour input are lower. There is no jump (it is a decrease in labour input growth, not in labour input level) so that the process is smooth.

2. Question on Picketty's findings and claims or predictions

Picketty uses the Solow model (with all its somewhat questionable aspects of perfect competition) but chooses to refer to savings net of depreciation, so that savings equal the *net* increment in the capital stock. Building on the Solow model, Picketty advances two relationships which he presents as long run: i) that the share of capital in income is given as $\alpha = \frac{rK}{Y}$, which is purely definitional since r is the return on K, and ii) that $\frac{K}{Y} = \frac{s_{ex-dep}}{g}$, which can be considered to be derived (as in Harrod-Domar) from imposing a steady-state condition on the national accounting identity so that $s_{ex-dep}Y = \Delta K = Kg$ and $g_K = g$. From the second relationship it would follow that $\alpha = \frac{rK}{Y} = \frac{rs_{ex-dep}}{g}$ so that, unless savings net of depreciation are zero, α would appear to rise inexorably if $r > g$. But we know from the fundamental equation that in Solow $\frac{K}{Y} = \frac{s}{g+n+\delta}$ so that as g and n both go to zero while the economy is on a balanced growth path we get $\alpha = \frac{s}{\delta}$. The implication in this case is that, if depreciation is positive, there must be an upper bound on the share of capital in income.

Underlying all this is an issue over whether savings should be considered gross or net of depreciation. Picketty, by taking net saving rather than gross saving to be constant (similar to Harrod-Domar) succeeds in removing the upper bound on the capital-output ratio, but he does generate a savings rate that goes to unity, i.e., which absorbs the entire income when $g = n = 0$. (His distributive results appear consistent with a zero-growth outlook if there is positive real interest: other than making good depreciation all income goes into pension pots and those without pensions get minimal labour income.)



Initial short question

Stylized facts of the neoclassical (Solow) growth model: (a) Input substitution in production (important for the model). (b) Correlation between labour input growth and population growth (not so important for the model but not observed in practice). The model predicts (i) Conditional rather than absolute convergence in capital per unit of labour input and hence in income per unit of labour income, conditional on exogenous savings rates, but only observed with **i**. Mankiw, Romer and Weil labour quality adjustment and only **ii**. rather slowly. (ii) Savings rate effect on the level of income per unit of labour input but not on the growth of income. (iii) Constancy of input shares in income. (iv) Capital should be observed to flow from higher income to lower income economies (though this proposition applies to capital as factor of production rather than as funds). *There are problems with each of these.*