Economic Growth
Supervision 2

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Question 1

(a)

We have a production function $F(\mathbf{x}, L_Y) = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha}$ where the marginal product of capital goods are independent of one another $\left(\partial^2 F/\partial x_a \partial x_b = 0 \text{ for } a \neq b\right)$. In other words there are no complementarities between capital goods.

The firm maximises

$$\pi_Y = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha} - w_Y L_Y - \sum_{j=1}^{A} p_j x_j$$

over L_Y and x_j (we suppress the firm index i). The first-order conditions are

$$\frac{\partial \pi_Y}{\partial L_Y} = (1-\alpha) L_Y^{-\alpha} \sum_{j=1}^{A} x_j^{\alpha} - w_Y = 0 \implies w_Y = (1-\alpha) \sum_{j=1}^{A} \left(\frac{x_j}{L_Y}\right)^{\alpha} \tag{1}$$

$$\frac{\partial \pi_Y}{\partial x_j} = \alpha \left(\frac{L_Y}{x_j}\right)^{1-\alpha} - p_j = 0 \implies p_j = \alpha \left(\frac{L_Y}{x_j}\right)^{1-\alpha} \tag{2}$$

(b)

An intermediate firm producing a capital good x_j maximises profits, which are given by

$$\pi_j = p_j(x_j) x_j - r x_j$$

The first order condition is

$$\frac{\partial \pi_j}{\partial x_j} = \frac{\partial p_j}{\partial x_j} x_j + p_j - r = 0$$

As a side note, this is the standard monopoly result; the equation can be re-arranged to yield $p_j = \frac{\varepsilon_j}{\varepsilon_j + 1} r$. Normalising the number of firms in the final goods sector to 1, we already have the inverse demand function for x_j; it appears in (2). So $p_j(x_j) = \alpha \left(\frac{L_Y}{x_j}\right)^{1-\alpha}$ and we have

$$\frac{\partial p_j}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\alpha \left(\frac{L_Y}{x_j}\right)^{1-\alpha}\right] = -(1-\alpha)\alpha \left(\frac{L_Y}{x_j}\right)^{1-\alpha} \frac{1}{x_j} = -(1-\alpha) p_j(x_j) \frac{1}{x_j}$$

Substituting this into the intermediate firm's first-order condition, we have the monopoly price

$$-(1-\alpha)p_j + p_j - r = 0 \implies p_j = \frac{r}{\alpha} \tag{3}$$

which implies a constant markup over the rental price of capital. Since this is true for any j, we must have $p_j = p$ and $x_j = x$ for all $j \in \{1, \ldots, A\}$. Then, output of the final good becomes

$$Y = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha} = A x^{\alpha} L_Y^{1-\alpha} \tag{4}$$

which, if we consider that aggregate output is $K = \sum_j x_j = Ax$, takes the Cobb-Douglas form employed in the textbook Solow model:

$$Y = A \left(\frac{K}{A}\right)^{\alpha} L_Y^{1-\alpha} = K^{\alpha}(AL_Y)^{1-\alpha} \tag{5}$$

We also have the equilibrium profits of the capital good producer

$$\pi_j = \underbrace{p_j x_j - r x_j = p_j x_j - \alpha p_j x_j}_{\text{using } p_j = \frac{r}{\alpha}} = \underbrace{(1-\alpha) p_j x_j = \alpha(1-\alpha) x_j^{\alpha} L_Y^{1-\alpha}}_{\text{using } p_j = \alpha\left(\frac{L_Y}{x_j}\right)^{1-\alpha}} = \alpha(1-\alpha)\frac{Y}{A} \tag{6}$$

which makes sense: profits in the intermediate good sector are driven down as the number of capital goods producers increases.

(c)

Buying a patent and selling it one period later will yield the buyer the profits made per firm in the intermediate good sector. Alternatively, investing the money used to buy the patent yields an interest payment. Therefore, the no-arbitrage condition is

$$rP_A = \pi + \dot{P}_A \implies r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A} \tag{7}$$

Now we digress a little. If we impose a balanced growth path, r must be constant. Again, from (3) the monopoly price of capital goods is $p = \frac{r}{\alpha}$. Equating this to (2), the inverse demand for capital goods, we have

$$\frac{r}{\alpha} = \alpha\left(\frac{L_Y}{x}\right)^{1-\alpha} \implies r = \alpha^2 \left(\frac{L_Y}{K/A}\right)^{1-\alpha} = \alpha^2 \frac{Y}{K}$$

and if r is a constant, Y and K must grow at the same rate. So $g_Y = g_K$. Then, taking growth rates of aggregate output given by (5), we have

$$g_Y = \alpha g_K + (1-\alpha)(g_A + g_{L_Y}) = \alpha g_Y + (1-\alpha)(g_A + g_{L_Y}) \implies g_Y = g_A + g_{L_Y} \tag{8}$$

Next, from (6) we found that profits were $\pi = \alpha(1-\alpha)\frac{Y}{A}$. So $g_\pi = g_Y - g_A = g_{L_Y} = n$.

We can finally look back at the no-arbitrage condition (7). Again, imposing a balanced growth path means r must be a constant, and both $\frac{\pi}{P_A}$ and $\frac{\dot{P}_A}{P_A}$ must be constant. Since $\frac{\pi}{P_A}$ is a constant and $g_\pi = n$, it must be that $\frac{\dot{P}_A}{P_A} = g_{P_A}$ is also n. At last, we have for the optimal price of a patent

$$r = \frac{\pi}{P_A} + n \implies P_A = \frac{\pi}{r - n} \tag{9}$$

which is analogous to the asset price formula in the Gordon growth model: n is the rate at which the patent price appreciates each period, and π is the next period's 'dividend' from holding the patent.

(d)

We've already expressed r as a function of parameters and $\frac{Y}{K}$ in (8): $r = \alpha^2 \frac{Y}{K}$. The higher the output-capital ratio, the higher the marginal product of capital, and the higher the rental price of capital (which in this model is equivalent to the interest rate; there is an implicit equivalence between physical and financial capital).

Question 2

It isn't strictly necessary for answering the questions later, but we can analytically solve for $K(t)$, which can help in verifying our results (and gives more exact solutions than the approximations taught in lectures). Aggregate output evolves according to the production function $Y(t) = F(K(t), L(t), Q(t)) = K(t)^\alpha L(t)^\beta Q(t)^{1-\alpha-\beta}$, and capital $K(t)$ evolves according to the following law of motion:

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{1}$$

Labour grows at rate n (which could be zero) and land is in fixed supply. Therefore, $L(t) = L_0 e^{nt}$ and $Q(t) = \bar{Q}$. From (1), we have

$$\begin{aligned}
\dot{K}(t) &= sK(t)^\alpha L(t)^\beta Q(t)^{1-\alpha-\beta} - \delta K(t) \\
\dot{K}(t) + \delta K(t) &= sK(t)^\alpha (L_0 e^{nt})^\beta \bar{Q}^{1-\alpha-\beta} \\
\dot{K}(t) K(t)^{-\alpha} + \delta K(t)^{1-\alpha} &= sL_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta nt} \\
(1-\alpha)\dot{K}(t)K(t)^{-\alpha} e^{\delta(1-\alpha)t} + \delta(1-\alpha)K(t)^{1-\alpha} e^{\delta(1-\alpha)t} &= s(1-\alpha)L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{[\beta n + \delta(1-\alpha)]t}
\end{aligned}$$

The left-hand side is equal to $\frac{\mathrm{d}}{\mathrm{d}t}\left[K(t)^{1-\alpha} e^{\delta(1-\alpha)t}\right]$. Integrating both sides, we have

$$\begin{aligned}
K(t)^{1-\alpha} e^{\delta(1-\alpha)t} &= \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{[\beta n + \delta(1-\alpha)]t} + C \\
K(t)^{1-\alpha} &= \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta nt} + Ce^{-\delta(1-\alpha)t} \\
K(t) &= \left[\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta nt} + Ce^{-\delta(1-\alpha)t}\right]^{\frac{1}{1-\alpha}}
\end{aligned}$$

We find C using the boundary condition $K(0) = K_0$:

$$K(0) = \left[\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} + C\right]^{\frac{1}{1-\alpha}} = K_0$$
$$C = K_0^{1-\alpha} - \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta}$$

With that we have our expression for $K(t)$:

$$K(t) = \left\{\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta n t} + \left[K_0^{1-\alpha} - \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta}\right] e^{-\delta(1-\alpha)t}\right\}^{\frac{1}{1-\alpha}}$$

And we can also divide by $L(t)$ to find $k(t) = \frac{K(t)}{L(t)}$:

$$\begin{aligned}
k(t) &= \frac{1}{L(t)} \left\{\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta n t} + \left[K_0^{1-\alpha} - \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta}\right] e^{-\delta(1-\alpha)t}\right\}^{\frac{1}{1-\alpha}} \\
&= \left\{L(t)^{-(1-\alpha)} \left\{\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta n t} + \left[K_0^{1-\alpha} - \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta}\right] e^{-\delta(1-\alpha)t}\right\}\right\}^{\frac{1}{1-\alpha}} \\
&= \left\{\left(L_0 e^{nt}\right)^{-(1-\alpha)} \left\{\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta} e^{\beta n t} + \left[K_0^{1-\alpha} - \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} L_0^\beta \bar{Q}^{1-\alpha-\beta}\right] e^{-\delta(1-\alpha)t}\right\}\right\}^{\frac{1}{1-\alpha}} \\
&= \left\{\frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} \left(\frac{\bar{Q}}{L_0}\right)^{1-\alpha-\beta} e^{-(1-\alpha-\beta)nt} + \left[\left(\frac{K_0}{L_0}\right)^{1-\alpha} - \frac{s(1-\alpha)}{\beta n + \delta(1-\alpha)} \left(\frac{\bar{Q}}{L_0}\right)^{1-\alpha-\beta}\right] e^{-(n+\delta)(1-\alpha)t}\right\}^{\frac{1}{1-\alpha}}
\end{aligned} \tag{2}$$

and we can see that the expression above depends on the model parameters and the initial capital/labour and land/labour ratios.

(a)

We can find the steady-state capital per capita with some simpler steps. In the steady state, we must have $g_k = 0$, so $g_K = g_L = 0$. Then, from (1), we have in the steady state

$$\begin{aligned}
\dot{K}(t) &= sY(t) - \delta K(t) \\
g_K &= s\frac{Y(t)}{K(t)} - \delta \\
&= sK(t)^{-(1-\alpha)} L(t)^\beta Q(t)^{1-\alpha-\beta} - \delta = 0 \\
\frac{\delta}{s} &= \left(\frac{K(t)}{L(t)}\right)^{-(1-\alpha)} \left(\frac{Q(t)}{L(t)}\right)^{1-\alpha-\beta} \\
k^* &= \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}} \left(\frac{\bar{Q}}{L_0}\right)^{\frac{1-\alpha-\beta}{1-\alpha}}
\end{aligned}$$

And looking at (2) confirms that it indeed is the steady-state level, and that the economy converges to this state no matter what the starting point is:

$$k(t) = \left\{ \frac{s}{\delta} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} + \left[\left(\frac{K_0}{L_0} \right)^{1-\alpha} - \frac{s}{\delta} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} \right] e^{-\delta(1-\alpha)t} \right\}^{\frac{1}{1-\alpha}} \quad \text{if } n = 0$$

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty} \left\{ \frac{s}{\delta} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} + \left[\left(\frac{K_0}{L_0} \right)^{1-\alpha} - \frac{s}{\delta} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} \right] e^{-\delta(1-\alpha)t} \right\}^{\frac{1}{1-\alpha}} = \left(\frac{s}{\delta} \right)^{\frac{1}{1-\alpha}} \left(\frac{\bar{Q}}{L_0} \right)^{\frac{1-\alpha-\beta}{1-\alpha}}$$

We can find the exact speed of convergence by calculating $g_k = \frac{\dot{k}(t)}{k(t)}$:

$$\begin{aligned}
k(t) &= \left\{ \frac{s}{\delta} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} + \left[\left(\frac{K_0}{L_0} \right)^{1-\alpha} - \frac{s}{\delta} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} \right] e^{-\delta(1-\alpha)t} \right\}^{\frac{1}{1-\alpha}} \\
&= \left[k^{*1-\alpha} + \left(k_0^{1-\alpha} - k^{*1-\alpha} \right) e^{-\delta(1-\alpha)t} \right]^{\frac{1}{1-\alpha}} \\
\dot{k}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left[k^{*1-\alpha} + \left(k_0^{1-\alpha} - k^{*1-\alpha} \right) e^{-\delta(1-\alpha)t} \right]^{\frac{1}{1-\alpha}} \\
&= \frac{1}{1-\alpha} \left[k^{*1-\alpha} + \left(k_0^{1-\alpha} - k^{*1-\alpha} \right) e^{-\delta(1-\alpha)t} \right]^{\frac{\alpha}{1-\alpha}} \left[-\delta(1-\alpha) \right] \left(k_0^{1-\alpha} - k^{*1-\alpha} \right) e^{-\delta(1-\alpha)t} \\
&= -\delta \left[k^{*1-\alpha} + \left(k_0^{1-\alpha} - k^{*1-\alpha} \right) e^{-\delta(1-\alpha)t} \right]^{\frac{\alpha}{1-\alpha}} \left[k_0^{1-\alpha} - k^{*1-\alpha} \right] e^{-\delta(1-\alpha)t} \\
g_k &= -\delta \frac{k_0^{1-\alpha} - k^{*1-\alpha}}{k^{*1-\alpha} + \left(k_0^{1-\alpha} - k^{*1-\alpha} \right) e^{-\delta(1-\alpha)t}} e^{-\delta(1-\alpha)t}
\end{aligned}$$

The above shows the exact growth rate of k across time. If we just want the instantaneous speed of convergence g_c at some point around the steady state, we can examine the growth rate at $t = 0$ at different levels of k_0:

$$g_c(k_0) = g_k(0) = -\delta \frac{k_0^{1-\alpha} - k^{*1-\alpha}}{k_0^{1-\alpha}}$$

which makes sense; when we start above the steady state, the growth rate is negative, and vice versa. The speed of convergence greater the further we are from the steady state.

We can also approximate the speed of convergence by taking a Taylor approximation of the growth rate of k around k^* (which, with no population growth, is equal to the growth rate of K):

$$\begin{aligned}
g_k &= g_K = \dot{K}/K \\
&= sK^{-(1-\alpha)} L_0^{\beta} \bar{Q}^{1-\alpha-\beta} - \delta \\
&= sk^{-(1-\alpha)} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} - \delta \\
g_k(k) &\approx g(k^*) + g_k'(k^*)(k - k^*) \\
&\approx -(1-\alpha) s k^{*-(1-\alpha)} \left(\frac{\bar{Q}}{L_0} \right)^{1-\alpha-\beta} \frac{k - k^*}{k^*} = -(1-\alpha)\delta \frac{k - k^*}{k^*}
\end{aligned}$$

where we used $g(k^*) = 0$ and $k^* = \left(\frac{s}{\delta} \right)^{\frac{1}{1-\alpha}} \left(\frac{\bar{Q}}{L_0} \right)^{\frac{1-\alpha-\beta}{1-\alpha}}$.

(b)

From (2), we can see that $k(t)$ always tends to 0 as $t \to \infty$. So the capital-labour ratio goes to zero, and so does the output-labour ratio since $y = k^\alpha \left(\frac{\bar{Q}}{L(t)}\right)^{1-\alpha-\beta}$.

(c)

Assuming factors of production are paid their marginal products, land rents r and wages w are

$$r = \frac{\partial}{\partial Q(t)} K(t)^\alpha L(t)^\beta Q(t)^{1-\alpha-\beta} = (1-\alpha-\beta)\frac{Y(t)}{\bar{Q}}$$
$$w = \frac{\partial}{\partial L(t)} K(t)^\alpha L(t)^\beta Q(t)^{1-\alpha-\beta} = \beta y(t)$$

Aggregate output grows to infinity as $t \to \infty$, so land rents go to infinity as well given that the supply of land is fixed. This is also expected since the marginal product of land is increasing in labour, and the labour force grows exponentially. Meanwhile, since output per cpaita goes to zero, wages are also driven to zero as $t \to \infty$.