1. This a version of the Kyland and Prescott 1982 'time to build' model with white noise multiplicative displacements Ψ_t in total factor productivity Z_t, viz. $Z_{t-1}^{\mu}\Psi_t$, and lags in the delivery of new capital (investment) orders.

(a) and (b) The household's problem is to maximize the discounted present value of lifetime expected consumption subject to its period-by-period budget constraint, i.e., $\max_{C_t,K_{t+1}} E_{t=0}(\sum_{t=0}^{\infty}\beta^t C_t)\ s.t.\ C_t + K_{t+1} - (1-\delta)K_t = W_tL_t + R_tK_t, \forall t$, while the firm's problem is to maximize profits Π subject to its production function by buying labour and renting capital from the household, i.e., $\max(Y_t - W_tL_t + R_tK_t)\ s.t. Y_t = Z_tK_t^{\alpha}L_t^{1-\alpha},\ Z_t = Z_{t-1}^{\mu}\Psi_t$. The Lagrangean $E_{t=0}\sum_{t=0}^{\infty}\beta^t[C_t - \gamma_t(C_t + K_{t+1} - (1-\delta)K_t - W_tL_t - R_tK_t)]$ generates as first order condition with regard to C_t $\frac{\partial\mathcal{L}}{\partial C_t} = \beta^t - \beta^t\gamma_t = 0$ so that the multiplier is $\gamma_t = 1$; for the first order condition with regard to K_{t+1} or K_t note that the Lagrangean is an infinite sum, each term of which contains both K_t and K_{t+1}, where the latter refers in any one period to the expectation of the capital stock for the next period. Thus in period t optimization involves differentiation with regard to K_{t+1} of

$$\beta^t[C_t - \gamma_t(C_t + K_{t+1} - (1-\delta)K_t - W_tL_t - R_tK_t)] + \beta^{t+1}E_t[C_{t+1} - \gamma_{t+1}(C_{t+1} + K_{t+2} - (1-\delta)K_{t+1} - W_{t+1}L_{t+1} - R_{t+1}K_{t+1})]$$

and the consequent first order condition is

$$-\beta^t\gamma_t + \beta^{t+1}E_t[\gamma_{t+1}(1-\delta) + R_{t+1}] = 0$$

so that

$$\gamma_t = \beta E_t[\gamma_{t+1}(1-\delta) + R_{t+1}]$$

and from the earlier $\gamma_t = 1$ we obtain

$$\beta E_t[(1-\delta) + R_{t+1}] = 1$$

which is the problem's Euler equation or $MRS = MRT$ efficiency condition (it is a version of what is known as a stochastic discount factor in asset pricing).

(c) To identify the equilibrium or steady state we note for the productivity shock $\bar{Z} = \bar{Z}^{\mu}\bar{\Psi}$ so that $\bar{Z} = \bar{Z}^{\mu}$ and it follows that we can write $\bar{Z} = 1$.

From the firm's maximization problem $\frac{\partial \Pi}{\partial K_t} = \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0$ and $\frac{\partial \Pi}{\partial L_t} = (1-\alpha) Z_t K_t^{\alpha} L_t^{-\alpha} - W_t = 0$ so that $R_t = \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha}$ while $W_t = (1-\alpha) Z_t K_t^{\alpha} L_t^{-\alpha}$; and if labour supply is inelastic $R_t = \alpha Z_t K_t^{\alpha-1}$ while $W_t = (1-\alpha) Z_t K_t^{\alpha}$

(d) Proceeding to substitute this into the Euler equation we have (*)

$$\beta E_t[(1-\delta) + \alpha Z_{t+1} K_{t+1}^{\alpha-1}] = 1$$

and in the steady state

$$\beta\left[(1-\delta) + \alpha \bar{Z} \bar{K}^{\alpha-1}\right] = 1$$

and thus with $\bar{Z} = 1$ we have $\beta\left[(1-\delta) + \alpha \bar{K}^{\alpha-1}\right] = 1$

From this it follows that

$$\bar{K} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}} \text{ (**) and } \bar{Y} = \bar{Z}\bar{K}^{\alpha} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}}$$

and from national accounting $\bar{C} = \bar{Y} - \delta \bar{K}$ we have

$$\bar{C} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

which must be in effect a version of Hall's 1978 consumption function given a specific (Cobb-Douglas) production function.

(e) Returning to the Euler equation (*) we see $\frac{\alpha\beta}{1-\beta(1-\delta)} = \frac{1}{E_t\left[Z_{t+1} K_{t+1}^{\alpha-1}\right]}$ or $E_t[Z_{t+1} K_{t+1}^{\alpha-1}] = \frac{1-\beta(1-\delta)}{\alpha\beta}$, that is, $E_t\left[Z_t^{\mu} \Psi_{t+1} K_{t+1}^{\alpha-1}\right] = \frac{1-\beta(1-\delta)}{\alpha\beta}$

We now note that Z_t is known at t and that K and Ψ are not correlated, so that we can write $Z_t^{\mu} E_t[\Psi_{t+1}] E_t[K_{t+1}^{\alpha-1}] = \frac{1-\beta(1-\delta)}{\alpha\beta}$, and hence, with $E_t[\Psi_{t+1}] = 1$ we obtain $Z_t^{\mu} E_t[K_{t+1}^{\alpha-1}] = \frac{1-\beta(1-\delta)}{\alpha\beta} = \bar{K}^{\alpha-1}$ from (**), and it now follows that $E_t[K_{t+1}^{\alpha-1}] = Z_t^{-\mu} \bar{K}^{\alpha-1}$. But at t we already know capital in the next period (since we are deciding upon it), so $K_{t+1}^{\alpha-1} = Z_t^{-\mu}\bar{K}^{\alpha-1}$ and $K_{t+1} = \bar{K}Z_t^{\frac{\mu}{1-\alpha}}$; hence output or income is given by $Y_t = Z_t K_t^{\alpha} Z_{t-1}^{\frac{\mu}{1-\alpha}}$

(f) From $Z_t = Z_{t-1}^{\mu}\Psi_t$ we obtain, by taking logarithms, $z_t = \mu z_{t-1} + \psi_t$ and likewise for the capital stock from $K_{t+1} = \bar{K}Z_t^{\frac{\mu}{1-\alpha}}$ we obtain $k_t = \bar{k} + \frac{\mu}{1-\alpha} z_t$ and, from $Z_t K_t^{\alpha} Z_{t-1}^{\frac{\mu}{1-\alpha}}$, $y_t = \alpha\bar{k} + z_t + \frac{\alpha\mu}{1-\alpha} z_{t-1}$

(g) Now, normalizing total factor productivity, $z_0 = 0$. **i**. For $\mu = 0$, $Z_t = \Psi_t$: the technology disturbance becomes white noise; thus $k_{t+1} = \bar{k} = k_t$ remaining constant; $y_t = \alpha\bar{k} + z_t$ and the log-output or growth-in-output process is white noise; and likewise $c_t = \ln(e^{y_t} + [(1-\delta)e^{k_t} - e^{k_{t+1}}])$ which is also white noise since the terms in square brackets are a constant. **ii**. For $\mu > 0$ the capital and productivity k_t, z_t processes are auto-regressive (order 1) while the y_t process is auto-regressive (order 2). Shock persistence derives only from technology shocks[1], not from consumption smoothing; this is essentially because consumer utility is linear with in effect infinite intertemporal elasticity of substitution.

2. "Business cycles are an economic phenomenon that is fully understood and explained by economic theory." This appears to have two components: theory and empirics, since a theory is only as good as the light that it might throw on empirical observations (providing those observations are reliable). ♦ Business cycle theory attempts to account for <u>stylized facts</u> concerning variation, and especially co-variation, of macro-variables. BCT starts from Wicksell and Frisch: exogenous shocks and a structure of endogenous responses [rocking horse analogy]. ♦ Pre-modern BCT [Samuelson multiplier-accelerator model, Hicks model] concentrated on structure of endogenous responses [non-linearity leading to regularity of fluctuation].

[1] This generates thus the characteristic real business cycle result, that trend growth is indistinguishable from the accumulation of productivity disturbances, a feature which empirically leads to the construction of the Hodrick-Prescott filter as the trend-cum-fluctuation trajectory of the economy.

♦ Modern BCT concentrates instead on shocks in exogenous variables. There are three main versions of modern BCT: **i**. monetary or nominal shocks to a competitive structure of optimizing responses [Lucas following Friedman], leading to disequilibrium behaviour and welfare-improving policy-scope; **ii**. real shocks to a competitive structure of optimizing responses [Kydland and Prescott] leading to equilibrium and no welfare-improving policy-scope; and **iii**. shocks real or nominal to an imperfectly competitive or 'New Keynesian' structure of optimizing responses [Mankiw, Calvo]. ♦ Modern BCT has led to empirical horse-races to see which type of model might win: as between the two disequilibrium models [new classical and new Keynesian] the empirical horse race appears to have been won by the new Keynesian. ♦ As between the equilibrium real and the disequilibrium models there is an impasse: while disequilibrium models can be specified in a way which is open to econometric testing, such is not the case for real equilibrium models – these models can only be calibrated with parameters taken from separate fields of study (rather than parameters estimated with the models themselves), and then examined in simulations; this raises a number of issues over which opinions are doctrinally divided. ♦ With regard to equilibrium models there is an additional obstacle in terms of examining stylized facts of fluctuation: how to separate trend from fluctuation when they are intrinsically intertwined? ♦ The device of the Hodrick-Prescott filter or moving average procedure, designed to this purpose, is also open to some objection. [Summers]

Two stylised facts in dispute: co-movement of productivity & output (equilibrium models)
co-movement of prices & output (disequilibrium models)