Economics of Uncertainty and Information
Supervision 4

Samuel Lee

Question 1

(a)

In this case, the limited partner is the principal and the general partner is the agent. The amount of oil discovered in a well (q) is drawn from a distribution with p.d.f. $f(q)$ and q has a support of $[0, \infty)$. Units of q are chosen such that one unit of oil brings in £1 of revenue. Both parties are risk-averse with a reservation utility of 0. The interactions proceed as such:

1. The principal decides on a fraction α of oil revenues (should there be any) that will accrue to the agent, with the remaining $1-\alpha$ going to the principal.
2. The principal pays £E towards the exploration for oil.
3. Exploring for oil reveals the amount of oil (q) that can potentially be extracted from a given well, but this information is privately known by the agent.
4. The agent decides whether to spend £C to complete the well.
   (a) Not completing the well costs the agent nothing, while the principal does not recoup the £E paid for exploration.
   (b) Completing the well nets the agent a utility of $\alpha q - C$, and the principal ends up with a utility of $(1-\alpha)q - E$.

For a given α, the agent will not complete the well if $q < \frac{C}{\alpha}$. This is even if $q > C + E$, and the principal can transfer some of the revenue to the agent, leaving both with a profit. Without the option of an ex-post renegotiation of α after the exploration, the principal has to choose α to maximize

$$\begin{aligned}
&\int_0^{\frac{C}{\alpha}} f(q)[-E]\mathrm{d}q + \int_{\frac{C}{\alpha}}^{\infty} f(q)[(1-\alpha)q - E]\mathrm{d}q \\
=&(1-\alpha)\int_{\frac{C}{\alpha}}^{\infty} f(q)\cdot q \ \mathrm{d}q - E
\end{aligned}$$

where $\int_0^\infty f(q)E \ \mathrm{d}q = E$. Differentiating the above with respect to α and setting it to zero:

$$\begin{aligned}
&(1-\alpha)\frac{\partial}{\partial\alpha}\left[\int_{\frac{C}{\alpha}}^{\infty} f(q)\cdot q \ \mathrm{d}q\right] - \int_{\frac{C}{\alpha}}^{\infty} f(q)\cdot q \ \mathrm{d}q \\
=&\frac{(1-\alpha)C^2}{\alpha^3} f\left(\frac{C}{\alpha}\right) - \int_{\frac{C}{\alpha}}^{\infty} f(q)\cdot q \ \mathrm{d}q = 0
\end{aligned}$$

using the Leibniz integration rule. The properties of this solution will depend on $f(q)$. However, in practice $f(q)$ may not have 'nice' properties; it could be discontinuous at $q = 0$ if, say, there is a significant chance of finding nothing at all but very low chance of finding 0.000000001 units of oil.

(b)

The principal could include in the contract a stipulation that the agent's assessments of the oil wells have to be made known. Alternatively, α could be negotiated after exploration (step 3), such that there will always be an incentive for the agent to consider completing an oil well as long as $q > C$. This may however incur higher transactions costs with the repeated negotiation of α. α could also be pre-negotiated as the share of the surplus $q - (C + E)$, although this likely oversimplifies the real-life problem since C and E are not going to be constant.

Question 2

(a)

If females know the healthiness of males, there is no advantage to flamboyance, so male tails will not be flamboyant at all.

(b)

In a separating equilibrium, unhealthy peacocks exhibit zero flamboyance and healthy peacocks exhibit some positive amount of flamboyance. For a separating equilibrium to be sustained,

1. It must be infeasible for an unhealthy peacock to exhibit the same flamboyance as healthy peacocks, even if this secures it a high-quality peahen.
2. The expected number of surviving peachicks for a healthy peacock must be higher when it exhibits some flamboyance and therefore only mates with higher-quality peahens than if it exhibits no flamboyance and mates with a mix of high and low-quality peahens.

(c)

We are looking for range of f (exhibited by healthy peacocks) which sustains a separating equilibrium. Pretending to be healthy allows the unhealthy peacock to mate with a peahen of quality $8 - f$, so the first condition implies that $(4 - 2f)(8 - f) \leq 16$, or $f \in [5 - \sqrt{17}, 5 + \sqrt{17}]$.

In a pooling equilibrium with $f = 0$ for both types, a peacock is believed to be healthy θ_H of the time, and believed to be unhealthy θ_U of the time. Assuming $\theta_U = 1 - \theta_H$, the second condition implies

$$(8 - f)^2 \geq 64\theta_H + 32(1 - \theta_H)$$

which implies $f \leq 8 - 4\sqrt{2(1 + \theta_H)}, f \geq 8 + 4\sqrt{2(1 + \theta_H)}$.

It must be the case that $\theta_H \in [0, 1]$, and $\theta_H = 0$ yields the most permissive range of values for f. But even at $\theta_H = 0$, the second inequality $f \geq 8 + 4\sqrt{2(1 + \theta_H)}$ never intersects the first condition, $f \in [5 - \sqrt{17}, 5 + \sqrt{17}]$. So the only relevant condition is $f \leq 8 - 4\sqrt{2(1 + \theta_H)}$.

We now have two range of values which f must reside in for a separating equilibrium to be sustained, but they only intersect when

$$\begin{aligned}
8-4\sqrt{2(1+\theta_H)} &\geq 5-\sqrt{17} \\
4\sqrt{2(1+\theta_H)} &\leq 3+\sqrt{17} \\
\sqrt{1+\theta_H} &\leq \frac{3+\sqrt{17}}{4\sqrt{2}} \\
1+\theta_H &\leq \frac{9+6\sqrt{17}+17}{32} \\
\theta_H &\leq \frac{13+3\sqrt{17}}{16}-1=\frac{3(\sqrt{17}-1)}{16}
\end{aligned}$$

which is less than 1 (around 0.586), so we know there are some values of θ_H which may be too high to sustain a pooling equilibrium. The intuition is that without any signals, high-quality peahens will just guess whether a peacock is healthy. If a peahen guesses (rightly or wrongly) that a peacock is healthy, the two will mate. When the peahens know that the proportion of healthy peacocks is high, this happens often enough that it is no longer worth it to have flamboyant tails that secure high-quality peahens: accepting the infrequent prospect of sometimes mating with a low-quality peahen is preferable to expending resources and risking predation just to guarantee a high-quality peahen.

So the separating equilibria are where the unhealthy peacocks exhibit $f=0$ and the healthy peacocks exhibit some $f \in \left[5-\sqrt{17}, \min\left\{8-4\sqrt{2(1+\theta_H)}, 2\right\}\right]$, provided $\theta_H \leq \frac{3(\sqrt{17}-1)}{16}$ (the upper bound is as such since $f \in [0,2]$). The best separating equilibrium for the healthy peacock is obviously where f is at the lower bound $5-\sqrt{17}$.

(d)

There are two effects that allow flamboyant tails to be a signal of health. The first is a marginal effect: increasing the flamboyance of tails is more deletrious to unhealthy peacocks than healthy peacocks. Hence the coefficient of 2 on f in the expected number of surviving peachicks. The second is that unhealthy peacocks are less likely to survive as a whole even without any flamboyant tails, so there is even less room to decrease that chance of survival further still. Past a certain point, only healthy peacocks can afford to have flamboyant tails.

Question 3

(a)

When the cow produces 2 gallons, there is no constraint and both firms should produce. When the cow produces 1 gallon, a choice has to be made. If $\varepsilon_c = 1$, then the utility of cream is higher given equal amounts of investment, so it is better to use the milk for cream. And vice versa if $\varepsilon_y = 1$ instead. So the socially efficient allocation of milk in each case is

Cow production	Cow produces 2 gallons		Cow produces 1 gallon	
Preference shock	$\varepsilon_c = 1$	$\varepsilon_y = 1$	$\varepsilon_c = 1$	$\varepsilon_y = 1$
Probability	$\frac{1}{2}(1-\mu)$		$\frac{\mu}{2}$	
Optimal allocation	Produce both		Produce cream	Produce yoghurt
Surplus	$I_c + I_y + 1$		$I_c + 1$	$I_y + 1$

The expected surplus, given investment levels I_c and I_c and provided the optimal allocation is attained in each state, is $(1-\mu)(I_c + I_y + 1) + \frac{\mu}{2}(I_c + I_y + 2) - I_c^2 - 2I_y^2$. To maximize this,

$$1 - \mu + \frac{\mu}{2} - 2I_c = 0 \implies I_c = \frac{2-\mu}{4}$$
$$1 - \mu + \frac{\mu}{2} - 4I_y = 0 \implies I_y = \frac{2-\mu}{8}$$

(b)

1. If the cow produces 2 gallons, the cream producer will definitely produce, netting himself $I_c + \varepsilon_c$, and as for the remaining milk he bargains for half the profits of the yoghurt producer $\left(\frac{I_y+\varepsilon_y}{2}\right)$.

2. If the cow produces 1 gallon, and $\varepsilon_c = 1$, the cream producer will keep the milk for himself since his surplus will be larger than whatever the yoghurt producer can make. He ends up with a surplus of $I_c + 1$.

3. If $\varepsilon_c = 0$, and the cream producer wants to sell the milk to the yoghurt producer, the cream producer's threat point is I_c: should negotiations break down, he can always keep the milk for himself and get a surplus of I_c. In this case, the result of bargaining maximizes the following:

$$(\pi_c - I_c)(I_y + 1 - \pi_c)$$

   where the yoghurt producer's threat point is 0 and his utility following the bargain is $I_y + 1$ minus whatever he pays to the cream producer as part of the bargain (π_c). Maximizing the above yields $\pi_c = \frac{I_c+I_y+1}{2}$. Since $I_c, I_y < 1$, getting π_c is always better than keeping the milk to make cream, and the milk will always be sold to the yoghurt producer.

Therefore, the payoffs that result are as such

Cow production	Cow produces 2 gallons		Cow produces 1 gallon	
Preference shock	$\varepsilon_c = 1$	$\varepsilon_y = 1$	$\varepsilon_c = 1$	$\varepsilon_y = 1$
Probability	$\frac{1}{2}(1-\mu)$		$\frac{\mu}{2}$	
Bargained surplus for cream producer	$I_c + 1 + \frac{I_y}{2}$	$I_c + \frac{I_y+1}{2}$	$I_c + 1$	$\frac{I_c+I_y+1}{2}$
Bargained surplus for yoghurt producer	$\frac{I_y}{2}$	$\frac{I_y+1}{2}$	0	$\frac{I_y+1-I_c}{2}$

Knowing this, the cream producer maximizes

$$\frac{1}{2}(1-\mu)\left(2I_c + I_y + \frac{3}{2}\right) + \frac{\mu}{2}\left(\frac{3I_c + I_y + 3}{2}\right) - I_c^2$$

yielding the best response $I_c = \frac{4-\mu}{8}$. This is an over-investment relative to the socially efficient case, albeit netting the cream producer higher profits.

Likewise, the yoghurt producer maximizes

$$\frac{1}{2}(1-\mu)\left(I_y + \frac{1}{2}\right) + \frac{\mu}{2} \cdot \frac{I_y + 1 - I_c}{2} - 2I_y^2$$

and this yields the best response $I_y = \frac{2-\mu}{16}$. This is an under-investment relative to the socially efficient case.

(c)

The reasoning is the same as before, so we skip to the resulting payoffs:

Cow production	Cow produces 2 gallons		Cow produces 1 gallon	
Preference shock	$\varepsilon_c = 1$	$\varepsilon_y = 1$	$\varepsilon_c = 1$	$\varepsilon_y = 1$
Probability	$\frac{1}{2}(1-\mu)$		$\frac{\mu}{2}$	
Bargained surplus for cream producer	$\frac{I_c+1}{2}$	$\frac{I_c}{2}$	$\frac{I_c+1-I_y}{2}$	0
Bargained surplus for yoghurt producer	$I_y + \frac{I_c+1}{2}$	$I_y + 1 + \frac{I_c}{2}$	$\frac{I_c+I_y+1}{2}$	$I_y + 1$

Knowing this, the cream producer maximizes

$$\frac{1}{2}(1-\mu)\left(I_c + \frac{1}{2}\right) + \frac{\mu}{2} \cdot \frac{I_c + 1 - I_y}{2} - I_c^2$$

yielding the best response $I_c = \frac{2-\mu}{8}$. This is an under-investment relative to the socially efficient case.

Likewise, the yoghurt producer maximizes

$$\frac{1}{2}(1-\mu)\left(2I_y + I_c + \frac{3}{2}\right) + \frac{\mu}{2} \cdot \frac{3I_y + I_c + 3}{2} - 2I_y^2$$

yielding the best response $I_y = \frac{4-\mu}{16}$. This is an over-investment relative to the socially efficient case, albeit netting the yoghurt producer higher profits.

(d)

The cases of over/under-investment were identified in (b) and (c). There is over-investment in cream for (b), for example, because investing improves the cream producer's bargaining power and allows him to extract some surplus even in the case where he normally would not produce, that is, the cow produces 1 gallon and $\varepsilon_c = 0$. This was hinted at when we mentioned that the result of bargaining maximizes $(\pi_c - I_c)(I_y + 1 - \pi_c)$. There is under-investment for the producer that does not own the cow since some of that investment is always being skimmed off by the cow owner.

(e)

The allocation of milk is the same in the socially efficient case and the case of private ownership; it is only the division of surplus and optimal investment levels which differ. This means that whoever owns the cow, the total social surplus is the expected surplus maximized in (a):

$$\begin{aligned} W &= (1-\mu)(I_c + I_y + 1) + \frac{\mu}{2}(I_c + I_y + 2) - I_c^2 - 2I_y^2 \\ &= (I_c + I_y)\left(1 - \frac{\mu}{2}\right) - I_c^2 - 2I_y^2 + 1 \end{aligned}$$

and we have

$$W_c - W_y = (\Delta I_c + \Delta I_y)\left(1 - \frac{\mu}{2}\right) - \Delta\left(I_c^2\right) - 2\Delta\left(I_y^2\right)$$

where

$$\begin{aligned} \Delta I_c &= \frac{4-\mu}{8} - \frac{2-\mu}{8} = \frac{1}{4} \\ \Delta I_y &= \frac{2-\mu}{16} - \frac{4-\mu}{16} = -\frac{1}{8} \\ \Delta\left(I_c^2\right) &= \left(\frac{4-\mu}{8}\right)^2 - \left(\frac{2-\mu}{8}\right)^2 = \frac{3-\mu}{16} \\ \Delta\left(I_y^2\right) &= \left(\frac{2-\mu}{16}\right)^2 - \left(\frac{4-\mu}{16}\right)^2 = \frac{\mu-3}{64} \end{aligned}$$

Therefore,

$$W_c - W_y = \frac{1}{8}\left(1 - \frac{\mu}{2}\right) - \frac{3-\mu}{16} - 2 \cdot \frac{\mu-3}{64} = \frac{1-\mu}{32}$$

The difference is positive, so if the cow has to be privately owned, it is more efficient (in terms of total social welfare) for the cream producer to own the cow. This difference decreases in μ and disappears as $\mu \to 1$.

Over-investment occurs because the cow owner can improve his bargaining position by investing more. However, when the cow produces 2 gallons, this advantage is irrelevant since the cow owner will always produce, and investing any more does not give him further leverage over the remaining gallon of milk to be sold to the other producer. In this case, there is only under-investment as the cow owner skims off the surplus of the other producer and makes the latter's investment less worthwhile.

Splitting half the surplus means that the one without the cow finds that investment is half as profitable as before, and with a quadratic cost function this means that optimal investment is cut in half. It is thus better to let the under-investment happen to the producer which has a lower socially optimal level of investment, so that the absolute (as opposed to relative) decrease in investment is smaller.

When the cow produces only 1 gallon, this effect is tempered by the incentive of the cow owner to over-invest. In this case the producer with higher investment costs finds it more costly to over-invest just for bargaining power, and will exceed the socially optimal level by less. This makes it less costly to give the cow to the producer with lower investment. As $\mu \to 1$, the two effects cancel out and both options are equally inefficient.