Real Business Cycles
Supervision 1

Samuel Lee

Question 1

(a)

Assuming households own capital, their budget constraint is

$$c_t + k_{t+1} - (1-\delta)k_t = r_t^K k_t + w_t$$

which says that labour and capital income are exhausted by consumption and net capital formation. Households maximise the expected discounted value of lifetime consumption subject to the budget constraint above. Since this is a linear optimisation problem with linear constraints, the following must hold for an equilibrium to be valid:

$$\beta \mathbb{E}_t[1 + r_{t+1}^K - \delta] = 1$$

which is a 'no-arbitrage' condition: we must not be expected to benefit from reducing consumption today, and using the capital income from the additional savings generated to increase consumption tomorrow.

(b)

Every period, the firm maximises profits

$$\pi_t = Y_t - r_t^K K_t - w_t L_t = Z_t K_t^\alpha L_t^{1-\alpha} - r_t^K K_t - w_t L_t$$

over K_t and L_t. The first-order conditions are

$$\left.\begin{aligned} \frac{\partial \pi_t}{\partial K_t} &= \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - r_t^K = 0 \\ \frac{\partial \pi_t}{\partial L_t} &= (1-\alpha) Z_t K_t^\alpha L_t^{-\alpha} - w_t = 0 \end{aligned}\right\} \implies \frac{r_t^K}{w_t} = \frac{\alpha}{1-\alpha}\frac{L_t}{K_t}$$

This simply means that factors of production are paid their marginal products, and the marginal rate of transformation is made equal to the factor price ratio.

(c)

Since labour is inelastically supplied, the market clearing conditions are

$$Y_t = C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{I_t} \tag{1}$$

$$L_t = 1 \tag{2}$$

where (1) and (2) are the goods and labour market clearing conditions. The other equilibrium conditions have already been derived: the equilibrium equations for the firm are

$$r_t^K = \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{3}$$

$$w_t = (1-\alpha) Z_t K_t^{\alpha} L_t^{-\alpha} \tag{4}$$

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha} \tag{5}$$

and the equilibrium conditions for the household are

$$\beta \mathbb{E}_t [1 + r_{t+1}^K - \delta] = 1 \tag{6}$$

$$c_t + k_{t+1} - (1-\delta)k_t = w_t + r_t^K k_t \tag{7}$$

The total population is normalised to one, so $c_t = C_t$ and $k_t = K_t$. We have 6 endogenous variables: C_t, K_t, L_t, Y_t, r_t^K, and w_t. By Walras's law, one equation can be suppressed. For one, the consumer budget constraint (7) is equal to the aggregate resource constraint (1) when we substitute in (3) and (4), since factors of production are paid their marginal products and the production function is homogeneous of degree one (Euler's homogeneous function theorem).

(d)

In a steady state, we drop the time subscripts, and assume that stochastic variables are at their unconditional mean. The only value of $\bar{Z}$ that is compatible with a steady state is 1, since

$$Z_t = (Z_{t-1})^{\mu} \times \Psi_t \implies \bar{Z} = \bar{Z}^{\mu}$$

when $\Psi_t = \mathbb{E}[\Psi_t] = 1$. Therefore, the rental rate of capital implied by (3) is

$$r^K = \alpha \bar{K}^{\alpha-1}$$

Combining this with the condition in (6), we get

$$\beta(1 + \alpha \bar{K}^{\alpha-1} - \delta) = 1 \implies \bar{K} = \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}$$

which makes sense: the steady state equilibrium capital stock is increasing in the output elasticity of capital and the discount factor, and decreasing in the rate of depreciation. The steady state capital stock is zero when households don't care about future periods ($\beta = 0$). Since labour is inelastically supplied, the steady state aggregate output given $\bar{K}$ and $\bar{Z}$ is

$$\bar{Y} = \bar{Z}\bar{K}^{\alpha} = \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}}$$

And since we know $\bar{K}$ and $\bar{Y}$, we can determine $\bar{C}$ from the goods market clearing condition (1):

$$\bar{C} = \bar{Y} - \delta\bar{K} = \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}$$

(e)

Combining the Euler equation (6) with (3),

$$\begin{aligned}\beta(1+\mathbb{E}_t[\alpha Z_{t+1}K_{t+1}^{\alpha-1}]-\delta) = 1 \implies K_{t+1} &= \frac{\alpha\beta}{1-\beta(1-\delta)}\mathbb{E}_t[Z_{t+1}] \\ &= \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\mathbb{E}_t[Z_t^{\mu}\Psi_{t+1}]\right]^{\frac{1}{1-\alpha}} \\ &= \left[\frac{\alpha\beta}{1-\beta(1-\delta)}Z_t^{\mu}\right]^{\frac{1}{1-\alpha}}\end{aligned}$$

Using this formula in the production function gives us

$$Y_t = Z_t\left[\frac{\alpha\beta}{1-\beta(1-\delta)}Z_{t-1}^{\mu}\right]^{\frac{\alpha}{1-\alpha}}$$

(f)

Taking logs, z_t is a first-order autoregressive process:

$$z_t = \log Z_t = \mu\log Z_{t-1} + \log\Psi_t = \mu z_{t-1} + \psi_t$$

The mean of ψ_t is not zero even though $\mathbb{E}[\Psi_t] = 1$; the logarithm is a concave function, and $\mathbb{E}[\psi_t] = \mathbb{E}[\log\Psi_t] \leq \log\mathbb{E}[\Psi_t] = 0$ by Jensen's inequality. But we can express z_t in the form of a zero-mean autoregressive process:

$$\begin{aligned}&z_t = \mathbb{E}[\psi_t] + \mu z_{t-1} + \psi_t - \mathbb{E}[\psi_t] \\ \implies &\underbrace{z_t - \frac{\mathbb{E}[\psi_t]}{1-\mu}}_{x_t} = \mu\underbrace{\left(z_{t-1} - \frac{\mathbb{E}[\psi_t]}{1-\mu}\right)}_{x_{t-1}} + \underbrace{\psi_t - \mathbb{E}[\psi_t]}_{\varepsilon_t}\end{aligned}$$

where $\mathbb{E}[\varepsilon]$ is now zero and z_t is stationary around a mean $\frac{\mathbb{E}[\psi_t]}{1-\mu}$. The expressions for k_{t+1} and y_t follow from the previous question:

$$\begin{aligned}k_{t+1} &= \frac{1}{1-\alpha}\ln\left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right] + \frac{\mu}{1-\alpha}z_t \\ y_t &= z_t + \frac{\alpha\mu}{1-\alpha}z_{t-1} + \frac{\alpha}{1-\alpha}\ln\left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]\end{aligned}$$

An unanticipated (positive) technology shock ψ_t raises y_{t+1}, given that $\frac{\alpha\mu}{1-\alpha}z_t$ features in the expression for y_{t+1}. Intuitively, the positive technology shock raises tomorrow's expected demand of capital by firms and hence puts upward pressure on tomorrow's expected rental rate of capital. This happens because of the persistence of shocks through the autoregressive structure; if Z_t were i.i.d. white noise with mean 1, the result would not follow. This upward pressure on tomorrow's rental rate of capital must be offset by increased savings today, otherwise the household Euler equation doesn't hold and one can raise the expected discounted value of lifetime consumption by shifting consumption to tomorrow. Since labour is inelastically supplied, tomorrow's expected output only varies with tomorrow's capital stock, and the increased savings raises y_{t+1}.

(g)

We can express y_t as the following

$$y_t = x_t + \frac{\alpha\mu}{1-\alpha} x_{t-1} + \text{constant}$$

where x_t is z_t demeaned. From our expression of x_t before, we get

$$x_t = \mu x_{t-1} + \varepsilon_t = \mu(\mu x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \ldots = \sum_{\tau=0}^{t} \mu^\tau \varepsilon_{t-\tau}$$

and so we can express y_t in terms of the demeaned shocks ε_t:

$$y_t = \sum_{\tau=0}^{t} \mu^\tau \varepsilon_{t-\tau} + \frac{\alpha\mu}{1-\alpha} \sum_{\tau=0}^{t-1} \mu^\tau \varepsilon_{t-1-\tau} + \text{constant}$$

We can trace the dynamic response of output y_t to a one-period technology shock (which is equivalent to a shock to ε_t since ε_t is just ψ_t minus its expected value). Using the impulse response function. The impulse response function traces the difference between the expected value of y_{t+h} given a shock $\varepsilon_t = \delta$ today (with future shocks set to zero) and the expected value of y_{t+h} given no shock today or in the future. Formally,

$$\begin{aligned}
I(h,\delta) &= \mathbb{E}_t[y_{t+h} \mid \varepsilon_t = \delta, \varepsilon_{t+1} = \varepsilon_{t+2} = \ldots = 0] - \mathbb{E}_t[y_{t+h} \mid \varepsilon_t = \varepsilon_{t+1} = \varepsilon_{t+2} = \ldots = 0] \\
&= \mu^h \times \delta + \frac{\alpha\mu}{1-\alpha} \mu^{h-1} \times \delta \\
&= \frac{1}{1-\alpha} \mu^h \times \delta
\end{aligned}$$

Therefore, a shock of size δ decays exponentially through time, and the rate of decay is faster the less persistent z_t is, that is, the lower μ is.

Question 2

The statement is obviously false if taken literally. If we limit our discussion to the family of real business cycle models like the one derived above, there are some stylised facts not explained by the qualitative results of such models. For instance, without introducing additional frictions, there is never any involuntary unemployment in these models. Fluctuations in labour supply are the result of optimising behaviour by households (though in the case above labour supply is inelastic). Some qualitative results are roughly coherent with empirical realisations: consumption and investment are both pro-cyclical in the baseline models. However, investment is not more volatile than output in some formulations of the model. In the model derived above, the coefficient on z_t in the expression for k_{t+1} is $\frac{\mu}{1-\alpha}$, while that in the expression for y_t is 1. This means investment is predicted to be more volatile than output only if $\mu > 1 - \alpha$. Likewise, in models which assume full depreciation and no labour-leisure trade-off for tractability, investment is predicted to be as volatile as consumption, when it should generally be more volatile.

More realistic assumptions and frictions could be introduced to these types of models possibly at the cost of mathematical tractability. We can still get at the dynamics of more complex models through simulation, and they may turn out to be closer in pattern to what is observed empirically, but a general criticism remains that much of the patterns and persistence exhibited in the models hinge on exogenous parameters like μ in the model derived before. Also, in many of these models the business cycle arises due to productivity shocks. Some might argue that the 2008 recession can be rationalised as a ‘productivity’ shock to the financial capabilities of the economy, but there surely comes a point where we cannot explain every downturn or upswing in terms of productivity shocks.

In general, it is probably safe to say that the origins of business cycles are still in debate. Justiniano, Primiceri, and Tambalotti (2010) note that attempts to answer this from a general equilibrium framework tend to point to ‘neutral technology shocks’ at the originator of economic fluctuations, whereas the structural VAR literature tends to emphasise other factors. A search on Google Scholar for ‘business cycles’ turns up, within the first 20 results, articles which point to the following variables as a driving force behind business cycles: microeconomic uncertainty shocks, ambiguity (‘Knightian’ uncertainty), investment shocks, anticipated shocks, disaster risk, news shocks about future technology, financial shocks, and household debt to GDP ratios. The articles are sorted by number of citations, so these are not fringe results lying on the periphery of a common consensus.