Time Series Models
Supervision 3

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Question 1

(a)

The OLS estimator is still unbiased, but is now inefficient and the standard errors are potentially biased and inconsistent.

(b)

We could use heteroscedasticity-robust standard errors to construct the confidence intervals in the usual way. Another possibility is to use bootstrapped confidence intervals using the empirical distribution of the OLS estimates from repeated resampling of the original sample.

(c)

If the model for conditional heteroscedasticity is true, we can estimate the following model

$$\frac{y_i}{\exp(x_i'\gamma/2)} = \frac{x_i'\beta}{\exp(x_i'\gamma/2)} + \underbrace{\frac{u_i}{\exp(x_i'\gamma/2)}}_{\eta_i}$$

which is a linear model with homoscedastic errors since $\mathrm{Var}[\eta_i|x_i] = 1$. However, it is more likely that γ is unknown and will have to be estimated. To do so, we can first get the squared residuals from the original OLS estimates, and run a regression of $\ln(\hat{u}_i{}^2)$ on x_i to get $\hat{\gamma}$. Substituting $\exp(x_i'\hat{\gamma}/2)$ for $\exp(x_i'\gamma/2)$ gives us consistent estimates for β which are asymptotically more efficient than OLS.

Question 2

(a)

The first regression is

$$x_i = \gamma w_i + \eta_i$$

so the fitted value from this regression will just be a re-scaling of w_i by some $\hat{\gamma}$.

Running a regression of y_i on a re-scaled w_i is just the same as running a regression of y_i on w_i and inversely scaling the estimated coefficient on w_i, and we can see that the coefficient here has little to do with β. We can also use the following algebra to show this:

The estimated coefficient from the first stage is

$$\hat{\gamma} = \min_{\gamma} \sum_{i=1}^{n} (x_i - \gamma w_i)^2$$

which we can solve for by the first-order condition

$$\sum_{i=1}^{n} w_i(x_i - \hat{\gamma} w_i) = 0 \implies \hat{\gamma} = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i^2}$$

If we follow the suggested procedure, the estimated β is

$$\begin{aligned}
\tilde{\beta} &= \frac{\sum_{i=1}^{n} y_i \left(\hat{x}_i - \bar{\hat{x}}\right)}{\sum_{i=1}^{n} \hat{x}_i \left(\hat{x}_i - \bar{\hat{x}}\right)} \\
&= \frac{\sum_{i=1}^{n} y_i \left(\hat{\gamma} w_i - \hat{\gamma}\bar{w}\right)}{\sum_{i=1}^{n} \hat{\gamma} w_i (\hat{\gamma} w_i - \hat{\gamma}\bar{w})} \\
&= \frac{1}{\hat{\gamma}} \frac{\sum_{i=1}^{n} y_i (w_i - \bar{w})}{\sum_{i=1}^{n} w_i (w_i - \bar{w})}
\end{aligned}$$

If $y_i = \alpha + \beta x_i + u_i$ is the true model albeit with $\mathrm{E}[x_i u_i] \neq 0$,

$$\begin{aligned}
\tilde{\beta} &= \frac{1}{\hat{\gamma}} \left[\frac{\sum_{i=1}^{n} \beta x_i (w_i - \bar{w})}{\sum_{i=1}^{n} w_i (w_i - \bar{w})} + \frac{\sum_{i=1}^{n} u_i (w_i - \bar{w})}{\sum_{i=1}^{n} w_i (w_i - \bar{w})} \right] \\
&= \frac{\hat{\rho}}{\hat{\gamma}} \beta + \frac{1}{\hat{\gamma}} \frac{\sum_{i=1}^{n} u_i (w_i - \bar{w})}{\sum_{i=1}^{n} w_i (w_i - \bar{w})}
\end{aligned}$$

where $\hat{\rho}$ is the estimated coefficient from the correct first-stage regression. Taking probability limits through fractions using the continuous mapping theorem, and applying the law of large numbers, the second term tends to 0 since $\mathrm{E}[w_i u_i] = 0$, and the first term tends to

$$\beta \cdot \frac{\mathrm{Cov}[x_i, w_i]}{\mathrm{Var}[w_i]} \cdot \frac{\mathrm{E}[w_i^2]}{\mathrm{E}[x_i w_i]}$$

which means

$$\lim_{n \to \infty} \tilde{\beta} = \beta \cdot \frac{\mathrm{E}[x_i w_i] - \mathrm{E}[x_i]\,\mathrm{E}[w_i]}{\mathrm{E}[w_i^2] - \mathrm{E}[w_i]^2} \cdot \frac{\mathrm{E}[w_i^2]}{\mathrm{E}[x_i w_i]}$$

which is not, in general, equal to β. One thing the representation above makes obvious is that the suggested estimator is consistent if $\mathrm{E}[w_i] = 0$.

(b)

The F-statistic for the first stage regression is

$$F = \frac{SST - SSR}{SSR/(100-2)} = \frac{1 - \frac{SSR}{SST}}{\frac{SSR}{SST}/98} = \frac{R^2}{(1-R^2)/98} = \frac{0.05}{(1-0.05)/98} \approx 5.1579$$

Using the rule of thumb that the F-statistic from the first stage regression should be more than 10, we might do well to consider that the instrument might be weak.

Question 3

(a)

Doing what we're told, we get

$$y_t = \beta_0 + \beta_1(x_t - e_t) + u_t = \beta_0 + \beta_1 x_t + \underbrace{u_t - \beta_1 e_t}_{v_t}$$

and it is apparent that $\mathrm{Cov}[x_t, v_t] = \mathrm{Cov}[x_t^* + e_t, u_t - \beta_1 e_t] = -\beta_1\sigma_e^2$. The OLS estimator of β_1 is therefore biased.

(b)

$\mathrm{E}[x_{t-1}v_t] = \mathrm{E}[x_{t-1}^* v_t + e_{t-1}v_t] = \mathrm{E}[x_{t-1}^* u_t - \beta_1 x_{t-1}^* e_t + e_{t-1}u_t + -\beta_1 e_{t-1}e_t] = 0$ since uncorrelatedness implies the expectation of a product equals the product of the expectations.

(c)

Considering the expression for $\mathrm{E}[x_t x_{t-1}]$:

$$\mathrm{E}[x_t x_{t-1}] = \mathrm{E}[(x_t^* + e_t)(x_{t-1}^* + e_{t-1})] = \mathrm{E}[x_t^* x_{t-1}^*] + \mathrm{E}[x_t^* e_{t-1}] + \mathrm{E}[x_{t-1}^* e_t] + \mathrm{E}[e_t e_{t-1}]$$

This means x_t and x_{t-1} are uncorrelated if the above is equal to $\mathrm{E}[x_t]\,\mathrm{E}[x_{t-1}] = 0$. Whether this is plausible probably depends on what x_t actually is. We might expect some sorts of errors to persist over time, such that $\mathrm{E}[e_t e_{t-1}] \neq 0$.

(d)

If x_t and x_{t-1} are correlated and $\mathrm{E}[e_t e_{t-1}] = 0$, x_{t-1} can instrument for x_t.

Question 4

No exogenous variables are excluded from the second equation, so the model is not identified.

Question 5

It is probably safe to say the second equation is the supply equation and the first is the demand equation. To get anywhere we should probably assume that rainfall and income are exogenous, but there are reasons to be wary at the county level. Corn supply could be endogenous if farmers locate in counties with favourable climates, and demand could be endogenous in counties where agriculture forms a large proportion of income.

Glossing over this, to estimate the demand equation, we first run a regression of *corn* on *rainfall* and $\textit{rainfall}^2$, and then run a regression of *corn* on the predicted values of the first regression and *income*. This yields the two-stage least squares estimates, and can be done for the supply equation as well by an analogous procedure.

To get more efficient estimates, we could estimate the covariance structure of u_1 and u_2 using the residuals from the two-stage least square regression, and rerun the second stage by feasible generalised least squares to yield the three-stage least square estimates.

The estimates can be obtained using the formulas given in the lecture notes, provided one has the will to write out the full matrix form with Kronecker products and all.