Game Theory
Supervision 5

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Question 1

In a double auction, both parties simultaneously announce a price. If the buyer's price p_b is weakly greater than the seller's price p_s, they split the difference and trade takes place at price $\frac{p_b+p_s}{2}$. In a Bayes–Nash equilibrium, the buyer's strategy given v_b is $p_b(v_b)$, which must solve

$$\max_p \left[v_b - \frac{p + \mathrm{E}[p_s|p_s \le p]}{2}\right] \Pr[p_s \le p]$$

Likewise, $p_s(v_s)$ must solve

$$\max_p \left[\frac{p + \mathrm{E}[p_b|p_b \ge p]}{2} - v_s\right] \Pr[p_b \ge p]$$

If both parties only choose from the set of affine strategies, we have $p_b \sim U[\alpha_b, \alpha_b + \beta_b]$ for some α_b and β_b, and $p_s \sim U[\alpha_s, \alpha_s + \beta_s]$ for some α_s and β_s. Then, $p_b(v_b)$ maximises

$$\left(v_b - \frac{p + \frac{\alpha_s+p}{2}}{2}\right)\frac{p-\alpha_s}{\beta_s} = \left(\frac{4v_b - \alpha_s - 3p}{4}\right)\frac{p-\alpha_s}{\beta_s} = \frac{(4v_b - \alpha_s - 3p)(p-\alpha_s)}{4\beta_s}$$

The first-order condition is

$$\frac{4v_b - \alpha_s - 3p}{4\beta_s} - \frac{3(p-\alpha_s)}{4\beta_s} = 0 \implies p_b(v_b) = \frac{\alpha_s}{3} + \frac{2}{3}v_b$$

Going through the same steps, we get

$$p_s(v_s) = \frac{\alpha_b + \beta_b}{3} + \frac{2}{3}v_s = \frac{\alpha_b}{3} + \frac{2}{9} + \frac{2}{3}v_s$$

This means $\frac{\alpha_b}{3} + \frac{2}{9} = 3\alpha_b \implies \alpha_b = \frac{1}{12}$, and $\alpha_s = \frac{1}{36} + \frac{2}{9} = \frac{1}{4}$. Again, this means the equilibrium affine strategies are

$$p_b(v_b) = \frac{1}{12} + \frac{2}{3}v_b$$
$$p_s(v_s) = \frac{1}{4} + \frac{2}{3}v_s$$

We have that $p_b \in \left[\frac{1}{12}, \frac{9}{12}\right]$, and $p_s \in \left[\frac{3}{12}, \frac{11}{12}\right]$. We can see that for $v_s \in \left(\frac{9}{12}, 1\right]$, the seller offers $p_s \in \left(\frac{9}{12}, \frac{11}{12}\right]$, where $p_s < v_s$. In this case the seller makes a loss if trade happens. If it happens that $v_s > \frac{9}{12}$, a weakly dominant strategy for the seller is to report the true value: either trade doesn't happen, or trade happens and the seller benefits. However, the beneficial trade can only happen if the buyer doesn't stick to the equilibrium strategy, since $p_b(v_b) \leq \frac{9}{12}$. A similar argument holds for the buyer if $v_b < \frac{3}{12}$. A reason this pair of strategies can still be an equilibrium is that, as mentioned, deviation brings no benefits since trade doesn't happen unless the other party deviates from the equilibrium strategy as well.

A strategy where no type uses a dominated strategy requires the buyer/seller to always submit a weakly smaller/greater offer than their valuation; otherwise, a weakly dominant strategy would be to report one's true value as above. We can slightly tweak the affine strategy above to get such a strategy, without having changed anything substantial about the possible outcomes of the auction. First, we know that any price above $\frac{9}{12}$ will never be accepted by the buyer. This also happens to be the point above which the seller would have to offer a price lower than their own valuation. We can make it such that the seller follows the equilibrium strategy above, except that they report their true value if $v_s > \frac{9}{12}$. The new strategy is still continuous in v_s since $p_s(v_s) = \frac{9}{12}$. Furthermore, the seller's strategy is still a best response to the buyer's strategy since any offered price above $\frac{9}{12}$ is irrelevant for outcomes. A similar argument holds for the buyer, and we let the buyer declare their true value for $v_b < \frac{3}{12}$. Therefore, we have the required equilibrium strategies:

$$p_b(v_b) = \begin{cases} v_b & \text{if } 0 \leq v_b < \frac{3}{12} \\ \frac{1}{12} + \frac{2}{3}v_b & \text{if } \frac{3}{12} \leq v_b \leq 1 \end{cases}$$

$$p_s(v_s) = \begin{cases} \frac{1}{4} + \frac{2}{3}v_s & \text{if } 0 \leq v_s \leq \frac{9}{12} \\ v_s & \text{if } \frac{9}{12} < v_s \leq 1 \end{cases}$$

Question 2

We denote by $U_s(t)$ the equilibrium expected utility of a type t seller with 'production cost' $v_s = 1-t$. We also denote by $U_s(\hat{t} \mid t_s = t)$ the expected utility of a type t seller when they choose instead to play the equilibrium strategy of a type $\hat{t}$ seller, given the buyer is playing their equilibrium strategy. This can be expressed as

$$U_s(\hat{t} \mid t_s = t) = p(\hat{t}) - (1-t)q(\hat{t})$$

where $p(\hat{t})$ is the expected payment received when announcing type $\hat{t}$, and $q(\hat{t})$ is the probability of trade when announcing type $\hat{t}$, given the equilibrium buyer strategy.

For the strategies to be an equilibrium, we must have $U_s(\hat{t} \mid t_s = t) \leq U_s(t)$, with equality at $\hat{t} = t$. This means that $U_s(t)$ represents an upper envelope curve on all the curves of $U_s(\hat{t} \mid t_s = t)$ give different values of t, and we have

$$U_s'(t) = \frac{\partial}{\partial t} U_s(t \mid t_s = t) = q(t)$$

by the envelope theorem. This also means $U_s(t)$ can be re-expressed as

$$U_s(t) = U_s(0) + \int_0^t q(\hat{t}) \mathrm{d}\hat{t}$$

With ex-post efficiency, trade always takes place whenever $v_s = 1 - t \leq v_b$. So we have

$$q(t) = \Pr[v_b \geq 1 - t] = t$$

which also means the lower the 'production cost' is, the higher the probability of trade. Therefore we have

$$U_s(t) = U_s(0) + \frac{t^2}{2} = U_s(0) + \frac{(1 - v_s)^2}{2}$$

as needed: the equilibrium expected payoff is equal to $0.5(1 - v_s)^2$ plus the equilibrium expected payoff of the type with the highest cost (lowest t).

The set of payoffs above is equivalent to that under a Groves mechanism. We first show that the Groves mechanism cannot be ex-ante budget-balanced and incentive-compatible at the same time. Suppose we have a double auction where the seller pays the buyer's declared value and vice versa. Then truth-telling is weakly dominant for both players and can be played by both in an efficient Bayes-Nash equilibrium. In such an equilibrium, both players get $v_b - v_s$ if trade takes place, so their expected payoffs are both equal to the expected surplus, which is

$$\int_0^1 \int_{v_s}^1 (v_b - v_s) \mathrm{d}v_b \mathrm{d}v_s = \int_0^1 \frac{1}{2} - v_s + \frac{v_s^2}{2} \mathrm{d}v_s = \left[\frac{v_s}{2} - \frac{v_s^2}{2} + \frac{v_s^3}{6}\right]_0^1 = \frac{1}{6}$$

This is a problem: the expected total payoff is twice the expected market surplus, so the mechanism is not budget-balanced on average. Addressing this by including a entry surcharge introduces the other problem of individual rationality: for one, a buyer whose private value is lower than the surcharge will never agree to participate.

To show that the payoffs derived above for any efficient equilibrium are equivalent to that under a Groves mechanism, the seller with value v_s has expected utility:

$$\int_{v_s}^1 (v_b - v_s) \mathrm{d}v_b = \frac{1}{2} - v_s + \frac{v_s^2}{2} = \frac{(1 - v_s)^2}{2}$$

which means the equilibrium expected payoff to the seller for any ex-post efficient equilibrium can be made equal to a Groves mechanism by including an entry surcharge equal to $-U_s(0)$. The same argument applies for the buyer and is covered in the lectures. Therefore, since any Groves mechanism cannot be simultaneously budget-balanced and incentive-compatible, neither can any ex-post efficient equilibrium.

Question 3

(a)

This equilibrium would involve the seller naming 3.5 in the first period and 3 in the first period. A good guess might be that in equilibrium, the buyer accepts the first price if $v = 4$ and the second price if $v = 3$. For the first action to be optimal, we must have

$$4 - 3.5 \geq \delta(4 - 3) \implies \delta \leq \frac{1}{2}$$

Assuming the buyer's strategy is to accept the offer whenever the payoff is greater than or equal to 0, the two prices are optimal for the seller if

$$0.8 \times 3.5 + 0.2 \times \delta \times 3 \geq 3 \implies \delta \geq \frac{1}{3}$$

since the seller can guarantee a payoff of 3 by offering 3 in both periods. Offering 3 in the first period only also yields a payoff of 3, and offering 3.5 in both periods yields an expected payoff of

$$0.8 \times 3.5 = 2.8$$

which is never optimal. Therefore the set of discount factors for which this equilibrium exists is $\frac{1}{3} \leq \delta \leq \frac{1}{2}$.

(b)

Setting the price to 3.5 in both periods is never optimal as mentioned before, so the equilibrium must involve setting the price to 3 for both periods. In equilibrium the buyer will never accept the second offer if it is identical to the first offer. If $\delta \leq \frac{1}{3}$, setting the price equal to 3 in both periods is better than setting it to 3.5 and 3 in the first and second periods. If the seller sets the price to 3 in the first period and 3.5 in the second period, the buyer will always accept the first offer, and the outcome is the same. We only need $\delta \leq \frac{1}{3}$ for this equilibrium to exist.