Game Theory
Supervision 1

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Question 1

Checking each pair of strategies will show that this game has one pure equilibrium: (D, R). We can check that there's no mixed strategy equilibrium. For easier notation, we refer to Player 2's strategy M as C, so we have the game

	L	C	R
U	$1, -2$	$-2, 1$	$0, 0$
M	$-2, 1$	$1, -2$	$0, 0$
D	$0, 0$	$0, 0$	$1, 1$

Denoting $\Pr(X)$ as p_X, the following are the payoffs to each player:

$$\begin{aligned}
U_1 &= p_U(p_L - 2p_C) + p_M(p_C - 2p_L) + (1 - p_U - p_M)(1 - p_L - p_C) \\
&= p_U(2p_L - p_C - 1) + p_M(2p_C - p_L - 1) + 1 - p_L - p_C \\
U_2 &= p_L(p_M - 2p_U) + p_C(p_U - 2p_M) + (1 - p_U - p_M)(1 - p_L - p_C) \\
&= p_L(2p_M - p_U - 1) + p_C(2p_U - p_M - 1) + 1 - p_U - p_M
\end{aligned}$$

Looking at U_1, we can see that the terms in parentheses cannot both be positive, since

$$2p_L - p_C - 1 + (2p_C - p_L - 1) = p_L + p_C - 2 < 0$$

Therefore at least one of $2p_L - p_C - 1$ and $2p_C - p_L - 1$ are less than 0. Assuming it's the former,

$$2p_L - p_C - 1 < 0 \underset{(A)}{\implies} p_C > 2p_L - 1 \underset{(B)}{\implies} 2p_C - p_L - 1 < -3(1 - p_L) \leq 0$$

where (A) is implied by re-arranging and (B) is implied by substituting p_C with something smaller.

Therefore p_U and p_M both contribute nothing or negatively to Player 1's utility, and should both be set to zero. We get the same result if we assume $2p_C - p_L - 1$ is negative. This means $p_D = 1 - p_U - p_M = 1$, making Player 2's utility equal to

$$U_2 = -p_L - p_C + 1$$

and it's obvious that this is maximised by setting $p_L = p_C = 0$.

An intuitive explanation is that either player can guarantee themselves a payoff of at least 0 by playing D or R. Any pair of strategies with both players not playing D or R involves one loser with -2 utility who would do better by securing the guaranteed minimum, following which it is also optimal for the other player to switch to D/R.

Question 2

(a)

A strategic game consists of

1. A set of players, in this case a set of firms $\mathcal{F} = \{1, \ldots, 19\}$
2. For each player, a set of strategies, in this case each firm's set of decisions

$$\mathcal{S}_i = \{(e, I) | \; e \in \{\text{enter}, \text{abstain}\}, \; I \in [0, 1]\}, i \in \mathcal{F}$$

3. A payoff function defined on all possible strategy profiles, in this case the firm's profits given what it does and what every other firm does: $\pi_i : \mathcal{S}_i \times \mathcal{S}_{-i} \to (-c, \max\{0, 0.25 - c\}]$

where $0.25 - c$ is an investor's best-case scenario; assuming no other firm invests, the net dividend given the investor's input I is $I(1 - I) - c$, which is maximised at $I = 0.5$. The payoff function is bounded below by $-c$.

(b)

When $c = 0$ a symmetric (pure) Nash equilibrium should involve everyone investing; there is no equilibrium in which no firm invests. In such an equilibrium, each firm will maximise

$$\pi_i = I_i(1 - I_{-i} - I_i)$$

over $I_i \in [0, 1]$. The solution to this is $I_i = \frac{1 - I_{-i}}{2}$, and assuming symmetry gives us $I_{-i} = 18I_i$. Therefore,

$$I_i = \frac{1 - 18I_i}{2} \implies I_i = \frac{1}{20}, \; i \in \mathcal{F}$$

which is a symmetric equilibrium: everyone is maximising $I_i \left(1 - 18 \times \frac{1}{20} - I_i\right)$ and the equilibrium payoff is $\frac{1}{20} \times \frac{1}{20} = \frac{1}{400} > 0$.

(c)

In a symmetric equilibrium where $c = 0.01$ and k firms abstain, no firm should profit. Because the investment decisions and payoffs are continuous, any abstaining firm can do better whenever profits are positive for the investing firms. The investing firms maximise

$$\pi_i = I_i(1 - I_{-i} - I_i) - c$$

and the solution to this is still $\frac{1 - I_{-i}}{2}$. This time, assuming symmetry gives us $I_{-i} = (18 - k)I_i$. Therefore,

$$I_i = \frac{1 - (18 - k)I_i}{2} \implies I_i = \frac{1}{2 + (18 - k)} = \frac{1}{20 - k}$$

The payoff to each investing firm is

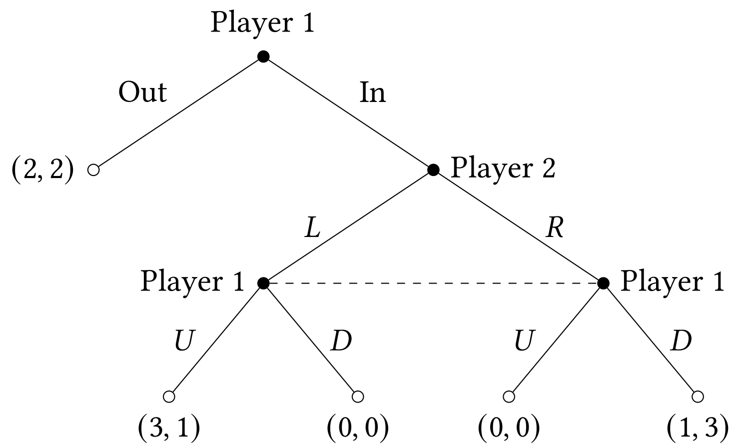
$$\pi_i = \frac{1}{20 - k}\left(1 - \frac{19}{20 - k}\right) = \frac{1 - k}{(20 - k)^2}, \; i \in \{k + 1, \ldots, 19\}$$

and the only possible value of k is 1; as promised, no firm makes positive profits. The solution in (b) where every firm invests is not sustainable here since firms would get $\frac{1}{400} - 0.01 = -\frac{3}{400} < 0$.

Question 3

(a)

We have the extensive form



For the strategic form, we have again

1. Two players 1 and 2
2. A set of strategies for each player
   (i) For Player 1, the strategy set $\mathcal{S}_1 = \{(\text{In}, U), (\text{In}, D), (\text{Out}, U), (\text{Out}, D)\}$
   (ii) For Player 2, the strategy set $\mathcal{S}_2 = \{L, R\}$ since Player 2 has just one information set
3. A payoff function for each player
   (i) For Player 1:

$$\pi_1(s_{1i}, s_{2i}) = \begin{cases} 3 & s_{1i} = (\text{In}, U), s_{2i} = L \\ 0 & s_{1i} = (\text{In}, D), s_{2i} = L \\ 0 & s_{1i} = (\text{In}, U), s_{2i} = R \\ 1 & s_{1i} = (\text{In}, D), s_{2i} = R \\ 2 & \text{otherwise} \end{cases}$$

   (ii) For Player 2:

$$\pi_2(s_{2i}, s_{1i}) = \begin{cases} 1 & s_{1i} = (\text{In}, U), s_{2i} = L \\ 0 & s_{1i} = (\text{In}, D), s_{2i} = L \\ 0 & s_{1i} = (\text{In}, U), s_{2i} = R \\ 3 & s_{1i} = (\text{In}, D), s_{2i} = R \\ 2 & \text{otherwise} \end{cases}$$

(b)

The pure-strategy Nash equilibria are $((\text{In}, U), L)$, $((\text{Out}, U), R)$, $((\text{Out}, D), L)$, and $((\text{Out}, D), R)$. Player 1 has an outside option which guarantees him a payoff of 2, and never gets a worse payoff in equilibrium.

(c)

A strategy is strictly dominated if there exists another strategy which delivers a strictly better outcome for every strategy that could be played by the other player. For Player 1, (In, D) gives him a payoff of at most 1, and is strictly dominated by playing Out. There is no other strategy which can be deleted; for Player 2, switching between L and R will not deliver a strictly better outcome if Player 1's strategy is Out. And if R is not ruled out, there is no strictly dominated strategy for Player 1. So the strategies which survive are

(i) For Player 1: (In, U), (Out, U), and (Out, D)

(ii) For Player 2: L and R

and all are played in at least one of the pure strategy Nash equilibria.

(d)

A strategy is weakly dominated if there exists another strategy which delivers a weakly better outcome for every strategy that could be played by the other player, with at least one scenario where the other strategy delivers a strictly better outcome.

As before, (In, D) is strictly dominated. After that, R is weakly dominated since L gives Player 2 the same utility if Player 1 plays Out and higher utility if Player 1 plays (In, U). With R being ruled out, any strategy where Player 1 plays Out is strictly dominated by (In, U), which yields a utility of 3. The strategy profiles that survive are (In, U) and L, which corresponds to a subgame-perfect Nash equilibrium.

Question 4

This mostly follows from the definition. For every combination of strategies by the other players, a weakly domininant strategy yields weakly higher payoffs compared to any other strategy, with at least one scenario where the payoff is strictly better. If s and s' are weakly dominant, we cannot have $\pi_i(s_i, s_{-i}) > \pi_i(s_i', s_{-i})$ for any s_{-i}, since that contradicts the assertion that s' is weakly dominant. So $\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i})$ for any s_{-i}. An analogous argument implies $\pi_i(s_i', s_{-i}) \geq \pi_i(s_i, s_{-i})$ for any s_{-i}, and we must have $\pi_i(s_i, s_{-i}) = \pi_i(s_i', s_{-i})$ for any s_{-i}.

Question 5

(a)

I'll play M, because I'm loss averse.

(b)

The pure Nash equilbria are (U, LL) and (D, RR). Taking both players' mixed stategies as given, the payoffs are

$$\begin{aligned}
\pi_1 &= p_U(100p_{LL} - 100p_L - 100p_R) + p_D(100p_L - 100p_{LL} + p_M + 100p_R) \\
&= p_U[100p_{LL} - 100p_L - 100(1 - p_{LL} - p_L - p_M)] \\
&\qquad + (1 - p_U)[100p_L - 100p_{LL} + p_M + 100(1 - p_{LL} - p_L - p_M)] \\
&= p_U[200p_{LL} - 200p_L - 200(1 - p_{LL} - p_L - p_M) - p_M] \\
&\qquad + 100p_L - 100p_{LL} + p_M + 100(1 - p_{LL} - p_L - p_M) \\
&= p_U(400p_{LL} + 199p_M - 200) + 100 - 200p_{LL} - 99p_M \\
\pi_2 &= p_{LL}(2p_U - 100p_D) + p_L(p_U - 49p_D) + p_R(2p_D - 100p_U) \\
&= p_{LL}(102p_U - 100) + p_L(50p_U - 49) + (1 - p_{LL} - p_L - p_M)(2 - 102p_U) \\
&= p_{LL}(204p_U - 102) + p_L(152p_U - 51) + p_M(102p_U - 2) + 2 - 102p_U
\end{aligned}$$

Each player faces a linear optimisation/programming problem. We start from the assumption that Player 1 plays a mixed strategy: for Player 1, we can see that a mixed strategy (non-corner solution) is only optimal when $400p_{LL} + 199p_M - 200 = 0$. Any larger, and the optimal strategy is to always play U. Any less, and the optimal strategy is to never play U. Furthermore, p_{LL} must be positive if Player 1 is to play a mixed strategy: if $p_{LL} = 0$, $400p_{LL} + 199p_M - 200$ will be less than 0. And we cannot have $p_{LL} = 1$, since that means $400p_{LL} + 199p_M - 200$ would be more than 0.

Since $0 < p_{LL} < 1$, we must have $204p_U - 102 = \max\{(152p_U - 51), (102p_U - 2), 0\}$. Intuitively this means that Player 2 cannot do any better by re-allocating p_{LL} to p_L, p_M, or p_R, or vice versa. It's clear that $204p_U - 102$ cannot be 0, since that would imply $p_U = 0.5$ and playing M would yield a utility of 49.

We then have two cases left:

(i) $204p_U - 102 = 152p_U - 51 \implies p_U = \frac{51}{52}$

(ii) $204p_U - 102 = 102p_U - 2 \implies p_U = \frac{50}{51}$

In the first case, $p_U = \frac{51}{52}$ implies $152p_U - 51 > 102p_U - 2 > 0$, so it is not ruled out. In the second case, $p_U = \frac{50}{51}$ implies the same relationship, which contradicts the assumption we imposed that $204p_U - 102 = \max\{(152p_U - 51), (102p_U - 2), 0\}$. So we have one mixed equilibrium: $p_U = \frac{51}{52}$, $p_{LL} = 0.5$, $p_L = 0.5$.

We started with the assumption that Player 1 plays a mixed strategy, but assuming otherwise would lead to Player 2 choosing a pure strategy as well, so we can ignore cases where Player 1 plays a pure strategy and Player 2 plays a mixed strategy; a similar argument follows for the converse.

To reiterate, we have the pure Nash equilibria (U, LL), (D, RR), and the mixed equilibrium $p_U = \frac{51}{52}, p_{LL} = 0.5, p_L = 0.5$.

My answer in (a) is not part of the above.

(c)

Assuming we have to check this for pure strategies only,
As Player 1:

(i) U is a best reply if we believe Player 2 will always play LL.

(ii) D is a best reply if we believe Player 2 will always play L.

As Player 2:

(i) LL is a best reply if we believe Player 1 will always play U.

(ii) L is a best reply if we believe Player 1 will play U with probability $\frac{50}{51}$.

(iii) M is a best reply if we believe Player 1 will play U with probability $\frac{48}{49}$.

(iv) R is a best reply if we believe Player 1 will always play D.

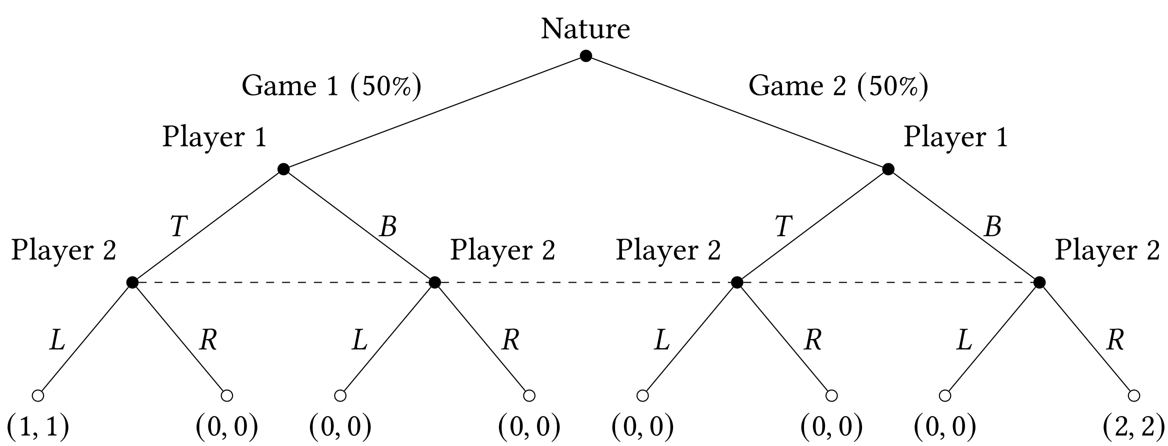
(d)

I would play M because I'm loss averse and don't trust others to do the calculations above.

Question 6

(a)

The game can be represented as a Bayesian game:



(b)

We denote Player 1's strategy by (X, Y), where $X \in \{T, B\}$ is the action at Game 1 and $Y \in \{T, B\}$ is the action at Game 2.

Player 2 has only two pure strategies available, since they only have one information set. If Player 2 plays L, Player 1's optimal strategy is (T, T) or (T, B). But (T, B) is not compatible with Player 2 playing L, since they could get a higher expected payoff by playing R:

$$\text{Player 2's expected payoff with } ((T, B), L) = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$$
$$\text{Player 2's expected payoff with } ((T, B), R) = \frac{1}{2} \times 0 + \frac{1}{2} \times 2 = 1$$

So only $((T, T), L)$ is a Bayes-Nash equilibrium.

If Player 2 plays R, Player 1's optimal strategy is (T, B) or (B, B). Both are compatible with Player 2 playing R in a Bayes-Nash equilibrium. Therefore, the Bayes-Nash equilibria are $((T, T), L)$, $((T, B), R)$, and $((B, B), R)$.

Question 7

A strategy is a best reply for one player if, for a given belief about the other player's strategies, that strategy would maximise the expected payoff if the belief were true. So a best reply s_i maximises

$$\mathrm{E}[\pi_i(s, p)] = \sum_{s_{-i} \in \mathcal{S}_{-i}} p(s_{-i}) \pi_i(s, s_{-i})$$

where p is i's belief about the probability distribution over the other player's strategies. A strictly dominated strategy can never be a best reply, since strict dominance implies that there is some strategy s_i' such that $\pi_i(s_i', s_{-i}) > \pi_i(s_i, s_{-i})$ for any s_{-i}. This means

$$\sum_{s_{-i} \in \mathcal{S}_{-i}} p(s_{-i}) \pi_i(s_i', s_{-i}) > \sum_{s_{-i} \in \mathcal{S}_{-i}} p(s_{-i}) \pi_i(s_i, s_{-i})$$

and s_i is not a best reply.

The second part follows from the first. If strictly dominated strategies are never a best reply, then they will also be eliminated when we carry out the iterated deletion of strategies that are never a best reply. If iterated deletion of strictly dominated strategies yields a unique prediction, this means all other strategies are strictly dominated at some point of the iterated deletion, and will also be deleted when we delete strategies which are never a best reply. Therefore we won't have to deal with cases where a strategy may not be strictly dominated but is never a best reply.