Mathematical Economics
Supervision 2

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Question 1

The Lagrange function for this problem is

$$\mathcal{L}(x, y, \lambda) = x^2 - y^2 - \lambda(x^2 + y^2 - 1)$$

and to assemble the bordered Hessian H,

$$\begin{aligned}
Dg(x, y) &= \begin{bmatrix} 2x & 2y \end{bmatrix} \\
D_{x,y}\mathcal{L} &= \begin{bmatrix} 2 - 2\lambda & 0 \\ 0 & -2 - 2\lambda \end{bmatrix} \\
H &= \begin{bmatrix} 0 & 2x & 2y \\ 2x & 2 - 2\lambda & 0 \\ 2y & 0 & -2 - 2\lambda \end{bmatrix} = 2\begin{bmatrix} 0 & x & y \\ x & 1 - \lambda & 0 \\ y & 0 & -1 - \lambda \end{bmatrix}
\end{aligned}$$

With 2 arguments and 1 constraint, the condition for a local minimizer is that $|H| < 0$.

$$\begin{aligned}
|H| &= 2\begin{vmatrix} 0 & x & y \\ x & 1 - \lambda & 0 \\ y & 0 & -1 - \lambda \end{vmatrix} \\
&= 2\{-x(x)(-1 - \lambda) + y[-y(1 - \lambda)]\} \\
&= 2[x^2(1 + \lambda) + y^2(\lambda - 1)]
\end{aligned}$$

Since $\bar{x} = 0, \bar{y} = \pm 1, \bar{\lambda} = -1$, $|H|$ evaluated at $x = \bar{x}, y = \bar{y}, \lambda = \bar{\lambda}$ is equal to $-4 < 0$. Thus $(0, -1, -1)$ and $(0, 1, -1)$ are local minimizers of $f(x, y) = x^2 - y^2$ on the constraint set $C = \{(x, y) \in \mathbb{R}^2 | g(x, y) = x^2 + y^2 = 1\}$.

Question 2

(a)

Since y^2 is always non-negative, for the constraint to be satisfied, $(x - 1)^3$ must be non-negative, that is, $x \geq 1$. Minimizing $x^2 + y^2$ subject to $(x - 1)^3 - y^2 = 0$ is equivalent to minimizing $f(x) = x^2 + (x - 1)^3$ with only the added constraint that $x \geq 1$. This function is strictly increasing in x since $f'(x) = 2x + 3(x - 1)^2 = 3x^2 - 4x + 3$; the discriminant of $f'(x)$ is $(-4)^2 - 4(3)(3) = -20 < 0$ which implies $f'(x) > 0\ \forall x$. Therefore, the solution for this minimization problem should be where the constraint $x \geq 1$ is binding, and the global minimizer for this problem is $(1, 0)$.

(b)

The method of Lagrange multipliers does not work here because $\nabla g(x, y) = \begin{bmatrix} 3(x-1)^2 \\ -2y \end{bmatrix}$, which is of full rank only when at least one entry is not equal to 0. The solution which was found in (a) happens to be the one which fails this criterion and the constraint qualification is not met here.

Question 3

(a)

The constraint here is $h(x, y) \geq 0$ or $-h(x, y) = x^2 + y^2 - 1 \leq 0$. The constraint set is compact, meaning a global constrained maximum and minimum exists owing to the Weierstrass theorem. The Lagrangian for this problem is

$$\mathcal{L}(x, y, \lambda) = x^2 - y - \lambda(x^2 + y^2 - 1)$$

and the Kuhn-Tucker conditions for a solution are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2x\lambda = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -1 - 2y\lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - x^2 - y^2 \geq 0,$$

$$\lambda \geq 0, \lambda \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

- Case 1: constraint is binding, $x^2 + y^2 = 1$.

  Condition (1) implies $x = 0$ or $\lambda = 1$ or both. Assuming $x = 0$, then $y = \pm 1$. From (2), $\lambda = -\frac{1}{2y}$. Since $\lambda \geq 0$, $y = 1$ would not be permissible. Therefore one possible solution is $(0, -1, \frac{1}{2})$.

  Assuming instead that $\lambda = 1$, (2) implies that $y = -\frac{1}{2}$. The assumption that the constraint is binding means $x = \pm\frac{\sqrt{3}}{2}$. So two additional possible solutions are $(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$.

- Case 2: constraint is not binding, $\lambda = 0$.

  Condition (2) is immediately violated in this case. Therefore there is nothing to be found in the case where the constraint is not binding.

The Lagrange method yielded 3 possible solutions: $(0, -1, \frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$. The constraint qualification is only violated when x and y are both 0, in which case $f(x, y) = 0$. We can evaluate the objective function at all 3 possible solutions, and if the largest value obtained by the solutions is greater than 0, we have found the global constrained maximum for this problem.

In this case $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$ yields the highest value of $\frac{5}{4}$, and therefore either one is a global constrained maximizer.

(b)

The function to be maximized is now $-f(x, y)$. Therefore, the Lagrangian for this problem is

$$\mathcal{L}(x, y, \lambda) = y - x^2 - \lambda(x^2 + y^2 - 1)$$

and the Kuhn-Tucker conditions for a solution are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= -2x - 2x\lambda = 0 && (3)\\
\frac{\partial \mathcal{L}}{\partial y} &= 1 - 2y\lambda = 0 && (4)\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 1 - x^2 - y^2 \geq 0\\
\lambda \geq 0, \lambda\frac{\partial \mathcal{L}}{\partial \lambda} &= 0
\end{aligned}$$

- Case 1: constraint is binding, $x^2 + y^2 = 1$.

  Now, (3) implies $x = 0$ or $\lambda = -1$ (which violates the condition for λ). Assuming $x = 0$, then $y = \pm 1$. Unlike in (a), (4) now means $y = -1$ is not admissible since it would lead to $\lambda < 0$. Therefore the one possible solution in this case is $(0, 1, \frac{1}{2})$. Logically this should be the global minimizer since any increase in the absolute value of x increases $f(x, y)$, while increases in y decreases the value of $f(x, y)$. There is no reason to consider the non-binding case since it would still lead to (4) being violated as in (a).

The objective function evaluated at $(0, 1, \frac{1}{2})$ is -1, and for the same reasons as in (a) this is the global constrained minimizer.

Question 4

The production function is homogeneous of degree 2, meaning it exhibits increasing returns to scale. Hence, even before starting, we should not expect to find any solution to this profit maximization problem.

(a)

The firm seeks to maximize

$$\pi(\mathbf{x}) = p_y \cdot g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} = p_y x_1(x_2 + x_3) - w_1x_1 - w_2x_2 - w_3x_3$$

subject to $x_i \geq 0$ (or $-x_i \leq 0$) $\forall i \in [1, 2, 3]$. The Lagrangian for this problem is

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = p_y x_1(x_2 + x_3) - w_1x_1 - w_2x_2 - w_3x_3 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3$$

and the Kuhn-Tucker conditions for a solution are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_i} &= \frac{\partial \pi}{\partial x_i} + \lambda_i = 0\\
\frac{\partial \mathcal{L}}{\partial \lambda_i} &= x_i \geq 0,\ \lambda_i \geq 0,\ \lambda_i x_i \geq 0
\end{aligned}$$

Since $\lambda_i \geq 0$, a neat trick is that we can simply reduce these two types of conditions to just one:

$$\frac{\partial \pi}{\partial x_i} = -\lambda_i \leq 0, \ (= 0 \text{ if } x_i > 0)$$

Therefore the equations that define the critical points of the Lagrangian are

$$\frac{\partial \pi}{\partial x_1} = p_y(x_2 + x_3) - w_1 \leq 0, \ (= 0 \text{ if } x_1 > 0) \tag{1}$$

$$\frac{\partial \pi}{\partial x_2} = p_y x_1 - w_2 \leq 0, \ (= 0 \text{ if } x_2 > 0) \tag{2}$$

$$\frac{\partial \pi}{\partial x_3} = p_y x_1 - w_3 \leq 0, \ (= 0 \text{ if } x_3 > 0) \tag{3}$$

and $\boldsymbol{\lambda} = -\nabla \pi(\bar{\mathbf{x}})$.

(b)

- Case 1: $x_1 = x_2 = x_3 = 0$

  In this case, all 3 inequalities are satisfied for $w_i > 0, i = 1, 2, 3$. So a critical point is $(0, 0, 0, w_1, w_2, w_3)$. The Lagrange multipliers here are odd; they suggest that relaxing the constraint (allowing negative inputs) will allow for a positive change in the objective function. This is a quirk that arises because at this particular solution, changing any x_i in any direction still results in 0 revenue, while allowing the firm to 'use' a negative amount of input i also means paying the firm w_i, with the net effect on profits being w_i.

- Case 2: $x_1 > 0$, $x_2 = x_3 = 0$

  Here, (1) is violated since $x_2 = x_3 = 0$ implies that $w_1 = 0$.

- Case 3: $x_2 > 0$, $x_1 = x_3 = 0$ or $x_3 > 0$, $x_1 = x_2 = 0$

  If $x_2 > 0$, (2) is violated since $x_1 = 0$ implies $-w_2 \leq 0$. A similar argument follows if only x_3 is not binding.

- Case 4: $x_1 > 0$, $x_2 > 0$, and $x_3 = 0$ or $x_1 > 0$, $x_3 > 0$, and $x_2 = 0$.

  If x_1 and x_2 are not binding, then (1) and (2) are equalities. Dividing (1) by (2) yields $\frac{w_1}{w_2} = \frac{x_2}{x_1}$ and substituting this back into (1) yields

  $$p_y \frac{w_1 x_1}{w_2} - w_1 = 0$$

  $$w_1 \left(\frac{p_y x_1}{w_2} - 1 \right) = 0$$

  which implies $p_y x_1 = w_2$ if $w_1 > 0$ (here the assumption that $p_y > 0$ is crucial). Thus,

  $$\lambda_3 = w_3 - p_y x_1 = w_3 - w_2$$

  which satisfies $\lambda_3 \geq 0$ only if $w_3 \geq w_2$. This makes intuitive sense; it implies that the firm chooses to produce using x_2 instead of x_3 if the latter is more expensive. x_2 and x_3 are symmetrical in this problem, so the same reasoning applies if $x_3 > 0$ and $x_2 = 0$.

  All the conditions are not violated in this case (save for the qualifications above regarding w_2 and w_3), but there are infinitely many solutions that could satisfy the equations. These are where $\frac{w_1}{w_2} = \frac{x_2}{x_1}$.

- Case 5: $x_1 = 0$, $x_2 > 0$, $x_3 > 0$.

  Again, (2) is violated.

- Case 6: all constraints are non-binding.

  Dividing (2) by (3) yields $w_2 = w_3$, and dividing (1) by (2) yields $\frac{w_1}{w_2} = \frac{w_1}{w_3} = \frac{x_2+x_3}{x_1}$. The Lagrange multipliers are not a problem since they are all equal to 0, and again, we are stuck with an equation that could be satisfied by infinitely many solutions.

Hence for all $(p_y, w_1, w_2, w_3) \in \mathbb{R}^4_{++}$, the Lagrangian has infinitely many critical points. If $w_2 < w_3$, then any combination of x_1 and x_2 that satisfies $\frac{w_1}{w_2} = \frac{x_2}{x_1}$ and $x_3 = 0$ is a critical point, and likewise if we replace the subscript 2 with 3. If they are equal then any combination that satisfies $\frac{w_1}{w_2} = \frac{w_1}{w_3} = \frac{x_2+x_3}{x_1}$ is a critical point.

(c)

None of the points could possibly identify a solution. For any arbitrary solution, for example one where $w_2 = w_3$ and the inputs satisfy $\frac{w_1}{w_2} = \frac{x_2+x_3}{x_1}$, it is always possible to increase profits while adhering to the condition by increasing x_1 and $x_2 + x_3$ by the same factor. As mentioned before this is because the production function exhibits increasing returns to scale and the constraint set is not bounded from above.

Question 5

(a)

Martha and Bob have additive constant loss aversion preferences, although in Martha's case her coefficient of loss aversion is 1 for both mugs and hats, that is, she exhibits no loss aversion. In Bob's case a loss of mugs is 4 times more painful than an equivalent gain in mugs, and a loss of hats is 2 times more painful than an equivalent gain in hats.

(b)

These people own 5 mugs (why?) and 1 hat. Thus their total disposable income (meaning Martha's total disposable income and Bob's total disposable income) is £8 if they receive £2, and £16 if they receive £10.

(c)

Martha exhibits no loss aversion, so her consumption should only depend on her disposable income and not the specific allocation that gives rise to that endowment of income.

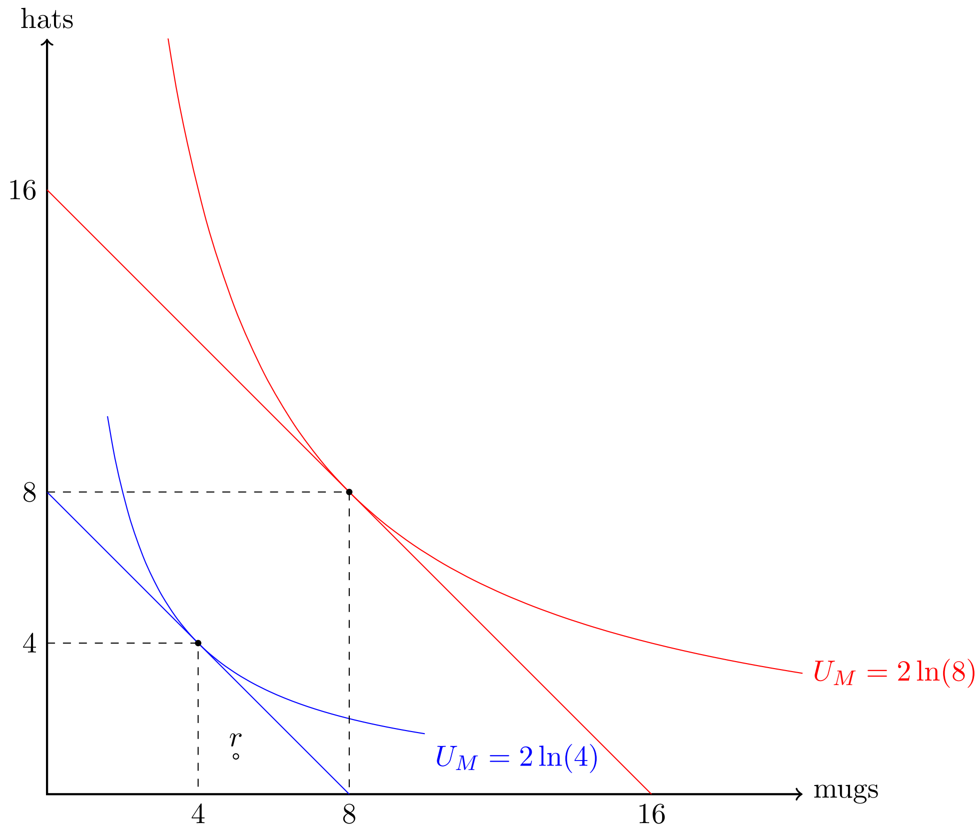
Martha seeks to maximize

$$U(x_1, x_2) = \ln x_1 - \ln 5 + \ln x_2 - \ln 1 = \ln x_1 + \ln x_2 - \ln 5$$

The $\ln 5$ should not matter since the order of her preferences are preserved under any affine transformation, so her utility maximization problem can simply be stated as

$$\max_{x_1, x_2} \{\ln x_1 + \ln x_2\} \text{ subject to } x_1 + x_2 = 8$$

if she only got a gift of £2. Her problem is quite simple; mugs and hats are symmetric in her preferences and she has convex preferences. Furthermore the prices of hats and mugs are the same. Thus she will just try to split her consumption equally between mugs and hats. With a gift of £2, her disposable income is £8 and she consumes 4 mugs and 4 hats. This is fewer mugs and more hats than she had before. With a gift of £10, her disposable income is £16 and she consumes 8 mugs and 8 hats, which is more mugs and hats than she had before.



(d)

For Bob, his utility function is

$$U_B(x_1, x_2) = \begin{cases} \ln x_1 - \ln 5 + \ln x_2 & \text{if } x_1 \geq 5, x_2 \geq 1 \\ 4(\ln x_1 - \ln 5) + \ln x_2 & \text{if } x_1 < 5, x_2 \geq 1 \\ \ln x_1 - \ln 5 + 2\ln x_2 & \text{if } x_1 \geq 5, x_2 < 1 \\ 4(\ln x_1 - \ln 5) + 2\ln x_2 & \text{if } x_1 < 5, x_2 < 1 \end{cases}$$

and his maximization problem is

$$\max_{x_1, x_2}\{U_B(x_1, x_2)\} \text{ subject to } x_1 + x_2 = 8$$

We can rule out $x_1 < 5, x_2 < 1$ since this is entirely within his budget set but not on the budget line. We know that in (c) there was a loss in mugs and gain in hats, so with Bob's preferences his optimal bundle may no longer be in that quadrant relative to r.

We can first try the region where there are no losses. The Lagrangian for this problem (again omitting the constant terms) is

$$\mathcal{L}(x_1, x_2, \lambda_0, \lambda_1, \lambda_2) = \ln x_1 + \ln x_2 - \lambda_0(x_1 + x_2 - 8) - \lambda_1(5 - x_1) - \lambda_2(1 - x_2)$$

and the Kuhn Tucker conditions are (applying the same shortcut for non-negativity constraints)

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda_0 \leq 0, (= 0 \text{ if } x_1 > 5) \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_2} - \lambda_0 \leq 0, (= 0 \text{ if } x_2 > 1) \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_0} = 8 - x_1 - x_2 = 0$$

and just to keep track,

$$\lambda_1 = \lambda_0 - \frac{1}{x_1}$$

$$\lambda_2 = \lambda_0 - \frac{1}{x_2}$$

$x_1 \geq 5$ and $x_2 \geq 1$ cannot be binding at the same time, so we can skip that case.

- Case 1: $x_1 = 5$, $x_2 > 1$.

  The budget constraint implies $x_2 = 3$ which is in the acceptable range. (2) implies $\lambda_0 = \frac{1}{3}$, and substituting this into (1) gives $\frac{1}{5} - \frac{1}{3} \leq 0$ which is also acceptable and yields a positive λ_1.

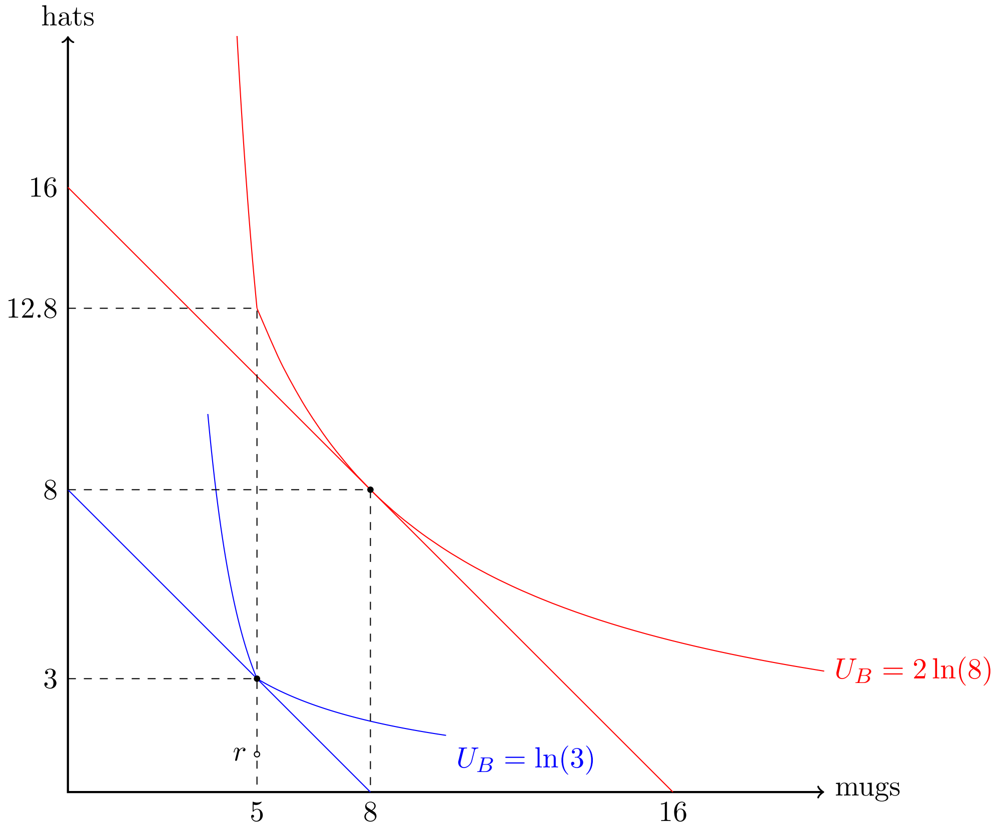
All other cases in this quadrant should not yield any solutions given that his preferences are convex. The solution we found, $(5, 3, \frac{1}{3}, \frac{2}{15}, 0)$, yields a utility of $\ln 3$ and was on the boundary between the quadrant where $x_1 \geq 5, x_2 \geq 1$ and the quadrant where $x_1 < 5, x_2 \geq 1$. Thus that is where we have to check next and see if the utility achieved in that quadrant exceeds $\ln 3$.

We could do the Lagrange problem for this again, or we could simply note that the solution we found would be the correct one as long as the budget line is not steeper than the indifference curve for $U_B = \ln 3$ in the adjacent quadrant. The slope of the budget line is -1, while the slope of the indifference curve in the other quadrant just before the kink would be

$$-\frac{\partial U_B/\partial x_1}{\partial U_B/\partial x_2}\bigg|_{x_1=5, x_2=3} = -\frac{4x_2}{x_1} = -\frac{12}{5}$$

which makes it steeper than the budget line throughout that whole quadrant since the indifference curve is convex. Thus Bob will consume 5 mugs and 3 hats. This differs from Martha's choice because Bob's aversion to losing 1 mug causes him to consume his endowment of mugs while spending his gift on hats. If he had never grown attached to his mugs in the first place (for example if he started out with nothing and received a gift of £8) he would consume the same bundle as Martha.

We don't have to repeat the exercise for the case where he receives £10. We know that the optimal bundle in the absence of loss aversion does not entail any losses for mugs or hats. Thus that bundle will remain the optimal bundle for Bob if he received £10.


hats
16
12.8
8
3
r
5
8
16
mugs
$U_B = 2\ln(8)$
$U_B = \ln(3)$