Economics of Uncertainty and Information
Supervision 1

Samuel Lee

Question 1

The utility function is concave in x, meaning the person is risk-averse; the utility of the expected value of a gamble is higher than the expected utility of a gamble. In this case, her expected utility from the investment is

$$\frac{1}{2}\ln(w+2) + \frac{1}{2}\ln(w-1) = \frac{1}{2}\ln[(w+2)(w-1)]$$

If she does not invest, she is guaranteed a utility of $\ln(w) = \frac{1}{2}\ln(w^2)$. Hence she will only accept the gamble if

$$\begin{aligned}(w+2)(w-1) &\geq w^2 \\ w^2 + w - 2 &\geq w^2 \\ w &\geq 2\end{aligned}$$

Question 2

The preference relation $\succeq$ on the space of simple lotteries $\mathcal{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in [0,1]$ we have that

$$L \succeq L' \text{ iff } \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$$

For a lottery L over the outcome space Ω, where $\omega \in \Omega$ occurs with probability $p(\omega)$, if $U(L)$ represents the utility derived from playing L, it can be shown that an expected utility function $u(\cdot)$ always exists such that $U(L)$ can be re-expressed as

$$U(L) = \sum_{\omega\in\Omega} p(\omega)u(\omega)$$

So for every L in the space of lotteries $\mathcal{L}$, there is a real value $U(L)$ associated with that lottery given by the expression above. When an expected utility maximizer is faced with $\mathcal{L}$, he will choose $L^* \in \mathcal{L}$ such that $U(L^*) \geq U(L')\ \forall L' \in \mathcal{L}$.

We now introduce an alternative lottery L''. We construct a new compound lottery L_C^* that yields the most preferred simple lottery L^* with probability α and the alternative simple lottery L'' with probability $1-\alpha$. Assuming the reduction of compound lotteries, the compound lottery L_C^* has the outcomes $\omega_C^* \in (\Omega^* \cup \Omega')$ that occur with probability $\alpha p^*(\omega_C^*) + (1-\alpha)p'(\omega_C^*)$. To break this down, Ω^* represents the possible outcomes from L^* and Ω' represents the possible outcomes from L'. So it stands to reason that the possible outcomes of L_C^* (a compound lottery made up of L^* and L') are represented by the union of these sets. $p^*(\cdot)$ maps the possible outcomes of L^* and $p'(\cdot)$ maps the possible outcomes of L' to a probability. Since the simple lotteries L^* and L' are played with probability α and $1-\alpha$ respectively in the compound lottery, the probability of any outcome in the compound lottery is its probability in each of the simple lotteries weighted by α and $1-\alpha$. To take a trivial example, the probabilty of an outcome ω^* that is present in L^* but not in L' is just $\alpha p^*(\omega^*)$ since $p'(\omega^*) = 0$.

We repeat these steps in constructing another compound lottery L'_C that yields the simple lottery L' with probability α and the simple lottery L'' with probability $1-\alpha$. Through the same logic we get that the compound lottery L'_C has the outcomes $\omega'_C \in (\Omega' \cup \Omega'')$ that occur with probability $\alpha p'(\omega'_c) + (1-\alpha)p''(\omega'_c)$.

From the expected utility hypothesis, we have that

$$\begin{aligned}
U(L^*_C) &= \sum_{\omega^*_C \in (\Omega^* \cup \Omega'')} \left[\alpha p^*(\omega^*_C) + (1-\alpha)p''(\omega^*_C)\right] u(\omega^*_C) \\
&= \alpha \sum_{\omega^*_C \in (\Omega^* \cup \Omega'')} p^*(\omega^*_C)u(\omega^*_C) + (1-\alpha) \sum_{\omega^*_C \in (\Omega^* \cup \Omega'')} p''(\omega^*_C)u(\omega^*_C) \\
&= \alpha \left[\sum_{\omega^* \in \Omega^*} p^*(\omega^*)u(\omega^*) + \sum_{\omega'' \in (\Omega'' \setminus \Omega^*)} p^*(\omega'')u(\omega'')\right] \\
&\quad + (1-\alpha) \left[\sum_{\omega'' \in \Omega''} p''(\omega'')u(\omega'') + \sum_{\omega^* \in (\Omega^* \setminus \Omega'')} p''(\omega^*)u(\omega^*)\right]
\end{aligned}$$

In the expression above, $p^*(\omega'')$ and $p''(\omega^*)$ must all equal to 0, since the events $\omega'' \in (\Omega'' \setminus \Omega^*)$ and $\omega^* \in (\Omega^* \setminus \Omega'')$ are by definition not in Ω^* and Ω'' respectively and their probabilities are 0. Therefore, the expression reduces to

$$U(L^*_C) = \alpha \sum_{\omega^* \in \Omega^*} p^*(\omega^*)u(\omega^*) + (1-\alpha) \sum_{\omega'' \in \Omega''} p''(\omega'')u(\omega'')$$

Repeating everything above for $U(L'_C)$ yields

$$U(L'_C) = \alpha \sum_{\omega' \in \Omega'} p'(\omega')u(\omega') + (1-\alpha) \sum_{\omega'' \in \Omega''} p''(\omega'')u(\omega'')$$

We wanted to show that an expected utility maximizer satisfies the independence axiom. We have shown that adding an alternative simple lottery L'' to L^* and L' with probability $1-\alpha$ leads to a utility of $U(L^*_C)$ and $U(L'_C)$ respectively. It is clear that the most preferred lottery L^* is preferred to L' if and only if $L^* \succeq L'_C$, since

$$\begin{aligned}
L^* \succeq L' \implies U(L^*) \geq U(L') \implies \sum_{\omega^* \in \Omega^*} p^*(\omega^*)u(\omega^*) \geq \sum_{\omega' \in \Omega'} p'(\omega')u(\omega') \\
\implies U(L^*_C) \geq U(L'_C) \text{ since } U(L^*_C) - U(L'_C) = \alpha \left(\sum_{\omega^* \in \Omega^*} p^*(\omega^*)u(\omega^*) - \sum_{\omega' \in \Omega'} p'(\omega')u(\omega')\right)
\end{aligned}$$

Therefore, an expected utility maximizer satisfies the independence axiom. To be accurate, this is strictly speaking a circular argument, since the existence of an expected utility formulation via the Von Neumann-Morgenstern utility theorem itself requires the independence axiom.

Question 3

55 of the balls in urn I are red while 45 of the balls in urn I are black. The person presumably has a belief that a proportion α of the balls in urn II are red while a proportion $1-\alpha$ are black. In the first situation, he chooses one of the urns and wins £100 if the ball drawn is black. Here he faces the two lotteries:

$$L_I : 0.55 \circ £0 \oplus 0.45 \circ £100$$
$$L_{II} : \alpha \circ £0 \oplus (1-\alpha) \circ £100$$

and strictly prefers L_I. If he is an expected utility maximizer, this means that for some expected utility function $u : \Omega \to \mathbb{R}$, where Ω is the outcome space $\{£0, £100\}$,

$$0.55u(£0) + 0.45u(£100) > \alpha u(£0) + (1-\alpha)u(£100) \tag{1}$$

In the second situation, he now gets £100 if the ball drawn is red. Now he faces the two lotteries

$$L'_I : 0.55 \circ £100 \oplus 0.45 \circ £0$$
$$L'_{II} : \alpha \circ £100 \oplus (1-\alpha) \circ £0$$

and strictly prefers L'_I. If he is an expected utility maximizer, this means that

$$0.55u(£100) + 0.45u(£0) > \alpha u(£100) + (1-\alpha)u(£0) \tag{2}$$

adding (1) and (2) implies

$$u(£0) + u(£100) > u(£0) + u(£100)$$

which is a violation no matter what α is. Thus there cannot exist any subjective probabilities α and $1-\alpha$ which would make his behaviour consistent with expected utility theory.

Question 4

An individual with current wealth £10 has to pay £2 to enter a competition, with a 1/3 chance of winning £19. The certainty equivalent (CE) of this gamble (L) is the amount which makes one indifferent between taking that amount for certain and taking the gamble. In other words, if the individual maximizes expected utility, $u(CE) = \mathrm{E}[u(L)]$. The risk premium (Π) of this gamble is the difference between the expected value of the gamble and the certainty equivalent of the gamble. That is, $\Pi = \mathrm{E}[L] - CE$. The expected value of the lottery $\mathrm{E}[L]$ is $\frac{1}{3} \cdot 27 + \frac{2}{3} \cdot 8 = \frac{43}{3}$.

(a)

For someone with the utility function $u(x) = \ln(x)$, we expect the certainty equivalent to be smaller than $\mathrm{E}[L] = \frac{43}{3}$ since the utility function is concave and the individual is risk-averse. The certainty equivalent CE must satisfy

$$\ln(CE) = \frac{1}{3}\ln 27 + \frac{2}{3}\ln 8 = \ln 3 + \ln 4 = \ln 12$$

which means $CE = 12$ which is smaller than $\frac{43}{3}$ as expected. The risk premium is $\Pi = \frac{43}{3} - 12 = \frac{7}{3}$.

(b)

The utility function $u(x) = x^2$ is convex and the individual is risk-seeking, so we expect the opposite result where the certainty equivalent is larger than $\frac{43}{3}$ and the risk premium is negative. We have that

$$
\begin{aligned}
(CE)^2 &= \frac{1}{3}(27)^2 + \frac{2}{3}(8)^2 = \frac{857}{3} \\
CE &\approx 16.90 \\
\Pi \approx \frac{43}{3} - 16.9 = -\frac{77}{30} < 0
\end{aligned}
$$

as expected.

(c)

The utility function is linear which means the individual is risk-neutral. The certainty equivalent will be equal to $\frac{43}{3}$ and the risk premium is 0.

If $u(x) = \ln(x)$, her expected utility is $\ln 12$. If she does not take the gamble, her expected utility is $\ln 10$. Therefore she will enter the competition.

Question 5

For a utility function $u(x)$, the coefficient of absolute risk aversion is $r(x) = -\frac{u''(x)}{u'(x)}$ and the coefficient of relative risk aversion is $\rho(x) = -\frac{u''(x)x}{u'(x)}$. If u is concave, $u''(x) \leq 0$. If $\rho(x)$ is decreasing in x, this implies

$$
\rho'(x) = -\frac{u'(x)[u'''(x)x + u''(x)] - [u''(x)]^2 x}{[u'(x)]^2} = \frac{[u''(x)]^2 x - u'(x)[u'''(x)x + u''(x)]}{[u'(x)]^2} < 0
$$

Differentiating $r(x)$ yields

$$
r'(x) = -\frac{u'(x)u'''(x) - [u''(x)]^2}{[u'(x)]^2} = \frac{[u''(x)]^2 - u'(x)u'''(x)}{[u'(x)]^2}
$$

This means that

$$
r'(x) = \rho'(x) + \frac{u'(x)u''(x)}{[u'(x)]^2} = \rho'(x) + \frac{u''(x)}{u'(x)}
$$

If we also assume that the utility function is increasing in x, that is, $u'(x) \geq 0$, then $r'(x) = \rho'(x) + \frac{u''(x)}{u'(x)} \leq \rho'(x) \leq 0$ since concavity implies $u''(x) \leq 0$. Therefore $r(x)$ is decreasing in x if $\rho(x)$ is decreasing in x.