Intertemporal Macroeconomics
Supervision 1

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Question 1

The person's utility in each period is $u_t(L_t, c_t) = \alpha\frac{c_t^{-\gamma}}{\gamma} + \beta\frac{L_t^{-\gamma}}{\gamma}$, where c_t and L_t are consumption and leisure in period t.

His lifetime utility over two periods is

$$u_1(L_1, c_1) + \theta \cdot u_2(L_2, c_2) = \alpha\frac{c_1^{-\gamma}}{\gamma} + \beta\frac{L_1^{-\gamma}}{\gamma} + \theta\left(\alpha\frac{c_2^{-\gamma}}{\gamma} + \beta\frac{L_2^{-\gamma}}{\gamma}\right)$$

With an initial wealth of W_0, price of consumption p_t, and interest earned on asset holdings ρ, his intertemporal budget constraint is

$$p_2c_2 = (1+\rho)(W_0 + w_1H_1 - p_1c_1) + w_2H_2$$

where H_t is the number of hours worked in period t, or

$$p_1c_1 + \frac{p_2c_2}{1+\rho} = W_0 + w_1H_1 + \frac{w_2H_2}{1+\rho}$$

which states that the present value of lifetime income & initial wealth equals the present value of lifetime spending.

Thus his optimization problem is

$$\max_{c_1, L_1, L_2} u_1(L_1, c_1) + \theta \cdot u_2(L_2, c_2) = \alpha\frac{c_1^{-\gamma}}{\gamma} + \beta\frac{L_1^{-\gamma}}{\gamma} + \theta\left(\alpha\frac{c_2^{-\gamma}}{\gamma} + \beta\frac{L_2^{-\gamma}}{\gamma}\right)$$

$$\text{s.t. } p_1c_1 + \frac{p_2c_2}{1+\rho} = W_0 + w_1H_1 + \frac{w_2H_2}{1+\rho}$$

Assuming $L_t = 24 - H_t$, the Lagrangian for this problem is

$$\begin{aligned}\mathcal{L} &= \alpha\frac{c_1^{-\gamma}}{\gamma} + \beta\frac{L_1^{-\gamma}}{\gamma} + \theta\left(\alpha\frac{c_2^{-\gamma}}{\gamma} + \beta\frac{L_2^{-\gamma}}{\gamma}\right) \\ &\quad - \lambda\left(p_1c_1 + \frac{p_2c_2}{1+\rho} - W_0 - w_1(24 - L_1) - \frac{w_2(24 - L_2)}{1+\rho}\right)\end{aligned}$$

and the first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_1} = -\alpha c_1^{-\gamma-1} \quad - \lambda p_1 \quad = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = -\theta\alpha c_2^{-\gamma-1} \quad - \frac{\lambda p_2}{1+\rho} = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial L_1} = -\beta L_1^{-\gamma-1} \quad - \lambda w_1 \quad = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial L_2} = -\theta\beta L_2^{-\gamma-1} - \frac{\lambda w_2}{1+\rho} = 0 \tag{4}$$

Dividing (1) by (2) yields

$$\left(\frac{c_1}{c_2}\right)^{-\gamma-1} = \frac{p_1}{p_2}(1+\rho)\theta$$

$$\frac{c_1}{c_2} = \left[\frac{p_1}{p_2}(1+\rho)\theta\right]^{-\frac{1}{1+\gamma}} \tag{5}$$

and dividing (3) by (4) yields

$$\left(\frac{L_1}{L_2}\right)^{-\gamma-1} = \frac{w_1}{w_2}(1+\rho)\theta$$

$$\frac{L_1}{L_2} = \left[\frac{w_1}{w_2}(1+\rho)\theta\right]^{-\frac{1}{1+\gamma}} \tag{6}$$

Equation (6) provides enough information to solve (iii): the person is induced to work more (less) in the second period than the first if his optimal choice of $\frac{L_1}{L_2}$ is greater (less) than unity, which will be the case when $\left[\frac{w_1}{w_2}(1+\rho)\theta\right]$ is less (greater) than 1. The term in square brackets can be interpreted as the cost of forgoing one hour of work in period 1 relative to the cost of forgoing one hour of work in period 2, subject to a discount rate which determines whether it is worth earning more now so as to consume more later.

When the term is less than one (for example, when w_2 is relatively high), it is worthwhile to work more in period 2, and when the term is greater than one (for example, when w_1 is relatively high and the discount rate θ is not so low as to make it less worthwhile to work more in period 1 in order to have more to spend in period 2), it is worthwhile to work more in period 1. The substitution effect dominates here: a wage increase in period 1 makes leisure in that period more expensive, and the person allocates relatively more hours to work. Whether the person works more in period 1 or 2 is independent of the price of consumption, which only affects the total hours worked.

Question 2

A household faces the following optimization problem:

$$
\begin{aligned}
&\max_{C_0, C_1, B_1} \{u(C_0) + u(C_1)\} \\
\text{subject to } & C_0 + B_1 = (1 + r_0)B_0 + Y_0 - T_0 && (1) \\
& C_1 = (1 + r_1)B_1 + Y_1 - T_1 && (2)
\end{aligned}
$$

By substituting the second equation into the first, the budget constraint can be expressed as

$$C_0 + \frac{C_1}{1 + r_1} = (1 + r_0)B_0 + Y_0 - T_0 + \frac{Y_1 - T_1}{1 + R_1}$$

The Lagrangian for this problem is

$$\mathcal{L} = u(C_0) + u(C_1) - \lambda \left[C_0 + \frac{C_1}{1 + r_1} - (1 + r_0)B_0 - (Y_0 - T_0) - \frac{Y_1 - T_1}{1 + r_1} \right]$$

and the first-order conditions are

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_0} &= u'(C_0) - \lambda &&= 0 && (3) \\
\frac{\partial \mathcal{L}}{\partial C_1} &= u'(C_1) - \lambda(\frac{1}{1 + r_1}) &&= 0 && (4)
\end{aligned}
$$

Dividing (4) by (3) yields

$$\frac{u'(C_1)}{u'(C_0)} = \frac{1}{1 + r_1}$$

which is the Euler equation for this problem. The intuition behind it is that the marginal benefit of increasing C_1 is equal to the marginal benefit of increasing C_0 multiplied by the amount of future consumption sacrificed with a unit increase of C_0. The equation makes it such that the lifetime utility sacrificed is equal whether one chooses to consume more in time period 1 or time period 2.

The government's budget constraints are given by

$$\begin{aligned} G_0 - D_1 &= T_0 - D_0(1 + r_0) \\ G_1 &= T_1 - D_1(1 + r_1) \end{aligned}$$

In equilibrium, $D_t = B_t$. Thus

$$\begin{aligned} G_0 - B_1 &= T_0 - B_0(1 + r_0) \qquad (5) \\ G_1 &= T_1 - B_1(1 + r_1) \qquad (6) \end{aligned}$$

Adding (5) to (1) and (6) to (2) yields

$$\begin{aligned} C_0 + G_0 &= Y_0 \\ C_1 + G_1 &= Y_1 \end{aligned}$$

If Y_t and G_t are exogenous,

$$\frac{u'(C_1)}{u'(C_0)} = \frac{u'(Y_1 - G_1)}{u'(Y_0 - G_0)} = \frac{1}{1 + r_1}$$

and the equilibrium real interest rate is

$$r_1 = \frac{u'(Y_0 - G_0)}{u'(Y_1 - G_1)} - 1$$

Given that $u'(\cdot) > 0$ and $u''(\cdot) < 0$, this implies that r_1 is increasing with respect to G_0 and decreasing with respect to G_1, since

$$\begin{aligned} \frac{dr_1}{dG_0} &= -\frac{u'(Y_1 - G_1) \cdot u''(Y_0 - G_0)}{[u'(Y_1 - G_1)]^2} > 0 \\ \frac{dr_1}{dG_1} &= \frac{u'(Y_0 - G_0) \cdot u''(Y_1 - G_1)}{[u'(Y_1 - G_1)]^2} < 0 \end{aligned}$$

The more the government spends in period 0, the higher the level of debt that rolls over to period 1 becomes. Because both income and the initial holdings of government bonds (B_0) are exogenous, r_1 has to increase for the equilibrium condition $D_t = B_t$ to hold, which leads to the result above.