Game Theory
Supervision 3

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Question 1

(a)

Assuming Bidder j's strategy is to use a strictly increasing differentiable bidding function $b(v_j)$ with inverse function $g(\cdot)$, Bidder i's expected payoff from submitting a bid of b_i is

$$\mathrm{E}[\pi_i|v_i] = \Pr[b_i > b(v_j)]v_i - b_i = \Pr[g(b_i) > v_j]v_i - b_i = g(b_i)v_i - b_i$$

The first-order condition is

$$\frac{\partial\,\mathrm{E}[\pi_i|v_i]}{\partial b_i} = g'(b_i)v_i - 1 = 0 \implies g'(b_i) = \frac{1}{v_i}$$

Assuming $b_i = b(v_i)$ is also optimal for Bidder i in that it solves the first-order condition:

$$g'[b(v_i)] = \frac{\frac{\mathrm{d}}{\mathrm{d}v_i} g[b(v_i)]}{b'(v_i)} = \frac{1}{b'(v_i)} = \frac{1}{v_i} \implies b(v_i) = \frac{v_i^2}{2} + C$$

since $\frac{\mathrm{d}}{\mathrm{d}v_i} g[b(v_i)] = g'[b(v_i)]b'(v_i)$ and $g[b(v_i)] = v_i$. We must have the boundary condition $b(0) = 0$ to satisfy the participation constraint, so $C = 0$. Therefore the strategy played in the Bayes-Nash equilibrium is $b(v) = \frac{v^2}{2}$.

Question 2

Under the four standard auctions we have the first-price sealed bid auction, the second-price sealed bid auction, the ascending (English) auction, and the descending (Dutch) auction. In all four auctions, because the lower bidder ends up with zero surplus, they will always be indifferent between not getting the good and purchasing it from the higher bidder for the amount they value it at. Since re-selling the good is always feasible, it might be instructive to consider when there is an incentive to re-sell.

(i) In the first-price sealed bid auction, playing $b(v) = \frac{v}{2}$ is not optimal when the bidders know each other's valuations and the losing bidder cannot reject when the winner tries to re-sell. Bidder j could bid $\frac{v_i}{2} + \varepsilon < v_i$ and sell the good to i for v_i. And if Bidder j tries that, Bidder i would rather buy the good at $\frac{v_i}{2} + \varepsilon + \delta < v_i$ and keep the good. It becomes apparent that

$$s_i = (\bar{v}, \text{ Re-sell if } v_i < \bar{v}), s_j = (\bar{v}, \text{ Re-sell if } v_j < \bar{v})$$

is a Nash equilibrium, where $\bar{v} = \max\{v_i, v_j\}$ and the losing bidder cannot reject when the winner tries to re-sell. There are also equilibria where neither, one, or both of the players commit to re-selling if $v_i = v_j = \bar{v}$, but these are only relevant in a pathological case which happens with probability 0.

On inspection, the equilibrium above also holds in a descending auction.

The seller always gets $\bar{v}$, so the expected revenue is

$$\mathrm{E}[\max\{v_i, v_j\}] = \int_0^1 2vf(v)F(v)\,\mathrm{d}\,v = \int_0^1 2v^2\,\mathrm{d}\,v = \frac{2}{3}$$

(ii) In the second-price sealed bid auction, unlike before, the usual equilibrium of bidding one's own true value is still an equilibrium when re-sale is possible. The bidder with the higher valuation gains nothing by lowering their bid. In addition, each bidder knows the other's valuation, so the bidder with the lower valuation could bid anything in $[0, \bar{v})$ and remain indifferent if their belief is that the other bidder will bid their true value. So we have a set of Nash equilibria $b_i = v_i, b_j \in [0, v_i)$ assuming Bidder i has the higher valuation. If Bidder i bids anything lower than v_i, then Bidder j could profit by outbidding b_i by a bit and re-selling the good.

On inspection, the equilibria above also hold in an ascending auction.

The expected revenue depends on what type of strategy the bidders will play if they turn out to value the good less. If we assume that everyone bids their true value, then the expected revenue is

$$\mathrm{E}[\min\{v_i, v_j\}] = \int_0^1 2v\,[1 - F(v)]\,f(v)\,\mathrm{d}\,v = \int_0^1 2v(1 - v)\,\mathrm{d}\,v = \frac{1}{3}$$

The stability of this equilibrium is contingent on the lower bidder believing that the higher bidder will bid $\bar{v}$ and that this belief is indeed true. If the higher bidder bids lower than $\bar{v}$ either by accident or just because they're feeling cheeky, they will not be better off, but we end up with the lower bidder making a suboptimal move. We might imagine that a more natural outcome is for both bidders to bid $\bar{v}$, such that whatever the other person does, the current strategy is still optimal. In other words, bidding one's true value is weakly dominated by bidding $\bar{v}$. If we assume the bidders play undominated strategies, then the expected revenue is $\mathrm{E}[\max\{v_i, v_j\}] = \frac{2}{3}$ as before.

There is a common thread in both cases, which is that both bidders end up with zero surplus (assuming bidders play undominated strategies). Essentially, when private values are known to one another, there is no longer any information rent. Furthermore, when re-sale is possible and costless, the good could have value to a bidder apart from their private valuation. Some version of the Coase theorem applies, and the good ends up with the one who values it most. In this case, we might denote Bidder i's willingness to pay $W_i = \max\{v_i, v_j\} = \bar{v}$, which will be the same for Bidder j.

Given this, we can solve for the revenue-maximising posted price:

$$E[\pi_i|p] = \Pr(\bar{v} \geq p)p = [1 - \Pr(v_i < p, v_2 < p)]\, p = p(1-p^2) \implies p^* = \frac{1}{\sqrt{3}}$$

which is the maximising price for the standard posted price mechanism. The expected revenue is $\frac{2}{3\sqrt{3}}$ which is lower than the expected revenue in the two cases we worked through above.

For both the first and second price auctions where bidders use undominated strategies, the seller captures the largest surplus available since they always get the maximum of the valuations. There should therefore be no gain in charging an entry fee. Another way to look at it is that the surplus to either bidder is already 0, so charging any entry fee without increasing their ex-ante surplus 1-for-1 would deter them from taking part.

Question 3

(a)

With n periods, we have a subgame-perfect Nash equilibrium where Player 1 always proposes $(v, 0)$ and Player 2 accepts in every period. Each period represents a subgame; at period n, given any offer, Player 2 cannot get a better outcome by rejecting, so accepting is a best response. Assuming Player 2 accepts, the offer that maximises Player 1's outcome is $(v, 0)$. Similarly, at period $n-1$, for any offer, Player 2 gets at best the outcome in the final subgame if they reject, which is 0. Knowing this, Player 1 still does best by proposing $(v, 0)$. The argument applies recursively and this Nash equilibrium is subgame-perfect.

(b)

We can have an equilibrium where Player 1 always proposes $(v, 0)$ and Player 2 always rejects; neither player can improve their outcome holding the other player's strategy fixed. However the equilibrium is not subgame-perfect; in the subgame where Player 1 makes a positive offer, a Nash equilibrium must involve Player 2 accepting.

(c)

A Nash equilibrium with $\left(\frac{v}{4}, \frac{3v}{4}\right)$ can be supported if Player 1 offers $\left(\frac{v}{4}, \frac{3v}{4}\right)$ and Player 2 rejects all offers below $\frac{3v}{4}$. Given Player 1's strategy, Player 2 cannot do better with a different rejection rule, and given Player 2's rejection rule, Player 1's achieves the best outcome by offering $\left(\frac{v}{4}, \frac{3v}{4}\right)$. This is again not subgame-perfect for the same reason as before.

Question 4

There seems a good chance that this is analogous to the usual Rubinstein bargaining model except with different discount rates where $\delta_2 = \delta_1^2$, since Player 1 has two consecutive periods to make offers every 3 periods and can essentially make Player 2 wait longer. In the Rubinstein bargaining model with different discount rates, the unique subgame-perfect Nash equilibrium involves Player 1 offering $\left(\frac{v(1-\delta_2)}{1-\delta_1\delta_2}, \frac{v\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ and Player 2 offering $\left(\frac{v\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{v(1-\delta_1)}{1-\delta_1\delta_2}\right)$. With $\delta_2 = \delta_1^2$ and $\delta_1 = \delta$, these are equivalent to $s_1 = \left(\frac{v(1+\delta)}{1+\delta+\delta^2}, \frac{v\delta^2}{1+\delta+\delta^2}\right)$ and $s_2 = \left(\frac{v\delta(1+\delta)}{1+\delta^+\delta^2}, \frac{v}{1+\delta+\delta^2}\right)$.

Therefore, a sensible guess might be that when $t \bmod 3 = 1$, Player 1 proposes $\left(\frac{v(1+\delta)}{1+\delta+\delta^2}, \frac{v\delta^2}{1+\delta+\delta^2}\right)$ and when $t \bmod 3 = 0$, Player 2 proposes $\left(\frac{v\delta(1+\delta)}{1+\delta^+\delta^2}, \frac{v}{1+\delta+\delta^2}\right)$. It might also be sensible to impose an indifference condition for $t \bmod 3 = 2$; Player 2 should not get a better deal when rejecting Player 1's first offer, so we have for the second offer by Player 1 $\left(\frac{v(1+\delta^2)}{1+\delta+\delta^2}, \frac{v\delta}{1+\delta+\delta^2}\right)$. This also makes Player 2 indifferent between rejecting Player 1's second offer and having their own offer to Player 1 being accepted one period later. So to reiterate,

(i) When $t \bmod 3 = 1$, Player 1 proposes $\left(\frac{v(1+\delta)}{1+\delta+\delta^2}, \frac{v\delta^2}{1+\delta+\delta^2}\right)$

(ii) When $t \bmod 3 = 2$, Player 1 proposes $\left(\frac{v(1+\delta^2)}{1+\delta+\delta^2}, \frac{v\delta}{1+\delta+\delta^2}\right)$

(iii) When $t \bmod 3 = 0$, Player 2 proposes $\left(\frac{v\delta(1+\delta)}{1+\delta^+\delta^2}, \frac{v}{1+\delta+\delta^2}\right)$

We now need to check if this is indeed a subgame-perfect Nash equilibrium. We assume that in each period, the non-proposing player rejects all offers below the amount that the proposing player offers. Since the game is continuous at infinity, we can consider one-shot deviations for all 6 subgames:

(i) When $t \bmod 3 = 1$:

   (a) If there is no profitable one-shot deviation where Player 1 offers a lower amount to Player 2 (which will be rejected), it must be that

$$\frac{v(1+\delta)}{1+\delta+\delta^2} \geq \delta \times \frac{v(1+\delta^2)}{1+\delta+\delta^2}$$

   which is always true. For this and subsequent cases, we don't need to consider cases where the proposer offers a higher amount to the other player, since that will never yield a higher payoff even if it will be accepted.

   (b) If there is no profitable one-shot deviation where Player 2 rejects Player 1's offer, it must be that

$$\frac{v\delta^2}{1+\delta+\delta^2} \geq \delta \times \frac{v\delta}{1+\delta+\delta^2}$$

   which we made true by construction.

(ii) When $t \bmod 3 = 2$:

(a) If there is no profitable one-shot deviation where Player 1 offers a lower amount to Player 2 (which will be rejected), it must be that

$$\frac{v(1+\delta^2)}{1+\delta+\delta^2} \geq \delta \times \frac{v\delta(1+\delta)}{1+\delta+\delta^2}$$

which is always true.

(b) If there is no profitable one-shot deviation where Player 2 rejects Player 1's offer, it must be that

$$\frac{v\delta}{1+\delta+\delta^2} \geq \delta \times \frac{v}{1+\delta+\delta^2}$$

which happens to hold with equality.

(iii) When $t \bmod 3 = 0$:

(a) If there is no profitable one-shot deviation where Player 1 rejects Player 2's offer, it must be that

$$\frac{v\delta(1+\delta)}{1+\delta+\delta^2} \geq \delta \times \frac{v(1+\delta)}{1+\delta+\delta^2}$$

which also happens to hold with equality.

(b) If there is no profitable one-shot deviation where Player 2 offers a lower amount to Player 1 (which will be rejected), it must be that

$$\frac{v}{1+\delta+\delta^2} \geq \delta \times \frac{v\delta^2}{1+\delta+\delta^2}$$

which is always true.

So there are no profitable one-shot deviations for the strategies we stumbled upon rather haphazardly, and they should (I'm a bit unsure) constitute a subgame-perfect Nash equilibrium.

Question 5

We assume that each player always makes the same offer in equilibrium. So Player 1 always offers $(x_1, v - x_1)$ and Player 2 always offers $(v - x_2, x_2)$. There will be no profitable one-shot deviations for Player 2 if $x_2 = \frac{v-x_1}{\delta_2}$, in which case Player 2 always offers $\left(\frac{x_1-(1-\delta_2)v}{\delta_2}, \frac{v-x_1}{\delta_2}\right)$. Assuming this, there will be no profitable one-shot deviations for Player 1 if

$$x_1 = \frac{x_1 - (1-\delta_2)v}{\delta_1\delta_2} \text{ or } x_1 = \frac{(1-\delta_2)v}{1-\delta_1\delta_2}$$

(a)

So we have a subgame-perfect Nash equilibrium where Player 1 always offers $\left(\frac{(1-\delta_2)v}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)v}{1-\delta_1\delta_2}\right)$ and rejects any offer below $\frac{\delta_1(1-\delta_2)v}{1-\delta_1\delta_2}$, while Player 2 always offers $\left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{(1-\delta_1)v}{1-\delta_1\delta_2}\right)$ and rejects any offer below $\frac{\delta_2(1-\delta_1)v}{1-\delta_1\delta_2}$.

(b)

Replacing δ_1 and δ_2 with $\delta_1(\Delta)$ and $\delta_2(\Delta)$ would not change the form of the equilibrium above since we're just choosing different constant values for δ_1 and δ_2. So the equilibrium payoff is $\left(\frac{[1-\delta_2(\Delta)]v}{1-\delta_1(\Delta)\delta_2(\Delta)}, \frac{\delta_2(\Delta)[1-\delta_1(\Delta)]v}{1-\delta_1(\Delta)\delta_2(\Delta)}\right)$. With L'Hôpital's rule, we have

$$\lim_{\Delta\to 0} \frac{[1-\delta_2(\Delta)]\,v}{1-\delta_1(\Delta)\delta_2(\Delta)} = \lim_{\Delta\to 0} \frac{r_2\delta_2(\Delta)v}{(r_1+r_2)\delta_1(\Delta)\delta_2(\Delta)} = \frac{r_2}{r_1+r_2}v$$

or the more impatient Player 2 is, the higher the payoff to Player 1. Going through the same steps gives us $\frac{r_1}{r_1+r_2}v$ for the equilibrium payoff to Player 2.

(c)

When $d_1 = d_2 = 0$, the Nash product becomes a standard Cobb-Douglas function which, at optimum, has the property $\frac{u_1}{v} = \alpha$ and $\frac{u_2}{v} = 1-\alpha$. So we just set $\alpha = \frac{r_2}{r_1+r_2}$ and we get the result in (b).