Mathematical Economics
Supervision 1

Samuel Lee

Question 1

(a)

A sequence $\{x^k\} \subset \mathbb{R}$ is bounded if there is an $r > 0$ such that $\|x^k\| < r$ for every positive integer k. This essentially means there exists some distance from the origin within which all members of the sequence x^k can be found, and there is no member of the sequence which is outside of this distance.

A sequence $\{x^k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ if $\forall \epsilon > 0, \exists K \in \mathbb{N}$ s.t. $\forall k > K, x^k \in B_\epsilon(x)$ where $B_\epsilon(x) = \{x' \in \mathbb{R} : |x' - x| < \epsilon\}$. This essentially means that if $\{x^k\}$ converges to some x, no matter how small a neighbourhood one looks at around x, there is some natural number K large enough such that every member of $\{x^k\}$ beyond the K^{th} member will lie inside this neighbourhood.

Choosing some $\epsilon > 0$, there is some K such that $\|x^k - x\| < \epsilon$ for all $k > K$. Using the Triangle Inequality ($\|a + b\| \le \|a\| + \|b\|$),

$$\|x^k\| = \|x^k - x + x\| \le \|x^k - x\| + \|x\| < \epsilon + \|x\|$$

for all $k > K$.

With that, we know that the Euclidean norm for any member of the sequence past the K^{th} member will never exceed $\epsilon + \|x\|$, meaning $\{x^k\}$ is bounded at least for $k > K$. With this the difficult part is dealt with: the part of the sequence that is infinite turns out to be bounded. For $k \le K$ it is much more simple; there is surely some maximum value for the finite set $\{\|x^k\| | 0 \le k \le K\}$. We can call this maximum value $\|x_m\|$. If this turns out to be smaller than $\epsilon + \|x\|$, then $\{x^k\}$ is bounded by the latter. If not, then $\{x^k\}$ is bounded by $\|x_m\|$.

(b)

From (a) we know that any converging sequence is bounded. For $\{y^k\}$ we can call this boundary M. Taking some $\epsilon > 0$, we define some $K \in \mathbb{N}$ such that $\forall k > K, \|x^k\| < \frac{\epsilon}{M}$. With that, for all $k > K$, $\|x^k\| < \frac{\epsilon}{M}$ and $\|y^k\| < M$ (since M is the boundary on $\{y^k\}$).

Then, $\|x^k y^k\| = \|x^k\| \cdot \|y^k\| < \left(\frac{\epsilon}{M}\right) M = \epsilon \; \forall k > K$. The definition for convergence will then follow: $\forall \epsilon > 0, \exists K \in \mathbb{N}$ s.t. $\forall k > K, \|x^k y^k\| < \epsilon$.

Question 2

A set $A \subset \mathbb{R}^n$ is open if $\forall x \in A, \exists \epsilon > 0$, such that $B_\epsilon(x) \subset A$. Essentially this means that for any member x of set A, there is always an ϵ neighbourhood around x such that the entire neighbourhood is still contained in A.

The definition of a closed set is just the converse: a set A is closed if its complement $\mathbb{R}^n \setminus A$ is open.

An alternative definition of a closed set is that a set $A \subset \mathbb{R}^n$ is closed if and only if for all sequences $\{x_k\}$ such that $x_k \in A$ for each k and $x_k \to x$, it is the case that $x \in A$. In other words any converging sequence made up of only points in A converge to a point that is also in A; the set contains all its limit points.

(a)

The complement of $[0,1]$ is $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$. If $[0,1]$ is closed, its complement must be open. As above, this means $\forall x \in \mathbb{R} \setminus [0,1], \exists \epsilon > 0$ such that $B_\epsilon \subset \mathbb{R} \setminus [0,1]$.

If $x \in \mathbb{R} \setminus [0,1]$, x is either greater than 1 or less than 0. Letting $\epsilon = -x$ if $x < 0$ and $\epsilon = x - 1$ if $x > 1$, we get if $x < 0$,

$$\begin{aligned} B_\epsilon(x) &= \{x' \in \mathbb{R} : |x' - x| < \epsilon\} \\ &= \{x' \in \mathbb{R} : |x' - x| < -x\} \\ &= \{x' \in \mathbb{R} : x + |x' - x| < 0\} \end{aligned}$$

and if $x > 1$,

$$\begin{aligned} B_\epsilon(x) &= \{x' \in \mathbb{R} : |x' - x| < \epsilon\} \\ &= \{x' \in \mathbb{R} : |x' - x| < x - 1\} \\ &= \{x' \in \mathbb{R} : x - |x' - x| > 1\} \end{aligned}$$

The above shows that there can always be some ϵ such that the open ϵ-neighbourhood around any $x \in (-\infty, 0) \cup (1, \infty)$ is always in $(-\infty, 0) \cup (1, \infty)$ i.e. not in $[0,1]$. If $x < 0$, then setting $\epsilon = -x$ ensures an open ϵ-neighbourhood such that every x' in $B_\epsilon(x)$ fulfills $x + |x' - x| < 0$, and it is also trivially true that $x - |x' - x| < 0$. Similarly for $x > 1$, setting $\epsilon = x - 1$ makes it such that $x - |x' - x| > 1$. Therefore, the complement of $[0,1]$ is open and $[0,1]$ is closed.

Alternatively, since $\mathbb{R} \setminus [0,1]$ is a union of open sets, $[0,1]$ is closed.

(b)

Similar to (a), for $x \in (0,1)$, define $\epsilon = \min\{1 - x, x\}$. Then $x - \epsilon \geq x - x = 0$ and $x + \epsilon \leq x + (1 - x) = 1$. Therefore there is always an open ϵ neighbourhood (which does not include the equality constraints above) that is within $(0,1)$, making $(0,1)$ an open set.

Question 3

A set $A \subset \mathbb{R}^n$ is convex if for all $x, y \in A$ and for all $\alpha \in [0,1]$, $\alpha x + (1 - \alpha) y \in A$.

(a)

The set $(0,1) \cup (2,3)$ is an open set (through the reasoning from question 2) but not convex. One of many examples can show this: for $x = \frac{1}{2}$, $y = \frac{3}{2}$, and $\alpha = \frac{1}{2}$, $\alpha x + (1 - \alpha) y = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{2} = 2 \notin (0,1) \cup (2,3)$.

(b)

A compact set is one that is closed and bounded. The set $A = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$ is a ring in $\mathbb{R}^2$ space centered on $(0,0)$ with outer radius 2 and inner radius 1, and contains all its limit points. While $(-2,0)$ and $(2,0)$ are in A, $\alpha(-2,0) + (1-\alpha)(2,0) = (2-4\alpha, 0)$ is not in A when $\alpha = \frac{1}{2}$. Thus A is a compact set that is not convex.

Question 4

(a)

$$f(\mathbf{x}) = \begin{bmatrix} x_1^2 - 3x_1x_2 + \ln x_2 \\ 5 \\ 2x_1 + x_2 \\ \frac{x_1+5x_2}{x_1x_2} \end{bmatrix}$$

$$Df(\mathbf{x}) = \begin{bmatrix} 2x_1 - 3x_2 & \frac{1}{x_2} - 3x_1 \\ 0 & 0 \\ 2 & 1 \\ \frac{x_1x_2 - x_2(x_1+5x_2)}{x_1^2x_2^2} & \frac{5x_1x_2 - x_1(x_1+5x_2)}{x_1^2x_2^2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 & \frac{1}{x_2} - 3x_1 \\ 0 & 0 \\ 2 & 1 \\ -\frac{5}{x_1^2} & -\frac{1}{x_2^2} \end{bmatrix}$$

(b)

$$Df(g(\mathbf{x})) = \begin{bmatrix} 2(x_1+3x_4) - 3\sqrt{x_2x_3} & \frac{1}{\sqrt{x_2x_3}} - 3(x_1+3x_4) \\ 0 & 0 \\ 2 & 1 \\ -\frac{5}{(x_1+3x_4)^2} & -\frac{1}{x_2x_3} \end{bmatrix}$$

(c)

$$Dg(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & \sqrt{\frac{x_3}{4x_2}} & \sqrt{\frac{x_2}{4x_3}} & 0 \end{bmatrix}$$

(d)

From the Chain Rule,

$$D_x f(g(\mathbf{x})) = Df(g(\mathbf{x})) \cdot Dg(\mathbf{x}) =$$

$$\begin{bmatrix} 2(x_1+3x_4) - 3\sqrt{x_2x_3} & \frac{1}{2x_2} - 3(x_1+3x_4)\sqrt{\frac{x_3}{4x_2}} & \frac{1}{2x_3} - 3(x_1+3x_4)\sqrt{\frac{x_2}{4x_3}} & 6(x_1+3x_4) - 9\sqrt{x_2x_3} \\ 0 & 0 & 0 & 0 \\ 2 & \sqrt{\frac{x_3}{4x_2}} & \sqrt{\frac{x_2}{4x_3}} & 6 \\ -\frac{5}{(x_1+3x_4)^2} & -\frac{1}{2x_2^{\frac{3}{2}}\sqrt{x_3}} & -\frac{1}{2\sqrt{x_2}x_3^{\frac{3}{2}}} & -\frac{15}{(x_1+3x_4)^2} \end{bmatrix}$$

Verifying from $f(g(\mathbf{x}))$,

$$f(g(\mathbf{x}))$$
$$= \left[(x_1+3x_4)^2 - 3(x_1+3x_4)\sqrt{x_2x_3} + \ln\sqrt{x_2x_3}, 5, 2(x_1+3x_4)+\sqrt{x_2x_3}, \frac{x_1+3x_4+5\sqrt{x_2x_3}}{(x_1+3x_4)\sqrt{x_2x_3}}\right]$$

$$\frac{\partial fg_3}{\partial x_1} = \frac{\partial}{\partial x_1}[2(x_1+3x_4)+\sqrt{x_2x_3}] = 2$$
$$\frac{\partial fg_3}{\partial x_2} = \frac{\partial}{\partial x_2}[2(x_1+3x_4)+\sqrt{x_2x_3}] = \sqrt{\frac{x_3}{4x_2}}$$
$$\frac{\partial fg_3}{\partial x_3} = \frac{\partial}{\partial x_3}[2(x_1+3x_4)+\sqrt{x_2x_3}] = \sqrt{\frac{x_2}{4x_3}}$$
$$\frac{\partial fg_3}{\partial x_4} = \frac{\partial}{\partial x_4}[2(x_1+3x_4)+\sqrt{x_2x_3}] = 6$$

Which are the elements of the 3^{rd} row of $Df_x(g(\mathbf{x}))$. It is also easy to see that this is true for the 2^{nd} row, and somewhat tedious but straightforward to see that this is true for the 1^{st} and 4^{th} rows (if $Df_x(g(\mathbf{x}))$ was calculated correctly).

Question 5

The Hessian matrix of a function $f(\mathbf{x})$ is $D^2f(\mathbf{x}) = D[\nabla f(\mathbf{x})]$, that is, the matrix where element i, j is $\frac{\partial^2 f}{\partial x_i x_j}$.

The $n \times n$ matrix M is negative semi-definite if, for all $z \in \mathbb{R}^n$, $zMz \leq 0$, and negative definite is the inequality is strict for all $z \neq 0$. The definitions for positive (semi)-definiteness follow from reversing the inequalities.

The function $f : A \to \mathbb{R}$ is concave (convex) if and only if its Hessian matrix is negative (positive) semi-definite for all $x \in A$. If the Hessian is negative (positive) definite for all $x \in A$, then the function is strictly concave (convex).

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + ax_2 - x_1 \\ 1 + ax_1 - x_2 \end{bmatrix}$$
$$D[\nabla f(\mathbf{x})] = \begin{bmatrix} -1 & a \\ a & -1 \end{bmatrix}$$

Since this is an unconstrained problem, without having to go through the whole game with permutations and determinants, the condition for concavity is just

$$\forall h, k \in \mathbb{R}, \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} -1 & a \\ a & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = -h^2 + 2hka - k^2 \leq 0$$

With the inequality being strict for strict concavity. We can manipulate the inequality as such:

$$
\begin{aligned}
-h^2 + 2hka - k^2 &\leq 0 \\
-\left(\frac{h}{k}\right)^2 + 2a\left(\frac{h}{k}\right) - 1 &\leq 0 \\
-\left[\left(\frac{h}{k}\right)^2 - 2a\left(\frac{h}{k}\right) + 1\right] &\leq 0 \\
-\left[\left(\frac{h}{k}\right)^2 - 2a\left(\frac{h}{k}\right) + a^2 - a^2 + 1\right] &\leq 0 \\
-\left[\left(\frac{h}{k} - a\right)^2 + 1 - a^2\right] &\leq 0 \\
\underbrace{-\left(\frac{h}{k} - a\right)^2}_{(-)} + a^2 - 1 &\leq 0
\end{aligned}
$$

Since $-\left(\frac{h}{k} - a\right)^2$ is non-positive for all $h, k,$ and a, $a^2 - 1$ must be non-positive to ensure concavity. The condition on a that ensures this is $-1 \leq a \leq 1$. For strict concavity, the condition is $-1 < a < 1$.

Question 6

These are constrained optimization problems with equality constraints. For such problems, the Lagrangian method will find the necessary conditions for constrained extrema at which the constraint qualification is met, if such points exist and are well-defined.

(a)

The Lagrange function for this problem is

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 2a^2)$$

and a constrained global extremum exists because the constraint set is compact. The critical points for this function are where

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda y = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 - 2a^2) = 0 \tag{3}$$

Dividing (1) by (2),

$$
\begin{aligned}
\frac{y}{x} &= \frac{x}{y} \\
x^2 &= y^2
\end{aligned}
$$

which implies $x = \pm y$. Substituting this into (3) gives

$$x^2 = a^2$$

yielding $\left(a, a, \frac{1}{2}\right), \left(a, -a, -\frac{1}{2}\right), \left(-a, a, -\frac{1}{2}\right), \left(-a, -a, \frac{1}{2}\right)$ as solutions. At these points the constraint qualification is met: for $g(x, y) = x^2 + y^2 - 2a^2$, $\nabla g(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ and $\nabla g(\bar{x}, \bar{y}) = \begin{bmatrix} \pm 2a \\ \pm 2a \end{bmatrix}$ or $\begin{bmatrix} \pm 2a \\ \mp 2a \end{bmatrix}$ which is of full rank when $a \neq 0$.

The second order conditions will help to determine whether these are necessary conditions for a maximizer or a minimizer at the different values of $\bar{x}, \bar{y}$. We have to know whether $D^2\mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})$ is negative/positive definite (which is a sufficient condition for a constrained local maximizer/minimizer) or semi-definite (which is instead a necessary condition) on the subspace $Z = \{z \in \mathbb{R}^2 : \nabla g(\bar{x}, \bar{y}, \bar{\lambda}) \cdot z = 0\}$.

To determine this, the bordered Hessian is required:

$$A_k^\pi = \begin{bmatrix} 0 & \nabla g(\bar{x}, \bar{y})_k^\pi \\ (\nabla g(\bar{x}, \bar{y})_k^\pi)^T & {}_k D^2\mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})_k^\pi \end{bmatrix}$$

And $D^2\mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})$ is negative definite on $Z = \{z \in \mathbb{R}^2 : \nabla g(\bar{x}, \bar{y}, \bar{\lambda}) \cdot z = 0\}$ if and only if $(-1)^k |A_k| > 0$ for every $k = m+1, \ldots, n$ where m is the number of constraints and n is the number of inputs. For positive definiteness, the factor $(-1)^k$ is replaced with $(-1)^m$.

If it turns out that it could be negative semi-definite, then we have the privilege of verifying this by checking that $(-1)^k |A_k^\pi \geq 0$ for every $k = m+1, \ldots, n$ and for every permutation π of $\{1, \ldots, n\}$, and likewise replacing $(-1)^k$ with $(-1)^m$ for positive semi-definiteness.

For this problem,

$$\begin{aligned} A &= \begin{bmatrix} 0 & 2\bar{x} & 2\bar{y} \\ 2\bar{x} & -2\bar{\lambda} & 1 \\ 2\bar{y} & 1 & -2\bar{\lambda} \end{bmatrix} \\ |A| &= 0 - 2\bar{x} \det \begin{bmatrix} 2\bar{x} & 1 \\ 2\bar{y} & -2\bar{\lambda} \end{bmatrix} + 2\bar{y} \det \begin{bmatrix} 2\bar{x} & -2\bar{\lambda} \\ 2\bar{y} & 1 \end{bmatrix} \\ &= -2\bar{x}(-4\bar{x}\bar{\lambda} - 2\bar{y}) + 2\bar{y}(2\bar{x} + 4\bar{y}\bar{\lambda}) \\ &= 8\bar{x}^2\bar{\lambda} + 4\bar{x}\bar{y} + 4\bar{x}\bar{y} + 8\bar{y}^2\bar{\lambda} \\ &= 8(\bar{x}^2\bar{\lambda} + \bar{x}\bar{y} + \bar{y}^2\bar{\lambda}) \end{aligned}$$

For $\left(a, a, \frac{1}{2}\right)$, $|A| > 0$; for $\left(a, -a, -\frac{1}{2}\right)$, $|A| < 0$; for $\left(-a, a, -\frac{1}{2}\right)$, $|A| < 0$; and for $\left(-a, -a, \frac{1}{2}\right)$, $|A| > 0$. Since the range of values for k is just $m+1 = n = 2$, and $m = 1$, this means the condition for negative definiteness is $|A| > 0$ and the condition for positive definiteness is $|A| < 0$. Therefore, (a, a) and $(-a, -a)$ are constrained local (as well as global) maximizers and $(a, -a)$ and $(-a, a)$ are constrained local (as well as global) minimizers.

(b)

The constraint for this problem is $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. With $\nabla g(x, y, z) = \begin{bmatrix} -\frac{1}{x^2} \\ -\frac{1}{y^2} \\ -\frac{1}{z^2} \end{bmatrix}$, this means the elements in the gradient vector cannot be 0, therefore the constraint qualification is met.

However, there is no global constrained maximum or minimum for this problem. The constraint set is not compact and $f(x, y, z)$ can be made arbitrarily large by increasing x indefinitely. The larger x is, the less an increase in x violates the constraint, and thus the constraint can be satisfied after an increase in x by a less than proportionate decrease in y or z, with the end result being a higher value of $f(x, y, z)$ while still meeting the constraint. The same argument can be used to make $f(x, y, z)$ arbitrarily small by making x negative. Therefore a global optimum does not exist whether or not one uses the Lagrange method.

This can be seen by substituting the budget constraint into the objective function:

$$\begin{aligned}
\frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= 1 \\
xy+xz+yz &= xyz \\
x &= \frac{yz}{yz-y-z} \\
f\left(\frac{yz}{yz-y-z}, y, z\right) &= \frac{yz}{yz-y-z}+y+z \\
&= \frac{yz}{(z-1)y-z}+y+z
\end{aligned}$$

Holding z constant at some level not equal to 0 (the budget constraint is undefined at $z=0$), the objective function can be made arbitrarily large or small by setting some $y=\frac{z-\epsilon}{z-1}$ or $y=\frac{z+\epsilon}{z-1}$ which makes $\frac{yz}{(z-1)y-z}$ tend to $\pm\infty$ as $\epsilon \to 0$ while still fulfilling the constraint since it has been substituted inside; z is held constant while y is made arbitrarily close to $\frac{z\pm\epsilon}{z-1}$, and x adjusts to satisfy the constraint.

Still, to find the local optima, the Lagrange function for this problem is

$$\mathcal{L}(x, y, \lambda) = x+y+z-\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

and the critical points for this function are where

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 1+\frac{\lambda}{x^2}=0 \\
\frac{\partial \mathcal{L}}{\partial y} &= 1+\frac{\lambda}{y^2}=0 \\
\frac{\partial \mathcal{L}}{\partial z} &= 1+\frac{\lambda}{z^2}=0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 1-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=0
\end{aligned}$$

From the first 3 conditions, it is apparent that $\frac{x^2}{y^2}$, $\frac{x^2}{z^2}$, and $\frac{y^2}{z^2}$ are all equal to 1, meaning x, y, and z have the same magnitude (but possibly with different signs).

Case 1: x, y, and z all have the same sign

$$\begin{aligned}
x &= y = z \\
\frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= \frac{3}{x} = 1 \\
x &= y = z = 3
\end{aligned}$$

Case 2: x has a different sign from y and z

$$\begin{aligned}
x &= -y = -z \\
\frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= -\frac{1}{x} = 3 \\
x &= -1, y = z = 1
\end{aligned}$$

Case 3: y has a different sign from x and z. With the same reasoning in case 2,

$$y = -1, x = z = 1$$

Case 4: z has a different sign from x and y

$$z = -1, x = y = 1$$

and the critical points of the Lagrangian (x, y, z, λ) are $(3, 3, 3, -9)$, $(-1, 1, 1, -1)$, $(1, -1, 1, -1)$, and $(1, 1, -1, -1)$.

To assemble the bordered Hessian,

$$\begin{aligned} D^2 f(x,y,z) &= 0 \\ D^2 g(x,y,z) &= \begin{bmatrix} \frac{2}{x^3} & 0 & 0 \\ 0 & \frac{2}{y^3} & 0 \\ 0 & 0 & \frac{2}{z^3} \end{bmatrix} \\ D^2_{x,y}\mathcal{L}(x,y,z,\lambda) &= \begin{bmatrix} -\frac{2\lambda}{x^3} & 0 & 0 \\ 0 & -\frac{2\lambda}{y^3} & 0 \\ 0 & 0 & -\frac{2\lambda}{z^3} \end{bmatrix} \end{aligned}$$

and the bordered Hessian H is

$$\begin{bmatrix} 0 & -\frac{1}{x^2} & -\frac{1}{y^2} & -\frac{1}{z^2} \\ -\frac{1}{x^2} & -\frac{2\lambda}{x^3} & 0 & 0 \\ -\frac{1}{y^2} & 0 & -\frac{2\lambda}{y^3} & 0 \\ -\frac{1}{z^2} & 0 & 0 & -\frac{2\lambda}{z^3} \end{bmatrix}$$

If at some critical point $(-1)^k|H_k| > 0$ for $k = 2, 3$ (meaning $|H_2| > 0, |H_3| < 0$), then that point is a constrained local maximizer. If $|H_k| < 0$ for $k = 2, 3$, then that point is a constrained local minimizer.

(Apologies, the following algebra is very messy but the numerical results are verified when I define the matrices and variables in GeoGebra and use it to calculate the determinants for me).

$$|H_2| = \begin{vmatrix} 0 & -\frac{1}{x^2} & -\frac{1}{y^2} \\ -\frac{1}{x^2} & -\frac{2\lambda}{x^3} & 0 \\ -\frac{1}{y^2} & 0 & -\frac{2\lambda}{y^3} \end{vmatrix} = \frac{1}{x^2} \cdot \frac{2\lambda}{x^2 y^3} - \frac{1}{y^2}\left(-\frac{2\lambda}{x^3 y^2}\right) = \frac{2\lambda}{x^4 y^3} + \frac{2\lambda}{x^3 y^4}$$

$$\begin{aligned} |H_3| = \begin{vmatrix} 0 & -\frac{1}{x^2} & -\frac{1}{y^2} & -\frac{1}{z^2} \\ -\frac{1}{x^2} & -\frac{2\lambda}{x^3} & 0 & 0 \\ -\frac{1}{y^2} & 0 & -\frac{2\lambda}{y^3} & 0 \\ -\frac{1}{z^2} & 0 & 0 & -\frac{2\lambda}{z^3} \end{vmatrix} &= \frac{1}{z^2}\begin{vmatrix} -\frac{1}{x^2} & -\frac{1}{y^2} & -\frac{1}{z^2} \\ -\frac{2\lambda}{x^3} & 0 & 0 \\ 0 & -\frac{2\lambda}{y^3} & 0 \end{vmatrix} - \frac{2\lambda}{z^3}\begin{vmatrix} 0 & -\frac{1}{x^2} & -\frac{1}{y^2} \\ -\frac{1}{x^2} & -\frac{2\lambda}{x^3} & 0 \\ -\frac{1}{y^2} & 0 & -\frac{2\lambda}{y^3} \end{vmatrix} \\ &= \frac{1}{z^2} \cdot \frac{2\lambda}{x^3}\left(-\frac{2\lambda}{y^3 z^2}\right) - \frac{2\lambda}{z^3}\left[\frac{1}{x^2} \cdot \frac{2\lambda}{x^2 y^3} - \frac{1}{y^2}\left(-\frac{2\lambda}{x^3 y^2}\right)\right] \\ &= -\frac{4\lambda^2}{x^3 y^3 z^4} - \frac{2\lambda}{z^3}\left(\frac{2\lambda}{x^4 y^3} + \frac{2\lambda}{x^3 y^4}\right) \\ &= -\left(\frac{4\lambda^2}{x^4 y^3 z^3} + \frac{4\lambda^2}{x^3 y^4 z^3} + \frac{4\lambda^2}{x^3 y^3 z^4}\right) \end{aligned}$$

For $(3, 3, 3, -9)$, $|H_2| = \frac{-36}{3^7} < 0$ and $|H_3| < 0$ (since x, y, and z are all positive, meaning the bracketed expression in $|H_3|$ is always positive). Thus $(3, 3, 3)$ is a local minimizer.

For any of the other points where the absolute value of x, y, and z are all equal to 1 and where one of the arguments has a different sign from the rest, $|H_2| = \frac{2\lambda}{x^4y^3} + \frac{2\lambda}{x^3y^4} \leq 0$ since $\lambda = -1 < 0$ in all the critical points, and x^3y^4 and x^4y^3 are of opposite signs if either x or y are negative (in which case $|H_2| = 0$) or are both equal to 1 if z is negative (in which case $|H_2| = -4 < 0$). Thankfully we don't have to check for semi-definiteness; applying a similar reasoning, $|H_3| > 0$ for any of the three remaining points which precludes any of these points from satisfying the second-order conditions.