Microeconometrics
Supervision 3

Samuel Lee

Question 1

(a)

Letting Y_i be individual i's choice, we have

$$\Pr[Y_i = j] = \Pr\left[\max_{k \in \{1,\ldots,J\}} U_{ik} = U_{ij}\right] = \Pr\left[\prod_{\substack{k=1 \\ k \neq j}}^{J} \mathbb{1}_{\{U_{ij} - U_{ik} > 0\}} = 1\right]$$

This implies that the signs of the differences in utility are sufficient to determine choice.

(b)

With two choices, we have

$$\begin{aligned}
\Pr[Y_i = 1 \mid \boldsymbol{\omega}, \boldsymbol{\alpha}, \sigma^2] &= \Pr[\alpha_1 + \boldsymbol{v}_{i1}'\boldsymbol{\omega} + \varepsilon_{i1} > \alpha_2 + \boldsymbol{v}_{i2}'\boldsymbol{\omega} + \varepsilon_{i2} \mid \boldsymbol{\omega}, \boldsymbol{\alpha}, \sigma^2] \\
&= \Pr\left[\frac{\varepsilon_{i1} - \varepsilon_{i2}}{\sigma\sqrt{2}} > \frac{(\alpha_2 - \alpha_1) + (\boldsymbol{v}_{i2} - \boldsymbol{v}_{i1})'\boldsymbol{\omega}}{\sigma\sqrt{2}} \,\middle|\, \boldsymbol{\omega}, \boldsymbol{\alpha}, \sigma^2\right] \\
&= 1 - \Phi\left[\frac{(\alpha_2 - \alpha_1) + (\boldsymbol{v}_{i2} - \boldsymbol{v}_{i1})'\boldsymbol{\omega}}{\sigma\sqrt{2}}\right] \\
&= \Phi\left[\frac{(\alpha_1 - \alpha_2) + (\boldsymbol{v}_{i1} - \boldsymbol{v}_{i2})'\boldsymbol{\omega}}{\sigma\sqrt{2}}\right]
\end{aligned}$$

since $\frac{\varepsilon_{i1} - \varepsilon_{i2}}{\sigma\sqrt{2}} \sim N(0,1)$. Likewise we have $\Pr[Y_i = 2 \mid \boldsymbol{\omega}, \boldsymbol{\alpha}, \sigma^2] = \Phi\left[\frac{(\alpha_2 - \alpha_1) + (\boldsymbol{v}_{i2} - \boldsymbol{v}_{i1})'\boldsymbol{\omega}}{\sigma\sqrt{2}}\right]$.

Given a vector of observed choices $\boldsymbol{y} = \begin{pmatrix} y_1 & \ldots & y_n \end{pmatrix}'$ and letting $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}'$, the log-likelihood function is

$$\begin{aligned}
\ell(\boldsymbol{\omega}, \boldsymbol{\alpha}, \sigma^2 \mid \boldsymbol{y}) = \sum_{i=1}^{n} \Bigg\{ &\mathbb{1}_{\{y_i = 1\}} \times \log \Phi\left[\frac{(\alpha_1 - \alpha_2) + (\boldsymbol{v}_{i1} - \boldsymbol{v}_{i2})'\boldsymbol{\omega}}{\sigma\sqrt{2}}\right] \\
&+ \mathbb{1}_{\{y_i = 2\}} \times \log \Phi\left[\frac{(\alpha_2 - \alpha_1) + (\boldsymbol{v}_{i2} - \boldsymbol{v}_{i1})'\boldsymbol{\omega}}{\sigma\sqrt{2}}\right] \Bigg\}
\end{aligned}$$

(c)

From the log-likelihood function, we can see that any $(\boldsymbol{\omega}, \boldsymbol{\alpha}, \sigma^2)$ is observationally equivalent to $(\lambda\boldsymbol{\omega}, \lambda\boldsymbol{\alpha}, \lambda^2\sigma^2)$ for some positive scalar λ.

(d)

We could interpret α_j as a product-specific attribute which is equally valued by all individuals. For example, with no income effects, α_j could be the price of some product j. The likelihood function only depends on α_1 and α_2 through their difference, so any $\boldsymbol{\alpha}$ and $\lambda\boldsymbol{\alpha}$ are observationally equivalent for $\lambda > 0$.

(e)

The probability of individual i choosing j is now

$$\begin{aligned}
&\Pr[Y_i = j \mid \boldsymbol{\beta}, \boldsymbol{\alpha}, \sigma^2] \\
=&\Pr[(\alpha_j - \alpha_k) + \boldsymbol{x}_i'(\boldsymbol{\beta}_j - \boldsymbol{\beta}_k) + (\varepsilon_{ij} - \varepsilon_{ik}) > 0 \; \forall\, k \in \Omega_J,\; k \neq j \mid \boldsymbol{\beta}, \boldsymbol{\alpha}, \sigma^2] \\
=&\Pr\left[\frac{\varepsilon_{ik} - \varepsilon_{ij}}{\sigma\sqrt{2}} < \frac{(\alpha_k - \alpha_j) + x_i'(\boldsymbol{\beta}_k - \boldsymbol{\beta}_j)}{\sigma\sqrt{2}} \; \forall\, k \in \Omega_J,\; k \neq j \;\middle|\; \boldsymbol{\beta}, \boldsymbol{\alpha}, \sigma^2\right]
\end{aligned}$$

where $\boldsymbol{\beta} = \begin{pmatrix}\boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_J\end{pmatrix}$. The likelihood function will be a function of the above, which only depends on $\boldsymbol{\beta}$, $\boldsymbol{\alpha}$, and σ^2 through the ratios $\frac{(\alpha_k - \alpha_j)}{\sigma}$ and $\frac{(\boldsymbol{\beta}_k - \boldsymbol{\beta}_j)}{\sigma}$. As long as we assume a strictly monotonic cdf for ε_{ij}, then the parameters are identified down to the two ratios above. This also means that when we normalise σ and some α_i and $\boldsymbol{\beta}_i$, the other parameters are identified.

Question 2

(a)

Again, we observe $\boldsymbol{y} = \begin{pmatrix}y_1 & \cdots & y_n\end{pmatrix}$ which is a vector of individual choices taking values in Ω_J. We assume ε_{ij} is i.i.d. with a zero-mean Gumbel distribution. Conditional on $\boldsymbol{\omega}$ and $\boldsymbol{v}_{ij}'$, we have

$$\begin{aligned}
\Pr[Y_i = j] &= \Pr[U_{ij} > U_{ik} \; \forall k \in \Omega_J,\; k \neq j] \\
&= \Pr[\boldsymbol{v}_{ij}'\boldsymbol{\omega} + \varepsilon_{ij} > \boldsymbol{v}_{ik}'\boldsymbol{\omega} + \varepsilon_{ik} \; \forall\, k \in \Omega_J,\; k \neq j] \\
&= \Pr\left[\frac{\varepsilon_{ik}}{\sigma} < \frac{(\boldsymbol{v}_{ij} - \boldsymbol{v}_{ik})'\boldsymbol{\omega} + \varepsilon_{ij}}{\sigma} \; \forall\, k \in \Omega_J,\; k \neq j\right] \\
&= \int_{-\infty}^{\infty} f(\varepsilon_{ij}) \prod_{\substack{k=1 \\ k \neq j}}^{J} F\left[\frac{(\boldsymbol{v}_{ij} - \boldsymbol{v}_{ik})'\boldsymbol{\omega} + \varepsilon_{ij}}{\sigma}\right] \, \mathrm{d}\varepsilon_{ij}
\end{aligned}$$

where f and F are the pdf and cdf of a standard Gumbel distribution. It is possible to play around with the integral to get the following

$$\Pr[Y_i = j] = \frac{e^{\boldsymbol{v}_{ij}' \frac{\boldsymbol{\omega}}{\sigma}}}{\sum_{k=1}^{J} e^{\boldsymbol{v}_{ik}' \frac{\boldsymbol{\omega}}{\sigma}}}$$

The above is not uniquely determined by values of $(\boldsymbol{\omega}, \sigma^2)$; any such point in the parameter space yields the same result as $(\lambda\boldsymbol{\omega}, \lambda^2\sigma^2)$ for $\lambda > 0$. Given $\boldsymbol{y}$, the likelihood function is

$$L(\boldsymbol{\omega}, \sigma^2) = \prod_{i=1}^{n} \Pr[Y_i = y_i]$$

so $(\boldsymbol{\omega}, \sigma^2)$ is observationally equivalent to $(\lambda\boldsymbol{\omega}, \lambda^2\sigma^2)$ for $\lambda > 0$ and $\boldsymbol{\omega}$ and σ are not separately identifiable. Since the cdf of $\frac{\varepsilon_{ik}-\varepsilon_{ij}}{\sigma\sqrt{2}}$ is monotonically increasing, $\boldsymbol{\omega}^*$ is identifiable.

(b)

i. We have the expression derived before. The parameter $\boldsymbol{\omega}$ is not individual- or product-specific, and individual choices are independent, so the likelihood function given a subsample of observations is proportional to that from the whole sample, and we can estimate $\boldsymbol{\omega}$ by maximum likelihood which is consistent under some regularity conditions including concavity of the likelihood function. Provided the model above is the true data generating process, the maximum likelihood estimate of $\boldsymbol{\omega}$ based on the whole sample and that based on a subsample both converge to the same limit. This has implications for the second part.

ii. As mentioned, the two estimators will converge to the same limit if the specification above is true. So we can employ a Hausman-like test using some function of $\hat{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}$ as a test statistic, where $\hat{\boldsymbol{\omega}}$ is the efficient estimator and $\tilde{\boldsymbol{\omega}}$ is the inefficient estimator, to check if there is truly no individual- or product-specific heterogeneity in $\boldsymbol{\omega}$.

(c)

i. We might want the variance of the errors to differ across individuals since some people may be richer than others, and we might expect the variance of the errors to be proportional to income. Or one unobserved factor that may lead to differences in σ^2 might be how seriously each individual takes the preference elicitation method.

ii.