Mathematical Economics
Supervision 3

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Question 1

(a)

As per Gul and Pesendorfer (2001), $u(x_b, x_s)$ represents his "commitment ranking" of options or "commitment utility". This is essentially the utility he gets from any option $(\bar{x}_b, \bar{x}_s) \in \mathbb{R}^2_{>0}$ without temptation. If $(\bar{x}_b, \bar{x}_s)$ is the only member of some set A, there is no temptation and his utility over A is

$$
\begin{aligned}
U(A) &= \max_{x_b, x_s \in A} \{u(x_b, x_s) + v(x_b, x_s)\} - \max_{y_b, y_s \in A} \{v(y_b, y_s)\} \\
&= u(\bar{x}_b, \bar{x}_s) + v(\bar{x}_b, \bar{x}_s) - v(\bar{x}_b, \bar{x}_s) = u(\bar{x}_b, \bar{x}_s)
\end{aligned}
$$

In this case, his preferences under self-control are characterized by a bliss point at $x_b = 2$, $x_s = 2$, and any deviation from this point lowers his utility. His committed self would like to consume on the point within his budget set closest to this point.

$v(x_b, x_s)$ represents the "temptation ranking" of options or "temptation utility" as it captures the disutility from resisting a tempting choice. In Gul and Pesendorfer (2001) $u(x) - v(y)$ is termed the "(utility) cost of self-control", where x is the choice that maximizes $u(x) + v(x)$ and y is the choice that maximizes $v(y)$. The function is set up such that we can identify temptation if removing an item from the choice set yields a higher utility over that set:

$$
\begin{aligned}
&\text{if } u(x) > u(y) \text{ and } v(x) < v(y), \\
&U(\{x, y\}) = u(y) + v(y) - v(y) < u(x) \\
\text{or } &U(\{x, y\}) = u(x) + v(x) - v(y) < u(x)
\end{aligned}
$$

In this case, the preferences of his "myopic" self are Cobb-Douglas with equal utility elasticities in burgers and salads. The tempted side of this footballer would like to spend all of his income and split it equally between burgers and salads.

(b)

His consumption will be the number of burgers and salads that maximizes $u(x_b, x_s) + v(x_b, x_s)$ subject to $x_b \geq 0, x_s \geq 0, x_b + x_s \leq m$. Everything in this problem is symmetric with respect to burgers and salads, so the amounts consumed of both are probably equal (but we'll see). The Lagrangian for this problem is

$$
\mathcal{L}(x_b, x_s, \lambda) = 10 - (x_b - 2)^2 - (x_s - 2)^2 + x_b x_s - \lambda(x_b + x_s - m)
$$

and the Kuhn-Tucker conditions that characterize a solution are

$$\frac{\partial \mathcal{L}}{\partial x_b} = -2(x_b - 2) + x_s - \lambda \le 0 \ (=0 \text{ if } x_b > 0) \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_s} = -2(x_s - 2) + x_b - \lambda \le 0 \ (=0 \text{ if } x_s > 0) \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - x_b - x_s \ge 0, \ \lambda \ge 0, \ \lambda(x_b + x_s - m) = 0$$

Obviously not all the constraints can be binding at once, so we can skip that case.

- Case 1: no constraints are binding.

  In this case $\lambda = 0$ and (1) and (2) are both equalities.

$$\begin{aligned} x_s &= 2(x_b - 2) \\ x_b &= 2(x_s - 2) \end{aligned}$$

  which implies $x_b = x_s = 4$. These values are valid only if $m \ge 8$.

- Case 2: only $x_b \ge 0$ is binding.

  In this case, $\lambda = x_b = 0$ and (2) is an equality. (2) suggests $x_s = 2$ but this violates (1). Thus this cannot be solution. The same argument would apply for $x_s > 0$ since the problem is symmetric for the two goods, so we can skip that.

- Case 3: only the budget constraint is binding.

  In this case, $x_b + x_s = m$ and (1) and (2) are equalities.

$$\begin{aligned} x_s - 2(x_b - 2) &= x_b - 2(x_s - 2) \\ x_s - 2x_b + 4 &= x_b - 2x_s + 4 \\ x_b &= x_s \end{aligned}$$

  and this just implies his income is split equally among burgers and salads. This will only be valid for $m \le 8$ since

$$\begin{aligned} \lambda &= x_s - 2(x_b - 2) \\ &= \frac{m}{2} - 2\left(\frac{m}{2} - 2\right) \\ &= 4 - \frac{m}{2} \end{aligned}$$

  which can only be non-negative when $m \le 8$. It is worth noting that for $m \le 4$, there is no confict between self-control and temptation: the footballer's committed side wishes to consume as close to $(2, 2)$ as possible, and the way to do so if he doesn't have enough money to get to that point is to split his entire income between the two goods. The footballer's tempted side wishes to do the same, and would do so for any amount of income.

  At this point we have possible solutions for all possible ranges of m, and it really seems like all the other cases where 2 constraints are binding would result in a violation of some sort, especially since they represent choices that neither the footballer's committed self nor his myopic self would prefer. Still,

- Case 3: only the budget constraint is non-binding.

  This results in $4 \leq 0$.

- Case 4: only the budget constraint and $x_b \geq 0$ are binding.

  Here, $x_s = m$ and (2) is an equality.

$$
\begin{aligned}
-2(m-2) - \lambda &= 0 \\
\lambda &= 2(2-m)
\end{aligned}
$$

  which is valid only if $m \leq 2$. But if we substitute this into (1),

$$m + 4 - 2(2-m) = 3m \leq 0$$

  which violates the initial assumptions unless $m = 0$ in which case the budget constraint is violated. This will also apply for $x_b > 0$.

So his optimal choice given m is

$$
(x_b^*, x_s^*) = \begin{cases} \left(\frac{m}{2}, \frac{m}{2}\right) & \text{if } m \leq 8 \\ (4,4) & \text{if } m > 8 \end{cases}
$$

We know that without temptation he would prefer to consume as close to $(2,2)$ as possible. As mentioned before, for $m \leq 4$ there is no struggle between temptation and self-control. However, as the footballer becomes richer, the tempted side of him just wants to spend all his money on burgers and salads equally and he continues splitting his income past the bliss point.

The forces here are symmetric: the footballer is tempted to consume as many burgers and salads as possible, and he finds greatest pleasure in consuming equal amounts of both. On the other hand, his committed self recognizes that increasing his consumption beyond the bliss point is least painful when he does it in equal proportions. Thus the optimal choice is still $\frac{m}{2}$. At $(4,4)$ the pain of further breaking his commitment becomes too great to be overcome by temptation.

One hint that the footballer is struggling with temptation is that he consumes more burgers and salads than he would if he were able to commit himself to maximizing just $u(x_b, x_s)$. However, in practice this could be conflated with a change in preferences over time. Theoretically we know there is temptation if he achieves a higher utility when certain options are taken off his menu; he violates the standard theory of revealed preferences in this case. If, instead of having m as income, he had a voucher that could only be exchanged for 2 burgers and 2 salads, he would be better off (and this can be shown in the next part where we calculate the utility he gains from income m).

(c)

His utility over a budget set with income m is

$$U(m) = \max_{x_b, x_s \in BS(m)} \{u(x_b, x_s) + v(x_b, x_s)\} - \max_{y_b, y_s \in BS(m)} \{v(y_b, y_s)\}$$

As previously discussed, the footballer is tempted to spend his entire income and split it equally between burgers and salads. So $\max_{y_b, y_s \in BS(m)} \{v(y_b, y_s)\} = \frac{m^2}{4}$, and his total utility is

$$
U(m) = \begin{cases} 10 - 2\left(\frac{m-4}{2}\right)^2 & \text{if } m \leq 8 \\ 2 & \text{if } m > 8 \end{cases}
$$

We can see that his utility is strictly increasing over $0 \leq m \leq 4$, strictly decreasing over $4 < m \leq 8$, and stays constant thereafter. So a poor footballer with less than £4 would do well with more income. Once he becomes a rich football star with a princely sum of £4 in his bank account, any more income leaves him worse off by tempting him away from his optimal allocation under self-control.

And we can briefly note that his utility with a voucher that is redeemable for 2 burgers and 2 salads as mentioned before would simply be $u(2,2) = 10$, which is weakly better than whatever he enjoys with $m \geq 4$ even though the money value of the voucher is less than or equal to m. This suggests he struggles with temptation over that range of income.

Question 2

(a)

Assuming the utility function is quasilinear in t, the regulator needs to maximize $\sqrt{q} - t$ subject to $t = \frac{\theta_i}{2} q$, $q \geq 0$. The inequality constraint is non-binding since the value function satisfies the Inada conditions while the cost function is linear; it is always optimal to produce the good. Thus first-order condition for a solution is an equality:

$$\begin{aligned} \frac{1}{2\sqrt{q}} &= \frac{\theta_i}{2} \\ q &= \frac{1}{\theta_i^2} \end{aligned}$$

and the optimal solution is $\left\{ \left(\frac{1}{2\theta_i}, \frac{1}{\theta_i^2} \right) \right\}_{i=1,2}$. The firm is paid its costs in full and this is the first-best scenario for the regulator.

(b)

The regulator now seeks to maximize the expected utility

$$p_1(\sqrt{q_1} - t_1) + (1 - p_1)(\sqrt{q_2} - t_2)$$

It cannot force any firm to make a loss, hence it is subject to

$$t_i - \frac{\theta_i}{2} q_i \geq 0 \ \forall i \in [1, 2]$$

and from the revelation principle we can restrict ourselves to mechanisms which are incentive-compatible, such that

$$t_i - \frac{\theta_i}{2} q_i \geq t_j - \frac{\theta_i}{2} q_j \ \forall i, j \in [1, 2]$$

Thus the Lagrangian for this problem is

$$\begin{aligned} \mathcal{L} = p_1(\sqrt{q_1} - t_1) + (1 - p_1)(\sqrt{q_2} - t_2) + \lambda_1 \left(t_1 - \frac{\theta_1}{2} q_1 \right) + \lambda_2 \left(t_2 - \frac{\theta_2}{2} q_2 \right) \\ + \mu_1 \left[t_1 - t_2 + \frac{\theta_1}{2}(q_2 - q_1) \right] + \mu_2 \left[t_2 - t_1 + \frac{\theta_2}{2}(q_1 - q_2) \right] \end{aligned}$$

and the Kuhn-Tucker conditions are (the first two multiplied by 2 to get rid of the ugly fractions)

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{p_1}{\sqrt{q_1}} - \theta_1\lambda_1 - \theta_1\mu_1 + \theta_2\mu_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{1-p_1}{\sqrt{q_2}} - \theta_2\lambda_2 + \theta_1\mu_1 - \theta_2\mu_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial t_1} = -p_1 + \lambda_1 + \mu_1 - \mu_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial t_2} = -(1-p_1) + \lambda_2 - \mu_1 + \mu_2 = 0$$

λ_1 can be set equal to 0: the first participation constraint already follows from the first incentive constraint and the second participation constraint. Then the third equation implies that $\mu_1 = \mu_2 + p_1 > 0$, meaning the first incentive constraint is binding. To restate what is known so far:

$$\frac{p_1}{\sqrt{q_1}} - \theta_1\mu_1 + \theta_2\mu_2 = 0 \tag{1}$$
$$\frac{1-p_1}{\sqrt{q_2}} - \theta_2\lambda_2 + \theta_1\mu_1 - \theta_2\mu_2 = 0 \tag{2}$$
$$-p_1 + \mu_1 - \mu_2 = 0 \tag{3}$$
$$-(1-p_1) + \lambda_2 - \mu_1 + \mu_2 = 0 \tag{4}$$
$$t_1 - \frac{\theta_1}{2}q_1 = t_2 - \frac{\theta_1}{2}q_2 \tag{5}$$

If the second incentive constraint were also binding, substituting one into the other and working through the first-order conditions would imply $q_1 = q_2 = 0$ where (1) and (2) are undefined. Even if the utility function took a different functional form such that we did not have this problem, adding (1) and (2) would show that $\lambda_2 > 0$ and $t_1 = t_2 = 0$ as long as $S'(q) > 0$. This surely cannot be the optimum: even just offering the contract one would offer to the less efficient firm in the first-best case would lead to a better result whether or not the firm has the opportunity to lie.

In that case we can assume the second incentive constraint is not binding, meaning $\mu_2 = 0$. Adding (3) and (4) then yields $\lambda_2 = 1 > 0$ and the second participation constraint is binding. (3) also means $p_1 = \mu_1$, and thus

$$\frac{\mu_1}{\sqrt{q_1}} - \theta_1\mu_1 = 0$$

and since we know μ_1 is non-zero

$$\frac{1}{\sqrt{q_1}} = \theta_1$$

Similarly, for a type 2 firm,

$$\frac{1-p_1}{\sqrt{q_2}} - \theta_2 + \theta_1\mu_1 = 0$$
$$\frac{1}{\sqrt{q_2}} = \frac{\theta_2 - \theta_1 p_1}{1-p_1}$$
$$= \theta_2 + \frac{p_1}{1-p_1}\Delta\theta$$

Which are the conditions for marginal benefit to be equal to marginal (virtual) cost. The optimal q can be found by squaring the reciprocals of these expressions, yielding $q_1^* = \frac{1}{\theta_1^2}$ as before and $q_2^* = \left(\frac{1-p_1}{\theta_2-\theta_1 p_1}\right)^2$ which is less than before. This is because there is no incentive for the less efficient firm to pretend to be more efficient: it would have to produce more with a smaller payment per unit produced. Therefore there is no need to distort the optimal quantity for the most efficient firm. Furthermore the quantity offered to the less efficient firm has to be distorted downward such that it will not be profitable for the more efficient firm to lie. If q_2^* is not low enough firm 1 could make a higher profit by receiving more payment per unit produced even though it will produce less in total.

The first incentive constraint and second participation constraint are binding, thus

$$\begin{aligned} t_2^* &= \frac{\theta_2}{2} q_2^* \\ t_1^* - \frac{\theta_1}{2} q_1^* &= t_2^* - \frac{\theta_1}{2} q_2^* \\ &= \frac{\theta_2}{2} q_2^* - \frac{\theta_1}{2} q_2^* \\ t_1^* &= \theta_1 q_1^* + \frac{\Delta\theta}{2} q_2^* \end{aligned}$$

In this case the less efficient firm still does not make any supernormal profits, while the more efficient firm now receives the information rent of $\Delta\theta q_2^*$. This is because the more efficient firm has to be paid more not to pretend to be less efficient.

As p_1 goes to 1, q_2^* goes to 0 and t_1^* approaches the first-best case. This is because there is practically no asymmetry of information as p_1 approaches 1, and furthermore the regulator knows the firm will almost surely be the more efficient firm, and there is no reason to offer another contract or pay any more than the marginal cost.

(c)

There is now an additional constraint that the firm will not reject the menu. It is always better to offer some menu which the firm will not reject since the utility function satisfies the Inada conditions. Specifically, the marginal utility to society approaches infinity as q approaches 0; there is always some menu which is worth offering compared to one that will be rejected from the start. Furthermore, we assume that the firm must choose a contract if it has already accepted the menu. Thus the former participation constraints don't apply. So the regulator seeks to maximize the expected utility

$$p_1(\sqrt{q_1} - t_1) + (1-p_1)(\sqrt{q_2} - t_2)$$

subject to

$$t_i - \frac{\theta_i}{2} q_i \geq t_j - \frac{\theta_i}{2} q_j \ \forall i,j \in [1,2]$$

If the firm is risk-neutral, where the utility of its expected returns is equal to the expected utility from accepting the menu, it will not reject the menu if

$$p_1 \left(t_1 - \frac{\theta_1}{2} q_1\right) + (1-p_1)\left(t_2 - \frac{\theta_2}{2} q_2\right) \geq 0$$

And the Lagrangian for this problem is

$$\mathcal{L} = p_1(\sqrt{q_1} - t_1) + (1-p_1)(\sqrt{q_2} - t_2) + \lambda\left[p_1\left(t_1 - \frac{\theta_1}{2}q_1\right) + (1-p_1)\left(t_2 - \frac{\theta_2}{2}q_2\right)\right]$$
$$+\mu_1\left[t_1 - t_2 + \frac{\theta_1}{2}(q_2 - q_1)\right] + \mu_2\left[t_2 - t_1 + \frac{\theta_2}{2}(q_1 - q_2)\right]$$

And the Kuhn-Tucker conditions are (again the first two are multiplied by 2)

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{p_1}{\sqrt{q_1}} - \lambda p_1\theta_1 - \mu_1\theta_1 + \mu_2\theta_2 = 0 \tag{6}$$
$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{1-p_1}{\sqrt{q_2}} - \lambda(1-p_1)\theta_2 + \mu_1\theta_1 - \mu_2\theta_2 = 0 \tag{7}$$
$$\frac{\partial \mathcal{L}}{\partial t_1} = -p_1 + \lambda p_1 + \mu_1 - \mu_2 = 0 \tag{8}$$
$$\frac{\partial \mathcal{L}}{\partial t_2} = -(1-p_1) + \lambda(1-p_1) - \mu_1 + \mu_2 = 0 \tag{9}$$

As in (b), adding (8) and (9) yields $\lambda = 1 > 0$. So we know the ex-ante participation constraint is binding; this is ideal for the regulator since offering a higher expected return lowers the utility to society. Now the last two equations imply $\mu_1 = \mu_2$. So we can restate the conditions and thereafter consider the cases where $\mu := \mu_1 = \mu_2 = 0$ or $\mu > 0$.

$$p_1\left(\frac{1}{\sqrt{q_1}} - \theta_1\right) + \mu\Delta\theta = 0$$
$$(1-p_1)\left(\frac{1}{\sqrt{q_2}} - \theta_2\right) - \mu\Delta\theta = 0$$

If $\mu = 0$, then the first-order conditions are the same as in the first-best case. If $\mu > 0$, the incentive constraints are both binding, meaning

$$t_1 - \frac{\theta_1}{2}q_1 = t_2 - \frac{\theta_1}{2}q_2$$
$$t_2 - \frac{\theta_2}{2}q_2 = t_1 - \frac{\theta_2}{2}q_1$$
$$\frac{\Delta\theta}{2}q_1 = \frac{\Delta\theta}{2}q_2$$

which, again, implies $q_1 = q_2 = 0$ and all that. So μ should be 0 and the optimal quantities ordered now should be back to that in the first-best case: $q_i^* = \frac{1}{\theta_i^2}$. Also, since the ex-ante participation constraint is binding,

$$p_1\left(t_1 - \frac{\theta_1}{2}q_1\right) + (1-p_1)\left(t_2 - \frac{\theta_2}{2}q_2\right) = 0$$
$$t_1 - \frac{\theta_1}{2}q_1 = -\frac{1-p_1}{p_1}\left(t_2 - \frac{\theta_2}{2}q_2\right)$$

If the left-hand side is 0, then $t_1 = \frac{\theta_1}{2}q_1$ and $t_2 = \frac{\theta_2}{2}q_2$. This will not be incentive compatible since

$$t_1 - \frac{\theta_1}{2}q_1 = 0 < t_2 - \frac{\theta_1}{2}q_2 = \frac{\Delta\theta}{2}q_2$$

So the optimum policy must involve a loss if the firm turns out to be inefficient and supernormal profits if the firm turns out to be efficient. Since we don't consider the possibility of reneging once the firm has accepted the menu, the asymmetry of information at this point is irrelevant. Thus we are back to the first-best allocation except that the payment to the inefficient firm is lower while the payment to the efficient firm is higher.

Question 3

This is almost equivalent to the model of regulation with 3 possible cost parameters. $V(t,q)$ is analogous to the utility function for society, but there is a difference here: the function is also concave but admits a maximum at $q = \frac{a}{2}$. It does not satisfy the Inada conditions, including the particularly important one ($\lim_{q\to 0} V'(q) = \infty$) which ensures a positive level of optimal production no matter what the marginal cost is, since $\left.\frac{\partial V}{\partial q}\right|_{q=0} = a$.

$U(t,q)$ is analogous to the profit earned by the firm, with $(\theta_P + \theta_C)$ taken together being equivalent to a cost parameter with 3 possible values: $2\theta_1$ with a probability of π_1^2, $\theta_1 + \theta_2$ with a probability of $2\pi_1(1-\pi_1)$, and $2\theta_2$ with a probability of $(1-\pi_1)^2$.

If $2\theta_1 \geq a$, then it is optimal not to tender for any contract at all. No one can market the good efficiently enough to have any money left over to pay the manufacturer. Otherwise, as in the 3-firm case for regulation, we can find the optimal quantities and transfer payments. We can work through the n-types case and then consider the specific instance where $n = 3$.

The problem for n retailers

We know that when the participation constraint for the most inefficient firm and the incentive constraints are taken together, all the other participation constraints are implied. A monotonicity constraint also makes the global incentive constraint redundant. We define $S(q) = aq - q^2$. For n types of retailers, the manufacturer seeks to maximize

$$\sum_{i=1}^{n} p_i[S(q_i) - t_i]$$

subject to

$$t_n - \theta_n q_n \geq 0, \qquad \text{(Participation constraint for most inefficient firm)}$$
$$t_i - \theta_i q_i \geq t_{i+1} - \theta_i q_{i+1}, \qquad \text{(Local upward incentive constraints)}$$

and the Lagrangian for this problem is

$$\mathcal{L} = \sum_{i=1}^{n} p_i[S(q_i) - t_i] - \lambda(\theta_n q_n - t_n) - \sum_{i=1}^{n-1} \mu_i[t_{i+1} - t_i + \theta_i(q_i - q_{i+1})]$$

The Kuhn-Tucker conditions are

$$\frac{\partial \mathcal{L}}{\partial q_1} = p_1 S'(q_1) - \mu_1\theta_1 \leq 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = p_i S'(q_i) - \mu_i\theta_i + \mu_{i-1}\theta_{i-1} \leq 0, \qquad i \in [2, \dots, n-1] \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial q_n} = p_n S'(q_n) - \lambda\theta_n + \mu_{n-1}\theta_{n-1} \leq 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial t_1} = -p_1 + \mu_1 = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial t_i} = -p_i + \mu_i - \mu_{i-1} = 0, \qquad i \in [2, \dots, n-1] \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial t_n} = -p_n + \lambda - \mu_{n-1} = 0 \tag{6}$$

(4) and (5) imply that $u_i > 0$ for $i \neq n$. This means that all the local upward incentive constraints are binding (the most inefficient retailer has no upward incentive constraint). Adding $\dfrac{\partial \mathcal{L}}{\partial t_{n-1}}$ to (6) yields

$$\lambda = p_n + p_{n-1} + \mu_{n-2} > 0$$

This means that the participation constraint for the most inefficient retailer has to be binding.

With that, we can make this problem more tractable. The participation constraint is binding, which implies $t_n = \theta_n q_n$. Also, the binding incentive constraints imply

$$t_{i+1} - t_i = \theta_i(q_{i+1} - q_i)$$

We have all we need to get an expression for t_i. We use the following result:

$$\begin{aligned} t_n &= \theta_n q_n \\ t_i &= t_n - (t_n - t_{n-1}) - (t_{n-1} - t_{n-2}) - \dots - (t_{i+1} - t_i) \\ &= \theta_n q_n - \theta_{n-1}(q_n - q_{n-1}) - \dots - \theta_i(q_{i+1} - q_i) \end{aligned}$$

Therefore, when the Kuhn-Tucker conditions are satisfied,

$$t_i^* = \theta_n q_n - \sum_{j=i}^{n-1} \theta_j (q_{j+1} - q_j)$$

for $i \neq n$. So now the problem becomes much simpler: we can just substitute t_i^* into the objective function and maximize that. This envelope theorem applies here, where the binding participation and incentive constraints means the other terms in the Lagrangian disappear. The only constraints we have now are the monotonicity constraints (and the non-negativity constraints we have so far ignored), and now we are maximizing

$$\begin{aligned} \Phi(\mathbf{q}) &= \sum_{i=1}^{n} p_i[S(q_i) - t_i] \\ &= \sum_{i=1}^{n-1} p_i \left[S(q_i) - \theta_n q_n + \sum_{j=i}^{n-1} \theta_j (q_{j+1} - q_j) \right] + p_n[S(q_n) - \theta_n q_n] \end{aligned}$$

and the first-order conditions are

$$
\begin{aligned}
\frac{\partial \Phi}{\partial q_1} &= p_1[S'(q_1) - \theta_1] \leq 0 \\
\frac{\partial \Phi}{\partial q_i} &= p_i S'(q_i) + \theta_{i-1} \sum_{j=1}^{i-1} p_j - \theta_i \sum_{j=1}^{i} p_j \leq 0 \\
\frac{\partial \Phi}{\partial q_n} &= p_n[S'(q_n) - \theta_n] - \theta_n \sum_{i=1}^{n-1} p_i + \theta_{n-1} \sum_{i=1}^{n-1} p_i \\
&= p_n[S'(q_n) - \theta_n] - (\theta_n - \theta_{n-1}) \sum_{i=1}^{n-1} p_i \leq 0
\end{aligned}
$$

The FOCs for q_i and q_n are actually identical, since

$$
\begin{aligned}
&p_i S'(q_i) + \theta_{i-1} \sum_{j=1}^{i-1} p_j - \theta_i \sum_{j=1}^{i} p_j \\
=&p_i S'(q_i) + \theta_{i-1} \sum_{j=1}^{i-1} p_j - \theta_i \sum_{j=1}^{i-1} p_j - \theta_i p_i \\
=&p_i[S'(q_i) - \theta_i] - (\theta_i - \theta_{i-1}) \sum_{j=1}^{i-1} p_j
\end{aligned}
$$

If we can pretend for the moment that we know all the first-order conditions are equalities, then

$$
\begin{aligned}
S'(q_1) &= \theta_1 \\
S'(q_i) &= \frac{(\theta_i - \theta_{i-1}) \sum_{j=1}^{i-1} p_j}{p_i} + \theta_i, \qquad\qquad i \neq 1
\end{aligned}
$$

So in this specific problem where $S'(q_i) = a - 2q_i$, $n = 3$, and the three types are $2\theta_1$, $\theta_1 + \theta_2$, and $2\theta_2$ (from now on we just redefine them as θ_1, θ_2, and θ_3), the optimal policy involves

$$
\begin{aligned}
a - 2q_1^* &= \theta_1 \\
q_1^* &= \frac{a - \theta_1}{2} \\
a - 2q_2^* &= \Delta\theta \frac{\pi_1^2}{2\pi_1(1 - \pi_1)} + \theta_2 \\
q_2^* &= \frac{1}{2} \left(a - \theta_2 - \Delta\theta \frac{\pi_1^2}{2\pi_1(1 - \pi_1)} \right) \\
q_3^* &= \frac{1}{2} \left(a - \theta_3 - \Delta\theta \frac{\pi_1^2 + 2\pi_1(1 - \pi_1)}{(1 - \pi_1)^2} \right) \\
&= \frac{1}{2} \left(a - \theta_3 - \Delta\theta \frac{\pi_1(2 - \pi_1)}{(1 - \pi_1)^2} \right)
\end{aligned}
$$

unless any or all of them are negative, in which case the negative quantities are set to 0. And if the monotone hazard rate condition is satisfied then the monotonicity constraint is satisfied. In this case, the condition for a monotone hazard rate amounts to

$$\begin{aligned} \frac{p_3}{p_1 + p_2} &\leq \frac{p_2}{p_1} \text{ , or} \\ \frac{(1-\pi_1)^2}{\pi_1^2 + 2\pi_1(1-\pi_1)} & \\ = \frac{(1-\pi_1)^2}{\pi_1(2-\pi_1)} &\leq \frac{2\pi_1(1-\pi_1)}{\pi_1^2} \end{aligned}$$

which is satisfied since

$$\pi_1^2(1-\pi_1)^2 \leq 2\pi_1^2(1-\pi_1)(2-\pi_1)$$

for all $\pi_1 \in [0,1]$. We don't have to worry about $q \geq \frac{a}{2}$ where $V(t,q)$ is decreasing in q as long as $\theta_1 > 0$. The monotone hazard rate also implies there is no bunching of contracts. We can find the optimal t_i^* to offer in the menu using the binding participation constraint for firm 3 and the binding local upward incentive constraints. This will be messy and it doesn't look like there will be anything mind-blowing. As usual we should expect the information rent to be 0 for the least efficient firm and increasing for more efficient firms since the monotone hazard rate condition is satisfied; the virtual cost of the firm is increasing in θ.