Game Theory, Welfare, and Applications
Supervision 2

Samuel Lee

Question 1

(a)

The marginal benefit of one more citizen contributing must exceed the 100 pounds he or she contributes. Therefore, the number the mayor must persuade is

$$\begin{aligned}
&\underset{n}{\operatorname{argmin}}\{(n+1)^2 - n^2 \geq 100 | n \in \mathbb{N}\} \\
=&\underset{n}{\operatorname{argmin}}\{2n+1 \geq 100 | n \in \mathbb{N}\} \\
=&\underset{n}{\operatorname{argmin}}\{n \geq 49.5 | n \in \mathbb{N}\} \\
=&50
\end{aligned}$$

When 50 people have been persuaded to contribute, the benefit to the 51^{st} person is $50^2 = 2500$ if he doesn't contribute and $51^2 - 100 = 2501$ if he does. The marginal benefit is increasing in n, so from then on everyone will contribute voluntarily.

(b)

The best response function of Citizen i contingent the number of contributors n before his decision is

$$b_i(n) = \begin{cases} 0 & \text{if } n < 50 \\ 1 & \text{otherwise} \end{cases}$$

For everyone to be playing their best response, it must be that $n = 0$ or $n = 100$. If $0 < n \leq 49$, anyone who is contributing is not playing their best response, and would benefit from withdrawing his contribution. If $50 \leq n < 100$, anyone not contributing can stand to gain by contributing.

Question 2

Firm i seeks to maximize

$$\pi_i(q_i) = p(q_i + \bar{q}_{-i}) \cdot q_i - c \cdot q_i$$

and its optimal quantity, if positive, satisifies

$$\begin{aligned}\frac{\partial \pi_i}{\partial q_i} = p'(q_i + \bar{q}_{-i}) \cdot q_i + p(q_i + \bar{q}_{-i}) - c = 0 \\ -q_i + a - (q_i + \bar{q}_{-i}) - c = 0 \\ q_i = \frac{a - c - \bar{q}_{-i}}{2}\end{aligned}$$

A Nash equilibrium will therefore have to satisfy

$$q_1 = \frac{a - c - q_2 - q_3}{2} \tag{1}$$

$$q_2 = \frac{a - c - q_1 - q_3}{2} \tag{2}$$

$$q_3 = \frac{a - c - q_1 - q_2}{2} \tag{3}$$

Substituting (2) into (1),

$$\begin{aligned}q_1 &= \frac{a - c - \frac{a-c-q_1-q_3}{2} - q_3}{2} \\ &= \frac{2a - 2c - (a - c - q_1 - q_3) - 2q_3}{4} \\ &= \frac{a - c + q_1 - q_3}{4} \\ &= \frac{a - c - q_3}{3} \qquad (4)\end{aligned}$$

By symmetry, substituting (2) into (3) yields

$$q_3 = \frac{a - c - q_1}{3} \tag{5}$$

and substituting (5) into (4) yields

$$\begin{aligned}q_1 &= \frac{a - c - \frac{a-c-q_1}{3}}{3} \\ &= \frac{3a - 3c - (a - c - q_1)}{9} \\ &= \frac{2a - 2c + q_1}{9} \\ &= \frac{a - c}{4}\end{aligned}$$

and the same will apply for q_2 and q_3. The equilibrium price is

$$p(q_1^* + q_2^* + q_3^*) = a - \frac{3(a-c)}{4} = \frac{a + 3c}{4}$$

and the equilibrium profit for each firm is

$$\frac{a+3c}{4} \cdot \frac{a-c}{4} - c \cdot \frac{a-c}{4} = \frac{a^2 + 2ac - 3c^2 - 4c(a-c)}{16} = \frac{a^2 - 2ac + c^2}{16} = \left(\frac{a-c}{4}\right)^2$$

If two of the firms merged, then a Nash equilibrium satisfies

$$q_1 = \frac{a - c - q_2}{2} \tag{6}$$

$$q_2 = \frac{a - c - q_1}{2} \tag{7}$$

Substituting (7) into (6):

$$\begin{aligned} q_1 &= \frac{a - c - \frac{a-c-q_1}{2}}{2} \\ &= \frac{2a - 2c - (a - c - q_1)}{4} \\ &= \frac{a - c + q_1}{4} \\ &= \frac{a - c}{3} \end{aligned}$$

and the same applies for q_2. The equilibrium price is now

$$a - \frac{2(a-c)}{3} = \frac{a + 2c}{3}$$

and the equilibrium profit for each firm is

$$\frac{a+2c}{3} \cdot \frac{a-c}{3} - c \cdot \frac{a-c}{3} = \frac{a^2 + ac - 2c^2 - 3c(a-c)}{9} = \frac{a^2 - 2ac + c^2}{9} = \left(\frac{a-c}{3}\right)^2$$

The two firms will find it profitable to merge if their profit after merger exceeds the combined profits they earned as separate entities, which is

$$2\left(\frac{a-c}{4}\right)^2 = \frac{(a-c)^2}{8} > \left(\frac{a-c}{3}\right)^2$$

Therefore there is no incentive to merge.

Question 3

A Bertrand game is where the firms choose prices rather than quantities. The three firms now compete in a market in which firm i faces the demand function

$$q_i = \begin{cases} a - p_i & \text{if } p_i < \min\{p_{-i}\} \\ \frac{a-p_i}{2} & \text{if } p_i = \min\{p_{-i}\} \\ 0 & \text{if } p_i > \min\{p_{-i}\} \end{cases}$$

Assuming $p_i < \min\{p_{-i}\}$, firm i's profit is simply

$$\pi_i = p_i \cdot q_i(p_i) - c \cdot q_i(p_i)$$

and the profit maximizing point under this condition satisfies

$$\begin{aligned} \frac{\partial \pi_i}{\partial p_i} = q_i(p_i) + p_i \cdot q'(p_i) - c \cdot q_i'(p_i) &= 0 \\ a - p_i - p_i + c &= 0 \\ p_i &= \frac{a+c}{2} \end{aligned}$$

Beyond this price, firm i would not have any incentive to raise the price further.

Therefore, for any firm i, its best response to the set of other firms' prices p_{-i} is

$$B_i(p_{-i}) = \begin{cases} \text{Any } p_i > \min\{p_{-i}\} & \text{if } \min\{p_{-i}\} < c \\ \text{Any } p_i \geq c & \text{if } \min\{p_{-i}\} = c \\ \text{Not well defined (undercut } \min\{p_{-i}\}) & \text{if } c < \min\{p_{-i}\} \leq \frac{a+c}{2} \\ \frac{a+c}{2} & \text{if } \min\{p_{-i}\} > \frac{a+c}{2} \end{cases}$$

On inspection, the only case where all firms are playing their best responses is when $p_i = c \forall i$. Profits are now driven down to 0 and prices are driven down to the unit cost of production. This is because the Bertrand game features strategic complements (one player's strategy reinforces another player's strategy) while the Cournot game features strategic subsitutes (one player's strategy offsets another player's strategy).

Question 4

Firm i maximizes

$$\pi_i = p(q_i + \bar{q}_{-i}) \cdot q_i - c_i q_i$$

and, if choosing to produce, seeks to choose q_i such that

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i} =& p'(q_i + \bar{q}_{-i}) \cdot q_i + p(q_i + \bar{q}_{-i}) - c_i = 0 \\ & - q_i + a - (q_i + \bar{q}_{-i}) - c_i \\ q_i =& \frac{a - c_i - \bar{q}_{-i}}{2} \end{aligned}$$

A Nash equilibrium where all firms produce a positive quantity must then satisfy

$$q_1 = \frac{a - c_1 - q_2}{2} \tag{1}$$
$$q_2 = \frac{a - c_2 - q_1}{2} \tag{2}$$

Substituting (2) into (1),

$$\begin{aligned} q_1 &= \frac{a - c_1 - \frac{a - c_2 - q_1}{2}}{2} \\ &= \frac{2a - 2c_1 - (a - c_2 - q_1)}{4} \\ &= \frac{a - 2c_1 + c_2 + q_1}{4} \\ &= \frac{a - 2c_1 + c_2}{3} \end{aligned}$$

and by symmetry

$$q_2 = \frac{a - 2c_2 + c_1}{3}$$

Assuming both firms produce positive quantities, the difference in their production is

$$\begin{aligned} q_1 - q_2 &= \frac{a - 2c_1 + c_2}{3} - \frac{a - 2c_2 + c_1}{3} \\ &= c_2 - c_1 < 0 \text{ since } c_1 > c_2 \end{aligned}$$

Therefore firm 2 (the more efficient firm) produces more in equilibrium.

We must now check the firms have an incentive to produce at all. The following condition should hold since $p(\mathbf{q}) = 0$ if $\sum_i q_i \geq a$:

$$\begin{aligned} q_1 + q_2 = \frac{a - 2c_1 + c_2}{3} + \frac{a - 2c_2 + c_1}{3} &< a \\ 2a - c_1 - c_2 &< 3a \\ -c_1 - c_2 &< a \end{aligned}$$

Which is trivially true since the left-hand side is negative. We must also check that firm 1 (and firm 2 by extension) has an incentive to produce a positive amount.

$$\begin{aligned} q_1 = \frac{a - 2c_1 + c_2}{3} &> 0 \\ \underset{(+)}{(a - c_1)} + \underset{(-)}{(c_2 - c_1)} &> 0 \end{aligned} \tag{3}$$

which is not always true. Since this is not always true, we must also check this for firm 2

$$\begin{aligned} q_2 = \frac{a - 2c_2 + c_1}{3} &> 0 \\ a - 2c_2 + c_1 &> 0 \\ (a - c_2) + (c_1 - c_2) &> 0 \end{aligned}$$

which is always true for $a > c_1 > c_2$.

Now it must be determined what happens when the optimal q_1 is non-positive (that is, when (3) is not met). Assuming for the moment that firm 1 doesn't produce no matter what, then firm 2 has a monopoly and its optimal quantity is where

$$\begin{aligned} \left.\frac{\partial \pi_2}{\partial q_2}\right|_{q_2^M} = p'(q_2^M) \cdot q_2^M + p(q_2^M) - c_2 &= 0 \\ -q_2^M + a - q_2^M - c_2 &= 0 \\ q_2^M &= \frac{a - c_2}{2} \end{aligned}$$

This monopoly quantity is less than or equal to the optimal q_2 found previously if (3) is not met, that is, the optimal q_1 is non-positive. This is because

$$q_2 = \frac{a - 2c_2 + c_1}{3} = \frac{2a - 4c_2 + 2c_1}{6} = \frac{3a - 3c_2}{6} + \frac{2c_1 - c_2 - a}{6} = q_2^M - \frac{(a - c_1) + (c_2 - c_1)}{6}$$

and (3) not being met implies $(a - c_1) + (c_2 - c_1) \leq 0$. Conversely the monopoly quantity is greater than the equilibrium q_2 found previously if the equilibrium q_1 is positive. There are now two possibilities. The first is where (3) is met, and both firms produce positive amounts with firm 2 producing less than the quantity it would produce if it had a monopoly. This is because there is less room for firm 2 to produce that much with firm 1 in the fray. The second is where (3) is not met, and firm 1 doesn't enter the market. However, firm 2's optimal quantity is now beyond the quantity it would produce if it had a monopoly.

One reason firm 2 wouldn't just stop producing at q_2^M could be that it needs to produce more to price firm 1 out of the market; producing at the monopoly quantity doesn't lower prices enough to make it unprofitable for firm 1 to enter. To check if this is the case, we can substitute q_2^M into the profit-maximizing q_1 to see whether q_1 would produce if firm 2 chose to just stop producing at the monopoly quantity, conditional on (3) not being met.

$$\begin{aligned}
q_1\Big|_{q_2=q_2^M} &= \frac{a-c_1-\frac{a-c_2}{2}}{2} \\
&= \frac{2a-2c_1-(a-c_2)}{4} \\
&= \frac{(a-c_1)+(c_2-c_1)}{4}
\end{aligned}$$

This is non-positive if (3) is not met (since $(a-c_1)+(c_2-c_1) \leq 0$), therefore there is no need for firm 2 to produce more than the monopoly quantity to price firm 1 out.

> A worked example to verify this: take $a = 10$, $c_1 = 9$, $c_2 = 2$. $a > c_1 > c_2$ is met, and $(a-c_1)+(c_2-c_1) = 1-7 = -6 \leq 0$. Assume firm 2 produces $q_2^M = \frac{a-c_2}{2} = 4$ units. This alone will drive the price down to $p(4) = a - 4 = 6$. This is lower than firm 1's unit cost of production and firm 1 will not produce. Firm 2's profit in this case is $p(4)\cdot 4 - c_2 \cdot 4 = 6\cdot 4 - 2\cdot 4 = 16$. There is no reason for firm 2 to produce $\frac{a-2c_1+c_1}{3} = \frac{10-2\cdot 2+9}{3} = 5$ units since that would lead to a profit of $p(5)\cdot 5 - c_2\cdot 5 = 5\cdot 5 - 2\cdot 5 = 15$.

With this, the equilibrium quantities are

$$q_1 = \begin{cases} 0 & \text{if } (a-c_1)+(c_2-c_1) \leq 0 \\ \frac{a-2c_1+c_2}{3} & \text{if } (a-c_1)+(c_2-c_1) > 0 \end{cases}$$

$$q_2 = \begin{cases} \frac{a-c_2}{2} & \text{if } (a-c_1)+(c_2-c_1) \leq 0 \\ \frac{a-2c_2+c_1}{3} & \text{if } (a-c_1)+(c_2-c_1) > 0 \end{cases}$$

Looking at the expressions above, a change in technology which lowers c_2 will always increase the amount produced by firm 2, and decrease the amount produced by firm 1. It could also have the effect of pushing the market from the case where (3) is met to the case where (3) is not met, essentially by making it cheap enough for firm 2 to flood the market with goods such that firm 1 cannot profitably produce.

Question 5

A mixed strategy Nash equilibrium is where

$$U_i(\alpha_i, \alpha_{-i}) \geq U_i(\hat{\alpha}_i, \alpha_{-i}) \forall \hat{\alpha}_i \in \Delta(A_i), \forall i$$

where α_i is a mixed strategy in the set $\Delta(A_i)$ which includes all valid α_i. This essentially means all players are playing a strategy (whether mixed or pure) such that they cannot do better by deviating.

If a given mixed strategy is a best response to some α_{-i}, then each of the actions in the mix must also be a best response when played as a pure strategy. In other words, the player must be indifferent over the actions he mixes over. If this were not true, against a given α_{-i}, there will be at least one action a_i in the mix that yields a lower payoff than the other pure actions in the mix when played as a pure strategy. But this would imply that dropping a_i from the mix will lead to a higher expected payoff, and the mix cannot have been a best response.

If p_i is the probability Player i plays H, the expected utility for Player 1 is

$$\begin{aligned}
&p_1 \cdot [p_2 \cdot U_1(H,H) + (1-p_2) \cdot U_1(H,T)] + (1-p_1) \cdot [p_2 \cdot U_1(T,H) + (1-p_2) \cdot U_1(T,T)] \\
=&p_1 \cdot [2p_2 - (1-p_2)] + (1-p_1) \cdot [-p_2 + 3(1-p_2)] \\
=&p_1 \cdot (3p_2 - 1) + (1-p_1) \cdot (3 - 4p_2)
\end{aligned}$$

Since we know from before that a mixed strategy can only be a best response when there is no strictly preferred pure strategy, it suffices to consider the conditions under which Player 1's pure strategy of choosing H is preferred to choosing T in order to rule out the values of p_2 for which the best response is not a mixed strategy.

The pure strategy of choosing H is a best response for Player 1 when

$$\begin{aligned}
3p_2 - 1 &> 3 - 4p_2 \\
7p_2 &> 4 \\
p_2 &> \frac{4}{7}
\end{aligned}$$

and T is a best response when $p_2 < \frac{4}{7}$. Player 1 is indifferent over all strategies (mixed or pure) when $p_2 = \frac{4}{7}$.

Likewise, the expected utility for Player 2 is

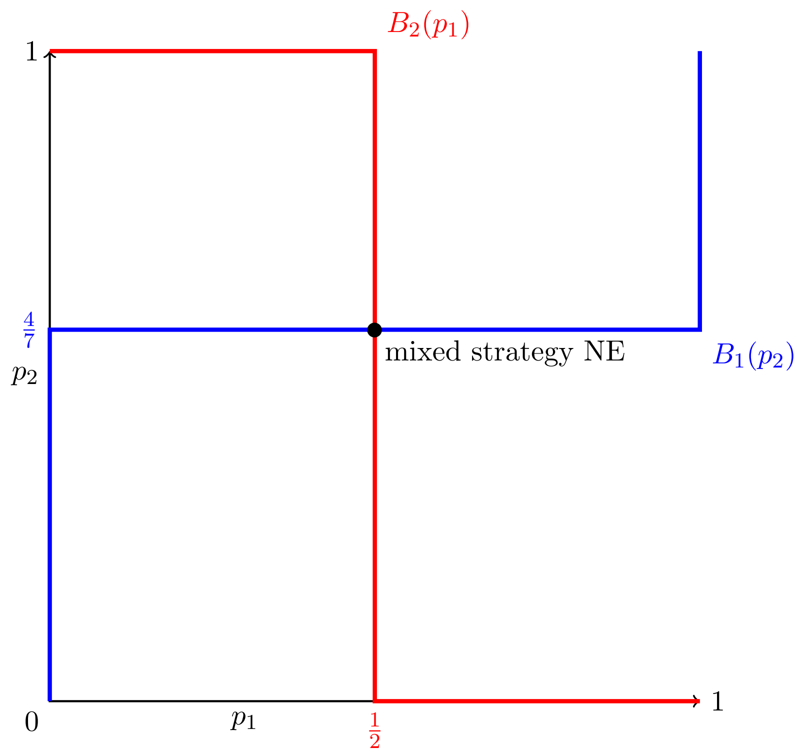
$$\begin{aligned}
&p_2 \cdot [p_1 \cdot U_2(H,H) + (1-p_1) \cdot U_2(T,H)] + (1-p_2) \cdot [p_1 \cdot U_2(H,T) + (1-p_1) \cdot U_2(T,T)] \\
=&p_2 \cdot [-p_1 + (1-p_1)] + (1-p_2) \cdot [p_1 - (1-p_1)] \\
=&p_2 \cdot (1 - 2p_1) + (1-p_2) \cdot (2p_1 - 1)
\end{aligned}$$

H is a best response for Player 2 when

$$\begin{aligned}
1 - 2p_1 &> 2p_1 - 1 \\
p_1 &< \frac{1}{2}
\end{aligned}$$

and T is a best response when $p_1 > \frac{1}{2}$. Player 2 is indifferent over all strategies (mixed or pure) when $p_2 = \frac{1}{2}$.

With that, the best response functions can be plotted as such



and the mixed strategy Nash equilibrium is $p_1 = \frac{1}{2}$, $p_2 = \frac{4}{7}$.

Question 6

Player 1's expected payoff with a pure strategy of some choice k_1 is

$$\begin{aligned}&\sum_{i \neq k_1}^{K} \left[p(k_2 = i) \cdot U_1(k_1 \neq k_2) \right] + p(k_2 = k_1) \cdot U_1(k_1 = k_2) \\ =&p(k_2 = k_1)\end{aligned}$$

and Player 2's expected payoff with a pure strategy of some choice k_2 is

$$\begin{aligned}&\sum_{i \neq k_2}^{K} \left[p(k_1 = i) \cdot U_2(k_2 \neq k_1) \right] + p(k_2 = k_1) \cdot U_2(k_2 = k_1) \\ =&-p(k_2 = k_1)\end{aligned}$$

A mixed strategy must have both players indifferent between playing any of the actions they mix over as a pure strategy. Furthermore, it makes sense that any $p(k_i = j) > 0 \; \forall i, j$. If Player 1 doesn't choose a certain integer with positive probability, a best response for Player 2 is to always choose that integer as a pure strategy. And if Player 2 doesn't choose a certain integer with positive probability, Player 1's best response must involve never choosing that integer, but doing so means Player 2 is not playing his best response (for the same reason as before).

Therefore, we know that any mixed strategy must include all integers up to K played with positive probability for both players. The condition for a Nash equilibrium is thus

$$p(k_2 = i) = p(k_2 = j) \;\forall i = 1, 2 \text{ and } \forall j \in \{1, \ldots, K\}$$

and since $\sum_{j=1}^{K} p(k_i = j) = 1 \; \forall i$, this implies that at the Nash equilibrium

$$\begin{aligned} K \cdot p(k_i = j) &= 1 \\ p(k_i = j) &= \frac{1}{K} \end{aligned}$$

for all $i = 1, 2$ and $j \in \{1, \ldots, K\}$.

Question 7

(a)

Country A will only find it profitable to outbid Country B if $v - k - b_B > 0$ or $b_B < v - k$. If $b_B = v - k$, Country A will only find it profitable to bid $b_A = b_B = v - k$ if

$$\begin{aligned} \frac{1}{2} \cdot v - k - b_A &\geq 0 \\ \frac{v}{2} - k - (v - k) &\geq 0 \\ -v &\geq 0 \end{aligned}$$

which can never be true. Therefore, if both countries can only play pure strategies, Country A's best response r_A to Country B is

$$r_A = \begin{cases} \text{Not well defined (outbid } B) & \text{if } 0 \leq b_B < v - k \\ \text{Not submit any proposal} & \text{if } b_B \geq v - k \end{cases}$$

and Country B has a similar best response to Country A. There is no situation in which both countries can be playing their best responses; they seek to outbid each other until one hits a bribe of $v - k$, forcing the other to not submit any proposal. But when one assumes it is a given that the other country doesn't submit a proposal the best response is to give a bribe very close to 0 rather than $v - k$. Therefore, there is no pure Nash equilibrium of this game.

(b)

Let the probability Country i submits a proposal be p_i. A mixed strategy Nash equilibrium must make both countries indifferent between not submitting a proposal and submitting a proposal with a bribe b_i for all $b_i \in [0, v - k]$. Therefore, in a mixed strategy Nash equilibrium, it must be that

$$\begin{aligned} p_A \cdot \{F_B(b_A) \cdot v + [1 - F_B(b_A)] \cdot 0\} - k - b_A &= (1 - p_A) \cdot 0 \\ p_A \cdot [F_B(b_A) \cdot v] &= k + b_A \\ F_B(b_A) &= \frac{k + b_A}{v \cdot p_A} \end{aligned}$$

$F_B(b_A)$ is a CDF with support $[0, v-k]$. Therefore,

$$\begin{aligned}
F_B(v-k) - F_B(0) &= 1 \\
\frac{k+v-k}{v \cdot p_A} - \frac{k}{v \cdot p_A} &= 1 \\
\frac{v-k}{v \cdot p_A} &= 1 \\
p_A &= \frac{v-k}{v}
\end{aligned}$$

and

$$\begin{aligned}
F_B(b_A) &= \frac{k+b_A}{v \cdot p_A} \\
&= \frac{k+b_A}{v \cdot \frac{v-k}{v}} \\
&= \frac{k+b_A}{v-k}
\end{aligned}$$

The game is symmetric so F and p can be found for the other country by simply swapping the A and B indices, and those probability mass functions/cumulative distribution functions are the ones that are played in the mixed strategy Nash equilibrium.