Integrales trigonométricas

Tipo 1: Potencias de trigonométricas

Teorema 1. *Si n es un entero positivo, entonces*

$$\int \operatorname{sen}^n(x)\,dx = -\frac{\operatorname{sen}^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \operatorname{sen}^{n-2}(x)\,dx.$$

Demostración. Dado que $\operatorname{sen}^n(x) = [\operatorname{sen}^{n-1}(x)](\operatorname{sen}(x))$, aplicamos la fórmula de integración por partes, como sigue:

$$\begin{array}{rclrcl} u(x) & = & \operatorname{sen}^{n-1}(x) & v'(x) & = & \operatorname{sen}(x)\,dx \\ u'(x) & = & (n-1)[\operatorname{sen}^{n-2}(x)](\cos(x))\,dx & v(x) & = & -\cos(x) \end{array}$$

$$\begin{aligned} \int \operatorname{sen}^n(x)\,dx &= \int [\operatorname{sen}^{n-1}(x)](\operatorname{sen}(x))\,dx \\ &= -\operatorname{sen}^{n-1}(x)\cos(x) + (n-1)\int \operatorname{sen}^{n-2}(x)\cos^2(x)\,dx \\ &= -\operatorname{sen}^{n-1}(x)\cos(x) + (n-1)\int \operatorname{sen}^{n-2}(x)[1-\operatorname{sen}^2(x)]\,dx \\ &= -\operatorname{sen}^{n-1}(x)\cos(x) + (n-1)\int \operatorname{sen}^{n-2}(x)\,dx - (n-1)\int \operatorname{sen}^n(x)\,dx \end{aligned}$$

Luego,

$$\int \operatorname{sen}^n(x)\,dx + (n-1)\int \operatorname{sen}^n(x)\,dx = -\operatorname{sen}^{n-1}(x)\cos(x) + (n-1)\int \operatorname{sen}^{n-2}(x)\,dx.$$

En consecuencia,

$$\int \operatorname{sen}^n(x)\,dx = -\frac{\operatorname{sen}^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \operatorname{sen}^{n-2}(x)\,dx.$$

$\square$

Para resolver $\int \operatorname{sen}^n(x)\,dx$ empleamos la identidad:

- $\operatorname{sen}^2(x) = \dfrac{1-\cos(2x)}{2}$, para n-par y
- $\operatorname{sen}^2(x) = 1-\cos^2(x)$, para n-impar.

Ejemplo 2. *Calcular* $\int \operatorname{sen}^2(x)\,dx$.

Solución. Como $n = 2$ es par, usamos la identidad $\operatorname{sen}^2(x) = \dfrac{1-\cos(2x)}{2}$ y hacemos el cambio de variable, $u = 2x \Longrightarrow du = 2\,dx$

$$\begin{aligned}
\int \operatorname{sen}^2(x)\,dx &= \int \frac{1-\cos(2x)}{2}\,dx \\
&= \frac{1}{2}\int 1-\cos(2x)\,dx \\
&= \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\,dx \\
&= \frac{1}{2}\int dx - \frac{1}{2}\left(\frac{1}{2}\right)\int \cos(u)\,du \\
&= \frac{1}{2}x - \frac{1}{4}\operatorname{sen}(u) + C \\
&= \frac{1}{2}x - \frac{1}{4}\operatorname{sen}(2x) + C.
\end{aligned}$$

$\square$

Ejemplo 3. *Calcular* $\int \operatorname{sen}^5(x)\,dx$

Solución. Como $n = 5$ es impar, sacamos como factor común al $\operatorname{sen}(x)$ y usamos la identidad $\operatorname{sen}^2(x) + \cos^2(x) = 1$.

Hacemos el cambio de variable

$$\begin{aligned} u &= \cos(x) \\ du &= -\operatorname{sen}(x)\,dx \end{aligned}$$

$$\begin{aligned} \int \operatorname{sen}^5(x)\,dx &= \int \operatorname{sen}^4(x)\operatorname{sen}(x)\,dx \\ &= \int [1-\cos^2(x)]^2\ \operatorname{sen}(x)dx \\ &= \int [1-2\cos^2(x)+\cos^4(x)]\operatorname{sen}(x)\,dx \\ &= -\int [1-2u^2+u^4]\,du \\ &= -\left(u-\frac{2}{3}u^3+\frac{u^5}{5}\right)+C \\ &= -\cos(x)+\frac{2}{3}\cos^3(x)-\frac{1}{5}\cos^5(x)+C. \end{aligned}$$

□

Teorema 4. *Si n es un número entero positivo, entonces*

$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\operatorname{sen}(x)}{n}+\frac{n-1}{n}\int \cos^{n-2}(x)\,dx.$$

Demostración. Dado que $\cos^n(x) = [\cos^{n-1}(x)](\cos(x))$, aplicamos la fórmula de integración por partes, como sigue:

$$\begin{aligned} u(x) &= \cos^{n-1}(x) & v'(x) &= \cos(x)\,dx \\ u'(x) &= (n-1)[\cos^{n-2}(x)](-\operatorname{sen}(x))\,dx & v(x) &= \operatorname{sen}(x) \end{aligned}$$

$$
\begin{aligned}
\int \cos^n(x)\,dx &= \int [\cos^{n-1}(x)](\cos(x))\,dx \\
&= \cos^{n-1}(x)\,\mathrm{sen}(x) + (n-1)\int \cos^{n-2}(x)\,\mathrm{sen}^2(x)\,dx \\
&= \cos^{n-1}(x)\,\mathrm{sen}(x) + (n-1)\int \cos^{n-2}(x)[1-\cos^2(x)]\,dx \\
&= \cos^{n-1}(x)\,\mathrm{sen}(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx
\end{aligned}
$$

Luego,

$$
\int \cos^n(x)\,dx + (n-1)\int \cos^n(x)\,dx = \cos^{n-1}(x)\,\mathrm{sen}(x) + (n-1)\int \cos^{n-2}(x)\,dx.
$$

En consecuencia,

$$
\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\,\mathrm{sen}(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.
$$

□

Para resolver $\int \cos^n(x)\,dx$ empleamos la identidad:

- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$, para n-par y
- $\cos^2(x) = 1 - \mathrm{sen}^2(x)$, para n-impar.

Ejemplo 5. *Calcular* $\int \cos^3(x)\,dx$.

Solución. Usaremos el cambio de variable

$$
u = \mathrm{sen}(x) \implies du = \cos(x)\,dx.
$$

$$
\begin{aligned}
\int \cos^3(x)\,dx &= \int \cos(x)\cos^2(x)\,dx \\
&= \int \cos(x)[1-\operatorname{sen}^2(x)]\,dx \\
&= \int \cos(x)\,dx - \int \cos(x)\operatorname{sen}^2(x)\,dx \\
&= \operatorname{sen}(x) - \int u^2 du \\
&= \operatorname{sen}(x) - \frac{1}{3}u^3 + C \\
&= \operatorname{sen}(x) - \frac{1}{3}\operatorname{sen}^3(x) + C.
\end{aligned}
$$

$\square$

Ejemplo 6. *Calcular* $\int \cos^4(x)\,dx$.

Solución. Hacemos $u = 2x \implies du = 2\,dx$.
$w = 2u \implies dw = 2\,du$

$$
\begin{aligned}
\int \cos^4(x)\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\,dx \\
&= \frac{1}{4}\int dx + \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx \\
&= \frac{1}{4}\int dx + \frac{1}{4}\int \cos(u)\,du + \frac{1}{8}\int \cos^2(u)\,du \\
&= \frac{1}{4}x + \frac{1}{4}\operatorname{sen}(u) + \frac{1}{8}\int \frac{1+\cos(2u)}{2}\,du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}x + \frac{1}{4}\operatorname{sen}(2x) + \frac{1}{16}\left[\int du + \int \cos(2u)\,du\right] \\
&= \frac{1}{4}x + \frac{1}{4}\operatorname{sen}(2x) + \frac{1}{16}\left[u + \frac{1}{2}\int \cos(w)\,dw\right] \\
&= \frac{1}{4}x + \frac{1}{4}\operatorname{sen}(2x) + \frac{1}{16}\left[u + \frac{1}{2}\operatorname{sen}(w)\right] + C \\
&= \frac{1}{4}x + \frac{1}{4}\operatorname{sen}(2x) + \frac{1}{16}u + \frac{1}{32}\operatorname{sen}(2u) + C \\
&= \frac{1}{4}x + \frac{1}{4}\operatorname{sen}(2x) + \frac{1}{16}2x + \frac{1}{32}\operatorname{sen}(4x) + C \\
&= \frac{3}{8}x + \frac{1}{4}\operatorname{sen}(2x) + \frac{1}{32}\operatorname{sen}(4x) + C.
\end{aligned}
$$

□

Teorema 7. *Si n es un número entero positivo, entonces*

$$
\int \tan^n(x)\,dx = \begin{cases} \ln|\sec(x)| + C, & \text{si } n = 1; \\ \dfrac{\tan^{n-1}(x)}{n-1} - \displaystyle\int \tan^{n-2}(x)\,dx, & \text{si } n \neq 1. \end{cases}
$$

Demostración.

- Para $n = 1$, se tiene que

$$
w = \cos(x) \Longrightarrow dw = -\operatorname{sen}(x)\,dx.
$$

$$
\begin{aligned}
\int \tan^n(x)\,dx &= \int \tan(x)\,dx = \int \frac{\operatorname{sen}(x)}{\cos(x)}\,dx \\
&= -\int \frac{dw}{w} = -\ln|w| + C \\
&= -\ln|\cos(x)| + C = \ln|\cos(x)|^{-1} + C \\
&= \ln|\sec(x)| + C.
\end{aligned}
$$

- Para $n \neq 1$, hacemos el cambio de variable

$$u = \tan(x) \Longrightarrow du = \sec^2(x)\, dx.$$

$$\begin{aligned}
\int \tan^n(x)\, dx &= \int \tan^2(x) \tan^{n-2}(x)\, dx \\
&= \int [\sec^2(x) - 1] \tan^{n-2}(x)\, dx \\
&= \int \sec^2(x) \tan^{n-2}(x)\, dx - \int \tan^{n-2}(x)\, dx \\
&= \int u^{n-2}\, du - \int \tan^{n-2}(x)\, dx \\
&= \frac{u^{(n-2)+1}}{(n-2)+1} - \int \tan^{n-2}(x)\, dx \\
&= \frac{u^{n-1}}{n-1} - \int \tan^{n-2}(x)\, dx \\
&= \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\, dx.
\end{aligned}$$

□

Para resolver $\int \tan^n(x)\, dx$ empleamos la identidad:

$$\tan^2(x) = \sec^2(x) - 1.$$

Mientras que, para resolver $\int \cot^n(x)\, dx$ usamos:

$$\cot^2(x) = \csc^2(x) - 1.$$

Ejemplo 8. *Calcular* $\int \tan^2(5x)\, dx.$

Solución. Usamos la identidad $\tan^2(5x) = \sec^2(5x) - 1$ y la sustitución $u = 5x \Longrightarrow du = 5\,dx$.

$$\begin{aligned}
\int \tan^2(5x)\,dx &= \int [\sec^2(5x) - 1]\,dx \\
&= \int \sec^2(5x)\,dx - \int dx \\
&= \frac{1}{5}\int \sec^2(u)\,du - \int dx \\
&= \frac{\tan(u)}{5} - x + C \\
&= \frac{\tan(5x)}{5} - x + C.
\end{aligned}$$

□

Ejemplo 9. *Calcular* $\int \tan^3(y)\,dy$.

Solución. Usamos la identidad $\tan^2(y) = \sec^2(y) - 1$. Hacemos los cambios de variables

$$u = \tan(y) \quad\Longrightarrow\quad du = \sec^2(y)\,dy.$$

$$w = \cos(y) \quad\Longrightarrow\quad dw = -\operatorname{sen}(y)\,dy.$$

$$\begin{aligned}
\int \tan^3(y)\,dy &= \int \tan^2(y)\tan(y)\,dy \\
&= \int [\sec^2(y) - 1]\tan(y)\,dy \\
&= \int \sec^2(y)\tan(y)\,dy - \int \tan(y)\,dy \\
&= \int u\,du - \int \frac{\operatorname{sen}(y)}{\cos(y)}\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u^2}{2} - (-)\int \frac{dw}{w} \\
&= \frac{u^2}{2} + \ln|w| + C \\
&= \frac{\tan^2(y)}{2} + \ln|\cos(y)| + C.
\end{aligned}
$$

□

Teorema 10. *Si n es un número entero positivo, entonces*

$$
\int \cot^n(x)\,dx = \begin{cases} \ln|\operatorname{sen}(x)| + C, & \text{si } n = 1; \\ -\dfrac{\cot^{n-1}(x)}{n-1} - \displaystyle\int \cot^{n-2}(x)\,dx, & \text{si } n \neq 1. \end{cases}
$$

Demostración.

- Para $n = 1$, se tiene que

$$
w = \operatorname{sen}(x) \Longrightarrow dw = \cos(x)\,dx.
$$

$$
\begin{aligned}
\int \cot^n(x)\,dx &= \int \cot(x)\,dx = \int \frac{\cos(x)}{\operatorname{sen}(x)}\,dx = \int \frac{dw}{w} \\
&= \ln|w| + C = \ln|\operatorname{sen}(x)| + C.
\end{aligned}
$$

- Para $n \neq 1$, empleamos el cambio de variable

$$
u = \cot(x) \Longrightarrow du = -\csc^2(x)\,dx.
$$

$$
\begin{aligned}
\int \cot^n(x)\,dx &= \int \cot^2(x)\cot^{n-2}(x)\,dx \\
&= \int [\csc^2(x) - 1]\cot^{n-2}(x)\,dx \\
&= \int \csc^2(x)\cot^{n-2}(x)\,dx - \int \cot^{n-2}(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\int u^{n-2}\,du - \int \cot^{n-2}(x)\,dx \\
&= -\frac{1}{(n-2)+1}u^{(n-2)+1} - \int \cot^{n-2}(x)\,dx \\
&= -\frac{1}{(n-2)+1}\cot^{(n-2)+1}(x) - \int \cot^{n-2}(x)\,dx \\
&= -\frac{1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)\,dx.
\end{aligned}
$$

□

Ejemplo 11. *Calcular* $\int \cot^4(2x)\,dx$.

Solución. Usaremos la identidad $\cot^2(x) = \csc^2(x) - 1$ y los cambios de variable:

$$u = 2x \implies du = 2\,dx.$$

$$w = \cot(u) \implies dw = -\csc^2(u)\,du.$$

$$
\begin{aligned}
\int \cot^4(2x)\,dx &= \frac{1}{2}\int \cot^4(u)\,du \\
&= \frac{1}{2}\int \cot^2(u)[\csc^2(u) - 1]\,du \\
&= \frac{1}{2}\int \cot^2(u)\csc^2(u) - \cot^2(u)\,du \\
&= \frac{1}{2}\int \cot^2(u)\csc^2(u)\,du - \frac{1}{2}\int \cot^2(u)\,du \\
&= -\frac{1}{2}\int w^2\,dw - \frac{1}{2}\int [\csc^2(u) - 1]\,du
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\int w^2\,dw - \frac{1}{2}\left[\int \csc^2(u)\,du - \int du\right] \\
&= -\frac{1}{2}\,\frac{w^3}{3} - \frac{1}{2}\left[-\cot(u) - u\right] + C \\
&= -\frac{\cot^3(u)}{6} + \frac{\cot(u)}{2} + \frac{u}{2} + C \\
&= -\frac{\cot^3(2x)}{6} + \frac{\cot(2x)}{2} + \frac{2x}{2} + C \\
&= -\frac{\cot^3(2x)}{6} + \frac{\cot(2x)}{2} + x + C.
\end{aligned}
$$

□

Teorema 12. *Si n es un número entero positivo, entonces*

$$
\int \sec^n(x)\,dx = \begin{cases} \ln|\sec(x)+\tan(x)| + C, & \textit{si } n = 1; \\[1ex] \dfrac{\sec^{n-2}(x)\tan(x)}{n-1} + \dfrac{n-2}{n-1}\displaystyle\int \sec^{n-2}(x)\,dx, & \textit{si } n \neq 1. \end{cases}
$$

Demostración.

- Para $n = 1$, hacemos

$$
u = \sec(x) + \tan(x) \implies du = \sec(x)\tan(x) + \sec^2(x)\,dx
$$

$$
\begin{aligned}
\int \sec^n(x)\,dx &= \int \sec(x)\,dx \\
&= \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)dx \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{du}{u} \\
&= \ln|u| + C = \ln|\sec(x)+\tan(x)| + C.
\end{aligned}
$$

- Para $n \neq 1$, ocupamos integración por partes, a saber:

$$\begin{aligned} u(x) &= \sec^{n-2}(x) & v'(x) &= \sec^2(x)\,dx \\ u'(x) &= (n-2)[\sec^{n-3}(x)](\sec(x)\tan(x))\,dx & v(x) &= \tan(x) \end{aligned}$$

$$\begin{aligned} \int \sec^n(x)\,dx &= \int \sec^{n-2}(x)\sec^2(x)\,dx \\ &= \sec^{n-2}(x)\tan(x) - \int (n-2)\sec^{n-2}(x)\tan^2(x)\,dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)[\sec^2(x)-1]\,dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)\,dx + (n-2)\int \sec^{n-2}(x)\,dx \end{aligned}$$

Luego,

$$\int \sec^n(x)\,dx + (n-2)\int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx$$

En consecuencia,

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx.$$

$\square$

Ejemplo 13. *Calcular* $\int \sec^4(x)\,dx$

Solución. Empleamos la identidad $\sec^2(x) = \tan^2(x) + 1$ y hacemos el cambio de variable $u = \tan(x) \Longrightarrow du = \sec^2(x)\,dx$.

$$\begin{aligned}
\int \sec^4(x)\,dx &= \int \sec^2(x)\sec^2(x)\,dx \\
&= \int \sec^2(x)[\tan^2(x)+1]\,dx \\
&= \int \sec^2(x)\tan^2(x)\,dx + \int \sec^2(x)\,dx \\
&= \left(\int u^2\,du\right) + \tan(x) \\
&= \frac{1}{3}u^3 + \tan(x) + C \\
&= \frac{1}{3}\tan^3(x) + \tan(x) + C
\end{aligned}$$

□

Ejemplo 14. *Calcular* $\int \sec^6(x)\,dx$

Solución. Empleamos la identidad $\sec^2(x) = \tan^2(x) + 1$ y hacemos el cambio de variable $u = \tan(x) \Longrightarrow du = \sec^2(x)\,dx$.

$$\begin{aligned}
\int \sec^6(x)\,dx &= \int [\sec^2(x)]^2\sec^2(x)\,dx \\
&= \int [\tan^2(x)+1]^2\sec^2(x)\,dx \\
&= \int [1+2\tan^2(x)+\tan^4(x)]\sec^2(x)\,dx \\
&= \int \sec^2(x)\,dx + 2\int \tan^2(x)\sec^2(x)\,dx + \int \tan^4(x)\sec^2(x)\,dx \\
&= \int \sec^2(x)\,dx + 2\int u^2\,du + \int u^4\,du \\
&= \tan(x) + \frac{2u^3}{3} + \frac{u^5}{5} + C = \tan(x) + \frac{2\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C.
\end{aligned}$$

□

Teorema 15. *Si n es un número entero positivo, entonces*

$$\int \csc^n(x)\,dx = \begin{cases} \ln|\csc(x)-\cot(x)|+C, & \textit{si } n=1; \\ -\dfrac{\csc^{n-2}(x)\cot(x)}{n-1}+\dfrac{n-2}{n-1}\displaystyle\int \csc^{n-2}(x)\,dx, & \textit{si } n\neq 1. \end{cases}$$

Demostración.

- Para $n=1$, hacemos

$$u=\csc(x)-\cot(x) \Longrightarrow du = -\csc(x)\cot(x)+csc^2(x)\,dx$$

$$\begin{aligned} \int \csc^n(x)\,dx &= \int \csc(x)\,dx \\ &= \int \csc(x)\left(\frac{\csc(x)-\cot(x)}{\csc(x)-\cot(x)}\right)dx \\ &= \int \frac{\csc^2(x)-\csc(x)\cot(x)}{\csc(x)-\cot(x)}\,dx = \int \frac{du}{u} \\ &= \ln|u|+C = \ln|\csc(x)-\cot(x)|+C. \end{aligned}$$

- Para $n\neq 1$, ocupamos integración por partes, a saber:

$$\begin{aligned} u(x) &= \csc^{n-2}(x) & v'(x) &= \csc^2(x)\,dx \\ u'(x) &= (n-2)[\csc^{n-3}(x)](-\csc(x)\cot(x))\,dx & v(x) &= -\cot(x) \end{aligned}$$

$$\begin{aligned} \int \csc^n(x)\,dx &= \int \csc^{n-2}(x)\csc^2(x)\,dx \\ &= -\csc^{n-2}(x)\cot(x) - \int (n-2)\csc^{n-2}(x)\cot^2(x)\,dx \\ &= -\csc^{n-2}(x)\cot(x) - (n-2)\int \csc^{n-2}(x)[\csc^2(x)-1]\,dx \\ &= -\csc^{n-2}(x)\cot(x) - (n-2)\int \csc^n(x)\,dx + (n-2)\int \csc^{n-2}(x)\,dx \end{aligned}$$

Luego,

$$\int \csc^n(x)\,dx + (n-2)\int \csc^n(x)\,dx = -\csc^{n-2}(x)\cot(x) + (n-2)\int \csc^{n-2}(x)\,dx$$

En consecuencia,

$$\int \csc^n(x)\,dx = -\frac{\csc^{n-2}(x)\cot(x)}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx.$$

□

Ejemplo 16. *Calcular* $\int \csc^4(2x)\,dx$.

Solución. Sea $w = 2x \implies dw = 2dx$ y $u = \cot(w) \implies du = -csc^2(w)\,dw$.

$$\begin{aligned}
\int \csc^4(2x)\,dx &= \frac{1}{2}\int \csc^2(w)\csc^2(w)\,dw \\
&= \frac{1}{2}\int [1+\cot^2(w)]\csc^2(w)\,dw \\
&= \frac{1}{2}\int \csc^2(w)\,dx + \frac{1}{2}\int \cot^2(w)\csc^2(w)\,dw \\
&= \frac{1}{2}\int \csc^2(w)\,dw - \frac{1}{2}\int u^2\,du \\
&= -\frac{1}{2}\cot(w) - \frac{1}{2}\left(\frac{u^3}{3}\right) + C \\
&= -\frac{1}{2}\cot(2x) - \frac{\cot^3(2x)}{6} + C.
\end{aligned}$$

□

Ejemplo 17. *Calcular* $\int \csc^3(x+1)\,dx$.

Solución. Primero hacemos el cambio de variable, $u = x + 1 \Longrightarrow du = dx$. Después aplicamos integración por partes como sigue:

$$\begin{aligned} w(u) &= \csc(u) & v'(u) &= \csc^2(u)\,du \\ w'(u) &= -\csc(u)\cot(u)\,du & v(u) &= -\cot(u) \end{aligned}$$

$$\begin{aligned} \int \csc^3(x+1)\,dx &= \int \csc^3(u)\,du = \int \csc^2(u)\csc(u)\,du \\ &= -\csc(u)\cot(u) - \int \csc(u)\cot^2(u)\,du \\ &= -\csc(u)\cot(u) - \int \csc(u)[\csc^2(u) - 1]\,du \\ &= -\csc(u)\cot(u) - \int \csc^3(u)\,du + \int \csc(u)\,du \\ &= -\csc(u)\cot(u) - \int \csc^3(u)\,du + \ln|\csc(u) - \cot(u)| \end{aligned}$$

Luego,

$$2\int \csc^3(u)\,du = -\csc(u)\cot(u) + \ln|\csc(u) - \cot(u)|$$

de donde,

$$\int \csc^3(u)\,du = -\frac{1}{2}\csc(u)\cot(u) + \frac{1}{2}\ln|\csc(u) - \cot(u)| + C.$$

Finalmente, regresamos a la variable original:

$$\int \csc^3(x+1)\,dx = -\frac{1}{2}\csc(x+1)\cot(x+1) + \frac{1}{2}\ln|\csc(x+1) - \cot(x+1)| + C.$$

□

Tipo 2: Productos de potencias

Teorema 18. *Si m y n son enteros pares positivos, entonces*

$$\int \operatorname{sen}^n(x)\cos^m(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)^{\frac{n}{2}} \left(\frac{1+\cos(2x)}{2}\right)^{\frac{m}{2}} dx.$$

Demostración. Como m y n son números pares positivos, los podemos escribir de la forma $m = 2p$ y $n = 2q$ para cualesquiera p y q enteros positivos. De manera que,

$$\cos^m(x) = \cos^{2p}(x) = (\cos^2(x))^p = \left(\frac{1+\cos(2x)}{2}\right)^p$$

$$\operatorname{sen}^n(x) = \cos^{2q}(x) = (\operatorname{sen}^2(x))^q = \left(\frac{1+\cos(2x)}{2}\right)^q.$$

Entonces,

$$\begin{aligned}
\int \operatorname{sen}^n(x)\cos^m(x)\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^q \left(\frac{1+\cos(2x)}{2}\right)^p dx \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^{\frac{n}{2}} \left(\frac{1+\cos(2x)}{2}\right)^{\frac{m}{2}} dx.
\end{aligned}$$

□

Teorema 19. *Si alguno de los enteros positivos m o n son impares, entonces*

$$\int \operatorname{sen}^n(x)\cos^m(x)\,dx = \begin{cases} \displaystyle\int \operatorname{sen}^n(x)[1-\operatorname{sen}^2(x)]^{\frac{m-1}{2}}\cos(x)\,dx, & \textit{si } m \textit{ es impar;} \\ \displaystyle\int \cos^m(x)[1-\cos^2(x)]^{\frac{n-1}{2}}\operatorname{sen}(x)\,dx, & \textit{si } n \textit{ es impar.} \end{cases}$$

Demostración.

- Si m es un número impar positivo, entonces $m = 2p+1$ para algún número positivo p. De modo que,

$$\begin{aligned}
\operatorname{sen}^n(x)\cos^m(x) &= \operatorname{sen}^n(x)\cos^{2p+1}(x) = \operatorname{sen}^n(x)\cos^{2p}(x)\cos(x) \\
&= \operatorname{sen}^n(x)(\cos^2(x))^p\cos(x) \\
&= \operatorname{sen}^n(x)[1-\operatorname{sen}^2(x)]^p\cos(x).
\end{aligned}$$

Por lo que,

$$\begin{aligned}\int \operatorname{sen}^n(x)\cos^m(x)\,dx &= \int \operatorname{sen}^n(x)[1-\operatorname{sen}^2(x)]^p\cos(x)\,dx\\ &= \int \operatorname{sen}^n(x)[1-\operatorname{sen}^2(x)]^{\frac{m-1}{2}}\cos(x)\,dx\end{aligned}$$

- Si n es impar, procedemos de forma similar con $n = 2p+1$.

$$\begin{aligned}\int \cos^m(x)\operatorname{sen}^n(x)\,dx &= \int \cos^m(x)[1-\cos^2(x)]^p\operatorname{sen}(x)\,dx\\ &= \int \cos^m(x)[1-\cos^2(x)]^{\frac{n-1}{2}}\operatorname{sen}(x)\,dx.\end{aligned}$$

□

Para resolver $\int \operatorname{sen}^m(x)\cos^n(x)\,dx$ empleamos la identidad:

- $\operatorname{sen}^2(x) = \dfrac{1-\cos(2x)}{2}$ y $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
  para m y n pares

- $\operatorname{sen}^2(x) + \cos^2(x) = 1$, para m o n impar.

Ejemplo 20. *Calcular* $\int \operatorname{sen}^2(x)\cos^5(x)\,dx$.

Solución. Hacemos $u = \operatorname{sen}(x) \implies du = \cos(x)\,dx$.

$$\int \operatorname{sen}^2(x)\cos^5(x)\,dx = \int \operatorname{sen}^2(x)[\cos^2(x)]^2\cos(x)\,dx$$

$$
\begin{aligned}
&= \int \operatorname{sen}^2(x)[1-\operatorname{sen}^2(x)]^2 \cos(x)\,dx \\
&= \int \operatorname{sen}^2(x)[1-2\operatorname{sen}^2(x)+\operatorname{sen}^4(x)] \cos(x)\,dx \\
&= \int \operatorname{sen}^2(x)\cos(x)dx - 2\int sen^4(x)\cos(x)dx \\
&\qquad\qquad + \int sen^6(x)\cos(x)dx \\
&= \int u^2\,du - 2\int u^4\,du + \int u^6\,du \\
&= \frac{u^3}{3} - 2\left(\frac{u^5}{5}\right) + \frac{u^7}{7} + C \\
&= \frac{\operatorname{sen}^3(x)}{3} - \frac{2\operatorname{sen}^5(x)}{5} + \frac{\operatorname{sen}^7(x)}{7} + C.
\end{aligned}
$$

□

Ejemplo 21. *Calcular* $\int \operatorname{sen}^3(x)\cos^3(x)\,dx$.

Solución. Usamos la identidad $\operatorname{sen}(x)\cos(x) = \dfrac{\operatorname{sen}(2x)}{2}$ y sean $w = 2x \implies dw = 2dx$, mientras que

$$u = \cos(w) \implies du = -\operatorname{sen}(w)\,dw.$$

$$
\begin{aligned}
\int \operatorname{sen}^3(x)\cos^3(x)\,dx &= \int \left(\frac{\operatorname{sen}(2x)}{2}\right)^3 dx \\
&= \frac{1}{8}\int \operatorname{sen}^3(2x)\,dx \\
&= \frac{1}{8}\int \operatorname{sen}(2x)\operatorname{sen}^2(2x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{8}\int \operatorname{sen}(2x)[1-\cos^2(2x)]\,dx \\
&= \frac{1}{8}\int \operatorname{sen}(2x)\,dx - \frac{1}{8}\int \operatorname{sen}(2x)\cos^2(2x)\,dx \\
&= \frac{1}{8}\left(\frac{1}{2}\right)\int \operatorname{sen}(w)\,dw - \frac{1}{8}\left(\frac{1}{2}\right)\int \operatorname{sen}(w)\cos^2(w)\,dw \\
&= \frac{1}{16}\int \operatorname{sen}(w)\,dw + \frac{1}{16}\int u^2\,du \\
&= -\frac{1}{16}\cos(w) + \frac{1}{16}\left(\frac{u^3}{3}\right) + C \\
&= -\frac{1}{16}\cos(2x) + \frac{\cos^3(w)}{48} + C \\
&= -\frac{1}{16}\cos(2x) + \frac{\cos^3(2x)}{48} + C.
\end{aligned}
$$

□

Teorema 22. *Si m y n son números impares positivos, entonces*

$$\int \sec^n(x)\tan^m(x)\,dx = \int u^{n-1}[u^2-1]^{\frac{m-1}{2}}\,du,$$

en donde $u = \sec(x)$.

Demostración. Como m y n son impares positivos, los podemos escribir como $m = 2p+1$ y $n = 2q+1$, para cualesquiera enteros positivos p y q. Así que,

$$
\begin{aligned}
\sec^n(x)\tan^m(x) &= \sec^{2q+1}(x)\tan^{2p+1}(x) \\
&= \sec^{2q}(x)(\tan^2(x))^p\sec(x)\tan(x).
\end{aligned}
$$

de manera que, si

$$u = \sec(x) \Longrightarrow du = \sec(x)\tan(x)\,dx$$

$$\begin{aligned}
\int \sec^n(x)\tan^m(x)\,dx &= \int \sec^{2q}(x)[\tan^2(x)]^p \sec(x)\tan(x)\,dx \\
&= \int \sec^{2q}(x)[\sec^2(x)-1]^p \sec(x)\tan(x)\,dx \\
&= \int u^{2q}(u^2-1)^p\,du \\
&= \int u^{n-1}[u^2-1]^{\frac{m-1}{2}}\,du.
\end{aligned}$$

□

Teorema 23. *Si alguno de los enteros positivos m o n es par, entonces*

$$\int \sec^n(x)\tan^m(x)\,dx = \begin{cases} \displaystyle\int u^m(u^2+1)^{\frac{n-2}{2}}\,du, & \text{si } n \text{ es par y } \ u=\tan(x) \\ \displaystyle\int \sec^n(x)[\sec^2(x)-1]^{\frac{m}{2}}\,dx, & \text{si } m \text{ es par.} \end{cases}$$

Demostración. Si n es un número par positivo, entonces $n-2$ es par, de modo que $n-2=2p$ para un entero positivo p. Luego, ponemos

$$u=\tan(x) \Longrightarrow du=\sec^2(x)\,dx.$$

$$\begin{aligned}
\int \sec^n(x)\tan^m(x)\,dx &= \int \sec^{n-2}(x)\tan^m(x)\sec^2(x)\,dx \\
&= \int \sec^{2p}(x)\tan^m(x)\sec^2(x)\,dx \\
&= \int (\sec^2(x))^p\tan^m(x)\sec^2(x)\,dx \\
&= \int [\tan^2(x)+1]^p\tan^m(x)\sec^2(x)\,dx \\
&= \int (u^2+1)^p u^m\,du \\
&= \int (u^2+1)^{\frac{n-2}{2}} u^m\,du
\end{aligned}$$

Si m es par positivo, entonces $m = 2q$ para un entero positivo q. Luego,

$$\begin{aligned}\int \sec^n(x)\tan^m(x)\,dx &= \int \sec^n(x)\tan^{2q}(x)\,dx \\ &= \int \sec^n(x)(\tan^2(x))^q\,dx \\ &= \int \sec^n(x)[\sec^2(x)-1]^q\,dx \\ &= \int \sec^n(x)[\sec^2(x)-1]^{\frac{m}{2}}\,dx.\end{aligned}$$

$\square$

Para resolver $\int \tan^m(x)\sec^n(x)\,dx,\quad \int \cot^m(x)\csc^n(x)\,dx$:

- Si n-par y $m \in \mathbb{R}$, se emplea

$$\sec^2(x) = 1 + \tan^2(x) \quad \text{o bien} \quad \csc^2(x) = 1 + \cot^2(x)$$

- Si m-impar y $n \in \mathbb{R}$, se usa

$$\tan^2(x) = \sec^2(x) - 1 \quad \text{o bien} \quad \cot^2(x) = \csc^2(x) - 1$$

Ejemplo 24. *Calcular* $\int \tan^3(3x)\sec(3x)\,dx$.

Solución. Primero, hacemos $u = \sec(3x) \implies du = 3\sec(3x)\tan(3x)\,dx$.

$$\int \tan^3(3x)\sec(3x)\,dx = \int \tan^2(3x)\tan(3x)\sec(3x)\,dx$$

$$
\begin{aligned}
&= \int [\sec^2(3x) - 1] \tan(3x) \sec(3x)\, dx \\
&= \int \sec^2(3x) \tan(3x) \sec(3x) dx - \int \tan(3x) \sec(3x) dx \\
&= \frac{1}{3} \int u^2\, du - \frac{1}{3} \int 3 \tan(3x) \sec(3x)\, dx \\
&= \frac{1}{9} u^3 - \frac{1}{3} \sec(3x) + C \\
&= \frac{1}{9} \sec^3(3x) - \frac{1}{3} \sec(3x) + C.
\end{aligned}
$$

□

Ejemplo 25. *Calcular* $\int \sec^3(x+1) \tan(x+1)\, dx$.

Hacemos $u = x + 1 \implies du = dx$ y $w = \sec(u) \implies dw = \sec(u) \tan(u)\, du$.

$$
\begin{aligned}
\int \sec^3(x+1) \tan(x+1)\, dx &= \int \sec^3(u) \tan(u)\, du \\
&= \int \sec^2(u) \sec(u) \tan(u)\, du \\
&= \int w^2\, dw = \frac{1}{3} w^3 + C \\
&= \frac{1}{3} \sec^3(u) + C \\
&= \frac{1}{3} \sec^3(x+1) + C.
\end{aligned}
$$

Ejemplo 26. *Calcular* $\displaystyle\int \tan^{\frac{3}{2}}(x) \sec^4(x)\, dx$.

Solución. Sea $u = \tan(x) \implies du = \sec^2(x)\, dx$.

$$\begin{aligned}
\int \tan^{\frac{3}{2}}(x)\sec^4(x)\,dx &= \int \tan^{\frac{3}{2}}(x)\sec^2(x)\sec^2(x)\,dx \\
&= \int \tan^{\frac{3}{2}}(x)[\tan^2(x)+1]\sec^2(x)\,dx \\
&= \int \tan^{\frac{3}{2}}(x)\sec^2(x)\,dx + \int \tan^{\frac{7}{2}}(x)\sec^2(x)\,dx \\
&= \int u^{\frac{3}{2}}\,du + \int u^{\frac{7}{2}}\,du \\
&= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{9}u^{\frac{9}{2}} + C \\
&= \frac{2}{5}\tan^{\frac{5}{2}}(x) + \frac{2}{9}\tan^{\frac{9}{2}}(x) + C.
\end{aligned}$$

□

Ejemplo 27. *Calcular* $\int \tan^4(x)\sec^4(x)\,dx$.

Solución. Sea $u = \tan(x) \implies du = \sec^2(x)\,dx$.

$$\begin{aligned}
\int \tan^4(x)\sec^4(x)\,dx &= \int \tan^4(x)\sec^2(x)\sec^2(x)\,dx \\
&= \int \tan^4(x)[\tan^2(x)+1]\sec^2(x)\,dx \\
&= \int \tan^4(x)\sec^2(x)\,dx + \int \tan^6(x)\sec^2(x)\,dx \\
&= \int u^4\,du + \int u^6\,du \\
&= \frac{u^5}{5} + \frac{u^7}{7} + C = \frac{\tan^5(x)}{5} + \frac{\tan^7(x)}{7} + C.
\end{aligned}$$

□

Teorema 28. *Si m y n son números impares positivos, entonces*

$$\int \csc^n(x)\cot^m(x)\,dx = -\int u^{n-1}[u^2-1]^{\frac{m-1}{2}}\,du,$$

en donde $u = \csc(x)$.

Demostración. Si m y n son números impares positivos, entonces $m = 2p+1$ y $n = 2q+1$, para p y q enteros positivos. Además, ponemos

$$u = \csc(x) \Longrightarrow du = -\csc(x)\cot(x)\,dx.$$

$$\begin{aligned}
\int \csc^n(x)\cot^m(x)\,dx &= \int \csc^{2q+1}(x)\cot^{2p+1}(x)\,dx \\
&= \int \csc^{2q}(x)[\cot^2(x)]^p\csc(x)\cot(x)\,dx \\
&= \int csc^{2q}(x)[\csc^2(x)-1]^p\csc(x)\cot(x)\,dx \\
&= -\int u^{2q}[u^2-1]^p\,du \\
&= -\int u^{n-1}[u^2-1]^p\,du.
\end{aligned}$$

□

Ejemplo 29. *Calcular* $\int \csc^4(x)\cot^3(x)\,dx.$

Solución. Sea $u = \cot(x) \Longrightarrow du = -csc^2(x)\,dx$.

$$\int \csc^4(x)\cot^3(x)\,dx = \int \cot^3(x)csc^2(x)\csc^2(x)\,dx$$

$$
\begin{aligned}
&= \int \cot^3(x)[1+\cot^2(x)]\csc^2(x)\,dx \\
&= \int \cot^3(x)\csc^2(x)\,dx + \int \cot^5(x)\csc^2(x)\,dx \\
&= -\int u^3\,du - \int u^5\,du \\
&= -\frac{u^4}{4} - \frac{u^6}{6} + C = -\frac{\cot^4(x)}{4} - \frac{\cot^6(x)}{6} + C.
\end{aligned}
$$

□

Ejemplo 30. *Calcular* $\int \cot(x)\csc^3(x)\,dx$.

Solución. Hacemos $u = \csc(x) \implies du = -\csc(x)\cot(x)\,dx.$

$$
\begin{aligned}
\int \cot(x)\csc^3(x)\,dx &= \int \csc^2(x)\csc(x)\cot(x)\,dx \\
&= -\int u^2\,du = -\frac{u^3}{3} + C \\
&= -\frac{\csc^3(x)}{3} + C.
\end{aligned}
$$

□

Ejemplo 31. *Calcular* $\displaystyle\int \csc^4(3x)\cot(3x)\,dx$.

Solución. Sea $u = \cot(3x) \implies du = -3csc^2(3x)\,dx$.

$$
\int \csc^4(3x)\cot(3x)\,dx = \int \csc^2(3x)\csc^2(3x)\cot(3x)\,dx
$$

$$
\begin{aligned}
&= \int \csc^2(3x)[1+\cot^2(3x)]\cot(3x)\,dx \\
&= \int \csc^2(3x)\cot(3x)\,dx + \int \csc^2(3x)\cot^3(3x)\,dx \\
&= -\frac{1}{3}\int u\,du - \frac{1}{3}\int u^3\,du \\
&= -\frac{u^2}{6} - \frac{u^4}{12} + C = -\frac{\cot^2(3x)}{6} - \frac{\cot^4(3x)}{12} + C.
\end{aligned}
$$

$\square$

Teorema 32.

$$
\int \csc^n(x)\cot^m(x)\,dx = \begin{cases} -\displaystyle\int (u^2+1)^{\frac{n-2}{2}} u^m\,du, & \textit{si } n \textit{ es par y } u=\cot(x) \\ \displaystyle\int \csc^n(x)[\csc^2(x)-1]^{\frac{m}{2}}\,dx, & \textit{si } m \textit{ es par.} \end{cases}
$$

Demostración. Si n es un número par positivo, entonces $n-2$ es par, de modo que $n-2=2p$ para un entero positivo p. Luego,

$$
\begin{aligned}
\csc^n(x)\cot^m(x) &= \csc^{n-2}(x)\cot^m(x)\csc^2(x) \\
&= \csc^{2p}(x)\cot^m(x)\csc^2(x) \\
&= (\csc^2(x))^p\cot^m(x)\csc^2(x) \\
&= [\cot^2(x)+1]^p\cot^m(x)\csc^2(x).
\end{aligned}
$$

y ponemos $u=\cot(x) \Longrightarrow du = -\csc^2(x)\,dx.$

$$
\begin{aligned}
\int \csc^n(x)\cot^m(x)\,dx &= \int [\cot^2(x)+1]^p\cot^m(x)\csc^2(x)\,dx \\
&= -\int (u^2+1)^p u^m\,du \\
&= -\int (u^2+1)^{\frac{n-2}{2}} u^m\,du
\end{aligned}
$$

Si m es par positivo, entonces $m=2q$ para un entero positivo q. Luego,

$$
\begin{aligned}
\int \csc^n(x)\cot^m(x)\,dx &= \int \csc^n(x)\cot^{2q}(x)\,dx \\
&= \int \csc^n(x)(\cot^2(x))^q\,dx \\
&= \int \csc^n(x)[\csc^2(x)-1]^q\,dx \\
&= \int \csc^n(x)[\csc^2(x)-1]^{\frac{m}{2}}\,dx.
\end{aligned}
$$

□

Tipo 3: Integrales de la forma $\int \operatorname{sen}(mx)\cos(nx)\,dx$

Teorema 33. *Si m y n son números distintos entre sí y no son simétricos, entonces*

$$
\int \operatorname{sen}(mx)\cos(nx)\,dx = -\frac{1}{2}\left(\frac{\cos([m+n]x)}{m+n}+\frac{\cos([m-n]x)}{m-n}\right)+C.
$$

Demostración. Se consideran las siguientes identidades trigonométricas:

$$
\operatorname{sen}(mx+nx) = \operatorname{sen}([m+n]x) = \operatorname{sen}(mx)\cos(nx)+\operatorname{sen}(nx)\cos(mx),
$$

$$
\operatorname{sen}(mx-nx) = \operatorname{sen}([m-n]x) = \operatorname{sen}(mx)\cos(nx)-\operatorname{sen}(nx)\cos(mx),
$$

Sumando ambas identidades, miembro a miembro, se obtiene:

$$
\operatorname{sen}(mx)\cos(nx) = \frac{1}{2}\operatorname{sen}([m+n]x)+\frac{1}{2}\operatorname{sen}([m-n]x).
$$

Por tanto, haciendo $u=[m+n]x \Longrightarrow du=(m+n)dx$ y $w=[m-n]x \Longrightarrow du=(m-n)dx$.

$$
\begin{aligned}
\int \operatorname{sen}(mx)\cos(nx)\,dx &= \int \frac{\operatorname{sen}([m+n]x)+\operatorname{sen}([m-n]x)}{2}\,dx \\
&= \frac{1}{2}\int \operatorname{sen}([m+n]x)\,dx + \frac{1}{2}\int \operatorname{sen}([m-n]x)\,dx \\
&= \frac{1}{2}\left(\frac{1}{m+n}\right)\int \operatorname{sen}(u)\,du + \frac{1}{2}\left(\frac{1}{m-n}\right)\int \operatorname{sen}(w)\,dw \\
&= -\frac{1}{2}\left(\frac{\cos(u)}{m+n}\right) - \frac{1}{2}\left(\frac{\cos(w)}{m-n}\right) + C \\
&= -\frac{1}{2}\left(\frac{\cos([m+n]x)}{m+n}\right) - \frac{1}{2}\left(\frac{\cos([m-n]x)}{m-n}\right) + C \\
&= -\frac{1}{2}\left(\frac{\cos([m+n]x)}{m+n} + \frac{\cos([m-n]x)}{m-n}\right) + C.
\end{aligned}
$$

□

Teorema 34. *Si m y n son números distintos entre sí y no son simétricos, entonces*

$$\int \operatorname{sen}(mx)\operatorname{sen}(nx)\,dx = \frac{1}{2}\left(\frac{\operatorname{sen}([m-n]x)}{m-n} - \frac{\operatorname{sen}([m+n]x)}{m+n}\right) + C.$$

Demostración. Se consideran las siguientes identidades trigonométricas:

$$\cos(mx-nx) = \cos(mx)\cos(nx) + \operatorname{sen}(nx)\operatorname{sen}(mx), \qquad (1)$$

$$\cos(mx+nx) = \cos(mx)\cos(nx) - \operatorname{sen}(nx)\operatorname{sen}(mx), \qquad (2)$$

Restando (1) menos (2) , miembro a miembro, se obtiene:

$$\operatorname{sen}(mx)\operatorname{sen}(nx) = \frac{1}{2}\cos([m-n]x) - \frac{1}{2}\cos([m+n]x).$$

Por tanto, análogamente haciendo los cambios de variable apropiados se tiene que:

$$
\begin{aligned}
\int \operatorname{sen}(mx)\operatorname{sen}(nx)\,dx &= \frac{1}{2}\int \cos([m-n]x)\,dx - \frac{1}{2}\int \cos([m+n]x)\,dx \\
&= \frac{1}{2}\left(\frac{\operatorname{sen}([m-n]x)}{m-n}\right) - \frac{1}{2}\left(\frac{\operatorname{sen}([m+n]x)}{m+n}\right) + C \\
&= \frac{1}{2}\left(\frac{\operatorname{sen}([m-n]x)}{m-n} - \frac{\operatorname{sen}([m+n]x)}{m+n}\right) + C.
\end{aligned}
$$

□

Teorema 35. *Si m y n son números distintos entre sí y no son simétricos, entonces*

$$\int \cos(mx)\cos(nx)\,dx = \frac{1}{2}\left(\frac{\operatorname{sen}([m+n]x)}{m+n} + \frac{\operatorname{sen}([m-n]x)}{m-n}\right) + C.$$

Demostración. Se consideran las siguientes identidades trigonométricas:

$$\cos(mx+nx) = \cos(mx)\cos(nx) - \operatorname{sen}(nx)\operatorname{sen}(mx),$$

$$\cos(mx-nx) = \cos(mx)\cos(nx) + \operatorname{sen}(nx)\operatorname{sen}(mx),$$

Sumando ambas identidades, miembro a miembro, se obtiene:

$$\cos(mx)\cos(nx) = \frac{1}{2}\cos([m+n]x) + \frac{1}{2}\cos([m-n]x).$$

Por tanto,

$$
\begin{aligned}
\int \cos(mx)\cos(nx)\,dx &= \frac{1}{2}\int \cos([m+n]x)\,dx + \frac{1}{2}\int \cos([m-n]x)\,dx \\
&= \frac{1}{2}\left(\frac{\operatorname{sen}([m+n]x)}{m+n}\right) + \frac{1}{2}\left(\frac{\operatorname{sen}([m-n]x)}{m-n}\right) + C \\
&= \frac{1}{2}\left(\frac{\operatorname{sen}([m+n]x)}{m+n} + \frac{\operatorname{sen}([m-n]x)}{m-n}\right) + C.
\end{aligned}
$$

□

Integrales de la forma $\int \operatorname{sen}(mx)\cos(nx)\,dx$

- $\operatorname{sen}(mx)\cos(nx) = \frac{1}{2}[\operatorname{sen}([m+n]x) + \operatorname{sen}([m-n]x)$
- $\operatorname{sen}(mx)\operatorname{sen}(nx) = -\frac{1}{2}[\cos([m+n]x) - \cos([m-n]x)$
- $\cos(mx)\cos(nx) = \frac{1}{2}[\cos([m+n]x) + \cos([m+n]x)$

Ejemplo 36. *Calcular* $\int \operatorname{sen}(2x)\cos(4x)\,dx$.

Solución. Hacemos $z = 6x \implies dz = 6\,dx$ y $w = 2x \implies dw = 2dx$.

$$\begin{aligned}
\int \operatorname{sen}(2x)\cos(4x)\,dx &= \frac{1}{2}\int \operatorname{sen}([2+4]x) + \operatorname{sen}([2-4]x)\,dx \\
&= \frac{1}{2}\int \operatorname{sen}(6x) + \operatorname{sen}(-2x)\,dx \\
&= \frac{1}{2}\int \operatorname{sen}(6x) - \operatorname{sen}(2x)\,dx \\
&= \frac{1}{2}\int \operatorname{sen}(6x)\,dx - \frac{1}{2}\int \operatorname{sen}(2x)\,dx \\
&= \frac{1}{2}\left(\frac{1}{6}\right)\int \operatorname{sen}(z)\,dz - \frac{1}{2}\left(\frac{1}{2}\right)\int \operatorname{sen}(w)\,dw \\
&= -\frac{1}{12}\cos(z) + \frac{1}{4}\cos(w) + C \\
&= -\frac{1}{12}\cos(6x) + \frac{1}{4}\cos(2x) + C.
\end{aligned}$$

□

Ejemplo 37. *Calcular* $\int \operatorname{sen}(4y)\cos(5y)\,dy$.

Hacemos $u = 9y \implies du = 9\,dy$ y $w = -y \Longrightarrow dw = -\,dy$.

$$\begin{aligned}
\int \operatorname{sen}(4y)\cos(5y)\,dy &= \int \frac{1}{2}[\operatorname{sen}([4+5]y) + \operatorname{sen}([4-5]y)\,dy \\
&= \int \frac{1}{2}[\operatorname{sen}(9y) + \operatorname{sen}(-y)]\,dy \\
&= \frac{1}{2}\int \operatorname{sen}(9y)\,dy + \frac{1}{2}\int \operatorname{sen}(-y)\,dy \\
&= \frac{1}{2}\left(\frac{1}{9}\right)\int \operatorname{sen}(u)\,du + \frac{1}{2}(-1)\int \operatorname{sen}(w)\,dw \\
&= \frac{1}{18}[-\cos(u)] - \frac{1}{2}[-\cos(w)] + C \\
&= -\frac{\cos(9y)}{18} + \frac{\cos(-y)}{2} + C.
\end{aligned}$$

Ejemplo 38. *Calcular* $\int \cos(\sqrt{2}\,x)\cos(x)\,dx$.

Hacemos $u = [\sqrt{2}+1]x \implies du = \sqrt{2}+1\,dx$ y $w = [\sqrt{2}-1]x \implies dw = \sqrt{2}-1\,dx$

$$\begin{aligned}
\int \cos(\sqrt{2}\,x)\cos(x)\,dx &= \frac{1}{2}\int \cos([\sqrt{2}+1]x) + \cos([\sqrt{2}-1]x)\,dx \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{2}+1}\right)\int \cos(u)\,du + \frac{1}{2}\left(\frac{1}{\sqrt{2}-1}\right)\int \cos(w)\,dw \\
&= \frac{1}{2(\sqrt{2}+1)}\operatorname{sen}(u) + \frac{1}{2(\sqrt{2}-1)}\operatorname{sen}(w) + C \\
&= \frac{1}{2(\sqrt{2}+1)}\operatorname{sen}([\sqrt{2}+1]x) + \frac{1}{2(\sqrt{2}-1)}\operatorname{sen}([\sqrt{2}-1]x) + C.
\end{aligned}$$

Ejemplo 39. *Calcular* $\displaystyle\int \cos(3x)\cos(2x)\,dx$.

Solución.

$$\begin{aligned}
\int \cos(3x)\cos(2x)\,dx &= \frac{1}{2}\int \cos([3-2]x)+\cos([3+2]x)\,dx \\
&= \frac{1}{2}\int \cos(x)+\cos(5x)\,dx \\
&= \frac{1}{2}\int \cos(x)\,dx + \frac{1}{2}\int \cos(5x)\,dx \\
&= \frac{1}{2}\operatorname{sen}(x) + \frac{1}{10}\operatorname{sen}(5x) + C.
\end{aligned}$$

□

Ejemplo 40. *Calcular* $\int \operatorname{sen}(5x)\operatorname{sen}(x)\,dx.$

Solución.

$$\begin{aligned}
\int \operatorname{sen}(5x)\operatorname{sen}(x)\,dx &= -\frac{1}{2}\int \cos([5+1]x)-\cos([5-1]x)\,dx \\
&= -\frac{1}{2}\int \cos(6x)-\cos(4x)\,dx \\
&= \frac{1}{2}\int \cos(4x)\,dx - \frac{1}{2}\int \cos(6x)\,dx \\
&= \frac{1}{8}\operatorname{sen}(4x) - \frac{1}{12}\operatorname{sen}(6x) + C.
\end{aligned}$$

□