Fracciones Parciales (Tema 6)

Def Una función racional es un cociente de la forma:

$$\frac{P(x)}{Q(x)} \quad \text{o} \quad \frac{ax+b}{(x^2+px+q)^n}$$

donde $a, b, p, q \in \mathbb{R}$ son constantes, $n \in \mathbb{N}$ y el polinomio x^2+px+q tiene no soluciones en $\mathbb{R}$, sin ser cero, $p^2-4q<0$.

Sea $f(x) = \frac{P(x)}{Q(x)}$ una función racional propia.

Caso 1 El polinomio $Q(x)$ sólo contiene factores lineales simples (no repetidos).

Por ejemplo:

$$f(x) = \frac{P(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \qquad (x \neq \pm 1)$$

Sistemáticamente,

$$f(x) = \frac{P(x)}{(x-a)(x-b)(x-c)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{A_3}{x-a_3} \qquad (x \neq 3)$$

y así sucesivamente.

Ejemplos

① $\int \frac{3x-5}{x^2-3x-2}\,dx = \int \frac{3x-5}{(x-2)(x+1)}\,dx = \int \frac{A}{x-2} + \frac{B}{x+1}\,dx$

$$\frac{3x-5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Método 1. Sólo nos fijamos en los numeradores:

$$3x-5 = A(x+1)+B(x-2)$$
$$= Ax+A+Bx-2B$$
$$= (A+B)x+(A-2B)$$

$\Rightarrow \begin{cases} A+B=3 \\ A-2B=-5 \end{cases}$ Lo resolvemos: $\begin{matrix} A=1 \\ B=2 \end{matrix}$

Método 2. $3x-5 = A(x+1)+B(x-2)$ ——— (*)

Sustituir en (*):

$x=-1 \Rightarrow 3(-1)-5 = A(-1+1)+B(-1-2)$
$\quad -3-5 = -3B$
$\quad -8 = -3B \qquad \therefore B=\frac{8}{3}$

$x=2 \Rightarrow 3(2)-5 = A(2+1)+B(2-2)$
$\quad 6-5 = 3A \qquad \therefore A=\frac{1}{3}$

$$= \int \frac{1}{x-2} + \frac{2}{x+1}\,dx = \int \frac{1}{x-2}\,dx + 2\int \frac{1}{x+1}\,dx$$

$\boxed{\begin{matrix} u=x-2 \\ du=dx \end{matrix}}$ $\boxed{\begin{matrix} u=x+1 \\ du=dx \end{matrix}}$

$$= 3\int \frac{1}{u}\,du + 2\int \frac{1}{u}\,du = 3\ln|u| + 2\ln|u| + C$$
$$= 3\ln|x-2| + 2\ln|x+1| + C$$

② $\int \frac{2x^2+3x-5}{x^3-x^2+x-1}\,dx = \int \frac{2x^2+3x-5}{(x-1)(x^2+1)}\,dx$

$$\frac{2x^2+3x-5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx}{x^2+1} + \frac{C}{x^2+1}$$

$$2x^2+3x-5 = A(x^2+1)+x(x-1)B + (x-1)C \qquad (*)$$
$$\quad 1 \qquad\qquad x \qquad\qquad x^2 \qquad\qquad 1$$

Método 1

$$2x^2+3x-5 = A(x^2+1)+x(x-1)B + (x-1)C$$
$$= Ax^2+A+Bx^2-Bx+Cx-C = (A+B)x^2+(-B+C)x+(A-C)$$
$$= (A+B)x^2+(-B+C)x+(A-C)$$

$$\begin{cases} A+B=2 \\ -B+C=3 \\ A-C=-5 \end{cases} \Rightarrow \begin{matrix} A=-1 \\ B=3 \\ C=2 \end{matrix}$$

Método 2. Dart en (*):

$x=1 \Rightarrow 2+3-5 = A(1+1)+B(1)(1-1)+C(1-1) \Rightarrow 0=2A \Rightarrow A=0$

$x=1 \Rightarrow 2(-1)+3(1)-5 = A(-1+1)+B(-1)(-1-1)+C(-1-1) \Rightarrow -4=4B \Rightarrow B=-1$

$x=0 \Rightarrow 2(0)+3(0)-5 = A(0+1)+B(0)(0-1)+C(0-1) \Rightarrow -5=A-C \Rightarrow C=-5$

$$= \int \frac{1}{x-1} + \frac{x}{x^2+1} - \frac{5}{x^2+1}\,dx$$
$$= \int \frac{1}{x-1}\,dx + \int \frac{x}{x^2+1}\,dx - 5\int \frac{1}{x^2+1}\,dx$$

$\boxed{\begin{matrix} u=x-1 \\ du=dx \end{matrix}}$ $\boxed{\begin{matrix} u=x^2+1 \\ du=2x\,dx \end{matrix}}$

$$= \int \frac{1}{u}\,du + \frac{1}{2}\int \frac{2x}{u}\,dx - 5\int \frac{1}{x^2+1}\,dx$$
$$= \ln|u| + \frac{1}{2}\ln|u| - 5\arctan(x) + C$$
$$= \ln|x-1| + \frac{1}{2}\ln|x^2+1| - 5\arctan(x) + C$$

Caso 2 El polinomio $Q(x)$ sólo tiene factores lineales, donde algunos de ellos están repetidos.

$$f(x) = \frac{P(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n} \dots$$

"La potencia n se descompone en n sumandos"

Por ejemplo:

$$\frac{P(x)}{(x-a)(x-b)^2(x-c)^3} = \frac{A_1}{x-a} + \frac{A_2}{x-b} + \frac{A_3}{(x-b)^2} + \frac{A_4}{x-c} + \frac{A_5}{(x-c)^2} + \frac{A_6}{(x-c)^3}$$

③ $\int \frac{3x^2+8x-2}{x(x+2)^2}\,dx$

$$\frac{3x^2+8x-2}{x(x+2)^2} = \frac{A_1}{x} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}$$

$$3x^2+8x-2 = A(x+2)^2 + B(x)(x+2) + Cx \qquad (*)$$
$$\quad 1 \qquad\qquad x \qquad\qquad x^2 \qquad\qquad 1$$

Sustituimos en (*):

$x=0 \Rightarrow -2 = 4A \Rightarrow A=-\frac{1}{2}$

$x=-2 \Rightarrow 3(4)+16-2 = 2C \Rightarrow 14=2C \Rightarrow C=7$

$x=1 \Rightarrow 3+8-2 = A(3)^2 + B(1)(3) + C(1)$
$\quad 9 = -\frac{9}{2} + 3B + \frac{7}{2} \Rightarrow \frac{27}{2} = 3B$
$\quad \therefore B=\frac{9}{2}$

$$= -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{9}{2}\int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{7}{(x+2)^2}$$

$\boxed{\begin{matrix} u=x+2 \\ du=dx \end{matrix}}$

$$= -\frac{1}{2}\ln|x| + \frac{9}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int \frac{7}{u^2}\,du$$
$$= -\frac{1}{2}\ln|x| + \frac{9}{2}\ln|x+2| + \frac{7}{2u} + C$$
$$= -\frac{1}{2}\ln|x| + \frac{9}{2}\ln|x+2| + \frac{7}{2(x+2)} + C$$

④ $\int \left[\frac{x}{x^2-x+2}\right]^2 dx = \int \frac{x^2}{(x^2-x+2)^2}\,dx$

$$= \int \frac{x^2}{(x-1)(x-2)^2}\,dx = \int \frac{A_1}{x-1} + \frac{A_2}{(x-2)} + \frac{A_3}{(x-2)^2}\,dx$$

$$x^2 = A_1(x-1)(x-2)^2 + B_1(x-2)^2 + C_1(x-1)(x-2) + D_1(x-1)^2$$
$$\quad 1 \qquad\qquad x \qquad\qquad x^2 \qquad\qquad 1$$

$x=1 \Rightarrow 1 = A_1 \Rightarrow A_1=1$

$x=2 \Rightarrow 4 = D_1(2-1)^2 \Rightarrow D_1=4$

$x=0 \Rightarrow 0 = A_1(0-2)^2 + B_1(0-2)^2 + C_1(0-1)(0-2) + D_1(0-1)^2$
$\quad = (1)(4) + 4B_1 + 2C_1 + 4$
$\quad = -4 + 4B_1 + 2C_1 - 8C = 4B_1$
$\quad \Rightarrow \boxed{4B_1 + 2C_1 = 8}$

$x=3 \Rightarrow 9 = A_1(3-1)(3-2)^2 + B_1(3-2)^2 + C_1(3-1)(3-2) + D_1(3-1)^2$
$\quad \Rightarrow \boxed{9 = A_1 + B_1 + 2C_1 + 4D_1}$

$$\begin{cases} 4B_1+2C_1=8 \\ 9=A_1+B_1+2C_1+4D_1 \end{cases} \Rightarrow \begin{matrix} A_1=1 \\ B_1=-3 \end{matrix}$$

$$= \int \frac{1}{x-1} - \frac{3}{x-2} + \frac{4}{(x-2)^2}\,dx$$
$$= \int \frac{1}{x-1}\,dx - 3\int \frac{1}{x-2}\,dx + 4\int \frac{1}{(x-2)^2}\,dx$$
$$= \ln|x-1| - 3\ln|x-2| - \frac{4}{x-2} + C$$

Caso 3 El polinomio $Q(x)$ contiene factores cuadráticos irreducibles, sin nuevos de otro capítulo.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(x^2+px+q)^n} = \dots + \frac{ax+b}{(x^2+px+q)} + \dots$$

Por ejemplo,

$$f(x) = \frac{P(x)}{(x-a)(x^2+x)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+x} \qquad (x \neq 0)$$

y

$$f(x) = \frac{P(x)}{(x+2)(x+1)^2(x^2+2x+2)} = \frac{A_1}{x+2} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2} + \frac{Bx+C}{x^2+2x+2}$$

(con $x \neq 0$)

⑤ $\int \frac{x^2-3x+5}{x(x^2+x)}\,dx$

$$\frac{x^2-3x+5}{x(x^2+x)} = \frac{A}{x} + \frac{Bx+C}{x^2+x}$$

$$x^2-3x+5 = A(x^2+x) + (Bx+C)x$$
$$= Ax^2+Ax+Bx^2+Cx$$
$$= (A+B)x^2 + (A+C)x + (0A)$$

$$\Rightarrow \begin{cases} A+B=1 \\ A+C=-3 \\ 0A=5 \end{cases} \Rightarrow \begin{matrix} B=4 \\ C=-7 \end{matrix} \qquad \Rightarrow \boxed{A=\frac{5}{0}}$$

$$= \int \frac{-\frac{5}{0}}{x} + \frac{\frac{5}{0}x-3}{x^2+x}\,dx$$
$$= -\frac{5}{0}\int \frac{1}{x} + \int \frac{\frac{5}{0}x-3}{x^2+x}\,dx$$

$\boxed{\begin{matrix} u=x^2+x \\ du=(2x+1)\,dx \end{matrix}}$

$$= \frac{5}{0}\int \frac{1}{u}\,du - \frac{3}{2}\int \frac{2x}{u}\,dx$$
$$= \frac{5}{0}\ln|u| - \frac{3}{2}\ln|x^2+x| + C$$
$$= -\frac{5}{0}\ln|x| - \frac{3}{2}\ln|x^2+x| + C$$

⑥ $\int \frac{x+2}{x(x^2+x+1)}\,dx = \int \frac{x+2}{(x+1)(x^2+x+1)}\,dx$

$$= \frac{x+2}{x(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

irreducible, irreducible

$$x+2 = A(x^2+x+1) + (Bx+C)(x+1)$$
$$= Ax^2+Ax+A + Bx^2+Bx+Cx+C$$
$$= (A+B)x^2 + (A+B+C)x + (A+C)$$

$$\Rightarrow \begin{cases} A+B=0 \\ A+B+C=1 \\ A+C=2 \end{cases} \quad \text{Con } x=1 \text{ se } y=3A \Rightarrow A=\frac{1}{3}$$

$$\Rightarrow \begin{cases} 1+B=0 \\ 1+B+C=1 \\ 1+C=2 \end{cases} \Rightarrow \begin{matrix} B=-1 \\ C=1 \end{matrix}$$

$$= \int \frac{\frac{1}{3}}{x+1} - \int \frac{x-1}{x^2+x+1}\,dx$$
$$= \frac{1}{3}\ln|x+1| - \int \frac{x-1}{x^2+x+1}\,dx$$
$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\int \frac{2x-2}{x^2+x+1}\,dx$$

$\boxed{\begin{matrix} u=x^2+x+1 \\ du=(2x+1)\,dx \end{matrix}}$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\int \frac{2x}{u}\,dx + \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$
$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$

$\boxed{\text{Completar cuadrados}}$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{1}{\left(\frac{2x+1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{1}{\frac{(2x+1)^2}{4}+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{4}{(2x+1)^2+3}\,dx$$

$\boxed{\begin{matrix} u=2x+1 \\ du=2\,dx \end{matrix}}$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{4}{u^2+3}\,dx$$
$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\int \frac{4}{u^2+\left(\sqrt{3}\right)^2}\,dx$$

$\boxed{\text{Ej arctan } \frac{u}{\sqrt{a}}}$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{3}{2}\cdot\frac{4}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \int \frac{x+4}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx + C$$

$$= \ln\left|\frac{x+4}{x^2+x+1}\right| - \frac{3}{2}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

5) $\displaystyle\int \frac{3x^2+2x+4}{(x-1)(x^2+x+1)}\,dx = \int \frac{3x^2+2x+4}{(x^2-1)(x^2+x+1)}\,dx$

$$= \int \frac{3x^2+2x+4}{(x-1)(x+1)(x^2+x+1)}\,dx$$

$$\frac{3x^2+2x+4}{(x-1)(x+1)(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$$

$$3x^2+2x+4 = A(x+1)(x^2+x+1) + B(x-1)(x^2+x+1) + (Cx+D)(x-1)(x+1)$$

$$x=-1 \Rightarrow 3-2+4 = B(-2) \Rightarrow -3 = -2B \Rightarrow B = \frac{3}{2}$$

$$x=1 \Rightarrow 3+2+4 = A(3) \Rightarrow 9 = 3A \Rightarrow A = 3$$

$$x=0 \Rightarrow 3+0+4 = A+B+D \Rightarrow 3+0+4 = 3-\frac{3}{2}+D \Rightarrow D = \frac{3}{2}$$

$$x=2 \Rightarrow 12+8+4 = 3A+3B+6(2C+D) \Rightarrow 24 = 9+\frac{9}{2}+12C+3 \Rightarrow 12C = 3 \Rightarrow C = \frac{1}{4}$$

$$= \int \frac{3}{x-1}\,dx + \int \frac{-\frac{3}{2}x+\frac{3}{2}}{x^2+x+1}\,dx$$

$$= 3\int \frac{1}{x-1}\,dx - \frac{3}{2}\int \frac{x-1}{x^2+x+1}\,dx$$

$$\boxed{\begin{array}{l} u = x-1 \\ du = dx \end{array}} \qquad \boxed{\begin{array}{l} u = x^2+x+1 \\ du = 2x+1\,dx \end{array}} \qquad \boxed{\begin{array}{l} 2x+1 = 2x-1+2 \\ 2x+1 = 2x-1+3 \end{array}}$$

$$= 3\int \frac{1}{u}\,du - \frac{3}{2}\int \frac{2x-\frac{3}{2}}{x^2+x+1}\,dx$$

$$= 3\ln|u| - \frac{3}{2}\ln|u| - \frac{3}{2}\int \frac{dx}{x^2+x+1} + \int \frac{dx}{x^2+x+1}$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\int \frac{dx}{x^2+x+1} + \int \frac{dx}{x^2+x+1}$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\ln|x^2+x+1| + \arctan(x) + C$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\ln|x^2+x+1| + \arctan(x) + C$$

6) $\displaystyle\int \frac{dx}{x^3+2x^2+x}$

$$\frac{1}{(x^3+2x^2+x)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$1 = (Ax+B)(x^2+x+1) + (Cx+D)(x^2+x)$$

$$= Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + Cx^3 + Cx^2 + Dx$$

$$= (A+C)x^3 + (A+B+C+D)x^2 + (A+B+D)x + B$$

$$\Rightarrow \begin{cases} A+C=0 & \Rightarrow A=-C \\ B+D=0 & \Rightarrow B=-D \\ A+C=0 & \Rightarrow 1+C+C=0 \Rightarrow -1=2C \Rightarrow C=-\frac{1}{2} \\ A+B=1 & \Rightarrow 1+B=1 \Rightarrow B=0 \end{cases}$$

$$A = 1$$
$$B = 0$$
$$C = -\frac{1}{2}$$
$$D = 0$$

$$= \int \frac{1}{x}\,dx - \int \frac{\frac{1}{2}x}{x^2+x+1}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{2x^2+2x+2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+x+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\int \frac{x}{x^2+x+1}\,dx$$

Nota: El polinomio que contiene factores cuadráticos irreducibles, siguiente de otro repetido.

$$\text{Ejm} = \frac{P(x)}{Q_n(x)} = \frac{P(x)}{\ldots(x^2+bx+c)^{n-1}} =$$

$$= \ldots + \frac{Ax+B}{x^2+bx+c} + \frac{Ax+B}{(x^2+bx+c)^2} + \ldots + \frac{Ax+B}{(x^2+bx+c)^n} + \ldots$$

Por ejemplo

$$\frac{P(x)}{(x^2+1)(x^2-1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-1} + \frac{Ex+F}{(x^2-1)^2} \qquad (x\neq 1)$$

7) $\displaystyle\int \frac{x^2+x}{(x^2+1)^2}\,dx$

$$\frac{x^2+x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$x^2+x = (Ax+B)(x^2+1) + (Cx+D)$$

$$= Ax^3 + Bx + Ax^2 + B + Cx + D$$

$$= Ax^3 + Bx + Ax^2 + B + Cx + D$$

$$A = 0$$
$$B = 1$$
$$C = 1$$
$$D = 0$$

$$= \int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx \qquad \boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$$

$$= \int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$\int \frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2}\int \frac{-2x}{(x^2+1)^2}\,dx = -\frac{1}{2}\int \frac{2x-2}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{2}\int \frac{2x}{(x^2+1)^2} - \frac{1}{2}\int \frac{2}{(x^2+1)^2}$$

$$= -\frac{1}{2}\int \frac{du}{u^2} - \frac{1}{2}\int \frac{2}{(x^2+1)^2}$$

$$\boxed{\begin{array}{l} u = 2x \\ du = 2\,dx \end{array}} \qquad \boxed{\begin{array}{l} u^2+1 = 1+\tan^2(\theta) \\ = \sec^2(\theta) \end{array}}$$

$$\int \frac{2}{(x^2+1)^2}\,dx = \int \frac{2\sec^2(\theta)}{(1+\tan^2(\theta))^2}\,d\theta = 2\int \frac{d\theta}{\sec^2(\theta)}$$

$$= 2\int \cos^2(\theta)\,d\theta = 2\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \int d\theta + \int \cos(2\theta)\,d\theta \qquad \boxed{\begin{array}{l} u = 2\theta \\ du = 2\,d\theta \end{array}}$$

$$= \theta + \frac{1}{2}\int \cos(u)\,du$$

$$= \theta + \frac{1}{2}\sin(u) + C$$

$$= \theta + \frac{1}{2}\sin(2\theta) + C$$

$$= \theta + \sin(\theta)\cos(\theta) + C$$

$$\tan(\theta) = \frac{x}{1}$$

$$\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$$

$$\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$$

$$= \arctan(x)\left(\frac{1}{\sqrt{x^2+1}}\right) + \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$

$$= \frac{\arctan(x)}{\sqrt{x^2+1}} + \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$$

$$= -\frac{1}{2}\left[-\frac{1}{2}\right] + \frac{1}{2}\left[\frac{\arctan(x)}{\sqrt{x^2+1}}\right] + \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$$

$$= \frac{1}{4(x^2+1)} + \frac{1}{2}\left[\frac{\arctan(x)}{\sqrt{x^2+1}}\right] + \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$$

$$= \frac{1}{2}\ln|x^2+1| + \frac{1}{4(x^2+1)} + \frac{1}{2}\arctan(x) + C$$

8) $\displaystyle\int \frac{x^3}{x^2+2}\,dx$

El grado menor multiplo entre 2 y 3 es los factores. 2, 3 $\Rightarrow$ 6

$$= \int \frac{x^3}{x^2+2}\,(x^2)\,dx \qquad \boxed{\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}} \qquad \boxed{\begin{array}{l} x^2 = 2+x^2-2 \\ x^2 = 2+x^2-2 \end{array}}$$

$$= \frac{1}{2}\int \frac{x^3}{x^2+2}\,2x\,dx$$

$$= \frac{1}{2}\int u^{\frac{3}{2}}\,du \qquad u^{\frac{3}{2}} = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}u^{\frac{5}{2}} = \frac{2}{5}(x^2)^{\frac{5}{2}} = \frac{2}{5}x^{\frac{10}{2}} = \frac{2}{5}x^5$$

$$= \frac{1}{2}\int u^{\frac{3}{2}}\,du = \frac{1}{2}\int \frac{u^{\frac{3}{2}}}{1}\,du$$

$$= \frac{1}{2}\int u^{\frac{3}{2}}\,du = \frac{1}{2}\int du + 2\int \frac{u^{\frac{3}{2}}}{u+2}\,du$$

$$\frac{u^{\frac{3}{2}}}{u+2} = \frac{A}{u^{\frac{1}{2}}} + \frac{B}{u+2}$$

$$u^{\frac{3}{2}} = Au^{\frac{1}{2}} + A(2) + Bu^{\frac{1}{2}}$$

$$= (A+B)u^{\frac{1}{2}} + (1+B+0)u + (B+0)$$

$$u^{0.5} \Rightarrow A+B = 0$$

$$u^{1} \Rightarrow 1+B+0 = 1 \Rightarrow B = 0$$

$$u^{0} \Rightarrow 2B = 0$$

$$\Rightarrow \begin{cases} A+B=0 & \Rightarrow B=-A \\ 1+B+0=1 & \\ B+0=0 & \Rightarrow C=-A \end{cases}$$

$$u^{0.5} \Rightarrow -A = A(u^{0.5}+2)B \Rightarrow A = \frac{-A}{u^{0.5}+2}$$

Luego, $B = C = 0$

$$= \frac{1}{2}u^{\frac{5}{2}} - 2u^{\frac{1}{2}} = \frac{1}{2}\left[\int du + 2\int \frac{u^{\frac{3}{2}}}{u+2}\right]$$

$$= \frac{1}{2}u^{\frac{5}{2}} - 2u^{\frac{1}{2}} = \frac{1}{2}\ln|u| + 2\int \frac{u^{\frac{3}{2}}}{u+2}\,du$$

$$\int \frac{u^{\frac{3}{2}}}{u+2}\,du = \frac{1}{2}\int \frac{2u^{\frac{3}{2}}}{u+2}\,du \qquad \boxed{\begin{array}{l} 2u^{\frac{3}{2}} = 2u+2 \\ 2u^{\frac{3}{2}} = 2u+2 \end{array}}$$

$$= \frac{1}{2}\int \frac{2u^{\frac{3}{2}}}{u+2}\,du = \frac{1}{2}\int \frac{2u}{u+2}\,du + \frac{1}{2}\int \frac{2}{u+2}\,du$$

$$= \frac{1}{2}\int \frac{2u}{u+2}\,du + \frac{1}{2}\int \frac{2}{u+2}\,du$$

$$\int \frac{2u}{u+2}\,du = \int \frac{2u}{(u+2)} - \frac{4}{u+2} = \int \frac{2u}{u+2} - \int \frac{4}{u+2}$$

$$= \int \frac{2u}{u+2} - 4\int \frac{1}{u+2} \qquad \boxed{\begin{array}{l} 2u = 2u+4-4 \\ 2u = 2u \end{array}}$$

$$= \frac{2u}{2} - 4\ln|u+2| \qquad \boxed{\begin{array}{l} 2u = 2u-4+4 \\ 2u = 2u \end{array}}$$

$$= \frac{2u}{2} - 4\ln|u+2|$$

$$= \frac{1}{2}\int \frac{2u}{u+

$$\int \frac{dx}{(x^2-x+2)^2} = \int \frac{dx}{\left[(x-\frac{1}{2})^2+\frac{7}{4}\right]^2} = 16\int \frac{dx}{\left[(2x-1)^2+3\right]^2}$$

$u = 2x-1$
$du = 2\,dx$

$$= \frac{1}{2}\int \frac{du}{(u^2+3)^2} = \dots \text{ (TEDIOSA)}$$

$$= \frac{1}{3^{4/2}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{4(2x-1)}{3\left[(2x-1)^2+3\right]} + C$$

$$= \frac{-1}{2(x^2-x+2)} + \frac{1}{3}\left[\frac{1}{3^{1/2}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{4(2x-1)}{3\left[(2x-1)^2+3\right]}\right] + C$$

$4x-2$

$$= 3\ln|x| + \frac{13}{15}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2(x^2-x+2)}$$

$$+ \frac{2\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{4x-2}{3\left[(2x-1)^2+3\right]} + C$$

(12) $\displaystyle\int \frac{7x-3}{x(x-1)(x+3)}\,dx$

$$\frac{7x-3}{x(x-1)(x+3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+3}$$

$$7x-3 = A(x-1)(x+3) + Bx(x+3) + Cx(x-1)$$

$x=1 \Rightarrow 7-3 = B(1)(4) \Rightarrow 4 = 4B \Rightarrow B=1$

$x=-3 \Rightarrow -21-3 = C(-3)(-4) \Rightarrow -24 = 12C \Rightarrow C=-2$

$x=0 \Rightarrow -3 = A(-1)(3) \Rightarrow -3 = -3A \Rightarrow A=1$

$$= \int \frac{1}{x} + \frac{1}{x-1} + \frac{-2}{x+3}\,dx$$

$$= \int \frac{dx}{x} + \int \frac{dx}{x-1} - 2\int \frac{dx}{x+3}$$

$$= \ln|x| + \ln|x-1| - 2\int \frac{dx}{x+3}$$

$$= \ln|x| + \ln|x-1| - 2\ln|x+3| + C$$

$$= \ln\left|\frac{x(x-1)}{(x+3)^2}\right| + C$$

(13) $\displaystyle\int \frac{x+2}{x^3-1}\,dx = \int \frac{x+2}{(x-1)(x^2+x+1)}\,dx$

$$\frac{x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$x+2 = A(x^2+x+1) + (Bx+C)(x-1)$$
$$= Ax^2+Ax+A + Bx^2 - Bx + Cx - C$$
$$= (A+B)x^2 + (A-B+C)x + (A-C)$$

$$\Rightarrow \begin{cases} A+B=0 \\ A-B+C=1 \\ A-C=2 \end{cases}$$

$x=1 \Rightarrow 3=3A \Rightarrow A=1$

$$\Rightarrow \begin{cases} 1+B=0 & \Rightarrow B=-1 \\ 1-B+C=1 & \\ 1-C=2 & \Rightarrow C=-1 \end{cases}$$

$$= \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+x+1}\,dx$$

$u = x-1$
$du = dx$

$$= \underset{\text{visar al cuadrado}}{\ln|x-1|} - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$z = x^2+x+1$
$dz = 2x+1\,dx$

$$= \ln|x-1| - \frac{1}{2}\int \frac{dz}{z} - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \ln|x-1| - \frac{1}{2}\ln|z| - \frac{1}{2}\underbrace{\left[\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right]}_{\text{ejercicio 12}} + C$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$= \ln\left|\frac{x-1}{\sqrt{x^2+x+1}}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

ver propiedades de ln

(14) $\displaystyle\int \frac{dx}{x(\sqrt{x-1}+1)} \overset{u}{=} \int \frac{2u}{(u^2+1)(u+1)}\,du$

$x-1 = u^2 \Rightarrow x = u^2+1$
$dx = 2u\,du$

$$\frac{2u}{(u^2+1)(u+1)} = \frac{Au+B}{u^2+1} + \frac{C}{u+1}$$

$$2u = (Au+B)(u+1) + C(u^2+1)$$

$u=-1 \Rightarrow -2 = 0 + C[(-1)^2+1] \Rightarrow -2 = 2C \Rightarrow C=-1$

Luego,

$$2u = Au^2 + Au + Bu + B + Cu^2 + C$$
$$= (A+C)u^2 + (A+B)u + (B+C)$$

$$\Rightarrow \begin{cases} A+C=0 & \Rightarrow A-1=0 & \Rightarrow A=1 \\ A+B=2 & \Rightarrow 1+B=2 & \Rightarrow B=1 \\ B+C=0 & & \end{cases}$$

$$= \int \frac{u+1}{u^2+1} - \frac{1}{u+1}\,du = \frac{1}{2}\int \frac{2(u+1)}{u^2+1}\,du - \int \frac{du}{u+1}$$

$$= \frac{1}{2}\int \frac{2u+2}{u^2+1}\,du - \ln|u+1|$$

$$= \frac{1}{2}\int \frac{2u}{u^2+1}\,du + \frac{1}{2}\int \frac{2}{u^2+1}\,du - \ln|u+1|$$

$w = u^2+1$
$dw = 2\,du$

$$= \frac{1}{2}\int \frac{dw}{w} + \int \frac{du}{u^2+1} - \ln|u+1|$$

$$= \frac{1}{2}\ln|w| + \arctan(u) - \ln|u+1| + C$$

$$= \frac{1}{2}\ln|u^2+1| + \arctan(u) - \ln|u+1| + C$$

$$= \frac{1}{2}\ln|x| + \arctan(\sqrt{x-1}) - \ln\left|\sqrt{x-1}+1\right| + C$$

(15) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$

$u = \sqrt{\frac{x-1}{x+1}} \Rightarrow x = \frac{u^2+1}{1-u^2}$
$dx = \frac{4u}{(u^2-1)^2}\,du$

$$= \int u\left(\frac{4u}{(u^2-1)^2}\right)du$$

$$= \int u\,\frac{4u}{[(u-1)(u+1)]^2}\,du = \int \frac{4u^2}{(u-1)^2(u+1)^2}\,du$$

$$\frac{4u^2}{(u-1)^2(u+1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2}$$

$$4u^2 = A(u-1)(u+1)^2 + B(u+1)^2 + C(u+1)(u-1)^2 + D(u-1)^2$$

$u=1 \Rightarrow 4=4B \Rightarrow B=1$

$u=-1 \Rightarrow 4=4D \Rightarrow D=1$

$u=0 \Rightarrow 0 = -A+B+C+D \Rightarrow 0 = -A+C+2 \Rightarrow A-C=2$

$u=2 \Rightarrow 16 = 9A+9B+3C+D \Rightarrow 6 = 9A+3C \Rightarrow 3A+C=2$

$$\therefore \begin{cases} A-C=2 \\ 3A+C=2 \end{cases} \underset{\text{(restando)}}{\Rightarrow} \begin{matrix} A=1 \\ C=1 \end{matrix}$$

$$= \int \frac{du}{u-1} + \int \frac{du}{(u-1)^2} + \int \frac{du}{u+1} + \int \frac{du}{(u+1)^2}$$

(Hacer los cambios de variable apropiados)

$$= \ln|u-1| - \frac{1}{u-1} + \ln|u+1| - \frac{1}{u+1} + C$$

$$= \ln\left|(u-1)(u+1)\right| - \left[\frac{u+1-1(u-1)}{(u-1)(u+1)}\right] + C$$

$$= \ln|u^2-1| - \frac{2u}{u^2-1} + C$$

$$= \ln\left|\frac{x-1}{x+1} - 1\right| - \frac{2\sqrt{\frac{x-1}{x+1}}}{\frac{x-1}{x+1}-1} + C$$

$$= \ln\left|\frac{x-1-(x+1)}{x+1}\right| - \frac{2\sqrt{\frac{x-1}{x+1}}}{\frac{x-1-1(x+1)}{x+1}} + C$$

$$= \ln\left|\frac{-2}{x+1}\right| - \frac{2\sqrt{x-1}\,(x+1)}{-2\sqrt{x+1}} + C$$

$$= \ln\left|\frac{-2}{x+1}\right| + \sqrt{x-1} + C$$