# Funciones Racionales (Parte 2)

## I. Integración de funciones racionales propias

Una función racional fraccionaria es el cociente de dos funciones polinomiales:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$$

Una función racional se llama propia si, con los datos, el grado del numerador es menor que el grado del denominador, y en el caso en que sean, se llama impropia.

**Ejemplos** de funciones propias (racional)

$f(x) = \frac{x-3}{x^2+2x+1}$ ; $g(x) = \frac{3x^2-4}{x^3+2x^2+3x-5}$

**Ejemplos** de funciones impropias (racional)

$f(x) = \frac{8x^3+7x^2+5x^4+2x+3}{x^2+3x^3+4x+5}$ ; $h(x) = \frac{x^5+3x^3+2x+1}{x^3-4x+5}$

**Obs.** Es sabemos integrar las funciones racionales propias, entonces sabemos más como integrar TODAS las funciones racionales.

### Funciones racionales propias SIMPLES

Las funciones racionales propias simples se pueden caracterizar en 4 tipos, a saber:

1. $f(x) = \frac{A}{x-a}$ $\quad$ A, a constantes
2. $f(x) = \frac{A}{(x-a)^n}$ $\quad$ A, a constantes $n \in \mathbb{N}$ con $n \geq 2$
3. $f(x) = \frac{Ax+B}{x^2+bx+c}$ $\quad$ A, B, b, c constantes $b^2-4ac<0$
4. $f(x) = \frac{Ax+B}{(x^2+bx+c)^n}$ $\quad$ A, B, b, c constantes $b^2-4ac<0$ $n \in \mathbb{N}$ con $n \geq 2$

**Caso** (Tipo 1)

$$\int \frac{A}{x-a}\,dx = A\int \frac{dx}{x-a} = A\int \frac{du}{u} = A\ln|u| + C$$

$$= A\ln|x-a| + C$$

$\boxed{\begin{array}{l} u = x-a \\ du = dx \end{array}}$

**Ejemplos**

① $\int \frac{3}{x-2}\,dx = 3\int \frac{dx}{x-2} = 3\int \frac{du}{u} = 3\ln|u| + C$

$\boxed{\begin{array}{l} u = x-2 \\ du = dx \end{array}}$

$= 3\ln|x-2| + C$

② $\int \frac{2x}{x-2}\,dx = 2\int \frac{dx}{x-2} = 2\int \frac{du}{u} = 2\ln|u| + C$

$\boxed{\begin{array}{l} u = x-2 \\ du = dx \end{array}}$

$= 2\ln|x-2| + C$

**Caso** (Tipo 2) $\quad n \neq 1 \quad n \in \mathbb{N}$

$$\int \frac{A}{(x-a)^n}\,dx = A\int \frac{dx}{(x-a)^n} = A\int \frac{du}{u^n} = A\int u^{-n}\,du$$

$\boxed{\begin{array}{l} u = x-a \\ du = dx \end{array}}$

$$= A\left[\frac{u^{-n+1}}{-n+1}\right] + C = A\left[\frac{(x-a)^{-n+1}}{1-n}\right] + C$$

**Ejemplos**

① $\int \frac{5}{(x+2)^3}\,dx = 5\int \frac{dx}{(x+2)^3} = 5\int \frac{du}{u^3} = 5\int u^{-3}\,du$

$\boxed{\begin{array}{l} u = x+2 \\ du = dx \end{array}}$

$= 5\left[-\frac{1}{2}\right] + C = \frac{-5}{2(x+2)^2} + C$

② $\int \frac{-13}{(x-4)^6}\,dx = 13\int \frac{dx}{(x-4)^6} = 13\int \frac{du}{u^6} = 13\int u^{-6}\,du$

$\boxed{\begin{array}{l} u = x-4 \\ du = dx \end{array}}$

$= 13\left[\frac{u^{-5}}{-5}\right] + C = -\frac{13}{5}\,\frac{1}{u^5} + C$

$= \frac{-13}{5}\left[\frac{1}{(x-4)^5}\right] + C$

**Caso** (Tipo 3)

$\boxed{\begin{array}{l} u = x^2+bx+c \\ du = (2x+b)\,dx \end{array}}$

$$\int \frac{Ax+B}{x^2+bx+c}\,dx$$

$$= \int \frac{x+B}{x^2+bx+c}\,dx \qquad (A=1)$$

$$= \frac{1}{2}\int \frac{2x+\frac{2B}{1}}{x^2+bx+c}\,dx \qquad k = \frac{2}{1}$$

$$= \frac{1}{2}\int \frac{2x+b-b}{x^2+bx+c}\,dx \qquad 0 = b-b$$

$$= \frac{1}{2}\int \frac{2x+b}{x^2+bx+c}\,dx + \frac{1}{2}\left(\frac{2}{1}B-b\right)\int \frac{dx}{x^2+bx+c}$$

$$= \frac{1}{2}\int \frac{du}{u} + \left(B-\frac{b}{2}\right)\underbrace{\int \frac{dx}{x^2+bx+c}}_{\text{TRIGO}}$$

$$= \frac{1}{2}\ln|u| + \left(B-\frac{b}{2}\right)\left[\frac{1}{\sqrt{4c-b^2}}\arctan\left(\frac{2x+b}{\sqrt{4c-b^2}}\right)\right] + C$$

$$= \frac{1}{2}\ln|x^2+bx+c| + \frac{(2B-b)}{\sqrt{4c-b^2}}\arctan\left(\frac{2x+b}{\sqrt{4c-b^2}}\right) + C$$

① $\int \frac{2x+3}{x^2+2x+2}\,dx$ $\quad \boxed{\begin{array}{l} u = x^2+2x+2 \\ du = (2x+2)\,dx \end{array}}$

$$= \int \frac{x+\frac{3}{2}}{x^2+2x+2}\,dx = 3\int \frac{x+\frac{3}{2}}{x^2+2x+2}\,dx$$

$$= \frac{3}{2}\int \frac{2(x+2)}{x^2+2x+2}\,dx = \frac{3}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \frac{3}{2}\int \frac{1}{x^2+2x+2}\,dx$$

$$= \frac{3}{2}\int \frac{du}{u} + 3\int \frac{dx}{(x+1)^2+1}$$

$\boxed{\begin{array}{l} u = x^2+2x \\ du = (2x+2)\,dx \end{array}}$

$$= \frac{3}{2}\ln|u| + 3\int \frac{dx}{u^2+1}$$

Sustitución trigonométrica

$$= \frac{3}{2}\ln|x^2+2x+2| + 3\arctan(x+1) + C$$

**Caso** (Tipo 4) $\quad \int \frac{Ax+B}{(x^2+bx+c)^n}\,dx \quad$ donde $b^2-4ac<0$, $n \geq 2$

Existe último caso no lo demostraremos pq requiere procedimientos complicados, que por ahora no son interesantes. No obstante, veremos como la idea usando los mismos trucos que en el tipo anterior.

**Ejemplos.**

① $\int \frac{3x+2}{(x^2+1)^2}\,dx$ $\quad \boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$

$$= 3\int \frac{x+\frac{2}{3}}{(x^2+1)^2}\,dx = \frac{3}{2}\int \frac{2x+\frac{4}{3}}{(x^2+1)^2}\,dx$$

$$= \frac{3}{2}\int \frac{2x}{(x^2+1)^2}\,dx + \int \frac{dx}{(x^2+1)^2}$$

$$= \frac{3}{2}\int \frac{du}{u^2} + \int \frac{dx}{(x^2+1)^2}$$

$$= -\frac{3}{2u} + \int \frac{dx}{(x^2+1)^2} = \frac{-3}{2(x^2+1)} + \int \frac{dx}{(x^2+1)^2}$$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2(t)\,dt}{(\tan^2(t)+1)^2} = \int \frac{\sec^2(t)}{\sec^4(t)}\,dt$$

$\boxed{\begin{array}{l} x = \tan(t) \\ dx = \sec^2(t)\,dt \end{array}}$ $\quad \boxed{\begin{array}{l} x^2+1 = \tan^2(t)+1 = \sec^2(t) \\ = 1+\cos^2(t) \end{array}}$

$$= \int \frac{dt}{\sec^2(t)} = \int \frac{1}{\sec^2(t)}\,dt = \int \frac{1+\cos(2t)}{2}\,dt$$

$$= \frac{1}{2}\int 1\,dt + \frac{1}{2}\int \cos(2t)\,dt$$

$\boxed{\begin{array}{l} u = 2t \\ du = 2\,dt \\ \frac{du}{2} = dt \end{array}}$

$$= \frac{1}{2}t + \frac{1}{2}\int \cos(u)\,\frac{du}{2} = \frac{1}{2}t + \frac{1}{4}\sin(2t) + C$$

$$= \frac{1}{2}t + \frac{1}{4}\,2\sin(t)\cos(t) + C$$

$$t = \arctan(x)$$

$$\sin(t) = \frac{x}{\sqrt{x^2+1}}$$

$$\cos(t) = \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$

$$= \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C$$

$$= \frac{-3}{2(x^2+1)} + \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C$$

② $\int \frac{x+3}{(x^2+4x+5)^2}\,dx$ $\quad \boxed{\begin{array}{l} u = x^2+4x+5 \\ du = (2x+4)\,dx \end{array}}$

$$= \frac{1}{2}\int \frac{2x+6}{(x^2+4x+5)^2}\,dx = \frac{1}{2}\int \frac{2x+4+2}{(x^2+4x+5)^2}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4}{(x^2+4x+5)^2}\,dx + \frac{1}{2}\int \frac{2}{(x^2+4x+5)^2}\,dx$$

$$= \frac{1}{2}\int \frac{du}{u^2} + \frac{1}{2}\int \frac{2}{(x^2+4x+5)^2}\,dx$$

$\boxed{\begin{array}{l} u = x^2+4x+5 \\ du = (2x+4)\,dx \end{array}}$

$$= -\frac{1}{2u} + \frac{1}{2}\int \frac{2}{(x^2+4x+5)^2}\,dx$$

$$= \frac{-1}{2(x^2+4x+5)} + \frac{1}{2}\int \frac{2}{(x^2+4x+5)^2}\,dx$$

$$\int \frac{2}{(x^2+4x+5)^2}\,dx = \int \frac{2}{(x^2+4x+4+1)^2}\,dx = \int \frac{2}{(x+2)^2+1)^2}\,dx$$

$\boxed{\begin{array}{l} x+2 = \tan(t) \\ dx = \sec^2(t)\,dt \end{array}}$ $\quad \boxed{\begin{array}{l} (x^2+4x+4)+1 = \sec^2(t) \end{array}}$

$$= \int \frac{2\sec^2(t)}{(\sec^2(t))^2}\,dt = \int \frac{2}{\sec^2(t)}\,dt = \int \cos^2(t)\,dt$$

$$= \int \left[\frac{1+\cos(2t)}{2}\right]dt = \frac{1}{2}\int 1 + \cos(2t) + \cos^2(2t)\,dt$$

$\boxed{\begin{array}{l} u = 2t \\ du = 2\,dt \end{array}}$

$$= \frac{1}{2}\int 1\,dt + \frac{1}{2}\int \cos(2t)\,dt + \frac{1}{2}\int \cos^2(2t)\,dt$$

$$= \frac{1}{2}t + \frac{1}{2}\int \cos(u)\,\frac{du}{2} + \frac{1}{2}\int \cos^2(2t)\,dt$$

$\boxed{\begin{array}{l} u = 2t \\ du = 2\,dt \end{array}}$

$$= \frac{1}{2}t + \frac{1}{4}\sin(u) + \frac{1}{2}\int \frac{1+\cos(4t)}{2}\,dt$$

$$= \frac{1}{2}t + \frac{1}{4}\sin(2t) + \frac{1}{4}\int 1 + \cos(4t)\,dt$$

$$= \frac{1}{2}t + \frac{1}{4}\sin(2t) + \frac{1}{4}t + \frac{1}{16}\sin(4t) + C$$

$$= \frac{3}{4}t + \frac{1}{4}\sin(2t) + \frac{1}{16}\sin(4t) + C$$

$$= \frac{3}{4}t + \frac{1}{4}\,2\sin(t)\cos(t) + \frac{1}{16}\,4\sin(2t)\cos(2t) + C$$

$$= \frac{3}{4}t + \frac{1}{4}\,2\sin(t)\cos(t) + \frac{1}{16}\,4\sin(t)\cos(t)\left[\cos^2(t)-\sin^2(t)\right] + C$$

$$= \frac{3}{4}t + \frac{1}{2}\sin(t)\cos(t) + \frac{1}{4}\sin(t)\cos(t)\cos^2(t) - \frac{1}{4}\sin^3(t)\cos(t) + C$$

$$t = \arctan(x)$$



$$\sin(t) = \frac{x}{\sqrt{x^2+1}}$$

$$\cos(t) = \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{3}{4}\arctan(x) + \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + \frac{1}{4}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)^2$$

$$+ \frac{1}{4}\left(\frac{x}{\sqrt{x^2+1}}\right)^3\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$

$$= \frac{3}{4}\arctan(x) + \frac{x}{2(x^2+1)} + \frac{x}{4(x^2+1)^2} + \frac{x^3}{4(x^2+1)^3} + C$$

$$= \frac{3}{4}\arctan(x) + \frac{2x(x^2+1)}{4(x^2+1)^2} + \frac{x}{4(x^2+1)^2} + \frac{x^3}{4(x^2+1)^3} + C$$

$$= \frac{3}{4}\arctan(x) + \frac{2x^3+2x+x}{4(x^2+1)^2} + \frac{x^3}{4(x^2+1)^3} + C$$

$$= \frac{-3}{4(x^2+4x+5)^2} + \frac{3}{4}\left[\frac{3}{4}\arctan(x+2) + \frac{2x^3+2x+x}{4(x^2+4x+5)^2}\right] + C$$