8 Predicate Logic

8.1 Symbols and Translation

Techniques were dev eloped in earlier chapters for evaluating two basically different kinds of arguments. The chapter on categorical syllogisms dealt with arguments such as the following:

> All student hookups are quickie sexual encounters.
> No quickie sexual encounters are committed relationships.
> Therefore, no student hookups are committed relationships.

In such arguments the fundamental components are *terms*, and the validity of the argument depends on the arrangement of the terms within the premises and conclusion.

The chapter on propositional logic, on the other hand, dealt with arguments such as this:

> If chronic stress is reduced, then relaxation increases and health improves.
> If health improves, then people live longer.
> Therefore, if chronic stress is reduced, then people live longer.

In such arguments the fundamental components are not terms but *statements*. The validity of these arguments depends not on the arrangement of the terms within the statements but on the arrangement of the statements themselves as simple units.

Not all arguments, however, can be assigned to one or the other of these two groups. There is a third type that is a kind of hybrid, sharing features with both categorical syllogisms and propositional arguments. Consider, for example, the following:

> Catherine Zeta-Jones is rich and beautiful.
> If a woman is either rich or famous, she is happy.
> Therefore, Catherine Zeta-Jones is happy.

The validity of this argument depends on both the arrangement of the terms and the arrangement of the statements. Accordingly, neither syllogistic logic nor propositional logic alone is sufficient to establish its validity. What is needed is a third kind of logic that combines the distinctive features of syllogistic logic and propositional logic. This third kind is called **predicate logic.**

The fundamental component in predicate logic is the **predicate,** symbolized by uppercase letters (*A*, *B*, *C*, . . . *X*, *Y*, *Z*), called **predicate symbols.** Here are some examples of bare predicates:

English predicate	Symbolic predicate
___ is a rabbit	*R* __
___ is gigantic	*G* __
___ is a doctor	*D* __
___ is helpless	*H* __

The blank space immediately following the predicate letter is not part of the predicate; rather, it indicates the place for some lowercase letter that will represent the subject of the statement. Depending on what lowercase letter is used, and on the additional symbolism involved, symbolic predicates may be used to translate three distinct kinds of statements: singular statements, universal statements, and particular statements.

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A **singular statement,** you may recall from Section 4.7, is a statement that makes an assertion about a specifically named person, place, thing, or time. Translating a singular statement involves writing a lowercase letter corresponding to the subject of the statement to the immediate right of the uppercase letter corresponding to the predicate. The letters that are allocated to serve as names of individuals are the first twenty-three letters of the alphabet (*a*, *b*, *c*, . . . *u*, *v*, *w*). These letters are called **individual constants.** Here are some examples of translated statements:

Statement	Symbolic translation
Socrates is mortal.	*Ms*
Tokyo is populous.	*Pt*
The *Sun-Times* is a newspaper.	*Ns*
King Lear is not a fairy tale.	*~Fk*
Berlioz was not a German.	*~Gb*

Compound arrangements of singular statements may be translated by using the familiar connectives of propositional logic. Here are some examples:

Statement	Symbolic translation
If Paris is beautiful, then Andre told the truth.	$Bp \supset Ta$
Irene is either a doctor or a lawyer.	$Di \vee Li$
Senator Wilkins will be elected only if he campaigns.	$Ew \supset Cw$
General Motors will prosper if either Nissan is crippled by a strike or Subaru declares bankruptcy.	$(Cn \vee Ds) \supset Pg$
Indianapolis gets rain if and only if Chicago and Milwaukee get snow.	$Ri \equiv (Sc \cdot Sm)$

Recall from Chapter 4 that a **universal statement** is a statement that makes an assertion about every member of its subject class. Such statements are either affirmative or negative, depending on whether the statement affirms or denies that the members of the subject class are members of the predicate class. The key to translating universal statements is provided by the Boolean interpretation of these statements (see Section 4.3):

Statement form	Boolean interpretation
All *S* are *P*.	If anything is an *S*, then it is a *P*.
No *S* are *P*.	If anything is an *S*, then it is not a *P*.

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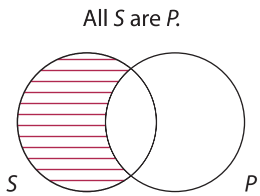
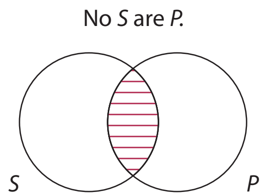
According to the Boolean interpretation, universal statements are translated as conditionals. We have a symbol (the horseshoe) to translate conditional statements, so we may use it to translate universal statements. What is still needed, however, is a symbol to indicate that universal statements make an assertion about *every* member of the *S* class. This symbol is called the **universal quantifier.** It is formed by placing a lowercase letter in parentheses, (x), and is translated as "for any x." The letters that are allocated for forming the universal quantifier are the last three letters of the alphabet (x, y, z). These letters are called **individual variables.**

The horseshoe operator and the universal quantifier are combined to translate universal statements as follows:

Statement form	Symbolic translation	Verbal meaning
All *S* are *P*.	$(x)(Sx \supset Px)$	For any x, if x is an *S*, then x is a *P*.
No *S* are *P*.	$(x)(Sx \supset \sim Px)$	For any x, if x is an *S*, then x is not a *P*.

An individual variable differs from an individual constant in that it can stand for any item at random in the universe. Accordingly, the expression $(x)(Sx \supset Px)$ means "If anything is an *S*, then it is a *P*," and $(x)(Sx \supset \sim Px)$ means "If anything is an *S*, then it is not a *P*." The fact that these expressions are equivalent to the Boolean interpretation of universal statements may be seen by recalling how the Boolean interpretation is represented by Venn diagrams (see Section 4.3). The Venn diagrams corresponding to the two universal statement forms are as follows:





Where shading designates emptiness, the diagram on the left asserts that if anything is in the *S* circle, it is also in the *P* circle, and the one on the right asserts that if anything is in the *S* circle, it is not in the *P* circle. This is exactly what is asserted by the symbolic expressions just given. These symbolic expressions may therefore be taken as being exactly synonymous with the Boolean interpretation of universal statements.

A possible source of confusion at this point concerns the fact that both *S* and *P* in the symbolic expressions are predicates, whereas in the original statement forms *S* is the subject and *P* is the predicate. Any problem in this regard vanishes, however, once one understands what happens when universal statements are converted into conditionals. When so converted, *S* becomes the predicate of the antecedent and *P* becomes the predicate of the consequent. In other words, in the conditional "If anything is an *S*, then it is a *P*," both *S* and *P* are predicates. Thus, using predicate symbolism to translate universal statements leads to no difficulties. When translating these statements, the point to remember is simply this: The subject of the original statement is represented by a capital letter in the antecedent, and the predicate by a capital letter in the consequent. Here are some examples:

Statement	**Symbolic translation**
All skyscrapers are tall.	$(x)(Sx \supset Tx)$
No frogs are birds.	$(x)(Fx \supset {\sim}Bx)$
All ambassadors are statesmen.	$(x)(Ax \supset Sx)$
No diamonds are rubies.	$(x)(Dx \supset {\sim}Rx)$

In these examples, the expressions $Sx \supset Tx$, $Fx \supset {\sim}Bx$, and so on are called statement functions. A **statement function** is the expression that remains when a quantifier is removed from a statement. It is a mere pattern for a statement. It makes no definite assertion about anything in the universe, has no truth value, and cannot be translated as a statement. The variables that occur in statement functions are called **free variables** because they are not bound by any quantifier. In contrast, the variables that occur in statements are called **bound variables**.

In using quantifiers to translate statements, we adopt a convention similar to the one adopted for the tilde operator. That is, the quantifier governs only the expression immediately following it. For example, in the statement $(x)(Ax \supset Bx)$ the universal

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quantifier governs the entire statement function in parentheses—namely, $Ax \supset Bx$. But in the expression $(x)Ax \supset Bx$, the universal quantifier governs only the statement function Ax. The same convention is adopted for the existential quantifier, which will be introduced presently.

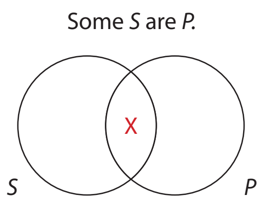
Recall from Chapter 4 that a **particular statement** is a statement that makes an assertion about one or more unnamed members of the subject class. As with universal statements, particular statements are either affirmative or negative, depending on whether the statement affirms or denies that members of the subject class are members of the predicate class. Also, as with universal statements, the key to translating particular statements is provided by the Boolean interpretation:

Statement form	Boolean interpretation
Some *S* are *P*.	At least one thing is an *S* and it is also a *P*.
Some *S* are not *P*.	At least one thing is an *S* and it is not a *P*.

In other words, particular statements are translated as conjunctions. Since we are already familiar with the symbol for conjunction (the dot), the only additional symbol that we need in order to translate these statements is a symbol for existence. This is provided by the **existential quantifier,** formed by placing a variable to the right of a backward E in parentheses, thus: $(\exists x)$. This symbol is translated "there exists an *x* such that." The existential quantifier is combined with the dot operator to translate particular statements as follows:

Statement form	Translation	Symbolic Verbal meaning
Some *S* are *P*.	$(\exists x)(Sx \bullet Px)$	There exists an *x* such that *x* is an *S* and *x* is a *P*.
Some *S* are not *P*.	$(\exists x)(Sx \bullet {\sim}Px)$	There exists an *x* such that *x* is an *S* and *x* is not a *P*.

As in the symbolic expression of universal statements, the letter *x* is an individual variable, which can stand for any item in the universe. Accordingly, the expression $(\exists x)(Sx \bullet Px)$ means "Something exists that is both an *S* and a *P*," and $(\exists x)(Sx \bullet {\sim}Px)$ means "Something exists that is an *S* and not a *P*." To see the equivalence of these expressions with the Boolean (and Aristotelian) interpretation of particular statements, it is again useful to recall how these statements are represented by Venn diagrams:



Where the X designates at least one existing item, the diagram on the left asserts that something exists that is both an *S* and a *P*, and the one on the right asserts that something exists that is an *S* and not a *P*. In other words, these diagrams assert exactly the same thing as the symbolic expressions just given. These symbolic expressions, therefore, exactly express the Boolean (and Aristotelian) interpretation of particular statements. Here are some examples:

Statement	Symbolic translation
Some men are paupers.	$(\exists x)(Mx \bullet Px)$
Some diseases are not contagious.	$(\exists x)(Dx \bullet {\sim}Cx)$
Some jobs are boring.	$(\exists x)(Jx \bullet Bx)$
Some vehicles are not motorcycles.	$(\exists x)(Vx \bullet {\sim}Mx)$

The general rule to follow in translating statements in predicate logic is always to make an effort to understand the meaning of the statement to be translated. If the statement makes an assertion about every member of its subject class, a universal quantifier should be used to translate it; but if it makes an assertion about only one or more members of this class, an existential quantifier should be used.

Many of the principles developed in syllogistic logic (see Section 4.7) may be carried over into predicate logic. Specifically, it should be understood that statements beginning with the words *only* and *none but* are exclusive propositions. When these statements are translated, the term occurring first in the original statement becomes the consequent in the symbolic expression, and the term occurring second becomes the antecedent. One of the few differences in this respect between predicate logic and syllogistic logic concerns singular statements. In syllogistic logic, singular statements are translated as universals, while in predicate logic, as we have seen, they are translated in a unique way. Here are some examples of a variety of statements:

Statement	Symbolic translation
There are happy marriages.	$(\exists x)(Mx \bullet Hx)$
Every pediatrician loses sleep.	$(x)(Px \supset Lx)$
Animals exist.	$(\exists x)Ax$
Unicorns do not exist.	${\sim}(\exists x)Ux$
Anything is conceivable	$(x)Cx$
Sea lions are mammals.	$(x)(Sx \supset Mx)$
Sea lions live in these caves.	$(\exists x)(Sx \bullet Lx)$
Egomaniacs are not pleasant companions.	$(x)(Ex \supset {\sim}Px)$
A few egomaniacs did not arrive on time.	$(\exists x)(Ex \bullet {\sim}Ax)$
Only close friends were invited to the wedding.	$(x)(Ix \supset Cx)$
None but citizens are eligible to vote.	$(x)(Ex \supset Cx)$
It is not the case that every Girl Scout sells cookies.	${\sim}(x)(Gx \supset Sx)$ or $(\exists x)(Gx \bullet {\sim}Sx)$
Not a single psychologist attended the convention.	${\sim}(\exists x)(Px \bullet Ax)$ or $(x)(Px \supset {\sim}Ax)$

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As these examples illustrate, the general procedure in translating statements in predicate logic is to render universal statements as conditionals preceded by a universal quantifier, and particular statements as conjunctions preceded by an existential quantifier. However, as the third and fifth examples indicate, there are exceptions to this procedure. A statement that makes an assertion about literally everything in the universe is translated in terms of a single predicate preceded by a universal quantifier, and a statement that asserts that some class of things simply exists is translated in terms of a single predicate preceded by an existential quantifier. The last two examples illustrate that a particular statement is equivalent to a negated universal, and vice versa. The first of these is equivalent to "Some Girl Scouts do not sell cookies" and the second to "No psychologists attended the convention." Actually, any quantified statement can be translated using either a universal or an existential quantifier, provided that one of them is negated. The equivalence of these two forms of expression will be analyzed further in Section 8.3.

More complex statements may be translated by following the basic rules just presented. Examples:

Statement	**Symbolic translation**
Only snakes and lizards thrive in the desert.	$(x)[Tx \supset (Sx \vee Lx)]$
Oranges and lemons are citrus fruits.	$(x)[(Ox \vee Lx) \supset Cx]$
Ripe apples are crunchy and delicious.	$(x)[(Rx \bullet Ax) \supset (Cx \bullet Dx)]$
Azaleas bloom if and only if they are fertilized.	$(x)[Ax \supset (Bx \equiv Fx)]$
Peaches are edible unless they are rotten.	$(x)[Px \supset (\sim Rx \supset Ex)]$ or $(x)[Px \supset (Ex \vee Rx)]$
Cats and dogs bite if they are frightened or harassed.	$(x)\{(Cx \vee Dx) \supset [(Fx \vee Hx) \supset Bx]\}$

Notice that the first example is translated in terms of the disjunction $Sx \vee Lx$ even though the English statement reads "snakes *and* lizards." If the translation were rendered as $(x)[Tx \supset (Sx \bullet Lx)]$ it would mean that anything that thrives in the desert is both a snake and a lizard (at the same time). And this is surely *not* what is meant. For the same reason, the second example is translated in terms of the disjunction $Ox \vee Lx$ even though the English reads "oranges *and* lemons." If the statement were translated $(x)[(Ox \bullet Lx) \supset Cx]$, it would mean that anything that is simultaneously an orange and a lemon (and there are none of these) is a citrus fruit. The same principle is used in translating the sixth example, which, incidentally, reads "If anything is a cat or a dog, then if it is frightened or harassed, it bites." The third example employs the conjunction $Rx \bullet Ax$ to translate ripe apples. This, of course, is correct, because such a thing is both ripe and an apple at the same time. The fifth example illustrates the fact that "unless" may be translated as either "if not" or "or."

The operators of propositional logic can be used to form compound arrangements of universal and particular statements, just as they can be used to form compound arrangements of singular statements. Here are some examples:

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Statement	Symbolic translation
If Elizabeth is a historian, then some women are historians.	$He \supset (\exists x)(Wx \cdot Hx)$
If some cellists are music directors, then some orchestras are properly led.	$(\exists x)(Cx \cdot Mx) \supset (\exists x)(Ox \cdot Px)$
Either everything is alive or Bergson's theory is not correct.	$(x)Ax \vee {\sim}Cb$
All novels are interesting if and only if some Steinbeck novels are not romances.	$(x)(Nx \supset Ix) \equiv (\exists x)[(Nx \cdot Sx) \cdot {\sim}Rx)]$
Green avocados are never purchased unless all the ripe ones are expensive.	$(x)[(Gx \cdot Ax) \supset {\sim}Px] \vee (x)[(Rx \cdot Ax) \supset Ex]$

We have seen that the general procedure is to translate universal statements as conditionals preceded by a universal quantifier, and to translate particular statements as conjunctions preceded by an existential quantifier. Let us see what happens to these translations when they are preceded by the wrong quantifier. Consider the false statement "No cats are animals." This is correctly translated $(x)(Cx \supset {\sim}Ax)$. If, however, it were translated $(\exists x)(Cx \supset {\sim}Ax)$, the symbolic statement would turn out to be true. This may be seen as follows. $(\exists x)(Cx \supset {\sim}Ax)$ is equivalent via material implication to $(\exists x)({\sim}Cx \vee {\sim}Ax)$, which in turn is equivalent via De Morgan's rule to $(\exists x){\sim}(Cx \bullet Ax)$. The latter statement, however, merely asserts that something exists that is not both a cat and an animal—for example, a dog—which is true. Again, consider the true statement "Some cats are animals." This is correctly translated $(\exists x)(Cx \bullet Ax)$. If, however, it were translated $(x)(Cx \bullet Ax)$, the symbolic statement would assert that everything in the universe is both a cat and an animal, which is clearly false. Thus, as these examples illustrate, it is imperative that the two quantifiers not be confused with each other.

One final observation needs to be made. It was mentioned earlier that the letters *x*, *y*, and *z* are reserved for use as variables for translating universal and particular statements. In accord with this convention, the other twenty-three lowercase letters (*a*, *b*, *c*, . . . *u*, *v*, *w*) may be used as names for translating singular statements. Thus, for example, "Albert is a scientist" is translated *Sa*. But a question naturally arises with statements such as "Xerxes was a king." Should this statement be translated *Kx*? The answer is no. Some other letter, for example the second letter in the name, should be selected instead of *x*. Maintaining this alphabetical convention will help us avoid mistakes in the next section when we use natural deduction to derive the conclusions of arguments.

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Exercise 8.1

Translate the following statements into symbolic form. Avoid negation signs preceding quantifiers. The predicate letters are given in parentheses.

★1. Elaine is a chemist. (*C*)

2. Nancy is not a sales clerk. (*S*)

3. Neither Wordsworth nor Shelley was Irish. (*I*)

★4. Rachel is either a journalist or a newscaster. (*J*, *N*)

5. Intel designs a faster chip only if Micron does. (*D*)

6. Belgium and France subsidize the arts only if Austria or Germany expand museum holdings. (*S*, *E*)

★7. All maples are trees. (*M*, *T*)

8. Some grapes are sour. (*G*, *S*)

9. No novels are biographies. (*N*, *B*)

★10. Some holidays are not relaxing. (*H*, *R*)

11. If Gertrude is correct, then the Taj Mahal is made of marble. (*C*, *M*)

12. Gertrude is not correct only if the Taj Mahal is made of granite. (*C*, *G*)

★13. Tigers exist. (*T*)

14. Anything that leads to violence is wrong. (*L*, *W*)

15. There are pornographic art works. (*A*, *P*)

★16. Not every smile is genuine. (*S*, *G*)

17. Every penguin loves ice. (*P*, *L*)

18. There is trouble in River City. (*T*, *R*)

★19. Whoever is a socialite is vain. (*S*, *V*)

20. Any caring mother is vigilant and nurturing. (*C*, *M*, *V*, *N*)

21. Terrorists are neither rational nor empathic. (*T*, *R*, *E*)

★22. Nobody consumed by jealousy is happy. (*C*, *H*)

23. Everything is imaginable. (*I*)

24. Ghosts do not exist. (*G*)

★25. A thoroughbred is a horse. (*T*, *H*)

26. A thoroughbred won the race. (*T*, *W*)

27. Not all mushrooms are edible. (*M*, *E*)

★28. Not any horse chestnuts are edible. (*H*, *E*)

29. A few guests arrived late. (*G*, *A*)

30. None but gentlemen prefer blondes. (*G*, *P*)

★31. A few cities are neither safe nor beautiful. (*C*, *S*, *B*)

32. There are no circular triangles. (*C*, *T*)

33. Snakes are harmless unless they have fangs. (*S*, *H*, *F*)

★34. Some dogs bite if and only if they are teased. (*D*, *B*, *T*)

35. An airliner is safe if and only if it is properly maintained. (*A*, *S*, *P*)

36. Some companies go bankrupt if sales decline. (*C*, *B*, *S*)

★37. Some children act up only if they are tired. (*C*, *A*, *T*)

38. The only musicians that are available are trombonists. (*M*, *A*, *T*)

39. Only talented musicians perform in the symphony. (*T*, *M*, *P*)

★40. Any well-made car runs smoothly. (*W*, *C*, *R*)

41. Not every foreign car runs smoothly. (*F*, *C*, *R*)

42. A good violin is rare and expensive. (*G*, *V*, *R*, *E*)

★43. Violins and cellos are stringed instruments. (*V*, *C*, *S*, *I*)

44. A room with a view is available. (*R*, *V*, *A*)

45. A room with a view is expensive. (*R*, *V*, *E*)

★46. Some French restaurants are exclusive. (*F*, *R*, *E*)

47. Some French cafés are not recommended. (*F*, *C*, *R*)

48. Hurricanes and earthquakes are violent and destructive. (*H*, *E*, *V*, *D*)

★49. Taylor is guilty if and only if all the witnesses committed perjury. (*G*, *W*, *C*)

50. If any witnesses told the truth, then either Parsons or Harris is guilty. (*W*, *T*, *G*)

51. If all mysteries are interesting, then *Rebecca* is interesting. (*M*, *I*)

★52. If there are any interesting mysteries, then *Rebecca* is interesting. (*M*, *I*)

53. Skaters and dancers are energetic individuals. (*S*, *D*, *E*, *I*)

54. Swiss watches are not expensive unless they are made of gold. (*S*, *W*, *E*, *M*)

★55. If all the buildings in Manhattan are skyscrapers, then the Chrysler building is a skyscraper. (*B*, *M*, *S*)

56. Experienced mechanics are well paid only if all the inexperienced ones are lazy. (*E*, *M*, *W*, *L*)

57. Balcony seats are never chosen unless all the orchestra seats are taken. (*B*, *S*, *C*, *O*, *T*)

★58. Some employees will get raises if and only if some managers are overly generous. (*E*, *R*, *M*, *O*)

59. The physicists and astronomers at the symposium are listed in the program if they either chair a meeting or read a paper. (*P*, *A*, *S*, *L*, *C*, *R*)

60. If the scientists and technicians are conscientious and exacting, then some of the mission directors will be either pleased or delighted. (*S*, *T*, *C*, *E*, *M*, *P*, *D*)

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8.2 Using the Rules of Inference

The chief reason for using truth-functional operators (the dot, wedge, horseshoe, and so on) in translating statements into the symbolism of predicate logic is to allow for the application of the eighteen rules of inference to derive the conclusion of arguments via natural deduction. Since, however, the first eight of these rules are applicable only to whole lines in an argument, as long as the quantifier is attached to a line these rules of inference cannot be applied—at least not to the kind of arguments we are about to consider. To provide for their application, four additional rules are required to remove quantifiers at the beginning of a proof sequence and to introduce them, when needed, at the end of the sequence. These four rules are called universal instantiation, universal generalization, existential instantiation, and existential generalization. The first two are used to remove and introduce universal quantifiers, respectively, and the second two to remove and introduce existential quantifiers.

Let us first consider **universal instantiation.** As an illustration of the need for this rule, consider the following argument:

All economists are social scientists.
Paul Krugman is an economist.
Therefore, Paul Krugman is a social scientist.

This argument, which is clearly valid, is symbolized as follows:

1.	$(x)(Ex \supset Sx)$	
2.	Ep	/ Sp

As the argument now stands, none of the first eight rules of inference can be applied; as a result, there is no way in which the two premises can be combined to derive the conclusion. However, if the first premise could be used to derive a line that reads $Ep \supset Sp$, this statement could be combined with the second premise to yield the conclusion via *modus ponens*. Universal instantiation serves exactly this purpose.

The first premise states that for any item x in the universe, if that item is an E, then it is an S. But since Paul Krugman is himself an item in the universe, the first premise implies that if Paul Krugman is an E, then Paul Krugman is an S. A line stating exactly this can be derived by universal instantiation (UI). In other words, universal instantiation provides us with an *instance* of the universal statement $(x)(Ex \supset Sx)$. In the completed proof, which follows, the p in line 3 is called the **instantial letter:**

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1.	$(x)(Ex \supset Sx)$	
2.	Ep	/ Sp
3.	$Ep \supset Sp$	1, UI
4.	Sp	2, 3, MP

At this point the question might arise as to why *modus ponens* is applicable to lines 2 and 3. In Chapter 7 we applied *modus ponens* to lines of the form $p \supset q$, but are we justified in applying it to a line that reads $Ep \supset Sp$? The answer is yes, because Ep and Sp are simply alternate ways of symbolizing simple statements. As so understood, these symbols do not differ in any material way from the p and q of propositional logic.

We may now give a general definition of instantiation. *Instantiation* is an operation that consists in deleting a quantifier and replacing every variable bound by that quantifier with the same instantial letter. For an example of an operation that violates the rule expressed in this definition, consider line 3 of the foregoing proof. If this line were instantiated as $Ep \supset Sx$, it would not be correct because the x in Sx was not replaced with the instantial letter p.

Let us now consider **universal generalization.** The need for this rule may be illustrated through reference to the following argument:

All psychiatrists are doctors.
All doctors are college graduates.
Therefore, all psychiatrists are college graduates.

This valid argument is symbolized as follows:

1. $(x)(Px \supset Dx)$
2. $(x)(Dx \supset Cx)$ / $(x)(Px \supset Cx)$

Once universal instantiation is applied to the two premises, we will have lines that can be used to set up a hypothetical syllogism. But then we will have to reintroduce a universal quantifier to derive the conclusion as written. This final step is obtained by universal generalization (UG). The justification for such a step lies in the fact that both premises are universal statements. The first states that if *anything* is a *P*, then it is a *D*, and the second states that if *anything* is a *D*, then it is a *C*. We may therefore conclude that if *anything* is a *P*, then it is a *C*. But because of the complete generality of this reasoning process, there is a special way in which we must perform the universal instantiation step. Instead of selecting a *specifically named* instance, as we did in the previous example, we must select a *variable* that can range over every instance in the universe. The variables at our disposal, you may recall from the previous section, are *x*, *y*, and *z*. Let us select *y*. The completed proof is as follows:

1.	$(x)(Px \supset Dx)$	
2.	$(x)(Dx \supset Cx)$	/ $(x)(Px \supset Cx)$
3.	$Py \supset Dy$	1, UI
4.	$Dy \supset Cy$	2, UI
5.	$Py \supset Cy$	3, 4, HS
6.	$(x)(Px \supset Cx)$	5, UG

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As noted earlier, the expressions in lines 3, 4, and 5 are called *statement functions*. As such, they are mere patterns for statements; they have no truth value and cannot be translated as statements. Yet if we take certain liberties, we might characterize line 5 as saying "If *it* is a *P*, then *it* is a *C*, where "it" designates any item at random in the universe. Line 6 can then be seen as reexpressing this sense of line 5.

As the two previous examples illustrate, we have two ways of performing universal instantiation. On the one hand, we may instantiate with respect to a *constant*, such as *a* or *b*, and on the other, with respect to a *variable*, such as *x* or *y*. The exact way in which this operation is to be performed depends on the kind of result intended. If we want some part of a universal statement to match a singular statement on another line, as in the first example, we instantiate with respect to a constant. But if, at the end of the proof, we want to perform universal generalization over some part of the statement we are instantiating, then we *must* instantiate by using a variable. This latter point leads to an important restriction governing universal generalization—namely, that we cannot perform this operation when the instantial letter is a constant. Consider the following *erroneous* proof sequence:

1.	Ta	
2.	$(x)Tx$	1, UG (invalid)

If *Ta* means "Albert is a thief," then on the basis of this information, we have concluded (line 2) that everything in the universe is a thief. Clearly, such an inference is invalid. This illustrates the fact that universal generalization can be performed only when the instantial letter (in this case *a*) is a variable.

Let us now consider **existential generalization**. The need for this operation is illustrated through the following argument:

All tenors are singers.
Placido Domingo is a tenor.
Therefore, there is at least one singer.

This argument is symbolized as follows:

1. $(x)(Tx \supset Sx)$
2. Tp / $(\exists x)Sx$

If we instantiate the first line with respect to *p*, we can obtain *Sp* via *modus ponens*. But if it is true that Placido Domingo is a tenor, then it certainly follows that there is at least one singer (namely, Placido Domingo). This last step is accomplished by existential generalization (EG). The proof is as follows:

1. $(x)(Tx \supset Sx)$
2. Tp / $(\exists x)Sx$
3. $Tp \supset Sp$ 1, UI
4. Sp 2, 3, MP
5. $(\exists x)Sx$ 4, EG

There are no restrictions on existential generalization, and the operation can be performed when the instantial letter is either a constant (as in the Domingo example) or a variable. As an instance of the latter, consider the following sequence:

1. $(x)(Px \supset Qx)$
2. $(x)Px$ / $(\exists x)Qx$
3. $Py \supset Qy$ 1, UI
4. Py 2, UI
5. Qy 3, 4, MP
6. $(\exists x)Qx$ 5, EG

Line 5 states in effect that everything in the universe is a *Q*. From this, the much weaker conclusion follows (line 6) that *something* is a *Q*. If you should wonder how an existential conclusion can be drawn from universal premises, the answer is that predicate logic assumes that at least one thing exists in the universe. Hence, line 2, which asserts that everything in the universe is a *P*, entails that at least one thing is a *P*. Without this assumption, universal instantiation in line 4 would not be possible.

We may now construct a definition of generalization that covers both varieties. *Generalization* in the inclusive sense is an operation that consists in (1) introducing a quantifier immediately prior to a statement, a statement function, or another quantifier, and (2) replacing one or more occurrences of a certain instantial letter in the statement or statement function with the same variable that appears in the quantifier. For universal generalization, *all* occurrences of the instantial letter must be replaced with the variable in the quantifier, and for existential generalization, *at least one* of the instantial letters must be replaced with the variable in the quantifier. Thus, both of the following cases of existential generalization are valid (although the one on the left is by far the more common version):

1. $Fa \cdot Ga$		1. $Fa \cdot Ga$	
2. $(\exists x)(Fx \cdot Gx)$	1, EG	2. $(\exists x)(Fx \cdot Ga)$	1, EG

On the other hand, only one of the following cases of universal generalization is valid:

1. $Fx \supset Gx$		1. $Fx \supset Gx$	
2. $(y)(Fy \supset Gy)$	1, UG	2. $(y)(Fy \supset Gx)$	1, UG (invalid)

The inference on the right is invalid because the x in Gx was not replaced with the variable in the quantifier (that is, y).

Of course, it may happen that the instantial letter is the same as the variable that appears in the quantifier. Thus, the operation "Gx, therefore $(x)Gx$" counts as a generalization. Cases of generalization where a quantifier is introduced prior to another quantifier will be presented in Section 8.6.

The need for **existential instantiation** can be illustrated through the following argument:

All attorneys are college graduates.
Some attorneys are golfers.
Therefore, some golfers are college graduates.

The symbolic formulation is as follows:

1.	$(x)(Ax \supset Cx)$	
2.	$(\exists x)(Ax \cdot Gx)$	/ $(\exists x)(Gx \cdot Cx)$

If both quantifiers can be removed, the conclusion can be derived via simplification, *modus ponens*, and conjunction. The universal quantifier can be removed by universal instantiation, but to remove the existential quantifier we need existential instantiation. Line 2 states that there is *something* that is both an A and a G. Existential instantiation consists in giving this something a *name*, for example, "David." We will call this name an "existential name" because it is obtained through existential instantiation. The completed proof is as follows:

1.	$(x)(Ax \supset Cx)$	
2.	$(\exists x)(Ax \cdot Gx)$	/ $(\exists x)(Gx \cdot Cx)$
3.	$Ad \cdot Gd$	2, EI
4.	$Ad \supset Cd$	1, UI
5.	Ad	3, Simp
6.	Cd	4, 5, MP
7.	$Gd \cdot Ad$	3, Com
8.	Gd	7, Simp
9.	$Gd \cdot Cd$	6, 8, Conj
10.	$(\exists x)(Gx \cdot Cx)$	9, EG

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Examination of this proof reveals an immediate restriction that must be placed on existential instantiation. The name that we have assigned to the particular something in line 2 that is both an *A* and a *G* is a hypothetical name. It would be a mistake to conclude that this something really has that name. Accordingly, we must introduce a restriction that prevents us from ending the proof with some line that includes the letter *d*. If, for example, the proof were ended at line 9, we would be concluding that the something that is a *G* and a *C* really does have the name *d*. This, of course, would not be legitimate, because *d* is an arbitrary name introduced into the proof for mere convenience. To prevent such a mistake, we require that the name selected for existential instantiation not appear to the right of the slanted line adjacent to the last premise that indicates the conclusion to be derived. Since the last line in the proof must be identical to this line, such a restriction prevents us from ending the proof with a line that contains the existential name.

Further examination of this proof indicates another important restriction on existential instantiation. Notice that the line involving existential instantiation is listed before the line involving universal instantiation. There is a reason for this. If the order were reversed, the existential instantiation step would rest on the illicit assumption that the something that is both an *A* and a *G* has the *same* name as the name used in the earlier universal instantiation step. In other words, it would involve the assumption that the something that is both an *A* and a *G* is the very same something named in the line $Ad \supset Cd$. Of course, no such assumption is legitimate. To keep this mistake from happening, we introduce the restriction that the name introduced by existential instantiation be a new name not occurring earlier in the proof sequence. The following defective proof illustrates what can happen if this restriction is violated:

1.	$(\exists x)(Fx \bullet Ax)$	
2.	$(\exists x)(Fx \bullet Ox)$	/ $(\exists x)(Ax \bullet Ox)$
3.	$Fb \bullet Ab$	1, EI
4.	$Fb \bullet Ob$	2, EI (invalid)
5.	$Ab \bullet Fb$	3, Com
6.	Ab	5, Simp
7.	$Ob \bullet Fb$	4, Com
8.	Ob	7, Simp
9.	$Ab \bullet Ob$	6, 8, Conj
10.	$(\exists x)(Ax \bullet Ox)$	9, EG

To see that this proof is indeed defective, let *F* stand for fruits, *A* for apples, and *O* for oranges. The argument that results is:

Some fruits are apples.
Some fruits are oranges.
Therefore, some apples are oranges.

Since the premises are true and the conclusion false, the argument is clearly invalid. The defect in the proof occurs on line 4. This line asserts that the something that is both an *F* and an *O* is the very same something that is both an *F* and an *A*. In other words, the restriction that the name introduced by existential instantiation be a new name not occurring earlier in the proof is violated.

The first restriction on existential instantiation requires that the existential name not occur in the line that indicates the conclusion to be derived, and the second restriction requires that this name be a new name that has not occurred earlier in the proof. These two restrictions can easily be combined into a single restriction that requires that the name introduced by existential instantiation be a new name that has not occurred in *any* previous line, including the line adjacent to the last premise that indicates the conclusion to be derived.

One further restriction that affects all four of these rules of inference requires that the rules be applied only to *whole lines* in a proof. The following sequence illustrates a violation of this restriction:

1. $(x)Px \supset (x)Qx$
2. $Py \supset Qy$ 1, UI (invalid)

In line 2 universal instantiation is applied to both the antecedent and consequent of the first line. To derive line 2 validly, the first line would have to read $(x)(Px \supset Qx)$. With this final restriction in mind, the four new rules of inference may now be summarized. In the formulation that follows, the symbols $\mathscr{F}x$ and $\mathscr{F}y$ represent any statement function—that is, any symbolic arrangement containing individual variables, such as $Ax \supset Bx$, $Cy \supset (Dy \vee Ey)$, or $Gz \bullet Hz$. And the symbol $\mathscr{F}a$ represents any **statement**; that is, any symbolic arrangement containing individual constants (or names), such as $Ac \supset Bc$, $Cm \supset (Dm \vee Em)$, or $Gw \bullet Hw$:

1. Universal instantiation (UI):

$$\frac{(x)\mathscr{F}x}{\mathscr{F}y} \qquad \frac{(x)\mathscr{F}x}{\mathscr{F}a}$$

2. Universal generalization (UG):

$$\frac{\mathscr{F}y}{(x)\mathscr{F}x} \qquad \text{not allowed:} \qquad \frac{\mathscr{F}a}{(x)\mathscr{F}x}$$

3. Existential instantiation (EI):

$$\frac{(\exists x)\mathscr{F}x}{\mathscr{F}a} \qquad \text{not allowed:} \qquad \frac{(\exists x)\mathscr{F}x}{\mathscr{F}y}$$

Restriction: The existential name a must be a new name that does not appear in any previous line (including the conclusion line).

4. Existential generalization (EG):

$$\frac{\mathscr{F}a}{(\exists x)\mathscr{F}x} \qquad \frac{\mathscr{F}y}{(\exists x)\mathscr{F}x}$$

The *not allowed* version of universal generalization recalls the already familiar fact that generalization is not possible when the instantial letter is a constant. In other words, the mere fact that the individual a is an $\mathscr{F}$ is not sufficient to allow us to conclude that everything in the universe is an $\mathscr{F}$. At present this is the only restriction

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needed for universal generalization. In Sections 8.4 and 8.6, however, two additional restrictions will be introduced. The *not allowed* version of existential instantiation merely recalls the fact that this operation is a naming process. Because variables (x, y, and z) are not names, they cannot be used as instantial letters in existential instantiation.

Let us now investigate some applications of these rules. Consider the following proof:

1.	$(x)(Hx \supset Ix)$	
2.	$(x)(Ix \supset Hx)$	/ $(x)(Hx \equiv Ix)$
3.	$Hx \supset Ix$	1, UI
4.	$Ix \supset Hx$	2, UI
5.	$(Hx \supset Ix) \cdot (Ix \supset Hx)$	3, 4, Conj
6.	$Hx \equiv Ix$	5, Equiv
7.	$(x)(Hx \equiv Ix)$	6, UG

Because we want to perform universal generalization on the last line of the proof, we instantiate lines 1 and 2 using a variable, not a constant. Notice that the variable selected for lines 3 and 4 is the same letter that occurs in lines 1 and 2. While a new letter (y or z) *could* have been selected, this is never necessary in such a step. It *is* necessary, however, since we want to combine lines 3 and 4, that the *same* variable be selected in deriving these lines. Another example:

1.	$(x)[(Ax \vee Bx) \supset Cx]$	
2.	$(\exists x)Ax$	/ $(\exists x)Cx$
3.	Am	2, EI
4.	$(Am \vee Bm) \supset Cm$	1, UI
5.	$Am \vee Bm$	3, Add
6.	Cm	4, 5, MP
7.	$(\exists x)Cx$	6, EG

In conformity with the restriction on existential instantiation, the EI step is performed *before* the UI step. The same letter is then selected in the UI step as was used in the EI step. In line 5, *Bm* is joined disjunctively via addition to *Am*. This rule applies in predicate logic in basically the same way that it does in propositional logic. Any statement or statement function we choose can be joined disjunctively to a given line.

Another example:

1.	$(\exists x)Kx \supset (x)(Lx \supset Mx)$	
2.	$Kc \cdot Lc$	/ Mc
3.	Kc	2, Simp
4.	$(\exists x)Kx$	3, EG
5.	$(x)(Lx \supset Mx)$	1, 4, MP
6.	$Lc \supset Mc$	5, UI
7.	$Lc \cdot Kc$	2, Com
8.	Lc	7, Simp
9.	Mc	6, 8, MP

Since the instantiation (and generalization) rules must be applied to whole lines, it is impossible to instantiate line 1. The only strategy that can be followed is to use some other line to derive the antecedent of this line and then derive the consequent via *modus ponens*. Once the consequent is derived (line 5), it is instantiated using the same letter that appears in line 2.

The next example incorporates all four of the instantiation and generalization rules:

1. $(x)(Px \supset Qx) \supset (\exists x)(Rx \bullet Sx)$	
2. $(x)(Px \supset Sx) \bullet (x)(Sx \supset Qx)$	/ $(\exists x)Sx$
3. $(x)(Px \supset Sx)$	2, Simp
4. $(x)(Sx \supset Qx) \bullet (x)(Px \supset Sx)$	2, Com
5. $(x)(Sx \supset Qx)$	4, Simp
6. $Py \supset Sy$	3, UI
7. $Sy \supset Qy$	5, UI
8. $Py \supset Qy$	6, 7, HS
9. $(x)(Px \supset Qx)$	8, UG
10. $(\exists x)(Rx \bullet Sx)$	1, 9, MP
11. $Ra \bullet Sa$	10, EI
12. $Sa \bullet Ra$	11, Com
13. Sa	12, Simp
14. $(\exists x)Sx$	13, EG

As with the previous example, line 1 cannot be instantiated. To instantiate the two conjuncts in line 2, they must first be separated (lines 3 and 5). Because UG is to be used in line 9, lines 3 and 5 are instantiated using a variable. On the other hand, a constant is used to instantiate line 10 because the statement in question is a particular statement.

Another example:

1. $[(\exists x)Ax \bullet (\exists x)Bx] \supset Cj$	
2. $(\exists x)(Ax \bullet Dx)$	
3. $(\exists x)(Bx \bullet Ex)$	/ Cj
4. $Am \bullet Dm$	2, EI
5. $Bn \bullet En$	3, EI
6. Am	4, Simp
7. Bn	5, Simp
8. $(\exists x)Ax$	6, EG
9. $(\exists x)Bx$	7, EG
10. $(\exists x)Ax \bullet (\exists x)Bx$	8, 9, Conj
11. Cj	1, 10, MP

When line 2 is instantiated (line 4), a letter other than *j*, which appears in line 1, is selected. Then, when line 3 is instantiated (line 5), another new letter is selected. The conclusion is derived, as in earlier examples, via *modus ponens* by deriving the antecedent of line 1.

The following examples illustrate *invalid* or *improper* applications of the instantiation and generalization rules:

1. $Fy \supset Gy$		
2. $(x)(Fx \supset Gy)$	1, UG	(invalid—every instance of *y* must be replaced with *x*)
1. $(x)Fx \supset Ga$		
2. $Fx \supset Ga$	1, UI	(invalid—instantiation can be applied only to whole lines)
1. $(x)Fx \supset (x)Gx$	1, UI	(invalid—instantiation can be applied only to whole lines)
2. $Fx \supset Gx$		
1. Fc		
2. $(\exists x)Gx$		
3. Gc	2, EI	(invalid—*c* appears in line 1)
1. $Fm \supset Gm$		
2. $(x)(Fx \supset Gx)$	1, UG	(invalid—the instantial letter must be a variable; *m* is a constant)
1. $(\exists x)Fx$		
2. $(\exists x)Gx$		
3. Fe	1, EI	
4. Ge	2, EI	(invalid—*e* appears in line 3)
1. $Fs \bullet Gs$		
2. $(\exists x)Fx \bullet Gs$	1, EG	(improper—generalization can be applied only to whole lines)
1. $\sim(x)Fx$		
2. $\sim Fy$	1, UI	(invalid—lines involving negated quantifiers cannot be instantiated; see Section 8.3)

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Exercise 8.2

I. Use the eighteen rules of inference to derive the conclusions of the following symbolized arguments. Do not use either conditional proof or indirect proof.

★(1) 1. $(x)(Ax \supset Bx)$
2. $(x)(Bx \supset Cx)$ / $(x)(Ax \supset Cx)$

(2) 1. $(x)(Bx \supset Cx)$
2. $(\exists x)(Ax \bullet Bx)$ / $(\exists x)(Ax \bullet Cx)$

(3) 1. $(x)(Ax \supset Bx)$
2. $\sim Bm$ / $(\exists x)\sim Ax$

★(4) 1. $(x)[Ax \supset (Bx \vee Cx)]$
2. $Ag \bullet \sim Bg$ / Cg

(5) 1. $(x)[(Ax \vee Bx) \supset Cx]$
2. $(\exists y)(Ay \bullet Dy)$ / $(\exists y)Cy$

(6) 1. $(x)[Jx \supset (Kx \bullet Lx)]$
2. $(\exists y){\sim}Ky$ / $(\exists z){\sim}Jz$

★(7) 1. $(x)[Ax \supset (Bx \vee Cx)]$
2. $(\exists x)(Ax \bullet {\sim}Cx)$ / $(\exists x)Bx$

(8) 1. $(x)(Ax \supset Bx)$
2. $Am \bullet An$ / $Bm \bullet Bn$

(9) 1. $(x)(Ax \supset Bx)$
2. $Am \vee An$ / $Bm \vee Bn$

★(10) 1. $(x)(Bx \vee Ax)$
2. $(x)(Bx \supset Ax)$ / $(x)Ax$

(11) 1. $(x)[(Ax \bullet Bx) \supset Cx]$
2. $(\exists x)(Bx \bullet {\sim}Cx)$ / $(\exists x){\sim}Ax$

(12) 1. $(\exists x)Ax \supset (x)(Bx \supset Cx)$
2. $Am \bullet Bm$ / Cm

★(13) 1. $(\exists x)Ax \supset (x)Bx$
2. $(\exists x)Cx \supset (\exists x)Dx$
3. $An \bullet Cn$ / $(\exists x)(Bx \bullet Dx)$

(14) 1. $(\exists x)Ax \supset (x)(Cx \supset Bx)$
2. $(\exists x)(Ax \vee Bx)$
3. $(x)(Bx \supset Ax)$ / $(x)(Cx \supset Ax)$

(15) 1. $(\exists x)Ax \supset (x)(Bx \supset Cx)$
2. $(\exists x)Dx \supset (\exists x){\sim}Cx$
3. $(\exists x)(Ax \bullet Dx)$ / $(\exists x){\sim}Bx$

II. Translate the following arguments into symbolic form. Then use the eighteen rules of inference to derive the conclusion of each. Do not use conditional or indirect proof.

★**1.** Oranges are sweet. Also, oranges are fragrant. Therefore, oranges are sweet and fragrant. (*O*, *S*, *F*)

2. Tomatoes are vegetables. Therefore, the tomatoes in the garden are vegetables. (*T*, *V*, *G*)

3. Apples and pears grow on trees. Therefore, apples grow on trees. (*A*, *P*, *G*)

★**4.** Carrots are vegetables and peaches are fruit. Furthermore, there are carrots and peaches in the garden. Therefore, there are vegetables and fruit in the garden. (*C*, *V*, *P*, *F*, *G*)

5. Beans and peas are legumes. There are no legumes in the garden. Therefore, there are no beans in the garden. (*B*, *P*, *L*, *G*)

6. There are some cucumbers in the garden. If there are any cucumbers, there are some pumpkins in the garden. All pumpkins are vegetables. Therefore, there are some vegetables in the garden. (*C*, *G*, *P*, *V*)

★7. All gardeners are industrious people. Furthermore, any person who is industrious is respected. Therefore, since Arthur and Catherine are gardeners, it follows that they are respected. (*G*, *I*, *P*, *R*)

8. Some huckleberries are ripe. Furthermore, some boysenberries are sweet. If there are any huckleberries, then the boysenberries are edible if they are sweet. Therefore, some boysenberries are edible. (*H*, *R*, *B*, *S*, *E*)

9. If there are any ripe watermelons, then the caretakers performed well. Furthermore, if there are any large watermelons, then whoever performed well will get a bonus. There are some large, ripe watermelons. Therefore, the caretakers will get a bonus. (*R*, *W*, *C*, *P*, *L*, *B*)

★10. If the artichokes in the kitchen are ripe, then the guests will be surprised. Furthermore, if the artichokes in the kitchen are flavorful, then the guests will be pleased. The artichokes in the kitchen are ripe and flavorful. Therefore, the guests will be surprised and pleased. (*A*, *K*, *R*, *G*, *S*, *F*, *P*)

III. The following dialogue contains nine arguments. Translate each into symbolic form and then use the eighteen rules of inference to derive the conclusion of each.

Where's the Beef?

"Have you decided what to order?" Paul says to Mindy, as he folds his menu and puts it on the table.

"I think I'll have the tofu stir-fry," she replies. "And you?"

"I'll have the rib steak," Paul says. "But are you sure you don't want something more substantial? The rib steak is really good here, and so is the pork tenderloin."

"Since this is our first date, it's understandable that you don't know me very well. I'm a vegetarian," she says.

"Oh. . . . And what made you decide to become a vegetarian?" Paul asks.

"For one thing," Mindy replies, "I think a vegetarian diet is healthier. People who eat meat increase their intake of cholesterol and carcinogens. Those who increase their intake of cholesterol run a higher risk of heart attack, and those who increase their intake of carcinogens run a higher risk of cancer. Anyone who runs a higher risk of heart attack and cancer is less healthy. Thus, people who eat meat are less healthy.

"I might add that if people who eat meat are less healthy, then if parents are responsible, then they will refrain from feeding meat to children. All parents love their children, and if they do, then they are responsible. But if parents refrain from feeding meat to children, then children will grow up to be vegetarians. And if that happens, then nobody will eat meat in the future. Thus, we can look forward to a future where everyone is a vegetarian."

"Well, I won't hold my breath on that," Paul says, as he offers Mindy a slice of bread from the basket on the table. "If children and teenagers fail to eat meat, then they

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become deficient in zinc. And if children become deficient in zinc, then they risk a weakened immune system. And if that happens, then they are less healthy. Also, if elderly people fail to eat meat, then they become deficient in iron. And if elderly people become deficient in iron, then they risk becoming anemic, and if that happens, they are less healthy. Therefore, if children and elderly people fail to eat meat, then they are less healthy."

"You have heard of zinc tablets and iron supplements, haven't you, Paul?" Mindy asks with a smile. "Don't you think they might do the trick? Anyway, there are also moral reasons for being a vegetarian. Consider this. Animals are sentient beings—they feel pain and are subject to fear and joy—and they have an interest in preserving their lives. But if animals are sentient, then if humans cause animals to suffer, then they act immorally. And if animals have an interest in preserving their lives, then if humans exploit animals, then they act immorally. But if humans kill animals for food, then they cause animals to suffer or they exploit them. Therefore, if humans kill animals for food, then they act immorally."

"I agree with you," Paul responds, "that animals should not be made to suffer. But if animals are raised under humane conditions, and some of them are, then they will not be caused to feel pain or distress. And if animals are not caused to feel pain, then we are morally justified in eating them. Thus, we are morally justified in eating some animals."

"But," Paul continues, "your argument also suggests that animals have rights. If animals have rights, then they have moral judgment. And if animals have moral judgment, then they respect the rights of other animals. But every animal pursues its own self-interest to the exclusion of other animals, and if that is so, then it does not respect the rights of other animals. Therefore, animals have no moral judgment and they also have no rights."

"Well," Mindy replies, as she takes a sip of water, "by that line of reasoning infants and mentally challenged adults have no rights. But everyone recognizes that they do have rights. And if infants have rights, then some people who lack the capacity for moral judgment have rights. But if this is true, then animals have rights, and if they do, then surely they have the right to life. But if animals have the right to life, then if humans are moral, then they must respect that right and they cannot kill animals for food. Thus, if humans are moral, they cannot kill animals for food."

"The question of infants and mentally challenged adults raises an interesting point," Paul says. "I think what it comes down to is this. Something is considered to have rights if and only if it looks human. Infants and mentally challenged adults look human, so they are considered to have rights. But animals do not look human, so they are not considered to have rights."

"That sounds awfully arbitrary," says Mindy. "But I think what it really comes down to is power. Something is considered to have rights if and only if it has as much power as humans. Animals do not have as much power as humans, so animals are not considered to have rights. But that seems terribly wrong to me. It shouldn't be a question of power. Anyway, now that our food has arrived, how's your steak?"

"It's great," Paul says, after taking a bite. "And how's your stir-fry?"

"Well, you better believe that it's healthier and more ethically proper than your steak," she says with a laugh.

8.3 Change of Quantifier Rule

The rules of inference developed thus far are not sufficient to derive the conclusion of every argument in predicate logic. For instance, consider the following:

$$\begin{array}{l} \sim(\exists x)(Px \bullet \sim Qx) \\ \underline{\sim(x)(\sim Rx \vee Qx)} \\ (\exists x)\sim Px \end{array}$$

Both premises have tildes preceding the quantifiers. As long as they remain, neither statement can be instantiated; and if these statements cannot be instantiated, the conclusion cannot be derived. What is needed is a rule that will allow us to remove the negation signs. This rule, which we will proceed to develop now, is called the **change of quantifier rule.**

As a basis for developing the change of quantifier rule, consider the following statements:

Everything is beautiful.
It is not the case that everything is beautiful.
Something is beautiful.
It is not the case that something is beautiful.

You should be able to see that these statements are equivalent in meaning to the following statements, respectively:

It is not the case that something is not beautiful.
Something is not beautiful.
It is not the case that everything is not beautiful.
Everything is not beautiful.

If we generalize these English equivalencies symbolically, we obtain:

$$\begin{array}{rcl} (x)\mathcal{F}x & :: & \sim(\exists x)\sim\mathcal{F}x \\ \sim(x)\mathcal{F}x & :: & (\exists x)\sim\mathcal{F}x \\ (\exists x)\mathcal{F}x & :: & \sim(x)\sim\mathcal{F}x \\ \sim(\exists x)\mathcal{F}x & :: & (x)\sim\mathcal{F}x \end{array}$$

These four expressions constitute the change of quantifier rule (CQ). Since they are stated as logical equivalences, they apply to parts of lines as well as to whole lines. They can be summarized as follows:

One type of quantifier can be replaced by the other type if and only if immediately before and after the new quantifier:

1. Tilde operators that were originally present are deleted.
2. Tilde operators that were not originally present are inserted.

To see how the change of quantifier rule is applied, let us return to the argument at the beginning of this section. The proof is as follows:

Eminent Logicians

Alfred North Whitehead 1861–1947
Bertrand Russell 1872–1970

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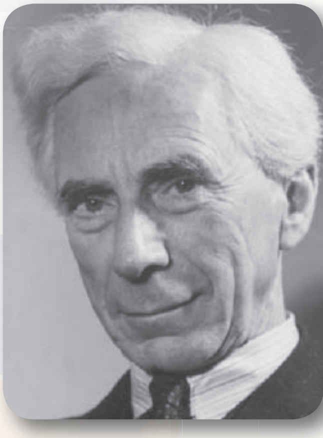

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Alfred North Whitehead and Bertrand Russell collaborated in writing *Principia Mathematica*, which is generally considered the most important logical endeavor of the twentieth century. It represents an attempt to reduce all of mathematics to logic. Published in three volumes between 1910 and 1913, the manuscript was so huge that it required a "four wheeler" to transport it to the printer. The combined work comprises over 1,900 pages, nearly all of them filled with highly complex and technical notation. The American philosopher Willard Van Orman Quine described the *Principia* as "one of the great intellectual monuments of all time."

Whitehead was born the son of an Anglican minister in Ramsgate, England. He entered Trinity College, Cambridge, with a scholarship in mathematics, and after graduating, he was elected a Fellow of Trinity. While there he published *A Treatise on Universal Algebra,* for which he was elected to the prestigious Royal Society. Whitehead's most distinguished student at Trinity was Bertrand Russell. After Russell graduated he and Whitehead became close friends. At age thirty-one Whitehead married Evelyn Wade, who bore him two sons and a daughter.

In 1910 Whitehead left Cambridge for London, where he taught at University College London and later at the Imperial College of Science and Technology. While in London he wrote books in the areas of physics and the philosophy of science. Then, in 1924 he was appointed professor of philosophy at Harvard University, where he wrote *Process and Reality*. The book became a cornerstone of what would later be called process philosophy and process theology. Whitehead died at age eighty-six in Cambridge, Massachusetts.

Bertrand Russell, one of the world's best-known intellectuals, was born in Wales to aristocratic parents. Both died when he was very young, and although they had requested that their son be raised an agnostic, the young Russell was brought up by a staunchly Victorian grandmother who fed him a steady dose of religion. As he grew older he became an atheist (or at best an agnostic) who considered religion little better than superstition. At Trinity College Russell studied mathematics and philosophy and befriended Ludwig Wittgenstein and G. E. Moore, as well as Whitehead. After graduating he was elected fellow of Trinity College and later fellow of the Royal Society. He was extremely prolific as a writer, and in 1950 he won the Nobel Prize in literature.

Russell was married four times, engaged in numerous love affairs with prominent women, was imprisoned for six months for opposing conscription in World War I, and was later imprisoned for opposing nuclear weapons. A staunch supporter of birth control, population control, democracy, free trade, and world government, he opposed communism, imperialism, and every form of mind control. He died at age ninety-seven at his home in Wales.

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1. $\sim(\exists x)(Px \cdot \sim Qx)$	
2. $\sim(x)(\sim Rx \vee Qx)$	/ $(\exists x)\sim Px$
3. $(x)\sim(Px \cdot \sim Qx)$	1, CQ
4. $(\exists x)\sim(\sim Rx \vee Qx)$	2, CQ
5. $\sim(\sim Ra \vee Qa)$	4, EI
6. $\sim(Pa \cdot \sim Qa)$	3, UI
7. $\sim\sim Ra \cdot \sim Qa$	5, DM
8. $Ra \cdot \sim Qa$	7, DN
9. $\sim Pa \vee \sim\sim Qa$	6, DM
10. $\sim Pa \vee Qa$	9, DN
11. $\sim Qa \cdot Ra$	8, Com
12. $\sim Qa$	11, Simp
13. $Qa \vee \sim Pa$	10, Com
14. $\sim Pa$	12, 13, DS
15. $(\exists x)\sim Px$	14, EG

Before either line 1 or line 2 can be instantiated, the tilde operators preceding the quantifiers must be removed. In accordance with the change of quantifier rule, tilde operators are introduced immediately after the new quantifiers in the expressions on lines 3 and 4.

Another example:

1. $(\exists x)(Hx \cdot Gx) \supset (x)Ix$	
2. $\sim Im$	/ $(x)(Hx \supset \sim Gx)$
3. $(\exists x)\sim Ix$	2, EG
4. $\sim(x)Ix$	3, CQ
5. $\sim(\exists x)(Hx \cdot Gx)$	1, 4, MT
6. $(x)\sim(Hx \cdot Gx)$	5, CQ
7. $(x)(\sim Hx \vee \sim Gx)$	6, DM
8. $(x)(Hx \supset \sim Gx)$	7, Impl

The statement that *m* is not an *I* (line 2) intuitively implies that not everything is an *I* (line 4); but existential generalization and change of quantifier are needed to get the desired result. Notice that lines 7 and 8 are derived via De Morgan's rule and material implication, even though the quantifier is still attached. Since these rules are rules of replacement, they apply to parts of lines as well as to whole lines. The following example illustrates the same point with respect to the change of quantifier rule:

1. $(\exists x)Jx \supset \sim(\exists x)Kx$	
2. $(x)\sim Kx \supset (x)\sim Lx$	/ $(\exists x)Jx \supset \sim(\exists x)Lx$
3. $(\exists x)Jx \supset (x)\sim Kx$	1, CQ
4. $(\exists x)Jx \supset (x)\sim Lx$	2, 3, HS
5. $(\exists x)Jx \supset \sim(\exists x)Lx$	4, CQ

The change of quantifier rule is applied to only the consequent of line 1, yielding line 3. Similarly, the change of quantifier rule is then applied to only the consequent of line 4, yielding line 5.

Exercise 8.3

I. Use the change of quantifier rule together with the eighteen rules of inference to derive the conclusions of the following symbolized arguments. Do not use either conditional proof or indirect proof.

★**(1)** 1. $(x)Ax \supset (\exists x)Bx$
2. $(x){\sim}Bx$ / $(\exists x){\sim}Ax$

(2) 1. $(\exists x){\sim}Ax \vee (\exists x){\sim}Bx$
2. $(x)Bx$ / ${\sim}(x)Ax$

(3) 1. ${\sim}(\exists x)Ax$ / $(x)(Ax \supset Bx)$

★**(4)** 1. $(\exists x)Ax \vee (\exists x)(Bx \bullet Cx)$
2. ${\sim}(\exists x)Bx$ / $(\exists x)Ax$

(5) 1. $(x)(Ax \bullet Bx) \vee (x)(Cx \bullet Dx)$
2. ${\sim}(x)Dx$ / $(x)Bx$

(6) 1. $(\exists x){\sim}Ax \supset (x)(Bx \supset Cx)$
2. ${\sim}(x)(Ax \vee Cx)$ / ${\sim}(x)Bx$

★**(7)** 1. $(x)(Ax \supset Bx)$
2. ${\sim}(x)Cx \vee (x)Ax$
3. ${\sim}(x)Bx$ / $(\exists x){\sim}Cx$

(8) 1. $(x)Ax \supset (\exists x){\sim}Bx$
2. ${\sim}(x)Bx \supset (\exists x){\sim}Cx$ / $(x)Cx \supset (\exists x){\sim}Ax$

(9) 1. $(\exists x)(Ax \vee Bx) \supset (x)Cx$
2. $(\exists x){\sim}Cx$ / ${\sim}(\exists x)Ax$

★**(10)** 1. ${\sim}(\exists x)(Ax \bullet {\sim}Bx)$
2. ${\sim}(\exists x)(Bx \bullet {\sim}Cx)$ / $(x)(Ax \supset Cx)$

(11) 1. ${\sim}(\exists x)(Ax \bullet {\sim}Bx)$
2. ${\sim}(\exists x)(Ax \bullet {\sim}Cx)$ / $(x)[Ax \supset (Bx \bullet Cx)]$

(12) 1. $(x)[(Ax \bullet Bx) \supset \mathrm{C}x]$
2. ${\sim}(x)(Ax \supset Cx)$ / ${\sim}(x)Bx$

★**(13)** 1. $(x)(Ax \bullet {\sim}Bx) \supset (\exists x)Cx$
2. ${\sim}(\exists x)(Cx \vee Bx)$ / ${\sim}(x)Ax$

(14) 1. $(\exists x){\sim}Ax \supset (x){\sim}Bx$
2. $(\exists x){\sim}Ax \supset (\exists x)Bx$
3. $(x)(Ax \supset Cx)$ / $(x)Cx$

(15) 1. ${\sim}(\exists x)(Ax \vee Bx)$
2. $(\exists x)Cx \supset (\exists x)Ax$
3. $(\exists x)Dx \supset (\exists x)Bx$ / ${\sim}(\exists x)(Cx \vee Dx)$

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II. Translate the following arguments into symbolic form. Then use the change of quantifier rules and the eighteen rules of inference to derive the conclusion of each. Do not use either conditional proof or indirect proof.

★**1.** If all the physicians are either hematologists or neurologists, then there are no cardiologists. But Dr. Frank is a cardiologist. Therefore, some physicians are not neurologists. (*P*, *H*, *N*, *C*)

2. Either Dr. Adams is an internist or all the pathologists are internists. But it is not the case that there are any internists. Therefore, Dr. Adams is not a pathologist. (*I*, *P*)

3. If some surgeons are allergists, then some psychiatrists are radiologists. But no psychiatrists are radiologists. Therefore, no surgeons are allergists. (*S*, *A*, *P*, *R*)

★**4.** Either some general practitioners are pediatricians or some surgeons are endocrinologists. But it is not the case that there are any endocrinologists. Therefore, there are some pediatricians. (*G*, *P*, *S*, *E*)

5. All physicians who did not attend medical school are incompetent. It is not the case, however, that some physicians are incompetent. Therefore, all physicians have attended medical school. (*P*, *A*, *I*)

6. It is not the case that some internists are not physicians. Furthermore, it is not the case that some physicians are not doctors of medicine. Therefore, all internists are doctors of medicine. (*I*, *P*, *D*)

★**7.** All pathologists are specialists and all internists are generalists. Therefore, since it is not the case that some specialists are generalists, it is not the case that some pathologists are internists. (*P*, *S*, *I*, *G*)

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8. If some obstetricians are not gynecologists, then some hematologists are radiologists. But it is not the case that there are any hematologists or gynecologists. Therefore, it is not the case that there are any obstetricians. (*O*, *G*, *H*, *R*)

9. All poorly trained allergists and dermatologists are untrustworthy specialists. It is not the case, however, that some specialists are untrustworthy. Therefore, it is not the case that some dermatologists are poorly trained. (*P*, *A*, *D*, *U*, *S*)

★**10.** It is not the case that some physicians are either on the golf course or in the hospital. All of the neurologists are physicians in the hospital. Either some physicians are cardiologists or some physicians are neurologists. Therefore, some cardiologists are not on the golf course. (*P*, *G*, *H*, *N*, *C*)

8.4 Conditional and Indirect Proof

Many arguments with conclusions that are either difficult or impossible to derive by the conventional method can be handled with ease by using either conditional or indirect proof. The use of these techniques on arguments in predicate logic is basically the same as it is on arguments in propositional logic. Arguments having conclusions expressed in the form of conditional statements or disjunctions (which can be derived from conditional statements) are immediate candidates for conditional proof. For these arguments, the usual strategy is to put the antecedent of the conditional statement to be obtained in the first line of an indented sequence, to derive the consequent as the last line, and to discharge the conditional sequence in a conditional statement that exactly matches the one to be obtained. Here is an example of such a proof:

1. $(x)(Hx \supset Ix)$	/ $(\exists x)Hx \supset (\exists x)Ix$
│ 2. $(\exists x)Hx$	ACP
│ 3. Ha	2, EI
│ 4. $Ha \supset Ia$	1, UI
│ 5. Ia	3, 4, MP
│ 6. $(\exists x)Ix$	5, EG
7. $(\exists x)Hx \supset (\exists x)Ix$	2–6, CP

In this argument the antecedent of the conclusion is a complete statement consisting of a statement function, *Hx*, preceded by a quantifier. This complete statement is assumed as the first line in the conditional sequence. The instantiation and generalization rules are used within an indented sequence (both conditional and indirect) in basically the same way as they are in a conventional sequence. When the consequent of the conclusion is derived, the conditional sequence is completed, and it is then discharged in a conditional statement having the first line of the sequence as its antecedent and the last line as its consequent.

The next example differs from the previous one in that the antecedent of the conclusion is a statement function, not a complete statement. With arguments such as this, only the statement function is assumed as the first line in the conditional sequence. The quantifier is added after the sequence is discharged.

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1. $(x)[(Ax \vee Bx) \supset Cx]$	/ $(x)(Ax \supset Cx)$
│ 2. Ax	ACP
│ 3. $Ax \vee Bx$	2, Add
│ 4. $(Ax \vee Bx) \supset Cx$	1, UI
│ 5. Cx	3, 4, MP
6. $Ax \supset Cx$	2–5, CP
7. $(x)(Ax \supset Cx)$	6, UG

This example leads to an important restriction on the use of universal generalization. You may recall that the *x* in line 2 of this proof is said to be *free* because it is not bound by any quantifier. (In contrast, the *x*'s in lines 1 and 7 are bound by quantifiers.) The restriction is as follows:

UG: $\dfrac{\mathscr{F}y}{(x)\mathscr{F}x}$ *Restriction*: UG must not be used within the scope of an indented sequence if the instantial variable y is free in the first line of that sequence.

The proof just given does not violate this restriction, because UG is not used within the scope of the indented sequence at all. It is used only after the sequence has been discharged, which is perfectly acceptable. If, on the other hand, UG had been applied to line 5 to produce a statement reading $(x)Cx$, the restriction would have been violated because the instantial variable x is free in the first line of the sequence.

To understand why this restriction is necessary, consider the following *defective* proof:

1. $(x)Rx \supset (x)Sx$	/ $(x)(Rx \supset Sx)$
2. Rx	ACP
3. $(x)Rx$	2, UG (invalid)
4. $(x)Sx$	1, 3, MP
5. Sx	4, UI
6. $Rx \supset Sx$	2–5, CP
7. $(x)(Rx \supset Sx)$	6, UG

If Rx means "x is a rabbit" and Sx means "x is a snake," then the premise translates "If everything in the universe is a rabbit, then everything in the universe is a snake." This statement is *true* because the antecedent is false; that is, it is *not* the case that everything in the universe is a rabbit. The conclusion, on the other hand, is *false*, because it asserts that all rabbits are snakes. The argument is therefore invalid. If the restriction on UG had been obeyed, UG could not have been used on line 3 and, as a result, the illicit conclusion could not have been derived.

It is interesting to see what happens when the premise and the conclusion of this defective argument are switched. The proof, which is perfectly legitimate, is as follows:

1. $(x)(Rx \supset Sx)$	/ $(x)Rx \supset (x)Sx$
2. $(x)Rx$	ACP
3. Rx	2, UI
4. $Rx \supset Sx$	1, UI
5. Sx	3, 4, MP
6. $(x)Sx$	5, UG
7. $(x)Rx \supset (x)Sx$	2–6, CP

Notice in this proof that UG *is* used within the scope of a conditional sequence, but the restriction is not violated because the instantial variable x is not free in the first line of the sequence.

Let us now consider some examples of *indirect* proof. We begin an indirect sequence by assuming the negation of the statement to be obtained. When a contradiction is derived, the indirect sequence is discharged by asserting the denial of the original assumption. In the examples that follow, the negation of the conclusion is assumed as the first line of the sequence, and the change of quantifier rule is then used to eliminate the tilde. When the resulting statement is then instantiated, a new letter, m, is selected that has not appeared anywhere in a previous line. The same letter is then selected for the universal instantiation of line 1:

1. $(x)[(Px \supset Px) \supset (Qx \supset Rx)]$	/ $(x)(Qx \supset Rx)$
2. $\sim (x)(Qx \supset Rx)$	AIP
3. $(\exists x){\sim}(Qx \supset Rx)$	2, CQ

4. $\sim(Qm \supset Rm)$	3, EI
5. $(Pm \supset Pm) \supset (Qm \supset Rm)$	1, UI
6. $\sim(Pm \supset Pm)$	4, 5, MT
7. $\sim(\sim Pm \vee Pm)$	6, Impl
8. $\sim\sim Pm \bullet \sim Pm$	7, DM
9. $Pm \bullet \sim Pm$	8, DN
10. $\sim\sim(x)(Qx \supset Rx)$	2–9, IP
11. $(x)(Qx \supset Rx)$	10, DN

The next example has a particular statement for its conclusion:

1. $(\exists x)Ax \vee (\exists x)Fx$	
2. $(x)(Ax \supset Fx)$	/ $(\exists x)Fx$
3. $\sim(\exists x)Fx$	AIP
4. $(\exists x)Fx \vee (\exists x)Ax$	1, Com
5. $(\exists x)Ax$	3, 4, DS
6. Ac	5, EI
7. $Ac \supset Fc$	2, UI
8. Fc	6, 7, MP
9. $(x)\sim Fx$	3, CQ
10. $\sim Fc$	9, UI
11. $Fc \bullet \sim Fc$	8, 10, Conj
12. $\sim\sim(\exists x)Fx$	3–11, IP
13. $(\exists x)Fx$	12, DN

Since indirect proof sequences are indented, they are subject to the same restriction on universal generalization as are conditional sequences. The following proof, which is similar to the previous one, violates this restriction because the instantial variable x is free in the first line of the sequence. The violation (line 4) allows a universal statement to be drawn for the conclusion, whereas only a particular statement is legitimate (as in the prior example):

1. $(\exists x)Ax \vee (\exists x)Fx$	
2. $(x)(Ax \supset Fx)$	/ $(x)Fx$
3. $\sim Fx$	AIP
4. $(x)\sim Fx$	3, UG (invalid)
5. $\sim(\exists x)Fx$	4, CQ
6. $(\exists x)Fx \vee (\exists x)Ax$	1, Com
7. $(\exists x)Ax$	5, 6, DS
8. Ac	7, EI
9. $Ac \supset Fc$	2, UI
10. Fc	8, 9, MP
11. $\sim Fc$	4, UI
12. $Fc \bullet \sim Fc$	10, 11, Conj
13. $\sim\sim Fx$	3–12, IP
14. Fx	13, DN
15. $(x)Fx$	14, UG

To see that this argument is indeed invalid, let *Ax* stand for "*x* is an apple" and *Fx* for "*x* is a fruit." The first premise then reads "Either an apple exists or a fruit exists" (which is true), and the second premise reads "All apples are fruits" (which is also true). The conclusion, however, reads "Everything in the universe is a fruit," and this, of course, is false.

As in propositional logic, conditional and indirect sequences in predicate logic may include each other. The following proof uses an indirect sequence within the scope of a conditional sequence.

1. $(x)[(Px \vee Qx) \supset (Rx \bullet Sx)]$	/ $(\exists x)(Px \vee Sx) \supset (\exists x)Sx$
2. $(\exists x)(Px \vee Sx)$	ACP
3. $\sim(\exists x)Sx$	AIP
4. $(x)\sim Sx$	3, CQ
5. $Pa \vee Sa$	2, EI
6. $\sim Sa$	4, UI
7. $Sa \vee Pa$	5, Com
8. Pa	6, 7, DS
9. $Pa \vee Qa$	8, Add
10. $(Pa \vee Qa) \supset (Ra \bullet Sa)$	1, UI
11. $Ra \bullet Sa$	9, 10, MP
12. $Sa \bullet Ra$	11, Com
13. Sa	12, Simp
14. $Sa \bullet \sim Sa$	6, 13, Conj
15. $\sim\sim(\exists x)Sx$	3–14, IP
16. $(\exists x)Sx$	15, DN
17. $(\exists x)(Px \vee Sx) \supset (\exists x)Sx$	2–16, CP

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The conditional sequence begins, as usual, by assuming the antecedent of the conditional statement to be derived. The objective, then, is to derive the consequent. This is accomplished by the indirect sequence, which begins with the negation of the consequent and ends (line 14) with a contradiction.

Exercise 8.4

I. Use either indirect proof or conditional proof to derive the conclusions of the following symbolized arguments.

★**(1)** 1. $(x)(Ax \supset Bx)$
2. $(x)(Ax \supset Cx)$ / $(x)[Ax \supset (Bx \bullet Cx)]$

(2) 1. $(\exists x)Ax \supset (\exists x)(Bx \bullet Cx)$
2. $(\exists x)(Cx \vee Dx) \supset (x)Ex$ / $(x)(Ax \supset Ex)$

(3) 1. $(\exists x)Ax \supset (\exists x)(Bx \bullet Cx)$
2. $\sim(\exists x)Cx$ / $(x)\sim Ax$

★**(4)** 1. $(x)(Ax \supset Cx)$
2. $(\exists x)Cx \supset (\exists x)(Bx \bullet Dx)$ / $(\exists x)Ax \supset (\exists x)Bx$

(5) 1. $(x)(Ax \supset Bx)$
2. $(x)[(Ax \bullet Bx) \supset Cx]$ / $(x)(Ax \supset Cx)$

(6) 1. $(\exists x)Ax \supset (x)Bx$
2. $An \supset {\sim}Bn$ / ${\sim}An$

★(7) 1. $(x)[(Ax \vee Bx) \supset Cx]$
2. $(x)[(Cx \vee Dx) \supset Ex]$ / $(x)(Ax \supset Ex)$

(8) 1. $(\exists x)(Ax \vee Bx) \supset {\sim}(\exists x)Ax$ / $(x){\sim}Ax$

(9) 1. $(x)(Ax \supset Bx)$
2. $(x)(Cx \supset Dx)$ / $(\exists x)(Ax \vee Cx) \supset (\exists x)(Bx \vee Dx)$

★(10) 1. $(x)(Ax \supset Bx)$
2. $Am \vee An$ / $(\exists x)Bx$

(11) 1. $(x)[(Ax \vee Bx) \supset Cx]$
2. $(x)[(Cx \vee Dx) \supset {\sim}Ax]$ / $(x){\sim}Ax$

(12) 1. $(\exists x)Ax \supset (x)(Bx \supset Cx)$
2. $(\exists x)Dx \supset (x){\sim}Cx$ / $(x)[(Ax \bullet Dx) \supset {\sim}Bx]$

★(13) 1. $(\exists x)Ax \supset (x)(Bx \supset Cx)$
2. $(\exists x)Dx \supset (\exists x)Bx$ / $(\exists x)(Ax \bullet Dx) \supset (\exists x)Cx$

(14) 1. $(\exists x)Ax \vee (\exists x)(Bx \bullet Cx)$
2. $(x)(Ax \supset Cx)$ / $(\exists x)Cx$

(15) 1. $(\exists x)Ax \supset (\exists x)(Bx \bullet Cx)$
2. $(\exists x)Cx \supset (x)(Dx \bullet Ex)$ / $(x)(Ax \supset Ex)$

★(16) 1. $(x)[(Ax \vee Bx) \supset Cx]$
2. $(\exists x)({\sim}Ax \vee Dx) \supset (x)Ex$ / $(x)Cx \vee (x)Ex$

(17) 1. $(x)Ax \equiv (\exists x)(Bx \bullet Cx)$
2. $(x)(Cx \supset Bx)$ / $(x)Ax \equiv (\exists x)Cx$

(18) 1. $(x)(Ax \equiv Bx)$
2. $(x)[Ax \supset (Bx \supset Cx)]$
3. $(\exists x)Ax \vee (\exists x)Bx$ / $(\exists x)Cx$

★(19) 1. $(x)[Bx \supset (Cx \bullet Dx)]$ / $(x)(Ax \supset Bx) \supset (x)(Ax \supset Dx)$

(20) 1. $(x)[Ax \supset (Bx \bullet Cx)]$
2. $(x)[Dx \supset (Ex \bullet Fx)]$ / $(x)(Cx \supset Dx) \supset (x)(Ax \supset Fx)$

(21) 1. $(\exists x)(Ax \vee Bx)$
2. $(\exists x)Ax \supset (x)(Cx \supset Bx)$
3. $(\exists x)Cx$ / $(\exists x)Bx$

II. Translate the following arguments into symbolic form. Then use conditional or indirect proof to derive the conclusion of each.

★**1.** All ambassadors are wealthy. Furthermore, all Republicans are clever. Therefore, all Republican ambassadors are clever and wealthy. (*A*, *W*, *R*, *C*)

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2. All senators are well liked. Also, if there are any well-liked senators, then O'Brien is a voter. Therefore, if there are any senators, then O'Brien is a voter. (*S*, *W*, *V*)

3. If all judges are wise, then some attorneys are rewarded. Furthermore, if there are any judges who are not wise, then some attorneys are rewarded. Therefore, some attorneys are rewarded. (*J*, *W*, *A*, *R*)

★4. All secretaries and undersecretaries are intelligent and cautious. All those who are cautious or vigilant are restrained and austere. Therefore, all secretaries are austere. (*S*, *U*, *I*, *C*, *V*, *R*, *A*)

5. All ambassadors are diplomats. Furthermore, all experienced ambassadors are cautious, and all cautious diplomats have foresight. Therefore, all experienced ambassadors have foresight. (*A*, *D*, *E*, *C*, *F*)

6. If there are any senators, then some employees are well paid. If there is anyone who is either an employee or a volunteer, then there are some legislative assistants. Either there are some volunteers or there are some senators. Therefore, there are some legislative assistants. (*S*, *E*, *W*, *V*, *L*)

★7. If there are any consuls, then all ambassadors are satisfied diplomats. If no consuls are ambassadors, then some diplomats are satisfied. Therefore, some diplomats are satisfied. (*C*, *A*, *S*, *D*)

8. If there are any voters, then all politicians are astute. If there are any politicians, then whoever is astute is clever. Therefore, if there are any voters, then all politicians are clever. (*V*, *P*, *A*, *C*)

9. Either no senators are present or no representatives are present. Furthermore, either some senators are present or no women are present. Therefore, none of the representatives who are present are women. (*S*, *P*, *R*, *W*)

★10. Either some governors are present or some ambassadors are present. If anyone is present, then some ambassadors are clever diplomats. Therefore, some diplomats are clever. (*G*, *P*, *A*, *C*, *D*)

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8.5 Proving Invalidity

In the previous chapter we saw that natural deduction could not be used with any facility to prove invalidity in propositional logic. The same thing can be said about natural deduction in predicate logic. But in predicate logic there is no simple and automatic technique such as truth tables or Venn diagrams to fall back on. However, there are two methods for proving invalidity in predicate logic that are just as effective as these other techniques, even though they may not be as convenient. One is the method used in Chapter 1 to prove the invalidity of various kinds of syllogisms—namely, the counterexample method. The other is what we will call the finite universe method. Both appeal to the basic idea underlying most proofs of invalidity: Any argument is proved invalid if it is shown that it is possible for it to have true premises and a false conclusion. Both methods are aimed at disclosing a situation that fulfills this requirement.

Counterexample Method

Recall from Chapter 1 that applying the **counterexample method** consists in finding a substitution instance of a given invalid argument form (or, equally well, a given invalid symbolized argument) that has actually true premises and a false conclusion. For an example of its use, consider the following invalid symbolized argument:

$(\exists x)(Ax \cdot \sim Bx)$
$(x)(Cx \supset Bx) \qquad / \ (\exists x)(Cx \cdot \sim Ax)$

In creating a substitution instance, beginning with the conclusion is often easiest. The conclusion is translated as "Some *C* are not *A*." Thus, to make this statement false, we need to find an example of a class (for *C*) that is included in another class (for *A*). Cats and animals will serve this purpose. A little ingenuity provides us with the following substitution instance:

Some animals are not mammals.
All cats are mammals.
Therefore, some cats are not animals.

When we produce such a substitution instance, the premises must turn out to be indisputably true in the actual world, and the conclusion indisputably false. Statements involving the names of animal classes are convenient for this purpose, because everyone agrees about cats, dogs, mammals, fish, and so on. Also, several different substitution instances can usually be produced that suffice to prove the argument invalid. Finally, any substitution instance that results in true premises and a true conclusion (or any arrangement other than true premises and false conclusion) proves nothing.

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Here is an example of an invalid symbolized argument that includes a singular statement:

$(x)(Ax \supset Bx)$
$\sim Ac \qquad / \ \sim Bc$

This argument form commits the fallacy of denying the antecedent. Producing a substitution instance is easy:

All cats are animals.
Lassie is not a cat.
Therefore, Lassie is not an animal.

In selecting the name of an individual for the second premise, it is again necessary to pick something that everyone agrees on. Since everyone knows that Lassie is a dog, this name serves our purpose. But if we had selected some other name, such as Trixie or Ajax, this would hardly suffice, because there is no general agreement as to what these names denote.

Here is a slightly more complex example:

$(x)[Ax \supset (Bx \vee Cx)]$
$(x)[Bx \supset (Cx \cdot Dx)] \quad / \; (x)(Ax \supset Dx)$

A little ingenuity produces the following substitution instance:

All dogs are either sharks or animals.
All sharks are animals that are fish.
Therefore, all dogs are fish.

The counterexample method is effective with most fairly simple invalid arguments in predicate logic. Since its application depends on the ingenuity of the user, however, it is not particularly well suited for complex arguments. For those, the finite universe method is probably a better choice.

Finite Universe Method

The **finite universe method** can be used to establish the invalidity of any invalid argument expressed in terms of a single variable. It depends on the idea that a valid argument remains valid no matter how things in the actual universe might be altered. Accordingly, if we are given a valid argument, then that argument remains valid if it should happen that the universe is contracted so that it contains only a single member. On the other hand, if it should turn out that an argument has true premises and false conclusion in a universe consisting of only one or a few members, then that argument has been proved invalid.

To see how this method works, we need to understand what happens to the meaning of universal and particular statements when the universe is shrunk in size. Accordingly, let us imagine that the universe contains only one thing instead of the billions of things that it actually contains. Let us name that one thing "Abigail." The statement "Everything in the universe is perfect" is then equivalent to "Abigail is perfect" (because Abigail is all that there is), and the statement "Something in the universe is perfect" is also equivalent to "Abigail is perfect" (because Abigail is that "something").

To represent this equivalence symbolically, we need a new metalogical symbol that asserts that the expressions on either side of it necessarily have the same truth value given a universe of a designated size. Although this equivalence bears a close resemblance to logical equivalence, it is not identical to it, because logical equivalence holds independently of any alterations in the universe. The concept that we need to represent is a kind of conditional logical equivalence. Accordingly, we will select the double colon superscribed with a "c" (for "conditional"). For our purpose here, $\overset{c}{::}$ has the same effect as **::**. Using the former symbol, we have for a universe consisting of one member:

$(x)Px \quad \overset{c}{::} \quad Pa$
$(\exists x)Px \quad \overset{c}{::} \quad Pa$

Proceeding, if we imagine that the universe contains exactly two things—let us name them "Abigail" and "Beatrice"—the statement "Everything in the universe is perfect" is equivalent to "Abigail is perfect *and* Beatrice is perfect." On the other hand, the

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statement "Something in the universe is perfect" is equivalent to "Abigail is perfect *or* Beatrice is perfect" (because "some" means at least one). In other words, the universal statement is equivalent to a *conjunction* of singular statements, and the particular statement is equivalent to a *disjunction* of singular statements. In symbols:

$$(x)Px \mathrel{::} (Pa \cdot Pb)$$
$$(\exists x)Px \mathrel{::} (Pa \vee Pb)$$

If the universe is increased to three—let us call the new member "Charmaine"—we have

$$(x)Px \mathrel{::} (Pa \cdot Pb \cdot Pc)$$
$$(\exists x)Px \mathrel{::} (Pa \vee Pb \vee Pc)$$

This equivalence continues indefinitely as more and more members are added to the universe.

Extending this treatment to the more typical kinds of universal and particular statements, we have, for a universe of three:

$$(x)(Px \supset Qx) \mathrel{::} [(Pa \supset Qa) \cdot (Pb \supset Qb) \cdot (Pc \supset Qc)]$$
$$(\exists x)(Px \cdot Qx) \mathrel{::} [(Pa \cdot Qa) \vee (Pb \cdot Qb) \vee (Pc \cdot Qc)]$$

For expressions involving combinations of quantified statements, each of the component statements is translated separately and the resulting statement groups are linked together by means of the connective appearing in the original statement. Here are two examples for a universe of three:

$$[(x)Px \supset (\exists x)Qx] \mathrel{::} [(Pa \cdot Pb \cdot Pc) \supset (Qa \vee Qb \vee Qc)]$$
$$[(x)(Px \supset Qx) \vee (\exists x)(Rx \cdot Sx)] \mathrel{::} \{[(Pa \supset Qa) \cdot (Pb \supset Qb) \cdot (Pc \supset Qc)] \vee [(Ra \cdot Sa) \vee (Rb \cdot Sb) \vee (Rc \cdot Sc)]\}$$

The method for proving an argument invalid consists in translating the premises and conclusion into singular statements, as per the above examples, and then testing the result with an indirect truth table (see Section 6.5). First a universe of one is tried. If it is possible for the premises to be true and the conclusion false in this universe, the argument is immediately identified as invalid. If, on the other hand, a contradiction results from this assumption, a universe of two is then tried. If, in this second universe, it is possible for the premises to be true and the conclusion false, the argument is invalid. If not, a universe of three is tried, and so on.

Consider the following symbolized argument:

$$(x)(Gx \supset Hx)$$
$$(\exists x)Hx \quad / \quad (\exists x)Gx$$

For a universe having one member—call this member "Abigail"—the argument translates into

$$Ga \supset Ha$$
$$Ha \quad / \quad Ga$$

Testing with an indirect truth table, we have

Ga	⊃	Ha	/	Ha	//	Ga
F	T	T		T		F

Because it is possible for the premises to be true and the conclusion false, the argument is invalid. Another example:

$$(x)(Jx \supset Kx)$$
$$(\exists x)Jx \qquad / \ (x)Kx$$

For a universe having one member, the indirect truth table is as follows:

Ja	$\supset$	Ka	/	Ja	/ /	Ka
T	T	F		T		F

Since it is impossible for the premises to be true and the conclusion false for this universe, we try a universe having two members, *a* and *b*:

(Ja	$\supset$	Ka)	$\cdot$	(Jb	$\supset$	Kb)	/	Ja	$\vee$	Jb	/ /	Ka	$\cdot$	Kb
T	T	T	T	F	T	F		T	T	F		T	F	F

Since it is possible for the premises to be true and the conclusion false for this universe, the argument is invalid.

Here is an example involving compound statements:

$$(\exists x)Hx \supset (x)(Fx \supset Gx)$$
$$(\exists x)Fx \qquad /(\exists x)Hx \supset (x)Gx$$

The indirect truth table for a universe having one member is as follows:

Ha	$\supset$	(Fa	$\supset$	Ga)	/	Fa	/ /	Ha	$\supset$	Ga
T	T	T	F	F		T		T	F	F

A contradiction results, so we try a universe having two members. The resulting indirect truth table proves the argument invalid:

(Ha	$\vee$	Hb)	$\supset$	[(Fa	$\supset$	Ga)	$\cdot$	(Fb	$\supset$	Gb)]	/	Fa	$\vee$	Fb	/ /	(Ha	$\vee$	Hb)	$\supset$	(Ga	$\cdot$	Gb)
	T		T	T	T	T	T	F	T	F		T	T	F			T		F	T	F	F

The next example involves singular statements:

$$(\exists x)Mx \cdot (\exists x)Nx$$
$$Md \qquad / \ Nd$$

The second premise asserts that something named *d* is an *M*. For this argument, the assumption that the universe contains only one member entails that this one member is named *d*. Here is the indirect truth table for such a universe:

Md	$\cdot$	Nd	/	Md	/ /	Nd
T	T	F		T		F

When the universe is expanded to include two members, we are free to give any name we wish to the second member. Let us call it *e*. The resulting indirect truth table, which follows, shows that the argument is invalid. Notice that the second premise and the conclusion remain the same:

(Md	$\vee$	Me)	$\cdot$	(Nd	$\vee$	Ne)	/	Md	/ /	Nd
T	T		T	F	T	T		T		F

The basic concept behind this method of proving invalidity rests on the fact that a valid argument is valid in all possible universes. Consequently, if an argument fails in a universe consisting of one, two, or any number of members, it is invalid.

While this method is intended primarily for proving arguments invalid, theoretically it can also be used to prove arguments valid. Several years ago a theorem was proved to the effect that an argument that does not fail in a universe of 2^n members, where n designates the number of different predicates, is valid.* According to this theorem, establishing the validity of an argument containing two different predicates requires a universe having four members, establishing the validity of an argument containing three different predicates requires a universe having eight members, and so on. For most arguments, however, a universe having four members is unwieldy at best, and a universe having eight members approaches the impossible (although a computer could handle it easily). Thus, while this method is usually quite convenient for proving invalidity, practical limitations impede its usefulness in establishing validity.

Exercise 8.5

I. Use the counterexample method to prove that the following symbolized arguments are invalid.

★**(1)** 1. $(x)(Ax \supset Bx)$
2. $(x)(Ax \supset {\sim}Cx)$ / $(x)(Cx \supset Bx)$

(2) 1. $(\exists x)(Ax \bullet Bx)$
2. $(x)(Cx \supset Ax)$ / $(\exists x)(Cx \bullet Bx)$

(3) 1. $(x)(Ax \supset Bx)$
2. Bc / Ac

★**(4)** 1. $(\exists x)(Ax \bullet Bx)$
2. $(\exists x)(Ax \bullet Cx)$ / $(\exists x)[Ax \bullet (Bx \bullet Cx)]$

(5) 1. $(x)[Ax \vee (Bx \vee Cx)]$ / $(x)Ax \vee [(x)Bx \vee (x)Cx]$

(6) 1. $(x)[Ax \supset (Bx \vee Cx)]$
2. $(x)[(Bx \bullet Cx) \supset Dx]$ / $(x)(Ax \supset Dx)$

★**(7)** 1. $(\exists x)Ax$
2. $(\exists x)Bx$
3. $(x)(Ax \supset {\sim}Cx)$ / $(\exists x)(Bx \bullet {\sim}Cx)$

(8) 1. $(x)[(Ax \vee Bx) \supset Cx]$
2. $(x)[(Cx \bullet Dx) \supset Ex]$ / $(x)(Ax \supset Ex)$

(9) 1. $(x)[(Ax \bullet Bx) \supset Cx]$
2. $(x)[(Ax \bullet Cx) \supset Dx]$ / $(x)[(Ax \bullet Dx) \supset Cx]$

*See Wilhelm Ackermann, *Solvable Cases of the Decision Problem* (Amsterdam: North-Holland Publishing Co., 1954), Chapter 4. This theorem, incidentally, holds only for monadic predicates.

★(10) 1. $(\exists x)(Ax \bullet Bx)$

2. $(\exists x)(Cx \bullet \sim Bx)$

3. $(x)(Ax \supset Cx)$ / $(\exists x)[(Cx \bullet Bx) \bullet \sim Ax]$

II. Use the finite universe method to prove that the following symbolized arguments are invalid.

★(1) 1. $(x)(Ax \supset Bx)$

2. $(x)(Ax \supset Cx)$ / $(x)(Bx \supset Cx)$

(2) 1. $(x)(Ax \vee Bx)$

2. $\sim An$ / $(x)Bx$

(3) 1. $(\exists x)Ax \vee (\exists x)Bx$

2. $(\exists x)Ax$ / $(\exists x)Bx$

★(4) 1. $(x)(Ax \supset Bx)$

2. $(\exists x)Ax$ / $(x)Bx$

(5) 1. $(x)[Ax \supset (Bx \vee Cx)]$

2. $(\exists x)Ax$ / $(\exists x)Bx$

(6) 1. $(\exists x)Ax$

2. $(\exists x)Bx$ / $(\exists x)(Ax \bullet Bx)$

★(7) 1. $(x)(Ax \supset Bx)$

2. $(\exists x)Bx \supset (\exists x)Cx$ / $(x)(Ax \supset Cx)$

(8) 1. $(\exists x)(Ax \bullet Bx) \equiv (\exists x)Cx$

2. $(x)(Ax \supset Bx)$ / $(x)Ax \equiv (\exists x)Cx$

(9) 1. $(\exists x)(Ax \bullet \sim Bx)$

2. $(\exists x)(Bx \bullet \sim Ax)$ / $(x)(Ax \vee Bx)$

★(10) 1. $(\exists x)(Ax \bullet Bx)$

2. $(\exists x)(\sim Ax \bullet \sim Bx)$ / $(x)(Ax \equiv Bx)$

III. Translate the following arguments into symbolic form. Then use either the counter-example method or the finite universe method to prove that each is invalid.

★**1.** Violinists who play well are accomplished musicians. There are some violinists in the orchestra. Therefore, some musicians are accomplished. (*V*, *P*, *A*, *M*, *O*)

2. Pianists and harpsichordists are meticulous. Alfred Brendel is a pianist. Therefore, everyone is meticulous. (*P*, *H*, *M*)

3. If there are any oboists, there are some bassoonists. If there are any clarinetists, there are some flutists. Amelia is both an oboist and a clarinetist. Therefore, some bassoonists are flutists. (*O*, *B*, *C*, *F*)

★**4.** All tympanists are haughty. If some tympanists are haughty, then some percussionists are overbearing. Therefore, all tympanists are overbearing. (*T*, *H*, *P*, *O*)

5. All cellists and violinists are members of the string section. Some violinists are not cellists. Also, some cellists are not violinists. Therefore, everyone is a member of the string section. (*C*, *V*, *M*)

8.6 Relational Predicates and Overlapping Quantifiers

Even the logical machinery developed thus far is not adequate for deriving the conclusions of a number of arguments. Consider, for example, the following:

> All dogs are animals. Therefore, whoever owns a dog owns an animal.
>
> If there are any butterflies, then if all butterflies are free, they are free. There are butterflies in the garden. Therefore, if all butterflies are free, something in the garden is free.

The first argument involves a relation—the relation of ownership—and we have yet to see how relations can be dealt with. The second argument, while not involving any relations, involves a quantifier that overlaps another quantifier. In this section the apparatus of predicate logic will be extended to cover examples such as these.

The predicates we have used thus far are called **monadic predicates,** or one-place predicates, because they are used to assign an attribute to individual things. A **relational predicate** (or relation) is a predicate that is used to establish a connection *between* or *among* individuals.

Relations occur in varying degrees of complexity, depending on the number of individuals related. The simplest, called *binary* (or *dyadic*) relations, establish a connection between two individuals. Some examples are the relation of being taller than, as expressed in the statement "Steve is taller than David," and the relation of being a friend, as expressed in "Sylvia is a friend of Olivia." *Trinary* (or *triadic*) relations establish a connection among three individuals: for example, the relation of being between, as in "St. Louis is between Chicago and New Orleans," and the relation of reading something to someone, as in "George read *Othello* to Madeline." *Quaternary* (or *tetradic*) relations link four individuals together—for example, the relation of reading something to someone at a certain time, as in "George read *Othello* to Madeline on Thursday." The complexity increases until we have what are called *n-ary* (or *n-adic*) relations, which link *n* things together. In this section we will restrict our attention to binary relations.

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Translating Relational Statements

Relations are symbolized like other predicates except that two lowercase letters, representing the two related individuals, are written to the immediate right of the uppercase letter representing the relation. Here are some examples of relational statements involving specifically named individuals:

Statement	Symbolic translation
Anthony is married to Cynthia.	*Mac*
Deborah loves physics.	*Ldp*
The Willis Tower is taller than the Empire State Building.	*Twe*
Donald is the father of Jim.	*Fdj*

Notice that the order in which the lowercase letters are listed often makes a difference. If the third statement were translated *Tew*, the symbolic statement would read "The Empire State Building is taller than the Willis Tower," which is false. Quantifiers are attached to relational predicates in the same way they are to monadic predicates. Some examples of relational statements involving quantifiers are as follows:

Statement	**Symbolic translation**
Thomas knows everything.	$(x)Ktx$
Thomas knows something.	$(\exists x)Ktx$
Everything is different from everything.	$(x)(y)Dxy$
Something is different from something.	$(\exists x)(\exists y)Dxy$
Everything is different from something (or other).	$(x)(\exists y)Dxy$
Something is different from everything.	$(\exists x)(y)Dxy$

The last four statements involve **overlapping quantifiers.** We may read these symbols as follows:

$(x)(y)$	For all *x* and for all *y* . . .
$(\exists x)(\exists y)$	There exists an *x* such that there exists a *y* such that . . .
$(x)(\exists y)$	For all *x* there exists a *y* such that . . .
$(\exists x)(y)$	There exists an *x* such that for all *y* . . .

Applying this phraseology to the last statement given, for example, we have "There exists an *x* such that for all *y*, *x* is different from *y*"—which is simply another way of saying "Something is different from everything."

When two quantifiers of the same sort appear adjacent to each other, the order in which they are listed is not significant. In other words, the statement $(x)(y)Dxy$ is logically equivalent to $(y)(x)Dxy$, and $(\exists x)(\exists y)Dxy$ is logically equivalent to $(\exists y)(\exists x)Dxy$. A little reflection on the meaning of these statements should justify this equivalence. But when different quantifiers appear adjacent to each other, the order *does* make a difference, sometimes even when the statement function is nonrelational. Accordingly, $(x)(\exists y)Dxy$ is not logically equivalent to $(\exists y)(x)Dxy$. This fact can be seen more clearly in terms of a different example. If *Lxy* means "*x* loves *y*" and we imagine the universe of discourse restricted to people, then $(x)(\exists y)Lxy$ means "Everyone loves someone (or other)," while $(\exists y)(x)Lxy$ means "There is someone whom everyone loves." Clearly these two statements are not equivalent.

Relational predicates can be combined with ordinary predicates to translate statements having varying degrees of complexity. In the examples that follow, *Px* means "*x* is a person." The meaning of the other predicates should be clear from the context:

Any heavyweight can defeat any lightweight.
$(x)[Hx \supset (y)(Ly \supset Dxy)]$

Some heavyweights can defeat any lightweight.
$(\exists x)[Hx \bullet (y)(Ly \supset Dxy)]$

No heavyweight can defeat every lightweight.
$(x)[Hx \supset (\exists y)(Ly \bullet {\sim}Dxy)]$
or
${\sim}(\exists x)[Hx \bullet (y)(Ly \supset Dxy)]$

Everyone cares for someone (or other).
$(x)[Px \supset (\exists y)(Py \bullet Cxy)]$

Someone does not care for anyone.
$(\exists x)[Px \bullet (y)(Py \supset {\sim}Cxy)]$

Anyone who cares for someone is cared for himself.
$(x)\{[Px \bullet (\exists y)(Py \bullet Cxy)] \supset (\exists z)(Pz \bullet Czx)\}$

Not everyone respects himself.
$(\exists x)(Px \bullet {\sim}Rxx)$
or
${\sim}(x)(Px \supset Rxx)$

Anyone who does not respect himself is not respected by anyone.
$(x)[(Px \bullet {\sim}Rxx) \supset (y)(Py \supset {\sim}Ryx)]$

The same general rule applies in translating these statements as applies in translating any other statement in predicate logic: Universal quantifiers go with implications and existential quantifiers go with conjunctions. Every one of the eight symbolic expressions given here follows this rule. For example, in the first statement, both quantifiers are universal and both operators are horseshoes. In the second statement, the main quantifier is existential and the subordinate quantifier universal; accordingly, the main operator is a dot and the subordinate operator is a horseshoe. Among these statements, the sixth is the most complex. The symbolic translation of this statement reads, "For all *x*, if *x* is a person and there exists a *y* such that *y* is a person and *x* cares for *y*, then there exists a *z* such that *z* is a person and *z* cares for *x*." It should be clear that this is simply another way of expressing the original English statement.

Another important rule to keep in mind when translating statements of this kind is that every variable must be bound by some quantifier. If a variable is left dangling outside the scope of its intended quantifier, the translation is defective. For example, if the second statement were translated $(\exists x)Hx \bullet (y)(Ly \supset Dxy)$, then the *x* in *Dxy* would not be bound by the existential quantifier. As a result, the translation would be defective. Instead, in our example brackets provide for the existential quantifier to range over *Dxy*.

The same techniques used to translate these eight statements are also used to translate certain statements involving monadic predicates throughout. Consider the following:

If anything is good and all good things are safe, then it is safe.

$(x)\{[Gx \bullet (y)(Gy \supset Sy)] \supset Sx\}$

If anything is good and some good things are dangerous, then it is dangerous.

$(x)\{[Gx \bullet (\exists y)(Gy \bullet Dy)] \supset Dx\}$

Since the "it" at the end of these statements refers to one of the "good" things mentioned at the beginning, the quantifier that binds the *x* in *Gx* must also bind the *x* in *Sx* and *Dx*. The set of braces in the symbolic expressions ensures this.

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Another point to notice regarding statements such as these is that the quantified expression inside the brackets is expressed in terms of a *new* variable. This procedure is essential to avoiding ambiguity. If instead of y, x had been used, the variable in this expression would appear to be bound by two different quantifiers at the same time.

In other statements, the one or more individuals mentioned at the end are *not* necessarily the same ones mentioned at the beginning. In such cases the quantifier that binds the individuals at the beginning should *not* bind those at the end. Compare the next pair of statements with those we have just considered.

If anything is good and all good things are safe, then something is safe.

$[(\exists x)Gx \cdot (y)(Gy \supset Sy)] \supset (\exists z)Sz$

If anything is good and some good things are dangerous, then something is dangerous.

$[(\exists x)Gx \cdot (\exists y)(Gy \cdot Dy)] \supset (\exists z)Dz$

In these cases the "something" at the end is not necessarily one of the "good" things mentioned at the beginning. Accordingly, the quantifier that binds the x in Gx does *not* range all the way to the end of the statement. Furthermore, the quantifier in question is now an *existential* quantifier. In the previous pair of statements the quantifier had to be universal because it ranged over the main operator, which was a horseshoe. In the new pair, however, no quantifier ranges over the implication symbol. As a result, the sense of these statements has shifted to mean "If *something* is good . . ."

Note that, although a different variable is used to express each of the three different components in the pair of statements just given, this is not required. Because in this case no quantifier ranges over any other quantifier, it would be perfectly appropriate to use the same variable throughout.

The next pair of statements involve relational predicates. As in the previous pair, no single quantifier ranges over the entire statement, because the individuals mentioned at the end are not necessarily the same ones mentioned at the beginning:

If everyone helps himself, then everyone will be helped.

$(x)(Px \supset Hxx) \supset (x)[(Px \supset (\exists y)Hyx)]$

If someone helps himself, then someone will be helped.

$(\exists x)(Px \cdot Hxx) \supset (\exists x)(\exists y)(Px \cdot Hyx)$

This completes our explanation of how to translate statements involving relational predicates and overlapping quantifiers. You may, if you wish, proceed directly to Exercise 8.6 Part I before completing the remainder of this section.

Using the Rules of Inference

Let us now see how the various quantifier rules apply to overlapping quantifiers. The change of quantifier rule is applied in basically the same way as it is with single quantifiers. The following short sequence illustrates its application:

1. ~$(x)(\exists y)Pxy$
2. $(\exists x)$~$(\exists y)Pxy$ 1, CQ
3. $(\exists x)(y)$~Pxy 2, CQ

As the tilde operator is moved past a quantifier, the quantifier in question is switched for its correlative. With the exception of a restriction on universal generalization, which we will introduce presently, the instantiation and generalization rules are also used in basically the same way as they are with single quantifiers. Example:

1. $(\exists x)(\exists y)Pxy$
2. $(\exists y)Pay$ 1, EI
3. Pab 2, EI
4. $(\exists x)Pxb$ 3, EG
5. $(\exists y)(\exists x)Pxy$ 4, EG

With each successive instantiation the outermost quantifier drops off. Generalization restores the quantifiers in the reverse order.

This proof demonstrates our earlier observation that the order of the quantifiers is not significant when the same kind of quantifier is used throughout. We also observed that the order does make a difference when different quantifiers appear together. Accordingly, the statement $(x)(\exists y)Pxy$ is not logically equivalent to $(\exists y)(x)Pxy$. As the instantiation and generalization rules now stand, however, it is quite possible, with a proof similar to the one just given, to establish the logical equivalence of these two expressions. Therefore, to keep this from happening we now introduce a new restriction on universal generalization:

UG: $\dfrac{\mathscr{F}y}{(x)\mathscr{F}x}$ *Restriction*: UG must not be used if the instantial variable y is free in any preceding line obtained by EI.

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To see how this restriction applies, let us attempt to derive $(\exists x)(y)Mxy$ from $(y)(\exists x)Mxy$:

1. $(y)(\exists x)Mxy$
2. $(\exists x)Mxy$ 1, UI
3. May 2, EI
4. $(y)May$ 3, UG (invalid)
5. $(\exists x)(y)Mxy$ 4, EG

The proof fails on line 4 because the instantial variable y (that is, y) occurs free in line 3, which was obtained by EI. To see that line 4 is invalid, suppose that the universe of discourse is restricted to people, and that Mxy means "x is the mother of y." Then line 1 asserts that for every person y there exists a person x such that x is the mother of y—which means that every person has a mother. On line 2 we select one of these people at random, and on line 3 we give the name Abigail to the mother of that person. Then on line 4 we draw the conclusion that Abigail is the mother of everyone in the universe. This inference is clearly invalid. The new restriction on UG prevents this kind of inference from being drawn.

In summary, we now have two restrictions on universal generalization. The first concerns only conditional and indirect sequences and prevents UG from occurring within the scope of such a sequence when the instantial variable is free in the first line. The second restriction concerns only arguments involving overlapping quantifiers. With these two restrictions in hand, we may now proceed to examine the use of natural deduction in arguments involving relational predicates and overlapping quantifiers. The example that follows does not include any relational predicates, but it does involve overlapping quantifiers:

1. $(\exists x)Ax \supset (\exists x)Bx$	/ $(\exists y)(x)(Ax \supset By)$
$\quad$ 2. Ax	ACP
$\quad$ 3. $(\exists x)Ax$	2, EG
$\quad$ 4. $(\exists x)Bx$	1, 3, MP
$\quad$ 5. Bc	4, EI
6. $Ax \supset Bc$	2–5, CP
7. $(x)(Ax \supset Bc)$	6, UG
8. $(\exists y)(x)(Ax \supset By)$	7, EG

Conditional and indirect proof are used in the same way with relational predicates and overlapping quantifiers as they are with monadic predicates and nonoverlapping quantifiers. The conditional proof just given begins, as usual, by assuming the antecedent of the conclusion. When line 7 is reached, we must be careful that neither of the restrictions against universal generalization is violated. While the instantial variable x is free in the first line of the conditional sequence, line 7 does not lie within that sequence, so the first restriction is obeyed. And while line 5 was obtained by EI, x is not free in that line. Thus, the second restriction is obeyed as well.

The next proof involves a relational predicate. The proof shows that while $(x)(\exists y)Dxy$ is not logically equivalent to $(\exists y)(x)Dxy$, it can be derived from that statement:

1. $(\exists y)(x)Dxy$	/ $(x)(\exists y)Dxy$
2. $(x)Dxm$	1, EI
3. Dxm	2, UI
4. $(\exists y)Dxy$	3, EG
5. $(x)(\exists y)Dxy$	4, UG

Notice that line 5 is derived by UG. Also notice that the instantial variable x occurs in line 2, which was derived by EI. However, x is not free in line 2. Thus, the second restriction on UG is obeyed.

The next example concludes with a line in which an individual is related to itself. Since there are no restrictions on universal instantiation, the procedure leading up to this line is perfectly legitimate. Notice in line 4 that tautology is used with relational predicates in the same way that it is with monadic predicates:

1. $(\exists y)(x)(Exy \vee Eyx)$	/ $(\exists z)Ezz$
2. $(x)(Exa \vee Eax)$	1, EI
3. $Eaa \vee Eaa$	2, UI
4. Eaa	3, Taut
5. $(\exists z)Ezz$	4, EG

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Sometimes the order in which instantiation steps are performed is critical. The following proof provides an example:

1. $(x)(\exists y)(Fxy \supset Gxy)$
2. $(\exists x)(y)Fxy$ / $(\exists x)(\exists y)Gxy$
3. $(y)Fmy$ 2, EI
4. $(\exists y)(Fmy \supset Gmy)$ 1, UI
5. $Fmo \supset Gmo$ 4, EI
6. Fmo 3, UI
7. Gmo 5, 6, MP
8. $(\exists y)Gmy$ 7, EG
9. $(\exists x)(\exists y)Gxy$ 8, EG

Line 2 must be instantiated before line 1 because the step introduces a new existential name. For the same reason, line 4 must be instantiated before line 3.

The next proof involves an indirect sequence. Such sequences often make use of the change of quantifier rule, as this proof illustrates:

1. $(\exists x)(\exists y)(Jxy \vee Kxy) \supset (\exists x)Lx$
2. $(x)(y)(Lx \supset \sim Ly)$ / $(x)(y)\sim Jxy$
    3. $\sim(x)(y)\sim Jxy$ AIP
    4. $(\exists x)\sim(y)\sim Jxy$ 3, CQ
    5. $(\exists x)(\exists y)\sim\sim Jxy$ 4, CQ
    6. $(\exists x)(\exists y)Jxy$ 5, DN
    7. $(\exists y)Jmy$ 6, EI
    8. Jmn 7, EI
    9. $Jmn \vee Kmn$ 8, Add
    10. $(\exists y)(Jmy \vee Kmy)$ 9, EG
    11. $(\exists x)(\exists y)(Jxy \vee Kxy)$ 10, EG
    12. $(\exists x)Lx$ 1, 11, MP
    13. Lo 12, EI
    14. $(y)(Lo \supset \sim Ly)$ 2, UI
    15. $Lo \supset \sim Lo$ 14, UI
    16. $\sim Lo$ 13, 15, MP
    17. $Lo \cdot \sim Lo$ 13, 16, Conj
18. $\sim\sim(x)(y)\sim Jxy$ 3–17, IP
19. $(x)(y)\sim Jxy$ 18, DN

Because line 1 cannot be instantiated, the only strategy is to derive the antecedent of the conditional with the aim of deriving the consequent via *modus ponens*. This is accomplished on line 11 via indirect proof. Notice on line 9 that addition is used with relational predicates in the same way that it is with monadic predicates.

A final word of caution is called for regarding universal instantiation and the two generalization rules. First, when UI is used to introduce variables into a proof, it is important that these variables end up free and that they not be captured in the process by other quantifiers. The following examples illustrate both correct and incorrect applications of this rule:

1. $(x)(\exists y)Pxy$		
2. $(\exists y)Pyy$	1, UI	(invalid—the instantial variable *y* has been captured by the existential quantifier)

1. $(x)(\exists y)Pxy$		
2. $(\exists y)Pxy$	1, UI	(valid—the instantial variable *x* is free)

1. $(x)(\exists y)Pxy$		
2. $(\exists y)Pzy$	1, UI	(valid—the instantial variable *z* is free)

An analogous caution applies to the two generalization rules. When UG and EG are used, it is important that the instantial letter be replaced by a variable that is captured by no previously introduced quantifier and that no other variables be captured by the newly introduced quantifier. The following examples illustrate both correct and incorrect applications of this rule:

1. $(\exists x)Pxy$		
2. $(x)(\exists x)Pxx$	1, UG	(invalid—the new *x* has been captured by the existential quantifier)

1. $(\exists x)Pxy$		
2. $(\exists x)(\exists x)Pxx$	1, EG	(invalid—the new *x* has been captured by the old existential quantifier)

1. $(\exists x)Pxy$		
2. $(\exists y)(\exists x)Pxy$	1, EG	(valid)

1. $(x)(\exists y)Lxy$		
2. $(\exists y)Lxy$	1, UI	
3. Lxa	2, EI	
4. $(\exists x)Lxx$	3, EG	(invalid—the quantifier has captured the *x* immediately adjacent to the *L*)

1. $(x)(\exists y)Lxy$		
2. $(\exists y)Lxy$	1, UI	
3. Lxa	2, EI	
4. $(\exists z)Lxz$	3, EG	(valid—the *x* remains free)

1. $(x)(y)Kxy$		
2. $(y)Kxy$	1, UI	
3. Kxx	2, UI	
4. $(x)Kxx$	3, UG	(valid)

To see that the fourth example is indeed invalid, let Lxy stand for "*x* is larger than *y*," and let the variables range over the real numbers. The statement $(x)(\exists y)Lxy$ then means that there is no smallest number—which is true. But the statement $(\exists x)Lxx$ means that there is a number that is larger than itself—which is false.

Exercise 8.6

I. Translate the following statements into symbolic form.

★1. Charmaine read *Paradise Lost*. (*Rxy*: *x* read *y*)

2. Whoever reads *Paradise Lost* is educated. (*Rxy*: *x* reads *y*; *Ex*: *x* is educated)

3. James is a friend of either Ellen or Connie. (*Fxy*: *x* is a friend of *y*)

★4. If James has any friends, then Marlene is one of them. (*Fxy*: *x* is a friend of *y*)

5. Dr. Jordan teaches only geniuses. (*Txy*: *x* teaches *y*; *Gx*: *x* is a genius)

6. Dr. Nelson teaches a few morons. (*Txy*: *x* teaches *y*; *Mx*: *x* is a moron)

★7. Every person can sell something or other. (*Px*: *x* is a person; *Sxy*: *x* can sell *y*)

8. Some people cannot sell anything.

9. No person can sell everything.

★10. Some people can sell anything.

11. The Royal Hotel serves only good drinks. (*Sxy*: *x* serves *y*; *Gx*: *x* is good; *Dx*:*x* is a drink)

12. The Clark Corporation advertises everything it produces. (*Axy*: *x* advertises *y*; *Pxy*: *x* produces *y*)

★13. Peterson can drive some of the cars in the lot. (*Dxy*: *x* can drive *y*; *Cx*: *x* is a car; *Lx*: *x* is in the lot)

14. Jones can drive any car in the lot.

15. Sylvia invited only her friends. (*Ixy*: *x* invited *y*; *Fxy*: *x* is a friend of *y*)

★16. Christopher invited some of his friends.

17. Some people break everything they touch. (*Px*: *x* is a person; *Bxy*: *x* breaks *y*; *Txy*: *x* touches *y*)

18. Some people speak to whoever speaks to them. (*Px*: *x* is a person; *Sxy*: *x* speaks to *y*)

★19. Every person admires some people he or she meets. (*Px*: *x* is a person; *Axy*: *x* admires *y*; *Mxy*: *x* meets *y*)

20. Some people admire every person they meet.

21. Some policemen arrest only traffic violators. (*Px*: *x* is a policeman; *Axy*: *x* arrests *y*; *Tx*: *x* is a traffic violator)

★22. Some policemen arrest every traffic violator they see. (*Px*: *x* is a policeman; *Axy*: *x* arrests *y*; *Tx*: *x* is a traffic violator; *Sxy*: *x* sees *y*)

23. If there are cheaters, then some cheaters will be punished. (*Cx*: *x* is a cheater; *Px*: *x* will be punished)

24. If there are any cheaters, then if all the referees are vigilant they will be punished. (*Cx*: *x* is a cheater; *Rx*: *x* is a referee; *Vx*: *x* is vigilant; *Px*: *x* will be punished)

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★**25.** Every lawyer will represent a wealthy client. (*Lx*: *x* is a lawyer; *Rxy*: *x* will represent *y*; *Wx*: *x* is wealthy; *Cx*: *x* is a client)

26. Some lawyers will represent any person who will not represent himself. (*Lx*: *x* is a lawyer; *Px*: *x* is a person; *Rxy*: *x* represents *y*)

27. Some children in the third grade can read any of the books in the library. (*Cx*: *x* is a child; *Tx*: *x* is in the third grade; *Rxy*: *x* can read *y*; *Bx*: *x* is a book; *Lx*: *x* is in the library)

★**28.** All children in the fourth grade can read any of the books in the library.

29. If there are any safe drivers, then if none of the trucks break down they will be hired. (*Sx*: *x* is safe; *Dx*: *x* is a driver; *Tx*: *x* is a truck; *Bx*: *x* breaks down; *Hx*: *x* will be hired)

30. If there are any safe drivers, then some safe drivers will be hired.

II. Derive the conclusion of the following symbolized arguments. Use conditional proof or indirect proof as needed.

★**(1)** 1. $(x)[Ax \supset (y)Bxy]$
2. Am / $(y)Bmy$

(2) 1. $(x)[Ax \supset (y)(By \supset Cxy)]$
2. $Am \bullet Bn$ / Cmn

(3) 1. $(\exists x)[Ax \bullet (y)(By \supset Cxy)]$
2. $(\exists x)Ax \supset Bj$ / $(\exists x)Cxj$

★**(4)** 1. $(x)(\exists y)(Ax \supset By)$ / $(x)Ax \supset (\exists y)By$

(5) 1. $(\exists x)Ax \supset (\exists y)By$ / $(\exists y)(x)(Ax \supset By)$

(6) 1. $(x)(y)(Ax \supset By)$
2. $(\exists x)(y)(Ay \supset Cx)$ / $(x)(\exists y)[Ax \supset (By \bullet Cy)]$

★**(7)** 1. $(\exists x)[Ax \bullet (y)(Ay \supset Bxy)]$ / $(\exists x)Bxx$

(8) 1. $(\exists x)[Ax \bullet (y)(By \supset Cxy)]$
2. $(x)(\exists y)(Ax \supset By)$ / $(\exists x)(\exists y)Cxy$

(9) 1. $(\exists x)(y)(Axy \supset Bxy)$
2. $(x)(\exists y){\sim}Bxy$ / ${\sim}(x)(y)Axy$

★**(10)** 1. $(x)(\exists y)Axy \supset (x)(\exists y)Bxy$
2. $(\exists x)(y){\sim}Bxy$ / $(\exists x)(y){\sim}Axy$

(11) 1. $(\exists x)\{Ax \bullet [(\exists y)By \supset Cx]\}$
2. $(x)(Ax \supset Bx)$ / $(\exists x)Cx$

(12) 1. $(\exists x)(y)[(Ay \bullet By) \supset Cxy]$
2. $(y)(Ay \supset By)$ / $(y)[Ay \supset (\exists x)Cxy]$

★**(13)** 1. $(\exists x)\{Ax \bullet (y)[(By \vee Cy) \supset Dxy]\}$
2. $(\exists x)Ax \supset (\exists y)By$ / $(\exists x)(\exists y)Dxy$

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(14) 1. $(x)\{Ax \supset [(\exists y)(By \bullet Cy) \supset Dx]\}$
2. $(x)(Bx \supset Cx)$ / $(x)[Ax \supset (Bx \supset Dx)]$

(15) 1. $(\exists x)(y)(Ayx \supset \sim Axy)$ / $\sim(x)Axx$

★(16) 1. $(x)(Ax \supset Bx)$
2. $(\exists x)Bx \supset \sim(\exists x)(\exists y)Cxy$ / $(\exists x)Ax \supset \sim Cmn$

(17) 1. $(\exists x)(y)(Axy \supset Byx)$
2. $(x)(\exists y)(Byx \supset \sim Axy)$ / $\sim(x)(y)Axy$

(18) 1. $(x)[Ax \supset (\exists y)(By \bullet Cxy)]$
2. $(\exists x)[Ax \bullet (y)(By \supset Dxy)]$ / $(\exists x)(\exists y)(Cxy \bullet Dxy)$

★(19) 1. $(\exists x)(y)Ayx \vee (x)(y)Bxy$
2. $(\exists x)(y)(Cy \supset \sim Byx)$ / $(x)(\exists y)(Cx \supset Axy)$

(20) 1. $(x)(y)[Axy \supset (Bx \bullet Cy)]$
2. $(x)(y)[(Bx \vee Dy) \supset \sim Axy]$ / $\sim(\exists x)(\exists y)Axy$

III. Translate the following arguments into symbolic form. Then derive the conclusion of each, using conditional proof or indirect proof when needed.

★**1.** Any professional can outplay any amateur. Jones is a professional but he cannot outplay Meyers. Therefore, Meyers is not an amateur. (*Px*: *x* is a professional; *Ax*: *x* is an amateur; *Oxy*: *x* can outplay *y*)

2. Whoever is a friend of either Michael or Paul will receive a gift. If Michael has any friends, then Eileen is one of them. Therefore, if Ann is a friend of Michael, then Eileen will receive a gift. (*Fxy*: *x* is a friend of *y*; *Rx*: *x* will receive a gift)

3. A horse is an animal. Therefore, whoever owns a horse owns an animal. (*Hx*: *x* is a horse; *Ax*: *x* is an animal; *Oxy*: *x* owns *y*)

★**4.** O'Brien is a person. Furthermore, O'Brien is smarter than any person in the class. Since no person is smarter than himself, it follows that O'Brien is not in the class. (*Px*: *x* is a person; *Sxy*: *x* is smarter than *y*; *Cx*: *x* is in the class)

5. If there are any honest politicians, then if all the ballots are counted they will be reelected. Some honest politicians will not be reelected. Therefore, some ballots will not be counted. (*Hx*: *x* is honest; *Px*: *x* is a politician; *Bx*: *x* is a ballot; *Cx*: *x* is counted; *Rx*: *x* will be reelected)

6. Dr. Rogers can cure any person who cannot cure himself. Dr. Rogers is a person. Therefore, Dr. Rogers can cure himself. (*Px*: *x* is a person; *Cxy*: *x* can cure *y*)

★**7.** Some people are friends of every person they know. Every person knows someone (or other). Therefore, at least one person is a friend of someone. (*Px*: *x* is a person; *Fxy*: *x* is a friend of *y*; *Kxy*: *x* knows *y*)

8. If there are any policemen, then if there are any robbers, then they will arrest them. If any robbers are arrested by policemen, they will go to jail. There are some policemen and Macky is a robber. Therefore, Macky will go to jail. (*Px*: *x* is a policeman; *Rx*: *x* is a robber; *Axy*: *x* arrests *y*; *Jx*: *x* will go to jail)

9. If anything is missing, then some person stole it. If anything is damaged, then some person broke it. Something is either missing or damaged. Therefore,

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some person either stole something or broke something. (*Mx*: *x* is missing; *Px*: *x* is a person; *Sxy*: *x* stole *y*; *Dx*: *x* is damaged; *Bxy*: *x* broke *y*)

★**10.** If there are any instructors, then if at least one classroom is available they will be effective. If there are either any textbooks or workbooks, there will be instructors and classrooms. Furthermore, if there are any classrooms, they will be available. Therefore, if there are any textbooks, then some instructors will be effective. (*Ix*: *x* is an instructor; *Cx*: *x* is a classroom; *Ax*: *x* is available; *Ex*: *x* is effective; *Tx*: *x* is a textbook; *Wx*: *x* is a workbook)

8.7 Identity

Many arguments in ordinary language involve a special relation called *identity,* and translating this relation requires special treatment. Consider, for example, the following argument:

> The only friend I have is Elizabeth. Elizabeth is not Nancy. Nancy is a Canadian. Therefore, there is a Canadian who is not my friend.

The peculiar feature of this argument is that it involves special statements about individuals. To translate such statements, we adopt a symbol from arithmetic, the equal sign (=), to represent the identity relation. We can use this symbol to translate a large variety of statements, including simple identity statements, existential assertions about individuals, statements involving "only," "the only," "no . . . except," and "all except," and statements involving superlatives, numerical claims, and definite descriptions. After seeing how the identity relation is used to translate such statements, we will see how natural deduction is used to derive the conclusions of arguments involving identity.

Simple Identity Statements

The simplest statements involving identity are those asserting that a named individual is identical to another named individual. Here are some examples:

Samuel Clemens is Mark Twain.	$s = m$
Whoopi Goldberg is Caryn Johnson.	$w = c$
Dr. Jekyll is Mr. Hyde.	$j = h$

The first statement asserts that Samuel Clemens is identically the same person as Mark Twain, the second that Whoopi Goldberg is the same person as Caryn Johnson, and the third that Dr. Jekyll is the same person as Mr. Hyde. In other words, the statements claim that the names "Samuel Clemens" and "Mark Twain" designate the same person, "Whoopi Goldberg" and "Caryn Johnson" designate the same person, and so on.

To translate a negated identity statement, we simply draw a slash through the identity symbol. Thus, "Beethoven is not Mozart" is translated $b \neq m$. The expression $b \neq m$ is just an abbreviated way of writing $\sim(b = m)$. Here are some additional examples:

William Wordsworth is not John Keats.	$w \neq j$
Natalie Portman is not Rosie O'Donnell.	$n \neq r$
Brian Williams is not Katie Couric.	$b \neq k$

The kinds of statements we will consider next are more complicated, and to facilitate their translation a set of conventions governing conjunctions, disjunctions, and simple identity statements will now be introduced. Many of our translations will involve lengthy strings of conjunctions, such as $Pm \bullet Km \bullet Pn \bullet Kn$. Instead of introducing parentheses and brackets into these expressions, we may simply write them as a string of conjuncts. Lengthy disjunctions may be treated the same way. In simple identity statements, the identity symbol controls only the letters to its immediate left and right. Accordingly, instead of writing $(c = n) \bullet (e = p) \bullet (s = t)$, we may write $c = n \bullet e = p \bullet s = t$, and instead of writing $P \supset (a = m)$, we may write $P \supset a = m$. Let us now use these conventions to translate some special kinds of statements involving identity.

"Only," "The Only," and "No . . . Except"

Section 4.7 explained that the words "only," "the only," and "no . . . except" signal an ordinary categorical proposition when the word that follows is a plural noun or pronoun. For example, the statement "Only relatives are invited" means simply "All invited people are relatives," and "The only animals in this canyon are skunks" means "All animals in this canyon are skunks." However, when the word that follows "only," "the only," or "no . . . except" designates an individual, something more is intended. Thus the statement "Only Nixon resigned the presidency" means (1) that Nixon resigned the presidency and (2) that if anyone resigned the presidency, that person is Nixon. Thus, the general form of such statements is that a designated individual has a stated attribute and anything having that attribute is identical to the designated individual. Here are some examples:

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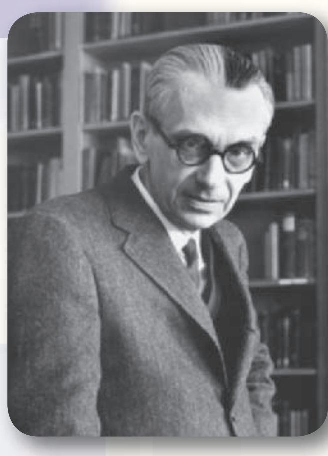
Only Nolan Ryan has struck out 5,000 batters.	$Sn \bullet (x)(Sx \supset x = n)$
The only opera written by Beethoven is *Fidelio*.	$Of \bullet Bf \bullet (x)[(Ox \bullet Bx) \supset x = f]$
No nation except Australia is a continent.	$Na \bullet Ca \bullet (x)[(Nx \bullet Cx) \supset x = a]$
The only presidents who were Whigs were Taylor and Fillmore.	$Pt \bullet Wt \bullet Pf \bullet Wf \bullet (x)[(Px \bullet Wx) \supset (x = t \vee x = f)]$

The first translation may be read as "Nolan Ryan has struck out 5,000 batters, and if anyone has struck out 5,000 batters, then he is identical to Nolan Ryan." The last part of the translation ensures that no other person has struck out 5,000 batters. The second translation may be read as "*Fidelio* is an opera, and *Fidelio* was written by Beethoven, and if anything is an opera written by Beethoven, then it is identical to *Fidelio*." Analogous remarks pertain to the other two statements. The third statement is equivalent to "The only nation that is a continent is Australia."

Eminent Logicians

Kurt Gödel 1906–1978



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Kurt Gödel, generally considered to be the most important logician of the contemporary period, was born in what is today Brno, Czechoslovakia, to a father who managed a textile factory and a mother who was educated and cultured. After excelling at the gymnasium in Brno, Gödel entered the University of Vienna, where he studied mathematics, physics, and philosophy. On completing his undergraduate degree he commenced graduate work in mathematics, earning his doctorate at age twenty-four. Four years later Gödel began teaching at the university as a Privatdozent. However, when the Nazis annexed Austria they abolished his teaching position in favor of one that required a political test, and one year later they found him qualified for military service.

In 1940, under threat of being drafted into the German army, Gödel and his wife Adel (whom he had married two years earlier) left for the United States, where he accepted a position at the famous Institute for Advanced Studies, in Princeton, New Jersey. Soon after arriving, he became best of friends with Albert Einstein, with whom he took daily walks. Gödel became a permanent member of the institute in 1946, and five years later he received the first Albert Einstein Award. In 1974 he was awarded the National Medal of Science.

Gödel is most famous for developing incompleteness theorems that relate to the efforts by logicians to reduce arithmetic to logic. Ordinary arithmetic rests on a set of axioms (called the Peano axioms), and for many years logicians thought (or hoped) that all of the theorems of arithmetic could be reduced to those axioms. Such a system would be complete in that every theorem would be linked to the axioms by a logical proof sequence. Quite to the surprise of these logicians, Gödel showed that every axiom system adequate to support arithmetic contains at least one assertion that is neither provable nor disprovable from the axioms. A second incompleteness theorem showed that the consistency of such a system cannot be proved within the system itself.

As a philosopher Gödel was a Platonic realist and a Leibnizian rationalist. He thought that abstract concepts (such as number and figure) represented objects in an ideal realm that were perfect, changeless, and eternal. As a result he thought that mathematics is a science that describes this ideal realm, and not, as many think, a mere invention of the human mind. Following Leibniz, Gödel conceived the visible world as fundamentally beautiful, perfect, and thoroughly ordered. To complete this perfect world he developed his own ontological argument for the existence of God.

Tragically, from an early age, Gödel was troubled by emotional afflictions, including depression, and as he grew older he suffered from psychotic delusions. In the middle of the winter he would open wide all the windows of his house because he thought that malevolent forces intended to poison him with gas. He also feared they wanted to poison his food, so he ate only his wife's cooking. When Adel became incapacitated from illness, Gödel stopped eating altogether. At age seventy-one he died from malnutrition, at which point his body weighed only 65 pounds.

"All Except"

Statements beginning with "all except" are similar to those beginning with "the only" in that they, too, assert something about a designated individual (or individuals). For example, the statement "All presidents except Washington had a predecessor" means that Washington did not have a predecessor but all other presidents did. Thus, the general form of such statements is that a designated individual lacks a stated attribute and that anything not identical to the designated individual has the stated attribute. Examples:

All painters except Jackson Pollock make sense.	$Pj \cdot \sim Mj \cdot (x)[(Px \cdot x \neq j) \supset Mx]$
All continents except Antarctica are heavily populated.	$Ca \cdot \sim Ha \cdot (x)[(Cx \cdot x \neq a) \supset Hx]$
All states except Alaska and Hawaii are contiguous with their sister states.	$Sa \cdot \sim Ca \cdot Sh \cdot \sim Ch \cdot (x)[(Sx \cdot x \neq a \cdot x \neq h) \supset Cx]$

The first translation may be read as "Jackson Pollock is a painter who does not make sense, and every painter not identical to Jackson Pollock makes sense."

Superlatives

Statements containing superlative adjectives are yet another kind of statement that can be translated by using the identity relation. These are statements asserting that, of all the members of a class, something is the largest, tallest, smallest, heaviest, lightest, and so on. To translate these statements, first give the designated item the class attribute, and then say that, if anything else has that attribute, it is somehow exceeded by the designated item. Here are some examples:

The largest planet is Jupiter.	$Pj \cdot (x)[(Px \cdot x \neq j) \supset Ljx]$
The deepest lake is Ozero Baykal.	$Lo \cdot (x)[(Lx \cdot x \neq o) \supset Dox]$
The highest peak in North America is Mt. McKinley.	$Pm \cdot Nm \cdot (x)[(Px \cdot Nx \cdot x \neq m) \supset Hmx]$

The first translation may be read as "Jupiter is a planet, and if anything is a planet and not identical to Jupiter, then Jupiter is larger than it." The second may be read as "Ozero Baykal is a lake, and if anything is a lake and not identical to Ozero Baykal, then Ozero Baykal is deeper than it."

Numerical Statements

One of the more interesting uses of the identity symbol is to translate certain kinds of numerical statements, such as "There are three people in this room." In particular, the identity symbol allows us to translate such statements without the use of numerals.

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There are three types of numerical statements: those that assert a property of *at most n* items, those that assert a property of *at least n* items, and those that assert a property of *exactly n* items.

The first group does not assert that there actually are any items that have the stated property but only that, if there are any with the stated property, then the maximum number is *n*. Accordingly, for "at most" statements we use universal quantifiers. Here are some examples:

There is at most one god.	$(x)(y)[(Gx \cdot Gy) \supset x = y]$
There is at most one U.S. representative from Alaska.	$(x)(y)[(Ux \cdot Ax \cdot Uy \cdot Ay) \supset x = y]$
There are at most two superpowers.	$(x)(y)(z)[(Sx \cdot Sy \cdot Sz) \supset (x = y \vee x = z \vee y = z)]$
There are at most two cities in Kuwait.	$(x)(y)(z)[(Cx \cdot Kx \cdot Cy \cdot Ky \cdot Cz \cdot Kz) \supset (x = y \vee x = z \vee y = z)]$

It can be seen from these examples that to translate "at most *n*" is to say that, if there are $n + 1$ items that have the stated property, then at least one of them is identical to at least one of the "others." The result is to limit the number of such items to *n*. Thus, to translate "at most one," we need two quantifiers; to translate "at most two," we need three quantifiers; and so on. We could use this procedure to translate statements about any number of items, but because such translations become rather lengthy, this discussion is limited to statements about one or two items at most.

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Unlike "at most" statements, statements that assert something about *at least n* items do claim that the items actually exist. Thus, to translate "at least" statements we need to use existential quantifiers. The number of quantifiers must be equal to the number of items asserted. Examples:

There is at least one city in Monaco.	$(\exists x)(Cx \cdot Mx)$
There are at least two women in *Hamlet*.	$(\exists x)(\exists y)(Wx \cdot Hx \cdot Wy \cdot Hy \cdot x \neq y)$
There are at least three satellites of Neptune.	$(\exists x)(\exists y)(\exists z)(Sx \cdot Sy \cdot Sz \cdot x \neq y \cdot x \neq z \cdot y \neq z)$

The first of these examples merely asserts that some city is in Monaco. Thus, it is translated without any inclusion of the identity relation. When the stated number is greater than one, however, the translation must incorporate one or more negative identity statements to ensure that the items referred to are distinct. Thus, in the second statement, if *x* and *y* should be identical, then there would actually be only one woman (at least) in *Hamlet*. To ensure that there are at least two distinct women, we must conjoin the assertion that *x* and *y* are not identical. Similarly, when we assert something about at least three items, we must conjoin the assertion that none of them is identical to either of the other two.

A statement about *exactly n* items can be seen to be the conjunction of a statement about at least *n* items and a statement about at most *n* items. For example, the statement "There are exactly three cars in the lot" means that there are at least three cars in the lot and at most three cars in the lot. Thus, a statement about exactly *n* items requires *n* existential quantifiers to ensure the existence of the items, one or more negated identity statements to ensure their distinctness (assuming *n* is greater than 1), and a universally quantified statement to limit the group to at most *n* items. Here are some examples:

There is exactly one city in Grenada.	$(\exists x)\{Cx \cdot Gx \cdot (y)[(Cy \cdot Gy) \supset x = y]\}$
There are exactly two houses of Congress.	$(\exists x)(\exists y)\{Hx \cdot Hy \cdot x \neq y \cdot (z)\ [Hz \supset (z = x \vee z = y)]\}$
There are exactly two sopranos in *La Boheme*.	$(\exists x)(\exists y)\{Sx \cdot Lx \cdot Sy \cdot Ly \cdot x \neq y \cdot (z)[(Sz \cdot Lz) \supset (z = x \vee z = y)]\}$

Definite Descriptions

The last form of phraseology considered here is the definite description. A *definite description* is a group of words of the form "the such-and-such" that identifies an individual person, place, or thing. Here are some examples:

The author of *Evangeline*
The capital of Nebraska
The mother of John F. Kennedy

The first designates Henry Wadsworth Longfellow, the second the city of Lincoln, and the third Rose Fitzgerald Kennedy. Definite descriptions are like names in that they identify only one thing, but unlike names they do so by describing a situation or relationship that only that one thing satisfies.

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Statements incorporating definite descriptions have given rise to disputes in logic, because alternate interpretations of such statements can lead to conflicts in truth value. Suppose, for example, we are given the statement "The queen of the United States is a woman." Should we consider this statement to be true, because every queen is a woman, or should we consider it to be false, because there is no queen of the United States? In response to this question, most logicians today accept an interpretation of definite descriptions originally proposed by Bertrand Russell. According to this interpretation, a statement that incorporates a definite description asserts three things: an item of a certain sort exists, there is only one such item, and that item has the attribute assigned to it by the statement. If we accept this interpretation, the statement about the queen of the United States is false, because no such person exists.

Here are some additional examples with their translations:

The inventor of the phonograph was an American.	$(\exists x)[Ixp \cdot (y)(Iyp \supset y = x) \cdot Ax]$
The author of *Middlemarch* was a Victorian freethinker.	$(\exists x)[Wxm \cdot (y)(Wym \supset y = x) \cdot Vx \cdot Fx]$
The painter of *The Starry Night* was Van Gogh.	$(\exists x)[Pxs \cdot (y)(Pys \supset y = x) \cdot x = v]$

The first translation may be read as "There is someone who invented the phonograph, and if anyone invented the phonograph, then that person is identical to the first, and the first person is an American." The second may be read as "There is someone who wrote *Middlemarch,* and if anyone wrote *Middlemarch,* then that person is identical to the first, and the first person is a Victorian freethinker." The third may be read as "There is someone who painted *The Starry Night,* and if anyone painted *The Starry Night,* then that person is identical to the first, and the first person is identical to Van Gogh."

This completes our explanation of how to translate statements involving the identity relation. At this point, you may, if you wish, proceed to Exercise 8.7 Part I before completing the remainder of this section.

Using the Rules of Inference

Now that we have seen how to translate statements involving the identity relation, let us use natural deduction to derive the conclusions of arguments that include statements of this sort. Before doing so, however, some special rules governing the identity relation must be introduced. These rules, which are collectively designated "Id," are as follows:

Id: (1) $\dfrac{\text{Prem.}}{\mathcal{a} = \mathcal{a}}$ (2) $\mathcal{a} = \mathcal{b} :: \mathcal{b} = \mathcal{a}$ (3) $\dfrac{\begin{array}{c}\mathcal{F}\mathcal{a}\\ \mathcal{a} = \mathcal{b}\end{array}}{\mathcal{F}\mathcal{b}}$

($\mathcal{a}$, $\mathcal{b}$ are any individual constants)

The first rule expresses the idea that anything is identical to itself; it asserts what is called the reflexive property of the identity relation. The rule allows us to insert a self-identity statement after any premise (that is, on any line in a proof).

The second rule is a rule of replacement; it expresses what is called the symmetric property of the identity relation. It states, very simply, that the letters on either side of the equal sign can be switched. An immediate use of this rule is to prove that $a \neq b$ is logically equivalent to $b \neq a$. Recall that $a \neq b$ is simply an abbreviation for $\sim(a = b)$. If we apply the rule to the latter expression, we obtain $\sim(b = a)$, which, in its abbreviated form, is $b \neq a$.

The third rule expresses the intuitively obvious idea that, if something is true of *x* and *x* is identical to *y*, then that something is true of *y*. This rule is the basis of what is called the transitive property of identity, which allows us to infer from $a = b$ and $b = c$ that $a = c$. If we suppose that the $\mathcal{F}$ in this rule stands for the expression "$a =$", that $\mathcal{a}$ is b, and that $\mathcal{b}$ is c, then the first line of the rule reads $a = b$, the second line reads $b = c$, and the conclusion is $a = c$. This inference is used often in the derivation of the conclusions of arguments.

In general, the rules of inference used earlier apply to arguments containing identity statements in the same way they apply to any other arguments. Also, conditional proof and indirect proof are used in the same way. We need only note that because $\mathcal{a}$ and $\mathcal{b}$ in these rules represent only individual constants ($a, b, \ldots v, w$) they cannot be applied to variables (x, y, z).

The following argument illustrates the first expression of the rule for identity.

No biologists are identical to Isabel. Therefore, Isabel is not a biologist.

If we use *Bx* to translate "*x* is a biologist," and *i* for Isabel, this argument becomes

1. $(x)(Bx \supset x \neq i)$ / $\sim Bi$

The fact that the conclusion contains *i* suggests that we instantiate line 1 with respect to that letter. The proof is as follows:

1.	$(x)(Bx \supset x \neq i)$	/ $\sim Bi$
2.	$Bi \supset i \neq i$	1, UI
3.	$i = i$	Id
4.	$\sim(i \neq i)$	3, DN
5.	$\sim Bi$	2, 4, MT

Line 3 comes merely from the first expression of the identity rule, which allows us to insert any self-identity statement after any premise. Thus, no numeral is included in the justification for that line. Also note that line 4 is simply another way of writing $\sim\sim(i = i)$.

Now let us return to the argument given at the beginning of this section:

The only friend I have is Elizabeth. Elizabeth is not Nancy. Nancy is a Canadian. Therefore, there is a Canadian who is not my friend.

If we use *Fx* to translate "*x* is my friend" and *Cx* to translate "*x* is a Canadian," this argument may be translated as follows:

1.	$Fe \cdot (x)(Fx \supset x = e)$	
2.	$e \neq n$	
3.	Cn	/ $(\exists x)(Cx \cdot \sim Fx)$

Inspecting the second line, we see a negated identity statement involving *e* and *n*. This suggests that we instantiate the universal statement in the first line with respect to *n*. The proof is as follows:

1.	$Fe \cdot (x)(Fx \supset x = e)$	
2.	$e \neq n$	
3.	Cn	/ $(\exists x)(Cx \cdot \sim Fx)$
4.	$(x)(Fx \supset x = e) \cdot Fe$	1, Com
5.	$(x)(Fx \supset x = e)$	4, Simp
6.	$Fn \supset n = e$	5, UI
7.	$n \neq e$	2, Id
8.	$\sim Fn$	6, 7, MT
9.	$Cn \cdot \sim Fn$	3, 8, Conj
10.	$(\exists x)(Cx \cdot \sim Fx)$	9, EG

Line 7 is justified by the second rule for identity. Also, since $n \neq e$ is simply an abbreviation for $\sim(n = e)$, line 8 follows directly from lines 6 and 7.

Here is another example:

> The only person who invested is Ms. Snyder. Cathy is one of the people who lost money. Some people who invested did not lose money. Therefore, Cathy is not Ms. Snyder.

The translation is as follows:

1. $Ps \cdot Is \cdot (x)[(Px \cdot Ix) \supset x = s]$
2. $Pc \cdot Lc$
3. $(\exists x)(Px \cdot Ix \cdot {\sim}Lx)$ $\quad$ / $c \neq s$

Cursory inspection reveals no easy way to obtain the conclusion. This suggests indirect proof:

1.	$Ps \cdot Is \cdot (x)[(Px \cdot Ix) \supset x = s]$	
2.	$Pc \cdot Lc$	
3.	$(\exists x)(Px \cdot Ix \cdot {\sim}Lx)$	/ $c \neq s$
4.	$c = s$	AIP
5.	$Pa \cdot Ia \cdot {\sim}La$	3, EI
6.	$(x)[(Px \cdot Ix) \supset x = s] \cdot Ps \cdot Is$	1, Com
7.	$(x)[(Px \cdot Ix) \supset x = s]$	6, Simp
8.	$Pa \cdot Ia \supset a = s$	7, UI
9.	$Pa \cdot Ia$	5, Simp
10.	$a = s$	8, 9, MP
11.	$s = c$	4, Id
12.	$a = c$	10, 11, Id
13.	${\sim}La \cdot Pa \cdot Ia$	5, Com
14.	${\sim}La$	13, Simp
15.	${\sim}Lc$	12, 14, Id
16.	$Lc \cdot Pc$	2, Com
17.	Lc	16, Simp
18.	$Lc \cdot {\sim}Lc$	15, 17, Conj
19.	$c \neq s$	4–18, IP

As usual, the existential statement is instantiated first, then the universal. Line 11 is derived by commuting line 4 by the second rule of identity, and line 12 is derived from lines 10 and 11 by applying the third rule of identity. Line 15 is derived by substituting *c* in the place of *a* in line 14 according to the third rule of identity. The indirect sequence is discharged in line 19 in the normal way.

In arguments involving identity, especially more complicated ones, it is often difficult or impossible to see by mere inspection how to obtain the conclusion. A good general procedure is to begin with instantiation. Always instantiate the existential statements first, then the universals. When instantiating the universal statements, normally pick the letter (or one of the letters) used to instantiate the existential statement(s). If there are no existential statements, pick one of the letters appearing in the singular statements. If the conclusion is still not apparent, try indirect proof. In general, whenever the conclusion is a complicated statement, it is best to start out with indirect proof. Developing facility in proving arguments involving identity requires a little practice, but adequate skill should not take too long to acquire.

Translation hints

Only i is F.	$Fi \cdot (x)[Fx \supset x = i]$
The only F that is G is i.	$Fi \cdot Gi \cdot (x)[(Fx \cdot Gx) \supset x = i]$
No F except i is G.	$Fi \cdot Gi \cdot (x)[(Fx \cdot Gx) \supset x = i]$
All F except i are G.	$Fi \cdot {\sim}Gi \cdot (x)[(Fx \cdot x \neq i) \supset Gx]$
i is the F that is most so-and-so.	$Fi \cdot (x)[(Fx \cdot x \neq i) \supset i$ is more so-and-so than $x]$
There is at most one F.	$(x)(y)[(Fx \cdot Fy) \supset x = y]$
There are at least two F's.	$(\exists x)(\exists y)[Fx \cdot Fy \cdot x \neq y]$
There are exactly two F's.	$(\exists x)(\exists y)\{Fx \cdot Fy \cdot x \neq y \cdot (z)[Fz \supset (z = x \vee z = y)]\}$
The F is G.	$(\exists x)[Fx \cdot (y)(Fy \supset y = x) \cdot Gx]$

Exercise 8.7

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I. Translate the following statements.

Simple identity statements

★1. Dr. Seuss is Theodore Geisel. (s, g)

2. Auguste Renoir is not Claude Monet. (r, m)

3. Marilyn Monroe is Norma Jean Baker. (m, b)

★4. Hermann Hesse is not André Gide. (h, g)

Statements involving "only," "the only," and "no . . . except"

★5. Only Linus Pauling has won two Nobel prizes. (Wx: x has won two Nobel prizes; p: Linus Pauling)

6. Only Don Larsen has pitched a perfect World Series game. (Px: x has pitched a perfect World Series game; l: Don Larsen)

7. The only national park in Maine is Acadia. (Nx: x is a national park; Mx: x is in Maine; a: Acadia)

★8. The only nation having a maple leaf flag is Canada. (Nx: x is a nation; Mx: x has a maple leaf flag; c: Canada)

9. The only U.S. presidents who were Federalists were Washington and Adams. (Ux: x is a U.S. president; Fx: x is a Federalist; w: Washington; a: Adams)

10. No state except Hawaii is surrounded by water. (Sx: x is a state; Wx: x is surrounded by water; h: Hawaii)

★11. No sport except hockey uses a puck. (Sx: x is a sport; Px: x uses a puck; h: hockey)

Superlative statements

★**12.** Hydrogen is the lightest element. (*Ex*: *x* is an element; *Lxy*: *x* is lighter than *y*; *h*: hydrogen)

13. The smallest planet in our solar system is Mercury. (*Px*: *x* is a planet in our solar system; *Sxy*: *x* is smaller than *y*; *m*: Mercury)

14. Harvard is the oldest American university. (*Ax*: *x* is American; *Ux*: *x* is a university; *Oxy*: *x* is older than *y*; *h*: Harvard)

★**15.** Death Valley is the lowest region in North America. (*Rx*: *x* is a region; *Nx*: *x* is in North America; *Lyx*: *x* is lower than *y*; *d*: Death Valley)

Statements involving "all except"

★**16.** Every city except Istanbul is situated on a single continent. (*Cx*: *x* is a city; *Sx*: *x* is situated on a single continent; *i*: Istanbul)

17. Every U.S. president except Ford won a national election. (*Ux*: *x* is a U.S. president; *Wx*: *x* won a national election; *f*: Ford)

18. All metals except mercury are solids at room temperature. (*Mx*: *x* is a metal; *Sx*: *x* is a solid at room temperature; *m*: mercury)

★**19.** Every pitcher except Cy Young has won fewer than 500 games. (*Px*: *x* is a pitcher; *Wx*: *x* has won fewer than 500 games; *c*: Cy Young)

Numerical statements

★**20.** There is at most one city in Belize. (*Cx*: *x* is a city; *Bx*: *x* is in Belize)

21. There are at most two national parks in South Dakota. (*Nx*: *x* is a national park; *Sx*: *x* is in South Dakota)

22. There is at most one national holiday in July. (*Nx*: *x* is a national holiday; *Jx*: *x* is in July)

★**23.** There are at most two cities in Malta. (*Cx*: *x* is a city; *Mx*: *x* is in Malta)

24. There is at least one quarterback on a football team. (*Qx*: *x* is a quarterback; *Fx*: *x* is on a football team)

25. There are at least two atoms in a water molecule. (*Ax*: *x* is an atom; *Wx*: *x* is in a water molecule)

★**26.** There are at least three carbon allotropes. (*Cx*: *x* is a carbon allotrope)

27. There is exactly one U.S. Supreme Court. (*Ux*: *x* is a U.S. Supreme Court)

28. There is exactly one natural satellite of the earth. (*Sx*: *x* is a satellite of the earth; *Nx*: *x* is natural)

★**29.** There are exactly two bright stars in Gemini. (*Sx*: *x* is a star; *Bx*: *x* is bright; *Gx*: *x* is in Gemini)

Statements containing definite descriptions

★**30.** The author of *Vanity Fair* was born in India. (*Wxy*: *x* wrote *y*; *Bx*: *x* was born in India; *v*: *Vanity Fair*)

31. The wife of Othello is Desdemona. (*Wxy*: *x* is the wife of *y*; *o*: Othello; *d*: Desdemona)

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32. The man who composed *The Nutcracker* was Russian. (*Mx: x* is a man; *Cxy: x* composed *y*; *Rx: x* was Russian; *n: The Nutcracker*)

★33. The artist who painted the *Allegory of Spring* was Botticelli. (*Ax: x* is an artist; *Pxy: x* painted *y*; *a*: the *Allegory of Spring*; *b*: Botticelli)

34. The capital of Georgia is not Savannah. (*Cxy: x* is the capital of *y*, *g*: Georgia; *s*: Savannah)

Assorted statements

★35. The smallest state is Rhode Island. (*Sx: x* is a state; *Sxy: x* is smaller than *y*; *r*: Rhode Island)

36. There is at least one newspaper in St. Louis. (*Nx: x* is a newspaper; *Sx: x* is in St. Louis)

37. Cat Stevens is Yusuf Islam. (*s*: Cat Stevens; *i*: Yusuf Islam)

★38. The only American president elected to a fourth term was Franklin D. Roosevelt. (*Ax: x* is an American president; *Ex: x* was elected to a fourth term; *r*: Franklin D. Roosevelt)

39. There are at least two cities in Qatar. (*Cx: x* is a city; *Qx: x* is in Qatar)

40. Only George Blanda has played 340 professional football games. (*Px*: *x* has played 340 professional football games; *b*: George Blanda)

★41. Hamlet had at most one sister. (*Sxy: x* is a sister of *y*; *h*: Hamlet)

42. No major league baseball player has hit 73 home runs except Barry Bonds. (*Mx*: *x* is a major league baseball player; *Hx*: *x* has hit 73 home runs; *b*: Barry Bonds)

43. There are at most two senators from New Hampshire. (*Sx*: *x* is a senator; *Nx*: *x* is from New Hampshire)

★44. Gustav Mahler is not Anton Bruckner. (*m*: Gustav Mahler; *b*: Anton Bruckner)

45. The explorer who discovered the North Pole was Admiral Peary. (*Ex*: *x* is an explorer; *Dxy: x* discovered *y*; *n*: the North Pole; *a*: Admiral Peary)

46. Hinduism is the oldest religion. (*Rx*: *x* is a religion; *Oxy*: *x* is older than *y*; *h*: Hinduism)

★47. There are exactly two tenors in *Carmen*. (*Tx*: *x* is a tenor; *Cx*: *x* is in *Carmen*)

48. Every Speaker of the House except Nancy Pelosi has been a man. (*Sx*: *x* is Speaker of the House; *Mx: x* is a man; *n*: Nancy Pelosi)

49. The person who discovered relativity theory was an employee in the Swiss patent office. (*Px: x* is a person; *Dxy: x* discovered *y*; *Ex: x* is an employee in the Swiss patent office; *r*: relativity theory)

★50. There are at least three stars in Orion. (*Sx: x* is a star; *Ox: x* is in Orion)

II. Derive the conclusion of the following symbolized arguments. Use conditional proof or indirect proof as needed.

★(1) 1. $(x)(x = a)$

2. $(\exists x)Rx$ / Ra

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(2) 1. Ke

2. $\sim Kn$ / $e \neq n$

(3) 1. $(x)(x = c \supset Nx)$ / Nc

★(4) 1. $(\exists x)(x = g)$

2. $(x)(x = i)$ / $g = i$

(5) 1. $(x)(Gx \supset x = a)$

2. $(\exists x)(Gx \bullet Hx)$ / Ha

(6) 1. $(x)(Ax \supset Bx)$

2. $Ac \bullet \sim Bi$ / $c \neq i$

★(7) 1. $(x)(x = a)$

2. Fa / $Fm \bullet Fn$

(8) 1. $(x)(x = r)$

2. $Hr \bullet Kn$ / $Hn \bullet Kr$

(9) 1. $(x)(Lx \supset x = e)$

2. $(x)(Sx \supset x = i)$

3. $(\exists x)(Lx \bullet Sx)$ / $i = e$

★(10) 1. $(x)(Px \supset x = a)$

2. $(x)(x = c \supset Qx)$

3. $a = c$ / $(x)(Px \supset Qx)$

(11) 1. $(x)(y)(Txy \supset x = e)$

2. $(\exists x)Txi$ / Tei

(12) 1. $(x)[Rx \supset (Hx \bullet x = m)]$ / $Rc \supset Hm$

★(13) 1. $(x)(Ba \supset x \neq a)$

2. Bc / $a \neq c$

(14) 1. $(\exists x)Gx \supset (\exists x)(Kx \bullet x = i)$ / $Gn \supset Ki$

(15) 1. $(x)(Rax \supset \sim Rxc)$

2. $(x)Rxx$ / $c \neq a$

★(16) 1. $(x)[Nx \supset (Px \bullet x = m)]$

2. $\sim Pm$ / $\sim Ne$

(17) 1. $(x)(Fx \supset x = e)$

2. $(\exists x)(Fx \bullet x = a)$ / $a = e$

(18) 1. $(x)[Ex \supset (Hp \bullet x = e)]$

2. $(\exists x)(Ex \bullet x = p)$ / He

★(19) 1. $(x)(\exists y)(Cxy \supset x = y)$

2. $(\exists x)(y)(Cxy \bullet x = a)$ / Caa

(20) 1. $(x)[Fx \supset (Gx \bullet x = n)]$

2. $Gn \supset (\exists x)(Hx \bullet x = e)$ / $Fm \supset He$

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III. Derive the conclusion of the following arguments. Use conditional proof or indirect proof as needed.

★**1.** Some of Jane Collier's novels are interesting. The only novel Jane Collier wrote is *The Cry*. Therefore, *The Cry* is interesting. (*Nx*: *x* is a novel; *Wxy*: *x* wrote *y*; *Ix*: *x* is interesting; *j*: Jane Collier; *c*: *The Cry*)

2. Ronald Reagan was the oldest U.S. president. Woodrow Wilson was a U.S. president. Woodrow Wilson is not Ronald Reagan. Therefore, Ronald Reagan was older than Woodrow Wilson. (*Ux*: *x* is a U.S. president; *Oxy*: *x* is older than *y*; *r*: Ronald Reagan; *w*: Woodrow Wilson)

3. The artist who painted the *Mona Lisa* was a Florentine. Leonardo is the artist who painted the *Mona Lisa*. Therefore, Leonardo was a Florentine. (*Ax*: *x* is an artist; *Pxy*: *x* painted *y*; *Fx*: *x* was a Florentine; *m*: the *Mona Lisa*; *l*: Leonardo)

★**4.** The novel on the table was written by Margaret Mitchell. The only novel Margaret Mitchell wrote is *Gone with the Wind*. Therefore, the novel on the table is *Gone with the Wind*. (*Nx*: *x* is a novel; *Tx*: *x* is on the table; *Wxy*: *x* wrote *y*; *m*: Margaret Mitchell; *g*: *Gone with the Wind*)

5. The author of *King Lear* was an English actor. John Milton was English but not an actor. Therefore, John Milton is not the author of *King Lear*. (*Wxy*:*x* wrote *y*; *Ex*: *x* is English; *Ax*: *x* is an actor; *k*: *King Lear*; *m*: John Milton)

6. The dog that bit the letter carrier is a large terrier. Ajax is a small dog. Therefore, Ajax did not bite the letter carrier. (*Dx*: *x* is a dog; *Bx*: *x* bit the letter carrier; *Lx*: *x* is large; *Tx*: *x* is a terrier; *a*: Ajax)

★**7.** Every member except Ellen sang a song. Every member except Nancy gave a speech. Ellen is not Nancy. Therefore, Ellen gave a speech and Nancy sang a song. (*Mx*: *x* is a member; *Sx*: *x* sang a song; *Gx*: *x* gave a speech; *e*: Ellen; *n*: Nancy)

8. The only person who ordered fish is Astrid. The only person who suffered indigestion is Ms. Wilson. Some person who ordered fish also suffered indigestion. Therefore, Astrid is Ms. Wilson. (*Px*: *x* is a person; *Ox*: *x* ordered fish; *Sx*: *x* suffered indigestion; *a*: Astrid; *w*: Ms. Wilson)

9. The highest mountain is in Tibet. Therefore, there is a mountain in Tibet that is higher than any mountain not in Tibet. (*Mx*: *x* is a mountain; *Hxy*: *x* is higher than *y*; *Tx*: *x* is in Tibet)

★**10.** The tallest building in North America is the Willis Tower. The tallest building in North America is located in Chicago. If one thing is taller than another, then the latter is not taller than the former. Therefore, the Willis Tower is located in Chicago. (*Bx*: *x* is a building in North America; *Txy*: *x* is taller than *y*; *Cx*: *x* is located in Chicago; *w*: the Willis Tower)

11. There are at least two philosophers in the library. Robert is the only French philosopher in the library. Therefore, there is a philosopher in the library who is not French. (*Px*: *x* is a philosopher; *Lx*: *x* is in the library; *Fx*: *x* is French; *r*: Robert)

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12. The only dogs that barked were Fido and Pluto. Fido is not Pluto. Every dog except Fido ran on the beach. Therefore, exactly one barking dog ran on the beach. (*Dx*: *x* is a dog; *Bx*: *x* barked; *Rx*: *x* ran on the beach; *f*: Fido; *p*: Pluto)

★13. There are at least two attorneys in the office. All attorneys are professionals. There are at most two professionals in the office. Therefore, there are exactly two professionals in the office. (*Ax*: *x* is an attorney; *Ox*: *x* is in the office; *Px*: *x* is a professional)

14. There are at most two scientists in the laboratory. At least two scientists in the laboratory are Russians. No Russians are Chinese. Therefore, if Norene is a Chinese scientist, then she is not in the laboratory. (*Sx*: *x* is a scientist; *Lx*: *x* is in the laboratory; *Rx*: *x* is Russian; *Cx*: *x* is Chinese; *n*: Norene)

15. Every candidate except Mary was elected. The only candidate who was elected is Ralph. Mary is not Ralph. Therefore, there were exactly two candidates. (*Cx*: *x* is a candidate; *Ex*: *x* was elected; *m*: Mary; *r*: Ralph)

★16. Every student except Charles and Norman passed the course. The only student who was dismissed was Norman. Every student retook the course if and only if he or she was not dismissed and did not pass. Charles is not Norman. Therefore, exactly one student retook the course. (*Sx*: *x* is a student; *Px*: *x* passed the course; *Dx*: *x* was dismissed; *Rx*: *x* retook the course; *c*: Charles; *n*: Norman)

Summary

8

Predicate Logic: Combines the use of these symbols:

- The five operators of propositional logic: ~, •, ∨, ⊃, ≡
- Symbols for predicates: *G*_, *H*_, *K*_, etc.
- Symbols for universal quantifiers: (*x*), (*y*), (*z*)
- Symbols for existential quantifiers: (∃*x*), (∃*y*), (∃*z*)
- Symbols for individual variables: *x*, *y*, *z*
- Symbols for individual constants: *a*, *b*, *c*, . . . *u*, *v*, *w*

Statements:

- Singular statements combine predicate symbols with constants: *Ga*, *Hc* • *Kc*, etc.
- Universal statements are usually translated as conditionals: (*x*)(*Px* ⊃ *Qx*), etc.
- Particular statements are usually translated as conjunctions: (∃*x*)(*Px* • *Qx*), etc.

Using the Rules of Inference (*modus ponens*, *modus tollens*, etc.):

- Rules are used in basically the same way as in propositional logic.
- Using the rules usually requires that quantifiers be removed or inserted:
    - Universal instantiation (UI): Removes universal quantifiers.

    - Universal generalization (UG): Introduces universal quantifiers.
    - Existential instantiation (EI): Removes existential quantifiers.
    - Existential generalization (EG): Introduces existential quantifiers.
- Restrictions:
    - For EI, the existential name that is introduced must not appear in any previous line, including the conclusion line.
    - The instantiation and generalization rules can be applied only to whole lines—not parts of lines.

Change of Quantifier Rule:

- Is used to remove or insert tildes preceding quantifiers.
- The instantiation rules cannot be applied if a tilde precedes the quantifier.
- One type of quantifier can be replaced by the other type if and only if immediately before and after the new quantifier:
    - Tildes that were originally there are deleted, and
    - Tildes that were not originally there are inserted.

Conditional Proof and Indirect Proof:

- Used in basically the same way as in propositional logic.
- Restriction:
    - UG must not be used within an indented sequence if the instantial variable is free in the first line of that sequence.

Proving Invalidity:

- Counterexample Method:
    - Produce a substitution instance of a symbolized argument that has indisputably true premises and an indisputably false conclusion.
- Finite Universe Method:
    - Reduce the universe of discourse to a finite number until an indirect truth table proves the resulting argument invalid.
    - Universal statements are rendered as conjunctions.
    - Particular statements are rendered as disjunctions.
    - Singular statements are kept as they are.

Relational Predicates:

- Symbols for relational predicates: *G*_ _, *H*_ _, *K*_ _ _, etc.
- Are used to translate relational statements.
- Example: "Paul is taller than Cathy": *Tpc*

Overlapping Quantifiers:

- Quantifiers that fall within the scope of another quantifier.
- Example: $(\exists x)[Mx \cdot (y)(Wy \supset Txy)]$

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- Restriction:
    - UG must not be used if the instantial variable is free in any preceding line obtained by EI.

Identity:

- The symbol for the identity relation is the equal sign: =
- Used to assert that one thing is identical to another.
- Used to translate statements involving "only," "the only," "no … except," and "all except" when these expressions are used with individuals.
- Used to translate superlative statements, numerical statements, and definite descriptions.
- The Rule for Identity (Id) allows the application of the other rules of inference.

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