7 Natural Deduction in Propositional Logic

7.1 Rules of Implication I

Natural deduction is a method for establishing the validity of propositional type arguments that is both simpler and more enlightening than the method of truth tables. By means of this method, the conclusion of an argument is actually derived from the premises through a series of discrete steps. In this respect natural deduction resembles the method used in geometry to derive theorems relating to lines and figures; but whereas each step in a geometrical proof depends on some mathematical principle, each step in a logical proof depends on a **rule of inference.** This chapter will present eighteen rules of inference.

The first eight rules of inference are called **rules of implication** because they consist of basic argument forms whose premises *imply* their conclusions. The following four rules of implication should be familiar from the previous chapter:

1. *Modus ponens* (MP):

$$\begin{array}{l} p \supset q \\ \underline{p \quad\quad} \\ q \end{array}$$

2. *Modus tollens* (MT):

$$\begin{array}{l} p \supset q \\ \underline{\sim q \quad\quad} \\ \sim p \end{array}$$

Additional resources are available on the Logic CourseMate website.

3. Pure hypothetical syllogism (HS):

$$\begin{array}{l} p \supset q \\ \underline{q \supset r} \\ p \supset r \end{array}$$

4. Disjunctive syllogism (DS):

$$\begin{array}{l} p \vee q \\ \underline{\sim p \quad} \\ q \end{array}$$

In constructing proofs, *modus ponens* allows us to assert the consequent of a conditional statement on a line by itself, and *modus tollens* allows us to assert the negation of the antecedent. Pure hypothetical syllogism ("hypothetical syllogism" for short) is used to derive a conditional statement from two other conditionals, and disjunctive syllogism allows us to assert the right-hand disjunct of a disjunctive statement on a line by itself.

These four rules will be sufficient to derive the conclusion of many simple arguments in propositional logic. Further, once we are supplied with all eighteen rules together with conditional proof, the resulting system will be sufficient to derive the conclusion of any valid argument in propositional logic. Conversely, since each rule is a valid argument form unto itself, any conclusion derived from their correct use results in a valid argument. The method of natural deduction is thus equal in power to the truth-table method as far as proving validity is concerned. However, since natural deduction cannot be used with any facility to prove invalidity, we still need the truth-table method for that purpose.

Applying the rules of inference rests on the ability to visualize more or less complex arrangements of simple propositions as substitution instances of the rules. For a fairly simple substitution instance of *modus ponens*, consider the following:

1. $\sim A \supset B$	$p \supset q$
2. $\sim A$	$\underline{p \quad}$
3. B	q

When $\sim A$ and B are mentally substituted, respectively, in place of the p and q of the *modus ponens* rule, then you should be able to see that the argument on the left is an instance of the rule. The fact that A is preceded by a tilde is irrelevant.

Here is a more complex example:

1. $(A \bullet B) \supset (C \vee D)$	$p \supset q$
2. $A \bullet B$	$\underline{p \quad}$
3. $C \vee D$	q

In this case, if you mentally substitute $A \bullet B$ and $C \vee D$, respectively, in place of p and q in the rule, you can see that the argument on the left is an instance of *modus ponens*. This example illustrates the fact that any pair of compound statements can be uniformly substituted in place of p and q to produce a substitution instance of the rule.

Finally, the order of the premises never makes a difference:

1. A	p
2. $A \supset (B \supset C)$	$\underline{p \supset q}$
3. $B \supset C$	q

In this case, if you mentally substitute A and $B \supset C$ in place of p and q, you can see, once again, that the argument on the left is an instance of *modus ponens*. The fact that the order of the premises is reversed makes no difference.

These arguments are all instances of ***modus ponens*** (MP)

$\sim F \supset (G \equiv H)$	$(A \vee B) \supset \sim(C \bullet D)$	$K \bullet L$
$\sim F$	$A \vee B$	$(K \bullet L) \supset [(R \supset S) \bullet (T \supset U)]$
$G \equiv H$	$\sim(C \bullet D)$	$(R \supset S) \bullet (T \supset U)$

Now let us use the rules of inference to construct a proof. Such a proof consists of a sequence of propositions, each of which is either a premise or is derived from preceding propositions by application of a rule of inference and the last of which is the conclusion of the original argument. Let us begin with the following example:

> If the Astros make the playoffs, then the Braves will not win the pennant.
> If the Cubs retain their manager, then the Braves will win the pennant.
> The Astros will make the playoffs. Therefore, the Cubs will not retain their manager.

The first step is to symbolize the argument, numbering the premises and writing the intended conclusion to the right of the last premise, separated by a slash mark:

1. $A \supset \sim B$
2. $C \supset B$
3. A / $\sim C$

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The next step is to derive the conclusion through a series of inferences. For this step, always begin by trying to "find" the conclusion in the premises. The conclusion to be derived is ~*C*, and we see that *C* appears in the antecedent of line 2. We could derive ~*C* from line 2 by *modus tollens* if we had ~*B*, so now we look for ~*B*. Turning our attention to line 1, we see that we could derive ~*B* by *modus ponens* if we had *A*, and we do have *A* on line 3. Thus, we have now thought through the entire proof, and we can begin to write it out. First, we derive ~*B* by *modus ponens* from lines 1 and 3:

1. $A \supset \sim B$
2. $C \supset B$
3. A / $\sim C$
4. $\sim B$ 1, 3, MP

The justification for line 4 is written to the right, directly beneath the slash mark. If you have trouble understanding how line 4 was derived, imagine substituting *A* and ~*B* in place of *p* and *q* in the *modus ponens* rule. Then you can see that lines 1, 3, and 4 constitute a substitution instance of that rule.

The final step is to derive ~*C* from lines 2 and 4 by *modus tollens*:

1. $A \supset \sim B$
2. $C \supset B$
3. A / $\sim C$
4. $\sim B$ 1, 3, MP
5. $\sim C$ 2, 4, MT

The proof is now complete. The justification for line 5 is written directly beneath the justification for line 4.

These arguments are all instances of ***modus tollens*** (MT)

$$\begin{array}{c}(D \vee F) \supset K \\ \sim K \\ \hline \sim(D \vee F)\end{array} \qquad \begin{array}{c}\sim G \supset \sim(M \vee N) \\ \sim\sim(M \vee N) \\ \hline \sim\sim G\end{array} \qquad \begin{array}{l}\sim T \\ [(H \vee K) \bullet (L \vee N)] \supset T \\ \hline \sim[(H \vee K) \bullet (L \vee N)]\end{array}$$

The next example is already translated into symbols:

1. $A \supset B$
2. $\sim A \supset (C \vee D)$
3. $\sim B$
4. $\sim C$ $\quad$ / D

Once again, to derive the conclusion, always begin by trying to "find" it in the premises. The intended conclusion is D, and after inspecting the premises we find D in line 2. If we had the consequent of that line, $C \vee D$, on a line by itself, we could derive D by disjunctive syllogism if we had $\sim C$. And we do have $\sim C$ on line 4. Also, we could derive $C \vee D$ by *modus ponens* if we had $\sim A$, so now we look for $\sim A$. Turning our attention to line 1, we see that we could derive $\sim A$ by *modus tollens* if we had $\sim B$, and we do have $\sim B$ on line 3. Thus, we have now thought through the entire proof, and we can write it out:

1.	$A \supset B$	
2.	$\sim A \supset (C \vee D)$	
3.	$\sim B$	
4.	$\sim C$	/ D
5.	$\sim A$	1, 3, MT
6.	$C \vee D$	2, 5, MP
7.	D	4, 6, DS

As usual, the justification for each line is written directly beneath the slash mark preceding the intended conclusion. If you have trouble understanding line 6, imagine substituting $\sim A$ and $C \vee D$ in place of p and q in the *modus ponens rule*. Then you can see that lines 2, 5, and 6 constitute a substitution instance of that rule.

These arguments are all instances of **pure hypothetical syllogism** (HS)

$$\begin{array}{l}A \supset (D \bullet F) \\ (D \bullet F) \supset \sim H \\ \hline A \supset \sim H\end{array} \qquad \begin{array}{l}\sim M \supset (R \supset S) \\ (C \vee K) \supset \sim M \\ \hline (C \vee K) \supset (R \supset S)\end{array} \qquad \begin{array}{l}(L \supset N) \supset [(S \vee T) \bullet K] \\ (C \equiv F) \supset (L \supset N) \\ \hline (C \equiv F) \supset [(S \vee T) \bullet K]\end{array}$$

Here is another example.

1. $F \supset G$
2. $F \vee H$
3. $\sim G$
4. $H \supset (G \supset I)$ $\quad$ / $F \supset I$

The intended conclusion is $F \supset I$. When we attempt to find it in the premises, we see no such statement. However, we do find $F \supset G$ on line 1, and $G \supset I$ in the consequent of line 4. This suggests that the conclusion should be derived by pure hypothetical syllogism. But first we must obtain $G \supset I$ on a line by itself. Examining line 4, we see that $G \supset I$ could be derived by *modus ponens*, if we had H on a line by itself, and examining line 2, we see that H could be derived by disjunctive syllogism if we had $\sim F$ on a line by itself. Turning to line 1, we see that $\sim F$ could be derived by *modus tollens* if we had ~G on a line by itself, and we do have $\sim G$ on line 3. Thus, we have now thought through the entire proof, and we can write it out:

1.	$F \supset G$	
2.	$F \vee H$	
3.	$\sim G$	
4.	$H \supset (G \supset I)$	/ $F \supset I$
5.	$\sim F$	1, 3, MT
6.	H	2, 5, DS
7.	$G \supset I$	4, 6, MP
8.	$F \supset I$	1, 7, HS

These arguments are all instances of **disjunctive syllogism** (DS)

$$\begin{array}{l} U \vee \sim(W \bullet X) \\ \underline{\sim U} \\ \sim(W \bullet X) \end{array} \qquad \begin{array}{l} \sim(E \vee F) \\ \underline{(E \vee F) \vee (N \supset K)} \\ N \supset K \end{array} \qquad \begin{array}{l} \sim B \vee [(H \supset M) \bullet (S \supset T)] \\ \underline{\sim\sim B} \\ (H \supset M) \bullet (S \supset T) \end{array}$$

The next example is more complex:

1.	$\sim(A \bullet B) \vee [\sim(E \bullet F) \supset (C \supset D)]$	
2.	$\sim\sim(A \bullet B)$	
3.	$\sim(E \bullet F)$	
4.	$D \supset G$	/ $C \supset G$

Again, when we attempt to find the intended conclusion in the premises, we see no such statement. But we do see $C \supset D$ on line 1 and $D \supset G$ on line 4. We could derive the conclusion by pure hypothetical syllogism if we could obtain $C \supset D$ on a line by itself. Examining line 1, we see that we could derive $C \supset D$ by *modus ponens* if we could obtain both $\sim(E \bullet F) \supset (C \supset D)$ and $\sim(E \bullet F)$ on lines by themselves, and we see that $\sim(E \bullet F)$ appears on line 3. Also, examining line 1, we see that we could derive $\sim(E \bullet F) \supset (C \supset D)$ by disjunctive syllogism if we had $\sim\sim(A \bullet B)$ on a line by itself, and we do have it on line 2. Thus, we can now write out the proof:

1.	$\sim(A \bullet B) \vee [\sim(E \bullet F) \supset (C \supset D)]$	
2.	$\sim\sim(A \bullet B)$	
3.	$\sim(E \bullet F)$	
4.	$D \supset G$	/ $C \supset G$
5.	$\sim(E \bullet F) \supset (C \supset D)$	1, 2, DS

6. $C \supset D$ 3, 5, MP
7. $C \supset G$ 4, 6, HS

If you have trouble seeing how lines 5 and 6 are derived, for line 5 imagine substituting $\sim(A \bullet B)$ and $\sim(E \bullet F) \supset (C \supset D)$, in place of the p and q of the disjunctive syllogism rule. Then you can see that lines 1, 2, and 5 constitute a substitution instance of that rule. For line 6, imagine substituting $\sim(E \bullet F)$ and $(C \supset D)$ in place of p and q in the *modus ponens* rule. Then you can see that lines 5, 3, and 6 constitute a substitution instance of that rule.

In applying the four rules of inference introduced in this section, we have noted that various expressions first had to be obtained on lines by themselves. If this procedure is not followed, the resulting proof will likely be invalid. For an example of an invalid application of *modus ponens*, consider the following:

1. $A \supset (B \supset C)$
2. B
3. C 1, 2, MP (invalid)

This inference is invalid because $B \supset C$ must first be obtained on a line by itself. In deriving the conclusion of an argument we always assume the premises are true. But if we assume line 1 of this proof is true, this does not entail that $B \supset C$ is true. What line 1 says is that *if* A is true, then $B \supset C$ is true. Thus, $B \supset C$ cannot be treated as a premise. We do not know if it is true or false.

Here are some additional examples of invalid inferences:

1. $(A \supset B) \supset C$
2. $\sim B$
3. $\sim A$ 1, 2, MT (invalid—$A \supset B$ must first be obtained on a line by itself)

1. $A \supset (B \supset C)$
2. $C \supset D$
3. $B \supset D$ 1, 2, HS (invalid—$B \supset C$ must first be obtained on a line by itself)

1. $(A \vee B) \supset C$
2. $\sim A$
3. B 1, 2, DS (invalid—$A \vee B$ must first be obtained on a line by itself)

We conclude this section with some strategies for applying the first four rules of inference.

Strategy 1: Always begin by attempting to "find" the conclusion in the premises.

Strategy 2: If the conclusion contains a letter that appears in the consequent of a conditional statement in the premises, consider obtaining that letter via *modus ponens:*

1. $A \supset B$
2. $C \vee A$
3. A / B
4. B 1, 3, MP

Strategy 3: If the conclusion contains a negated letter and that letter appears in the antecedent of a conditional statement in the premises, consider obtaining the negated letter via *modus tollens:*

1. $C \supset B$
2. $A \supset B$
3. $\sim B$ / $\sim A$
4. $\sim A$ 2, 3, MT

Strategy 4: If the conclusion is a conditional statement, consider obtaining it via pure hypothetical syllogism:

1. $B \supset C$
2. $C \supset A$
3. $A \supset B$ / $A \supset C$
4. $A \supset C$ 1, 3, HS

Strategy 5: If the conclusion contains a letter that appears in a disjunctive statement in the premises, consider obtaining that letter via disjunctive syllogism:

1. $A \supset B$
2. $A \lor C$
3. $\sim A$ / C
4. C 2, 3, DS

Of course, these strategies apply to deriving any line prior to the conclusion, just as they apply to deriving the conclusion.

Exercise 7.1

I. For each of the following lists of premises, derive the conclusion and supply the justification for it. There is only one possible answer for each problem.

★(1) 1. $G \supset F$
2. $\sim F$
3. ________ ____

(2) 1. S
2. $S \supset M$
3. ________ ____

(3) 1. $R \supset D$
2. $E \supset R$
3. ________ ____

★(4) 1. $B \lor C$
2. $\sim B$
3. ________ ____

(5) 1. N
2. $N \lor F$
3. $N \supset K$
4. ________ ____

(6) 1. $\sim J \lor P$
2. $\sim J$
3. $S \supset J$
4. ________ ____

★(7) 1. $H \supset D$
2. $F \supset T$
3. $F \supset H$
4. ________ ____

(8) 1. $S \supset W$
2. $\sim S$
3. $S \lor N$
4. ________ ____

(9) 1. $F \supset \sim A$
2. $N \supset A$
3. $\sim F$
4. $\sim A$
5. ________ ____

★(10) 1. $H \supset A$
2. A
3. $A \vee M$
4. $G \supset H$
5. ________ ____

(11) 1. $W \vee B$
2. W
3. $B \supset T$
4. $W \supset A$
5. ________ ____

(12) 1. $K \supset \sim R$
2. $\sim R$
3. $R \vee S$
4. $R \supset T$
5. ________ ____

★(13) 1. $\sim C \supset \sim F$
2. $L \supset F$
3. $\sim\sim F$
4. $F \vee \sim L$
5. ________ ____

(14) 1. $N \supset \sim E$
2. $\sim\sim S$
3. $\sim E \vee \sim S$
4. $\sim S \vee N$
5. ________ ____

(15) 1. $\sim R \supset \sim T$
2. $\sim T \vee B$
3. $C \supset \sim R$
4. $\sim C$
5. ________ ____

★(16) 1. $\sim K$
2. $\sim K \supset \sim P$
3. $\sim K \vee G$
4. $G \supset Q$
5. ________ ____

(17) 1. $F \vee (A \supset C)$
2. $A \vee (C \supset F)$
3. A
4. $\sim F$
5. ________ ____

(18) 1. $(R \supset M) \supset D$
2. $M \supset C$
3. $D \supset (M \vee E)$
4. $\sim M$
5. ________ ____

★(19) 1. $(S \vee C) \supset L$
2. $\sim S$
3. $\sim L$
4. $S \supset (K \supset L)$
5. ________ ____

(20) 1. $(A \vee W) \supset (N \supset Q)$
2. $Q \supset G$
3. $\sim A$
4. $(Q \supset G) \supset (A \vee N)$
5. ________ ____

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II. The following symbolized arguments are missing a premise. Write the premise needed to derive the conclusion (last line), and supply the justification for the conclusion. Try to construct the simplest premise needed to derive the conclusion.

★(1) 1. $B \vee K$
2. ________
3. K ____

(2) 1. $N \supset S$
2. ________
3. S ____

(3) 1. $K \supset T$
2. ________
3. $\sim K$ ____

★(4) 1. $C \supset H$
2. ________
3. $R \supset H$ ____

(5) 1. $F \supset N$
2. $N \supset T$
3. ________
4. $\sim F$ ____

(6) 1. $W \vee T$
2. $A \supset W$
3. ______
4. $A \supset T$ ____

★(7) 1. $M \supset B$
2. $Q \supset M$
3. ______
4. M ____

(8) 1. $C \vee L$
2. $L \supset T$
3. ______
4. L ____

(9) 1. $E \supset N$
2. $T \vee \sim E$
3. $S \supset E$
4. ______
5. E ____

★(10) 1. $H \supset A$
2. $S \supset H$
3. $\sim M \vee H$
4. ______
5. $\sim H$ ____

(11) 1. $T \supset N$
2. $G \supset T$
3. $H \vee T$
4. ______
5. $F \supset T$ ____

(12) 1. $G \supset C$
2. $M \vee G$
3. $T \vee \sim G$
4. ______
5. G ____

★(13) 1. $\sim S \supset \sim B$
2. $R \vee \sim B$
3. $\sim B \supset \sim S$
4. ______
5. $\sim\sim B$ ____

(14) 1. $\sim R \supset D$
2. $\sim J \supset \sim R$
3. $N \vee \sim R$
4. ______
5. $\sim F \supset \sim R$ ____

(15) 1. $\sim S \vee \sim P$
2. $\sim K \supset P$
3. $\sim P \supset F$
4. ______
5. $\sim P$ ____

★(16) 1. $J \supset E$
2. $B \vee \sim J$
3. $\sim Z \supset J$
4. ______
5. J ____

(17) 1. $(H \supset C) \supset A$
2. $N \supset (F \supset K)$
3. $(E \bullet R) \supset K$
4. ______
5. $H \supset K$ ____

(18) 1. $(S \supset M) \supset G$
2. $S \supset (M \bullet G)$
3. $G \supset (R \supset \sim S)$
4. ______
5. $\sim S$ ____

★(19) 1. $(W \vee \sim F) \supset H$
2. $(H \vee G) \supset \sim F$
3. $T \supset (F \supset G)$
4. ______
5. $\sim F$ ____

(20) 1. $(H \bullet A) \vee T$
2. $\sim S \supset (P \supset T)$
3. $(N \vee T) \supset P$
4. ______
5. T ____

III. Use the first four rules of inference to derive the conclusions of the following symbolized arguments.

★(1) 1. $\sim C \supset (A \supset C)$
2. $\sim C$ / $\sim A$

(2) 1. $F \vee (D \supset T)$
2. $\sim F$
3. D / T

(3) 1. $(K \bullet B) \vee (L \supset E)$
2. $\sim(K \bullet B)$
3. $\sim E$ / $\sim L$

★(4) 1. $P \supset (G \supset T)$
2. $Q \supset (T \supset E)$
3. P
4. Q / $G \supset E$

(5) 1. $\sim W \supset [\sim W \supset (X \supset W)]$
2. $\sim W$ / $\sim X$

(6) 1. $J \supset (K \supset L)$
2. $L \vee J$
3. $\sim L$ / $\sim K$

★(7) 1. $\sim S \supset D$
2. $\sim S \vee (\sim D \supset K)$
3. $\sim D$ / K

(8) 1. $A \supset (E \supset \sim F)$
2. $H \vee (\sim F \supset M)$
3. A
4. $\sim H$ / $E \supset M$

(9) 1. $\sim G \supset (G \vee \sim A)$
2. $\sim A \supset (C \supset A)$
3. $\sim G$ / $\sim C$

★(10) 1. $N \supset (J \supset P)$
2. $(J \supset P) \supset (N \supset J)$
3. N / P

(11) 1. $G \supset [\sim O \supset (G \supset D)]$
2. $O \vee G$
3. $\sim O$ / D

(12) 1. $\sim M \vee (B \vee \sim T)$
2. $B \supset W$
3. $\sim\sim M$
4. $\sim W$ / $\sim T$

★(13) 1. $R \supset (G \vee \sim A)$
2. $(G \vee \sim A) \supset \sim S$
3. $G \supset S$
4. R / $\sim A$

(14) 1. $(L \equiv N) \supset C$
2. $(L \equiv N) \vee (P \supset \sim E)$
3. $\sim E \supset C$
4. $\sim C$ / $\sim P$

(15) 1. $\sim J \supset [\sim A \supset (D \supset A)]$
2. $J \vee \sim A$
3. $\sim J$ / $\sim D$

★(16) 1. $(B \supset \sim M) \supset (T \supset \sim S)$
2. $B \supset K$
3. $K \supset \sim M$
4. $\sim S \supset N$ / $T \supset N$

(17) 1. $H \vee (Q \vee F)$
2. $R \vee (Q \supset R)$
3. $R \vee \sim H$
4. $\sim R$ / F

(18) 1. $\sim A \supset (B \supset \sim C)$
2. $\sim D \supset (\sim C \supset A)$
3. $D \vee \sim A$
4. $\sim D$ / $\sim B$

★(19) 1. $\sim G \supset [G \vee (S \supset G)]$
2. $(S \vee L) \supset \sim G$
3. $S \vee L$ / L

(20) 1. $H \supset [\sim E \supset (C \supset \sim D)]$
2. $\sim D \supset E$
3. $E \vee H$
4. $\sim E$ / $\sim C$

(21) 1. $\sim B \supset [(A \supset K) \supset (B \vee \sim K)]$
2. $\sim J \supset K$
3. $A \supset \sim J$
4. $\sim B$ / $\sim A$

★(22) 1. $(C \supset M) \supset (N \supset P)$
2. $(C \supset N) \supset (N \supset M)$
3. $(C \supset P) \supset \sim M$
4. $C \supset N$ / $\sim C$

(23) 1. $(R \supset F) \supset [(R \supset \sim G) \supset (S \supset Q)]$
2. $(Q \supset F) \supset (R \supset Q)$
3. $\sim G \supset F$
4. $Q \supset \sim G$ / $S \supset F$

(24) 1. $\sim A \supset [A \vee (T \supset R)]$
2. $\sim R \supset [R \vee (A \supset R)]$
3. $(T \vee D) \supset \sim R$
4. $T \vee D$ / D

★(25) 1. $\sim N \supset [(B \supset D) \supset (N \vee \sim E)]$
2. $(B \supset E) \supset \sim N$
3. $B \supset D$
4. $D \supset E$ / $\sim D$

IV. Translate the following arguments into symbolic form and use the first four rules of inference to derive the conclusion of each. The letters to be used for the simple statements are given in parentheses after each exercise. Use these letters in the order in which they are listed.

★**1.** If the average child watches more than five hours of television per day, then either his power of imagination is improved or he becomes conditioned to expect constant excitement. The average child's power of imagination is not improved by watching television. Also, the average child does watch more than five hours of television per day. Therefore, the average child is conditioned to expect constant excitement. (*W*, *P*, *C*)

2. If a ninth planet exists, then its orbit is perpendicular to that of the other planets. Either a ninth planet is responsible for the death of the dinosaurs, or its orbit is not perpendicular to that of the other planets. A ninth planet is not responsible for the death of the dinosaurs. Therefore, a ninth planet does not exist. (*E*, *O*, *R*)

3. If quotas are imposed on textile imports only if jobs are not lost, then the domestic textile industry will modernize only if the domestic textile industry is not destroyed. If quotas are imposed on textile imports, the domestic textile industry will modernize. The domestic textile industry will modernize only if jobs are not lost. Therefore, if quotas are imposed on textile imports, the domestic textile industry will not be destroyed. (*Q*, *J*, *M*, *D*)

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★**4.** If teachers are allowed to conduct random drug searches on students only if teachers are acting in loco parentis, then if teachers are acting in loco parentis, then students have no Fourth Amendment protections. Either students have no Fourth Amendment protections or if teachers are allowed to conduct random drug searches on students, then teachers are acting in loco parentis. It is not the case that students have no Fourth Amendment protections. Therefore, teachers are not allowed to conduct random drug searches on students. (*R*, *L*, *F*)

5. Either funding for nuclear fusion will be cut or if sufficiently high temperatures are achieved in the laboratory, nuclear fusion will become a reality. Either the supply of hydrogen fuel is limited, or if nuclear fusion becomes a reality, the world's energy problems will be solved. Funding for nuclear fusion will not be cut. Furthermore, the supply of hydrogen fuel is not limited. Therefore, if sufficiently high temperatures are achieved in the laboratory, the world's energy problems will be solved. (*C*, *H*, *R*, *S*, *E*)

6. Either the continents are not subject to drift or if Antarctica was always located in the polar region, then it contains no fossils of plants from a temperate climate. If the continents are not subject to drift, then Antarctica contains no fossils of plants from a temperate climate. But it is not the case that Antarctica contains no fossils of plants from a temperate climate. Therefore, Antarctica was not always located in the polar region. (*D*, *L*, *F*)

★**7.** If terrorists take more hostages, then terrorist demands will be met if and only if the media give full coverage to terrorist acts. Either the media will voluntarily limit the flow of information or if the media will recognize they are being exploited by terrorists, they will voluntarily limit the flow of information. Either the media will recognize they are being exploited by terrorists or terrorists will take more hostages. The media will not voluntarily limit the flow of information. Therefore, terrorist demands will be met if and only if the media give full coverage to terrorist acts. (*H*, *D*, *A*, *V*, *R*)

8. Either we take recycling seriously or we will be buried in garbage. If we incinerate our garbage only if our health is jeopardized, then we do not take recycling seriously. If our landfills are becoming exhausted, then if we incinerate our garbage, then toxic ash will be produced. If toxic ash is produced, then our health is jeopardized. Our landfills are becoming exhausted. Therefore, we will be buried in garbage. (*R*, *B*, *I*, *H*, *L*, *T*)

9. If the drug interdiction program is strengthened only if cocaine becomes more readily available, then either the number of addicts is decreasing or the war on drugs is failing. If the drug interdiction program is strengthened, then smugglers will shift to more easily concealable drugs. If smugglers shift to more easily concealable drugs, then cocaine will become more readily available. Furthermore, the number of addicts is not decreasing. Therefore, the war on drugs is failing. (*D*, *C*, *N*, *W*, *S*)

★**10.** If the death penalty is not cruel and unusual punishment, then either it is cruel and unusual punishment or if society is justified in using it, then it will deter other criminals. If the death penalty is cruel and unusual punishment, then it is both cruel and unusual and its use degrades society as a whole. It is not the case that both the death penalty is cruel and unusual and its use degrades society as a whole. Furthermore, the death penalty will not deter other criminals. Therefore, society is not justified in using the death penalty. (*C*, *J*, *D*, *U*)

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7.2 Rules of Implication II

Four additional rules of implication are listed here. Constructive dilemma should be familiar from Chapter 6. The other three are new.*

5. Constructive dilemma (CD):
$(p \supset q) \bullet (r \supset s)$
$p \vee r$
$\overline{\quad\quad\quad}$
$q \vee s$

6. Simplification (Simp):
$p \bullet q$
$\overline{\quad\quad}$
p

7. Conjunction (Conj):
p
q
$\overline{\quad\quad}$
$p \bullet q$

8. Addition (Add):
p
$\overline{\quad\quad}$
$p \vee q$

*Some textbooks include a rule called *absorption* by which the statement form $p \supset (q \bullet p)$ is deduced from $p \supset q$. This rule is necessary only if conditional proof is not presented. This textbook opts in favor of conditional proof, to be introduced shortly.

Like the previous four rules, these are fairly easy to understand, but if there is any doubt about them their validity may be proven by means of a truth table.

Constructive dilemma can be understood as involving two *modus ponens* steps. The first premise states that if we have p then we have q, and if we have r then we have s. But since, by the second premise, we do have either p or r, it follows by *modus ponens* that we have either q or s. Constructive dilemma is the only form of dilemma that will be included as a rule of inference. By the rule of transposition, which will be presented in Section 7.4, any argument that is a substitution instance of the destructive dilemma form can be easily converted into a substitution instance of constructive dilemma. Destructive dilemma, therefore, is not needed as a rule of inference.

These arguments are both instances of **constructive dilemma** (CD)

$\sim M \vee N$	$[(K \supset T) \supset (A \bullet B)] \bullet [(H \supset P) \supset (A \bullet C)]$
$(\sim M \supset S) \bullet (N \supset \sim T)$	$(K \supset T) \vee (H \supset P)$
$S \vee \sim T$	$(A \bullet B) \vee (A \bullet C)$

Simplification states that if two propositions are given as true on a single line, then each of them is true separately. According to the strict interpretation of the simplification rule, only the left-hand conjunct may be stated in the conclusion. Once the commutativity rule for conjunction has been presented, however (see Section 7.3), we will be justified in replacing a statement such as $H \bullet K$ with $K \bullet H$. Once we do this, the K will appear on the left, and the appropriate conclusion is K.

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These arguments are all instances of **simplification** (Simp)

$\sim F \bullet (U \equiv E)$	$(M \vee T) \bullet (S \supset R)$	$[(X \supset Z) \bullet M] \bullet (G \supset H)$
$\sim F$	$(M \vee T)$	$[(X \supset Z) \bullet M]$

Conjunction states that two propositions—for example, H and K—asserted separately on different lines may be conjoined on a single line. The two propositions may be conjoined in whatever order we choose (either $H \bullet K$ or $K \bullet H$) without appeal to the commutativity rule for conjunction.

These arguments are all instances of **conjunction** (Conj)

$\sim E$	$C \supset M$	$R \supset (H \bullet T)$
$\sim G$	$D \supset N$	$K \supset (H \bullet O)$
$\sim E \bullet \sim G$	$(C \supset M) \bullet (D \supset N)$	$[R \supset (H \bullet T)] \bullet [K \supset (H \bullet O)]$

Addition states that whenever a proposition is asserted on a line by itself it may be joined disjunctively with any proposition we choose. In other words, if G is asserted to be true by itself, it follows that $G \vee H$ is true. This may appear somewhat puzzling at first, but once one realizes that $G \vee H$ is a much weaker statement than G by itself, the puzzlement should disappear. The new proposition must, of course, always be joined disjunctively (not conjunctively) to the given proposition. If G is stated on a line by itself, we are *not* justified in writing $G \bullet H$ as a consequence of addition.

These arguments are all instances of **addition** (Add)

$$\frac{S}{S \vee \sim T} \qquad \frac{(C \bullet D)}{(C \bullet D) \vee (K \bullet \sim P)} \qquad \frac{W \equiv Z}{(W \equiv Z) \vee [A \supset (M \supset O)]}$$

The use of these four rules may now be illustrated. Consider the following argument:

1.	$A \supset B$	
2.	$(B \vee C) \supset (D \bullet E)$	
3.	A	/ D

As usual, we begin by looking for the conclusion in the premises. D appears in the consequent of the second premise, which we can derive via simplification if we first obtain $B \vee C$. This expression as such does not appear in the premises, but from lines 1 and 3 we see that we can derive B by itself via *modus ponens*. Having obtained B, we can derive $B \vee C$ via addition. The proof has now been thought through and can be written out as follows:

1.	$A \supset B$	
2.	$(B \vee C) \supset (D \bullet E)$	
3.	A	/ D
4.	B	1, 3, MP
5.	$B \vee C$	4, Add
6.	$D \bullet E$	2, 5, MP
7.	D	6, Simp

Another example:

1.	$K \supset L$	
2.	$(M \supset N) \bullet S$	
3.	$N \supset T$	
4.	$K \vee M$	/ $L \vee T$

Seeing that $L \vee T$ does not appear as such in the premises, we look for the separate components. Finding L and T as the consequents of two distinct conditional statements causes us to think that the conclusion can be derived via constructive dilemma. If a constructive dilemma can be set up, it will need a disjunctive statement as its second premise, and such a statement appears on line 4. Furthermore, each of the components of this statement, K and M, appears as the antecedent of a conditional statement, exactly as they both should for a dilemma. The only statement that is missing now is $M \supset T$. Inspecting line 2 we see that we can obtain $M \supset N$ via simplification, and putting this together with line 3 gives us $M \supset T$ via hypothetical syllogism. The completed proof may now be written out:

1.	$K \supset L$	
2.	$(M \supset N) \cdot S$	
3.	$N \supset T$	
4.	$K \vee M$	/ $L \vee T$
5.	$M \supset N$	2, Simp
6.	$M \supset T$	3, 5, HS
7.	$(K \supset L) \cdot (M \supset T)$	1, 6, Conj
8.	$L \vee T$	4, 7, CD

Another example:

1.	$\sim M \cdot N$	
2.	$P \supset M$	
3.	$Q \cdot R$	
4.	$(\sim P \cdot Q) \supset S$	/ $S \vee T$

When we look for $S \vee T$ in the premises we find S in the consequent of line 4 but no T at all. This signals an important principle: Whenever the conclusion of an argument contains a letter not found in the premises, addition must be used to introduce the missing letter. Addition is the *only* rule of inference that can introduce new letters. To introduce T by addition, however, we must first obtain S on a line by itself. S can be derived from line 4 via *modus ponens* if we first obtain $\sim P \cdot Q$. This, in turn, can be derived via conjunction, but first $\sim P$ and Q must be obtained individually on separate lines. Q can be derived from line 3 via simplification and $\sim P$ from line 2 via *modus tollens*, but the latter step requires that we first obtain $\sim M$ on a line by itself. Since this can be derived from line 1 via simplification, the proof is now complete. It may be written out as follows:

1.	$\sim M \cdot N$	
2.	$P \supset M$	
3.	$Q \cdot R$	
4.	$(\sim P \cdot Q) \supset S$	/ $S \vee T$
5.	$\sim M$	1, Simp
6.	$\sim P$	2, 5, MT
7.	Q	3, Simp
8.	$\sim P \cdot Q$	6, 7, Conj
9.	S	4, 8, MP
10.	$S \vee T$	9, Add

Addition is used together with disjunctive syllogism to derive the conclusion of arguments having inconsistent premises. As we saw in Chapter 6, such arguments are always valid. The procedure is illustrated as follows:

1.	S	
2.	$\sim S$	/ T
3.	$S \vee T$	1, Add
4.	T	2, 3, DS

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With arguments of this sort the conclusion is always introduced via addition and then separated via disjunctive syllogism. Since addition can be used to introduce any letter or arrangement of letters we choose, it should be clear from this example that inconsistent premises validly entail any conclusion whatever.

To complete this presentation of the eight rules of implication, let us consider some of the typical ways in which they are *misapplied*. Examples are as follows:

1. $P \vee (S \bullet T)$	
2. S	1, Simp (invalid—$S \bullet T$ must first be obtained on a line by itself)
1. K	
2. $K \bullet L$	1, Add (invalid—the correct form of addition is "$K \vee L$")
1. $M \vee N$	
2. M	1, Simp (invalid—simplification is possible only with a conjunctive premise; line 1 is a disjunction)
1. $G \supset H$	
2. $G \supset (H \vee J)$	1, Add (improper—J must be added to the whole line, not just to the consequent: $(G \supset H) \vee J$)
1. $L \supset M$	
2. $L \supset N$	
3. $M \bullet N$	1, 2, Conj (invalid—M and N must first be obtained on lines by themselves)
1. $\sim(P \bullet Q)$	
2. $\sim P$	1, Simp (invalid—parentheses must be removed first)
1. $\sim(P \vee Q)$	
2. $\sim P$	
3. Q	1, 2, DS (invalid—parentheses must be removed first)

The use of addition in the $G \supset H$ example is called "improper" because the letter that is added is not added to the whole line. It turns out, however, that even though the addition rule is not correctly applied here, the inference is still valid. Hence, this inference is not called "invalid," as the others are. As for the last two examples, a rule will be presented in the next section (De Morgan's rule) that will allow us to remove parentheses preceded by negation signs. But even after the parentheses have been removed from these examples, the inferences remain invalid.

Like the previous section, this one ends with a few strategies for applying the last four rules of implication:

Strategy 6: If the conclusion contains a letter that appears in a conjunctive statement in the premises, consider obtaining that letter via simplification:

1. $A \supset B$	
2. $C \bullet B$	
3. $C \supset A$	/ C
4. C	2, Simp

Strategy 7: If the conclusion is a conjunctive statement, consider obtaining it via conjunction by first obtaining the individual conjuncts:

1.	$A \supset C$	
2.	B	
3.	$\sim C$	/ $B \bullet \sim C$
4.	$B \bullet \sim C$	2, 3, Conj

Strategy 8: If the conclusion is a disjunctive statement, consider obtaining it via constructive dilemma or addition:

1.	$(A \supset B) \bullet (C \supset D)$	
2.	$B \supset C$	
3.	$A \vee C$	/ $B \vee D$
4.	$B \vee D$	1, 3, CD

1.	$A \vee C$	
2.	B	
3.	$C \supset D$	/ $B \vee D$
4.	$B \vee D$	2, Add

Strategy 9: If the conclusion contains a letter not found in the premises, addition *must* be used to introduce that letter.

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Strategy 10: Conjunction can be used to set up constructive dilemma:

1.	$A \supset B$	
2.	$C \supset D$	
3.	$A \vee C$	/ $B \vee D$
4.	$(A \supset B) \bullet (C \supset D)$	1, 2, Conj
5.	$B \vee D$	3, 4, CD

Exercise 7.2

I. For each of the following lists of premises, derive the indicated conclusion and complete the justification. In problems 4 and 8 you can add any statement you choose.

★(1) 1. $S \vee H$
2. $B \bullet E$
3. $R \supset G$
4. ______ ____, Simp

(2) 1. $(N \supset T) \bullet (F \supset Q)$
2. $(N \supset R) \vee (F \supset M)$
3. $N \vee F$
4. ________________ ____, CD

(3) 1. D
2. W
3. _____ ____, Conj

★(4) 1. H
2. _____ ____, Add

(5) 1. $R \bullet (N \vee K)$
2. $(G \bullet T) \vee S$
3. $(Q \bullet C) \supset (J \bullet L)$
4. ______________ ____, Simp

(6) 1. $\sim R \vee P$
2. $(P \supset \sim D) \bullet (\sim R \supset S)$
3. $(\sim R \supset A) \bullet (P \supset \sim N)$
4. ______________ _____, CD

★(7) 1. $(Q \vee K) \bullet \sim B$
2. $(M \bullet R) \supset D$
3. $(W \bullet S) \vee (G \bullet F)$
4. ______________ _____, Simp

(8) 1. $E \bullet G$
2. ______ _____, Add

(9) 1. $\sim B$
2. $F \vee N$
3. _____ _____, Conj

★(10) 1. $S \vee \sim C$
2. $(S \supset \sim L) \bullet (\sim C \supset M)$
3. $(\sim N \supset S) \bullet (F \supset \sim C)$
4. ______________ _____, CD

II. In the following symbolized arguments, derive the line needed to obtain the conclusion (last line), and supply the justification for both lines.

★(1) 1. $G \supset N$
2. $G \bullet K$
3. ______ _____
4. $G \vee T$ _____

(2) 1. $\sim A$
2. $A \vee E$
3. ______ _____
4. $\sim A \bullet E$ _____

(3) 1. $B \supset N$
2. $B \vee K$
3. $K \supset R$
4. ______ _____
5. $N \vee R$ _____

★(4) 1. T
2. $T \supset G$
3. $(T \vee U) \supset H$
4. ____________ _____
5. H _____

(5) 1. $S \supset E$
2. $E \vee (S \bullet P)$
3. $\sim E$
4. ________ _____
5. S _____

(6) 1. N
2. $N \supset F$
3. $(N \supset A) \bullet (F \supset C)$
4. ______________ _____
5. $A \vee C$ _____

★(7) 1. J
2. $\sim L$
3. $F \supset L$
4. ______ _____
5. $\sim F \bullet J$ _____

(8) 1. $(E \supset B) \bullet (Q \supset N)$
2. $K \supset E$
3. $B \supset K$
4. ______________ _____
5. $E \supset K$ _____

(9) 1. $G \vee N$
2. $\sim G$
3. $\sim G \supset (H \bullet R)$
4. ____________ _____
5. H _____

★(10) 1. M
2. $(M \bullet E) \supset D$
3. E
4. __________ _____
5. D _____

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III. Use the first eight rules of inference to derive the conclusions of the following symbolized arguments:

★(1)
1. $\sim M \supset Q$
2. $R \supset \sim T$
3. $\sim M \vee R$ / $Q \vee \sim T$

(2)
1. $N \supset (D \bullet W)$
2. $D \supset K$
3. N / $N \bullet K$

(3)
1. $E \supset (A \bullet C)$
2. $A \supset (F \bullet E)$
3. E / F

★(4)
1. $(H \vee \sim B) \supset R$
2. $(H \vee \sim M) \supset P$
3. H / $R \bullet P$

(5)
1. $G \supset (S \bullet T)$
2. $(S \vee T) \supset J$
3. G / J

(6)
1. $(L \vee T) \supset (B \bullet G)$
2. $L \bullet (K \equiv R)$ / $L \bullet B$

★(7)
1. $(\sim F \vee X) \supset (P \vee T)$
2. $F \supset P$
3. $\sim P$ / T

(8)
1. $(N \supset B) \bullet (O \supset C)$
2. $Q \supset (N \vee O)$
3. Q / $B \vee C$

(9)
1. $(U \vee W) \supset (T \supset R)$
2. $U \bullet H$
3. $\sim R \bullet \sim J$ / $U \bullet \sim T$

★(10)
1. $(D \vee E) \supset (G \bullet H)$
2. $G \supset \sim D$
3. $D \bullet F$ / M

(11)
1. $(B \vee F) \supset (A \supset G)$
2. $(B \vee E) \supset (G \supset K)$
3. $B \bullet \sim H$ / $A \supset K$

(12)
1. $(P \supset R) \supset (M \supset P)$
2. $(P \vee M) \supset (P \supset R)$
3. $P \vee M$ / $R \vee P$

★(13)
1. $(C \supset N) \bullet E$
2. $D \vee (N \supset D)$
3. $\sim D$ / $\sim C \vee P$

(14)
1. $F \supset (\sim T \bullet A)$
2. $(\sim T \vee G) \supset (H \supset T)$
3. $F \bullet O$ / $\sim H \bullet \sim T$

(15)
1. $(\sim S \vee B) \supset (S \vee K)$
2. $(K \vee \sim D) \supset (H \supset S)$
3. $\sim S \bullet W$ / $\sim H$

★(16)
1. $(C \vee \sim G) \supset (\sim P \bullet L)$
2. $(\sim P \bullet C) \supset (C \supset D)$
3. $C \bullet \sim R$ / $D \vee R$

(17)
1. $[A \vee (K \bullet J)] \supset (\sim E \bullet \sim F)$
2. $M \supset [A \bullet (P \vee R)]$
3. $M \bullet U$ / $\sim E \bullet A$

(18)
1. $\sim H \supset (\sim T \supset R)$
2. $H \vee (E \supset F)$
3. $\sim T \vee E$
4. $\sim H \bullet D$ / $R \vee F$

★(19)
1. $(U \bullet \sim\sim P) \supset Q$
2. $\sim O \supset U$
3. $\sim P \supset O$
4. $\sim O \bullet T$ / Q

(20)
1. $(M \vee N) \supset (F \supset G)$
2. $D \supset \sim C$
3. $\sim C \supset B$
4. $M \bullet H$
5. $D \vee F$ / $B \vee G$

(21)
1. $(F \bullet M) \supset (S \vee T)$
2. $(\sim S \vee A) \supset F$
3. $(\sim S \vee B) \supset M$
4. $\sim S \bullet G$ / T

★(22)
1. $(\sim K \bullet \sim N) \supset$ $[(\sim P \supset K) \bullet (\sim R \supset G)]$
2. $K \supset N$
3. $\sim N \bullet B$
4. $\sim P \vee \sim R$ / G

(23)
1. $(\sim A \vee D) \supset (B \supset F)$
2. $(B \vee C) \supset (A \supset E)$
3. $A \vee B$
4. $\sim A$ / $E \vee F$

(24) 1. $(J \supset K) \cdot (\sim O \supset \sim P)$
2. $(L \supset J) \cdot (\sim M \supset \sim O)$
3. $\sim K \supset (L \vee \sim M)$
4. $\sim K \cdot G$ / $\sim P$

★(25) 1. $(\sim M \cdot \sim N) \supset [(\sim M \vee H) \supset (K \cdot L)]$
2. $\sim M \cdot (C \supset D)$
3. $\sim N \cdot (F \equiv G)$ / $K \cdot \sim N$

(26) 1. $(P \vee S) \supset (E \supset F)$
2. $(P \vee T) \supset (G \supset H)$
3. $(P \vee U) \supset (E \vee G)$
4. P / $F \vee H$

(27) 1. $(S \supset Q) \cdot (Q \supset \sim S)$
2. $S \vee Q$
3. $\sim Q$ / $P \cdot R$

★(28) 1. $(D \supset B) \cdot (C \supset D)$
2. $(B \supset D) \cdot (E \supset C)$
3. $B \vee E$ / $D \vee B$

(29) 1. $(R \supset H) \cdot (S \supset I)$
2. $(\sim H \cdot \sim L) \supset (R \vee S)$
3. $\sim H \cdot (K \supset T)$
4. $H \vee \sim L$ / $I \vee M$

(30) 1. $(W \cdot X) \supset (Q \vee R)$
2. $(S \vee F) \supset (Q \vee W)$
3. $(S \vee G) \supset (\sim Q \supset X)$
4. $Q \vee S$
5. $\sim Q \cdot H$ / R

IV. Translate the following arguments into symbolic form and use the first eight rules of inference to derive the conclusion of each. Use the letters in the order in which they are listed.

★**1.** If topaz is harder than quartz, then it will scratch quartz and also feldspar. Topaz is harder than quartz and it is also harder than calcite. Therefore, either topaz will scratch quartz or it will scratch corundum. (*T*, *Q*, *F*, *C*, *O*)

2. If clear-cutting continues in primary forests and the Endangered Species Act is not repealed, then either the Endangered Species Act will be repealed or thousands of animal species will become extinct. Clear-cutting continues in primary forests. The Endangered Species Act will not be repealed. Therefore, thousands of animal species will become extinct. (*C*, *E*, *T*)

3. If either executive salaries are out of control or exorbitant bonuses are paid, then either shareholders will be cheated or ordinary workers will be paid less. Executive salaries are out of control and the rich are getting richer. If shareholders are cheated, then future investors will stay away; also, if ordinary workers are paid less, then consumer spending will decline. If either future investors stay away or consumer spending declines, then the economy will suffer. Therefore, the economy will suffer. (*S*, *B*, *C*, *P*, *R*, *F*, *D*, *E*)

★**4.** Either animals are mere mechanisms or they feel pain. If either animals feel pain or they have souls, then they have a right not to be subjected to needless pain and humans have a duty not to inflict needless pain on them. It is not the case that animals are mere mechanisms. Therefore, animals have a right not to be subjected to needless pain. (*M*, *P*, *S*, *R*, *D*)

5. If half the nation suffers from depression, then if either the insurance companies have their way or the psychiatrists have their way, then everyone will be taking antidepressant drugs. If either half the nation suffers from depression or sufferers want a real cure, then it is not the case that everyone will be taking antidepressant drugs. Half the nation suffers from depression. Therefore, it is not the case that either the insurance companies or the psychiatrists will have their way. (*H*, *I*, *P*, *E*, *W*)

6. If either parents get involved in their children's education or the school year is lengthened, then if the children learn phonics, their reading will improve and if they are introduced to abstract concepts earlier, their math will improve. If either parents get involved in their children's education or nebulous subjects are dropped from the curriculum, then either the children will learn phonics or they will be introduced to abstract concepts earlier. Parents will get involved in their children's education, and writing lessons will be integrated with other subjects. Therefore, either the children's reading or their math will improve. (*P*, *S*, *L*, *R*, *I*, *M*, *N*, *W*)

★7. If either manufacturers will not concentrate on producing a superior product or they will not market their product abroad, then if they will not concentrate on producing a superior product, then the trade deficit will worsen. Either manufacturers will concentrate on producing a superior product or the trade deficit will not worsen. Manufacturers will not concentrate on producing a superior product. Therefore, today's business managers lack imagination. (*C*, *M*, *T*, *B*)

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8. If either medical fees or malpractice awards escape restrictions, then health care costs will soar and millions of poor will go uninsured. If the lawyers get their way, then malpractice awards will escape restrictions. If the doctors get their way, then medical fees will escape restrictions. Either the doctors or the lawyers will get their way, and insurance companies will resist reform. Therefore, health care costs will soar. (*F*, *A*, *H*, *P*, *L*, *D*, *I*)

9. If we are less than certain the human fetus is a person, then we must give it the benefit of the doubt. If we are certain the human fetus is a person, then we must accord it the right to live. If either we must give the fetus the benefit of the doubt or accord it the right to live, then we are not less than certain the fetus is human and it is not merely a part of the mother's body. Either we are less than certain the human fetus is a person or we are certain about it. If we are certain the human fetus is a person, then abortion is immoral. Therefore, abortion is immoral. (*L*, *G*, *C*, *A*, *M*, *I*)

★10. If the assassination of terrorist leaders violates civilized values and also is not effective in the long run, then if it prevents terrorist atrocities, then it is effective in the long run. If the assassination of terrorist leaders violates civilized values, then it is not effective in the long run. The assassination of terrorist leaders violates civilized values and is also illegal. If the assassination of terrorist leaders is not effective in the long run, then either it prevents terrorist atrocities or it justifies acts of revenge by terrorists. Therefore, the assassination of terrorist leaders justifies acts of revenge by terrorists and also is not effective in the long run. (*V*, *E*, *P*, *I*, *J*)

7.3

Rules of Replacement I

Unlike the rules of implication, which are basic argument forms, the ten **rules of replacement** are expressed in terms of pairs of logically equivalent statement forms, either of which can replace the other in a proof sequence. To express these rules, a new symbol, called a **double colon** (::), is used to designate logical equivalence. This symbol is a *metalogical* symbol in that it makes an assertion not about things but about symbolized statements: It asserts that the expressions on either side of it have the same truth value regardless of the truth values of their components. Underlying the use of the rules of replacement is an **axiom of replacement,** which asserts that within the context of a proof, logically equivalent expressions may replace each other. The first five rules of replacement are as follows:

9. De Morgan's rule (DM):
   $\sim(p \cdot q) :: (\sim p \vee \sim q)$
   $\sim(p \vee q) :: (\sim p \cdot \sim q)$

10. Commutativity (Com):
    $(p \vee q) :: (q \vee p)$
    $(p \cdot q) :: (q \cdot p)$

11. Associativity (Assoc):
    $[p \vee (q \vee r)] :: [(p \vee q) \vee r]$
    $[p \cdot (q \cdot r)] :: [(p \cdot q) \cdot r]$

12. Distribution (Dist):
    $[p \cdot (q \vee r)] :: [(p \cdot q) \vee (p \cdot r)]$
    $[p \vee (q \cdot r)] :: [(p \vee q) \cdot (p \vee r)]$

13. Double negation (DN):
    $p :: \sim\sim p$

De Morgan's rule (named after the nineteenth-century logician Augustus De Morgan) was discussed in Section 6.1 in connection with translation. There it was pointed out that "Not both *p* and *q*" is logically equivalent to "Not *p* or not *q*," and that "Not either *p* or *q*" is logically equivalent to "Not *p* and not *q*." When applying De Morgan's rule, one should keep in mind that it holds only for conjunctive and disjunctive statements (not for conditionals or biconditionals). The rule may be summarized as follows: When moving a tilde inside or outside a set of parentheses, a dot switches with a wedge and vice versa.

Commutativity asserts that the truth value of a conjunction or disjunction is unaffected by the order in which the components are listed. In other words, the component statements may be commuted, or switched for one another, without affecting the truth value. The validity of this rule should be immediately apparent. You may recall from arithmetic that the commutativity rule also applies to addition and multiplication and asserts, for example, that 3 + 5 equals 5 + 3, and that 2 × 3 equals 3 × 2. However, it does *not* apply to division; 2 ÷ 4 does not equal 4 ÷ 2. A similar lesson applies in logic: The commutativity rule applies only to conjunction and disjunction; it does *not* apply to implication.

Associativity states that the truth value of a conjunctive or disjunctive statement is unaffected by the placement of parentheses when the same operator is used throughout. In other words, the way in which the component propositions are grouped, or associated with one another, can be changed without affecting the truth value. The validity of this rule is quite easy to see, but if there is any doubt about it, it may be readily checked by means of a truth table. You may recall that the associativity rule also applies to addition and multiplication and asserts, for example, that $3 + (5 + 7)$ equals $(3 + 5) + 7$, and that $2 \times (3 \times 4)$ equals $(2 \times 3) \times 4$. But it does *not* apply to division: $(8 \div 4) \div 2$ does not equal $8 \div (4 \div 2)$. Analogously, in logic, the associativity rule applies only to conjunctive and disjunctive statements; it does *not* apply to conditional statements. Also note, when applying this rule, that the order of the letters remains unchanged; only the placement of the parentheses changes.

Distribution, like De Morgan's rule, applies only to conjunctive and disjunctive statements. When a proposition is conjoined to a disjunctive statement in parentheses or disjoined to a conjunctive statement in parentheses, the rule allows us to put that proposition together with each of the components inside the parentheses, and also to go in the reverse direction. In the first form of the rule, a statement is distributed through a disjunction, and in the second form, through a conjunction. While the rule may not be immediately obvious, it is easy to remember: The operator that is at first outside the parentheses goes inside, and the operator that is at first inside the parentheses goes outside. Note also how distribution differs from commutativity and associativity. The latter two rules apply only when the *same* operator (either a dot or a wedge) is used throughout a statement. Distribution applies when a dot and a wedge appear *together* in a statement.

Double negation is fairly obvious and needs little explanation. The rule states simply that pairs of tildes immediately adjacent to one another may be either deleted or introduced without affecting the truth value of the statement.

There is an important difference between the rules of implication, treated in the first two sections of this chapter, and the rules of replacement. The rules of implication derive their name from the fact that each is a simple argument form in which the premises imply the conclusion. To be applicable in natural deduction, certain lines in a proof must be interpreted as substitution instances of the argument form in question. Stated another way, the rules of implication are applicable only to *whole lines* in a proof. For example, step 3 in the following proof is not a legitimate application of *modus ponens*, because the first premise in the *modus ponens* rule is applied to only a *part* of line 1.

1. $A \supset (B \supset C)$
2. B
3. C 1, 2, MP (invalid)

The rules of replacement, on the other hand, are not rules of implication but rules of logical equivalence. Since, by the axiom of replacement, logically equivalent statement forms can always replace one another in a proof sequence, the rules of replacement can be applied either to a whole line or to any part of a line. Step 2 in the following proof is a quite legitimate application of De Morgan's rule, even though the rule is applied only to the consequent of line 1:

1. $S \supset \sim(T \bullet U)$	
2. $S \supset (\sim T \vee \sim U)$	1, DM (valid)

Another way of viewing this distinction is that the rules of implication are "one-way" rules, whereas the rules of replacement are "two-way" rules. The rules of implication allow us to proceed only from the premise lines of a rule to the conclusion line, but the rules of replacement allow us to replace either side of an equivalence expression with the other side.

Application of the first five rules of replacement may now be illustrated. Consider the following argument:

1. $A \supset \sim(B \bullet C)$	
2. $A \bullet C$	/ $\sim B$

Examining the premises, we find B in the consequent of line 1. This leads us to suspect that the conclusion can be derived via *modus ponens*. If this is correct, the tilde would then have to be taken inside the parentheses via De Morgan's rule and the resulting $\sim C$ eliminated by disjunctive syllogism. The following completed proof indicates that this strategy yields the anticipated result:

1. $A \supset \sim(B \bullet C)$	
2. $A \bullet C$	/ $\sim B$
3. A	2, Simp
4. $\sim(B \bullet C)$	1, 3, MP
5. $\sim B \vee \sim C$	4, DM
6. $C \bullet A$	2, Com
7. C	6, Simp
8. $\sim\sim C$	7, DN
9. $\sim C \vee \sim B$	5, Com
10. $\sim B$	8, 9, DS

The rationale for line 6 is to get C on the left side so that it can be separated via simplification. Similarly, the rationale for line 9 is to get $\sim C$ on the left side so that it can be eliminated via disjunctive syllogism. Line 8 is required because, strictly speaking, the negation of $\sim C$ is $\sim\sim C$—not simply C. Thus, C must be replaced with $\sim\sim C$ to set up the disjunctive syllogism. If your instructor permits it, you can combine commutativity and double negation with other inferences on a single line, as the following shortened proof illustrates. However, we will avoid this practice throughout the remainder of the book.

1. $A \supset \sim(B \bullet C)$	
2. $A \bullet C$	/ $\sim B$
3. A	2, Simp
4. $\sim(B \bullet C)$	1, 3, MP
5. $\sim B \vee \sim C$	4, DM
6. C	2, Com, Simp
7. $\sim B$	5, 6, Com, DN, DS

Another example:

1. $D \cdot (E \vee F)$
2. $\sim D \vee \sim F$ $\quad$ / $D \cdot E$

The conclusion requires that we get D and E together. Inspection of the first premise suggests distribution as the first step in achieving this. The completed proof is as follows:

1.	$D \cdot (E \vee F)$	
2.	$\sim D \vee \sim F$	/ $D \cdot E$
3.	$(D \cdot E) \vee (D \cdot F)$	1, Dist
4.	$(D \cdot F) \vee (D \cdot E)$	3, Com
5.	$\sim(D \cdot F)$	2, DM
6.	$D \cdot E$	4, 5, DS

Some proofs require that we use distribution in the reverse manner. Consider this argument:

1. $(G \cdot H) \vee (G \cdot J)$
2. $(G \vee K) \supset L$ $\quad$ / L

The conclusion can be obtained from line 2 via *modus ponens* if we first obtain $G \vee K$ on a line by itself. Since K does not occur in the first premise at all, it must be introduced by addition. Doing this requires in turn that we obtain G on a line by itself. Distribution applied to line 1 provides the solution:

1.	$(G \cdot H) \vee (G \cdot J)$	
2.	$(G \vee K) \supset L$	/ L
3.	$G \cdot (H \vee J)$	1, Dist
4.	G	3, Simp
5.	$G \vee K$	4, Add
6.	L	2, 5, MP

Application of the associativity rule is illustrated in the next proof:

1.	$M \vee (N \vee O)$	
2.	$\sim O$	/ $M \vee N$
3.	$(M \vee N) \vee O$	1, Assoc
4.	$O \vee (M \vee N)$	3, Com
5.	$M \vee N$	2, 4, DS

Before O can be eliminated via disjunctive syllogism from line 1, it must be moved over to the left side. Associativity and commutativity together accomplish this objective.

In some arguments the attempt to "find" the conclusion in the premises is not immediately successful. When confronted with such an argument, one should often begin by "deconstructing" the conclusion using the rules of replacement. In other words, one should first apply the rules of replacement to the conclusion to see how it is put together. After this is done, how the premises entail the conclusion may be evident. This procedure is justified by the fact that the rules of replacement are two-way rules. As a result, after the conclusion is deconstructed, it can be derived by using the same rules in reverse order. Here is an example of such an argument:

1. $K \supset (F \vee B)$
2. $G \bullet K$ $\quad$ / $(F \bullet G) \vee (B \bullet G)$

If immediate inspection does not reveal how the conclusion should be derived, we may begin by applying the rules of replacement to the conclusion. The form of the conclusion suggests the distribution rule, but first we must use commutativity to move the *G*'s to the left-hand side. The deconstruction proceeds as follows:

$(F \bullet G) \vee (B \bullet G)$	
$(G \bullet F) \vee (B \bullet G)$	Com
$(G \bullet F) \vee (G \bullet B)$	Com
$G \bullet (F \vee B)$	Dist

Now we see that if we can obtain *G* on a line by itself, and $F \vee B$ on a line by itself, we can combine them on a single line via the conjunction rule. We can then derive the conclusion via distribution and commutativity. Inspection of the premises reveals that *G* can be derived from line 2 of the premises by simplification, and $F \vee B$ can be derived from line 1 by *modus ponens*. The completed proof is as follows:

1. $K \supset (F \vee B)$	
2. $G \bullet K$	/ $(F \bullet G) \vee (B \bullet G)$
3. G	2, Simp
4. $K \bullet G$	2, Com
5. K	4, Simp
6. $F \vee B$	1, 5, MP
7. $G \bullet (F \vee B)$	3, 6, Conj
8. $(G \bullet F) \vee (G \bullet B)$	7, Dist
9. $(F \bullet G) \vee (G \bullet B)$	8, Com
10. $(F \bullet G) \vee (B \bullet G)$	9, Com

Here are some strategies for applying the first five rules of replacement. Most of them show how these rules may be used together with other rules.

Strategy 11: Conjunction can be used to set up De Morgan's rule:

1. $\sim A$	
2. $\sim B$	
3. $\sim A \bullet \sim B$	1, 2, Conj
4. $\sim(A \vee B)$	3, DM

Strategy 12: Constructive dilemma can be used to set up De Morgan's rule:

1. $(A \supset \sim B) \bullet (C \supset \sim D)$	
2. $A \vee C$	
3. $\sim B \vee \sim D$	1, 2, CD
4. $\sim(B \bullet D)$	3, DM

Strategy 13: Addition can be used to set up De Morgan's rule:

1. $\sim A$	
2. $\sim A \vee \sim B$	1, Add
3. $\sim(A \bullet B)$	2, DM

7

Willard Van Orman Quine 1908–2000

AP Photo/Julia Malakie

Prior to his death in the year 2000, Willard Van Orman Quine was widely considered to be, as Stuart Hampshire put it, "the most distinguished and influential of living philosophers." At that time, over 2000 scholarly articles had been written about his work.

Quine was born in Akron, Ohio, in 1908 to a father who founded a heavy equipment company and a mother who taught elementary school. He earned his bachelor's degree in mathematics from Oberlin College, where he graduated *summa cum laude* in 1930. He then entered Harvard University, where he switched to philosophy so he could study under Alfred North Whitehead. He earned his Ph.D. in a record two years. Except for four years during World War II, when he served in the Navy decoding messages from German submarines, Quine remained affiliated with Harvard for the remainder of his life.

Quine wrote twenty-two books, the first five of which dealt with mathematical logic. One of the goals of the earlier books was to show how the foundations of mathematics could be laid out in less than a fourth of the space taken by Whitehead and Russell's *Principia Mathematica*. One of his most famous publications was "Two Dogmas of Empiricism," which shook the pillars of analytic philosophy by undermining the sacrosanct distinction between analytic and synthetic statements. As a result of this work, even the truths of logic and mathematics became subject to the dictates of empirical experience.

As a boy, Quine had a fascination with collecting stamps and drawing maps, which, as an adult, he translated into a zest for world travel. He visited 118 countries, became fluent in six different languages, delivered lectures all over the world, and was awarded the first Schock Prize (Stockholm, 1993) and the Kyoto Prize (Tokyo, 1996). He was married twice, raised two children from each marriage, loved Dixieland jazz, and played the banjo, mandolin, and piano. He was singularly unpretentious, had an unfailing curiosity about a vast range of topics, and delighted in teaching freshman logic as well as advanced courses in philosophy. He died in Boston at the age of 92.

Strategy 14: Distribution can be used in two ways to set up disjunctive syllogism:

1.	$(A \vee B) \cdot (A \vee C)$	
2.	$\sim A$	
3.	$A \vee (B \cdot C)$	1, Dist
4.	$B \cdot C$	2, 3, DS

1.	$A \cdot (B \vee C)$	
2.	$\sim(A \cdot B)$	
3.	$(A \cdot B) \vee (A \cdot C)$	1, Dist
4.	$A \cdot C$	2, 3, DS

Strategy 15: Distribution can be used in two ways to set up simplification:

1. $A \vee (B \bullet C)$
2. $(A \vee B) \bullet (A \vee C)$ — 1, Dist
3. $A \vee B$ — 2, Simp

1. $(A \bullet B) \vee (A \bullet C)$
2. $A \bullet (B \vee C)$ — 1, Dist
3. A — 2, Simp

Strategy 16: If inspection of the premises does not reveal how the conclusion should be derived, consider using the rules of replacement to deconstruct the conclusion. (See the final example in this section.)

Exercise 7.3

I. For each of the following lists of premises, derive the indicated conclusion and complete the justification. For double negation, avoid the occurrence of triple tildes.

★(1) 1. $\sim(E \supset H)$
2. $\sim(N \vee G)$
3. $\sim A \vee D$
4. ________ _____, DM

(2) 1. $G \supset (N \supset K)$
2. $R \vee (D \supset F)$
3. $S \bullet (T \vee U)$
4. ____________ _____, Dist

(3) 1. $M \vee (G \vee T)$
2. $P \bullet (S \supset N)$
3. $D \bullet (R \vee K)$
4. ____________ _____, Assoc

★(4) 1. $B \supset W$
2. $G \equiv F$
3. $S \bullet A$
4. _______ _____, Com

(5) 1. $\sim\sim R \vee T$
2. $\sim N \vee \sim B$
3. $\sim A \supset \sim H$
4. __________ _____, DN

(6) 1. $(F \vee N) \vee (K \bullet D)$
2. $(H \bullet Z) \vee (H \bullet W)$
3. $(P \supset H) \vee (P \supset N)$
4. __________________ _____, Dist

★(7) 1. $\sim(G \bullet \sim Q)$
2. $\sim(K \equiv \sim B)$
3. $\sim T \supset \sim F$
4. __________ _____, DM

(8) 1. $G \supset (\sim L \supset T)$
2. $L \equiv (\sim R \supset \sim C)$
3. $J \supset (S \vee \sim N)$
4. ____________ _____, Com

(9) 1. $S \supset (M \supset D)$
2. $(K \bullet G) \vee B$
3. $(E \bullet H) \bullet Q$
4. ___________ _____, Assoc

★(10) 1. $\sim R \vee \sim P$
2. $\sim F \supset \sim W$
3. $G \bullet \sim A$
4. _________ _____, DM

(11) 1. $\sim B \vee E$
2. $\sim E \bullet \sim A$
3. $\sim C \supset \sim R$
4. _________ _____, DN

(12) 1. $\sim G \bullet (S \supset A)$
2. $\sim S \supset (B \bullet K)$
3. $\sim Q \vee (T \bullet R)$
4. ___________ _____, Dist

★(13) 1. $F \supset (\sim S \vee M)$
2. $H \supset (\sim L \bullet \sim D)$
3. $N \supset (\sim G \supset \sim C)$
4. ___________ _____, DM

(14) 1. $F \supset (P \supset \sim E)$
2. $C \vee (S \bullet \sim B)$
3. $M \bullet (R \bullet \sim T)$
4. ____________ _____, Assoc

(15) 1. $(D \vee \sim K) \bullet (D \vee \sim W)$
2. $(S \vee \sim Z) \vee (P \vee \sim T)$
3. $(Q \supset \sim N) \bullet (Q \supset \sim F)$
4. ___________________ _____, Dist

II. In the following symbolized arguments, derive the line needed to obtain the conclusion (last line), and supply the justification for both lines.

★(1) 1. $K \vee C$
2. $\sim C$
3. _____ _____
4. K _____

(2) 1. $G \supset (R \vee N)$
2. $\sim R \bullet \sim N$
3. ___________ _____
4. $\sim G$ _____

(3) 1. $H \cdot T$
2. ________ ______
3. T ______

★(4) 1. $(L \cdot S) \cdot F$
2. ____________ ______
3. L ______

(5) 1. $\sim B \vee K$
2. ____________ ______
3. $\sim(B \cdot \sim K)$ ______

(6) 1. $C \supset \sim A$
2. A
3. ________ ______
4. $\sim C$ ______

★(7) 1. $(D \cdot M) \vee (D \cdot N)$
2. ________________ ______
3. D ______

(8) 1. $(U \vee T) \supset R$
2. $T \vee U$
3. ____________ ______
4. R ______

(9) 1. $\sim L \vee M$
2. L
3. ________ ______
4. M ______

★(10) 1. $D \vee (N \cdot H)$
2. ____________ ______
3. $D \vee N$ ______

(11) 1. $(K \vee E) \cdot (K \vee G)$
2. $\sim K$
3. ________________ ______
4. $E \cdot G$ ______

(12) 1. $(N \supset T) \cdot (F \supset Q)$
2. $F \vee N$
3. ________________ ______
4. $T \vee Q$ ______

★(13) 1. $(M \vee G) \vee T$
2. $\sim M$
3. ____________ ______
4. $G \vee T$ ______

(14) 1. $(\sim A \supset T) \cdot (\sim S \supset K)$
2. $\sim(A \cdot S)$
3. ________________ ______
4. $T \vee K$ ______

(15) 1. $\sim R$
2. ________ ______
3. $\sim(R \cdot T)$ ______

III. Use the first thirteen rules of inference to derive the conclusions of the following symbolized arguments.

★(1) 1. $(\sim M \supset P) \cdot (\sim N \supset Q)$
2. $\sim(M \cdot N)$ / $P \vee Q$

(2) 1. $\sim S$ / $\sim(F \cdot S)$

(3) 1. $J \vee (K \cdot L)$
2. $\sim K$ / J

★(4) 1. $\sim(N \cdot T)$
2. T / $\sim N$

(5) 1. $H \supset \sim A$
2. A / $\sim(H \vee \sim A)$

(6) 1. $R \supset \sim B$
2. $D \vee R$
3. B / D

★(7) 1. $T \supset (B \vee E)$
2. $\sim E \cdot T$ / B

(8) 1. $(O \vee M) \supset S$
2. $\sim S$ / $\sim M$

(9) 1. $Q \vee (L \vee C)$
2. $\sim C$ / $L \vee Q$

★(10) 1. $(K \cdot H) \vee (K \cdot L)$
2. $\sim L$ / H

(11) 1. $\sim(\sim E \cdot \sim N) \supset T$
2. $G \supset (N \vee E)$ / $G \supset T$

(12) 1. $H \cdot (C \cdot T)$
2. $\sim(\sim F \cdot T)$ / F

★(13) 1. $(E \cdot I) \vee (M \cdot U)$
2. $\sim E$ / $\sim(E \vee \sim M)$

(14) 1. $\sim(J \vee K)$
2. $B \supset K$
3. $S \supset B$ / $\sim S \cdot \sim J$

(15) 1. $(G \cdot H) \vee (M \cdot G)$
2. $G \supset (T \cdot A)$ / A

★(16) 1. $(Q \bullet N) \vee (N \bullet T)$
2. $(Q \vee C) \supset {\sim}N$ / T

(17) 1. ${\sim}(U \vee R)$
2. $({\sim}R \vee N) \supset (P \bullet H)$
3. $Q \supset {\sim}H$ / ${\sim}Q$

(18) 1. ${\sim}(F \bullet A)$
2. ${\sim}(L \vee {\sim}A)$
3. $D \supset (F \vee L)$ / ${\sim}D$

★(19) 1. $[(I \vee M) \vee G] \supset {\sim}G$
2. $M \vee G$ / M

(20) 1. $E \supset {\sim}B$
2. $U \supset {\sim}C$
3. ${\sim}({\sim}E \bullet {\sim}U)$ / ${\sim}(B \bullet C)$

(21) 1. ${\sim}(K \vee F)$
2. ${\sim}F \supset (K \vee C)$
3. $(G \vee C) \supset {\sim}H$ / ${\sim}(K \vee H)$

★(22) 1. $S \vee (I \bullet {\sim}J)$
2. $S \supset {\sim}R$
3. ${\sim}J \supset {\sim}Q$ / ${\sim}(R \bullet Q)$

(23) 1. $(J \vee F) \vee M$
2. $(J \vee M) \supset {\sim}P$
3. ${\sim}F$ / ${\sim}(F \vee P)$

(24) 1. $(K \bullet P) \vee (K \bullet Q)$
2. $P \supset {\sim}K$ / $Q \vee T$

★(25) 1. $E \vee {\sim}(D \vee C)$
2. $(E \vee {\sim}D) \supset C$ / E

(26) 1. $A \bullet (F \bullet L)$
2. $A \supset (U \vee W)$
3. $F \supset (U \vee X)$ / $U \vee (W \bullet X)$

(27) 1. $(T \bullet R) \supset P$
2. $({\sim}P \bullet R) \bullet G$
3. $({\sim}T \vee N) \supset H$ / H

★(28) 1. $P \vee (I \bullet L)$
2. $(P \vee I) \supset {\sim}(L \vee C)$
3. $(P \bullet {\sim}C) \supset (E \bullet F)$ / $F \vee D$

(29) 1. $B \vee (S \bullet N)$
2. $B \supset {\sim}S$
3. $S \supset {\sim}N$ / $B \vee W$

(30) 1. $({\sim}M \vee E) \supset (S \supset U)$
2. $({\sim}Q \vee E) \supset (U \supset H)$
3. ${\sim}(M \vee Q)$ / $S \supset H$

★(31) 1. $({\sim}R \vee D) \supset {\sim}(F \bullet G)$
2. $(F \bullet R) \supset S$
3. $F \bullet {\sim}S$ / ${\sim}(S \vee G)$

(32) 1. ${\sim}Q \supset (C \bullet B)$
2. ${\sim}T \supset (B \bullet H)$
3. ${\sim}(Q \bullet T)$ / B

(33) 1. ${\sim}(A \bullet G)$
2. ${\sim}(A \bullet E)$
3. $G \vee E$ / ${\sim}(A \bullet F)$

★(34) 1. $(M \bullet N) \vee (O \bullet P)$
2. $(N \vee O) \supset {\sim}P$ / N

(35) 1. $(T \bullet K) \vee (C \bullet E)$
2. $K \supset {\sim}E$
3. $E \supset {\sim}C$ / $T \bullet K$

IV. Translate the following arguments into symbolic form and then use the first thirteen rules of inference to derive the conclusion of each. Use the translation letters in the order in which they are listed.

★**1.** Either health care costs are skyrocketing and they are attributable to greedy doctors, or health care costs are skyrocketing and they are attributable to greedy hospitals. If health care costs are skyrocketing, then both the government should intercede and health care may have to be rationed. Therefore, health care costs are skyrocketing and health care may have to be rationed. (*S*, *D*, *H*, *I*, *R*)

2. Either the ancient Etruscans were experienced city planners and they invented the art of writing or they were highly skilled engineers and they invented the art of writing. If the ancient Etruscans were bloodthirsty numskulls (as scholars once thought), they did not invent the art of writing. Therefore, the ancient

Etruscans were not bloodthirsty numskulls (as scholars once thought). (*C*, *I*, *H*, *B*)

3. It is not the case that either the earth's molten core is stationary or that it contains no iron. If it is not the case that both the earth's molten core is stationary and has a regular topography, then either the earth's core contains no iron or the direction of the earth's magnetic field is subject to change. Therefore, the direction of the earth's magnetic field is subject to change. (*S*, *C*, *R*, *D*)

★4. Either mosquito genes can be cloned or mosquitoes will become resistant to all insecticides and the incidence of encephalitis will increase. If either mosquito genes can be cloned or the incidence of encephalitis increases, then mosquitoes will not become resistant to all insecticides. Therefore, either mosquito genes can be cloned or mosquitoes will multiply out of control. (*G*, *R*, *E*, *M*)

5. Protein engineering will prove to be as successful as genetic engineering, and new enzymes will be developed for producing food and breaking down industrial wastes. If protein engineering proves to be as successful as genetic engineering and new enzymes are developed for breaking down industrial wastes, then it is not the case that new enzymes will be developed for producing food but not medicines. Therefore, protein engineering will prove to be as successful as genetic engineering and new enzymes will be developed for producing medicines. (*E*, *P*, *B*, *M*)

6. If workers have a fundamental right to a job, then unemployment will be virtually nonexistent but job redundancy will become a problem. If workers have no fundamental right to a job, then production efficiency will be maximized but job security will be jeopardized. Workers either have or do not have a fundamental right to a job. Therefore, either unemployment will be virtually nonexistent or production efficiency will be maximized. (*F*, *U*, *R*, *P*, *S*)

★7. If Japan is to reduce its huge trade surplus, then it must either convince its citizens to spend more or it must move its manufacturing facilities to other countries. It is not the case that Japan will either increase its imports or convince its citizens to spend more. Furthermore, it is not the case that Japan will either allow foreign construction companies to compete on an equal footing or move its manufacturing facilities to other countries. Therefore, Japan will not reduce its huge trade surplus. (*R*, *C*, *M*, *I*, *A*)

8. If women are by nature either passive or uncompetitive, then it is not the case that there are lawyers who are women. If men are by nature either insensitive or without the ability to nurture, then it is not the case that there are kindergarten teachers who are men. There are lawyers who are women and kindergarten teachers who are men. Therefore, it is not the case that either women by nature are uncompetitive or men by nature are without the ability to nurture. (*P*, *U*, *L*, *I*, *W*, *K*)

9. It is not the case that either the sun's interior rotates faster than its surface or Einstein's general theory of relativity is wrong. If the sun's interior does not rotate faster than its surface and eccentricities in the orbit of Mercury can be

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explained by solar gravitation, then Einstein's general theory of relativity is wrong. Therefore, eccentricities in the orbit of Mercury cannot be explained by solar gravitation. (*S*, *E*, *M*)

★**10.** Either school dropout programs are not as effective as they could be, or they provide basic thinking skills and psychological counseling to their students. Either school dropout programs are not as effective as they could be, or they adequately prepare their students for getting a job and working effectively with others. Either school dropout programs do not provide psychological counseling to their students or they do not provide adequate preparation for working effectively with others. Therefore, school dropout programs are not as effective as they could be. (*E*, *B*, *P*, *G*, *W*)

V. The following dialogue contains eight arguments. Translate each into symbolic form and then use the first thirteen rules of inference to derive the conclusion of each.

With This Ring

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"Hi. I didn't expect to see you here," says Ken as he catches sight of Gina on the steps of the church. "You must be friends with the bride."

"I am," she says, "and are you a friend of the groom?"

"A friend of a friend of the groom," he replies. "So I don't know too many people here."

"Well, I'll be happy to keep you company until the ceremony starts," Gina says. "And it looks like things are running late, so we'll have a few minutes."

"Every time I attend a wedding," Gina continues, "I feel sad for a lesbian couple I know who would give just about anything to get married. Unfortunately this state doesn't allow same-sex marriage."

"Well, I don't think that's unfortunate," says Ken. "If marriage is sacred, then we shouldn't tamper with it; and if that's the case, then we shouldn't allow same-sex marriage. And I do think marriage is sacred, so we shouldn't allow same-sex marriage."

"Also," Ken continues, "the Bible condemns homosexuality. If either Leviticus or Romans is true, then homosexuality is an abomination and it must be avoided; and if it must be avoided or it's contrary to nature, then if it's a sin, then same-sex marriage must not be allowed. Now it's certainly the case that Romans is true, and if the Bible condemns homosexuality, then it's a sin. Thus, same-sex marriage must not be allowed."

"Obviously you're injecting religion into the issue," Gina responds, as she waves to a friend in the gathering crowd. "But as you know the First Amendment to the Constitution says that the state must not act either to establish a religion or to interfere with religious practices. But if the state bars same-sex marriage for your reasons, it acts to establish a religion. Thus, it must not bar same-sex marriage for your reasons."

"Also," Gina continues, "our country is based on the principle of equality. If straight couples can get married, and obviously they can, then either same-sex couples can get married or same-sex couples are not equal to straight couples. And if same-sex couples can get married, then our law has to change and other state laws have to change as well. Now if our country is based on the principle of equality, then same-sex couples are equal to straight couples. Thus, our law has to change."

"Okay," says Ken, "let's look at this another way. Marriage has always been between a man and a woman. And if that is so and tradition is worth preserving, then if we allow same-sex marriage, then the very concept of marriage will change and gender roles will switch. Now tradition is worth preserving and gender roles must not switch. Thus, we cannot allow same-sex marriage."

"Ha!" says Gina. "I can see why you don't want gender roles to switch. You can't see yourself in the kitchen preparing meals and washing dishes."

"Well, I don't really relish the idea," Ken says. "I think God wants us to keep things the way they are. But here's another reason. One of the chief purposes of marriage is raising children, and if that is so, then it's important that the children grow up well adjusted. But if the children are to be well adjusted, then they must have both a male role model and a female role model. But if the parents are both men, then the children will have no female role model, and if they are both women, then they will have no male role model. Clearly if the marriage is a same-sex marriage, then the parents are either both men or they are both women. Therefore, the marriage must not be a same-sex marriage."

"Your reasoning is a bit shortsighted," Gina says. "In a same-sex marriage with children, the parents are either both men or they are both women. This much I grant you. But if they are both men, then surely they have close female friends, and if that is so, then the marriage has both male and female role models. If the parents are both women, then surely they have close male friends, and if that is so, then the marriage has both female and male role models. If a marriage has both male and female role models, then the children will be well adjusted. Therefore, in a same-sex marriage with children, the children will be well adjusted. What do you think of that?"

"Well," says Ken, as he scratches his head, "I wonder if those surrogate role models would be as effective as male and female parents. But in either event there is the option of civil unions. Why won't that satisfy you?"

"There are many ways that civil unions fall short of marriage," Gina replies. "Three of them are that they are valid only in the state in which they are performed, and they do not allow the partners either to file a joint federal tax return or to receive social security survivor benefits. If they are valid only in the state in which they are performed, then if the partners move to a different state, then if one partner is hospitalized, the other partner may have no visitation rights. If the partners cannot file a joint federal tax return, then they must file as single taxpayers. And if that is so, then they might pay much more in taxes. If the partners do not receive social security survivor benefits, then if a partner receiving social security benefits dies, then the other will not receive anything as a survivor. Now let's suppose for the sake of the argument that two partners in a civil union move to a different state, and that one partner, who receives social security benefits, is hospitalized and eventually dies. The conclusion is that the other partner will not have either visitation rights while the hospitalized partner is alive or survivor benefits after that partner dies, and the partners might pay much more in taxes. Does that seem like a fair substitute for marriage?"

"Okay, I can see your point," Ken says, as he stretches to see above the crowd. "But I still think there's something unnatural about same-sex marriage. Anyway, I see the bride and groom have finally arrived, so let's go inside."

"Good, let's go," Gina replies, as they both turn toward the door of the church.

7.4 Rules of Replacement II

The remaining five rules of replacement are as follows:

14. Transposition (Trans):
    $(p \supset q) :: (\sim q \supset \sim p)$
15. Material implication (Impl):
    $(p \supset q) :: (\sim p \vee q)$
16. Material equivalence (Equiv):
    $(p \equiv q) :: [(p \supset q) \bullet (q \supset p)]$
    $(p \equiv q) :: [(p \bullet q) \vee (\sim p \bullet \sim q)]$
17. Exportation (Exp):
    $[(p \bullet q) \supset r] :: [p \supset (q \supset r)]$
18. Tautology (Taut):
    $p :: (p \vee p)$
    $p :: (p \bullet p)$

Transposition asserts that the antecedent and consequent of a conditional statement may switch places if and only if tildes are inserted before both or tildes are removed from both. The rule is fairly easy to understand and is easily proved by a truth table.

Material implication is less obvious than transposition, but it can be illustrated by substituting actual statements in place of the letters. For example, the statement "If you bother me, then I'll punch you in the nose" ($B \supset P$) is logically equivalent to "Either you stop bothering me or I'll punch you in the nose" ($\sim B \vee P$). The rule states that a horseshoe may be replaced by a wedge if the left-hand component is negated, and the reverse replacement is allowed if a tilde is deleted from the left-hand component.

Material equivalence has two formulations. The first is the same as the definition of material equivalence given in Section 6.1. The second formulation is easy to remember through recalling the two ways in which $p \equiv q$ may be true. Either p and q are both true or p and q are both false. This, of course, is the meaning of $[(p \bullet q) \vee (\sim p \bullet \sim q)]$.

Exportation is also fairly easy to understand. It asserts that the statement "If we have both p and q, then we have r" is logically equivalent to "If we have p, then if we have q, then we have r." As an illustration of this rule, the statement "If Bob and Sue told the truth, then Jim is guilty" is logically equivalent to "If Bob told the truth, then if Sue told the truth, then Jim is guilty."

Tautology, the last rule introduced in this section, is obvious. Its effect is to eliminate redundancy in disjunctions and conjunctions.

The following proofs illustrate the use of these five rules.

1. $\sim A$	/ $A \supset B$

In this argument the conclusion contains a letter not found in the premise. Obviously, addition must be used to introduce the B. The material implication rule completes the proof:

1. $\sim A$	/ $A \supset B$
2. $\sim A \vee B$	1, Add
3. $A \supset B$	2, Impl

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Here is another example:

1.	$F \supset G$	
2.	$F \vee G$	/ G

To derive the conclusion of this argument, some method must be found to link the two premises together and eliminate the *F*. Hypothetical syllogism provides the solution, but first the second premise must be converted into a conditional. Here is the proof:

1.	$F \supset G$	
2.	$F \vee G$	/ G
3.	$\sim\sim F \vee G$	2, DN
4.	$\sim F \supset G$	3, Impl
5.	$\sim F \supset \sim\sim G$	4, DN
6.	$\sim G \supset F$	5, Trans
7.	$\sim G \supset G$	1, 6, HS
8.	$\sim\sim G \vee G$	7, Impl
9.	$G \vee G$	8, DN
10.	G	9, Taut

Another example:

1.	$J \supset (K \supset L)$	/ $K \supset (J \supset L)$

The conclusion can be obtained by simply rearranging the components of the single premise. Exportation provides the simplest method:

1.	$J \supset (K \supset L)$	/ $K \supset (J \supset L)$
2.	$(J \cdot K) \supset L$	1, Exp
3.	$(K \cdot J) \supset L$	2, Com
4.	$K \supset (J \supset L)$	3, Exp

Another example:

1.	$M \supset N$	
2.	$M \supset O$	/ $M \supset (N \cdot O)$

As with the *F* and *G* example, some method must be found to link the two premises together. In this case, however, hypothetical syllogism will not work. The solution lies in setting up a distribution step:

1.	$M \supset N$	
2.	$M \supset O$	/ $M \supset (N \cdot O)$
3.	$\sim M \vee N$	1, Impl
4.	$\sim M \vee O$	2, Impl
5.	$(\sim M \vee N) \cdot (\sim M \vee O)$	3, 4, Conj
6.	$\sim M \vee (N \cdot O)$	5, Dist
7.	$M \supset (N \cdot O)$	6, Impl

Another example:

1.	$P \supset Q$	
2.	$R \supset (S \cdot T)$	
3.	$\sim R \supset \sim Q$	
4.	$S \supset (T \supset P)$	/ $P \equiv R$

The conclusion is a biconditional, and there are only two ways that a biconditional can be obtained from such premises—namely, via the two formulations of the material equivalence rule. The fact that the premises are all conditional statements suggests the first formulation of this rule. Accordingly, we must try to obtain $P \supset R$ and $R \supset P$. Again, the fact that the premises are themselves conditionals suggests hypothetical syllogism to accomplish this. Premises 1 and 3 can be used to set up one hypothetical syllogism; premises 2 and 4 provide the other. Here is the proof:

1.	$P \supset Q$	
2.	$R \supset (S \cdot T)$	
3.	$\sim R \supset \sim Q$	
4.	$S \supset (T \supset P)$	/ $P \equiv R$
5.	$Q \supset R$	3, Trans
6.	$P \supset R$	1, 5, HS
7.	$(S \cdot T) \supset P$	4, Exp
8.	$R \supset P$	2, 7, HS
9.	$(P \supset R) \cdot (R \supset P)$	6, 8, Conj
10.	$P \equiv R$	9, Equiv

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As we saw in Section 7.3, if it is not readily apparent how the conclusion should be derived, we can use the rules of replacement to deconstruct the conclusion. This will usually provide insight on how best to proceed. Again, this technique is justified because the rules of replacement are two-way rules. As a result, they can be applied in reverse order in the completed proof. Here is an example:

1.	$\sim S \supset K$	
2.	$S \supset (R \vee M)$	/ $\sim R \supset (\sim M \supset K)$

In deconstructing the conclusion, the form of the conclusion suggests exportation, and the result of this step suggests De Morgan's rule. For further insight, we apply transposition to the latter step. Each step follows from the one preceding it:

$\sim R \supset (\sim M \supset K)$	
$(\sim R \cdot \sim M) \supset K$	Exp
$\sim(R \vee M) \supset K$	DM
$\sim K \supset \sim\sim(R \vee M)$	Trans
$\sim K \supset (R \vee M)$	DN

Now, examining the premises in light of the deconstruction suggests that we begin by setting up a hypothetical syllogism. This will give us the last step in the deconstruction. We can then obtain the conclusion by repeating the deconstruction steps in reverse order. The completed proof is as follows:

1. $\sim S \supset K$
2. $S \supset (R \vee M)$ $\quad / \sim R \supset (\sim M \supset K)$
3. $\sim K \supset \sim\sim S$ $\quad$ 1, Trans
4. $\sim K \supset S$ $\quad$ 3, DN
5. $\sim K \supset (R \vee M)$ $\quad$ 2, 4, HS
6. $\sim(R \vee M) \supset \sim\sim K$ $\quad$ 5, Trans
7. $\sim(R \vee M) \supset K$ $\quad$ 6, DN
8. $(\sim R \cdot \sim M) \supset K$ $\quad$ 7, DM
9. $\sim R \supset (\sim M \supset K)$ $\quad$ 8, Exp

Here is another example:

1. $K \supset M$
2. $L \supset M$ $\quad / \ (K \vee L) \supset M$

In deconstructing the conclusion, the form of the premises suggests that we use some procedure that will combine *M* separately with *K* and *L*. This, in turn, suggests distribution; but before we can use distribution, we must eliminate the horseshoe via material implication. The deconstruction is as follows:

$(K \vee L) \supset M$	
$\sim(K \vee L) \vee M$	Impl
$(\sim K \cdot \sim L) \vee M$	DM
$M \vee (\sim K \cdot \sim L)$	Com
$(M \vee \sim K) \cdot (M \vee \sim L)$	Dist
$(\sim K \vee M) \cdot (M \vee \sim L)$	Com
$(\sim K \vee M) \cdot (\sim L \vee M)$	Com
$(K \supset M) \cdot (\sim L \vee M)$	Impl
$(K \supset M) \cdot (L \supset M)$	Impl

Now, examining the premises in light of the last line of the deconstruction suggests that we begin by joining the premises together via the conjunction rule. The conclusion can then be obtained by reversing the steps of the deconstruction:

1. $K \supset M$
2. $L \supset M$ $\quad / \ (K \vee L) \supset M$
3. $(K \supset M) \cdot (L \supset M)$ $\quad$ 1, 2, Conj
4. $(\sim K \vee M) \cdot (L \supset M)$ $\quad$ 3, Impl
5. $(\sim K \vee M) \cdot (\sim L \vee M)$ $\quad$ 4, Impl
6. $(M \vee \sim K) \cdot (\sim L \vee M)$ $\quad$ 5, Com
7. $(M \vee \sim K) \cdot (M \vee \sim L)$ $\quad$ 6, Com
8. $M \vee (\sim K \cdot \sim L)$ $\quad$ 7, Dist
9. $(\sim K \cdot \sim L) \vee M$ $\quad$ 8, Com
10. $\sim(K \vee L) \vee M$ $\quad$ 9, DM
11. $(K \vee L) \supset M$ $\quad$ 10, Impl

Note that whenever we use this strategy of working backward from the conclusion, the rules of replacement are the *only* rules we may use. We may not use the rules of implication, because these rules are one-way rules.

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This section ends with some strategies that show how the last five rules of replacement can be used together with various other rules.

Strategy 17: Material implication can be used to set up hypothetical syllogism:

1.	$\sim A \vee B$	
2.	$\sim B \vee C$	
3.	$A \supset B$	1, Impl
4.	$B \supset C$	2, Impl
5.	$A \supset C$	3, 4, HS

Strategy 18: Exportation can be used to set up *modus ponens:*

1.	$(A \bullet B) \supset C$	
2.	A	
3.	$A \supset (B \supset C)$	1, Exp
4.	$B \supset C$	2, 3, MP

Strategy 19: Exportation can be used to set up *modus tollens:*

1.	$A \supset (B \supset C)$	
2.	$\sim C$	
3.	$(A \bullet B) \supset C$	1, Exp
4.	$\sim(A \bullet B)$	2, 3, MT

Strategy 20: Addition can be used to set up material implication:

1.	A	
2.	$A \vee \sim B$	1, Add
3.	$\sim B \vee A$	2, Com
4.	$B \supset A$	3, Impl

Strategy 21: Transposition can be used to set up hypothetical syllogism:

1.	$A \supset B$	
2.	$\sim C \supset \sim B$	
3.	$B \supset C$	2, Trans
4.	$A \supset C$	1, 3, HS

Strategy 22: Transposition can be used to set up constructive dilemma:

1.	$(A \supset B) \bullet (C \supset D)$	
2.	$\sim B \vee \sim D$	
3.	$(\sim B \supset \sim A) \bullet (C \supset D)$	1, Trans
4.	$(\sim B \supset \sim A) \bullet (\sim D \supset \sim C)$	3, Trans
5.	$\sim A \vee \sim C$	2, 4, CD

Strategy 23: Constructive dilemma can be used to set up tautology:

1.	$(A \supset C) \bullet (B \supset C)$	
2.	$A \vee B$	
3.	$C \vee C$	1, 2, CD
4.	C	3, Taut

Strategy 24: Material implication can be used to set up tautology:

1.	$A \supset \sim A$	
2.	$\sim A \vee \sim A$	1, Impl
3.	$\sim A$	2, Taut

Strategy 25: Material implication can be used to set up distribution:

1.	$A \supset (B \cdot C)$	
2.	$\sim A \vee (B \cdot C)$	1, Impl
3.	$(\sim A \vee B) \cdot (\sim A \vee C)$	2, Dist

Exercise 7.4

I. For each of the following lists of premises, derive the indicated conclusion and complete the justification.

★(1) 1. $H \vee F$
2. $N \vee \sim S$
3. $\sim G \vee Q$
4. _______ _____, Impl

(2) 1. $R \supset (S \supset N)$
2. $T \supset (U \vee M)$
3. $K \bullet (L \supset W)$
4. ____________ _____, Exp

(3) 1. $G \equiv R$
2. $H \supset P$
3. $\sim F \vee T$
4. _______ _____, Trans

★(4) 1. $(B \supset N) \bullet (N \supset B)$
2. $(R \vee F) \bullet (F \vee R)$
3. $(K \supset C) \vee (C \supset K)$
4. __________________ _____, Equiv

(5) 1. $E \vee \sim E$
2. $A \vee A$
3. $G \bullet \sim G$
4. _______ _____, Taut

(6) 1. $S \vee \sim M$
2. $\sim N \bullet \sim T$
3. $\sim L \supset Q$
4. ________ _____, Trans

★(7) 1. $\sim C \supset \sim F$
2. $D \vee \sim P$
3. $\sim R \bullet Q$
4. _________ _____, Impl

(8) 1. $E \supset (R \bullet Q)$
2. $(G \bullet N) \supset Z$
3. $(S \supset M) \supset P$
4. _____________ _____, Exp

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(9) 1. $(D \bullet H) \lor ({\sim}D \bullet {\sim}H)$
2. $(F \supset J) \bullet ({\sim}F \supset {\sim}J)$
3. $(N \lor T) \bullet ({\sim}N \lor {\sim}T)$
4. ____________________ _____, Equiv

★(10) 1. $L \supset (A \supset A)$
2. $K \supset (R \lor {\sim}R)$
3. $S \supset (G \bullet G)$
4. ____________ _____, Taut

(11) 1. $K \bullet (S \lor B)$
2. ${\sim}F \supset {\sim}J$
3. ${\sim}E \lor {\sim}M$
4. ____________ _____, Trans

(12) 1. $H \supset (K \bullet J)$
2. $(N \lor E) \supset B$
3. $C \supset (H \supset A)$
4. ____________ _____, Exp

★(13) 1. $(A \supset {\sim}C) \bullet (C \supset {\sim}A)$
2. $(W \supset {\sim}T) \bullet ({\sim}T \supset W)$
3. $(M \supset {\sim}E) \bullet ({\sim}M \supset E)$
4. ____________________ _____, Equiv

(14) 1. $({\sim}K \lor M) \equiv S$
2. $T \lor (F \bullet G)$
3. $R \equiv (N \bullet {\sim}H)$
4. ____________ _____, Impl

(15) 1. $(S \lor S) \supset D$
2. $K \supset (T \bullet {\sim}T)$
3. $(Q \supset Q) \supset M$
4. ____________ _____, Taut

II. In the following symbolized arguments, derive the line needed to obtain the conclusion (last line), and supply the justification for both lines.

★(1) 1. ${\sim}J \lor M$
2. $M \supset B$
3. _______ _____
4. $J \supset B$ _____

(2) 1. $(J \bullet F) \supset N$
2. J
3. _______ _____
4. $F \supset N$ _____

(3) 1. $C \supset A$
2. $A \supset C$
3. ______ _____
4. $C \equiv A$ _____

★(4) 1. $(G \supset K) \bullet (T \supset K)$
2. $G \lor T$
3. __________________ _____
4. K _____

(5) 1. $(G \supset B) \bullet ({\sim}C \supset {\sim}H)$
2. $G \lor H$
3. ____________________ _____
4. $B \lor C$ _____

(6) 1. $J \supset (M \supset Q)$
2. $J \bullet M$
3. ____________ _____
4. Q _____

★(7) 1. $H \supset (\sim C \vee R)$
2. ________ ____
3. $(H \bullet C) \supset R$ ____

(8) 1. $\sim G \supset \sim T$
2. $G \supset N$
3. ________ ____
4. $T \supset N$ ____

(9) 1. $K \supset (A \supset F)$
2. $\sim F$
3. ________ ____
4. $\sim(K \bullet A)$ ____

★(10) 1. $H \supset \sim H$
2. ________ ____
3. $\sim H$ ____

(11) 1. $\sim S$
2. ________ ____
3. $S \supset K$ ____

(12) 1. $M \supset (M \supset D)$
2. ________ ____
3. $M \supset D$ ____

★(13) 1. $(N \supset A) \bullet (\sim N \supset \sim A)$
2. ________ ____
3. $N \equiv A$ ____

(14) 1. $E \bullet R$
2. ________ ____
3. $E \equiv R$ ____

(15) 1. $Q \supset (\sim W \supset \sim G)$
2. ________ ____
3. $(Q \bullet G) \supset W$ ____

III. Use the eighteen rules of inference to derive the conclusions of the following symbolized arguments.

★(1) 1. $(S \bullet K) \supset R$
2. K / $S \supset R$

(2) 1. $T \supset (F \vee F)$
2. $\sim(F \bullet F)$ / $\sim T$

(3) 1. $G \supset E$
2. $H \supset \sim E$ / $G \supset \sim H$

★(4) 1. $S \equiv Q$
2. $\sim S$ / $\sim Q$

(5) 1. $\sim N \vee P$
2. $(N \supset P) \supset T$ / T

(6) 1. $F \supset B$
2. $B \supset (B \supset J)$ / $F \supset J$

★(7) 1. $(B \supset M) \bullet (D \supset M)$
2. $B \vee D$ / M

(8) 1. $Q \supset (F \supset A)$
2. $R \supset (A \supset F)$
3. $Q \bullet R$ / $F \equiv A$

(9) 1. $T \supset (\sim T \vee G)$
2. $\sim G$ / $\sim T$

★(10) 1. $(B \supset G) \bullet (F \supset N)$
2. $\sim(G \bullet N)$ / $\sim(B \bullet F)$

(11) 1. $(J \bullet R) \supset H$
2. $(R \supset H) \supset M$
3. $\sim(P \vee \sim J)$ / $M \bullet \sim P$

(12) 1. T / $S \supset T$

★(13) 1. $K \supset (B \supset {\sim}M)$
2. $D \supset (K \bullet M)$ / $D \supset {\sim}B$

(14) 1. $(O \supset C) \bullet ({\sim}S \supset {\sim}D)$
2. $(E \supset D) \bullet ({\sim}E \supset {\sim}C)$ / $O \supset S$

(15) 1. ${\sim}(U \bullet W) \supset X$
2. $U \supset {\sim}U$ / ${\sim}(U \vee {\sim}X)$

★(16) 1. $T \supset R$
2. $T \supset {\sim}R$ / ${\sim}T$

(17) 1. $S \vee {\sim}N$
2. ${\sim}S \vee Q$ / $N \supset Q$

(18) 1. $M \supset (U \supset H)$
2. $(H \vee {\sim}U) \supset F$ / $M \supset F$

★(19) 1. ${\sim}R \vee P$
2. $R \vee {\sim}P$ / $R \equiv P$

(20) 1. ${\sim}H \supset B$
2. ${\sim}H \supset D$
3. ${\sim}(B \bullet D)$ / H

(21) 1. $J \supset (G \supset L)$ / $G \supset (J \supset L)$

★(22) 1. $S \supset (L \bullet M)$
2. $M \supset (L \supset R)$ / $S \supset R$

(23) 1. $F \supset (A \bullet K)$
2. $G \supset ({\sim}A \bullet {\sim}K)$
3. $F \vee G$ / $A \equiv K$

(24) 1. $(I \supset E) \supset C$
2. $C \supset {\sim}C$ / I

★(25) 1. $T \supset G$
2. $S \supset G$ / $(T \vee S) \supset G$

(26) 1. $H \supset U$ / $H \supset (U \vee T)$

(27) 1. $Q \supset (W \bullet D)$ / $Q \supset W$

★(28) 1. $P \supset ({\sim}E \supset B)$
2. ${\sim}(B \vee E)$ / ${\sim}P$

(29) 1. $(G \supset J) \supset (H \supset Q)$
2. $J \bullet {\sim}Q$ / ${\sim}H$

(30) 1. $I \vee (N \bullet F)$
2. $I \supset F$ / F

★(31) 1. $K \equiv R$
2. $K \supset (R \supset P)$
3. ${\sim}P$ / ${\sim}R$

(32) 1. $C \supset ({\sim}L \supset Q)$
2. $L \supset {\sim}C$
3. ${\sim}Q$ / ${\sim}C$

(33) 1. $(E \supset A) \cdot (F \supset A)$
2. $E \vee G$
3. $F \vee {\sim}G$ / A

★(34) 1. $(F \cdot H) \supset N$
2. $F \vee S$
3. H / $N \vee S$

(35) 1. $T \supset (H \cdot J)$
2. $(H \vee N) \supset T$ / $T \equiv H$

(36) 1. $T \supset {\sim}(A \supset N)$
2. $T \vee N$ / $T \equiv {\sim}N$

★(37) 1. $(D \supset E) \supset (E \supset D)$
2. $(D \equiv E) \supset {\sim}(G \cdot {\sim}H)$
3. $E \cdot G$ / $G \cdot H$

(38) 1. $(O \supset R) \supset S$
2. $(P \supset R) \supset {\sim}S$ / ${\sim}R$

(39) 1. $(L \vee P) \supset U$
2. $(M \supset U) \supset I$
3. P / I

★(40) 1. $A \equiv W$
2. ${\sim}A \vee {\sim}W$
3. $R \supset A$ / ${\sim}(W \vee R)$

(41) 1. $(S \vee T) \supset (S \supset {\sim}T)$
2. $(S \supset {\sim}T) \supset (T \supset K)$
3. $S \vee T$ / $S \vee K$

(42) 1. $G \equiv M$
2. $G \vee M$
3. $G \supset (M \supset T)$ / T

★(43) 1. $O \supset (Q \cdot N)$
2. $(N \vee E) \supset S$ / $O \supset S$

(44) 1. $H \equiv I$
2. $H \supset (I \supset F)$
3. ${\sim}(H \vee I) \supset F$ / F

★(45) 1. $P \supset A$
2. $Q \supset B$ / $(P \vee Q) \supset (A \vee B)$

IV. Translate the following arguments into symbolic form and then use the eighteen rules of inference to derive the conclusion of each. Use the translation letters in the order in which they are listed.

★**1.** If sports shoe manufacturers decline to use kangaroo hides in their products, then Australian hunters will cease killing millions of kangaroos yearly. It is not the case that both Australian hunters will cease killing millions of kangaroos yearly and the kangaroo will not be saved from extinction. Therefore, if sports shoe manufacturers decline to use kangaroo hides in their products, then the kangaroo will be saved from extinction. (*D*, *C*, *S*)

2. If there is a direct correlation between what a nation spends for health care and the health of its citizens, then America has the lowest incidence of disease and the lowest mortality rates of any nation on earth. But America does not have the lowest mortality rates of any nation on earth. Therefore, there is not a direct correlation between what a nation spends for health care and the health of its citizens. (*C*, *D*, *M*)
3. It is not the case that strict controls exist on either the manufacture or the sale of handguns. Therefore, if strict controls exist on the sale of handguns, then the use of handguns in the commission of crimes has decreased. (*M*, *S*, *U*)

★4. If birth control devices are made available in high school clinics, then the incidence of teenage pregnancy will decrease. Therefore, if both birth control information and birth control devices are made available in high school clinics, then the incidence of teenage pregnancy will decrease. (*D*, *P*, *I*)

5. If Congress enacts a law that either establishes a religion or prohibits the free exercise of religion, then that law is unconstitutional. Therefore, if Congress enacts a law that establishes a religion, then that law is unconstitutional. (*E*, *P*, *U*)
6. If cigarette smokers are warned of the hazards of smoking and they continue to smoke, then they cannot sue tobacco companies for any resulting lung cancer or emphysema. Cigarette smokers are warned of the hazards of smoking. Therefore, if cigarette smokers continue to smoke, they cannot sue tobacco companies for any resulting lung cancer or emphysema. (*W*, *C*, *S*)

★7. If grade-school children are assigned daily homework, then their achievement level will increase dramatically. But if grade-school children are assigned daily homework, then their love for learning may be dampened. Therefore, if grade-school children are assigned daily homework, then their achievement level will increase dramatically but their love for learning may be dampened. (*G*, *A*, *L*)

8. If a superconducting particle collider is built, then the data yielded will benefit scientists of all nations and it deserves international funding. Either a superconducting particle collider will be built, or the ultimate nature of matter will remain hidden and the data yielded will benefit scientists of all nations. Therefore, the data yielded by a superconducting particle collider will benefit scientists of all nations. (*S*, *D*, *I*, *U*)
9. If parents are told that their unborn child has Tay-Sachs disease, then if they go ahead with the birth, then they are responsible for their child's pain and suffering. Therefore, if parents are not responsible for their child's pain and suffering, then if they go ahead with the birth, then they were not told that their unborn child had Tay-Sachs disease. (*T*, *G*, *R*)

★10. Vitamin E is an antioxidant and a useless food supplement if and only if it does not reduce heart disease. It is not the case either that vitamin E does not reduce heart disease or is not an antioxidant. Therefore, vitamin E is not a useless food supplement. (*A*, *U*, *R*)

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V. The following dialogue contains ten arguments. Translate each into symbolic form and then use the eighteen rules of inference to derive the conclusion of each.

Is This the End?

"I'm sorry for your loss," Brian says to Molly, as he gives her a sympathetic hug in the funeral home.

"Thank you," she says, dabbing a tear from her cheek. "This was such a senseless death—falling off a cliff while rock climbing. If it weren't so sad, it would be almost laughable."

"I know you were quite close to Karl," Brian says. "But do you think in some sense Karl could still be with us? I mean do you think there could be such a thing as post-mortem persistence of consciousness—life after death, as most people say?"

"I wish there were," Molly replies, "and that's what makes death so tragic. As I see it, the mind is totally dependent on the brain, and if that is so, when the brain dies, the mind dies. And of course, if the mind dies, then consciousness dies, too. Thus, if the brain dies, then consciousness dies—which means there is no persistence of consciousness after death."

"But what makes you think that the mind is totally dependent on the brain?" Brian asks.

"Our day-to-day experience provides lots of evidence," Molly replies, as she leans over to sniff a flower. "If you drink alcohol, your mind is affected. Also, if you smoke marijuana, your mind is affected. And if your mind is affected by these things, then you have first-hand experience that the mind is dependent on the brain. Thus, if you either smoke marijuana or drink alcohol, then you have first-hand experience that the mind is dependent on the brain."

"Molly, I think all your argument proves is that the mind is affected by the brain. And anyone with ordinary sensation knows that," Brian retorts. "If your eye receives a visual stimulus, then that stimulus is sent to the brain and your mind is affected. If your ear receives an auditory stimulus, then your mind is affected. Thus, if either your eye or your ear receives a stimulus, then your mind is affected. But that doesn't prove that the mind is *necessarily* dependent on the brain. And there are lots of reasons for saying that it isn't."

"What reasons are those?" Molly asks.

"Well, we learned about Plato in Introduction to Philosophy," Brian replies. "And Plato held that the mind can conceive ideal objects such as perfect justice and perfect triangularity. Now, if either of these concepts came through the senses, then perfect ideals exist in nature. But no perfect ideals exist in nature. And if the concept of triangularity did not come through the senses, then the mind produced it independently of the brain. But if that is the case or the concept of triangularity is innate, then the mind is not necessarily dependent on the brain. The conclusion is obvious."

"Very interesting," Molly replies, "but I question whether the mind is really capable of conceiving ideal objects such as perfect justice and perfect triangularity. For me, these things are just words. But there are additional reasons for thinking that the mind is *necessarily* dependent on the brain. For example, stimulation of the visual cortex, which is part of the brain, is associated with visual experience. If the visual cortex is not stimulated, there is no visual sensation. But if visual sensation occurs only if the visual cortex is stimulated, and if the visual cortex is part of the brain, then visual sensation is dependent on the brain. And if that is true and visual sensation is a function of the mind, then the mind is necessarily dependent on the brain. Therefore, if visual sensation is a function of the mind, then the mind is necessarily dependent on the brain."

"Furthermore," Molly continues, "there are many cases where strokes have caused loss of memory, and also loss of speech. But if remembering is a mental function, then if the mind is not necessarily dependent on the brain, then strokes do not cause loss of memory. Therefore, if remembering is a mental function, then the mind is necessarily dependent on the brain."

"It may indeed be the case," Brian replies, "that memory—or at least certain kinds of memory—are dependent on the brain. And the same may be true of sensation. But that doesn't prove that consciousness as such is brain dependent. It seems to me that consciousness as such is a nonmaterial process, and that it can inhere only in a nonmaterial entity, such as a soul. And if those two claims are true and the soul is immortal, then consciousness survives the death of the body. Thus, if the soul is immortal, then consciousness survives the death of the body."

"If memory goes with the brain," Molly replies, "then I wonder if the consciousness you speak of is in any way *your* consciousness. But setting that aside, are there any reasons for thinking that the soul is immortal?"

"I think there are," Brian replies. "If the soul is nonmaterial, then it has no parts, and if it has no parts, then it cannot come 'a-part'—in other words, it cannot disintegrate. And if it cannot disintegrate, then if nothing can destroy it, then it is immortal. But the soul can be destroyed only if God destroys it, and God does not destroy souls. Therefore, if the soul is nonmaterial, then it is immortal. I think Leibniz invented that argument."

"Fine," Molly says. "But what makes you think that you have a nonmaterial soul in the first place?"

"Well," Brian replies, "according to Descartes, I am essentially either a mind or a body. But if I can doubt that I have a body, then I am not *essentially* a body. And I can doubt that I have a body. For example, I can imagine that I am in the Matrix, and that all of my sensations are illusions. If I am essentially a mind, then if the essence of mind is to be nonextended, then I am a nonextended substance. But the essence of mind, being different from the essence of body, is to be nonextended. And if I am a nonextended substance, then I am (or have) a nonmaterial soul. Therefore, I am (or have) a nonmaterial soul."

"Your argument is so abstruse that I don't find it very persuasive," says Molly, as she scratches her head. "I think the evidence is overwhelming that humans are the product of biological evolution, and if that is true and humans have souls, then there is a point in the course of evolution where humans either received or developed a soul. But there is no evidence that humans ever received a soul. Also, there is no evidence that humans ever developed a soul. Therefore, humans do not have souls."

"Wow, that sounds pretty radical," Brian replies. "Well, it looks like the service is ready to start, so we'll have to hang this up. But maybe we can continue it at a later date."

"Maybe we can," Molly replies.

7.5

Conditional Proof

Conditional proof is a method for deriving a conditional statement (either the conclusion or some intermediate line) that offers the usual advantage of being both shorter and simpler to use than the direct method. Moreover, some arguments have conclusions that cannot be derived by the direct method, so some form of conditional proof must be used on them. The method consists of assuming the antecedent of the required conditional statement on one line, deriving the consequent on a subsequent line, and then "discharging" this sequence of lines in a conditional statement that exactly replicates the one that was to be obtained.

Any argument whose conclusion is a conditional statement is an immediate candidate for conditional proof. Consider the following example:

1.	$A \supset (B \cdot C)$	
2.	$(B \vee D) \supset E$	/ $A \supset E$

Using the direct method to derive the conclusion of this argument would require a proof having at least twelve lines, and the precise strategy to be followed in constructing it might not be immediately obvious. Nevertheless, we need only give cursory inspection to the argument to see that the conclusion does indeed follow from the premises. The conclusion states that if we have *A*, we then have *E*. Let us suppose, for a moment, that we do have *A*. We could then derive $B \cdot C$ from the first premise via *modus ponens*. Simplifying this expression we could derive *B*, and from this we could get $B \vee D$ via addition. *E* would then follow from the second premise via *modus ponens*. In other words, if we assume that we have *A*, we can get *E*. But this is exactly what the conclusion says. Thus, we have just proved that the conclusion follows from the premises.

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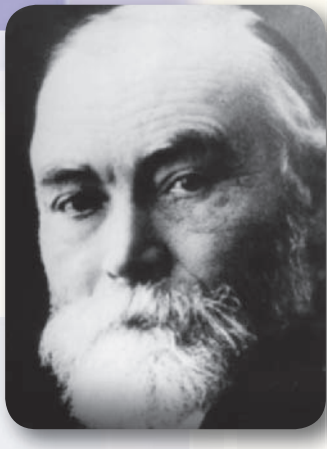
The method of conditional proof consists of incorporating this simple thought process into the body of a proof sequence. A conditional proof for this argument requires only eight lines and is substantially simpler than a direct proof:

1.	$A \supset (B \cdot C)$	
2.	$(B \vee D) \supset E$	/ $A \supset E$
	| 3. A	ACP
	| 4. $B \cdot C$	1, 3, MP
	| 5. B	4, Simp
	| 6. $B \vee D$	5, Add
	| 7. E	2, 6, MP
8.	$A \supset E$	3–7, CP

Lines 3 through 7 are indented to indicate their hypothetical character: They all depend on the assumption introduced in line 3 via ACP (assumption for conditional proof). These lines, which constitute the conditional proof sequence, tell us that if we assume *A* (line 3), we can derive *E* (line 7). In line 8 the conditional sequence is discharged in the conditional statement $A \supset E$, which simply reiterates the result of the conditional sequence. Since line 8 is not hypothetical, it is written adjacent to the original margin, under lines 1 and 2. A vertical line is added to the conditional sequence to emphasize the indentation.

Eminent Logicians

Gottlob Frege 1848–1925



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The German mathematician, logician, and philosopher Gottlob Frege (pronounced fray-ga) was born in Wismar, a small town in northern Germany on the Baltic Sea. His parents taught at a private girls' school, which his father had helped to found. Frege attended the local gymnasium, where he studied mathematics, and then the University of Jena, where he studied mathematics, philosophy, and chemistry. After two years he transferred to the University of Göttingen, earning a doctor's degree in mathematics at age twenty-four. He then returned to the University of Jena, where he taught until retiring in 1917. While there he married Margaret Liesburg, who bore him at least two children. The children died young, but years later the couple adopted a son, Alfred.

Frege spent his entire life analyzing the concept of number, developing theories of logic and language, and attempting to reduce arithmetic to logic. In 1879 he published the *Begriffschrift* ("Concept-script"), a work written in the tradition of Leibniz that develops a purely formal symbolic language to express any proposition in any area of human discourse. Five years later he published *Die Grundlagen der Arithmetik* ("The Foundations of Arithmetic"), a less technical work containing few symbols that outlined his goal of reducing arithmetic to logic. Then, nine years later he published Volume I of *Grundgesetze der Arithmetik* ("Basic Laws of Arithmetic"), which attempted to accomplish the first phase of this reduction.

None of these works were well received, for several reasons: They were ahead of their time, the symbolic notation of the technical works struck readers as bizarre, and they were written in German, while most of the new work in logic was being done by English speakers. In fact, the last of these works was so badly reviewed that Frege was forced to publish Volume II at his own expense. To make matters worse, in 1902, while Volume II was in proof, Frege received a letter from Bertrand Russell that left him "thunderstruck." He was later to remark that it had destroyed his entire life's work.

Basic Law V of the *Grundgesetze* provides for the creation of classes of things merely by describing the properties of their members. So Russell invited Frege to create the class of all classes that are not members of themselves, and he then asked whether this very class is a member of itself. If it is a member of itself, then it is one of those classes that are not members of themselves; but if is not a member of itself, then, again, it is one of those classes, and it is a member of itself. The derivation of this contradiction (which has come to be called Russell's paradox) meant that the axioms of the *Grundgesetze* were fatally inconsistent. Frege attempted a last-minute modification of his system, but the change proved unworkable.

Despite this setback, Frege is universally recognized today as one of the most important logicians and philosophers of all time. Singlehandedly he developed quantification theory and predicate logic, and his analysis of the concept of number led to a general theory of meaning that introduced the important distinction between *Sinn* (sense) and *Beduetung* (reference). Also, his work on concept clarification initiated the current movement known as analytic philosophy.

7

The first step in constructing a conditional proof is to decide what should be assumed on the first line of the conditional sequence. While any statement whatsoever *can* be assumed on this line, only the right statement will lead to the desired result. The clue is always provided by the conditional statement to be obtained in the end. The antecedent of this statement is what must be assumed. For example, if the statement to be obtained is $(K \bullet L) \supset M$, then $K \bullet L$ should be assumed on the first line. This line is always indented and tagged with the designation "ACP." Once the initial assumption has been made, the second step is to derive the consequent of the desired conditional statement at the end of the conditional sequence. To do this, we simply apply the ordinary rules of inference to any previous line in the proof (including the assumed line), writing the result directly below the assumed line. The third and final step is to discharge the conditional sequence in a conditional statement. The antecedent of this conditional statement is whatever appears on the first line of the conditional sequence, and the consequent is whatever appears on the last line. For example, if $A \vee B$ is on the first line and $C \bullet D$ is on the last, the sequence is discharged by $(A \vee B) \supset (C \bullet D)$. This discharging line is always written adjacent to the original margin and is tagged with the designation "CP" (conditional proof) together with the numerals indicating the first through the last lines of the sequence.

Conditional proof can also be used to derive a line other than the conclusion of an argument. The following proof, which illustrates this fact, incorporates two conditional sequences one after the other within the scope of a single direct proof:

1. $G \supset (H \bullet I)$	
2. $J \supset (K \bullet L)$	
3. $G \vee J$	/ $H \vee K$
│ 4. G	ACP
│ 5. $H \bullet I$	1, 4, MP
│ 6. H	5, Simp
7. $G \supset H$	4–6, CP
│ 8. J	ACP
│ 9. $K \bullet L$	2, 8, MP
│ 10. K	9, Simp
11. $J \supset K$	8–10, CP
12. $(G \supset H) \bullet (J \supset K)$	7, 11, Conj
13. $H \vee K$	3, 12, CD

The first conditional proof sequence gives us $G \supset H$, and the second $J \supset K$. These two lines are then conjoined and used together with line 3 to set up a constructive dilemma, from which the conclusion is derived.

This proof sequence provides a convenient opportunity to introduce an important rule governing conditional proof. The rule states that after a conditional proof sequence has been discharged, no line in the sequence may be used as a justification for a subsequent line in the proof. If, for example, line 5 in the proof just given were used as a justification for line 9 or line 12, this rule would be violated, and the corresponding inference would be invalid. Once the conditional sequence is discharged, it is sealed off from the remaining part of the proof. The logic behind this rule is easy to understand. The lines in a conditional sequence are hypothetical in that they depend on the assumption stated in the first line. Because no mere assumption can provide any genuine support for anything, neither can any line that depends on such an assumption. When a conditional sequence is discharged, the assumption on which it rests is expressed as the antecedent of a conditional statement. This conditional statement *can* be used to support subsequent lines because it makes no claim that its antecedent is true. The conditional statement merely asserts that *if* its antecedent is true, then its consequent is true, and this, of course, is what has been established by the conditional sequence from which it is obtained.

Just as a conditional sequence can be used within the scope of a direct proof to derive a desired statement, one conditional sequence can be used within the scope of another to derive a desired statement. The following proof provides an example:

1. $L \supset [M \supset (N \vee O)]$	
2. $M \supset \sim N$	/ $L \supset (\sim M \vee O)$
| 3. L	ACP
| 4. $M \supset (N \vee O)$	1, 3, MP
| | 5. M	ACP
| | 6. $N \vee O$	4, 5, MP
| | 7. $\sim N$	2, 5, MP
| | 8. O	6, 7, DS
| 9. $M \supset O$	5–8, CP
| 10. $\sim M \vee O$	9, Impl
11. $L \supset (\sim M \vee O)$	3–10, CP

The rule introduced in connection with the previous example applies unchanged to examples of this sort. No line in the sequence 5–8 could be used to support any line subsequent to line 9, and no line in the sequence 3–10 could be used to support any line subsequent to line 11. Lines 3 or 4 could, of course, be used to support any line in the sequence 5–8.

One final reminder regarding conditional proof is that every conditional proof must be discharged. It is absolutely improper to end a proof on an indented line. If this rule is ignored, any conclusion one chooses can be derived from any set of premises. The following invalid proof illustrates this mistake:

1. P	/ $Q \supset R$
| 2. $\sim Q$	ACP
| 3. $\sim Q \vee R$	2, Add
| 4. $Q \supset R$	2, Impl

Exercise 7.5

I. Use conditional proof and the eighteen rules of inference to derive the conclusions of the following symbolized arguments. Having done so, attempt to derive the conclusions without using conditional proof.

★**(1)** **1.** $N \supset O$
2. $N \supset P$ / $N \supset (O \bullet P)$

(2) 1. $F \supset E$
2. $(F \bullet E) \supset R$ / $F \supset R$

(3) 1. $G \supset T$
2. $(T \vee S) \supset K$ / $G \supset K$

★(4) 1. $(G \vee H) \supset (S \bullet T)$
2. $(T \vee U) \supset (C \bullet D)$ / $G \supset C$

(5) 1. $A \supset \sim(A \vee E)$ / $A \supset F$

(6) 1. $J \supset (K \supset L)$
2. $J \supset (M \supset L)$
3. $\sim L$ / $J \supset \sim(K \vee M)$

★(7) 1. $M \vee (N \bullet O)$ / $\sim N \supset M$

(8) 1. $P \supset (Q \vee R)$
2. $(P \supset R) \supset (S \bullet T)$
3. $Q \supset R$ / T

(9) 1. $H \supset (I \supset N)$
2. $(H \supset \sim I) \supset (M \vee N)$
3. $\sim N$ / M

★(10) 1. $C \supset (A \bullet D)$
2. $B \supset (A \bullet E)$ / $(C \vee B) \supset A$

(11) 1. $M \supset (K \supset L)$
2. $(L \vee N) \supset J$ / $M \supset (K \supset J)$

(12) 1. $F \supset (G \bullet H)$ / $(A \supset F) \supset (A \supset H)$

★(13) 1. $R \supset B$
2. $R \supset (B \supset F)$
3. $B \supset (F \supset H)$ / $R \supset H$

(14) 1. $(F \bullet G) \equiv H$
2. $F \supset G$ / $F \equiv H$

(15) 1. $C \supset (D \vee \sim E)$
2. $E \supset (D \supset F)$ / $C \supset (E \supset F)$

★(16) 1. $Q \supset (R \supset S)$
2. $Q \supset (T \supset \sim U)$
3. $U \supset (R \vee T)$ / $Q \supset (U \supset S)$

(17) 1. $N \supset (O \bullet P)$
2. $Q \supset (R \bullet S)$ / $(P \supset Q) \supset (N \supset S)$

(18) 1. $E \supset (F \supset G)$
2. $H \supset (G \supset I)$
3. $(F \supset I) \supset (J \vee \sim H)$ / $(E \bullet H) \supset J$

★(19) 1. $P \supset [(L \vee M) \supset (N \bullet O)]$
2. $(O \vee T) \supset W$ / $P \supset (M \supset W)$

(20) 1. $A \supset [B \supset (C \bullet \sim D)]$
2. $(B \vee E) \supset (D \vee E)$ / $(A \bullet B) \supset (C \bullet E)$

II. Translate the following arguments into symbolic form, using the letters in the order in which they are listed. Then use conditional proof and the eighteen rules of inference to derive the conclusion of each. Having done so, attempt to derive the conclusion without using conditional proof.

★**1.** If high-tech products are exported to Russia, then domestic industries will benefit. If the Russians can effectively utilize high-tech products, then their standard of living will improve. Therefore, if high-tech products are exported to Russia and the Russians can effectively utilize them, then their standard of living will improve and domestic industries will benefit. (*H*, *D*, *U*, *S*)

2. If the police take you into custody, then if they inform you that you have the right to remain silent, then whatever you say will be used against you. If the police inform you that you have the right to remain silent, then if whatever you say will be used against you, then you should not say anything. Therefore, if the police take you into custody, then if they inform you that you have the right to remain silent, then you should not say anything. (*P*, *I*, *W*, *S*)

3. A doctor must disconnect a dying patient from a respirator if and only if the fact that patients are self-determining implies that the doctor must follow the patient's orders. If a dying patient refuses treatment, then the doctor must disconnect the patient from a respirator and the patient will die peacefully. Patients are self-determining. Therefore, if a dying patient refuses treatment, then the doctor must follow the patient's orders. (*D*, *S*, *F*, *R*, *P*)

★**4.** If jails are overcrowded, then dangerous suspects will be released on their own recognizance. If jails are overcrowded and dangerous suspects are released on their own recognizance, then crime will increase. If no new jails are built and crime increases, then innocent victims will pay the price of increased crime. Therefore, if jails are overcrowded, then if no new jails are built, then innocent victims will pay the price of increased crime. (*J*, *D*, *C*, *N*, *I*)

5. If astronauts attempt interplanetary space travel, then heavy shielding will be required to protect them from solar radiation. If massive amounts of either fuel or water are carried, then the spacecraft must be very large. Therefore, if heavy shielding is required to protect the astronauts from solar radiation only if massive amounts of fuel are carried, then if astronauts attempt interplanetary space travel, then the spacecraft must be very large. (*A*, *H*, *F*, *W*, *L*)

7.6 Indirect Proof

Indirect proof is a technique similar to conditional proof that can be used on any argument to derive either the conclusion or some intermediate line leading to the conclusion. It consists of assuming the negation of the statement to be obtained, using this assumption to derive a contradiction, and then concluding that the original assumption is false. This last step, of course, establishes the truth of the statement to be obtained. The following proof sequence uses indirect proof to derive the conclusion:

1.	$(A \vee B) \supset (C \bullet D)$	
2.	$C \supset {\sim}D$	/ ${\sim}A$
	│ 3. A	AIP
	│ 4. $A \vee B$	3, Add
	│ 5. $C \bullet D$	1, 4, MP
	│ 6. C	5, Simp
	│ 7. ${\sim}D$	2, 6, MP
	│ 8. $D \bullet C$	5, Com
	│ 9. D	8, Simp
	│ 10. $D \bullet {\sim}D$	7, 9, Conj
11.	${\sim}A$	3–10, IP

The indirect proof sequence (lines 3–10) begins by assuming the negation of the conclusion. Since the conclusion is a negated statement, it shortens the proof to assume *A* instead of ~~*A*. This assumption, which is tagged "AIP" (assumption for indirect proof), leads to a contradiction in line 10. Since any assumption that leads to a contradiction is false, the indirect sequence is discharged (line 11) by asserting the negation of the assumption made in line 3. This line is then tagged with the designation "IP" (indirect proof) together with the numerals indicating the scope of the indirect sequence from which it is obtained.

Indirect proof can also be used to derive an intermediate line leading to the conclusion. Example:

1.	$E \supset [(F \vee G) \supset (H \bullet J)]$	
2.	$E \bullet {\sim}(J \vee K)$	/ ${\sim}(F \vee K)$
3.	E	2, Simp
4.	$(F \vee G) \supset (H \bullet J)$	1, 3, MP
5.	${\sim}(J \vee K) \bullet E$	2, Com
6.	${\sim}(J \vee K)$	5, Simp
7.	${\sim}J \bullet {\sim}K$	6, DM
	│ 8. F	AIP
	│ 9. $F \vee G$	8, Add
	│ 10. $H \bullet J$	4, 9, MP
	│ 11. $J \bullet H$	10, Com
	│ 12. J	11, Simp
	│ 13. ${\sim}J$	7, Simp
	│ 14. $J \bullet {\sim}J$	12, 13, Conj
15.	${\sim}F$	8–14, IP
16.	${\sim}K \bullet {\sim}J$	7, Com
17.	${\sim}K$	16, Simp
18.	${\sim}F \bullet {\sim}K$	15, 17, Conj
19.	${\sim}(F \vee K)$	18, DM

The indirect proof sequence begins with the assumption of *F* (line 8), leads to a contradiction (line 14), and is discharged (line 15) by asserting the negation of the assumption. One should consider indirect proof whenever a line in a proof appears difficult to obtain.

As with conditional proof, when an indirect proof sequence is discharged, no line in the sequence may be used as a justification for a subsequent line in the proof. In reference to the last proof, this means that none of the lines 8–14 could be used as a justification for any of the lines 16–19. Occasionally, this rule requires certain priorities in the derivation of lines. For example, for the purpose of deriving the contradiction, lines 6 and 7 could have been included as part of the indirect sequence. But this would not have been advisable, because line 7 is needed as a justification for line 16, which lies outside the indirect sequence. If lines 6 and 7 had been included within the indirect sequence, they would have had to be repeated after the sequence had been discharged to allow ~*K* to be derived on a line outside the sequence.

Just as a conditional sequence may be constructed within the scope of another conditional sequence, so a conditional sequence may be constructed within the scope of an indirect sequence, and, conversely, an indirect sequence may be constructed within the scope of either a conditional sequence or another indirect sequence. The next example illustrates the use of an indirect sequence within the scope of a conditional sequence:

1. $L \supset [\sim M \supset (N \cdot O)]$	
2. $\sim N \cdot P$	/ $L \supset (M \cdot P)$
│ 3. L	ACP
│ 4. $\sim M \supset (N \cdot O)$	1, 3, MP
│ │ 5. $\sim M$	AIP
│ │ 6. $N \cdot O$	4, 5, MP
│ │ 7. N	6, Simp
│ │ 8. $\sim N$	2, Simp
│ │ 9. $N \cdot \sim N$	7, 8, Conj
│ 10. $\sim\sim M$	5–9, IP
│ 11. M	10, DN
│ 12. $P \cdot \sim N$	2, Com
│ 13. P	12, Simp
│ 14. $M \cdot P$	11, 13, Conj
15. $L \supset (M \cdot P)$	3–14, CP

The indirect sequence (lines 5–9) is discharged (line 10) by asserting the negation of the assumption made in line 5. The conditional sequence (lines 3–14) is discharged (line 15) in the conditional statement that has the first line of the sequence as its antecedent and the last line as its consequent.

Indirect proof provides a convenient way for proving the validity of an argument having a tautology for its conclusion. In fact, the only way in which the conclusion of many such arguments can be derived is through either conditional or indirect proof.

For the following argument, indirect proof is the easier of the two:

1. S	/ $T \vee \sim T$
│ 2. $\sim(T \vee \sim T)$	AIP
│ 3. $\sim T \cdot \sim\sim T$	2, DM
4. $\sim\sim(T \vee \sim T)$	2–3, IP
5. $T \vee \sim T$	4, DN

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Here is another example of an argument having a tautology as its conclusion. In this case, since the conclusion is a conditional statement, conditional proof is the easier alternative:

1. S	/ $T \supset T$
2. T	ACP
3. $T \vee T$	2, Add
4. T	3, Taut
5. $T \supset T$	2–4, CP

The similarity of indirect proof to conditional proof may be illustrated by returning to the first example presented in this section. In the proof that follows, conditional proof—not indirect proof—is used to derive the conclusion:

1. $(A \vee B) \supset (C \cdot D)$	
2. $C \supset \sim D$	/ $\sim A$
3. A	ACP
4. $A \vee B$	3, Add
5. $C \cdot D$	1, 4, MP
6. C	5, Simp
7. $\sim D$	2, 6, MP
8. $D \cdot C$	5, Com
9. D	8, Simp
10. $D \vee \sim A$	9, Add
11. $\sim A$	7, 10, DS
12. $A \supset \sim A$	3–11, CP
13. $\sim A \vee \sim A$	12, Impl
14. $\sim A$	13, Taut

This example illustrates how a conditional proof can be used to derive the conclusion of *any* argument, whether or not the conclusion is a conditional statement. Simply begin by assuming the negation of the conclusion, derive contradictory statements on separate lines, and use these lines to set up a disjunctive syllogism yielding the negation of the assumption as the last line of the conditional sequence. Then, discharge the sequence and use tautology to derive the negation of the assumption outside the sequence.

Indirect proof can be viewed as a variety of conditional proof in that it amounts to a modification of the way in which the indented sequence is discharged, resulting in an overall shortening of the proof for many arguments. The indirect proof for the argument just given is repeated as follows, with the requisite changes noted in the margin:

1. $(A \vee B) \supset (C \cdot D)$	
2. $C \supset \sim D$	/ $\sim A$
3. A	AIP
4. $A \vee B$	3, Add
5. $C \cdot D$	1, 4, MP
6. C	5, Simp
7. $\sim D$	2, 6, MP
8. $D \cdot C$	5, Com

| 9. D 8, Simp
| 10. $D \bullet \sim D$ 7, 9, Conj
11. $\sim A$ 3–10, IP   changed

The reminder at the end of the previous section regarding conditional proof pertains to indirect proof as well: It is essential that every indirect proof be discharged. No proof can be ended on an indented line. If this rule is ignored, indirect proof, like conditional proof, can produce any conclusion whatsoever. The following invalid proof illustrates such a mistake:

1. P / Q
| 2. Q AIP
| 3. $Q \vee Q$ 2, Add
| 4. Q 3, Taut

Exercise 7.6

I. Use either indirect proof or conditional proof (or both) and the eighteen rules of inference to derive the conclusions of the following symbolized arguments. Having done so, attempt to derive the conclusions without using indirect proof or conditional proof.

★**(1)** **1.** $(S \vee T) \supset \sim S$ / $\sim S$

(2) **1.** $(K \supset K) \supset R$
2. $(R \vee M) \supset N$ / N

(3) **1.** $(C \bullet D) \supset E$
2. $(D \bullet E) \supset F$ / $(C \bullet D) \supset F$

★**(4)** **1.** $H \supset (L \supset K)$
2. $L \supset (K \supset \sim L)$ / $\sim H \vee \sim L$

(5) **1.** $S \supset (T \vee \sim U)$
2. $U \supset (\sim T \vee R)$
3. $(S \bullet U) \supset \sim R$ / $\sim S \vee \sim U$

(6) **1.** $\sim A \supset (B \bullet C)$
2. $D \supset \sim C$ / $D \supset A$

★**(7)** **1.** $(E \vee F) \supset (C \bullet D)$
2. $(D \vee G) \supset H$
3. $E \vee G$ / H

(8) **1.** $\sim M \supset (N \bullet O)$
2. $N \supset P$
3. $O \supset \sim P$ / M

(9) **1.** $(R \vee S) \supset T$
2. $(P \vee Q) \supset T$
3. $R \vee P$ / T

★(10) 1. K / $S \supset (T \supset S)$

(11) 1. $(A \vee B) \supset C$
2. $({\sim}A \vee D) \supset E$ / $C \vee E$

(12) 1. $(K \vee L) \supset (M \bullet N)$
2. $(N \vee O) \supset (P \bullet {\sim}K)$ / ${\sim}K$

★(13) 1. $[C \supset (D \supset C)] \supset E$ / E

(14) 1. F / $(G \supset H) \vee ({\sim}G \supset J)$

(15) 1. $B \supset (K \bullet M)$
2. $(B \bullet M) \supset (P \equiv {\sim}P)$ / ${\sim}B$

★(16) 1. $(N \vee O) \supset (C \bullet D)$
2. $(D \vee K) \supset (P \vee {\sim}C)$
3. $(P \vee G) \supset {\sim}(N \bullet D)$ / ${\sim}N$

(17) 1. $(R \bullet S) \equiv (G \bullet H)$
2. $R \supset S$
3. $H \supset G$ / $R \equiv H$

(18) 1. $K \supset [(M \vee N) \supset (P \bullet Q)]$
2. $L \supset [(Q \vee R) \supset (S \bullet {\sim}N)]$ / $(K \bullet L) \supset {\sim}N$

★(19) 1. $A \supset [(N \vee {\sim}N) \supset (S \vee T)]$
2. $T \supset {\sim}(F \vee {\sim}F)$ / $A \supset S$

(20) 1. $F \supset [(C \supset C) \supset G]$
2. $G \supset \{[H \supset (E \supset H)] \supset (K \bullet {\sim}K)\}$ / ${\sim}F$

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II. Translate the following arguments into symbolic form, using the letters in the order in which they are listed. Then use indirect proof and the eighteen rules of inference to derive the conclusion of each. Having done so, attempt to derive the conclusion without using indirect proof.

★**1.** If government deficits continue at their present rate and a recession sets in, then interest on the national debt will become unbearable and the government will default on its loans. If a recession sets in, then the government will not default on its loans. Therefore, either government deficits will not continue at their present rate or a recession will not set in. (C, R, I, D)

2. If either the sea turtle population continues to decrease or rescue efforts are commenced to save the sea turtle from extinction, then nesting sanctuaries will be created and the indiscriminate slaughter of these animals will be halted. If either nesting sanctuaries are created or poachers are arrested, then if the indiscriminate slaughter of these animals is halted, then the sea turtle population will not continue to decrease. Therefore, the sea turtle population will not continue to decrease. (C, R, N, I, P)

3. If asbestos workers sue their employers, then if punitive damages are awarded, then their employers will declare bankruptcy. If asbestos workers sue their employers, then punitive damages will be awarded. If asbestos workers contract asbestosis, then either they will sue their employers or their employers will declare bankruptcy. Therefore, either asbestos workers will not contract asbestosis or their employers will declare bankruptcy. (*S*, *P*, *B*, *C*)

★**4.** If astronauts spend long periods in zero gravity only if calcium is resorbed in their bodies, then astronauts on a Mars voyage will arrive with brittle bones. If astronauts attempt a voyage to Mars only if they spend long periods in zero gravity, then astronauts on a Mars voyage will arrive with brittle bones. Therefore, astronauts on a Mars voyage will arrive with brittle bones. (*Z*, *C*, *B*, *V*)

5. Either deposits should be required on beer and soft drink containers, or these containers will be discarded along highways and the countryside will look like a dump. If these containers will be discarded either in parks or along highways, then deposits should be required on soft drink containers. Therefore, deposits should be required on soft drink containers. (*B*, *S*, *H*, *C*, *P*)

7 7.7 Proving Logical Truths

Both conditional and indirect proof can be used to establish the truth of a logical truth (tautology). Tautological statements can be treated as if they were the conclusions of arguments having no premises. Such a procedure is suggested by the fact that any argument having a tautology for its conclusion is valid regardless of what its premises are. As we saw in the previous section, the proof for such an argument does not use the premises at all but derives the conclusion as the exclusive consequence of either a conditional or an indirect sequence. Using this strategy for logical truths, we write the statement to be proved as if it were the conclusion of an argument, and we indent the first line in the proof and tag it as being the beginning of either a conditional or an indirect sequence. In the end, this sequence is appropriately discharged to yield the desired statement form.

Tautologies expressed in the form of conditional statements are most easily proved via a conditional sequence. The following example uses two such sequences, one within the scope of the other:

		/ $P \supset (Q \supset P)$
| 1.	P	ACP
| | 2.	Q	ACP
| | 3.	$P \vee P$	1, Add
| | 4.	P	3, Taut
| 5.	$Q \supset P$	2–4, CP
6.	$P \supset (Q \supset P)$	1–5, CP

Notice that line 6 restores the proof to the original margin—the first line is indented because it introduces the conditional sequence.

Here is a proof of the same statement using an indirect proof. The indirect sequence begins, as usual, with the negation of the statement to be proved:

		/ $P \supset (Q \supset P)$
1.	$\sim[P \supset (Q \supset P)]$	AIP
2.	$\sim[\sim P \vee (Q \supset P)]$	1, Impl
3.	$\sim[\sim P \vee (\sim Q \vee P)]$	2, Impl
4.	$\sim\sim P \bullet \sim(\sim Q \vee P)$	3, DM
5.	$P \bullet \sim(\sim Q \vee P)$	4, DN
6.	$P \bullet (\sim\sim Q \bullet \sim P)$	5, DM
7.	$P \bullet (\sim P \bullet \sim\sim Q)$	6, Com
8.	$(P \bullet \sim P) \bullet \sim\sim Q$	7, Assoc
9.	$P \bullet \sim P$	8, Simp
10.	$\sim\sim[P \supset (Q \supset P)]$	1–9, IP
11.	$P \supset (Q \supset P)$	10, DN

More complex conditional statements are proved by merely extending the technique used in the first proof. In the following proof, notice how each conditional sequence begins by asserting the antecedent of the conditional statement to be derived:

		/ $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$
1.	$P \supset (Q \supset R)$	ACP
2.	$P \supset Q$	ACP
3.	P	ACP
4.	$Q \supset R$	1, 3, MP
5.	Q	2, 3, MP
6.	R	4, 5, MP
7.	$P \supset R$	3–6, CP
8.	$(P \supset Q) \supset (P \supset R)$	2–7, CP
9.	$[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	1–8, CP

Tautologies expressed as equivalences are usually proved using two conditional sequences, one after the other. Example:

		/ $P \equiv [P \bullet (Q \supset P)]$
1.	P	ACP
2.	$P \vee \sim Q$	1, Add
3.	$\sim Q \vee P$	2, Com
4.	$Q \supset P$	3, Impl
5.	$P \bullet (Q \supset P)$	1, 4, Conj
6.	$P \supset [P \bullet (Q \supset P)]$	1–5, CP
7.	$P \bullet (Q \supset P)$	ACP
8.	P	7, Simp
9.	$[P \bullet (Q \supset P)] \supset P$	7–8, CP
10.	{line 6} $\bullet$ {line 9}	6, 9, Conj
11.	$P \equiv [P \bullet (Q \supset P)]$	10, Equiv

Exercise 7.7

Use conditional proof or indirect proof and the eighteen rules of inference to establish the truth of the following tautologies.

★1. $P \supset [(P \supset Q) \supset Q]$

2. $(\sim P \supset Q) \vee (P \supset R)$

3. $P \equiv [P \vee (Q \bullet P)]$

★4. $(P \supset Q) \supset [(P \bullet R) \supset (Q \bullet R)]$

5. $(P \vee \sim Q) \supset [(\sim P \vee R) \supset (Q \supset R)]$

6. $P \equiv [P \bullet (Q \vee \sim Q)]$

★7. $(P \supset Q) \vee (\sim Q \supset P)$

8. $(P \supset Q) \equiv [P \supset (P \bullet Q)]$

9. $[(P \supset Q) \bullet (P \supset R)] \supset [P \supset (Q \bullet R)]$

★10. $[\sim(P \bullet \sim Q) \bullet \sim Q] \supset \sim P$

11. $(P \supset Q) \vee (Q \supset P)$

12. $[P \supset (Q \supset R)] \equiv [Q \supset (P \supset R)]$

★13. $(P \supset Q) \supset [(P \supset \sim Q) \supset \sim P]$

14. $[(P \supset Q) \supset R] \supset [(R \supset \sim R) \supset P]$

15. $(\sim P \vee Q) \supset [(P \vee \sim Q) \supset (P \equiv Q)]$

★16. $\sim[(P \supset \sim P) \bullet (\sim P \supset P)]$

17. $P \supset [(Q \bullet \sim Q) \supset R]$

18. $[(P \bullet Q) \vee R] \supset [(\sim R \vee Q) \supset (P \supset Q)]$

★19. $P \equiv [P \vee (Q \bullet \sim Q)]$

20. $P \supset [Q \equiv (P \supset Q)]$

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Summary

Natural Deduction in Propositional Logic:

- A step-by-step method for proving the validity of propositional type arguments.
- Shows exactly how the conclusion "comes out" of the premises.
- Consists in applying eighteen rules of inference to the premises and deriving the conclusion as the last line in a sequence of lines.
- Success in using this method requires much practice.

Rules of Inference:

- Rules of Implication: These are "one-way" rules:
    - The premise(s) can be used to derive the conclusion.
    - The conclusion cannot be used to derive the premise(s).
- Rules of Replacement: These are "two-way" rules:
    - Expressed as logical equivalencies.
    - Either side of the equivalence can replace the other.
    - Can be used to "deconstruct" the conclusion for insight into how to derive it.

Conditional Proof:

- A method for deriving a conditional statement.
- Assume the antecedent of the desired conditional on an indented line.
- Derive the consequent of the desired conditional statement.
- Discharge the indented sequence in a conditional statement having the first line of the sequence as the antecedent and the last line as the consequent.
- This method can greatly simplify many proofs.

Indirect Proof:

- A method for deriving any kind of statement.
- Assume the negation of the desired statement (often this is the conclusion) on an indented line.
- Derive a contradiction.
- Any assumption that necessarily leads to a contradiction is false.
- Discharge the indented sequence in a statement consisting of the negation of the first line of the sequence.

Proving Logical Truths (Tautologies):

- Use conditional proof to derive conditionals and biconditionals.
    - Assume the antecedent of the conditional statement on an indented line.
    - Derive the consequent.
    - Discharge the indented sequence in the usual way.
    - Biconditionals require two indented sequences.
- Use indirect proof to derive any logical truth:
    - Assume the negation of the logical truth on an indented line.
    - Derive a contradiction.
    - Discharge the indented sequence in the usual way.