CHAPTER 6

Isometries

Contents

In these notes we assume working knowledge of eigenvalues and eigenvectors. Have a look at Lecture 9 of your Algebra course from last semester.

6.1 Isometries

Definition. An *isometry* is a map $\phi : \mathbb{E}^n \to \mathbb{E}^n$ which preserves distances, i.e.

$$d(\phi(P), \phi(Q)) = d(P, Q)$$

for any points $P, Q \in \mathbb{E}^n$.

- One can show that isometries are affine transformations, i.e. they are elements in $\mathrm{AGL}(\mathbb{E}^n)$.

Proposition 6.1. Let $\phi \in \mathrm{AGL}(\mathbb{E}^n)$ be an affine transformation given by $\phi(\mathbf{x}) = A\mathbf{x} + b$ with respect to some orthonormal coordinate system. The following are equivalent:

1. ϕ is an isometry
2. $A^{-1} = A^t$.

Proposition 6.2. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a matrix such that $A^t A = I_n$. Then $\det(A) \in \{\pm 1\}$.

Definition. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that $A^t A = I_n$ is called *orthogonal*. The set of all such matrices are denoted by $O(n)$. The set of matrices in $O(n)$ with determinant 1 is denoted by $SO(n)$. Such matrices are called *special orthogonal*.

The set $O(n)$ is a subgroup of $\mathrm{AGL}(\mathbb{R}^n)$ and $SO(n)$ is a normal subgroup of $O(n)$:

$$SO(n) \trianglelefteq O(n) \leq \mathrm{AGL}(\mathbb{R}^n).$$

Let $\phi \in \mathrm{AGL}(\mathbb{E}^n)$ be given by $\phi(\mathbf{x}) = A\mathbf{x}+b$ with respect to some orthonormal coordinate system. Then ϕ is called a *displacement*, or a *direct isometry*, if $A \in SO(n)$.

6.1.1 Rotations in dimension 2

Proposition 6.3. A matrix A is in $SO(2)$ if and only if A has the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$.

Corollary 6.4. A direct isometry ϕ of $\mathbb{E}^2$ that fixes a point is either the identity or a rotation. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\mathrm{tr}(\mathrm{lin}(\phi))}{2}.$$

6.1.2 Rotations in dimension 3

Theorem 6.5 (Euler)**.** A direct isometry ϕ of $\mathbb{E}^3$ that fixes a point is either the identity or a rotation around an axis that passes through that point. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\mathrm{tr}(\mathrm{lin}(\phi)) - 1}{2}.$$

6.1.3 Classification of isometries

Theorem 6.6 (Chasles)**.** An isometry of the plane $\mathbb{E}^2$ is either a direct isometry, in which case it is

- the identity, or
- a translation, or
- a rotation;

or an indirect isometry, in which case it is

- a reflection, or

- a glidereflection.

Theorem 6.7 (Euler)**.** Any isometry of the 3-dimensional Euclidean space $\mathbb{E}^3$ is either a direct isometry, in which case it is

- the identity, or
- a translation, or
- a rotation around an axis, or
- a gliderotation (also called helical displacement);

or an indirect isometry, in which case it is

- a reflection, or
- a glidereflection, or
- a rotation-reflection.

6.2 Spectral theorem

Proposition 6.8. Any set of mutually orthogonal vectors is linearly independent. In particular, any set of n mutually orthogonal vectors is a basis of $\mathbb{V}^n$.

Proposition 6.9. Let $\boldsymbol{e} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ and $\boldsymbol{f} = \{\mathbf{f}_1,\dots,\mathbf{f}_n\}$ be two bases of the vector space $\mathbb{V}^n$, and suppose that $\boldsymbol{e}$ is orthonormal. The basis $\boldsymbol{f}$ is orthonormal if and only if the base change matrix $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}}$ is orthogonal.

- Proposition 6.9, says that if we change the coordinate system from an orthonormal basis $\boldsymbol{e}$ to an orthonormal basis $\boldsymbol{f}$ then the base change matrix $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}}$ is in $O(n)$, i.e. the base change is an isometry.
- For a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and a matrix $M \in O(n)$, since $M^{-1} = M^t$ we have
$$M^{-1}AM = M^tAM.$$
- The above observation is of funamental importance for the isometric classification of quadrics. Suppose you have a quadratic equation
$$2x^2 - 6xy - 1y^2 = c \tag{6.1}$$
for some constant $c \in \mathbb{R}$. Then, you may write it in matrix form like this:
$$\begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -3 \\ -3 & -1 \end{bmatrix}}_{=:A} \begin{bmatrix} x \\ y \end{bmatrix} = c$$

and if we change coordinates with a change of basis matrix M then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$$

and thus, Equation (6.1) becomes

$$\begin{bmatrix} x' & y' \end{bmatrix} \left(M^t A M \right) \begin{bmatrix} x' \\ y' \end{bmatrix} = c$$

Now if M is orthogonal, i.e. $M \in O(n)$ (which by Proposition 6.9 happens when we go from one orthonormal coordinate system to another), then the above equation becomes

$$\begin{bmatrix} x' & y' \end{bmatrix} \left(M^{-1} A M \right) \begin{bmatrix} x \\ y \end{bmatrix} = c$$

The matrix A is the matrix of a linear map ϕ_A, and $M^{-1}AM$ is the matrix of the same linear map with respect to a different (orthonormal) basis. Thus, if we find an orthonormal basis with respect to which the matrix of ϕ_A is 'nice' (spoiler: diagonal), then with respect to that basis Equation (6.1) becomes

$$\lambda_1 x^2 + \lambda_2 y^2 = c$$

for some values $\lambda_1, \lambda_2 \in \mathbb{R}$ which are the eigenvalues of ϕ_A and of A.

- Notice that if we change the order of two vectors in the basis $\boldsymbol{f}$ then we change the sign of $\det(M_{\boldsymbol{e},\boldsymbol{f}})$. Notice also that if we change the sign of one vector in $\boldsymbol{f}$ we change the sign of $\det(M_{\boldsymbol{e},\boldsymbol{f}})$. So, changing orthonormal coordinate systems can be performed with matrices in $SO(n)$ if we give ourselves the freedom of interchanging two axes or of changing the direction of one axis.

- The above observation is of interest because coordinate changes performed with matrices in $SO(n)$ are displacements. Such transformations are close to our intuition since they correspond to the usual movements that we do/see in our surroundings.

- The next statements show how a symmetric operator (a linear map with a symmetric matrix) can be diagonalized with matrices in $SO(n)$.

Lemma 6.10. The characteristic polynomial of a symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ has only real roots.

Theorem 6.11. (Spectral Theorem) Let $T : \mathbb{V}^n \to \mathbb{V}^n$ be a symmetric operator. There is an orthonormal basis of $\mathbb{V}^n$ with respect to which the matrix of T is diagonal.

Theorem 6.12. For every real symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ there is an orthogonal matrix $M \in O(n)$ such that $M^{-1}AM$ is diagonal.

Proposition 6.13. Let $T : \mathbb{V}^n \to \mathbb{V}^n$ be a symmetric operator on a Euclidean vector space. If λ and μ are two distinct eigenvalues of T then every eigenvector with eigenvalue λ is orthogonal to every eigenvector with eigenvalue μ.