5 Quantiles

Definition 5.1. *Let X be a random variable with cumulative distribution function $F : \mathbb{R} \to \mathbb{R}$ and let $\alpha \in (0, 1)$. A **quantile (percentile) of order $\boldsymbol{\alpha}$** is a number q_α satisfying the conditions*

$$\begin{aligned} P(X < q_\alpha) &\leq \alpha \\ P(X > q_\alpha) &\leq 1 - \alpha, \end{aligned} \tag{5.1}$$

or, equivalently,

$$P(X < q_\alpha) \leq \alpha \leq P(X \leq q_\alpha),$$

i.e.

$$F(q_\alpha - 0) \leq \alpha \leq F(q_\alpha). \tag{5.2}$$

To interpret (5.1), a quantile is a number with the property that it exceeds at most $100\alpha\%$ of the data, and is exceeded by at most $100(1 - \alpha)\%$ of the data.

Of all quantiles, the most important are:
The **median**, the number $M = q_{1/2}$; there are at most 50% of the data to the left of the median and at most 50% to its right.
The **quartiles** are the numbers

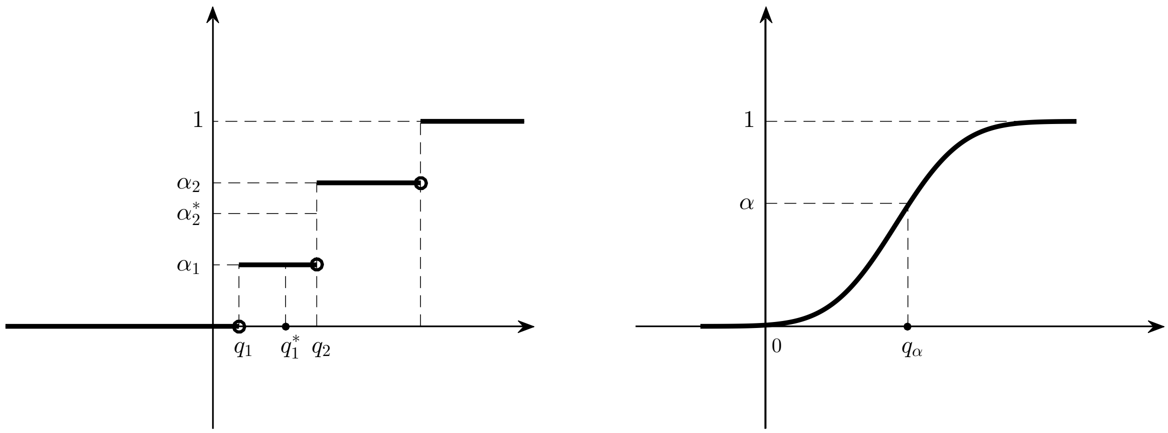
$$Q_1 = q_{1/4},\ Q_2 = M = q_{1/2},\ Q_3 = q_{3/4}.$$

Remark 5.2.
1. Quantiles are useful in statistical analysis of data. The median roughly locates the “middle” of a set of data, while the quartiles approximately locate every 25 % of a set of data. These will be discussed again in the next chapter.
2. If X is discrete, then a quantile can take an infinite number of values, if the line $y = \alpha$ and the curve $y = F(x)$ have in common a segment line (see Figure 1a).

The case when X is continuous is more interesting and the one we will use in Statistics. If X is continuous, then for each $\alpha \in (0, 1)$, there is a *unique* quantile q_α, given by

$$F(q_\alpha) = \alpha,$$

since F is a continuous function, $F(q_\alpha - 0) = \alpha = F(q_\alpha)$. In this case, for $F : \mathbb{R} \to \mathbb{R}$ there always exists $A \subset \mathbb{R}$ such that $F : A \to [0, 1]$ is both injective and surjective, hence invertible. Thus, in



(a) Discrete cdf

(b) Continuous cdf

Fig. 1: Quantiles

this case the unique quantile q_α is found by

$$q_\alpha \;=\; F^{-1}(\alpha). \tag{5.3}$$

Now, as an interpretation, let us recall that for continuous random variables, the cdf is expressed as an integral, which means as an *area*. So we have

$$\alpha \;=\; F(q_\alpha) \;=\; \int_{-\infty}^{q_\alpha} f(x)\,dx,$$

which is the area underneath the graph of the pdf f, above the x-axis and to the left of q_α. This is illustrated in Figure 2.

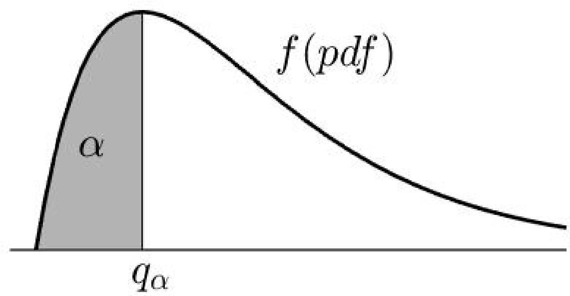


Fig. 2: Quantile of order α

6 Covariance and Correlation Coefficient

So far we have discussed numerical characteristics associated with one random variable. But oftentimes it is important to know if there is some kind of relationship between two (or more) random variables. So we need to define numerical characteristics that somehow measure that relationship.

Definition 6.1. *Let X and Y be random variables. The* ***covariance*** *of X and Y is the number*

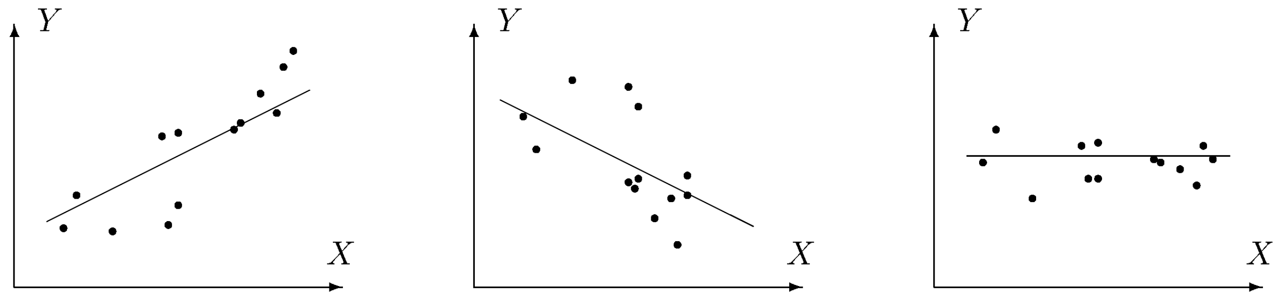
$$\operatorname{cov}(X,Y) \;=\; E\Big((X - E(X)) \cdot (Y - E(Y))\Big), \tag{6.1}$$

if it exists. The ***correlation coefficient*** *of X and Y is the number*

$$\rho(X,Y) \;=\; \frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)V(Y)}} \;=\; \frac{\operatorname{cov}(X,Y)}{\sigma(X)\sigma(Y)}, \tag{6.2}$$

if $\operatorname{cov}(X,Y), V(X), V(Y)$ *exist and* $V(X) \neq 0, V(Y) \neq 0$.

Notice the similarity between the definition of the covariance and that of the variance. The covariance measures the variation of two random variables *with respect to each other*. Just like with variance, large values (in absolute value) of the covariance show a strong relationship between X and Y, while small absolute values suggest a weak relationship. Unlike variance, covariance can also be negative. A negative value means that as the values of one variable increase, the values of the other decrease (see Figure 3).



(a) $\mathrm{Cov}(X,Y) > 0$ (b) $\mathrm{Cov}(X,Y) < 0$ (c) $\mathrm{Cov}(X,Y) = 0$

Fig. 3: Covariance

Theorem 6.2. *(Properties of covariance) Let X, Y and Z be random variables. Then the following properties hold:*

a) $\mathrm{cov}(X,X) = V(X)$.

b) $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$.

c) *If X and Y are independent, then* $\mathrm{cov}(X,Y) = \rho(X,Y) = 0$ (*we say that X and Y are* ***uncorrelated***).

d) $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab\,\mathrm{cov}(X,Y)$, *for all* $a, b \in \mathbb{R}$.

e) $\mathrm{cov}(X+Y,Z) = \mathrm{cov}(X,Z) + \mathrm{cov}(Y,Z)$.

Proof.
a) This follows directly from Definition 6.1.
b) A straightforward computation leads to

$$\begin{aligned}
\mathrm{cov}(X,Y) &= E\Big(XY - E(X)Y - E(Y)X + E(X)E(Y)\Big) \\
&= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

c) This follows from b), keeping in mind that X and Y are independent, so $E(XY) = E(X)E(Y)$.

d)

$$\begin{aligned}
V(aX+bY) &= E\Big(aX+bY-aE(X)-bE(Y)\Big)^2 \\
&= E\Big[a\Big(X-E(X)\Big)+b\Big(Y-E(Y)\Big)\Big]^2 \\
&= E\Big[a^2\Big(X-E(X)\Big)^2+2ab\Big(X-E(X)\Big)\Big(Y-E(Y)\Big) \\
&+ b^2\Big(Y-E(Y)\Big)^2\Big] \\
&= a^2V(X)+b^2V(Y)+2ab\,\mathrm{cov}(X,Y).
\end{aligned}$$

e)

$$\begin{aligned}
\mathrm{cov}(X+Y,Z) &= E\Big((X+Y-E(X)-E(Y))(Z-E(Z))\Big) \\
&= E\Big((X-E(X))(Z-E(Z)) \\
&+ (Y-E(Y))(Z-E(Z))\Big) \\
&= \mathrm{cov}(X,Z)+\mathrm{cov}(Y,Z).
\end{aligned}$$

□

Remark 6.3.
1. Property d) of Theorem 6.2 can be generalized to any number of variables:

$$V\left(\sum_{i=1}^n a_iX_i\right)=\sum_{i=1}^n a_i^2V(X_i)+2\sum_{1\le i<j\le n} a_ia_j\,\mathrm{cov}(X_i,X_j).$$

2. A consequence of a) and e) of Theorem 6.2 is the following property:

$$\mathrm{cov}(aX+b,X)=aV(X),\ \text{for all}\ \ a,b\in\mathbb{R}.$$

3. The converse of Theorem 6.2c) is *not* true. Independence is a much stronger condition.

Theorem 6.4. *Let X and Y be random variables. Then the following properties hold:*

a) $|\rho(X,Y)|\le 1$, *i.e.* $-1\le\rho(X,Y)\le 1$.

b) $|\rho(X,Y)|=1$ *if and only if there exist* $a,b\in\mathbb{R}$, $a\ne 0$, *such that* $Y=aX+b$.

Remark 6.5. As Theorem 6.4 states, the correlation coefficient $\rho(X,Y)$ measures the linear "trend" between the variables X and Y. When $\rho=\pm1$, there is "perfect linear correlation", so all the points

(X, Y) are on a straight line (see Figure 4). The closer its value is to ± 1, the "more linear" the relationship between X and Y is. This notion will be revisited in the next chapter.

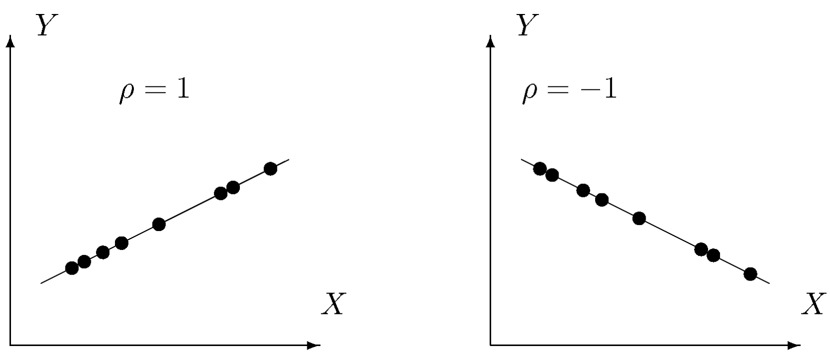


Fig. 4: Perfect correlation

7 Inequalities

Inequalities can be useful in estimation theory, for approximating probabilities or numerical characteristics associated with a random variable.

Proposition 7.1 (Hölder's Inequality). *Let X and Y be random variables and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$E(|XY|) \leq (E(|X|^p))^{\frac{1}{p}} \cdot (E(|Y|^q))^{\frac{1}{q}}. \tag{7.1}$$

Remark 7.2.

1. One important particular case of Hölder's inequality is for $p = q = 2$,

$$E(|XY|) \leq \sqrt{E(X^2)} \cdot \sqrt{E(Y^2)}, \tag{7.2}$$

known as **Schwarz's inequality**.

2. A particular case of the above inequality is for $Y = 1$,

$$E(|X|) \leq \sqrt{E(X^2)}, \tag{7.3}$$

known as **Cauchy-Buniakowsky's inequality**.

Proposition 7.3 (**Minkowsky's Inequality**). *Let X and Y be random variables and let $p > 1$. Then*

$$(E(|X+Y|^p))^{\frac{1}{p}} \leq (E(|X|^p))^{\frac{1}{p}} + (E(|Y|^p))^{\frac{1}{p}}. \tag{7.4}$$

Proposition 7.4 (**Lyapunov's Inequality**). *Let X be a random variable, let $0 < a < b$ and $c \in \mathbb{R}$. Then*

$$(E(|X-c\,|^a))^{\frac{1}{a}} \leq \left(E(|X-c\,|^b)\right)^{\frac{1}{b}}. \tag{7.5}$$

The next two inequalities are *specific* to random variables. They have many applications in statistical analysis.

Proposition 7.5 (**Markov's Inequality**). *Let X be a random variable and let $a > 0$. Then*

$$P(|X| \geq a) \leq \frac{1}{a}E(|X|). \tag{7.6}$$

Proof. Let $A = \{e \in S \mid |X(e)| \geq a\}$, with the indicator function

$$I_A(e) \;=\; \begin{cases} 0, & |X(e)| < a \\ 1, & |X(e)| \geq a. \end{cases}$$

Then

$$a\;I_A(e) \;=\; \begin{cases} 0, & |X(e)| < a \\ a, & |X(e)| \geq a. \end{cases}$$

Now, if $|X(e)| < a$, then

$$aI_A(e) = 0 \leq |X(e)|$$

and if $|X(e)| \geq a$, then

$$aI_A(e) = a \leq |X(e)|.$$

So, either way, $aI_A(e) \leq |X(e)|$, $\forall e \in S$. That means, as random variables, $aI_A \leq |X|$, which means the same thing is true for their expected values, $E(aI_A) \leq E(|X|)$. Now, the pdf of aI_A is

$$aI_A \begin{pmatrix} 0 & a \\ 1 - P(|X| \geq a) & P(|X| \geq a) \end{pmatrix},$$

so

$$E(aI_A) = aP(|X| \geq a).$$

Thus,

$$aP(|X| \geq a) \leq E(|X|),$$

i.e.

$$P(|X| \geq a) \leq \frac{1}{a}E(|X|).$$

$\square$

Proposition 7.6 (Chebyshev's Inequality). *Let X be a random variable and let $\varepsilon > 0$. Then*

$$P\Big(|X - E(X)| \geq \varepsilon\Big) \leq \frac{1}{\varepsilon^2}V(X), \tag{7.7}$$

or, equivalently,

$$P\Big(|X - E(X)| < \varepsilon\Big) \geq 1 - \frac{1}{\varepsilon^2}V(X), \tag{7.8}$$

Proof.
Apply Markov's inequality (7.6) to $\Big(X - E(X)\Big)^2$ and $a = \varepsilon^2$, to get

$$P\Big((X - E(X))^2 \geq \varepsilon^2\Big) \leq \frac{1}{\varepsilon^2}\, E\Big((X - E(X))^2\Big),$$

i.e.

$$P\Big(|X - E(X)| \geq \varepsilon\Big) \leq \frac{1}{\varepsilon^2}\, V(X),$$

and, equivalently,

$$1 - P\Big(|X - E(X)| < \varepsilon\Big) \leq \frac{1}{\varepsilon^2}V(X),$$
$$P\Big(|X - E(X)| < \varepsilon\Big) \geq 1 - \frac{1}{\varepsilon^2}V(X).$$

$\square$

Example 7.7. Suppose the number of errors in a new software, X, has expectation $E(X) = 20$. Find a bound for the probability that there are at least 30 errors if the standard deviation is
a) $\sigma(X) = 2$;
b) $\sigma(X) = 5$.

Solution. According to (7.7),

$$P(|X - 20| \geq \varepsilon) \leq \frac{(\sigma(X))^2}{\varepsilon^2}.$$

So,

$$
\begin{aligned}
P(X \geq 30) &= P(X - 20 \geq 10) \\
&\leq P\Big((X - 20 \geq 10) \cup (X - 20 \leq -10)\Big) \\
&= P(|X - 20| \geq 10) \\
&\leq \frac{(\sigma(X))^2}{100}
\end{aligned}
$$

a) If $\sigma(X) = 2$, we can estimate that

$$P(X \geq 30) \leq 0.04.$$

b) However, for a larger standard deviation of $\sigma(X) = 5$, the estimation is

$$P(X \geq 30) \leq 0.25.$$

∎

8 Central Limit Theorem

Central Limit Theorems are also results that can help approximate characteristics of random variables. First, a little bit of preparation.

Given the special nature of random variables, as opposed to numerical variables, there are various types of convergence that can be defined for sequences of such variables, having to do with probability-related notions (convergence in probability, convergence in mean, convergence almost surely, etc.)

Definition 8.1. *Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables with cumulative distribution functions $F_n = F_{X_n}$, $n \in \mathbb{N}$ and let X be a random variable with cdf $F = F_X$. Then X_n* ***converges in distribution*** *to X, denoted by $X_n \xrightarrow{d} X$, if*

$$\lim_{n \to \infty} F_n(x) = F(x), \tag{8.1}$$

for every $x \in \mathbb{R}$, a point of continuity of F.

Remark 8.2. Convergence in distribution is especially important, because the cdf of a random variable is used to compute probabilities. Knowing the limiting cdf of a sequence of random variables makes possible the computation of probabilities (and other characteristics) in the "long run". So such results can be helpful in estimating characteristics of random variables as n gets larger.

A statement about the limit in distribution of a sequence of random variable is called a **limit theorem**. If the limit variable has a Normal distribution, then such a result is called a **central limit theorem**. So, there are *many* such results, the name "Central Limit Theorem" is just generic.

We want to discuss a central limit theorem that applies to the following case: Suppose $X_1, X_2, \ldots,$ X_n are **independent, identically distributed (iid)** random variables (this is a case that will be used oftenly in Statistics). Having the same pdf, they have the same expectation $\mu = E(X_i)$ and the same standard deviation $\sigma = \mathrm{Std}(X_i) = \sqrt{V(X_i)}$. We are interested in the random variable

$$S_n \;=\; X_1 + \ldots + X_n.$$

This case appears in many applications and in many statistical procedures. We see right away that

$$\begin{aligned} E(S_n) &= n\mu, \\ V(S_n) &= n\sigma^2. \end{aligned}$$

How does S_n behave for large n?

The *pure* sum S_n diverges. In fact, this should be anticipated because

$$V(S_n) \;=\; n\sigma^2 \to \infty,$$

so the variability of S_n grows unboundedly, as n goes to infinity.

The *average* S_n/n converges. Indeed, in this case, we have

$$V(S_n/n) \;=\; \frac{1}{n^2}V(S_n) \;=\; \frac{\sigma^2}{n} \to 0,$$

so the variability of S_n/n vanishes as $n \to \infty$.

An interesting case is the variable $S_n/\sqrt{n}$,

$$\begin{aligned} E(S_n/\sqrt{n}) &= \sqrt{n}\mu, \\ V(S_n/\sqrt{n}) &= \sigma^2, \end{aligned}$$

which neither diverges nor converges. In fact, it behaves like some random variable. The following theorem (CLT) states that this variable has approximately Normal distribution for large n. In fact, the result is for its *reduced (standardized)* variable

$$\frac{\frac{S_n}{\sqrt{n}} - E\left(\frac{S_n}{\sqrt{n}}\right)}{\mathrm{Std}\left(\frac{S_n}{\sqrt{n}}\right)} = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

Theorem 8.3. [Central Limit Theorem (CLT)]
Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with expectation $\mu = E(X_i)$ and standard deviation $\sigma = \sigma(X_i)$ and let

$$S_n = X_1 + \ldots + X_n. \tag{8.2}$$

Then, as $n \to \infty$, the reduced sum

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z \in N(0,1), \tag{8.3}$$

which means

$$F_{Z_n} \to F_{N(0,1)}, \text{ i.e. } P(Z_n \le x) \to P(Z \le x),\ \forall x \in \mathbb{R}, \text{ as } n \to \infty.$$

Remark 8.4.
1. This result can be very helpful, since $F_{N(0,1)}(x) = \Phi(x)$ is Laplace's function (see equation (6.6) in Lecture 5), whose values are known.
2. The CLT can be used as an approximation tool for n "large". In practice, it has been determined that that means $n > 30$.

Example 8.5. A disk has free space of 330 megabytes. Is it likely to be sufficient for 300 independent images, if each image has expected size of 1 megabyte with a standard deviation of 0.5 megabytes?

Solution. For each $i = 1, 2, \ldots, n$ (i.e. for each image), let X_i denote the space it takes, in megabytes. Then the *total* space taken by all 300 images will be

$$S_n = X_1 + X_2 + \ldots + X_n$$

and there will be sufficient space on the disk if

$$S_n \le 330.$$

We have $n = 300, \mu = 1, \sigma = 0.5$. The number of images n is large enough, so the CLT applies to their total size S_n. Then

$$
\begin{aligned}
P(\text{sufficient space}) &= P(S_n \leq 330) \\
&= P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \frac{330 - n\mu}{\sigma\sqrt{n}}\right) \\
&= P\left(Z_n \leq \frac{330 - 300 \cdot 1}{0.5 \cdot 10\sqrt{3}}\right) \\
&= P(Z_n \leq 3.46) \\
&\overset{\text{CLT}}{\approx} P(Z \leq 3.46) \;=\; \Phi(3.46) \;=\; 0.9997,
\end{aligned}
$$

a very high probability, hence, the available disk space is very likely to be sufficient.

■