Seminar 7

The dimension of a space is given by the number of vectors in its (canonical) basis.

If $f : V \to V$, then $dim(V) = dim(ker(f)) + dim(Im(f))$.

If $ker(f) = \{0\}$, then $dim(ker(f)) = 0$.

If $A \subseteq B$, then $dim(A) \leq dim(B)$.

We have: $dim(A) + dim(B) = dim(A + B) + dim(A \cap B)$.

$\bar{S}$ is a complement of S, where $S \bigoplus \bar{S} = V$, the whole space.

1. $A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} = \{(x, y, 0) \mid x, y \in \mathbb{R}\} = \{(x, 0, 0) + (0, y, 0) \mid x, y \in \mathbb{R}\} = \{x(1, 0, 0)+y(0, 1, 0) \mid x, y \in \mathbb{R}\} =< (1, 0, 0), (0, 1, 0) >$ which is a basis if the vectors are linearly independent $\iff a(1, 0, 0)+ b(0, 1, 0) = (0, 0, 0) \iff (a, b, 0) = (0, 0, 0) \Rightarrow a = b = 0$ (true) $\Rightarrow< (1, 0, 0), (0, 1, 0) >$ is a base of A. So, $dim(A) = 2$.

   $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} = \{(x, y, z) \in \mathbb{R}^3 \mid z = -x - y\} = \{(x, y, -x - y) \mid x, y \in \mathbb{R}\} = \{(x, 0, -x) + (0, y, -y) \mid x, y \in \mathbb{R}\} = \cdots =< (1, 0, -1), (0, 1, -1) >$ which is a basis if the vectors are linearly independent $\iff a(1, 0, -1) + b(0, 1, -1) = (0, 0, 0) \Rightarrow a = b = 0 \Rightarrow$ is a base of B. So, $dim(B) = 2$.

   $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\} = \{(x, x, x) \mid x \in \mathbb{R}\} = \cdots = < (1, 1, 1) >$ which is a basis, as we have one vector (so linearly independent). So, $dim(C) = 1$.

2. (i) S is a subspace of K^n if $S \neq \emptyset$ (which is true, as $(0, 0, \ldots, 0) \in S$) and $\forall a, b \in K, \forall x, y \in S \Rightarrow ax + by \in S$. So, $ax + by = a(x_1, \ldots, x_n) + b(y_1, \ldots, y_n) = \cdots = (ax_1 + by_1, \ldots, ax_n + by_n)$, which is in S if $ax_1 + by_1 + \cdots + ax_n + by_n = 0 \iff a(x_1 + \ldots x_n) + b(y_1 + \ldots y_n) = a \cdot 0 + b \cdot 0 = 0 \Rightarrow S$ is a subspace of K^n.

   (ii) From $x_1 + \cdots + x_n = 0 \Rightarrow x_n = -x_1 - x_2 \cdots - x_{n-1}$.
   So, $S = \{(x_1, \ldots, x_{n-1}, -x_1 - \cdots - x_{n-1}) \mid x_1, \ldots, x_{n-1} \in K\} = \{(x_1, 0, \ldots, 0, -x_1)+\cdots+(0, 0, \ldots, 0, x_{n-1}, -x_{n-1}) \mid x_1, \ldots, x_{n-1} \in K\} = \cdots =< (1, 0, \ldots, 0, -1), \ldots, (0, 0, \ldots, 0, 1, -1) >$ which is a basis if the vectors are linearly independent (you can prove this yourselves). So, $dim(S) = n - 1$.

3. We know that $(\mathbb{C}, +)$ is an Abelian group. Now, for $\mathbb{C}$ to be a vector space, we need to see if the 4 conditions hold:

(a) $(k_1 + k_2)z = k_1 z + k_2 z$ (true)
(b) $k(z_1 + z_2) = kz_1 + kz_2$ (true)
(c) $(k_1 k_2)z = k_1(k_2 z)$ (true)
(d) $1 \cdot z = z$ (true)

Now, $\forall z \in \mathbb{C}, \exists a, b \in \mathbb{R}$ such that $z = a \cdot 1 + b \cdot i \Rightarrow \mathbb{C} =< 1, i >$ which is a basis, as 1 and i are linearly independent and $dim(\mathbb{C}) = 2$.

4. f is an $\mathbb{R}$-linear map if $\forall a, b \in \mathbb{R}, \forall x, y \in \mathbb{R}^3 : f(ax + by) = af(x) + bf(y)$.

   So, $f(a(x_1, x_2, x_3) + b(y_1, y_2, y_3)) = f(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) = \cdots = a(x_2, -x_1) + b(y_2, -y_1) = af(x) + bf(y)$.

   $ker(f) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = (0, 0)\} = \{(x, y, z) \in \mathbb{R}^3 \mid (y, -x) = (0, 0)\} = \{(0, 0, z) \mid z \in \mathbb{R}\} =< (0, 0, 1) >$.

   So, $dim(ker(f)) = 1$.

   $Im(f) = \{(y, -x) \in \mathbb{R}^2 \mid f(x, y, z) = (y, -x)\} = \{(y, 0) + (0, -x) \mid x, y \in \mathbb{R}\} =< (1, 0), (0, -1) >$. So, $dim(Im(f)) = 2$.

5. $ker(f) = \{(x, y, z) \in \mathbb{R}^3 \mid (-y+5z, x, y-5z) = (0, 0, 0)\} \Rightarrow -y+5z = 0$ and $x = 0$ and $y - 5z = 0 \Rightarrow x = 0$ and $y = 5z \Rightarrow ker(f) = \{(0, 5z, z) \mid z \in \mathbb{R}\} =< (0, 5, 1) >$, with $dim(ker(f)) = 1$.

   $Im(f) = \{(-y+5z, x, y-5z) \in \mathbb{R}^3 \mid f(x, y, z) = (-y, 0, y)+(5z, 0, -5z)+(0, x, 0)\} =< (-1, 0, 1), (5, 0, -5), (0, 1, 0) >$, but $(5, 0, -5) = -5(-1, 0, 1)$ (i.e. they are not linearly independent) so a basis for $Im(f) =< (-1, 0, 1), (0, 1, 0) >$, with $dim(Im(f)) = 2$.

6. For A we need to find a third vector in the basis, which is linearly independent with the other two. So, $a(1, 0, 0) + b(0, 1, 0) + c(x, y, z) = (0, 0, 0) \iff a = b = c = 0 \Rightarrow a + cx = 0$ and $b + cy = 0$ and $cz = 0 \Rightarrow x, y, z \in \mathbb{R}$ (not all zero) $\Rightarrow (x, y, z) = (0, 0, 1)$.

   For B the same as above $\Rightarrow a(1, 0, -1) + b(0, 1, -1) + c(x, y, z) = (0, 0, 0) \iff a = b = c = 0 \Rightarrow a + cx = 0$ and $b + cy = 0$ and $-a-b+cz = 0 \Rightarrow c(x+y+z) = 0 \Rightarrow x+y+z \neq 0 \Rightarrow (x, y, z) = (1, 1, 0)$.

   For C we need to find two vectors in the basis, which are linearly independent with the third one. So, we can add the vectors $(a, b, c) = (1, 1, 0)$ and $(x, y, z) = (1, 0, 1)$.

7. (i) We can easily see that $A =< (-2, 1, 0), (-3, 0, 1) >$. So, we need to complete this generator to a basis in $\mathbb{R}^3$.

   Let (a, b, c) be the vector we need to put there:

   $$\Rightarrow \begin{vmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ a & b & c \end{vmatrix} \neq 0 \Rightarrow a + 2b + 3c \neq 0.$$

   So, we can take $a = 0, b + 0, c + 1 \Rightarrow (0, 0, 1)$ generates the complement of A.

   $\bar{A} =< (0, 0, 1) >= \{(0, 0, z) \mid z \in \mathbb{R}\}$.

   (ii) $B =< (1, 0, 0), (0, 0, 1) >$ so, to complete it to $\mathbb{R}_3[X]$ we need to add another vector. It is easy to see that we can take a vector from the canonical basis $(0, 1, 0)$. So, $\bar{B} =< (0, 1, 0) >= \{cX^2 \mid c \in \mathbb{R}\}$.

8. $dim(S) + dim(U) = dim(S \cap U) + dim(S + U) = dim(T \cap U) + dim(T + U) = dim(T) + dim(U) \Rightarrow dim(S) = dim(T)$. As $S \subseteq T$, we know $dim(S) \leq dim(T)$. So, if their dimensions are equal $\Rightarrow S = T$.

9. First, rewrite S as a generated subset and T as a set.

   $S =< (0, 1, 0), (0, 0, 1) >$ and $T = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$ (you can simply find those).

   Now, $S \cap T = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } x - y + z = 0\} = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } y = z\} =< (0, 1, 1) >$, with $dim(S \cap T) = 1$.

   As, $S, T \subseteq \mathbb{R}^3 \Rightarrow S + T \subseteq \mathbb{R}^3 \Rightarrow dim(S + T) \leq dim(\mathbb{R}^3) = 3$.

   From $dim(S) + dim(T) = dim(S \cap T) + dim(S + T)$ and $dim(S) = dim(T) = 2 \Rightarrow 2 + 2 = 1 + dim(S + T) \Rightarrow dim(S + T) = 4 - 1 = 3 = dim(\mathbb{R}^3) \Rightarrow S + T = \mathbb{R}^3$.

10. For S we need to see if the vetors are linearly independent.

    $$a \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

    We obtain the system: $\begin{cases} a + b = 0 \\ a = 0 \\ b = 0 \end{cases} \Rightarrow$ they are linearly independent, so $dim(S) = 2$.

    The same goes for $T \Rightarrow a \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} b \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

We obtain the system: $\begin{cases} a = 0 \\ a + b = 0 \\ b + 0 \end{cases}$ $\Rightarrow$ they are linearly independent, so $dim(T) = 2$.

We know that $dim(S+T) = dim(S \cup T)$, so we need to see how many vectors (from both generators) are linearly independent.

$a \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

We obtain the system: $\begin{cases} a + b = 0 \Rightarrow a = -b \\ a + c = 0 \Rightarrow c = -a \Rightarrow c = b \\ b + c + d = 0 \Rightarrow b + b - b = 0 \Rightarrow b = 0 \\ b + d = 0 \Rightarrow d = -b \end{cases}$

$\Rightarrow a = b = c = d = 0 \Rightarrow$ they are all linearly independent. So, $dim(S+T) = 4$.

Since $dim(S) + dim(T) = dim(S+T) + dim(S \cap T) \Rightarrow dim(S \cap T) = 0$.