Seminar Nr.2, Classical Probability; Rules of Probability; Conditional Probability; Independent Events

Theory Review

Classical Probability: $P(A) = \dfrac{\text{nr. of favorable outcomes}}{\text{total nr. of possible outcomes}} = \dfrac{N_f}{N_t}$.

Mutually Exclusive Events: $A,\ B$ **m. e. (disjoint, incompatible)** $<=> P(A \cap B) = 0$.

Rules of Probability:

$P(\overline{A}) = 1 - P(A)$;
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
$P(A \setminus B) = P(A) - P(A \cap B)$.

Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$.

Independent Events: $A,\ B$ **ind.** $<=> P(A \cap B) = P(A)P(B) <=> P(A|B) = P(A)$.

Total Probability Rule: $\{A_i\}_{i \in I}$ a partition of S, then $P(E) = \sum\limits_{i \in I} P(A_i)P(E|A_i)$.

Multiplication Rule: $P\left(\bigcap\limits_{i=1}^{n} A_i\right) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2)\ \ldots\ P\left(A_n | \bigcap\limits_{i=1}^{n-1} A_i\right)$.

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1. The faces of a cube are painted each in a different color (the cube is transparent on the inside). Then the cube is broken into 1000 smaller, equally-sized cubes and one such cube is randomly picked. Find the probability of the following events:
a) A: the cube picked has exactly three colored faces;
b) B: the cube picked has exactly two colored faces;
c) C: the cube picked has exactly one colored face;
d) D: the cube picked has no colored faces.

Solution:

This is classical probability. First, we compute the denominator, N_t, since it is the same for all events A, B, C and D. Obviously,

$$N_t = 1000.$$

Then, for each event, first we locate the cubes and then we count them.
a) These are the cubes on corners. There are 8 corners, so $N_f = 8$ and

$$P(A) = \frac{8}{1000}.$$

b) These are the cubes on edges, but not on corners. There are 12 edges and on each edge there are 8 such cubes (without the corner ones!). Thus $N_f = 12 \cdot 8 = 96$ and

$$P(B) = \frac{96}{1000}.$$

c) These are the cubes on faces, but not on edges. There are 6 faces and $8 \cdot 8 = 64$ cubes on each face, making $N_f = 6 \cdot 64 = 384$ and

$$P(C) = \frac{384}{1000}.$$

d) We could compute this probability the same way as the previous ones (these would be the interior cubes), but let us use something else instead. Notice that in this experiment, the only possible numbers of colored faces are $0, 1, 2$ or 3. In other words, events A, B, C and D cover all possibilities, $S = A \cup B \cup C \cup D$. They are also mutually exclusive, obviously. Then they form a partition of the sample space. Recall that the sum of probabilities of all events in a partition is equal to 1 (the probability of the sure event). Since we have already computed the other ones, we have

$$P(D) = 1 - P(A) - P(B) - P(C) = \frac{512}{1000}.$$

2. (Pigeonhole Principle) A postman distributes n letters in N mailboxes. What is the probability of the event A: there are m letters in a given (fixed) mailbox ($0 \leq m \leq n$)?

<u>Solution</u>:

For the number of possible outcomes N_t:
The 1^{st} letter can be distributed in any of the N mailboxes, so there are N choices. The same thing is true for the 2^{nd} letter, so another N choices. So, if we had 2 letters, we would have N^2 cases. Since the same thing is true for each of the n letters, we have

$$N_t = N^n.$$

For the number of favorable outcomes N_f:
First, the m letters can be chosen in C_n^m ways. Then, once those m letters are determined, the other $n - m$ letters should be distributed in the remaining $N - 1$ mailboxes. From above, this can be done in $(N-1)^{n-m}$ ways. Since these two problems (choosing the m letters and distributing the other $n - m$ letters) are independent, the number of favorable outcomes is

$$N_f = C_n^m (N-1)^{n-m}$$

and

$$P(A) = \frac{C_n^m (N-1)^{n-m}}{N^n}.$$

3. (Breaking Passwords) An account uses 8-character passwords, consisting of letters (distinguishing between lower-case and capital letters) and digits. A spy program can check about 1 million passwords per second.
a) On the average, how long will it take the spy program to guess your password?
b) What is the probability that the spy program will break your password within a week (event A)?
c) Same questions, if capital letters are not used.

Solution:
a) First, let us see how long it will take the spy program to check **all** passwords.
There are 62 characters to choose from (26 lower-case letters, 26 capital letters and 10 digits) and we need to choose 8, with repetitions, so there are

$$(62)^8 = 2.183 \cdot 10^{14}$$

8-character possible passwords.
Now, how fast does the spy program check them? A speed of 1 million per second, means
$10^6 \cdot 60 \cdot 60 \cdot 24 = 8.64 \cdot 10^{10}$ per day,
$8.64 \cdot 10^{10} \cdot 7 = 6.048 \cdot 10^{11}$ per week,
$6.048 \cdot 10^{11} \cdot 52 = 3.145 \cdot 10^{13}$ per year.
So it will take it

$$\frac{21.83}{3.145} = 6.9412,$$

about 7 years, to check *all* the passwords and, hence, to guess yours. Since it's just as probable to try (and guess) a combination at *any time* in the interval $[0, 6.9412]$, on average, it will take about half of that, i.e. 3 and a half years for the spy program to crack your password.

b) This is just classical probability with $N_t = 2.183 \cdot 10^{14}$ and $N_f = 6.048 \cdot 10^{11}$, so

$$p_1 = P(A) = \frac{6.048 \cdot 10^{11}}{2.183 \cdot 10^{14}} \simeq 0.00277,$$

very small!

c) Without capital letters, there are fewer characters to choose from (only $26 + 10 = 36$), so only $(36)^8 = 2.8211 \cdot 10^{12}$ passwords to test. The spy program works at the same speed, so nothing changes there. Now it will take only

$$\frac{2.8211 \cdot 10^{12}}{8.64 \cdot 10^{10}} = 32.6516$$

DAYS to test them all. On the average, in this case, the spyware will break your password in about 16 days.
For the second part, now $N_t = 2.821 \cdot 10^{12}$ and N_f is the same. So the probability that the password

will be cracked within a week is

$$p_2 = P(A) = \frac{6.048 \cdot 10^{11}}{2.821 \cdot 10^{12}} \simeq 0.2144,$$

much bigger!

4. (System Reliability) Compute the reliability of the system in Figure 1 if each of the five components is operable with probability 0.92, independently of each other.

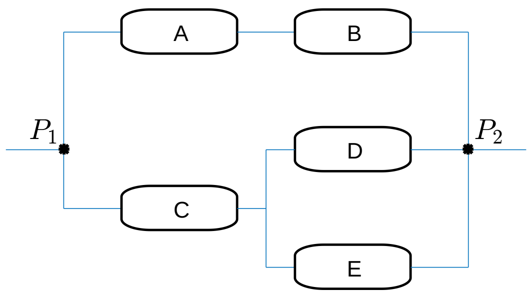

Figure 1: System Reliability

Solution:

The *reliability* is the probability of the system being operable. Now, how do we interpret two components being linked **sequentially**, versus being connected **in parallel**? For sequential connection, *both* components have to be operable (that means the word "and" would be used), whereas for parallel connection one OR the other would have to work. So now we know which set operations to use.

Let us denote by B1 the upper branch connecting points P_1 and P_2 and by B2 the lower branch. Then the reliability is

$$R = P(\text{the system is operable}) = P(B1 \cup B2),$$

since $B1$ and $B2$ are linked in parallel. We can compute this two ways: the direct way, or using the contrary event. Let us do it directly now and we will use the other way later. We have

$$P(B1 \cup B2) = P(B1) + P(B2) - P(B1 \cap B2) = P(B1) + P(B2) - P(B1)P(B2),$$

because the independence of all events implies the independence of any combinations/operations of/with them (so $B1$ and $B2$ are also independent).

Now, let us compute the probabilities $P(B1)$ and $P(B2)$. We have

$$P(B1) = P(A \cap B) = P(A) \cdot P(B) = (0.92)^2 = 0.8464,$$

since A and B are linked sequentially and operate independently of each other.

Similarly, we go further:

$$P(B2) = P(C \cap (D \cup E)) = P(C) \cdot P(D \cup E),$$

since C and $D \cup E$ are also independent.
For $P(D \cup E)$, we will use the complementary event. Thus,

$$\begin{aligned} P(D \cup E) &= 1 - P(\overline{D \cup E}) = 1 - P(\overline{D} \cap \overline{E}) \\ &= 1 - P(\overline{D}) \cdot P(\overline{E}) = 1 - (1 - 0.92)^2 = 0.9936. \end{aligned}$$

Then

$$P(B2) = P(C) \cdot P(D \cup E) = 0.92 \cdot 0.9936 = 0.9141.$$

Finally,

$$R = P(B1 \cup B2) = 0.8464 + 0.9141 - 0.8464 \cdot 0.9141 = 0.9868.$$

5. Among employees of a certain firm, 70% know C/C++, 60% know Fortran and 50% know both. What portion of programmers

a) does not know Fortran (event A_1)?
b) does not know C/C++ and does not know Fortran (event A_2)?
c) knows C/C++, but not Fortran (event A_3)?
d) Are "knowing C/C++" and "knowing Fortran" independent of each other?
e) What is the probability that someone who knows Fortran, also knows C/C++ (event A_4)?
f) What is the probability that someone who knows C/C++, does not also know Fortran (event A_5)?

Solution:

Denote the events:
C: employee knows C/C++,
F: employess knows Fortran.
We know

$$P(C) = 0.7,\ P(F) = 0.6 \text{ and } P(C \cap F) = 0.5.$$

Then we express (carefully!) all the events with the right operations.

a) $P(A_1) = P(\overline{F}) = 1 - P(F) = 0.4;$

b) $P(A_2) = P(\overline{C} \cap \overline{F}) = P(\overline{C \cup F}) = 1 - P(C \cup F) = 1 - \Big(P(C) + P(F) - P(C \cap F)\Big)$
$= 1 - (0.7 + 0.6 - 0.5) = 0.2;$

c) $P(A_3) = P(C \cap \overline{F}) = P(C \setminus F) = P(C) - P(C \cap F) = 0.7 - 0.5 = 0.2;$

d) Since

$$P(C \cap F) = 0.5 \neq 0.42 = P(C) \cdot P(F),$$

the answer is “NO”, they are not;

e) $P(A_4) = P(C|F) = \dfrac{P(C \cap F)}{P(F)} = \dfrac{0.5}{0.6} = \dfrac{5}{6} = 0.833;$

f) $P(A_5) = P(\overline{F}|C) = \dfrac{P(\overline{F} \cap C)}{P(C)} = \dfrac{P(A_3)}{P(C)} = \dfrac{0.2}{0.7} = \dfrac{2}{7} = 0.286.$
Alternatively, we can use the probability of the contrary event. The same formula holds for conditional probabilities:
$P(A_5) = P(\overline{F}|C) = 1 - P(F|C) = 1 - \dfrac{P(F \cap C)}{P(C)} = 1 - \dfrac{0.5}{0.7} = \dfrac{2}{7} = 0.286.$

6. Three shooters aim at a target. The probabilities that they hit the target are 0.4, 0.5 and 0.7, respectively. Find the probability that the target is hit exactly once.

<u>Solution</u>:
Refer to Problem 5 in Seminar 1. In fact, the hard work was done last time! Denote the events
A: the target is hit exactly once,
A_i: the i^{th} shooter hits the target, $i = 1, 2, 3$.
Then $P(A_1) = 0.4, P(A_2) = 0.5, P(A_3) = 0.7$ and, from last time,

$$A = \left(A_1 \cap \overline{A_2} \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap A_2 \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap \overline{A_2} \cap A_3\right)$$

Since the union is disjoint and the events A_1, A_2 and A_3 are independent, we have

$$P(A) = 0.4 \cdot 0.5 \cdot 0.3 + 0.6 \cdot 0.5 \cdot 0.7 + 0.6 \cdot 0.5 \cdot 0.7 = 0.36.$$

<u>Note.</u> This is a classical example of a Poisson probabilistic model.

7. Under good weather conditions, 80% of flights arrive on time. During bad weather, only 50% of flights arrive on time. Tomorrow, the chance of good weather is 60%. What is the probability that your flight will arrive on time?

<u>Solution</u>: Denote the events
A: the flight arrives on time,
G: there’s good weather,
B: there’s bad weather.
Then, in fact, $B = \overline{G}$, we have $P(G) = 0.6, P(\overline{G}) = 0.4$ and $\{G, \overline{G}\}$ form a partition of the sample space.

We are given $P(A|G) = 0.8$ and $P(A|\overline{G}) = 0.5$ and we want to compute $P(A)$, *without* any condition. By the Total Probability Rule,

$$P(A) = P(A|G)P(G) + P(A|\overline{G})P(\overline{G}) = 0.8 \cdot 0.6 + 0.5 \cdot 0.4 = 0.68.$$