Changing reference frames.

1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 0.1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

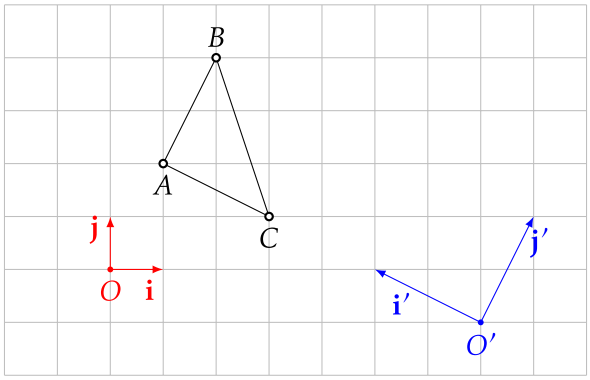

Figure 0.1: Coordinate systems 2D.

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

3. Consider the tetrahedron $ABCD$ (see Fig. 0.2) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

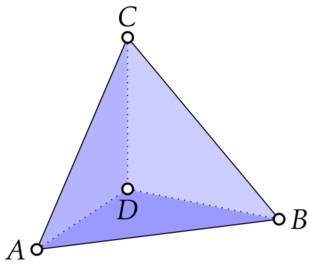

Figure 0.2: Tetrahedron

4. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 0.3) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

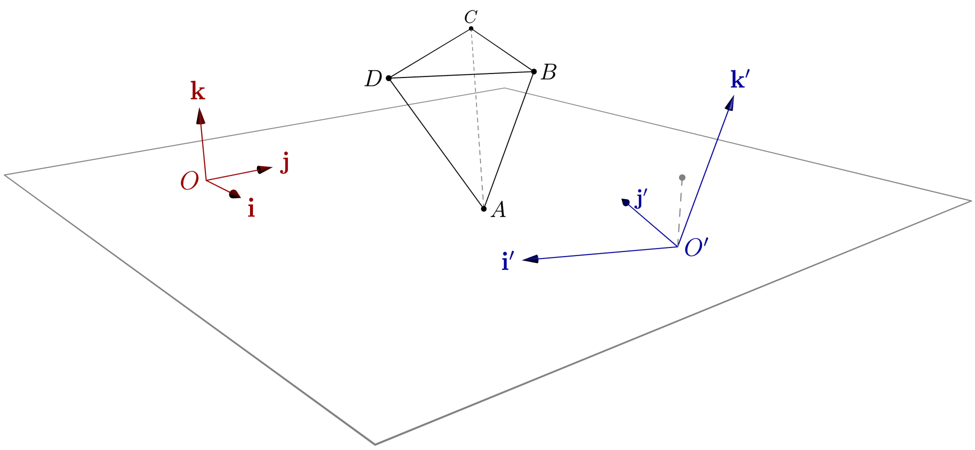

Figure 0.3: Coordinate systems 3D.

5. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

Projections and reflections on/in hyperplanes.

6. Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

7. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

8. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

9. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

10. Give Cartesian equations for the line passing through the point $M(1,0,7)$, parallel to the plane $\pi : 3x - y + 2z - 15 = 0$ and intersecting the line

$$\ell : \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

11. In $\mathbb{E}^3$, show that the orthogonal reflection $\mathrm{Ref}_{\pi}^{\perp}(x)$ in the plane $\pi : \langle n, x \rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2} n$.

12. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$