Seminar 5

$V = A \oplus B$ if $V = A + B$ and $A \cap B = \{0\}$. Or $\forall v \in V, \exists! s \in S, t \in T$ such that $v = s + t$.

$f : A \to B$ **endomorphism** if $A = B$ and f homomorphism.

$ker(f) = \{x \in R \mid f(x) = 0\}$ and $Im(f) = \{f(x) \mid x \in R\}$.

1. (i) $< 1, X, X^2 >= \{a + bX + cX^2 \mid a, b, c \in \mathbb{R}\} = \mathbb{R}_2[X]$.

   (ii) $< \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} >=$
   $\{a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b, c, d \in \mathbb{R}\} =$
   $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}\} = M_2(\mathbb{R})$.

2. (i) $(0, a, b) = (0, a, 0) + (0, 0, b) = a \cdot (0, 1, 0) + b \cdot (0, 0, 1) \Rightarrow$
   $A =< (0, 1, 0), (0, 0, 1) >$.

   (ii) $a + b + c = 0 \Rightarrow a = -b - c = -(b + c) \Rightarrow (-(b + c), b, c) = (-b, b, -0) + (-c, 0, c) = b(-1, 1, 0) + c(-1, 0, 1) \Rightarrow B =< (-1, 1, 0), (-1, 0, 1) >$.

   (iii) $(a, a, a) = a(1, 1, 1) \Rightarrow C =< (1, 1, 1) >$.

3. In order for those two to be equal, we may show that, for example, the vectors c, d, e can be written as a linear combination of the vectors a, b.

   It is easy to see that: $\begin{cases} c = a + b \\ d = a - b \\ e = 3a - b \end{cases}$

4. $S =< (-1, 1, 0), (-1, 0, 1) >\Rightarrow s_1 = (-1, 1, 0)$ and $s_2 = (-1, 0, 1)$.

   $T =< (1, 1, 1) >\Rightarrow t = (1, 1, 1)$.

   From *Seminar4*, we know that S, T are subspaces of $\mathbb{R}^{\mathbb{R}}$. To prove that $\mathbb{R}^3 = S \oplus T$, we prove that $S + T = \mathbb{R}^3$ and $S \cap T = \{0_3\}$.

   $\forall v \in \mathbb{R}^3, \exists! s \in S, \exists! t \in T$ such that $v = s + t \iff (v_1, v_2, v_3) = a \cdot s_1 + b \cdot s_2 + c \cdot t \iff (v_1, v_2, v_3) = (-a, a, 0) + (-b, 0, b) + (c, c, c) \iff$

   $\begin{cases} v_1 = -a - b + c \\ v_2 = a + c \\ v_3 = b + c \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{3}v_1 + \frac{2}{3}v_2 - \frac{1}{3}v_3, \\ b = -\frac{1}{3}v_1 - \frac{1}{3}v_2 + \frac{2}{3}v_3 \\ c = \frac{1}{3}(v_1 + v_2 + v_3) \end{cases}$, so they are unique.

5. Remember:

$$f:\mathbb{R}\to\mathbb{R},\ \text{f-odd} \Rightarrow \forall x\in\mathbb{R}, f(-x)=-f(x)$$

$$f:\mathbb{R}\to\mathbb{R},\ \text{f-even} \Rightarrow f(-x)=f(x)$$

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$S\neq\emptyset$, as $\theta(x)=0\in S$ and $t\neq\emptyset$, as $f(x)=-x\in T$.

Take $f,g\in S, a,b\in\mathbb{R}\Rightarrow (af+bg)(-x)=(af)(-x)+(bg)(-x)=-af(x)-bg(x)=-(af+bg)(x)\in S\Rightarrow S\leq\mathbb{R}^{\mathbb{R}}$.

Take $f,g\in T, a,b\in\mathbb{R}\Rightarrow (af+bg)(-x)=(af)(-x)+(bg)(-x)=af(x)+bg(x)=(af+bg)(x)\in T\Rightarrow T\leq\mathbb{R}^{\mathbb{R}}$.

Take $f:\mathbb{R}\to\mathbb{R}, g\in S, h\in T$, as $f(x)=g(x)+h(x)$. Then $f(-x)=g(-x)+h(-x)=-g(x)+h(x)\Rightarrow g(x)=\frac{1}{2}(f(x)+f(-x))\in S$ and $h(x)=\frac{1}{2}(f(x)-f(-x))\in R$. So, g,h are unique functions, with which we can write any function $f:\mathbb{R}\to\mathbb{R}$. Now, for the intersection: if $f(-x)=-f(x)$ and $f(-x)=f(x)\Rightarrow f(x)=-f(x)\Rightarrow f(x)=\theta(x)$. So $S\cap T=\{\theta(x)=0\}$.

6. $f((x_1,y_1)+(x_2,y_2))=f(x_1+x_2,y_1+y_2)=(x_1+x_2+y_1+y_2,x_1+x_2-y_1-y_2)=(x_1+y_1,x_1-y_1)+(x_2+y_2,x_2-y_2)=f(x_1,y_1)+f(x_2,y_2)$

   $f(k(x,y))=f(kx,ky)=(kx+ky,kx-ky)=(k(x+y),k(x-y))=k(x+y,x-y)=kf(x,y)$

   $\Rightarrow f$ endomorphism.

   $g((x_1,y_1)+(x_2,y_2))=g(x_1+x_2,y_1+y_2)=(2x_1+2x_2-y_1-y_2,4x_1+4x_2-2y_1-2y_2)=g(x_1,y_1)+g(x_2,y_2)$

   $g(k(x,y))=(2kx-ky,4kx-2ky)=(k(2x-y),k(4x-2y))=kg(x,y)$

   $\Rightarrow g$ endomorphism.

   $h((x_1,y_1,z_1)+(x_2,y_2,z_2))=h(x_1+x_2,y_1+y_2,z_1+z_2)=(x_1+x_2-y_1-y_2,y_1+y_2-z_1-z_2,z_1+z_2-x_1-x_2)=h(x_1,y_1,z_1)+h(x_2,y_2,z_2)$

   $h(k(x,y))=(kx-ky,ky-kz,kz-kx)=(k(x-y),k(y-z),k(z-x))=kh(x,y,z)$

   $\Rightarrow h$ endomorphism.

7. (i) $f(x,y)=(ax+by,cx+dy)$

   $f(x_1+x_2,y_1+y_2)=(ax_1+ax_2+by_1+by_2,cx_1+cx_2+dy_1+dy_2)=(ax_1+by_1,cx_1+dy_1)+(ax_2+by_2,cx_2+dy_2)=f(x_1,y_1)+f(x_2,y_2)$

$f(k(x,y)) = (kax+kby, kcx+kdy) = k(ax+by, cx+dy) = kf(x,y)$
$\Rightarrow f$ endomorphism.

(ii) $g(x,y) = (a+x, b+y)$
For $a = b = 0 \Rightarrow g(x,y) = (x,y)$ - endomorphism of $\mathbb{R}^2$. But $\forall a,b \in \mathbb{R}^* \Rightarrow g(x_1+x_2, y_1+y_2) = (a+x_1+x_2, b+y_1+y_2) = (a+x_1, b+y_1) + (x_2,y_2) = g(x_1,y_1) + (x_2,y_2) \Rightarrow g$ is NOT an endomorphism.

8. $\forall (x,y), (m,n) \in \mathbb{R}^2, \forall k \in \mathbb{R}$ we have:

$$f((x,y)+(m,n)) = f(x+m, y+n) = f(x,y) + f(m,n)$$

$$f(k(x,y)) = f(kx,ky) = kf(x,y)$$

(Homework)

9. $ker(f) = \{(x,y) \mid (x+y, x-y) = (0,0)\} \Rightarrow x+y = 0$ and $x-y = 0 \Rightarrow x = y$ and $2y = 0 \Rightarrow x = y = 0 \Rightarrow ker(f) = \{(0,0)\}$.

$Im(f) = \{(x+y, x-y) \mid x,y \in \mathbb{R}\} = \{(x,x) + (y,-y) \mid x,y \in \mathbb{R}\} = \{x(1,1) + y(1,-1) \mid x,y \in \mathbb{R}\} \Rightarrow Im(f) =< (1,1), (1,-1) >$.

$ker(g) = \{(x,y) \mid (2x-y, 4x-2y) = (0,0)\} \Rightarrow 2x-y = 0$ and $4x-2y = 0 \Rightarrow 2x = y$. So, take $x = a \in \mathbb{R} \Rightarrow y = 2a \in \mathbb{R} \Rightarrow ker(g) = \{(a,2a) \mid a \in \mathbb{R}\} =< (1,2) >$

$Im(g) = \{(2a-b, 4a-2b) \mid x,y \in \mathbb{R}\} = \{(2a,4a) + (-b,-2b) \mid x,y \in \mathbb{R}\} = \{a(2,4) + b(-1,-2) \mid x,y \in \mathbb{R}\} \Rightarrow Im(g) =< (2,4), (-1,-2) >$

$ker(h) = \{(x,y,z) \mid (x-y, y-z, z-x) = (0,0,0)\} \Rightarrow x-y = 0, y-z = 0, z-x = 0 \Rightarrow x = y = z \Rightarrow ker(h) = \{(x,x,x) \mid x \in \mathbb{R}\} =< (1,1,1) >$

$Im(h) = \{(a-b, b-c, c-a) \mid a,b,c \in \mathbb{R}\} = \{(a,0,a) + (-b,b,0) + (0,-c,c) \mid a,b,c \in \mathbb{R}\} = \{a(1,0,-1) + b(-1,1,0) + c(0,-1,1) \mid a,b,c \in \mathbb{R}\} \Rightarrow Im(h) =< (1,0,-1), (-1,1,0), (0,-1,1) >$.

10. $S \neq \emptyset$, as $f(0) = 0 \in S$.

$\forall x,y \in S \Rightarrow x+y = f(x) + f(y) = f(x+y) \in S$, as f is an endomorphism.

$\forall a \in K, \forall x \in S \Rightarrow ax = af(x) = f(ax) \in S$, as f is an endomorphism.

So, $S \leq V$.