CHAPTER 5

Affine morphisms

Contents

Let $\phi : V \to W$ be a linear map between the $\mathbb{R}$-vector spaces V and W. Let $\boldsymbol{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V and let $\boldsymbol{w} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be a basis for W. In your Algebra course, Lecture 8, you used the notation $[\phi]_{\boldsymbol{v},\boldsymbol{w}}$ for the matrix of the linear map ϕ with respect to the bases $\boldsymbol{v}$ and $\boldsymbol{w}$. We will use the notation

$$\mathrm{M}_{\boldsymbol{w},\boldsymbol{v}}(\phi) = [\phi]_{\boldsymbol{v},\boldsymbol{w}}$$

Notice that the indices $\boldsymbol{v}, \boldsymbol{w}$ are *reversed*.

You have also learned that if $\psi : W \to U$ is another linear map, to some vector space U with basis $\boldsymbol{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, then

$$\mathrm{M}_{\boldsymbol{u},\boldsymbol{v}}(\psi \circ \phi) = \mathrm{M}_{\boldsymbol{u},\boldsymbol{w}}(\psi) \cdot \mathrm{M}_{\boldsymbol{w},\boldsymbol{v}}(\phi).$$

(See Algebra, Lecture 8, Theorem 3.4.9). In particular if $V = W$ then

$$\mathrm{M}_{\boldsymbol{w},\boldsymbol{w}}(\phi) = \mathrm{M}_{\boldsymbol{w},\boldsymbol{v}}(\mathrm{Id}_V) \cdot \mathrm{M}_{\boldsymbol{v},\boldsymbol{v}}(\phi) \cdot \mathrm{M}_{\boldsymbol{v},\boldsymbol{w}}(\mathrm{Id}_V) = \mathrm{M}_{\boldsymbol{v},\boldsymbol{w}}(\mathrm{Id}_V)^{-1} \cdot \mathrm{M}_{\boldsymbol{v},\boldsymbol{v}}(\phi) \cdot \mathrm{M}_{\boldsymbol{v},\boldsymbol{w}}(\mathrm{Id}_V).$$

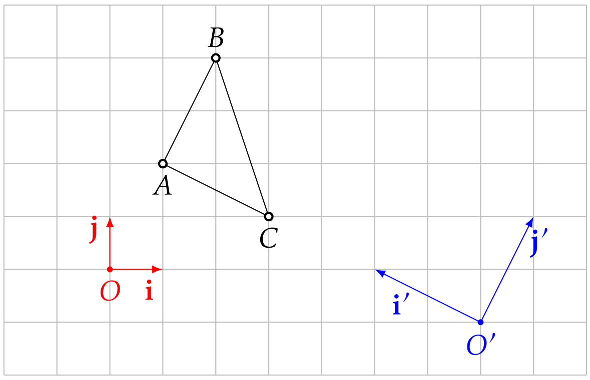
So, the matrix of ϕ with respect to the basis $\boldsymbol{w}$ is obtained from the matrix of ϕ with respect to the basis $\boldsymbol{v}$ by conjugating with the matrix $\mathrm{M}_{\boldsymbol{v},\boldsymbol{w}}(\mathrm{Id}_V)$. We call the matrix $\mathrm{M}_{\boldsymbol{v},\boldsymbol{w}} := \mathrm{M}_{\boldsymbol{v},\boldsymbol{w}}(\mathrm{Id}_V)$ the *change of basis matrix from the basis $\boldsymbol{w}$ to the basis $\boldsymbol{v}$*. If $\mathcal{K}$ and $\mathcal{K}'$ are two coordinate systems, we write $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ for the base change matrix w.r.t. the corresponding bases.

5.1 Changing coordinate systems

Example (In dimension 2)**.** Let $\mathcal{K} = O\mathbf{ij}$ and $\mathcal{K}' = O'\mathbf{i}'\mathbf{j}'$ be two coordinate systems (reference frames) of $\mathbb{E}^2$. Suppose that we know O', $\mathbf{i}'$ and $\mathbf{j}'$ relative to $\mathcal{K}$:

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{i}' = -2\mathbf{i} + \mathbf{j} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{j}' = \mathbf{i} + 2\mathbf{j} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{K}}.$$

How can we translate the coordinates of points from $\mathcal{K}$ to $\mathcal{K}'$?



We can do this in two steps: (a) first we change the origin, i.e. we go from $O\mathbf{ij}$ to $O'\mathbf{ij}$ and (b) we change the directions of the coordinate axes, i.e. we go from $O'\mathbf{ij}$ to $O'\mathbf{i}'\mathbf{j}'$. The first step is just a translation and the second step corresponds to the usual base change from linear algebra.

(a) Change the origin.

(b) Change the direction of the axes.

For the first step

$$[\overrightarrow{O'A}]_{\mathcal{K}'} = [\overrightarrow{O'A}]_{\mathcal{K}} = [\overrightarrow{OA}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}}.$$

For the second step let $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ denote the base change matrix from the basis in $\mathcal{K}'$ to the basis in $\mathcal{K}$. Then

$$[\overrightarrow{OA}]_{\mathcal{K}} = \mathrm{M}_{\mathcal{K},\mathcal{K}'}[\overrightarrow{OA}]_{\mathcal{K}'} \quad \text{and} \quad [\overrightarrow{OA}]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OA}]_{\mathcal{K}}.$$

Hence composing the two operations, (a) and (b), we obtain

$$[\overrightarrow{OA}]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([\overrightarrow{OA}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OA}]_{\mathcal{K}} - \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OO'}]_{\mathcal{K}} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OA}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}'}$$

Hence, the formula for changing coordinates from the system $\mathcal{K}$ to the system $\mathcal{K}'$ is

$$[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([A]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[A]_{\mathcal{K}} + [O]_{\mathcal{K}'} \tag{5.1}$$

Suppose now that a point A is given and that the coordinates of A in the frame $\mathcal{K}$ (relative to the coordinate frame/coordinate system $\mathcal{K}$) are $(1,2)$. Then the coordinates of A relative to $\mathcal{K}'$ are

$$[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{K}} + [O]_{\mathcal{K}'}$$

Since we know $\mathbf{i}'$ and $\mathbf{j}'$ with respect to $\mathbf{i}$ and $\mathbf{j}$, we can write down the matrix $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ and then $\mathrm{M}_{\mathcal{K}',\mathcal{K}} = \mathrm{M}^{-1}_{\mathcal{K},\mathcal{K}'}$. Since we know the coordinates of O' with respect to $\mathcal{K}$, it is more convenient to use the first equality in (5.1)

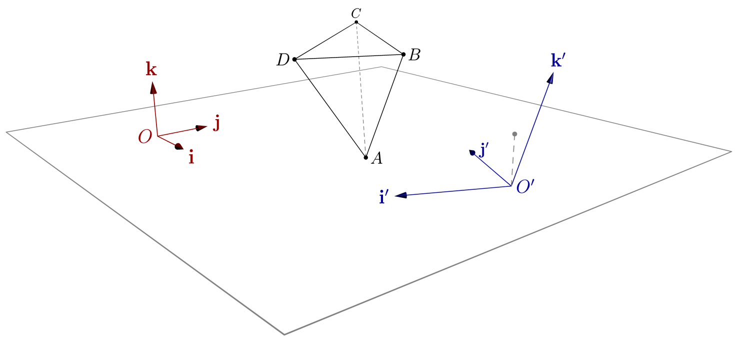
$$[A]_{\mathcal{K}'} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{K}} - \begin{bmatrix} 7 \\ -1 \end{bmatrix}_{\mathcal{K}} \right) = \frac{1}{-5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{K}} + \begin{bmatrix} 7 \\ -1 \end{bmatrix}_{\mathcal{K}} \right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Theorem 5.1. Let $\mathcal{K} = O\mathbf{e}_1 \dots \mathbf{e}_n$ and $\mathcal{K}' = O'\mathbf{f}_1, \dots, \mathbf{f}_n$ be two coordinate systems of $\mathbb{E}^n$. Denote by $\boldsymbol{e} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and by $\boldsymbol{f} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ the two bases of $\mathbb{V}^n$. Let $\mathrm{M}_{\mathcal{K}',\mathcal{K}}$ be the change of basis matrix from the basis $\boldsymbol{e}$ to the basis $\boldsymbol{f}$ and let $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ be the change of basis matrix from the basis $\boldsymbol{f}$ to the basis $\boldsymbol{e}$. For any point $P \in \mathbb{E}^n$ we have

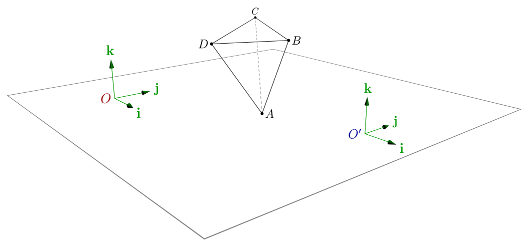
$$[P]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}} \cdot ([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}^{-1}_{\mathcal{K},\mathcal{K}'} \cdot ([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}} \cdot [P]_{\mathcal{K}} + [O]_{\mathcal{K}'}. \tag{5.2}$$

Example (In dimension 3). Let $\mathcal{K} = O\mathbf{ijk}$ and $\mathcal{K}' = O'\mathbf{i}'\mathbf{j}'\mathbf{k}'$ be two coordinate systems (reference frames) of $\mathbb{E}^3$. Suppose that we know O', $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ relative to $\mathcal{K}$:

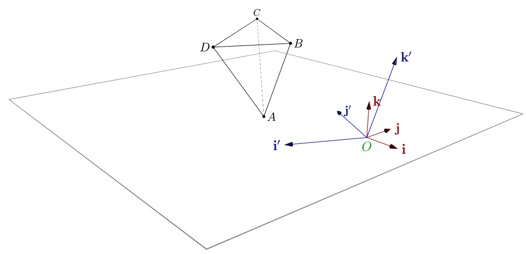
$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{i}' = -\mathbf{i} - 2\mathbf{j} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{j}' = -2\mathbf{i} + \mathbf{j} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{k}' = \mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{K}}.$$



The coordinates with respect to $\mathcal{K}'$ can be obtained from the coordinates with respect to $\mathcal{K}$ in two steps:



(a) Change the origin.



(b) Change the direction of the axes.

If B is the point with coordinates $(1,5,1)$ with respect to $\mathcal{K}$, then

$$[B]_{\mathcal{K}'} = \mathrm{M}^{-1}_{\mathcal{K},\mathcal{K}'} \cdot ([B]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} \right) = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

5.2 Affine morphisms

Definition. An affine morphism $\phi : \mathbb{E}^n \to \mathbb{E}^m$ is a map which relative to some reference frame $\mathcal{K}$ of $\mathbb{E}^n$ and some reference frame $\mathcal{K}'$ of $\mathbb{E}^m$ can be expressed as

$$[\phi(P)]_{\mathcal{K}'} = A \cdot [P]_{\mathcal{K}} + b \tag{5.3}$$

with $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $b \in \mathrm{Mat}_{m\times 1}(\mathbb{R})$.

- In (5.3) both A and b depend on the choice of the coordinate systems $\mathcal{K}$ and $\mathcal{K}'$.
- If $n = m$, so, if $\phi : \mathbb{E}^n \to \mathbb{E}^n$ then ϕ is called an *affine endomorphisms*. The set of all such endomorphisms is denoted by $\mathrm{End}_{\mathrm{aff}}(\mathbb{E}^n)$.
- The morphism ϕ is invertible if and only if the matrix A is invertible.
- If $n = m$ and ϕ is invertible then ϕ is called an *affine automorphism* or *affine transformation*. The set of all affine transformations of $\mathbb{E}^n$ is denoted by $\mathrm{AGL}(\mathbb{E}^n)$
- If $\mathcal{K} = \mathcal{K}'$ in (5.3) and b is the zero vector, then ϕ is a linear map. In particular

$$\mathrm{GL}(\mathbb{V}^n) \subseteq \mathrm{AGL}(\mathbb{E}^n).$$

5.3 Projections and reflections in a hyperplane

Definition. A hyperplane in $\mathbb{E}^n$ is a subset of points H satisfying a linear equation,

$$H : a_1x_1 + \cdots + a_nx_n + a_{n+1} = 0$$

with respect to some coordinate system $O\mathbf{e}_1 \ldots \mathbf{e}_n$.

- Hyperplanes in $\mathbb{E}^2$ are lines.
- Hyperplanes in $\mathbb{E}^3$ are planes.
- Consider a line $\ell \subseteq \mathbb{E}^n$ passing through a point $Q(q_1,\dots,q_n)$ and having $\mathbf{v}(v_1,\dots,v_n)$ as direction vector:
$$\ell = \{Q + t\mathbf{v} : t \in \mathbb{R}\}. \tag{5.4}$$
Consider a hyperplane $H \subseteq \mathbb{E}^n$ given by the Cartesian equation
$$H : a_1x_1 + \cdots + a_nx_n + a_{n+1} = 0 \tag{5.5}$$
The intersection $\ell \cap H$ can be described as follows
$$Q + t\mathbf{v} \in \ell \cap H \quad \Leftrightarrow \quad a_1(q_1 + tv_1) + \cdots + a_n(q_n + tv_n) + a_{n+1} = 0$$
So, the intersection point (if it exists) is
$$Q' = Q - \frac{a_1q_1 + \cdots + a_nq_n + a_{n+1}}{v_1q_1 + \cdots + v_nq_n}\mathbf{v}. \tag{5.6}$$

5.3.1 Tensor products

Definition. Let $\mathbf{v}(v_1,\dots,v_n)$ and $\mathbf{w}(w_1,\dots,w_n)$ be two vectors. The *tensor product* $\mathbf{v}\otimes\mathbf{w}$ is the $n\times n$ matrix defined by $(\mathbf{v}\times\mathbf{w})_{ij} = v_iw_j$. In other words

$$\mathbf{v}\otimes\mathbf{w} = \mathbf{v}\cdot\mathbf{w}^t = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1,\dots,w_n \end{bmatrix} = \begin{bmatrix} v_1w_1 & \dots & v_1w_n \\ \vdots & & \vdots \\ v_nw_1 & \dots & v_nw_n \end{bmatrix}.$$

Proposition 5.2. The map $\mathbb{R}^n\times\mathbb{R}^n \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ given by $(\mathbf{v},\mathbf{w}) \mapsto \mathbf{v}\otimes\mathbf{w}$ has the following properties:

1. It is linear in both arguments,
2. $(\mathbf{v}\otimes\mathbf{w})^t = \mathbf{w}\otimes\mathbf{v}$.

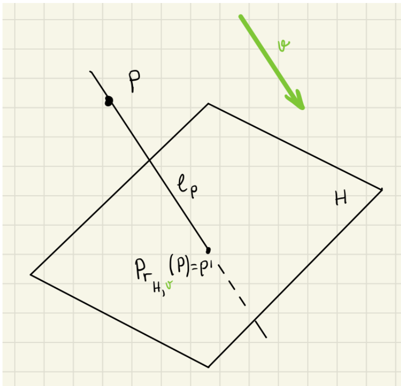
- The tensor product of two vectors is also called outer product - to be compared with the inner product which is
$$\langle\mathbf{v},\mathbf{w}\rangle = \mathbf{v}^t\cdot\mathbf{w} = \begin{bmatrix} v_1,\dots,v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1w_n + \cdots + v_nw_n.$$
- It is easy to show that for three vectors $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}^n$ we have
$$(\mathbf{u}\otimes\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}.$$

5.3.2 Parallel projection on a hyperplane

Definition. Let H be a hyperplane and let $\mathbf{v}$ be a vector in $\mathbb{V}^n$ which is not parallel to H. For any point $P \in \mathbb{E}^n$ there is a unique line ℓ_P passing through P and having $\mathbf{v}$ as direction vector. The line ℓ_P is not parallel to H, hence, it intersects H in a unique point P'. We denote P' by $\mathrm{Pr}_{H,\mathbf{v}}(P)$ and call it the *projection of the point P on the hyperplane H parallel to* $\mathbf{v}$. This gives a map

$$\mathrm{Pr}_{H,\mathbf{v}} : \mathbb{E}^n \to \mathbb{E}^n$$

called, the *projection on the hyperplane H parallel to* $\mathbf{v}$.



- With respect to the reference frame $\mathcal{K}$, the hyperplane H has an equation as in (5.5).
- By (5.6), $\mathrm{Pr}_{H,\mathbf{v}}(P) = P - \frac{\varphi(P)}{\operatorname{lin}\varphi(\mathbf{v})}\mathbf{v}$ where $\phi(P(p_1,\dots,p_n)) = a_1p_1 + \cdots + a_np_n + p_{n+1}$.
- Hence, if we denote by $p_1',\dots,p_n'$ the coordinates of the projected point $\mathrm{Pr}_{H,\mathbf{v}}(P)$ then

$$\begin{cases} p_1' = p_1 + \mu v_1 \\ \quad\vdots \\ p_n' = p_n + \mu v_n \end{cases} \quad \text{where} \quad \mu = -\frac{a_1p_1 + \cdots + a_np_n + a_{n+1}}{a_1v_1 + \cdots + a_nv_n}.$$

- In matrix form, we can rearrange this as follows

$$\begin{bmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \end{bmatrix} = \underbrace{\frac{1}{\operatorname{lin}\varphi(\mathbf{v})}\begin{bmatrix} \sum_{i=1}^{n,i\neq 1} a_iv_i & -a_2v_1 & -a_3v_1 & \dots & -a_nv_1 \\ -a_1v_2 & \sum_{i=1}^{n,i\neq 2} a_iv_i & -a_3v_2 & \dots & -a_nv_2 \\ \vdots & & & & \vdots \\ -a_1v_n & -a_2v_n & \dots & -a_{n-1}v_n & \sum_{i=1}^{n,i\neq n} a_iv_i \end{bmatrix}}_{\operatorname{lin}\mathrm{Pr}_{H,\mathbf{v}}}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} - \frac{a_{n+1}}{\operatorname{lin}\varphi(\mathbf{v})}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- It is possible to give a more compact description of the above matrix form if we use tensor products:

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \frac{1}{\operatorname{lin}\varphi(\mathbf{v})}\left((\mathbf{v}^t\cdot\mathbf{a})\,\mathrm{Id}_n - \underbrace{\mathbf{v}\cdot\mathbf{a}^t}_{\mathbf{v}\otimes\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\operatorname{lin}\varphi(\mathbf{v})}[\mathbf{v}]_{\mathcal{K}}$$

  where $\mathbf{a} = (a_1,\dots,a_n)^t$.

- If we further notice that $\operatorname{lin}\varphi(\mathbf{v}) = \mathbf{v}^t\cdot\mathbf{a}$ then

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

- Parallel projections on hyperplanes are affine morphisms. Obviously, they are not bijective, so

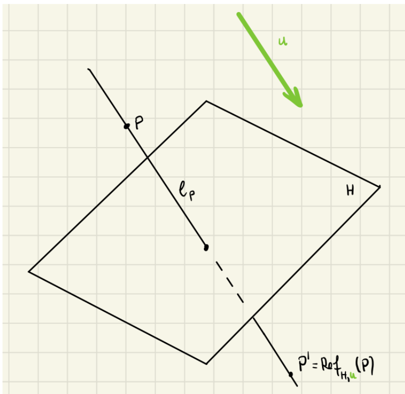
$$\mathrm{Pr}_{H,\mathbf{v}} \in \mathrm{End}_{\mathrm{aff}}(\mathbb{E}^n) \quad \text{but} \quad \mathrm{Pr}_{H,\mathbf{v}} \notin \mathrm{AGL}(\mathbb{E}^n).$$

5.3.3 Parallel reflection in a hyperplane

Definition. Let H be a hyperplane and let $\mathbf{v}$ be a vector in $\mathbb{V}^n$ which is not parallel to H. For any point $P \in \mathbb{E}^n$ there is a unique point P' such that $\mathrm{Pr}_{H,\mathbf{v}}(P)$ is the midpoint of the segment $[PP']$. We denote P' by $\mathrm{Ref}_{H,\mathbf{v}}(P)$ and call it the *reflection of the point P in the hyperplane H parallel to* $\mathbf{v}$. This gives a map

$$\mathrm{Ref}_{H,\mathbf{v}} : \mathbb{E}^n \to \mathbb{E}^n$$

called, the *reflection in the hyperplane H parallel to* $\mathbf{v}$.



- With respect to the reference frame $\mathcal{K}$, the hyperplane H has an equation as in (5.5).
- Since $\mathrm{Pr}_{H,\mathbf{v}}(P)$ is the midpoint of the segment $[PP']$, with respect to $\mathcal{K}$ we have

$$[P]_{\mathcal{K}} - \frac{\varphi(P)}{\operatorname{lin}\varphi(\mathbf{v})}[\mathbf{v}]_{\mathcal{K}} = \frac{[P]_{\mathcal{K}} + [P']_{\mathcal{K}}}{2} \quad\Leftrightarrow\quad 2[P]_{\mathcal{K}} - 2\frac{\varphi(P)}{\operatorname{lin}\varphi(\mathbf{v})}[\mathbf{v}]_{\mathcal{K}} = [P]_{\mathcal{K}} + [P']_{\mathcal{K}}.$$

Therefore

$$[\mathrm{Ref}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = [P]_{\mathcal{K}} - 2\frac{\varphi(P)}{\mathrm{lin}\,\varphi(\mathbf{v})}[\mathbf{v}]_{\mathcal{K}}.$$

- As in the case of $\mathrm{Pr}_{H,\mathbf{v}}$, it is possible to give a compact description of the matrix form if we use tensor products:

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - 2\frac{\mathbf{v}\otimes\mathbf{a}}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - 2\frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

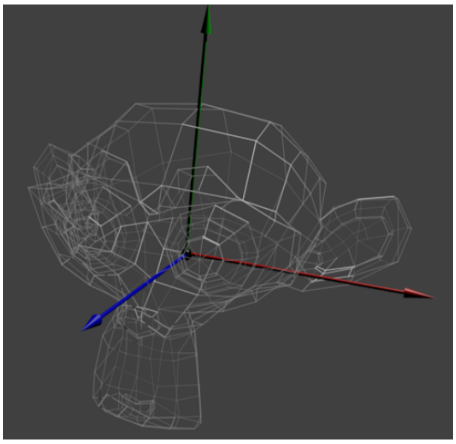
where $\mathbf{a} = (a_1,\ldots,a_n)^t$.

- Parallel reflections in hyperplanes are affine morphisms. Obviously, they are bijective, so

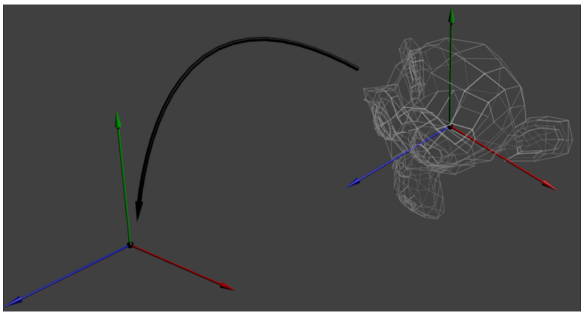
$$\mathrm{Ref}_{H,\mathbf{v}} \in \mathrm{AGL}(\mathbb{E}^n) \subseteq \mathrm{End}_{\mathrm{aff}}(\mathbb{E}^n).$$

5.4 Applications

- [Changing reference frames] Changing coordinates is so common in real life applications that it is impossible to do any meaningful modeling without coordinate changes. Take for example the OpenGL-pipeline (a process under which objects are rendered on the screen in order to simulate a 3D scene). In this process there are two dimensional changes of coordinates for instance when a PNG-image is transformed into an OpenGL-texture, or when an OpenGL texture is translated into device coordinates before being displayed on the screen.



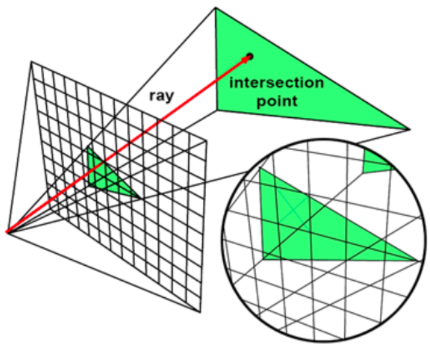
(a) Model space.



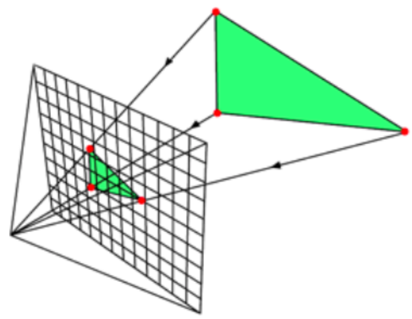
(b) World space.

Three dimensional changes of coordinates are needed in the OpenGL-pipeline as well as in any modeling software (like Blender or Maya). Complex 3D-models are constructed by several people, even several teams. The different pieces are constructed separately, each in its own coordinate system. When they are ready, the whole object is assembled by putting all pieces in a common coordinate system. In this context one calls the coordinate system of a single piece 'model space' and the coordinate system where the whole object is placed in, is called the 'world space'. This is important when modeling machines such as spacecrafts or cars. It is also important when constructing computer games. As an example, above is Suzanne, the Blender monkey.

- [Projecting] There are two main types of algorithms which project a 3D scene in computer graphics: ray-tracing algorithms and rasterization algorithms. Ray-tracing algorithms intersect rays with planes determined by the triangles in the scene while rasterization algorithms try to project the triangles on the display screen.



(a) Ray-tracing.



(b) Rasterization.

It is clearly much more efficient to construct a projection map like $\mathrm{Pr}_{H,\ell}$ and project all the triangles, however, this only works for parallel projections. In order to simulate perspective, rasterization algorithms use a projective transformation before using a projection like the one described in the previous paragraphs. Ray-tracing algorithms are conceptually much simpler but they are much more resource intensive.