INTRODUCTORY

1. THE CONCEPT OF A FUNCTION. Underlying the formal cal-
culi which we shall develop is the concept of a function, as it
appears in various branches of mathematics, either under that
name or under one of the synonymous names, "operation" or "trans-
formation." The study of the general properties of functions,
independently of their appearance in any particular mathematical
(or other) domain, belongs to formal logic or lies on the boun-
dary line between logic and mathematics. This study is the orig-
inal motivation for the calculi — but they are so formulated
that it is possible to abstract from the intended meaning and
regard them merely as formal systems.

A <u>function</u> is a rule of correspondence by which when any-
thing is given (as <u>argument</u>) another thing (the <u>value</u> of the
function for that argument) may be obtained. That is, a func-
tion is an operation which may be applied on one thing (the ar-
gument) to yield another thing (the value of the function). It
is not, however, required that the operation shall necessarily
be applicable to everything whatsoever; but for each function
there is a class, or range, of possible arguments -- the class
of things to which the operation is significantly applicable --
and this we shall call the <u>range of arguments</u>, or <u>range of the
independent variable</u>, for that function. The class of all values
of the function, obtained by taking all possible arguments, will
be called the <u>range of values</u>, or <u>range of the dependent variable</u>.

If f denotes a particular function, we shall use the nota-
tion $(f\alpha)$ for the value of the function f for the argument
α. If α does not belong to the range of arguments of f, the
notation $(f\alpha)$ shall be meaningless.

It is, of course, not excluded that the range of arguments
or range of values of a function should consist wholly or partly
of functions. The derivative, as this notion appears in the el-

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ementary differential calculus, is a familiar mathematical exam-
ple of a function for which both ranges consist of functions.
Or, turning to the integral calculus, if in the expression
$\int_0^1 (f x) dx$ we take the function f as independent variable, we
are led to a function for which the range of arguments consists
of functions and the range of values, of numbers. Formal logic
provides other examples; thus the existential quantifier, accord-
ing to the present account, is a function for which the range of
arguments consists of propositional functions, and the range of
values consists of truth-values.

In particular it is not excluded that one of the elements of
the range of arguments of a function f should be the function
f itself. This possibility has frequently been denied, and in-
deed, if a function is defined as a correspondence between two
previously given ranges, the reason for the denial is clear.
Here, however, we regard the operation or rule of correspondence,
which constitutes the function, as being first given, and the
range of arguments then determined as consisting of the things to
which the operation is applicable. This is a departure from the
point of view usual in mathematics, but it is a departure which
is natural in passing from consideration of functions in a spec-
ial domain to the consideration of function in general, and it
finds support in consistency theorems which will be proved below.

The identity function I is defined by the rule that $(I x)$
is x, whatever x may be; then in particular (II) is I. If
a function H is defined by the rule that (Hx) is I, what-
ever x may be, then in particular (HH) is I. If Σ is the
existential quantifier, then $(\Sigma \Sigma)$ is the truth-value truth.

The functions I and H may also be cited as examples of
functions for which the range of arguments consists of all things
whatsoever.

2. EXTENSION AND INTENSION. The foregoing discussion
leaves it undetermined under what circumstances two functions
shall be considered the same.

The most immediate and, from some points of view, the best
way to settle this question is to specify that two functions f
and g are the same if they have the same range of arguments and,
for every element a that belongs to this range, (fa) is the

same as $(g\,a)$. When this is done we shall say that we are deal-
ing with <u>functions in extension</u>.

It is possible, however, to allow two functions to be dif-
ferent on the ground that the rule of correspondence is differ-
ent in meaning in the two cases although always yielding the same
result when applied to any particular argument. When this is
done we shall say that we are dealing with <u>functions in inten-
sion</u>. The notion of difference in meaning between two rules of
correspondence is a vague one, but, in terms of some system of
notation, it can be made exact in various ways. We shall not at-
tempt to decide what is the true notion of difference in meaning
but shall speak of functions in intension in any case where a
more severe criterion of identity is adopted than for functions
in extension. There is thus not one notion of function in inten-
sion, but many notions, involving various degrees of intensional-
ity.

In the calculus of λ-conversion and the calculus of re-
stricted λ-K-conversion, as developed below, it is possible, if
desired, to interpret the expressions of the calculus as denoting
functions in extension. However, in the calculus of λ-δ-conver-
sion, where the notion of identity of functions is introduced in-
to the system by the symbol δ, it is necessary, in order to
preserve the finitary character of the transformation rules, so
to formulate these rules that an interpretation by functions in
extension becomes impossible. The expressions which appear in
the calculus of λ-δ-conversion are interpretable as denoting
functions in intension of an appropriate kind.

3. FUNCTIONS OF SEVERAL VARIABLES. So far we have tacitly
restricted the term "function" to functions of one variable (or,
of one argument). It is desirable, however, for each positive
integer n, to have the notion of a function of n variables.
And, in order to avoid the introduction of a separate primitive
idea for each n, it is desirable to find a means of explaining
functions of n variables as particular cases of functions of
one variable. For our present purpose, the most convenient and
natural method of doing this is to adopt an idea of Schönfinkel
[49], according to which a function of two variables is regarded
as a function of one variable whose values are functions of one

variable, a function of three variables as a function of one va-
riable whose values are functions of two variables, and so on.

  Thus if f denotes a particular function of two variables,
the notation $((fa)b)$ -- which we shall frequently abbreviate as
(fab) or fab -- represents the value of f for the arguments
$a,b.$ The notation (fa) -- which we shall frequently abbreviate
as fa -- represents a function of one variable, whose value for
any argument x is $fax.$ The function f has a range of argu-
ments, and the notation fa is meaningful only when a belongs
to that range; the function fa again has a range of arguments,
which is, in general, different for different elements $a,$ and
the notation fab is meaningful only when b belongs to the
range of arguments of $fa.$

  Similarly, if f denotes a function of three variables,
$(((fa)b)c)$ or $fabc$ denotes the value of f for the arguments
$a,b,c,$ fa denoting a certain function of two variables, and .
$((fa)b)$ or fab denoting a certain function of one variable --
and so on.

  (According to another scheme, which is the better one for
certain purposes, a function of two variables is regarded as a
function (of one variable) whose arguments are ordered pairs, a
function of three variables as a function whose arguments are
ordered triads, and so on. This other concept of a function of
several variables is not however, excluded here. For, as will
appear below, the notions of ordered pair, ordered triad, etc.,
are definable by means of abstraction (§4) and the Schönfinkel
concept of a function of severable variables; and thus functions
of several variables in the other sense are also provided for.)

  An example of a function of two variables (in the sense of
Schönfinkel) is the <u>constancy function</u> $K,$ defined by the rule
that Kxy is $x,$ whatever x and y may be. We have, for in-
stance that KII is $I,$ KHI is $H,$ and so on. Also KI is
H (where H is the function defined above in §1). Similarly
KK is a function whose value is constant and equal to $K.$

  Another example of a function of two variables is the func-
tion whose value for the arguments f, x is (fx); for rea-
sons which will appear later we designate this function by the
symbol 1. The function 1, regarded as a function of one vari-
able, is a kind of identity function, since the notation $(1f)$

whenever significant, denotes the same function as f; the func-
tions I and 1 are not, however, the same function, since the
range of arguments consists in one case of all things whatever,
in the other case merely of all functions.

Other examples of functions of two or more variables are the
function H, already defined, and the functions T, J, B, C, W,
S, defined respectively by the rules that Txf is (fx), $Jfxyz$
is $fx(fzy)$, $Bfgx$ is $f(gx)$, $Cfxy$ is (fyx), Wfx is (fxx),
$Snfx$ is $f(nfx)$.

Of these, B and C may be more familiar to the reader un-
der other names, as the product or resultant of two transforma-
tions f and g, and as the converse of a function of two vari-
ables f. To say that BII is I is to say that the product
of the identity transformation by the identity transformation is
the identity transformation, whatever the domain within which
transformations are being considered; to say that $B11$ is 1
is to say that within any domain consisting entirely of functions
the product of the identity transformation by itself is the iden-
tity transformation. BI is 1, since it is the operation of
composition with the identity transformation, and thus an iden-
tity operation, but one applicable only to transformations.

The reader may further verify that CK is H, CT is 1,
$C1$ is T, CI is T -- that 1 and I have the same converse
is explained by the fact that, while not the same function, they
have the same effect in all cases where they can significantly
be applied to two arguments. The function BCC, the converse
of the converse, has the effect of an identity when applied to a
function of two variables, but when applied to a function of one
variable it has the effect of so restricting the range of argu-
ments as to transform the function into a function of two vari-
ables (if possible); thus $BCCI$ is 1.

There are many similar relations between these functions,
some of them quite complicated.

4. ABSTRACTION. For our present purpose it is necessary
to distinguish carefully between a symbol or expression which
denotes a function and an expression which contains a variable
and denotes ambiguously some value of the function -- a distinc-
tion which is more or less obscured in the usual language of

mathematical function theory.

To take an example from the theory of functions of natural numbers, consider the expression $(x^2+x)^2$. If we say, "$(x^2+x)^2$ is greater than 1,000," we make a statement which depends on x and actually has no meaning unless x is determined as some particular natural number. On the other hand, if we say, "$(x^2+x)^2$ is a primitive recursive function," we make a definite statement whose meaning in no way depends on a determination of the variable x (so that in this case x plays the rôle of an apparent, or bound, variable). The difference between the two cases is that in the first case the expression $(x^2+x)^2$ serves as an ambiguous, or variable, denotation of a natural number, while in the second case it serves as the denotation of a particular function. We shall hereafter distinguish by using $(x^2+x)^2$ when we intend an ambiguous denotation of a natural number, but $(\lambda x(x^2+x)^2)$ as the denotation of the corresponding function -- and likewise in other cases.

(It is, of course, irrelevant here that the notation $(x^2+x)^2$ is commonly used also for a certain function of real numbers, a certain function of complex numbers, etc. In a logically exact notation the functions, addition of natural numbers, addition of real numbers, addition of complex numbers, would be denoted by different symbols, say $+_n$, $+_r$, $+_c$; and the three functions, square of a natural number, square of a real number, square of a complex number, would be similarly distinguished. The uncertainty as to the exact meaning of the notation $(x^2+x)^2$, and the consequent uncertainty as to the range of arguments of the function $(\lambda x(x^2+x)^2)$, would then disappear.)

In general, if M is an expression containing a variable x (as a free variable, i.e., in such a way that the meaning of M depends on a determination of x), then (λxM) denotes a function whose value, for an argument a, is denoted by the result of substituting (a symbol denoting) a for x in M. The range of arguments of the function (λxM) consists of all objects a such that the expression M has a meaning when (a symbol denoting) a is substituted for x.

If M does not contain the variable x (as a free variable), then (λxM) might be used to denote a function whose value is constant and equal to (the thing denoted by) M, and whose

range of arguments consists of all things. This usage is contemplated below in connection with the calculi of λ-K-conversion, but is excluded from the calculi of λ-conversion and λ-δ-conversion -- for technical reasons which will appear.

Notice that, although x occur as a free variable in M, nevertheless, in the expression $(\lambda x M)$, x is a bound, or apparent, variable. Example: the equation $(x^2+x)^2 = (y^2+y)^2$ expresses a relation between the natural numbers denoted by x and y and its truth depends on a determination of x and of y (in fact, it is true if and only if x and y are determined as denoting the same natural number); but the equation $(\lambda x(x^2+x)^2) = (\lambda y(y^2+y)^2)$ expresses a particular proposition -- namely that $(\lambda x(x^2+x)^2)$ is the same function as $(\lambda y(y^2+y)^2)$ -- and it is true (there is no question of a determination of x and y).

Notice also that λ, or λx, is not the name of any function or other abstract object, but is an <u>incomplete symbol</u> -- i.e., the symbol has no meaning alone, but appropriately formed expressions containing the symbol have a meaning. We call the symbol λx an <u>abstraction operator</u>, and speak of the function which is denoted by $(\lambda x M)$ as obtained from the expression M by <u>abstraction</u>.

The expression $(\lambda x(\lambda y M))$, which we shall often abbreviate as $(\lambda xy.M)$, denotes a function whose value, for an argument denoted by x, is denoted by $(\lambda y M)$ -- thus a function whose values are functions, or a function of two variables. The expression $(\lambda y(\lambda x M))$, abbreviated as $(\lambda yx.M)$, denotes the converse function to that denoted by $(\lambda xy.M)$. Similarly $(\lambda x(\lambda y(\lambda z M)))$, abbreviated as $(\lambda xyz.M)$, denotes a function of three variables, and so on.

Functions introduced in previous sections as examples can now be expressed, if desired, by means of abstraction operators. For instance, I is (λxx); J is $(\lambda fxyz.fx(fzy))$; S is $(\lambda nfx.f(nfx))$; H is (λxI), or $(\lambda x(\lambda yy))$, or $(\lambda xy.y)$; K is $(\lambda xy.x)$; 1 is $(\lambda fx.fx)$.