Chapter IV

COMBINATIONS, GÖDEL NUMBERS

12. COMBINATIONS. If s is any set of well-formed formulas, the class of s-combinations is defined by the two following rules, a formula being an s-combination if and only if it is determined as such by these rules:

(1) Any formula of the set s, and any variable standing alone, is an s-combination.

(2) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are s-combinations, $\boldsymbol{AB}$ is an s-combination.

In the cases in which we shall be interested the formulas of s will contain no free variables and will none of them be of the form $\boldsymbol{AB}$. In such a case it is possible to distinguish the terms of an s-combination, each occurrence of a free variable or of one of the formulas of s being a term.

If s is the null set, the s-combinations will be called combinations of variables.

If s consists of the two formulas I, J, where

$$I \longrightarrow \lambda aa,$$
$$J \longrightarrow \lambda abcd.ab(adc),$$

the s-combinations will be called simply combinations.

We shall prove that every well-formed formula is convertible into a combination. This theorem is taken from Rosser [47], the present proof of it from Church [8]; the ideas involved go back to Schönfinkel [49] and Curry [18, 21].

Let:

$$\tau \longrightarrow JII.$$

Then τ conv $\lambda ab.ba$, and hence $\tau\boldsymbol{AB}$ conv $\boldsymbol{BA}$.

If M is any combination containing x as a free variable, we define an associated combination $\lambda_x M|$, which does not contain x as a free variable but otherwise contains the same free variables as M. This definition is by recursion, according to the following rules:

(1) $\lambda_x x|$ is I.
(2) If B contains x as a free variable and A does not, $\lambda_x AB|$ is $J\tau\lambda_x B|(JIA)$.
(3) If A contains x as a free variable and B does not, $\lambda_x AB|$ is $J\tau B\lambda_x A|$.
(4) If both A and B contain x as a free variable, $\lambda_x AB|$ is $J\tau\tau(JI(J\tau\tau(J\tau\lambda_x B|(J\tau\lambda_x A|J))))$.

12 I. If M is a combination containing x as a free variable, $\lambda_x M|$ conv $\lambda x M$.

We prove this by induction with respect to the number of terms of M.

If M has one term, then M is x, and $\lambda_x M|$ is I, which is convertible into λxx.

If M is AB and B contains x as a free variable and A does not, then $\lambda_x M|$ is $J\tau\lambda_x B|(JIA)$, which (see definitions of I, J, τ) is convertible into $\lambda a.A(\lambda_x B|a)$, which, by hypothesis of induction, is convertible into $\lambda a.A((\lambda x B)a)$ which finally is convertible into $\lambda x.AB$.

If M is AB and A contains x as a free variable and B does not, then $\lambda_x M|$ is $J\tau B\lambda_x A|$, which is convertible into $\lambda a.\lambda_x A|aB$, which, by hypothesis of induction is convertible into $\lambda a.(\lambda x A)aB$, which finally is convertible into $\lambda x.AB$.

If M is AB and both A and B contain x as a free variable, then $\lambda_x M|$ is $J\tau\tau(JI(J\tau\tau(J\tau\lambda_x B|(J\tau\lambda_x A|J))))$, which is convertible into $\lambda a.\lambda_x A|a(\lambda_x B|a)$, which, by hypothesis of induction, is convertible into $\lambda a.(\lambda x A)a((\lambda x B)a)$, which finally is convertible into $\lambda x.AB$.

The foregoing tacitly assumes that A and B do not contain a as a free variable. The modification necessary for the contrary case is, however, obvious.

This completes the proof of 12 I. We define the combination belonging to a well-formed formula, by recursion as follows:

(1) The combination belonging to x is x (where x is any variable).

(2) The combination belonging to $\boldsymbol{FA}$ is $\boldsymbol{F'A'}$, where $\boldsymbol{F'}$ and $\boldsymbol{A'}$ are the combinations belonging to $\boldsymbol{F}$ and $\boldsymbol{A}$ respectively.

(3) The combination belonging to $\lambda x\boldsymbol{M}$ is $\lambda_x \boldsymbol{M'}|$, where $\boldsymbol{M'}$ is the combination belonging to $\boldsymbol{M}$.

12 II. Every well-formed formula is convertible into the combination belonging to it.

Using 12 I, this is proved by induction with respect to the length of the formula. The proof is straightforward and details are left to the reader.

12 III. The combination belonging to $\boldsymbol{X}$ and the combination belonging to $\boldsymbol{Y}$ are identical if and only if $\boldsymbol{X}$ conv-I $\boldsymbol{Y}$.

13. PRIMITIVE SETS OF FORMULAS. A set s of well-formed formulas is called a <u>primitive set</u>, if the formulas of s contain no free variables and are none of them of the form $\boldsymbol{AB}$, and every well-formed formula is convertible into an s-combination. (When necessary to distinguish this idea from the analogous idea in the calculus of λ-K-conversion, the calculus of λ-δ-conversion, etc. -- see Chapter V -- we may speak of primitive sets of λ-formulas, primitive sets of λ-K-formulas, primitive sets of λ-δ-formulas, etc.)

It was proved in §12 that the formulas I, J are a primitive set. Another primitive set of formulas, suggested by the work of Curry, consists of the four formulas B, C, W, I, where:

$$B \rightarrow \lambda abc.a(bc).$$

$$C \rightarrow \lambda abc.acb.$$

$$W \rightarrow \lambda ab.abb.$$

In order to prove this it is sufficient to express J as a $\{B, C, W, I\}$-combination, as follows:

$$J \text{ conv } B(BC(BC))(B(W(BBB))C).$$

Still another primitive set of formulas consists of the four formulas $B, T, D, I,$ where:

$$T \rightarrow \lambda ab.ba.$$

$$D \rightarrow \lambda a.aa.$$

In order to prove this it is sufficient to express C and W as $\{B, T, D, I\}$-combinations, as follows:

$$C \text{ conv } B(T(BBT))(BBT).$$

$$W \text{ conv } B(B(T(BD(B(TT)(B(BBB)T))))(BBT))(B(T(B(TI)(TI)))B).$$

A primitive set of formulas is said to be <u>independent</u> if it ceases to be a primitive set upon omission of any one of the formulas. It seems plausible that each of the three primitive sets which have been named is independent. -- In the case of the set $\{I, J\}$, the independence of J follows (using 7 XVII) from the fact that any combination all of whose terms are I is convertible into I; and the independence of I follows (using 7 XXVIII) from the fact that if $\mathbf{A}$ imr $\mathbf{B}$ and $\mathbf{B}$ contains a (well-formed) part convertible-I into I then $\mathbf{A}$ must contain a (well-formed) part convertible-I into I.

14. AN APPLICATION OF THE THEORY OF COMBINATIONS. We prove now the following theorems, due to Kleene [34, 35, 37]:

14 I. If $\mathbf{A}_1$ and $\mathbf{A}_2$ contain no free variables, a formula $\mathbf{L}$ can be found such that $\mathbf{L}1$ conv $\mathbf{A}_1$ and $\mathbf{L}2$ conv $\mathbf{A}_2$.

For, by 12 II, $\mathbf{A}_1$ and $\mathbf{A}_2$ are convertible into combinations $\mathbf{A}_1'$ and $\mathbf{A}_2'$ respectively. We take $\mathbf{A}_1'$ to be the combination belonging to $\mathbf{A}_1$, unless that combination fails to contain an occurrence of J, in which case we take $\mathbf{A}_1'$ to be $JIIII$; and $\mathbf{A}_2'$ is similarly determined relatively to $\mathbf{A}_2$. Let $\mathbf{A}_1''$ and $\mathbf{A}_2''$ be the result of replacing all occurrences of J by the variable j in $\mathbf{A}_1'$ and $\mathbf{A}_2'$ respectively, and let $\mathbf{B}_1$ and

$\boldsymbol{B}_2$ be $\lambda j \boldsymbol{A}_1''$ and $\lambda j \boldsymbol{A}_2''$ respectively. Then $\boldsymbol{B}_1 J$ conv $\boldsymbol{A}_1$, and $\boldsymbol{B}_2 J$ conv $\boldsymbol{A}_2$, and $\boldsymbol{B}_1 I$ conv I, and $\boldsymbol{B}_2 I$ conv I. Consequently a formula $\boldsymbol{L}$ having the required property is:

$$\lambda n . n(\lambda x . x(\lambda y . y I \boldsymbol{B}_2))(\lambda z . z I I) \boldsymbol{B}_1 J .$$

14 II. If $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ contain no free variables, a formula $\boldsymbol{L}$ can be found such that $\boldsymbol{L}1$ conv $\boldsymbol{A}_1$, $\boldsymbol{L}2$ conv $\boldsymbol{A}_2$, ..., $\boldsymbol{L}\boldsymbol{N}$ conv $\boldsymbol{A}_n$ ($\boldsymbol{N}$ being the formula which represents n).

For the case that n is 1 or 2, this follows from 14 I. For larger values of n, we prove it by induction.

Let $\boldsymbol{L}_2$ be a formula such that $\boldsymbol{L}_2 1$ conv $\boldsymbol{A}_1$, and let $\boldsymbol{L}_1$ be a formula such that $\boldsymbol{L}_1 1$ conv $\boldsymbol{A}_2$, $\boldsymbol{L}_1 2$ conv $\boldsymbol{A}_3$, ..., $\boldsymbol{L}_1 \boldsymbol{M}$ conv $\boldsymbol{A}_n$ (where $\boldsymbol{M}$ represents n-1). Also let $\boldsymbol{C}$ be a formula such that $\boldsymbol{C}1$ conv $\boldsymbol{L}_1$ and $\boldsymbol{C}2$ conv $\boldsymbol{L}_2$. Then a formula $\boldsymbol{L}$ having the required property is:

$$\lambda i . \boldsymbol{C}[3 \dot{-} i](P i).$$

14 III. If $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n, \boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$ contain no free variables, a formula $\boldsymbol{E}$ can be found which represents an enumeration of the least set of formulas which contains $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ and is closed under each of the operations of forming $\boldsymbol{F}_\alpha \boldsymbol{X}\boldsymbol{Y}$ from the formulas $\boldsymbol{X}, \boldsymbol{Y}$ ($\alpha = 1, 2, \ldots, n$), in the sense that every formula of this set is convertible into one of the formulas in the infinite sequence

$$\boldsymbol{E}1, \boldsymbol{E}2, \ldots,$$

and every formula in this infinite sequence is convertible into one of the formulas of the set.

We prove this first for the case m = 1, using a device due to Kleene for obtaining formulas satisfying arbitrary conversion conditions of the general kind illustrated in (1) below.

Using 14 II, let $\boldsymbol{U}$ be a formula such that

$$\mathbf{U}1 \text{ conv } I,$$

$$\mathbf{U}2 \text{ conv } \lambda xy.\mathbf{F}_1(y(S[\mathbf{N}' \dot{-} Zx])[Zx \dot{-} \mathbf{N}]y)(y(S[\mathbf{N}' \dot{-} Z'x])[Z'x \dot{-} \mathbf{N}]y),$$

$$\mathbf{U}3 \text{ conv } \lambda xy.yx\mathbf{A}_1,$$

$$\mathbf{U}4 \text{ conv } \lambda xy.yx\mathbf{A}_2,$$

$$\cdots\cdots\cdots\cdots$$

$$\mathbf{UN}' \text{ conv } \lambda xy.yx\mathbf{A}_n,$$

where $\mathbf{N}$ represents n and $\mathbf{N}'$ represents n+2, and Z and Z' are the formulas introduced in §9. Let $\mathbf{E}$ be the formula,

$$\lambda i.\mathbf{U}(S[\mathbf{N}' \dot{-} i])[i \dot{-} \mathbf{N}]\mathbf{U}.$$

Then we have:

$$\mathbf{E}1 \text{ conv } \mathbf{A}_n,$$

$$\mathbf{E}2 \text{ conv } \mathbf{A}_{n-1},$$

(1) $$\cdots\cdots\cdots\cdots$$

$$\mathbf{EN} \text{ conv } \mathbf{A}_1,$$

$$\mathbf{EK} \text{ conv } \mathbf{F}_1(\mathbf{E}(Z[\mathbf{K} \dot{-} \mathbf{N}]))(\mathbf{E}(Z'[\mathbf{K} \dot{-} \mathbf{N}])),$$

$\mathbf{K}$ being any formula which represents an integer greater than n. From this it follows that $\mathbf{E}$ is a formula of the kind required.

Consider now the case m > 1. Let $\mathbf{M}$ represent m and let $\mathbf{F}$ be a formula such that $\mathbf{F}1$ conv $\mathbf{F}_1$, $\mathbf{E}2$ conv $\mathbf{F}_2$, ..., $\mathbf{FM}$ conv $\mathbf{F}_m$. By the preceding proof for the case m = 1, a formula $\mathbf{E}'$ can be found which represents an enumeration of the least set of formulas which contains $[1, \mathbf{A}_1]$, $[2, \mathbf{A}_1]$, ..., $[\mathbf{M}, \mathbf{A}_1]$, $[1, \mathbf{A}_2]$, $[2, \mathbf{A}_2]$, ..., $[\mathbf{M}, \mathbf{A}_2]$, ..., $[1, \mathbf{A}_n]$, $[2, \mathbf{A}_n]$, ..., $[\mathbf{M}, \mathbf{A}_n]$ and is closed under the operation of forming $\mathbf{Y}(\lambda xy[x, \mathbf{XF}y])$ from the formulas $\mathbf{X}$, $\mathbf{Y}$. Then a formula $\mathbf{E}$ of the kind required is:

$$\lambda i.2_2(\mathbf{E}'i).$$

It is immaterial that the enumeration so obtained contains repetitions. (Notice that $2_2[\mathbf{B}, \mathbf{C}]$ conv $\mathbf{C}$ if $\mathbf{B}$ is any formula such that $\mathbf{BI}$ conv I, in particular if $\mathbf{B}$ is any formula representing a positive integer; the case considered in §9 that $\boldsymbol{B}$ and $\boldsymbol{C}$ both represent positive integers is thus only a special case.)

14 IV. If $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n, \boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m, \boldsymbol{F}_{m+1}, \boldsymbol{F}_{m+2}, \ldots, \boldsymbol{F}_{m+r}$ contain no free variables, a formula $\boldsymbol{E}$ can be found which represents an enumeration of the least set of formulas which contains $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ and is closed under each of the operations of forming $\boldsymbol{F}_\alpha \boldsymbol{X}\boldsymbol{Y}$ from the formulas $\boldsymbol{X}, \boldsymbol{Y}$ $(\alpha = 1, 2, \ldots, m)$ and of forming $\boldsymbol{F}_{m+\beta}\boldsymbol{X}$ from the formula $\boldsymbol{X}$ $(\beta = 1, 2, \ldots, r)$ -- in the sense that every formula of this set is convertible into one of the formulas in the infinite sequence

$$\boldsymbol{E}1, \boldsymbol{E}2, \ldots,$$

and every formula in this infinite sequence is convertible into one of the formulas of the set.

(The case is not excluded that $m = 0$ or that $r = 0$, provided that m and r are not both 0.)

By the method used in the proof of 14 I, find formulas $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_n, \boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_{m+r}$ such that $\boldsymbol{B}_1 J$ conv $\boldsymbol{A}_1$, $\boldsymbol{B}_2 J$ conv $\boldsymbol{A}_2$, ..., $\boldsymbol{B}_n J$ conv $\boldsymbol{A}_n$, $\boldsymbol{C}_1 J$ conv $\boldsymbol{F}_1$, $\boldsymbol{C}_2 J$ conv $\boldsymbol{F}_2$, ..., $\boldsymbol{C}_{m+r} J$ conv $\boldsymbol{F}_{m+r}$, and $\boldsymbol{B}_1 I$ conv I, $\boldsymbol{B}_2 I$ conv I, ..., $\boldsymbol{B}_n I$ conv I, $\boldsymbol{C}_1 I$ conv I, $\boldsymbol{C}_2 I$ conv I, ..., $\boldsymbol{C}_{m+r} I$ conv I. By 14 III, a formula $\boldsymbol{E}'$ can be found which represents an enumeration of the least set of formulas which contains $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_n$ and is closed under each of the operations of forming $\lambda x.\boldsymbol{C}_\alpha x(\boldsymbol{X}x)(\boldsymbol{Y}x)$ from the formulas $\boldsymbol{X}, \boldsymbol{Y}$ $(\alpha = 1, 2, \ldots, m)$ and of forming $\lambda x.\boldsymbol{Y}I\boldsymbol{C}_{m+\beta}x(\boldsymbol{X}x)$ from the formulas $\boldsymbol{X}, \boldsymbol{Y}$ $(\beta = 1, 2, \ldots, r)$. Then a formula $\boldsymbol{E}$ of the kind required is:

$$\lambda i.\boldsymbol{E}'iJ.$$

15. A COMBINATORY EQUIVALENT OF CONVERSION. It is desirable to have a set of operations (upon combinations) which have the property that they always change a combination into a combination and which constitute an equivalent of conversion in the sense that a combination $\boldsymbol{X}$ can be changed into a combination

$\boldsymbol{Y}$ by a sequence of (0 or more of) these operations if and only if $\boldsymbol{X}$ conv $\boldsymbol{Y}$. Such a set of operations is the following (OI - OXXXVIII) -- where $\boldsymbol{F}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are arbitrary combinations, β, γ, ω are defined as indicated below, and the sign $\vdash$ is used to mean that the combination which precedes $\vdash$ is changed by the operation into the combination which follows:

OI. $I\boldsymbol{A} \vdash \boldsymbol{A}$.

OII. $\boldsymbol{A} \vdash I\boldsymbol{A}$.

OIII. $\boldsymbol{F}(I\boldsymbol{A}) \vdash \boldsymbol{FA}$.

OIV. $\boldsymbol{FA} \vdash \boldsymbol{F}(I\boldsymbol{A})$.

OV. $\boldsymbol{F}(I\boldsymbol{AB}) \vdash \boldsymbol{F}(\boldsymbol{AB})$.

OVI. $\boldsymbol{F}(\boldsymbol{AB}) \vdash \boldsymbol{F}(I\boldsymbol{AB})$.

OVII. $\boldsymbol{F}(J\boldsymbol{ABCD}) \vdash \boldsymbol{F}(\boldsymbol{AB}(\boldsymbol{ADC}))$.

OVIII. $\boldsymbol{F}(\boldsymbol{AB}(\boldsymbol{ADC})) \vdash \boldsymbol{F}(J\boldsymbol{ABCD})$.

OIX. $\boldsymbol{F}J \vdash \boldsymbol{F}(\omega(\beta\gamma(\beta(\beta(\beta\gamma))(\beta(\beta(\beta\beta\beta))I)))))$.

OX. $\boldsymbol{F}(\omega(\beta\gamma(\beta(\beta(\beta\gamma))(\beta(\beta(\beta\beta\beta))I))))) \vdash \boldsymbol{F}J$.

OXI. $\boldsymbol{F}\beta \vdash \boldsymbol{F}(\beta(\beta(\beta I))\beta)$.

OXII. $\boldsymbol{F}(\beta(\beta(\beta I))\beta) \vdash \boldsymbol{F}\beta$.

OXIII. $\boldsymbol{F}\gamma \vdash \boldsymbol{F}(\beta(\beta(\beta I))\gamma)$.

OXIV. $\boldsymbol{F}(\beta(\beta(\beta I))\gamma) \vdash \boldsymbol{F}\gamma$.

OXV. $\boldsymbol{F}I \vdash \boldsymbol{F}(\beta II)$.

OXVI. $\boldsymbol{F}(\beta II) \vdash \boldsymbol{F}I$.

OXVII. $\boldsymbol{F}(\gamma(\beta\beta(\beta\beta\beta))\beta) \vdash \boldsymbol{F}(\beta(\beta\beta)\beta)$.

OXVIII. $\boldsymbol{F}(\beta(\beta\beta)\beta) \vdash \boldsymbol{F}(\gamma(\beta\beta(\beta\beta\beta))\beta)$.

OXIX. $\boldsymbol{F}(\gamma(\beta\beta(\beta\beta\beta))\gamma) \vdash \boldsymbol{F}(\beta(\beta\gamma)(\beta\beta\beta))$.

OXX. $\boldsymbol{F}(\beta(\beta\gamma)(\beta\beta\beta)) \vdash \boldsymbol{F}(\gamma(\beta\beta(\beta\beta\beta))\gamma)$.

OXXI. $\boldsymbol{F}(\gamma(\beta\beta\beta)\omega) \vdash \boldsymbol{F}(\beta(\beta\omega)(\beta\beta\beta))$.

OXXII. $\boldsymbol{F}(\beta(\beta\omega)(\beta\beta\beta)) \vdash \boldsymbol{F}(\gamma(\beta\beta\beta)\omega)$.

OXXIII. $\boldsymbol{F}(\gamma\beta I) \vdash \boldsymbol{F}(\beta(\beta I)I)$.

OXXIV. $\boldsymbol{F}(\beta(\beta I)I) \vdash \boldsymbol{F}(\gamma\beta I)$.

OXXV. $\boldsymbol{F}(\beta\beta\gamma) \vdash \boldsymbol{F}(\beta(\beta(\beta\gamma)\gamma)(\beta\beta))$.

OXXVI. $\boldsymbol{F}(\beta(\beta(\beta\gamma)\gamma)(\beta\beta)) \vdash \boldsymbol{F}(\beta\beta\gamma)$.

OXXVII. $\boldsymbol{F}(\beta\beta\omega) \vdash \boldsymbol{F}(\beta(\beta(\beta(\beta(\beta\omega)\omega)(\beta\gamma))(\beta(\beta\beta)))\beta)$.

OXXVIII. $\boldsymbol{F}(\beta(\beta(\beta(\beta(\beta\omega)\omega)(\beta\gamma))(\beta(\beta\beta)))\beta) \vdash \boldsymbol{F}(\beta\beta\omega)$.

OXXIX. $\boldsymbol{F}(\beta\gamma\gamma) \vdash \boldsymbol{F}(\beta(\beta I))$.

OXXX. $\boldsymbol{F}(\beta(\beta I)) \vdash \boldsymbol{F}(\beta\gamma\gamma)$.

OXXXI. $\boldsymbol{F}(\beta(\beta(\beta\gamma)\gamma)(\beta\gamma)) \vdash \boldsymbol{F}(\beta(\beta\gamma(\beta\gamma))\gamma)$.

OXXXII. $\boldsymbol{F}(\beta(\beta\gamma(\beta\gamma))\gamma) \vdash \boldsymbol{F}(\beta(\beta(\beta\gamma)\gamma)(\beta\gamma))$.

OXXXIII. $\boldsymbol{F}(\beta\gamma\omega) \vdash \boldsymbol{F}(\beta(\beta(\beta\omega)\gamma)(\beta\gamma))$.
OXXXIV. $\boldsymbol{F}(\beta(\beta(\beta\omega)\gamma)(\beta\gamma)) \vdash \boldsymbol{F}(\beta\gamma\omega)$.
OXXXV. $\boldsymbol{F}(\beta\omega\gamma) \vdash \boldsymbol{F}\omega$.
OXXXVI. $\boldsymbol{F}\omega \vdash \boldsymbol{F}(\beta\omega\gamma)$.
OXXXVII. $\boldsymbol{F}(\beta\omega\omega) \vdash \boldsymbol{F}(\beta\omega(\beta\omega))$.
OXXXVIII. $\boldsymbol{F}(\beta\omega(\beta\omega)) \vdash \boldsymbol{F}(\beta\omega\omega)$.

$$\gamma \longrightarrow J\tau(J\tau)(J\tau).$$
$$\beta \longrightarrow \gamma(JI\gamma)(JI).$$
$$\omega \longrightarrow \gamma(\gamma(\beta\gamma(\gamma(\beta J\tau)\tau))\tau).$$

(Note that τ, γ, β, ω are convertible respectively into T, C, B, W.)

These thirty-eight operations have characteristics of simplicity not possessed by the operations I, II, III of §6, namely: (1) they are one-valued, i.e., given the combination operated on and the particular one of the thirty-eight operations which is applied, the combination resulting is uniquely determined; (2) they do not involve the idea of substitution at an arbitrary place, but only that of substitution at a specified place. This has the effect of rendering some of the developments in §16 much simpler than they otherwise might be.

The proof of the equivalence of OI-OXXXVIII to conversion is too long to be included here. It may be found in Rosser's dissertation [47] (cf. Section H therein). Many of the important ideas and methods involved derive from Curry [17, 18, 20, 21]; in fact, Curry has results which may be thought of as constituting an approximate equivalent to the one in question here but which are nevertheless sufficiently different so that we are unable to use them directly.

16. GÖDEL NUMBERS. The Gödel number of a combination is defined by induction as follows:

(1) The Gödel number of I is 1.
(2) The Gödel number of J is 3.
(3) The Gödel number of the nth variable in alphabetical order (see §5) is $2n+5$.
(4) If m and n are the Gödel numbers of $\boldsymbol{A}$ and $\boldsymbol{B}$ respec-

tively, the Gödel number of $\boldsymbol{AB}$ is $(m+n)(m+n-1)-2n+2$.

The <u>Gödel number belonging to</u> a formula is defined to be the Gödel number of the combination belonging to the formula. (Notice that the Gödel number belonging to a combination is thus in general not the same as the Gödel number of the combination.)

It is left to the reader to verify that the Gödel numbers of two combinations $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same; and that the Gödel numbers belonging to two formulas $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same if and only if $\boldsymbol{A}$ conv-I $\boldsymbol{B}$ (cf. 12 III). (Notice that the Gödel number of $\boldsymbol{AB}$, according to (4), is twice the number of the ordered pair $[m, n]$ in the enumeration of ordered pairs described at the end of §9.)

The usefulness of Gödel numbers arises from the fact that our formalism contains no notations for formulas -- i.e., for sequences of symbols. (It is not possible to use formulas as notations for themselves, because interconvertible formulas must denote the same thing although they are not the same formula, and because formulas containing free variables cannot denote any [fixed] thing.) The Gödel number belonging to a formula serves in many situations as a substitute for a notation for the formula and often enables us to accomplish things which might have been thought to be impossible without a formal notation for formulas.

This use of Gödel numbers is facilitated by the existence of a formula, form, such that, if $\boldsymbol{N}$ represents the Gödel number belonging to $\boldsymbol{A}$, and $\boldsymbol{A}$ contains no free variables, then, form $\boldsymbol{N}$ conv $\boldsymbol{A}$. In order to obtain this formula, first notice that par $\boldsymbol{N}$ conv 2 if $\boldsymbol{N}$ represents the Gödel number of a combination having more than one term, and par $\boldsymbol{N}$ conv 1 if $\boldsymbol{N}$ represents the Gödel number of a combination having only one term; also that if $\boldsymbol{N}$ represents the Gödel number of a combination $\boldsymbol{AB}$, then $Z(H\boldsymbol{N})$ is convertible into the formula representing the Gödel number of $\boldsymbol{A}$, and $Z'(H\boldsymbol{N})$ is convertible into the formula representing the Gödel number of $\boldsymbol{B}$ (see §9). We introduce the abbreviations:

$$\boldsymbol{N}_1 \rightarrow Z(H\boldsymbol{N}).$$

$$\boldsymbol{N}_2 \rightarrow Z'(H\boldsymbol{N}).$$

Subscripts used in this way may be iterated, so that, for instance,

$$\boldsymbol{N}_{122} \rightarrow Z'(H(Z'(H(Z(H\boldsymbol{N}))))).$$

By the method of §14, find a formula $\mathfrak{D}$ such that

$$\mathfrak{D}1 \text{ conv } \lambda x.x12,$$
$$\mathfrak{D}2 \text{ conv } I,$$
$$\mathfrak{D}3 \text{ conv } \lambda x.x12J,$$

and a formula $\mathfrak{U}$ such that

$$\mathfrak{U}1 \text{ conv } \mathfrak{D},$$
$$\mathfrak{U}2 \text{ conv } \lambda xy.y(\text{par } x_1)x_1y(y(\text{par } x_2)x_2y),$$

(these formulas $\mathfrak{D}$ and $\mathfrak{U}$ can be explicitly written down by referring to the proofs of 14 I and 14 II).

Let

$$\text{form} \longrightarrow \lambda n.\mathfrak{U}(\text{par } n)n\mathfrak{U}.$$

Then

$$\text{form } 1 \text{ conv } I,$$
$$\text{form } 3 \text{ conv } J, \text{ and}$$
$$\text{form } \boldsymbol{N} \text{ conv form } \boldsymbol{N}_1(\text{form } \boldsymbol{N}_2)$$

if $\boldsymbol{N}$ represents an even positive integer. From this it follows that form has the property ascribed to it above; for if $\boldsymbol{N}$ represents the Gödel number of a combination $\boldsymbol{A}'$ belonging to a formula $\boldsymbol{A}$, containing no free variables, then form $\boldsymbol{N}$ conv $\boldsymbol{A}'$, and $\boldsymbol{A}'$ conv $\boldsymbol{A}$.

Let:

$$\begin{aligned}
\sigma \rightarrow \lambda n\,.\ & [\text{par}n+\text{par}n_1+\text{eq}\overline{24812}n_{11}+[3\dot{-}\text{eq}\overline{156}n_{12}]+\text{par}n_2+\text{eq}\overline{12}n_{21}\dot{-}\overline{10}] \\
& + [2\times[\text{par}n+\text{par}n_1+\text{eq}\overline{24812}n_{11}+[3\dot{-}\text{min}(\text{par}n_2)(\text{eq}\overline{12}n_{21})]\dot{-}6]] \\
& + [3\times[\text{par}n+\text{eq}\overline{623375746}n_1+\text{par}n_2+\text{eq}\overline{12}\,n_{21}+\text{par}n_{22} \\
& \qquad +\text{eq}\overline{623375746}n_{221}+\text{par}n_{222}+\text{par}n_{2221}+\text{eq}\overline{24812}n_{22211} \\
& \qquad +\text{par}n_{2222}+\text{par}n_{22221}+\text{eq}\overline{24812}n_{222211}+\text{eq}3n_{22222}\dot{-}\overline{24}]] \\
& \dot{-}\ 5.
\end{aligned}$$

Noting that the Gödel numbers of JI, τ, $J\tau$, $J\tau\tau$ are respectively 12, 156, 24812, 623375746, the reader may verify that:

$\sigma\boldsymbol{N}$ conv 1, 2, 3, or 4 if $\boldsymbol{N}$ represents a positive integer;

σN conv 2 if N represents the Gödel number of a combination of the form $J\tau B(JIA)$, with B different from τ;

σN conv 3 if N represents the Gödel number of a combination of the form $J\tau BA$ but not of the form $J\tau B(JIA)$;

σN conv 4 if N represents the Gödel number of a combination of the form $J\tau\tau(JI(J\tau\tau(J\tau B(J\tau AJ))))$;

σN conv 1 if N represents the Gödel number of a combination not of one of these three forms.

Again using §14, we find a formula u such that

u1 conv $\lambda xy.y5x$,
u2 conv $\lambda xy.y(\sigma x_{12})x_{12}y$,
u3 conv $\lambda xy.y(\sigma x_2)x_2 y$,
u4 conv $\lambda xy.\min(y(\sigma x_{22212})x_{22212}y)(y(\sigma x_{222212})x_{222212}y)$,
u5 conv $\lambda x.3\dot{-}x$,

and we let

$$o \rightarrow \lambda n.u(\sigma n)nu.$$

Then o λ-defines a function of positive integers whose value is 2 for an argument which is the Gödel number of a combination of the form $\lambda_x M|$, and 1 for an argument which is the Gödel number of a combination not of this form -- or, as we shall say briefly, o λ-defines the property of a combination of being of the form $\lambda_x M|$.

By similar constructions, involving lengthy detail but nothing new in principle, the following formulas may be obtained:

1) A formula, occ; such that, if N represents a positive integer n, we have that occ N λ-defines the property of a combination of containing the nth variable in alphabetical order, as a free variable (i.e., as a term).

2) A formula ϵ, such that, N representing a positive integer n, if C represents the Gödel number of a combination not of the form $\lambda_x M|$, then ϵNC conv C, and if C represents the Gödel number of a combination $\lambda_x M|$, then ϵNC is convertible into the formula representing the Gödel number of the combination obtained from M by substituting for all free occurrences of x in M the nth variable in alphabetical order.

3) A formula $\mathfrak{C}$, such that, if C represents the Gödel

number of a combination not of the form $\lambda_{x}\boldsymbol{M}|$, then $\mathfrak{C}\boldsymbol{C}$ conv $\boldsymbol{C}$, and if $\boldsymbol{C}$ represents the Gödel number of a combination $\lambda_{x}\boldsymbol{M}|$, then $\mathfrak{C}\boldsymbol{C}$ is convertible into the formula representing the Gödel number of the combination obtained from $\boldsymbol{M}$ by substituting for all free occurrences of $\boldsymbol{x}$ in $\boldsymbol{M}$ the first variable in alphabetical order which does not occur in $\boldsymbol{M}$ as a free variable.

4) A formula $\mathfrak{r}$ which λ-defines the property of a combination, that there is a formula to which it belongs.

5) A formula Λ which λ-defines the property of a combination of belonging to a formula of the form $\lambda\boldsymbol{x}\boldsymbol{M}$.

6) A formula, prim, which λ-defines the property of a combination of containing no free variables.

7) A formula, norm, which λ-defines the property of a combination of belonging to a formula which is in normal form.

8) A formula 0_1 which corresponds to the operation OI of §15, in the sense that, if $\boldsymbol{C}$ represents the Gödel number of a combination of such a form that OI is not applicable to it, then $0_1\boldsymbol{C}$ conv $\boldsymbol{C}$, and if $\boldsymbol{C}$ represents the Gödel number of a combination $\boldsymbol{M}$ to which OI is applicable, then $0_1\boldsymbol{C}$ is convertible into the formula representing the Gödel number of the combination obtained from $\boldsymbol{M}$ by applying OI.

9) Formulas 0_2, 0_3, ..., 0_{38} which correspond respectively to the operations OII, OIII, ..., OXXXVIII of §15, in the same sense.

By 14 III, a formula, cb, can be found which represents an enumeration of the least set of formulas which contains 1 and 3 and is closed under the operation of forming $(\lambda ab\,.\,2 \times \mathrm{nr}\,ab)\boldsymbol{X}\boldsymbol{Y}$ from the formulas $\boldsymbol{X}$, $\boldsymbol{Y}$. But if $\boldsymbol{X}$, $\boldsymbol{Y}$ represent the Gödel numbers of combinations $\boldsymbol{A}$, $\boldsymbol{B}$ respectively, then $(\lambda ab\,.\,2 \times \mathrm{nr}\,ab)\boldsymbol{X}\boldsymbol{Y}$ is convertible into the formula which represents the Gödel number of $\boldsymbol{A}\boldsymbol{B}$. Hence the formula, cb, enumerates the Gödel numbers of combinations containing no free variables, in the sense that every formula representing such a Gödel number is convertible into one of the formulas in the infinite sequence

$$\mathrm{cb}\ 1,\ \mathrm{cb}\ 2,\ \ldots,$$

and every formula in this infinite sequence is convertible into a formula representing such a Gödel number.

If now we let

$$\text{ncb} \rightarrow \lambda n \,.\, \text{cb}\,(\mathfrak{P}(\lambda x \,.\, \text{norm}\,(\text{cb}\; x))n),$$

then ncb enumerates, in the same sense, the Gödel numbers of combinations which belong to formulas in normal form and contain no free variables (cf. 10 II).

By 14 IV, a formula O can be found which represents an enumeration of the least set of formulas which contains I and is closed under each of the thirty-eight operations of forming $(\lambda ab.\mathrm{O}_\beta(ab))\boldsymbol{X}$ from the formula $\boldsymbol{X}$ ($\beta = 1, 2, \ldots, 38$). Let

$$\text{cnvt} \rightarrow \lambda ab.\mathrm{O}ba.$$

Then if $\boldsymbol{C}$ represents the Gödel number of a combination $\boldsymbol{M}$, the formula, cnvt $\boldsymbol{C}$, enumerates (again in the same sense as in the two preceding paragraphs) the Gödel numbers of combinations obtainable from $\boldsymbol{M}$ by conversion -- cf. §15.

Let

$$\text{nf} \rightarrow \lambda n \,.\, \text{cnvt}\; n(\mathfrak{p}(\lambda x \,.\, \text{norm}\,(\text{cnvt}\; nx))1).$$

Then nf λ-defines the operation <u>normal form of</u> a formula, in the sense that (1) if $\boldsymbol{C}$ represents the Gödel number of a combination $\boldsymbol{M}$, then nf $\boldsymbol{C}$ is convertible into the formula representing the Gödel number belonging to the normal form of $\boldsymbol{M}$; and hence (2) if $\boldsymbol{C}$ represents the Gödel number belonging to a formula $\boldsymbol{M}$, then nf $\boldsymbol{C}$ is convertible into the formula representing the Gödel number belonging to the normal form of $\boldsymbol{M}$. If $\boldsymbol{C}$ represents the Gödel number of a combination (or belonging to a formula) which has no normal form, then nf $\boldsymbol{C}$ has no normal form (cf. 10 I).

Let $\mathfrak{i}$ and $\mathfrak{s}$ be the formulas representing the Gödel numbers belonging 1 and S respectively. Then the formulas

$$Z'(H(1(\lambda x \,.\, 2 \times \text{nr}\,\mathfrak{s}x)\mathfrak{i})),\; Z'(H(2(\lambda x \,.\, 2 \times \text{nr}\;\mathfrak{s}x)\mathfrak{i})),$$
$$Z'(H(3(\lambda x \,.\, 2 \times \text{nr}\;\mathfrak{s}x)\mathfrak{i})), \ldots,$$

are convertible respectively into formulas representing Gödel numbers belonging to

$$1,\; S1,\; S(S1),\; \ldots\,.$$

Hence a formula $\mathfrak{v}$ which λ-defines the property of a combination of belonging to a formula in normal form which represents a positive integer, may be obtained by defining:

$$\nu \longrightarrow \lambda n \,.\, \pi(\mathrm{eq}\ n)(\lambda m \,.\, \mathrm{eq}\ n(\mathrm{nf}(Z'(H(m(\lambda x \,.\, 2 \times \mathrm{nr}\ {}_{\mathrm{s}}x)\,{}_{\mathrm{i}})))))).$$

(It is necessary, in order to see this, to refer to 10 III, and to observe that the Gödel number belonging to a formula in normal form representing a positive integer is always greater than that positive integer.)