Chapter III

LAMBDA-DEFINABILITY

8. LAMBDA-DEFINABILITY OF FUNCTIONS OF POSITIVE INTEGERS.

We define,

$$1 \rightarrow \lambda ab.ab,$$

$$2 \rightarrow \lambda ab.a(ab),$$

$$3 \rightarrow \lambda ab.a(a(ab)),$$

and so on, each numeral (in the Arabic decimal notation) being introduced as an abbreviation for a corresponding formula of the indicated form. But where a numeral consists of more than one digit, a bar is used over it, in order to avoid confusion with other notations; thus,

$$\overline{11} \rightarrow \lambda ab.a(a(a(a(a(a(a(a(a(a(ab)))))))))),$$

but 11, without the bar, is an abbreviation for

$$(\lambda ab.ab)(\lambda ab.ab).$$

In connection with these definitions an interpretation of the calculus of λ-conversion is contemplated under which each of the formulas abbreviated as a numeral is interpreted as denoting the corresponding positive integer. Since it is intended at the same time to retain the interpretation of the formulas of the calculus (which have a normal form) as denoting certain functions in accordance with the ideas of Chapter I, this means that the positive integers are identified with certain functions. For example, the number 2 is identified with the function which, when applied to the function f as argument, yields the product of f by itself (product in the sense of the product, or resultant, of two transformations); similarly the number 14 is identified with the function which, when applied to the function f as argument, yields the fourteenth power of f (power in the sense of power of a transformation). This is allowable on the ground that abstract number theory requires of the positive integers only that they form a progression and, subject to this condition, the integers may be identified with any entities whatever; as a matter of fact, logical constructions of the positive integers by identifying them with entities thought to be logically more fundamental are possible in many different ways (the present method should be compared with that familiar in the works of Frege and Russell, according to which the non-negative integers are identified with classes of similar finite classes).

A function F of positive integers -- i.e., a function of one variable for which the range of arguments and the range of values each consist of positive integers -- is said to be <u>λ-definable</u> if there is a formula $\boldsymbol{F}$ such that (1) whenever m and n are positive integers, and $Fm = n$, and $\boldsymbol{M}$ and $\boldsymbol{N}$ are the formulas which represent (denote) the integers m and n respectively, then $\boldsymbol{FM}$ conv $\boldsymbol{N}$, and (2) whenever the function F has no value for the positive integer m as argument, and $\boldsymbol{M}$ represents m, then $\boldsymbol{FM}$ has no normal form. Similarly the function F of two integer variables is said to be <u>λ-definable</u> if there is a formula $\boldsymbol{F}$ such that (1) if l, m, n are positive integers, and $Flm = n$, and $\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}$ represent the integers l, m, n respectively, then $\boldsymbol{FLM}$ conv $\boldsymbol{N}$, and (2) if the function F has no value for the positive integers l, m as arguments, and $\boldsymbol{L}, \boldsymbol{M}$ represent l, m respectively, then $\boldsymbol{FLM}$ has no normal form. And so on, for functions of any number of variables.

We shall say also, under the circumstances described, that the formula $\boldsymbol{F}$ <u>λ-defines</u> the function F (we use the word "λ-defines rather than "denotes" or "represents" only because the function which $\boldsymbol{F}$ denotes, in general has other elements than positive integers in its range -- or ranges -- of arguments).

The successor function of positive integers (i.e., the function $x+1$) is λ-defined by the formula $\boldsymbol{S}$, where

$$\boldsymbol{S} \rightarrow \lambda abc.b(abc).$$

It is left to the reader to verify this, and also to verify that addition, and multiplication, and exponentiation of positive integers are λ-defined by the formulas $\lambda mn.m+n$, and $\lambda mn.m \times n$, and $\lambda mn.m^n$ respectively (see definitions in §5).

These λ-definitions of addition, multiplication, and exponentiation are due to Rosser (see Kleene [35]). The definition of multiplication depends on the observation that the product of two positive integers in the sense of the product of transformations is the same as their product in the arithmetic sense, and the definition of exponentiation then follows because, when the positive integer n is taken of any function f as argument, there results the nth power of f in the sense of the product of transformations.

The reader may also verify that, for any formulas $\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}$ (whether representing positive integers or not):

$$[\boldsymbol{L}+\boldsymbol{M}]+\boldsymbol{N} \text{ conv } \boldsymbol{L}+[\boldsymbol{M}+\boldsymbol{N}],$$

$$[\boldsymbol{L}\times\boldsymbol{M}]\times\boldsymbol{N} \text{ conv } \boldsymbol{L}\times[\boldsymbol{M}\times\boldsymbol{N}],$$

$$[\boldsymbol{L}+\boldsymbol{M}]\times\boldsymbol{N} \text{ conv } [\boldsymbol{L}\times\boldsymbol{N}]+[\boldsymbol{M}\times\boldsymbol{N}],$$

$$\boldsymbol{L}^{\boldsymbol{M}+\boldsymbol{N}} \text{ conv } \boldsymbol{L}^{\boldsymbol{M}}\times\boldsymbol{L}^{\boldsymbol{N}},$$

$$\boldsymbol{L}^{\boldsymbol{M}\times\boldsymbol{N}} \text{ conv } [\boldsymbol{L}^{\boldsymbol{N}}]^{\boldsymbol{M}},$$

$$S\boldsymbol{M} \text{ conv } 1+\boldsymbol{M}.$$

9. ORDERED PAIRS AND TRIADS, THE PREDECESSOR FUNCTION. We now introduce formulas which may be thought of as representing ordered pairs and ordered triads, as follows:

$$[\boldsymbol{M}, \boldsymbol{N}] \rightarrow \lambda a.a\boldsymbol{M}\boldsymbol{N},$$

$$[\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}] \rightarrow \lambda a.a\boldsymbol{L}\boldsymbol{M}\boldsymbol{N},$$

$$2_1 \rightarrow \lambda a.a(\lambda bc.cIb),$$

$$2_2 \rightarrow \lambda a.a(\lambda bc.bIc),$$

$$3_1 \rightarrow \lambda a.a(\lambda bcd.cIdIb),$$

$$3_2 \rightarrow \lambda a.a(\lambda bcd.bIdIc),$$

$$3_3 \rightarrow \lambda a.a(\lambda bcd.bIcId).$$

If $\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}$ are formulas representing positive integers, then $2_1[\boldsymbol{M}, \boldsymbol{N}]$ conv $\boldsymbol{M}$, $2_2[\boldsymbol{M}, \boldsymbol{N}]$ conv $\boldsymbol{N}$, $3_1[\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}]$ conv $\boldsymbol{L}$, $3_2[\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}]$ conv $\boldsymbol{M}$, and $3_3[\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}]$ conv $\boldsymbol{N}$.

Verification of this depends on the observation that, if $\boldsymbol{M}$ is a formula representing a positive integer, $\boldsymbol{M}I$ conv I (the mth power of the identity is the identity).

By the predecessor function of positive integers we mean the function whose value for the argument 1 is 1 and whose value for any other positive integer argument x is $x-1$. This function is λ-defined by

$$P \rightarrow \lambda a . 3_3(a(\lambda b[S(3_1 b), 3_1 b, 3_2 b])[1, 1, 1]).$$

For if $\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{M}$ represent positive integers,

$$(\lambda b[S(3_1 b), 3_1 b, 3_2 b])[\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{M}] \text{ conv } [S\boldsymbol{K}, \boldsymbol{K}, \boldsymbol{L}],$$

and hence if $\boldsymbol{A}$ represents a positive integer,

$$\boldsymbol{A}(\lambda b[S(3_1 b), 3_1 b, 3_2 b])[1, 1, 1] \text{ conv } [S\boldsymbol{A}, \boldsymbol{A}, \boldsymbol{B}],$$

where $\boldsymbol{B}$ represents the predecessor of the positive integer represented by $\boldsymbol{A}$. (The method of λ-definition of the predecessor function due to Kleene [35] is here modified by employment of a different formal representation of ordered triads.)

A kind of subtraction of positive integers, which we distinguish by placing a dot above the sign of subtraction, and which differs from the usual kind in that $x \dot{-} y = 1$ if $x \leqq y$, may now be shown to be λ-definable:

$$[\boldsymbol{M} \dot{-} \boldsymbol{N}] \rightarrow \boldsymbol{N}P\boldsymbol{M}.$$

The functions the lesser of the two positive integers x and y and the greater of the two positive integers x and y are λ-definable respectively by

$$\min \rightarrow \lambda ab . Sb \dot{-} . Sb \dot{-} a,$$

$$\max \rightarrow \lambda ab . [a+b] \dot{-} \min ab.$$

The parity of a positive integer, i.e., the function whose value is 1 for an odd positive integer and 2 for an even positive integer, is λ-defined by

$$\text{par} \rightarrow \lambda a.a(\lambda b.3 \dot{-} b)2.$$

Using ordered pairs in a way similar to that in which ordered triads were used to obtain a λ-definition of the predecessor function, we give a λ-definition of the function <u>the least integer not less than half of</u> x -- or, in other words, the quotient upon dividing $x+1$ by 2, in the sense of division with a remainder:

$$H \rightarrow \lambda a.P(2_1(a(\lambda b[P[2_1 b + 2_2 b], 3 \dot{-} 2_2 b])[1, 2])).$$

Of course this H is unrelated to the -- entirely different -- function H which was introduced for illustration in §1.

If we let

$$\mathfrak{L} \rightarrow \lambda b.b(\lambda c \lambda d[dPc(\lambda e.e1I)(\lambda fg.fgS)c,$$
$$dPc(\lambda h.h1IS)(\lambda ijk.kij(\lambda l.l1))d]),$$

$$\mathfrak{U} \rightarrow \lambda a.a\mathfrak{L}[1, 1],$$

$$Z \rightarrow \lambda a.2_2(\mathfrak{U}a),$$

$$Z' \rightarrow \lambda a.\mathfrak{U}a(\lambda bc.b \dot{-} c),$$

then, if $\mathbf{M}, \mathbf{N}$ represent the positive integers m, n respectively, $\mathfrak{L}[\mathbf{M}, \mathbf{N}]$ conv $[S\mathbf{M}, 1]$ if $m \dot{-} n = 1$ and conv $[\mathbf{M}, S\mathbf{N}]$ if $m \dot{-} n > 1$; hence $\mathfrak{U}1, \mathfrak{U}2, \ldots$ are convertible respectively into

$$[2, 1], [3, 1], [3, 2], [4, 1], [4, 2], [4, 3], [5, 1], \ldots ;$$

hence $Z1, Z2, \ldots$ are convertible respectively into

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots ,$$

and $Z'1$, $Z'2$, ... are convertible respectively into

$$1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, \ldots .$$

Thus the infinite sequence of ordered pairs,

$$[Z1, Z'1], [Z2, Z'2], [Z3, Z'3], \ldots,$$

contains all ordered pairs of positive integers, with no repetitions. The function whose value for the arguments x, y is the number of the ordered pair $[x, y]$ in this enumeration is λ-defined by

$$\mathrm{nr} \rightarrow \lambda ab \,.\, S(H[[a+b] \times P[a+b]]) \dot{-} b.$$

10. PROPOSITIONAL FUNCTIONS; THE KLEENE $\mathfrak{p}$-FUNCTION. By a *propositional function* we shall mean a function (of one or more variables) whose values are *truth values* — i.e., truth and falsehood. A *property* is a propositional function of one variable; a *relation* is a propositional function of two variables. The *characteristic function* associated with a propositional function is the function whose value is 2 when (i.e., for an argument or arguments for which) the value of the propositional function is truth, whose value is 1 when the value of the propositional function is falsehood, and which has no value otherwise.

A propositional function of positive integers will be said to be λ-definable if the associated characteristic function is a λ-definable function. (It can readily be shown that the choice of the particular integers 2 and 1 in the definition of *characteristic function* is here non-essential; the class of λ-definable propositional functions of positive integers remains unaltered if any other pair of distinct positive integers is substituted.)

In particular, the relations $>$ and $=$ between positive integers are λ-definable, as is shown by giving λ-definitions of the associated characteristic functions:

$$\text{exc} \longrightarrow \lambda ab \,.\, \text{min } 2\ [Sa \dot{-} b].$$

$$\text{eq} \longrightarrow \lambda ab \,.\, 4 \dot{-} \,.\, \text{exc } ab + \text{exc } ba.$$

From this follows the λ-definability of a great variety of properties and relations of positive integers which are expressible by means of equations and inequalities; conjunction, disjunction, and negation of equations and inequalities can be provided for by using min, max, and $\lambda a.3 \dot{-} a$ respectively.

We prove also the two following theorems from Kleene [35], and a third closely related theorem:

10 I. If R is a λ-definable propositional function of $n+1$ positive integer arguments, then the function F is λ-definable (1) whose value for the positive integer arguments $x_1, x_2, \ldots, x_n$ is the least positive integer y such that $Rx_1x_2\ldots x_ny$ holds (i.e., has the value truth), provided that there is such a least positive integer y and that, for every positive integer z less than this y, $Rx_1x_2\ldots x_nz$ has a value, truth or falsehood, and (2) which has no value otherwise.

In the case that R has a value for every set of $n+1$ positive integer arguments, F may be described simply by saying that $Fx_1x_2\ldots x_n$ is the least positive integer y such that $Rx_1x_2\ldots x_ny$ holds.

Let

$$\mathfrak{G} \longrightarrow \lambda n.n(\lambda r.r(\lambda s.s\,1II(\lambda xgt.g1(tx)Ix)))$$
$$(\lambda f.fI1II)(\lambda xgt.g(t(Sx))(Sx)gt).$$

Then

$$\mathfrak{G}1 \text{ red } \lambda xgt.g(t(Sx))(Sx)gt,$$

$$\mathfrak{G}2 \text{ red } \lambda xgt.g1(tx)Ix.$$

Hence if $\boldsymbol{N}$ represents a positive integer and $\boldsymbol{TN}$ conv either 1 or 2, we have (using 7 XXVIII to show that $\boldsymbol{TN}$ red 1 or 2),

$$\mathfrak{G}1\boldsymbol{N}\mathfrak{G}\boldsymbol{T} \text{ red } \mathfrak{G}(\boldsymbol{T}(S\boldsymbol{N}))(S\boldsymbol{N})\mathfrak{G}\boldsymbol{T},$$

$$\mathfrak{G}2\boldsymbol{N}\mathfrak{G}\boldsymbol{T} \text{ red } \boldsymbol{N}.$$

Hence if we let

$$\mathfrak{p} \longrightarrow \lambda tx.\mathfrak{G}(tx)x\mathfrak{G}t,$$

we have $\mathfrak{p}\boldsymbol{TN}$ red $\boldsymbol{N}$ if $\boldsymbol{TN}$ conv 2, and $\mathfrak{p}\boldsymbol{TN}$ conv $\mathfrak{p}\boldsymbol{T}(S\boldsymbol{N})$ if $\boldsymbol{TN}$ conv 1, and (by 7 XXXI, 7 XXXII) $\mathfrak{p}\boldsymbol{TN}$ has no normal form if $\boldsymbol{TN}$ has no normal form.

If $\boldsymbol{N}$ represents the positive integer n and $\boldsymbol{T}$ λ-defines the characteristic function associated with the property T of positive integers, it follows that $\mathfrak{p}\boldsymbol{TN}$ is convertible into the formula which represents the least positive integer y, not less than n, for which Ty holds, provided that there is such a least positive integer y and that, for every positive integer z less than this y and not less than n, Tz has a value, truth or falsehood; and that in any other case $\mathfrak{p}\boldsymbol{TN}$ has no normal form (in the case that Ty has the value falsehood for all positive integers y not less than n, we have

$$\mathfrak{p}\boldsymbol{TN} \text{ red } \mathfrak{G}(\boldsymbol{TN})\boldsymbol{N}\mathfrak{G}\boldsymbol{T} \text{ red } \mathfrak{G}(\boldsymbol{T}(S\boldsymbol{N}))(S\boldsymbol{N})\mathfrak{G}\boldsymbol{T} \text{ red } \mathfrak{G}(\boldsymbol{T}(S(S\boldsymbol{N})))(S(S\boldsymbol{N}))\mathfrak{G}\boldsymbol{T}$$
$$\text{red } \ldots$$

to infinity, and hence no normal form by 7 XXXI).

Let $\boldsymbol{R}$ be a formula which λ-defines the characteristic function associated with the propositional function R referred to in 10 I. Then F is λ-defined by

$$\lambda \boldsymbol{x}_1 \boldsymbol{x}_2 \ldots \boldsymbol{x}_n . \mathfrak{p}(\boldsymbol{R}\boldsymbol{x}_1 \boldsymbol{x}_2 \ldots \boldsymbol{x}_n)1.$$

10 II. If T is a λ-definable property of positive integers, the function F is λ-definable (1) whose value for the positive integer argument x is the xth positive integer y (in the order of magnitude of the positive integers) such that Ty holds, provided that there is such a positive integer y and that, for every positive integer z less than y, Tz has a value, truth or falsehood, and (2) which has no value otherwise.

For let $\mathbf{T}$ be a formula which λ-defines the characteristic function associated with T, and let

$$\mathbf{P} \rightarrow \lambda tx.P(x(\lambda n.S(\mathfrak{p}\, t\, n))1).$$

Then $\mathbf{PT}$ λ-defines F.

10 III. If R_1 and R_2 are λ-definable propositional functions each of $n+1$ positive integer arguments, then the propositional function R is λ-definable

(1) whose value for the positive integer arguments x_1, x_2, ..., x_n is falsehood if there is a positive integer y such that $R_1x_1x_2\ldots x_ny$ holds and $R_1x_1x_2\ldots x_nz$ and $R_2x_1x_2\ldots x_nz$ both have the value falsehood for every positive integer z less than y, and

(2) whose value for the positive integer arguments x_1, x_2. ..., x_n is truth if there is a positive integer y such that $R_2x_1x_2\ldots x_ny$ holds and $R_1x_1x_2\ldots x_ny$ has the value falsehood and $R_1x_1x_2\ldots x_nz$ and $R_2x_1x_2\ldots x_nz$ both have the value falsehood for every positive integer z less than y, and

(3) which has no value otherwise.

Let

$$\text{alt} \rightarrow \lambda xyn.\text{par}\,n(\lambda a.a(\lambda b.b1\mathbf{I}y))(\lambda c.c(\lambda def.fde))x(\mathbf{H}n).$$

$$\pi \rightarrow \lambda xy.\text{par}(\mathfrak{p}(\text{alt}\ xy)1).$$

If $\mathbf{F}$ and $\mathbf{G}$ are functions of positive integers, each being a function of one argument and including the integer 1 in its range of arguments, and if $\mathbf{F}$ and $\mathbf{G}$ λ-define F and G respectively, then alt $\mathbf{FG}$ λ-defines the function whose value for the odd integer $2x-1$ is Fx and whose value for the even integer $2x$ is Gx.

If $\mathbf{R}_1$ and $\mathbf{R}_2$ λ-define the characteristic functions associated with R_1 and R_2 respectively, then the characteristic function associated with R is λ-defined by

$$\lambda \mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_n . \pi(\mathbf{R}_1 \mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_n)(\mathbf{R}_2 \mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_n)$$

-- this completes the proof of 10 III.

Formulas having the essential properties of $\mathfrak{p}$ and $\mathfrak{P}$ were first obtained by Kleene. These formulas λ-define (in a sense which will be readily understood without explicit definition) certain functions of functions of positive integers, as already indicated.

As a further application of the formula $\mathfrak{p}$, we give λ-definitions of subtraction of positive integers in the ordinary sense (so that $x-y$ has no value if $x \leq y$) and exact division (so that $x \div y$ has no value unless x is a multiple of y):

$$[\mathbf{M}-\mathbf{N}] \rightarrow \mathfrak{p}(\lambda \boldsymbol{\alpha} . \text{ eq } \mathbf{M}\ [\mathbf{N}+\boldsymbol{\alpha}])1.$$

$$[\mathbf{M}\div\mathbf{N}] \rightarrow \mathfrak{p}(\lambda \boldsymbol{\alpha} . \text{ eq } \mathbf{M}\ [\mathbf{N}\times\boldsymbol{\alpha}])1.$$

11. DEFINITION BY RECURSION. A function F of n positive integer arguments is said to be defined by <u>composition</u> in terms of the functions G and $H_1, H_2, \ldots, H_m$ of positive integers (of the indicated numbers of arguments) by the equation,

$$Fx_1x_2\ldots x_n = G(H_1x_1x_2\ldots x_n)(H_2x_1x_2\ldots x_n)\ldots(H_mx_1x_2\ldots x_n).$$

(The case is not excluded that m or n or both are 1.)

A function F of n+1 positive integer arguments is said to be defined by <u>primitive recursion</u> in terms of the functions G_1 and G_2 of positive integers (of the indicated numbers of arguments) by the pair of equations:

$$Fx_1x_2\ldots x_n1 = G_1x_1x_2\ldots x_n,$$

$$Fx_1x_2\ldots x_n(y+1) = G_2x_1x_2\ldots x_ny(Fx_1x_2\ldots x_ny).$$

(The case is not excluded that n = 0, the function G_1 being replaced in that case by a given positive integer $\mathfrak{a}$.)

The class of <u>primitive recursive functions</u> of positive integers is defined by the three following rules, a function being primitive recursive if and only if it is determined as such by these rules:

(1) The function C such that $Cx = 1$ for every positive integer x, the successor function of positive integers, and the functions U_i^n (where n is any positive integer and i is any positive integer not greater than n) such that $U_i^n x_1 x_2 \dots x_n = x_i$, are primitive recursive.

(2) If the function F of n arguments is defined by composition in terms of the functions G and $H_1, H_2, \dots, H_m$ and if $G, H_1, H_2, \dots, H_m$ are primitive recursive, then F is primitive recursive.

(3) If the function F of n+1 arguments is defined by primitive recursion in terms of the functions G_1 and G_2 and if G_1 and G_2 are primitive recursive, then F is primitive recursive; or in the case that n = 0, if F is defined by primitive recursion in terms of the integer a and the function G_2 and if G_2 is primitive recursive, then F is primitive recursive.

In order to show that every primitive recursive function of positive integers is λ-definable, we must show that all the functions mentioned in (1) are λ-definable; that if F is defined by composition in terms of G and $H_1, H_2, \dots, H_m$ and $G, H_1, H_2, \dots, H_m$ are λ-definable, then F is λ-definable; and that if F is defined by primitive recursion in terms of G_1 and G_2 (or, in the case n = 0, in terms of a and G_2) and if G_1 and G_2 are λ-definable (or, in the case n = 0, if G_2 is λ-definable), then F is λ-definable.

Only the last of these three things makes any difficulty. Suppose that F is defined by primitive recursion in terms of G_1 and G_2, and that G_1 and G_2 are λ-defined respectively by $\mathbf{G}_1$ and $\mathbf{G}_2$. Then in order to obtain a formula $\mathbf{F}$ which λ-defines F we employ ordered triads:

$$\mathbf{F} \longrightarrow \lambda \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_n \mathbf{y} . \mathfrak{Z}_3(\mathbf{y}(\lambda \mathbf{z}[S(\mathfrak{Z}_1 \mathbf{z}),$$
$$\mathbf{G}_2 \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_n (\mathfrak{Z}_1 \mathbf{z})(\mathfrak{Z}_2 \mathbf{z}),\ \mathfrak{Z}_2 \mathbf{z}])[1, \mathbf{G}_1 \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_n,\ 1])$$

($\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \mathbf{y}, \mathbf{z}$ being any n+2 distinct variables). In the case n = 0, this reduces to:

$$F \rightarrow \lambda y. \mathfrak{Z}_3(y(\lambda z[S(\mathfrak{Z}_1 z), G_2(\mathfrak{Z}_1 z)(\mathfrak{Z}_2 z), \mathfrak{Z}_2 z])[1, A, 1]),$$

where A represents the positive integer a.

(The λ-definition of the predecessor function given in §9 may be regarded as a special case of the foregoing in which a is 1 and G_2 is U_1^2. The extension of the method used for the predecessor function to the general case of definition by primitive recursion is due to Paul Bernays, in a letter of May 27th, 1935 -- where, however, the matter is stated within the context of the calculus of λ-K-conversion and ordered pairs are consequently used instead of ordered triads. As remarked by Bernays, this method of dealing with definition by primitive recursion has the advantage that it shows also, for each n, the λ-definability of the function ρ of functions of positive integers whose value for the arguments G_1 and G_2 is the function F defined by primitive recursion in terms of G_1 and G_2 -- i.e., essentially, the function ρ of Hilbert [31].)

Thus we have:

11 I. Every primitive recursive function of positive integers is λ-definable.

The class of primitive recursive functions is known to include substantially all the ordinarily used numerical functions -- cf., e.g., Skolem [50], Gödel [27], Péter [41] (it is readily seen to be a non-essential difference that some of these authors deal with primitive recursive functions of non-negative integers rather than of positive integers). Primitive recursive, in particular, are functions corresponding to the quotient and remainder in division, the greatest common divisor, the xth prime number, and many related functions; λ-definitions of these functions can consequently be obtained by the method just given.

The two schemata, of definition by composition and by primitive recursion, have this property in common, that -- on the hypothesis that all particular values are known of the functions in terms of which F is defined -- the given equations make possible the calculation of any required particular value of F by a series of steps each consisting of a substitution, either of a (symbol for a) particular number for (all occurrences of) a variable, or of one thing for another known to be equal to it. By allowing additional, or more general, schemata having this property, various more extensive notions of recursiveness are obtainable (cf. Hilbert [31], Ackermann [1], Pēter [41, 42, 43, 44]). If the definition of primitive recursiveness is modified by allowing, in place of (2) and (3), <u>any</u> definition by a set of equations having this property, the functions obtained are called <u>general recursive</u> -- if it is required of all functions defined that they have a value for every set of the relevant number of positive integer arguments -- or <u>partial recursive</u> if this is not required. For a more exact statement (which may be made in any one of several equivalent ways), the reader is referred to Gödel [28], Church [9], Kleene [36, 39], Hilbert and Bernays [33].

That every general recursive function of positive integers is λ-definable can be proved in consequence of 10 I and 11 I by using the result of Kleene [36], that every general recursive function of n positive integer arguments $x_1, x_2, \ldots, x_n$ can be expressed in the form $F(\epsilon y(Rx_1x_2\ldots x_n y))$, where F is a primitive recursive function of positive integers, R is a propositional function of positive integers whose associated characteristic function is primitive recursive, and "ϵy" is to be read "the least positive integer y such that." (Cf. Kleene [37]). The converse proposition, that every λ-definable function of positive integers, having a value for every set of the relevant number of positive integer arguments, is general recursive, is proved by the method of Church [9] or Kleene [37] (the proof makes use of the fact that, by 7 XXXI, the process of reduction to normal form provides a method of calculating explicitly any required particular value of a function whose λ-definition is given, and proceeds by setting up a set of recursion equations which in effect describe this process of calculation).

These proofs may be extended to the case of partial recursive functions without major modifications (cf. Kleene [39]). Hence are obtained the following theorems (proofs omitted here):

11 II. **Every partial recursive function of positive integers**

is λ-definable.

11 III. Every λ-definable function of positive integers is partial recursive.

The notion of a method of effective calculation of the values of a function, or the notion of a function for which such a method of calculation exists, is of not uncommon occurrence in connection with mathematical questions, but it is ordinarily left on the intuitive level, without attempt at explicit definition. The known theorems concerning λ-definability, or recursiveness, strongly suggest that the notion of an effectively calculable function of positive integers be given an exact definition by identifying it with that of a λ-definable function, or equivalently of a partial recursive function. As in all cases where a formal definition is offered of what was previously an intuitive or empirical idea, no complete proof is possible; but the writer has little doubt of the finality of the identification. (Concerning the origin of this proposal, see Church [9], footnotes 3, 18.)

An equivalent definition of effective calculability is to identify it with calculability within a formalized system of logic whose postulates and rules have appropriate properties of recursiveness -- cf. Church [9], §7, Hilbert and Bernays [33], Supplement II.

Another equivalent definition, having a more immediate intuitive appeal is that of Turing [55], who calls a function computable if (roughly speaking) it is possible to make a finite calculating machine capable of computing any required value of the function. The machine is supplied with a tape on which computations are printed (the analogue of the paper used by a human calculator), and no upper limit is placed on the length of tape or on the time required for computation of a particular value of the function, except that it be finite in each case. Further restrictions imposed on the character of the machine are more or less clearly either non-essential or necessarily contained in the requirement of finiteness. The equivalence of computability to λ-definability and general recursiveness (attention being confined to functions of one argument for which the range of arguments consists of all positive integers) is proved in Turing [57].

Mention should also be made of the notion of a finite combinatory process introduced by Post [46]. This again is equivalent to the other concepts of effective calculability.

Examples of functions which are not effectively calculable can now be given in various ways. In particular, it is proved in Church [9] that if the set of well-formed formulas of the calculus of λ-conversion be enumerated in a straightforward way (any one of the particular enumerations which immediately suggest themselves may be employed), and if F is the function such that F is 2 or 1 according as the xth formula in this enumeration has or has not a normal form, then F is not λ-definable. This may be taken as the exact meaning of the somewhat vague statement made at the end of §6, that the condition of having a normal form is not effective.

In the explicit proofs of many of the theorems which have been stated without proof in this section, use is made of the notion of the Gödel number of a formula or formal expression. In the published papers referred to, this notion is introduced by a method closely similar to that employed by Gödel [27]. In the case of well-formed formulas of the calculus of λ-conversion, however, it would be equally possible to use the somewhat different method of our next chapter.