Chapter Title: THE CALCULI OF λ-K-CONVERSION AND λ-δ-CONVERSION

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Chapter V

# THE CALCULI OF $\lambda$-$K$-CONVERSION AND $\lambda$-$\delta$-CONVERSION

17. THE CALCULUS OF $\lambda$-$K$-CONVERSION. The calculus of $\lambda$-$K$-conversion is obtained if a single change is made in the construction of the calculus of $\lambda$-conversion which appears in §§ 5,6: namely, in the definition of well-formed formula (§5) deleting the words "and contains at least one free occurrence of **x**" from the rule 3. The rules of conversion, I, II, III, in §6 remain unchanged, except that well-formed is understood in the new sense.

Typical of the difference between the calculi of $\lambda$-conversion and $\lambda$-$K$-conversion is the possibility of defining in the latter the constancy function,

$$K \longrightarrow \lambda a(\lambda b a),$$

and the integer zero, by analogy with definitions of the positive integers in §8,

$$0 \longrightarrow \lambda a(\lambda b b).$$

Many of the theorems of §7 hold also in the calculus of $\lambda$-$K$-conversion. But obvious minor modifications must be made in 7 III and 7 V, and the following theorems fail: 7 XVII, clause (3) of 7 XXVI, and 7 XXXI, and 7 XXXII. Instead of 7 XXXI, the following weaker theorem can be proved, which is sufficient for certain purposes, in particular for the definition of $\mathfrak{p}$ (see §10):

17 I. Let a reduction be called of order one if the application of Rule II involved is a contraction of the initial $(\lambda \mathbf{x}\mathbf{M})\mathbf{N}_1$ in a formula of the form

$$(\lambda \mathbf{x}\mathbf{M})\mathbf{N}_1\mathbf{N}_2\ldots\mathbf{N}_r \qquad (r = 1, 2, \ldots).$$

> Then if $\boldsymbol{A}$ has a normal form, there is a number m such that at most m reductions of order one can occur in a sequence of reductions on $\boldsymbol{A}$.

A notion of <u>λ-*K*-definability</u> of functions of non-negative integers may be introduced, analogous to that of λ-definability of functions of positive integers, and the developments of Chapter III may then be completely paralleled in the calculus of λ-*K*-conversion. The same definitions may be employed for the successor function and for addition and multiplication as in Chapter III. Many of the developments are simplified by the presence of the zero: in particular, ordered pairs may be employed instead of ordered triads in the definition of the predecessor function, and the definition of $\mathfrak{p}$ may be simplified as in Turing [58].

It can be proved (see Kleene [37], Turing [57]) that a function $F$ of one non-negative integer argument is λ-*K*-definable if and only if $\lambda x \,.\, F(x-1)+1$ is λ-definable -- and similarly for functions of more than one argument.

The calculus of λ-*K*-conversion has obvious advantages over the calculus of λ-conversion, including the possibility of defining the constancy function and of introducing the integer zero in a simpler and more natural way. However, for many purposes -- in particular for the development of a system of symbolic logic such as that sketched in §21 below -- these advantages are more than offset by the failure of 7 XXXII. Indeed if we regard those and only those formulas as meaningful which have a normal form, it becomes clearly unreasonable that $\boldsymbol{FN}$ should have a normal form and $\boldsymbol{N}$ have no normal form (as may happen in the calculus of λ-*K*-conversion); or even if we impose a more stringent condition of meaningfulness, Rule III of the calculus of λ-*K*-conversion can be objected to on the ground that if $\boldsymbol{M}$ is a meaningful formula containing no free variables, the substitution of $(\lambda x\boldsymbol{M})\boldsymbol{N}$ for $\boldsymbol{M}$ ought not to be possible unless $\boldsymbol{N}$ is meaningful. This way of putting the matter involves the meanings of the formulas, and thus an appeal to intuition, but corresponding difficulties do appear in the formal developments in certain directions.

§18. THE CALCULUS OF RESTRICTED λ-*K*-CONVERSION. In order to avoid the difficulty just described, Bernays [4] has proposed a modification of the calculus of λ-*K*-conversion which consists in adding to Rules II and III the proviso that **N** shall be in normal form (notice that the condition of being in normal form is effective, although that of having a normal form is not). We shall call the calculus so obtained the calculus of restricted λ-*K*-conversion. In it, as follows by the methods of §7, a formula which in the calculus of λ-*K*-conversion had a normal form and had no parts without normal form will continue to have the same normal form; in particular, no possibility of conversion into a normal form is lost which existed in the calculus of λ-conversion. On the other hand, all of the theorems 7 XXVIII - 7 XXXII remain valid in the calculus of restricted λ-*K*-conversion -- and are much more simply proved than in the calculus of λ-conversion. (It should be added that the content of the theorems 7 XXVIII - 7XXXXII for the calculus of restricted λ-*K*-conversion is in a certain sense much less than the content of these theorems for the calculus of λ-conversion, and in fact cannot be regarded as sufficient to establish the satisfactoriness of the calculus of restricted λ-*K*-conversion from an intuitive viewpoint without addition of such a theorem as that asserting the equivalence to the calculus of (unrestricted) λ-*K*-conversion in the case of formulas all of whose parts have normal forms.)

The development of the calculus of restricted λ-*K*-conversion may follow closely that of the calculus of λ-conversion (as in Chapters II-IV), with such modifications as are indicated in §17 for the calculus of λ-*K*-conversion. Many of the theorems must have added hypotheses asserting that certain of the formulas involved have normal forms.

§19. TRANSFINITE ORDINALS. Church and Kleene [15] have extended the concept of λ-definability to ordinal numbers of the second number class and functions of such ordinal numbers. There results from this on the one hand an extension of the notion of effective calculability to the second number class (cf. Church [13], Kleene [39], Turing [59]), and on the other hand a method of introducing some theory of ordinal numbers into the system of symbolic logic of §21 below.

Instead of reproducing here this development within the calculus of λ-conversion, we sketch briefly an analogous development within the calculus of restricted λ-$K$-conversion.

According to the idea underlying the definitions of §8, the positive integers (or the non-negative integers) are certain functions of functions, namely the finite powers of a function in the sense of iteration. This idea might be extended to the ordinal numbers of the second number class by allowing them to correspond in the same way to the transfinite powers of a function, provided that we first fixed upon a limiting process relative to which the transfinite powers should be taken. Thus the ordinal $\omega$ could be taken as the function whose value for a function $f$ as argument is the function $g$ such that $gx$ is the limit of the sequence, $x, fx, f(fx), \ldots$ . Then $\omega+1$ would be $\lambda x.f(\omega f x)$, and so on.

Or, instead of fixing upon a limiting process, we may introduce the limiting process as an additional argument $a$ (for instance taking the ordinal $\omega$ to be the function whose value for $a$ and $f$ as arguments is the function $g$ such that $gx$ is the limit of the sequence $x, fx, f(fx), \ldots$ , relative to the limiting process $a$). This leads to the following definitions in the calculus of restricted λ-$K$-conversion, the subscript o being used to distinguish these notations from similar notations used in other connections:

$$0_o \rightarrow \lambda a(\lambda b(\lambda c c)),$$

$$1_o \rightarrow \lambda abc.bc,$$

$$2_o \rightarrow \lambda abc.b(bc), \quad \text{and so on.}$$

$$S_o \rightarrow \lambda dabc.b(dabc).$$

$$L_o \rightarrow \lambda rabc.a(\lambda d.rdabc).$$

$$\omega_o \rightarrow \lambda abc.a(\lambda d.dabc).$$

We prescribe that $0_o$ shall represent the ordinal 0; if $\mathbf{N}$ represents the ordinal $n$, the principal normal form of $S_o\mathbf{N}$ shall represent the ordinal $n+1$; if $\mathbf{R}$ represents the monotone increasing infinite sequence of ordinals, $n_0, n_1, n_2, \ldots$, in the sense that $\mathbf{R}0_o, \mathbf{R}1_o, \mathbf{R}2_o, \ldots$ are convertible into formulas representing $n_0, n_1, n_2, \ldots$, respectively, then the principal normal form of $L_0 \boldsymbol{R}$ shall represent the upper limit of this infinite sequence of ordinals. The transfinite ordinals which are represented by formulas then turn out to constitute a certain segment of the second number class, which may be described as consisting of those ordinals which can be effectively built up to from below (in a sense which we do not make explicit here).

The formula representing a given ordinal of the second number class is not unique: for example, the ordinal $\omega$ is represented not only by $\omega_0$ but also by the principal normal form of $L_0 S_0$, and by many other formulas. Hence the formulas representing ordinals are not to be taken as denoting ordinals but rather as denoting certain things which are in many-one correspondence with ordinals.

A function $F$ of ordinal numbers is said to be <u>λ-K-defined</u> by a formula $\boldsymbol{F}$ if (1) whenever $Fm = n$ and $\boldsymbol{M}$ represents $m$, the formula $\boldsymbol{FM}$ is convertible into a formula representing $n$, and (2) whenever an ordinal $m$ is not in the range of $F$ and $\boldsymbol{M}$ represents $m$, the formula $\boldsymbol{FM}$ has no normal form.

The foregoing account presupposes the classical second number class. By suitable modifications (cf. Church [13]), this presupposition may be eliminated, with the result that the calculus of restricted λ-K-conversion is used to obtain a definition of a (non-classical) <u>constructive</u> second number class, in which each classical ordinal is represented, if at all, by an infinity of elements.

20. THE CALCULUS OF λ-δ-CONVERSION. <u>The calculus of λ-δ-conversion</u> is obtained by making the following changes in the construction of the calculus of λ-conversion which appears in §§5, 6: adding to the list of primitive symbols a symbol δ, which is neither an improper symbol nor a variable, but is classed with the variables as a <u>proper symbol</u>; adding to the rule 1 in the definition of <u>well-formed formula</u> that the symbol δ is a well-formed formula; and adding to the rules of conversion in §6 four additional rules, as follows:

IV. To replace any part $\delta \boldsymbol{MN}$ of a formula by 1, provided that $\boldsymbol{M}$ and $\boldsymbol{N}$ are in δ-normal form and contain no free variables and $M$ is not convertible-I into $N$.

V. To replace any part 1 of a formula by $\delta MN$, provided that $M$ and $N$ are in $\delta$-normal form and contain no free variables and $M$ is not convertible-I into $N$.

VI. To replace any part $\delta MM$ of a formula by 2, provided that $M$ is in $\delta$-normal form and contains no free variables.

VII. To replace any part 2 of a formula by $\delta MM$, provided that $M$ is in $\delta$-normal form and contains no free variables.

Here a formula is said to be in <u>$\delta$-normal</u> form if it contains no part of the form $(\lambda x P)Q$ and contains no part of the form $\delta RS$ with $R$ and $S$ containing no free variables. It is necessary to observe that both the condition of being in $\delta$-normal form and the condition that $M$ is not convertible-I into $N$ are effective.

A <u>conversion</u> (or a <u>$\lambda$-$\delta$-conversion</u>) is a finite sequence of applications of Rules I-VII. A $\lambda$-$\delta$-conversion is called a <u>reduction</u> (or a <u>$\lambda$-$\delta$-reduction</u>) if it contains no application of Rules III, V, VII and exactly one application of one of the Rules II, IV, VI. $A$ is said to be <u>immediately reducible to</u> $B$ if there is a reduction of $A$ into $B$, and $A$ is said to be <u>reducible</u> to $B$ if there is a conversion of $A$ into $B$ which consists of one or more successive reductions.

All the theorems of §7 hold also in the calculus of $\lambda$-$\delta$-conversion, if some appropriate modifications are made (see Church and Rosser [16]). The <u>residuals</u> of $(\lambda x_p M_p)N_p$ after an application of Rule I or II are defined in the same way as before, and after an application of IV or VI they are defined as what $(\lambda x_p M_p)N_p$ becomes (this is always something of the form $(\lambda x_p M'_p)N'_p$). The residuals of $\delta M_p N_p$ after an application of I, II, IV, or VI are defined only in the case that $M_p$ and $N_p$ are in $\delta$-normal form and contain no free variables. In that case the residuals of $\delta M_p N_p$ are whatever part or parts of the entire resulting formula $\delta M_p N_p$ becomes, except that after an application of IV or VI in which $\delta M_p N_p$ itself is contracted (i.e., replaced by 1 or 2), $\delta M_p N_p$ has no residual. Thus residuals of $\delta M_p N_p$

are always of the form $\delta MN$, where $M$ and $N$ are in $\delta$-normal form and contain no free variables. A sequence of contractions on a set of parts $(\lambda x_j M_j)N_j$ and $\delta R_1 S_1$ of $A_1$, where $R_1$ and $S_1$ are in $\delta$-normal form and contain no free variables, is defined by analogy with the definition in §7. Similarly a terminating sequence of such contractions. In 7 XXV, the set of parts of $A$ on which a sequence of contractions is taken is allowed to include not only parts of the form $(\lambda x_j M_j)N_j$, but also parts of the form $\delta R_1 S_1$ in which $R_1$ and $S_1$ are in $\delta$-normal form and contain no free variables. The modified 7 XXV may then be proved by an obvious extension of the proof given in §7, and thereupon 7 XXVI - 7 XXXII follow as before. In 7 XXVI - 7 XXXII "conv-I-II" must be replaced throughout by "conv-I-II-IV-VI" and in 7 XXVI the case must also be considered that $A$ imr $B$ by a contraction of the part $\delta MN$ of $A$. For 7 XXX, there must be supplied a definition of principal $\delta$-normal form of a formula, analogous to the definition in §6 of the principal ($\lambda$-)normal form.

In connection with the calculus of $\lambda$-$\delta$-conversion we shall use both of the terms $\lambda$-conversion and $\lambda$-$\delta$-conversion, the former meaning a finite sequence of applications of Rules I-III, the latter a finite sequence of applications of Rules I-VII. The term conversion will be used to mean a $\lambda$-$\delta$-conversion, as already explained.

Similarly we shall use both of the terms $\lambda$-normal form of a formula and $\delta$-normal form of a formula. A formula will be called a $\lambda$-normal form of another if it is in $\lambda$-normal form and can be obtained from the other by $\lambda$-conversion. A formula will be called a $\delta$-normal form of another if it is in $\delta$-normal form and can be obtained from the other by $\lambda$-$\delta$-conversion. By 7 XXIX applied to the calculus of $\lambda$-conversion, the $\lambda$-normal form of a formula (in the calculus of $\lambda$-$\delta$-conversion), if it exists, is unique to within applications of Rule I. By the analogue of 7 XXIX for the calculus of $\lambda$-$\delta$-conversion, the $\delta$-normal form of a formula, if it exists, is unique to within applications of Rule I.

In order to see that the calculus of $\lambda$-$\delta$-conversion requires an intensional interpretation (cf. §2), it is sufficient to observe that, for example, although $1$ and $\lambda ab.\delta ab1ab$ correspond to the same function in extension, they are nevertheless not interchangeable, since $\delta 11$ conv 2 but $\delta 1(\lambda ab.\delta ab1ab)$ conv 1.

A constancy function $\kappa$ may be defined:

$$\kappa \rightarrow \lambda ab.\delta bbIa.$$

Then $\kappa AB$ conv $A$, if $B$ has a δ-normal form and contains no free variables, and in that case only (the conversion properties of $\kappa$ are thus weaker than those of the formula $K$ in either of the calculi of λ-$K$-conversion).

The entire theory of λ-definability of functions of positive integers carries over into the calculus of λ-δ-conversion, since the calculus of λ-conversion is contained in that of λ-δ-conversion as a part. It only requires proof that the notion of λ-δ-definability of functions of positive integers is not more general than that of λ-definability, and this can be supplied by known methods (e.g., those of Kleene [37]).

The theory of combinations carries over into the calculus of λ-δ-conversion, provided that we redefine a combination to mean an {I, J, δ}-combination. In defining the combination belonging to a formula, it is necessary to add the provision that the combination belonging to δ is δ.

If $A_1$ is a well-formed formula of the calculus of λ-δ-conversion and contains no free variables, a formula $B_1$ can be found such that $B_1J$ conv $A_1$ and $B_1I$ conv $I$. For let $A_1'$ be the combination belonging to $A_1$, unless that combination fails to contain an occurrence of either $J$ or δ, in which case let $A_1'$ be $JIIII$. Let $A_1''$ be obtained from $A_1'$ be replacing $J$ and δ throughout by $j$ and $\delta Ij(\lambda x.x(\lambda y.yIII))(\lambda z.zI)\delta$ respectively. Then $B_1$ may be taken as $\lambda jA_1''$.

Hence 14 I, and the remaining theorems of §14, may be proved for the calculus of λ-δ-conversion in the same way as for the calculus of λ-conversion.

In order to obtain a combinatory equivalent of λ-δ-conversion, analogous to the combinatory equivalent of λ-conversion given in §15, it is necessary to add to OI-OXXXVIII the following four additional operations -- where $F$, $A$, $B$, $C$ are combinations, and $A$ and $B$ belong to formulas in δ-normal form, contain no free variables, and are not the same, and $C$ belongs to the formula which represents the Gödel number of $A$:

OXXXIX. $\boldsymbol{F}(\delta \boldsymbol{AB}) \vdash \boldsymbol{F}(\beta I)$.

OXL. $\boldsymbol{FAB}(\beta I) \vdash \boldsymbol{FAB}(\delta \boldsymbol{AB})$.

OXLI. $\boldsymbol{F}(\delta \boldsymbol{AA}) \vdash \boldsymbol{F}(\omega\beta)$.

OXLII. $\boldsymbol{FC}(\omega\beta) \vdash \boldsymbol{FC}(\delta \boldsymbol{AA})$.

The reader should verify that the conditions on $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ -- although complex in character -- are effective (§6).

In order to see that these four operations are equivalent, in the presence of OI - OXXXVIII, to the rules of conversion IV - VII, it is necessary to observe that $\beta I$ and $\omega\beta$ are $\lambda$-convertible into 1 and 2 respectively.

To show that OXLII provides an equivalent to Rule VII, we must show that it enables us to change $\boldsymbol{C}(\omega\beta)$ into $\boldsymbol{C}(\delta\boldsymbol{AA})$. Since OI - OXXXVIII are equivalent to $\lambda$-conversion, this can be done as follows: $\boldsymbol{C}(\omega\beta)$ is $\lambda$-convertible into $\gamma(\tau I)\boldsymbol{CC}(\omega\beta)$, and this becomes, by OXLII, $\gamma(\tau I)\boldsymbol{CC}(\delta\boldsymbol{AA})$, and this in turn is $\lambda$-convertible into $\boldsymbol{C}(\delta\boldsymbol{AA})$.

Similarly, to show that OXL provides an equivalent of Rule V, we must show that it enables us to change $\boldsymbol{C}(\beta I)$ into $\boldsymbol{C}(\delta\boldsymbol{AB})$. This can be done as follows: $\boldsymbol{C}(\beta I)$ is $\lambda$-convertible into $\gamma(\gamma(\tau I)\boldsymbol{C})(\beta I)(\omega\beta)$; and this can be changed by the method of the preceding paragraph into $\gamma(\gamma(\tau I)\boldsymbol{C})(\beta I)(\delta\boldsymbol{BB})$; and this is $\lambda$-convertible into $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)\boldsymbol{C})(\beta I)))(\delta\boldsymbol{BB})(\omega\beta)$; and this can be changed by the method of the preceding paragraph into $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)\boldsymbol{C})(\beta I)))(\delta\boldsymbol{BB})(\delta\boldsymbol{AA})$; and this is $\lambda$-convertible into $\gamma(\beta(\gamma(\beta\gamma(\gamma(\gamma(\beta(\beta\beta)(\omega\delta))I)(\gamma(\gamma(\tau I)\boldsymbol{C}))))(\omega\delta))\boldsymbol{AB}(\beta I)$; and this becomes, by OXL, $\gamma(\beta(\gamma(\beta\gamma(\gamma(\gamma(\beta(\beta\beta)(\omega\delta))I)(\gamma(\gamma(\tau I)\boldsymbol{C}))))(\omega\delta))\boldsymbol{AB}(\delta\boldsymbol{AB})$; and this is $\lambda$-convertible into $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)\boldsymbol{C})(\delta\boldsymbol{AB})))(\delta\boldsymbol{BB})(\delta\boldsymbol{AA})$; and this becomes, by OXLI, $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)\boldsymbol{C})(\delta\boldsymbol{AB})))(\delta\boldsymbol{BB})(\omega\beta)$; and this is $\lambda$-convertible into $\gamma(\gamma(\tau I)\boldsymbol{C})(\delta\boldsymbol{AB})(\delta\boldsymbol{BB})$; and this becomes, by OXLI, $\gamma(\gamma(\tau I)\boldsymbol{C})(\delta\boldsymbol{AB})(\omega\beta)$; and this, finally, is $\lambda$-convertible into $\boldsymbol{C}(\delta\boldsymbol{AB})$.

Only minor modifications are necessary in §16 in order to carry over its results to the calculus of $\lambda$-$\delta$-conversion. In the definition of the Gödel number of a combination the clause must be added: (2a) The Gödel number of $\delta$ is 5. In the construction of the formula, form, it is only necessary to impose on $\mathfrak{D}$ the further condition that $\mathfrak{D}\,5$ conv $\lambda x.x12\delta$, so insuring that form 5 conv $\delta$. The construction of o remains unchanged. The formulas occ, $\epsilon$, $\mathfrak{C}$, r, $\Lambda$, prim, norm, and $0_1$ - $0_{38}$ may then be obtained, having the properties described in §16 (norm λ-defines the property of a combination of belonging to a formula which is in λ-normal form). The formulas cb, ncb, 0, cnvt, nf (the λ-normal form of), and $\nu$ may then also be obtained as before. The formula, cb, represents an enumeration of the least set of formulas which contains 1, 3, and 5 and is closed under the operation of forming $(\lambda ab \,.\, 2 \times \mathrm{nr}\ ab)\boldsymbol{XY}$ from the formulas $\boldsymbol{X}$, $\boldsymbol{Y}$.

Besides norm it is also possible to obtain a formula, dnorm, which λ-defines the property of a combination of belonging to a formula in δ-normal form. Details of this are left to the reader.

Formulas $0_{39}$ - $0_{42}$ may be obtained, related to the operations OXXXIX - OXLII in the same way that $0_1$ - $0_{38}$ are related to OI - OXXXVIII. We give details in the case of $0_{40}$ and $0_{42}$. Let $\mathfrak{I}_{40}$ be a formula such that $\mathfrak{I}_{40}1$ conv $I$ and $\mathfrak{I}_{40}2$ conv $\lambda x \,.\, 2 \times \mathrm{nr}\ x_1[2 \times \mathrm{nr}[2 \times \mathrm{nr}\ 5x_{112}]x_{12}]$; then let

$$0_{40} \longrightarrow \lambda x \,.\, \mathfrak{I}_{40}[\mathrm{par}\ x + \mathrm{par}\ x_1 + \mathrm{par}\ x_{11} + \mathrm{prim}\ x_{112} + \mathrm{dnorm}\ x_{112} + \mathrm{prim}\ x_{12} + \mathrm{dnorm}\ x_{12} + \mathrm{eq}\ \eta x_2 \dot{-} \mathrm{eq}\ x_{112}x_{12} \dot{-} \overline{13}]x,$$

$\eta$ being the formula representing the Gödel number of $\beta I$. Let $\mathfrak{I}_{42}$ be a formula such that $\mathfrak{I}_{42}1$ conv $I$ and $\mathfrak{I}_{42}2$ conv $\lambda x \,.\, 2 \times \mathrm{nr}\ x_1[2 \times \mathrm{nr}\ [2 \times \mathrm{nr}\ 5(\mathrm{form}\ x_{12})](\mathrm{form}\ x_{12})]$; then let

$$0_{42} \longrightarrow \lambda x \,.\, \mathfrak{I}_{42}[\mathrm{par}\ x + \mathrm{par}\ x_1 + \mathrm{h}(\nu x_{12})x_{12} + \mathrm{eq}\ \zeta x_2 \dot{-} 6]x,$$

where $\zeta$ is the formula representing the Gödel number of $\omega\beta$, and h is such a formula that h1 conv $\lambda x . x1$ and h2 conv $\lambda x.\mathrm{min}\ (\mathrm{prim}\ (\mathrm{form}\ x))(\mathrm{dnorm}\ (\mathrm{form}\ x))$.

Then a formula, do, may be obtained, analogous to 0 but involving all of $0_1$ - $0_{42}$ instead of only $0_1$ - $0_{38}$. Let

$$\mathrm{dcnvt} \longrightarrow \lambda ab \,.\, \mathrm{do}\ ba.$$

Then, if $\boldsymbol{C}$ represents the Gödel number of a combination $\boldsymbol{M}$, the formula, dcnvt $\boldsymbol{C}$, enumerates the Gödel numbers of combinations obtainable from $\boldsymbol{M}$ by λ-δ-conversion (whereas cnvt $\boldsymbol{C}$

enumerates merely the Gödel numbers of combinations obtainable from $\mathbf{M}$ by λ-conversion).

It is also possible, by using the formula, dnorm, to obtain a formula, dnf, which λ-defines the operation δ-normal form of a formula, and a formula, dncb, which enumerates the Gödel numbers of combinations which belong to formulas in δ-normal form and contain no free variables. The definitions parallel those of nf and ncb.

Finally, in the calculus of λ-δ-conversion, a formula, met, may be obtained which provides a kind of inverse of the function, form: if $\mathbf{M}$ is a formula which contains no free variables and has a δ-normal form, then met $\mathbf{M}$ is convertible into the formula representing the Gödel number belonging to the δ-normal form of $\mathbf{M}$. The definition is as follows:

$$\text{met} \rightarrow \lambda x \,.\, \text{dncb}\,(\mathfrak{p}(\lambda n \,.\, \delta(\text{form}\,(\text{dncb}\; n))x)1).$$

21. A SYSTEM OF SYMBOLIC LOGIC. If we identify the truth values, truth and falsehood, with the positive integers 2 and 1 respectively, we may base a system of symbolic logic on the calculus of λ-δ-conversion. This system has one primitive formula or axiom, namely the formula 2, and seven rules of inference, namely the rules I - VII of λ-δ-conversion; the provable formulas, or theses, of the system are the formulas which can be derived from the formula 2 by sequences of applications of the rules of inference. (As a matter of fact, the rules of inference II, IV, VI are superfluous, in the sense that their omission would not decrease the class of provable formulas, as follows from 7 XXVII, or rather from the analogue of this theorem for the calculus of λ-δ-conversion.)

The identification of the truth values, truth and falsehood, with the positive integers 2 and 1 is, of course, artificial, but apparently it gives rise to no actual formal difficulty. If it be thought objectionable, the artificiality may be avoided by a minor modification in the system, which consists in introducing a symbol ⊢ and writing ⊢2, instead of 2, as the primitive formula; all the theses of the system will then be preceded by the sign ⊢, which may be interpreted as asserting that that which follows is equal to 2.

In this system of symbolic logic the fundamental operations of the propositional calculus -- negation, conjunction, disjunction -- may be introduced by the following definitions:

$$[\sim \boldsymbol{A}] \longrightarrow \pi(\lambda \boldsymbol{a}.\boldsymbol{a} I(\delta 2 \boldsymbol{A}))(\lambda \boldsymbol{a}.\boldsymbol{a} I(\delta 1 \boldsymbol{A})).$$

$$[\boldsymbol{A} \& \boldsymbol{B}] \longrightarrow 4 \dot{-} . [\sim \boldsymbol{A}] + [\sim \boldsymbol{B}].$$

$$[\boldsymbol{A} \vee \boldsymbol{B}] \longrightarrow \sim . [\sim \boldsymbol{A}] \,\&\, [\sim \boldsymbol{B}].$$

It follows from these definitions that $\boldsymbol{A} \vee \boldsymbol{B}$ cannot be a thesis unless either $\boldsymbol{A}$ or $\boldsymbol{B}$ is a thesis -- and this situation apparently cannot be altered by any suitable change in the definitions. Since this property is known to fail for classical systems of logic, e.g., that of Whitehead and Russell's Principia Mathematica, it is clear that the present system therefore differs from the classical systems in a direction which may be regarded as finitistic in character.

Functions of positive integers are of course represented in the system by the formulas $\lambda$-defining these functions, and properties of and relations between positive integers are represented by the formulas $\lambda$-defining the corresponding characteristic functions. The propositional function to be a positive integer is represented in the system as a formula $N$, defined as follows (referring to §§16, 20):

$$N \longrightarrow \lambda x.\nu(\text{met } x).$$

The general relation of equality or identity (in intension) is represented by $\delta$.

An existential quantifier $\Sigma$ may be introduced:

$$\iota \longrightarrow \lambda f \,.\, \text{form } (Z'(H(\text{dcnvt } \alpha(\mathfrak{p}(\lambda n \,.\, \delta f$$
$$(\text{form } (Z(H(\text{dcnvt } \alpha n)))))1)))),$$

where $\alpha$ is the formula representing the Gödel number belonging to the formula 2;

$$\Sigma \longrightarrow \lambda f.f(\iota f).$$

Here $\iota$ represents a general selection operator. Given a formula $\boldsymbol{F}$; if there is any formula $\boldsymbol{A}$ such that $\boldsymbol{FA}$ conv 2, then $\iota\boldsymbol{F}$ is one of the formulas $\boldsymbol{A}$ having this property; and in the contrary case $\iota\boldsymbol{F}$ has no normal form. Consequently $\Sigma$ represents an existential quantifier without a negation: $\Sigma \boldsymbol{F}$ conv 2 if there is a formula $\boldsymbol{A}$ such that $\boldsymbol{FA}$ conv 2, and in the contrary case $\Sigma \boldsymbol{F}$ has no normal form.

The operator $\iota$ should be compared with Hilbert's operator $\epsilon$ [31 and elsewhere], or, perhaps better, the $\eta$-operator of Hilbert and Bernays [33]. The $\iota$ should be used with the caution that the equivalence of propositional functions represented in the system by $\boldsymbol{F}$ and $\boldsymbol{G}$ need not imply the equality of $\iota \boldsymbol{F}$ and $\iota \boldsymbol{G}$.

The interpretation of $\iota$ as a selection operator and of $\Sigma$ as an existential quantifier depends on an identification of formal provability in the system with truth. But this is justified by a completeness property which the system possesses: a formula which is not provable, unless it is convertible into a principal normal form other than 2 and hence is disprovable, must have no normal form, and hence be meaningless.

For convenience in the further development of the system, or for the sake of comparison with more usual notations, we may introduce the abbreviations:

$$[\imath x \boldsymbol{M}] \longrightarrow \iota(\lambda x \boldsymbol{M}).$$

$$[\exists x \boldsymbol{M}] \longrightarrow \Sigma(\lambda x \boldsymbol{M}).$$

The problem of introducing universal quantifiers into the system, or, equivalently, of introducing existential quantifiers having a negation, is beyond the scope of the present treatise. It follows by the methods of Gödel [27] that any universal quantifier introduced by definition will have a certain character of incompleteness; this is in effect the same incompleteness property which, in accordance with the results of Gödel, almost any consistent and satisfactorily adequate system of formal logic must have, except that it here appears transferred from the realm of provability to the realm of meaning of the quantifiers.

The consistency of the system of symbolic logic just outlined is a corollary of 7 XXX, or rather of the analogue of this theorem for the calculus of $\lambda$-$\delta$-conversion. This consistency proof is of a strictly constructive or finitary nature.

(The failure in this system of the known paradoxes of set theory depends, in some of the simpler cases, merely on the fact that the formula which would otherwise lead to the paradox fails to have a normal form. Thus, in the case of Russell's paradox, we find that $(\lambda x.\sim(xx))(\lambda x.\sim(xx))$ has no normal form; and in the case of Grelling's paradox concerning heterological words, or, as we shall put it, concerning heterological Gödel numbers, we find that $(\lambda x.\sim(\text{form } xx))(\text{met } (\lambda x.\sim(\text{form } xx)))$ has no normal form. In more complicated cases, where the expression of the paradox requires a universal quantifier, the failure may depend on the above indicated incompleteness property of the quantifier.)