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THE DECISION PROBLEM FOR SOME CLASSES OF SENTENCES WITHOUT QUANTIFIERS

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In this paper we shall be concerned with questions regarding decision problems for certain classes of sentences (without quantifiers) of various kinds of algebra. In the first section we shall establish a result of a general nature, which enables one, in a number of cases, to reduce a decision problem of the type considered to a somewhat simpler problem. In the second section we formulate a rather broad sufficient condition for the existence of a decision method for sentences without quantifiers; and in the last section we show that this condition holds of lattices.

By an *algebra*[2] we shall mean an ordered set $\Gamma = (K, O_1, \cdots, O_n)$ where: (1) K is a class; (2) for each i from 1 to n, there is a positive integer n_i such that O_i is an n_i-ary operation; (3) each operation O_i is defined over K; and (4) if $x_1, \cdots, x_{n_i}$ are any elements of K, then $O_i\ (x_1, \cdots, x_{n_i})$ is also an element of K. Two algebras $\Gamma = (K, O_1, \cdots, O_n)$ and $\Gamma' = (K', O_1', \cdots, O_m')$ are called *similar*, if they have the same number of operations ($n = m$), and if each two of the operations O_i and O_1' are operations on the same number of terms.

We now turn aside from mathematical notions to introduce some metamathematical terms.

As *unary operation variables* we use the signs

$$R_1^{(1)}, R_2^{(1)}, R_3^{(1)}, \cdots .$$

As *binary operation variables* we use the signs

$$R_1^{(2)}, R_2^{(2)}, R_3^{(2)}, \cdots .$$

And so on.

As *individual variables* we use the signs

$$x_1, x_2, x_3, \cdots .$$

As *terms* we understand all individual variables, together with all meaningful expressions formed by composition from individual variables by means of operation variables. Thus, for example,

$$x_7, \quad R_3^{(1)}(x_5), \quad R_4^{(2)}[R_3^{(1)}(x_5), x_1]$$

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[2] This term is sometimes used in a more general sense (see, for example, Garrett Birkhoff's paper, *On the combination of subalgebras*, ***Proceedings of the Cambridge Philosophical Society,*** vol. 29 (1933), pp. 441–464). Thus we could allow more than one class K, or we could allow there to be infinitely many operations, or operations not of a finite character. For our present purpose, however, such generality would lead only to superfluous complications.

are terms. But

$$R_4^{(3)}[R_3^{(1)}(x_5), x_2]$$

is not a term, since $R_4^{(3)}$ has here only two arguments instead of three.

By an *equation* we mean any expression of the form

$$\alpha = \beta$$

where α and β are terms.

We use the signs

$$\sim, \mathbf{v}, \&, \text{and} \rightarrow$$

respectively for *negation*, *disjunction*, *conjunction*, and material *implication*. We use the signs

$$\prod \text{and} \sum$$

respectively for *universal quantification* and *existential quantification*.

By *sentences* we understand equations, together with all meaningful expressions which can be built up from equations by means of negation, disjunction, conjunction, and implication signs, and by application of the quantifiers with an individual variable written beneath. We use the terms *free variable* and *bound variable* in the usual way. Thus the expression

$$(x_1 = x_3) \mathbf{v} \sim\prod_{x_2}[R_1^{(1)}(x_3) = R_2^{(2)}(x_2, x_3)]$$

is a sentence in which x_1 and x_3 occur free, and x_2 occurs bound

We call a sentence *closed* if it contains no free individual variables. We call a sentence *open* if it contains no bound variables. Thus the sentence

$$(1) \qquad \prod_{x_1}\prod_{x_2}[R_1^{(1)}(x_2) = x_1]$$

is closed. And the sentence

$$(2) \qquad R_1^{(1)}(x_2) = x_1$$

is open. Some sentences are neither closed nor open; for example the conjunction of the two sentences (1) and (2).

By the *closure* of a sentence, is meant the sentence obtained from it by putting in front of it, in alphabetical order, the universal quantifiers with the individual variables which occur free in it (a closed sentence coincides with its own closure). Thus sentence (1) above is the closure of sentence (2).

By a *conditional equation*, we shall mean a sentence of the form

$$(\varepsilon_1 \ \& \ \varepsilon_2 \ \& \cdots \& \ \varepsilon_n) \rightarrow \varepsilon_{n+1}$$

where $\varepsilon_1, \cdots, \varepsilon_{n+1}$ are equations.

We now assign to each algebra in a unique way a sequence of operation variables. If $(K, O_1, \cdots, O_n)$ is an algebra with n operations, where O_i is of degree n_i (for $i = 1, \cdots, n$), then we assign to it the sequence of n distinct operation variables $(R_1^{(n_1)}, R_2^{(n_2)}, \cdots, R_n^{(n_n)})$. Thus, for example, if K is the class of all sets of real numbers, and if O_1, O_2, and O_3 are respectively the operations of complementation, union, and intersection, then we assign to the algebra (K, O_1, O_2, O_3) the sequence of operation variables $(R_1^{(1)}, R_2^{(2)}, R_3^{(2)})$, since O_1 is a unary operation, and O_2 and O_3 are binary operations.

It is clear that if Γ_1 and Γ_2 are similar algebras, then this process assigns the same sequence of operation variables to Γ_1 as to Γ_2. Thus we can speak of the sequence of operation variables which *corresponds* to any *class* of similar algebras.

We say that a *sentence* α, *corresponds* to an *algebra* Γ (or to a class $\mathfrak{A}$ of similar algebras), if every operation variable occurring in α is one of the variables in the sequence of variables corresponding to Γ (or to $\mathfrak{A}$). Under these conditions we also sometimes say, that α is a sentence *of* Γ (or of $\mathfrak{A}$).

It is clear that we can define[3] what is meant by saying that a closed sentence α which corresponds to an algebra Γ is *true* of Γ. Thus consider the algebra (K, O_1, O_2, O_3) introduced above. We see that the sentence

$$\prod_{x_1}\prod_{x_2}[R_2^{(2)}(x_1, x_2) = R_2^{(2)}(x_2, x_1)]$$

is true of this algebra, since the union of two sets is independent of the order in which they are added. On the other hand, the sentence

$$\prod_{x_1}\left\{\sum_{x_2}\sim[R_2^{(2)}(x_1, x_2) = x_2] \rightarrow \sum_{x_3}\sum_{x_4}([\sim R_2^{(2)}(x_3, x_4) = x_4] \,\&\, [R_2^{(2)}(x_1, x_3) = x_1])\right\}$$

is not true of this algebra, since there are sets ("atoms") in the class K which consist of but a single element.

If α is an arbitrary sentence (not necessarily closed) of an algebra Γ, then we say that α is true of Γ, if the closure of α is true of Γ.

We say that a sentence α is *true of a class* $\mathfrak{A}$ of similar algebras, if it is true of every algebra Γ which belongs to $\mathfrak{A}$.

We say that a (finite) set $\alpha_1, \cdots, \alpha_n$ of sentences is an *axiom system* for a class $\mathfrak{A}$ of similar algebras, if each of the sentences is true of $\mathfrak{A}$, and if every algebra, for which $\alpha_1, \cdots, \alpha_n$ are all true, belongs to $\mathfrak{A}$. A class of algebras is called *axiomatizable* if there exists an axiom system for it. An axiomatizable class of algebras will sometimes be referred to as a *kind* of algebra. It will be noticed that most classes of algebras are not axiomatizable; since there are only a countable infinity of possible axiom systems, whereas the number of possible classes of algebras is uncountable.

A class of similar algebras is said to be *equationally definable* if there exists an axiom system for it which consists exclusively of equations. Many of the most familiar kinds of algebra are equationally definable. For instance, the class of lattices (in terms of the operations of meet and join), the class of Boolean algebras (in terms of the operations of complementation, union, and intersection), and the class of groups (in terms of the operations of multiplication, and the formation of inverses).

By a *decision method* for a class $\mathfrak{A}$ of algebras, is meant a method whereby,

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[3] As in A. Tarski's paper, *Der Wahrheitsbegriff in den formalisierten Sprachen*, **Studia philosophica,** vol. 1 (1936), pp. 261–405.

given a sentence α, one can always decide in a finite number of steps whether α is true of $\mathfrak{A}$. More generally, we can speak of a decision method for an arbitrary class A of sentences corresponding to a class $\mathfrak{A}$ of algebras; by this we mean a method whereby, given a sentence α of A, one can always decide in a finite number of steps whether α is true of $\mathfrak{A}$. In this paper we shall be concerned mostly with decision methods for classes of open sentences. By the *decision problem* for a class A, is meant the problem to find a decision method for A.

If $\mathfrak{A}$ is an axiomatizable class of algebras, then the decision problem for $\mathfrak{A}$ can be reduced to a problem regarding the restricted function calculus: if α is any sentence corresponding to $\mathfrak{A}$, then we can construct a sentence α^* of the restricted function calculus such that α is true of $\mathfrak{A}$ if and only if α^* is provable in the restricted function calculus. To see this, suppose that $\alpha_1, \cdots, \alpha_n$ are the closures of axioms for $\mathfrak{A}$, and let β_1 be the sentence

$$(\alpha_1 \;\&\; \cdots \;\&\; \alpha_n) \rightarrow \alpha .$$

It is now easily seen that α is true of $\mathfrak{A}$ if and only if β_1 is true of every algebra similar to the algebras of $\mathfrak{A}$.

It is well known that operations can always be eliminated in favor of predicates. Let β_2 be the sentence which results from β_1 by this transformation. Then β_2 is a sentence built up from expressions of the form

$$x_{i_1} = x_{i_2}$$

and expressions of the form

$$P_i(x_{i_1}, \cdots, x_{i_n})$$

(where P_i is a predicate variable and $x_{i_1}, \cdots, x_{i_n}$ are individual variables), by means of the operations of the restricted function calculus; and β_2 is universally valid if and only if α is true of $\mathfrak{A}$. Let β_3 be the sentence that results from β_2 by replacing the equality sign in β_2 by a two-place predicate variable P which does not occur in β_2. Let β_4 be the sentence consisting of the conjunction of the sentences

$$\prod_{x_1} P(x_1, x_1) ,$$

$$\prod_{x_1}\prod_{x_2} [P(x_1, x_2) \rightarrow P(x_2, x_1)] ,$$

$$\prod_{x_1}\prod_{x_2}\prod_{x_3} [P(x_1, x_2) \;\&\; P(x_2, x_3) \rightarrow P(x_1, x_3)] .$$

Suppose that the predicate variables in β_3 other than P are $P_1, \cdots, P_n$, and let β_5 be the conjunction of the closures of all sentences of the form

$$P_i(x_1, \cdots, x_k, \cdots, x_{n_i}) \;\&\; P(x_k, y) \rightarrow P_i(x_1, \cdots, y, \cdots, x_{n_i}) ,$$

for $i = 1, \cdots, n$, and $k = 1, \cdots, n_i$. Let α^* be the sentence

$$\beta_4 \;\&\; \beta_5 \rightarrow \beta_3 .$$

It is easily shown that α^* is universally valid if and only if β_2 is universally valid, and hence if and only if α is true of $\mathfrak{A}$. Since α^* is a sentence of the restricted function calculus, it is universally valid if and only if it is provable.[4] Hence α is true of $\mathfrak{A}$ if and only if α^* is provable in the restricted function calculus, as was to be shown.

I. A reduction of the decision problem for open sentences to a simpler problem. If $\Gamma = (K, O_1, \cdots, O_n)$ and $\Gamma' = (K', O'_1, \cdots, O'_n)$ are two similar algebras, then by the *direct union* of Γ and Γ' is meant the algebra $\Gamma'' = (K'', O''_1, \cdots, O''_n)$, where K'' is the set of all ordered couples $\langle x, x' \rangle$, in which x is an element of K and x' is an element of K', and where each of the operations O''_i $(i = 1, \cdots, n)$ is defined as follows:

$$O''_i(\langle x_1, x'_1 \rangle, \cdots, \langle x_{n_i}, x'_{n_i} \rangle) = \langle O_i(x_1, \cdots, x_{n_i}), O'_i(x'_1, \cdots, x'_{n_i}) \rangle .$$

Analogously we can define the direct union of any number of similar algebras.

We also use the notion of isomorphism of two algebras in the usual way. Two similar algebras $(K, O_1, \cdots, O_n)$ and $(K', O'_1, \cdots, O'_n)$ are said to be *isomorphic* if there is a one-to-one correspondence between the elements of K and those of K' such that (for $i = 1, \cdots, n$), when $x_1, \cdots, x_{n_i}$ are elements of K corresponding respectively to the elements $x'_1, \cdots, x'_{n_i}$ of K', then the element $O_i(x_1, \cdots, x_{n_i})$ of K corresponds to the element $O'_i(x'_1, \cdots, x'_{n_i})$ of K'.

We say that a class $\mathfrak{A}$ of similar algebras is *closed under direct union*, if, for every two algebras Γ and Γ' of $\mathfrak{A}$, there exists in $\mathfrak{A}$ an algebra Γ'', which is isomorphic to the direct union of Γ and Γ'. It is well known that many of the familiar kinds of algebras are closed under direct union. Thus the direct union of two groups is a group, and the direct union of two Boolean algebras is a Boolean algebra. On the other hand, the class of fields is not closed under direct union, since the direct union of two fields is in general not a field.

It can be shown quite generally that if $\mathfrak{A}$ is a class of algebras for which there exists a set of axioms each of which is either an equation, the negation of a conjunction of equations, or a conditional equation, then $\mathfrak{A}$ is closed under direct union. For it is easily verified that if an equation, or a sentence of one of the forms

$$\sim(\varepsilon_1 \ \& \cdots \& \ \varepsilon_n) \ \text{ or } \ (\varepsilon_1 \ \& \cdots \& \ \varepsilon_n) \rightarrow \varepsilon_{n+1}$$

(where $\varepsilon_1, \cdots, \varepsilon_{n+1}$ are equations) is true of each of two algebras, then it is also true of their direct union.

This general remark makes one understand why so many familiar kinds of algebra are closed under direct union. It also affords an "explanation" of the fact that the class of integrity domains, for example, is not closed under direct union; for the usual formulation of the axioms for integrity domains includes the axiom

$$(x_1 \cdot x_2 = 0) \rightarrow (x_1 = 0 \lor x_2 = 0) ,$$

[4] See Kurt Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, ***Monatshefte für Mathematik und Physik***, vol. 37 (1930), pp. 349–360; or Hilbert and Ackermann, ***Grundzüge der theoretischen Logik***, 2nd edn., Berlin 1938, pp. 74–81.

which is not a conditional equation, since it involves a disjunction in the consequence.

We prove now a simple, but rather interesting, theorem about classes of algebras closed under direct union.

THEOREM 1. Let $\mathfrak{A}$ be a class of algebras which is closed under direct union, and suppose that the sentence

$$(\varepsilon_1 \mathbin{\&} \cdots \mathbin{\&} \varepsilon_r) \rightarrow (\varepsilon_1' \vee \cdots \vee \varepsilon_s')$$

(where $\varepsilon_1, \cdots, \varepsilon_r, \varepsilon_1', \cdots, \varepsilon_s'$ are equations) is true of $\mathfrak{A}$. Then there exists an $i \leqq s$ such that the sentence

$$(\varepsilon_1 \mathbin{\&} \cdots \mathbin{\&} \varepsilon_r) \rightarrow \varepsilon_i'$$

is true of $\mathfrak{A}$.

Proof. Let the variables in the sentence

$$(\varepsilon_1 \mathbin{\&} \cdots \mathbin{\&} \varepsilon_r) \rightarrow (\varepsilon_1' \vee \cdots \vee \varepsilon_s')$$

be $x_1, \cdots, x_n$. It is convenient to represent our sentence in the more explicit form

$$[\varepsilon_1(x_1, \cdots, x_n) \mathbin{\&} \cdots \mathbin{\&} \varepsilon_r(x_1, \cdots, x_n)] \rightarrow \varepsilon_1'(x_1, \cdots, x_n) \vee \cdots \vee \varepsilon_s'(x_1, \cdots, x_n)] .$$

Suppose that no one of the sentences

$$[\varepsilon_1(x_1, \cdots, x_n) \mathbin{\&} \cdots \mathbin{\&} \varepsilon_r(x_1, \cdots, x_n)] \rightarrow \varepsilon_i'(x_1, \cdots, x_n)$$

(for $i = 1, \cdots, s$) is true of $\mathfrak{A}$. Then for each $i \leqq s$ there is an algebra Γ_i in $\mathfrak{A}$, which contains elements $a_{i,1}, \cdots, a_{i,n}$ such that

$$(1) \qquad [\varepsilon_1(a_{i,1}, \cdots, a_{i,n}) \mathbin{\&} \cdots \mathbin{\&} \varepsilon_r(a_{i,1}, \cdots, a_{i,n})] \rightarrow \varepsilon_i'(a_{i,1}, \cdots, a_{i,n})$$

is false of Γ_i. From the falsity of (1) we conclude that each of the equations

$$\varepsilon_1(a_{i,1}, \cdots, a_{i,n}), \cdots, \varepsilon_r(a_{i,1}, \cdots, a_{i,n})$$

is true of Γ_i, while the equation

$$\varepsilon_i'(a_{i,1}, \cdots, a_{i,n})$$

is false of Γ_i.

Consider now the elements

$$a_1 = \langle a_{1,1}, \cdots, a_{s,1}\rangle, \cdots, a_n = \langle a_{1,n}, \cdots, a_{s,n}\rangle$$

in the direct union of the algebras $\Gamma_1, \cdots, \Gamma_s$. It is easily seen that each of the equations

$$\varepsilon_1(a_1, \cdots, a_n), \cdots, \varepsilon_r(a_1, \cdots, a_n)$$

is true in this direct union, and that each of the equations

$$\varepsilon_1'(a_1, \cdots, a_n), \cdots, \varepsilon_s'(a_1, \cdots, a_n)$$

is false. Hence the sentence

$$[\varepsilon_1(a_1, \cdots, a_n) \,\&\, \cdots \,\&\, \varepsilon_r(a_1, \cdots, a_n)] \rightarrow [\varepsilon_1'(a_1, \cdots, a_n) \vee \cdots \vee \varepsilon_s'(a_1, \cdots, a_n)]$$

is false. Thus the sentence

$$(\varepsilon_1 \,\&\, \cdots \,\&\, \varepsilon_r) \rightarrow (\varepsilon_1' \vee \cdots \vee \varepsilon_s')$$

is not true of the direct union of the algebras $\Gamma_1, \cdots, \Gamma_s$; hence, since $\mathfrak{A}$ is closed under direct union, it is not true of $\mathfrak{A}$.

Our theorem now follows by contraposition.

It should be noticed that the above theorem holds in particular for the case where the only equation in the antecedent is the equation

$$x_1 = x_1.$$

Since an implication of this sort is true if and only if its consequent is true, we see that if a class $\mathfrak{A}$ of algebras is closed under direct union, and if a disjunction

$$\varepsilon_1' \vee \cdots \vee \varepsilon_s'$$

of equations if true of $\mathfrak{A}$, then at least one of the equations ε_i' $(i = 1, \cdots, s)$ is true of $\mathfrak{A}$.

THEOREM 2. Let $\mathfrak{A}$ be a class of algebras which is closed under direct union. Let A be the class of all open sentences corresponding to $\mathfrak{A}$, let B be the class of conditional equations, and let C be the class of disjunctions of inequalities (i.e., sentences of the form

$$\sim\varepsilon_1 \vee \cdots \vee \sim\varepsilon_n$$

where $\varepsilon_1, \cdots, \varepsilon_n$ are equations). Then to solve the decision problem for A is equivalent to solving the decision problem for B and for C.

Proof. Since B and C are subclasses of A, it is clear that a solution of the decision problem for A implies a solution for B and for C.

Suppose, on the other hand, that we have a solution of the decision problem for B and for C. And let α be any element of A (i.e., let α be an arbitrary open sentence corresponding to $\mathfrak{A}$). By the well-known theorem of the sentential calculus, α can be reduced to conjunctive normal form; that is to say, there are sentences $\alpha_1, \cdots, \alpha_n$ each of which is a disjunction of equations and inequalities, and such that α has always the same truth value as

$$\alpha_1 \,\&\, \cdots \,\&\, \alpha_n .$$

Since a conjunction is true if and only if each conjunct is true, we therefore see that α is true of $\mathfrak{A}$ if and only if each of the sentences $\alpha_1, \cdots, \alpha_n$ is true of $\mathfrak{A}$. Thus it suffices, in order to solve the decision problem for A, to solve it for an arbitrary disjunction of equations and inequalities.

Now let β be any disjunction of equations and inequalities. If β consists only of inequalities, then β belongs to the class C, and hence is decidable by hypothesis. If β consists only of equations, then it is equivalent to the sentence

$$\beta \vee \sim(x_1 = x_1),$$

which contains an inequality. There remains therefore only the case that β contains both equations and inequalities. Then β is of the form

$$\varepsilon_1' \vee \cdots \vee \varepsilon_s' \vee \sim\varepsilon_1 \vee \cdots \vee \sim\varepsilon_r$$

where $\varepsilon_1, \cdots, \varepsilon_r$ and $\varepsilon_1', \cdots, \varepsilon_s'$ are equations. But this is equivalent to the sentence

$$(\varepsilon_1 \,\&\, \cdots \,\&\, \varepsilon_r) \rightarrow (\varepsilon_1' \vee \cdots \vee \varepsilon_s').$$

By Theorem 1, this sentence is true of $\mathfrak{A}$ if and only if at least one of the sentences

$$(\varepsilon_1 \,\&\, \cdots \,\&\, \varepsilon_r) \rightarrow \varepsilon_i' \qquad (i = 1, \cdots, s)$$

is true of $\mathfrak{A}$. Since each of the latter sentences belongs to B, it is decidable by hypothesis. Thus we can decide whether β is true of $\mathfrak{A}$, as was to be shown.

The condition in our last theorem, that we be able to decide regarding disjunctions of inequalities, is in many cases easily taken care of. It frequently happens in practice that one can show that *no* disjunctions of inequalities at all are true. If, for example, the class $\mathfrak{A}$ of algebras contains an algebra $\Gamma = (K, O_1, \cdots, O_n)$ for which K consists of a single element, then it is seen that no disjunction of inequalities is true of Γ, nor therefore of $\mathfrak{A}$. The fact that $\mathfrak{A}$ contains an algebra whose class K contains a single element, is implied, in particular, if we know that every homomorphic image of an algebra of $\mathfrak{A}$ is itself an algebra of $\mathfrak{A}$. Thus we see that the decision problem for open sentences in such kinds of algebras as groups, Abelian groups, or lattices, is equivalent to the decision problem for conditional equations in these systems. We notice, furthermore, since the decision problem for conditional equations in Abelian groups has been solved,[5] that we are provided with a decision method for all open sentences in Abelian groups.

It also may happen in some cases, that one can show of some particular equation corresponding to a class $\mathfrak{A}$ of algebras, that it is identically false, for all elements of every algebra of $\mathfrak{A}$. If ε is such an equation, then any disjunction of inequalities

$$\sim\varepsilon_1 \vee \cdots \vee \sim\varepsilon_n$$

is equivalent to

$$\sim\varepsilon_1 \vee \cdots \vee \sim\varepsilon_n \vee \varepsilon$$

[5] See Kurt Reidemeister, ***Einführung in die kombinatorische Topologie,*** Braunschweig 1932, p. 56. Reidemeister solves here the problem of deciding whether an arbitrary equation is true of an Abelian group with a finite number of generating relations. It is easily shown that this is equivalent to the decision problem for conditional equations in Abelian groups. The problem solved here by Reidemeister is usually called the "word problem," though Church uses the term in a somewhat wider sense, since he admits also infinitely many generating relations (see his abstract, *Combinatory logic as a semigroup*, ***Bulletin of the American Mathematical Society,*** vol. 43 (1937), p. 333). The word problem for general groups is unsolved.

and hence to the conditional equation

$$(\varepsilon_1 \,\&\, \cdots \,\&\, \varepsilon_n) \rightarrow \varepsilon.$$

Thus (supposing $\mathfrak{A}$ to be closed under direct union) we again see, by a slight modification of the proof of Theorem 2, that the decision problem for open sentences reduces to the decision problem for conditional equations. This condition is satisfied when one has, for example, such an axiom as

$$\sim[R_1^{(1)}(x_1) = x_1],$$

which could be taken as one of the axioms for Boolean algebra (supposing we want to impose the condition that a Boolean algebra has always at least two elements).

II. A class of cases in which the decision problem for open sentences can be solved. An algebra $\Gamma = (K, O_1, \cdots, O_n)$ is called *finite*, if K is a finite class.

If $\mathfrak{A}$ is a class of algebras, and if A is a class of sentences corresponding to $\mathfrak{A}$, then we say that A is *finitely reducible* with respect to $\mathfrak{A}$ if the following is true: every sentence α of A, which is true of every finite algebra in $\mathfrak{A}$, is true of $\mathfrak{A}$ (i.e., is true of every algebra in $\mathfrak{A}$). Thus if A is finitely reducible with respect to $\mathfrak{A}$, and if α is a sentence which corresponds to $\mathfrak{A}$, but is not true of $\mathfrak{A}$, then there exists a finite algebra Γ in $\mathfrak{A}$ for which α is not true.

In the following theorem we formulate a principle[6] which will be employed later to give a decision method for open sentences of some special kinds of algebra.

THEOREM 3. If A is a class of sentences which is finitely reducible with respect to an axiomatizable class $\mathfrak{A}$ of algebras, then there is a decision method for A.

Proof. We first notice that if $\Gamma = (K, O_1, \cdots, O_n)$ is any finite algebra, and if α is any sentence that corresponds to Γ, then one can always decide in a finite number of steps whether α is true of Γ.[7] For if K contains the s members $a_1, \cdots, a_s$, then a sentence of the form

$$\prod_{x_1} \alpha(x_1)$$

is true of Γ if and only if the sentence

$$\alpha(a_1) \,\&\, \cdots \,\&\, \alpha(a_s)$$

is true of Γ. And a sentence of the form

$$\sum_{x_1} \alpha(x_1)$$

is true of Γ if and only if the sentence

$$\alpha(a_1) \vee \cdots \vee \alpha(a_s)$$

[6] This principle has been known for a long time, though perhaps not in quite the general form in which it is here formulated. It was employed by L. Löwenheim to solve the decision problem for the sentences of the restricted function calculus which contain only one-place predicates (see Hilbert and Ackermann, loc. cit., p. 95). The author also made use of it in his paper, *A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology*, this JOURNAL, vol. 6 (1941), pp. 117–134.

[7] See Hilbert and Ackermann, loc. cit., p. 92.

is true of Γ. Thus the question whether α is true of Γ reduces to the question whether a sentence without variables is true of Γ; and this last question is clearly decidable in a finite number of steps.

Hence if $\mathfrak{A}$ is an axiomatizable class of algebras, one can always decide in a finite number of steps whether a given finite algebra Γ belongs to $\mathfrak{A}$. For if $\alpha_1, \cdots, \alpha_t$ are axioms for $\mathfrak{A}$, then one can decide whether each α_i $(i \leqq t)$ is true of Γ.

If Γ and Γ' are mutually isomorphic algebras, it is clear that a sentence α is true of Γ if and only if it is true of Γ'. Hence the question whether a sentence α is true of all finite algebras, is equivalent to the question whether it is true of every algebra $(K, O_1, \cdots, O_n)$, where K is the set consisting of the first p natural numbers.

Let $\mathfrak{A}_m$ be the class of algebras which contains all algebras $\Gamma = (K, O_1, \cdots, O_n)$ which are isomorphic to algebras of $\mathfrak{A}$, and such that K consists of the natural numbers $1, \cdots, m$. Then every sentence of A which is true of each of the classes

$$\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m, \cdots$$

is true of every finite algebra in $\mathfrak{A}$, and hence, since by hypothesis $\mathfrak{A}$ is finitely reducible, is true of $\mathfrak{A}$.

Since there are but a finite number of algebras $\Gamma = (K, O_1, \cdots, O_n)$ where K consists of the elements $1, \cdots, m$, and since we can decide in a finite number of steps whether such an algebra belongs to $\mathfrak{A}_m$, we see that for each m we can construct the class A_m.

It is well-known that the set of all proofs of the restricted function calculus can be effectively enumerated. Let

$$P_1, P_2, \cdots, P_m, \cdots$$

be some such enumeration.

Now suppose that α is any sentence of A. Let α^* be the sentence of the restricted function calculus (the construction of which was described in the introduction) such that α is true of $\mathfrak{A}$ if and only if α^* is provable in the restricted function calculus. To decide whether α is true of $\mathfrak{A}$, we now proceed as follows. We examine the first proof P_1 in the restricted function calculus, to see whether it is a proof of α^*. If it is, then α is true of $\mathfrak{A}$. If it is not, then we examine the class $\mathfrak{A}_1$ of algebras, to see whether α is true of $\mathfrak{A}_1$. If α is not true of $\mathfrak{A}_1$, then it is not true of $\mathfrak{A}$. If α is true of $\mathfrak{A}_1$, then we look to see whether P_2 is a proof of α^*. And so on. This process must come to an end, giving us either a proof of α^*, or a counterexample for α. For if α is true of $\mathfrak{A}$, then α^* is provable in the restricted function calculus, and hence there will be found some m such that P_m is a proof of α^*. And if α is not true of $\mathfrak{A}$, then there is an algebra Γ in $\mathfrak{A}$ for which α is not true; and hence, since $\mathfrak{A}$ is finitely reducible, an $\mathfrak{A}_m$ for which α is not true.

The decision method provided by the above theorem is about as impractical of application as any decision method could very well be. The number of steps necessary to decide about a given sentence α would in general be very great. And the method, it should be added, suffers from an even worse defect. Given a sentence α, when we start out to apply this method we cannot even tell in advance any sort of upper bound to the number of steps that must be taken to decide about α.

In applying the principle furnished by the theorem, however, one is often able to avoid this last objectionable feature in the following way. One associates with each sentence α an integer m_α in some simple way (so that m_α is, for example, the number of symbols in α, or the square of the number of equations in α, etc.) and one shows that α is true of $\mathfrak{A}$ if and only if it is true of every $\mathfrak{A}_r$ where $r \leqq m_\alpha$. With this improvement it is unnecessary to examine the proofs of the restricted function calculus at all. And one can at least roughly foresee how long it may take to decide whether a given sentence α is true of $\mathfrak{A}$.

As an example of the method at its worst, it will perhaps be of interest to point out that the author has found a proof that the class of open sentences of Abelian groups is finitely reducible, but without being able to find an upper bound for the number of elements in the Abelian groups which must be examined in order to refute or confirm a given sentence. This will not be shown here however, since, as mentioned at the end of the first section, a less laborious decision method for open sentences of Abelian groups can be found.

III. Lattices. The sort of notation we have used heretofore is suitable when one wants to speak of arbitrary algebras. When one comes to special cases, however, where there are only two or three operations, each of which is either unary or binary, then it becomes more convenient to use the familiar kind of notation—using, for example, such symbols as "$\times$" and "$+$" for binary operations, and writing the symbol between the two arguments, instead of in front of them. In what follows we shall therefore use this ordinary notation, since we shall henceforth be concerned with special kinds of algebras. Thus in giving axiom systems we shall write

$$\alpha\times\beta \text{ instead of } R_1^{(2)}(\alpha, \beta),$$

and

$$\alpha+\beta \text{ instead of } R_2^{(2)}(\alpha, \beta).$$

We shall also use as individual variables other symbols than the symbols

$$x_1, x_2, x_3, \cdots$$

introduced previously. In particular, we shall use the letters

$$x, y, \text{ and } z.$$

In the present section, we shall be concerned with lattices of various sorts. It is convenient first to list some axioms:[8]

Π_1 $$(x\times x = x) \;\&\; (x+x = x),$$

Π_2 $$(x\times y = y\times x) \;\&\; (x+y = y+x),$$

Π_3 $$[x\times(y\times z) = (x\times y)\times z] \;\&\; [x+(y+z) = (x+y)+z],$$

[8] These axioms are given by Garrett Birkhoff. See his book, ***Lattice theory,*** New York 1940, p. 19, and p. 74.

Π_4 $$[x\times(x+y) = x] \mathbin{\&} [x+(x\times y) = x],$$

Π_5 $$[x\times(y+z) = (x\times y)+(x\times z)] \mathbin{\&} [x+(y\times z) = (x+y)\times(x+z)].$$

We shall denote by $\mathfrak{L}_1$ the class of all algebras $\Gamma = (K, \times, +)$, which satisfy axioms Π_1, Π_2, Π_3, and Π_4; we call $\mathfrak{L}_1$ the class of *lattices*. We denote by $\mathfrak{L}_2$ the class of all algebras which satisfy axioms Π_1, Π_2, Π_3, Π_4, and Π_5; we call $\mathfrak{L}_2$ the class of *distributive lattices*.

$\mathfrak{L}_1$ and $\mathfrak{L}_2$ are of course axiomatizable; each of them, indeed, is equationally definable. Hence each $\mathfrak{L}_i$ $(i = 1, 2)$ is closed under direct union. Moreover each $\mathfrak{L}_i$ contains an algebra $(K, \times, +)$ with just one member in the class K. Thus we see that no inequalities are true of $\mathfrak{L}_i$. Moreover, if a sentence of the form

$$(\varepsilon_1 \mathbin{\&} \cdots \mathbin{\&} \varepsilon_m) \rightarrow (\varepsilon_1' \vee \cdots \vee \varepsilon_n')$$

is true of $\mathfrak{L}_i$ (where $\varepsilon_1, \cdots, \varepsilon_m$ and $\varepsilon_1', \cdots, \varepsilon_n'$ are equations), then at least one of the sentences

$$(\varepsilon_1 \mathbin{\&} \cdots \mathbin{\&} \varepsilon_m) \rightarrow \varepsilon_j' \qquad (j \leq n)$$

is true of $\mathfrak{L}_i$.

Thus the decision problem for open sentences in $\mathfrak{L}_i$ reduces to the decision problem for conditional equations. We shall not make use of this fact, however, in giving a decision method[9] for open sentences in $\mathfrak{L}_i$, but shall instead make direct application of Theorem 3.

LEMMA 1. Let $(K, \times, +)$ be a lattice, and let K_1 be a finite subclass of K such that: (i) whenever $x \in K_1$ and $y \in K_1$, then $x\times y \in K_1$; (ii) there is an element a of K_1 such that $a\times x = x$, for every element x of K_1. Then there is an operation $+_1$ such that:

(1) $(K_1, \times, +_1)$ is a lattice,

(2) if $x \in K_1$, $y \in K_1$, and $x+y \in K_1$, then $x+_1y = x+y$.

Proof. In order to carry out the proof, it is convenient first to define a binary relation $\prec$ over K as follows: $x \prec y$ is true if and only if $x\times y = x$.

It is immediately seen from Π_1 that $x \prec x$ is always true. Moreover, from Π_2 we see that whenever $x \prec y$ and $y \prec x$ are both true, then $x = y$. And from Π_3 we see that the relation is transitive; i.e., whenever $x \prec y$ and $y \prec z$, then $x \prec z$.

From Π_2 and Π_4 we can conclude that $x \prec y$ is true when and only when $x+y = y$. For suppose first that $x \prec y$; then, by definition, $x\times y = x$, and hence $y+(x\times y) = y+x$; or, using Π_2 and Π_4, $y = x+y$. On the other hand, if $x+y = y$, then $x\times(x+y) = x\times y$; and hence, by Π_4, $x = x\times y$, or $x \prec y$, as was to be shown.

[9] The results of this section include, but are more general than, some previously established results. A decision method for equations in lattices was given by Philip M. Whitman in his paper: *Free lattices*, ***Annals of mathematics,*** ser. 2 vol. 42 (1941), pp. 325–330. Some special cases of the decision problem for open sentences in distributive lattices were solved in Th. Skolem, *Untersuchungen über die Axiome des Klassenkalkuls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen*, ***Skrifter utgit av Videnskapsselskapet i Kristiania,*** I. Matematisk-naturvidenskabelig klasse 1919, no. 3 (1919), 37 pp.

We notice also, that if $x \prec y$ and $x \prec z$, then $x \prec y \times z$. For if $x \prec y$ and $x \prec z$, then $x \times y = x$ and $x \times z = x$, and hence $x \times (y \times z) = (x \times x) \times (y \times z) = (x \times y) \times (x \times z) = x \times x = x$, so that $x \prec y \times z$. By mathematical induction we conclude that if $x \prec x_1, \cdots, x \prec x_r$, then $x \prec x_1 \times \cdots \times x_r$.

Finally we notice that $x \times y \prec x$. For $(x \times y) \times x = (x \times x) \times y = x \times y$.

We return now to the proof of our lemma.

We say that an element x of K is *covered* by an element y of K, if $x \prec y$, and $y \in K_1$. By condition (ii) we see that there is an element a of K_1 which covers every element x of K_1.

If x and y are any elements of K_1, and if $u_1, \cdots, u_n$ are the elements which cover both x and y, then we set

$$x +_1 y = u_1 \times \cdots \times u_n.$$

We are to show that the algebra $(K_1, \times, +_1)$ satisfies conditions (1) and (2).

It is convenient first to show that condition (2) is satisfied. Suppose that x, y, and z are elements of K_1 such that $x+y = z$. Multiplying both sides of this equation by x, we have $x \times z = x \times (x+y) = x$, so that $x \prec z$. Similarly $y \prec z$. Since $z \in K_1$, we therefore see that z covers x and y. Suppose now that u is any element which covers both x and y. Then we have $x \prec u$ and $y \prec u$, and hence $x+u = u$ and $y+u = u$. Adding corresponding sides of these last equations, we have $(x+u)+(y+u) = u+u$, or $(x+y)+u = u$, or $z+u \prec u$; thus $z \prec u$, and hence $z \times u = z$. Hence if the elements which cover x and y are $z, u_1, \cdots, u_n$, we have

$$\begin{aligned} x +_1 y &= z \times u_1 \times u_2 \times \cdots \times u_n \\ &= z \times u_2 \times \cdots \times u_n \\ &\;\vdots \\ &= z \\ &= x+y, \end{aligned}$$

as was to be shown.

We shall now show that $(K_1, \times, +_1)$ is a lattice. Since $(K, \times, +)$ is a lattice by hypothesis, we see that all the postulates which involve only $\times$ are true of $(K_1, \times, +_1)$.

To see that $x +_1 x = x$ is true, we notice that $x+x = x$, and hence, by condition (2), $x +_1 x = x+x = x$.

To see that $x +_1 y = y +_1 x$, we need only observe that the elements which cover x and y are the same as the elements which cover y and x.

Before continuing the proof that the axioms for lattices are satisfied by $(K_1, \times, +_1)$ it is convenient first to establish the following: if x, y, and z are any elements of K_1, then z covers $x +_1 y$ if and only if it covers both x and y. For suppose first that z covers both x and y; then if $z_1, \cdots, z_n$ are the other elements which cover x and y, we have

$$x +_1 y = z \times z_1 \times \cdots \times z_n;$$

thus $x +_1 y \prec z$, so that z covers $x +_1 y$. To show on the other hand that every element which covers $x +_1 y$, also covers both x and y, it suffices to show that

$x+_1y$ covers both x and y. Suppose that $u_1, \cdots, u_n$ are the elements which cover both x and y. Then we have

$$x+_1y = u_1\times\cdots\times u_n.$$

Since $x < u_1, \cdots, x < u_n$, we see that $x < u_1\times\cdots\times u_n$, or $x < x+_1y$, so that x is covered by $x+_1y$. Similarly y is covered by $x+_1y$.

To show now that $x+_1(y+_1z) = (x+_1y)+_1z$, it will clearly suffice to show that the elements which cover x and $y+_1z$ are the same as those which cover $x+_1y$ and z. But if u covers x and $y+_1z$, then u covers x, y, and z, and hence u covers x and $y+_1z$.

We now show that $x\times(x+_1y) = x$. Since x is covered by $x+_1y$, we have $x+_1y = x+(x+_1y)$. Thus $x\times(x+_1y) = x\times[x+(x+_1y)] = x$.

Finally it is necessary to show that $x+_1(x\times y) = x$. If an element u covers x, then it clearly also covers $x\times y$, since $x\times y < x$. Thus the elements which cover both x and $x\times y$ are the same as those which cover x; so that $x+_1(x\times y) = x+_1x = x$, as was to be shown.

This completes the proof of our lemma.

If α is any open sentence, then by $T(\alpha)$ is meant the number of distinct terms which occur in α. Thus, for instance, if α is the sentence

$$[(x+y)\times z = (x+y)+z] \rightarrow [x+x = y+y],$$

then $T(\alpha) = 8$, since the following eight different terms occur in α:

$$x,\ y,\ z,\ x+x,\ y+y,\ x+y,\ (x+y)\times z,\ (x+y)+z.$$

THEOREM 4. Let α be any open sentence which is not true of every lattice; then there is a finite lattice with at most $2^{T(\alpha)+1}$ elements, for which α is not true.

Proof. Let the variables in α be $x_1, \cdots, x_n$; and let the distinct terms occurring in α be $\beta_1, \cdots, \beta_{T(\alpha)}$. It is convenient to write $\alpha(x_1, \cdots, x_n)$ instead of α, and $\beta_1(x_1, \cdots, x_n), \cdots, \beta_{T(\alpha)}(x_1, \cdots, x_n)$ instead of $\beta_1, \cdots, \beta_{T(\alpha)}$, respectively. Since by hypothesis α is not true of every lattice, there is some lattice $(K, \times, +)$ where K contains elements $a_1, \cdots, a_n$ such that $\alpha(a_1, \cdots, a_n)$ is false.

Let $b_1, \cdots, b_r$ be the distinct elements of the set $\beta_1(a_1, \cdots, a_n), \cdots, \beta_{T(\alpha)}(a_1, \cdots, a_n)$. It is clear that $r \leqq T(\alpha)$.

Let

$$b_0 = b_1+\cdots+b_r.$$

Let K_1 be the class of elements which can be obtained from $b_0, b_1, \cdots, b_r$ by multiplying them together two at a time, three at a time, $\cdots$, $r+_1$ at a time. It is easily seen that K_1 contains at most 2^{r+1} elements, and hence at most $2^{T(\alpha)+1}$ elements. Moreover it will be noticed that if x and y are any elements of K_1, then $x\times y$ is also an element of K_1. Finally, if x is any element of K_1, then $b_0\times x = x$. For since x is a product of some of the elements $b_0, \cdots, b_r$, we have $x < b_i$ for some i, or $x = x\times b_i$. Hence $b_0\times x = b_0\times(x\times b_i) = (b_i\times b_0)\times x = b_i\times(b_1+\cdots+b_r)\times x = b_i\times x = x$, as was to be shown.

Thus we see that K_1 satisfies the hypothesis of Lemma 1. Hence there is a lattice $(K_1, \times, +_1)$ such that, whenever x, y, and $x+y$ are all in K, then $x+_1y = x+y$. It is now easily seen that $\beta_1(a_1, \cdots, a_n), \cdots, \beta_{T(\alpha)}(a_1, \cdots, a_n)$ have the same values in the new lattice $(K_1, \times, +_1)$ as in the original lattice $(K, \times, +)$. Hence $\alpha(a_1, \cdots, a_n)$ has the same truth value in $(K_1, \times, +_1)$ as in $(K, \times, +)$. Since α is not true of the lattice $(K, \times, +)$ we therefore see that α is not true of the lattice $(K_1, \times, +_1)$, which contains at most $2^{T(\alpha)+1}$ elements. This completes our proof.

THEOREM 5. There is a decision method for the class A of all open sentences corresponding to the class $\mathfrak{L}_1$ of all lattices.

Proof. By Theorems 3 and 4.

In order to extend this result to the class of distributive lattices, it is necessary to prove an additional lemma.

LEMMA 2. Let $(K, \times, +)$ be a distributive lattice, and let F be a finite subclass of K containing just n elements. Then there exists a finite subclass K_1 of K such that:

(1) $(K_1, \times, +)$ is a distributive lattice,

(2) F is a subclass of K_1,

(3) K_1 contains at most 2^{2^n} elements.

Proof. Let the elements of F be $a_1, \cdots, a_n$. Let K_1 be the subclass of K which consists of all sums of products of the elements $a_1, \cdots, a_n$. By the first parts of Π_1, Π_2, and Π_3, we see that there are at most as many products of the elements $a_1, \cdots, a_n$ as there are combinations (any number at a time) of n things; hence there are at most 2^n products. Then by the second parts of these same axioms we see that there are at most 2^{2^n} sums of products.

It is clear that F is a subclass of K_1.

To show that $(K_1, \times, +)$ is a distributive lattice, it suffices to show that, whenever x and y are in K_1, then $x+y$ and $x\times y$ are in K_1. But if x and y are sums of products of the elements $a_1, \cdots, a_n$, then clearly $x+y$ is a sum of such products. Moreover, by the first part of Π_5, we see that if x and y are sums of products, then $x\times y$ is also a sum of products.

This completes the proof of our lemma.

THEOREM 6. Let α be any open sentence which is not true of every distributive lattice; then there is a finite distributive lattice with at most 2^{2^n} elements, for which α is not true, where n is the number of different variables occurring in α.

Proof. Suppose that the variables occurring in α are $x_1, \cdots, x_n$. It is convenient to write "$\alpha(x_1, \cdots, x_n)$" instead of "α". Since $\alpha(x_1, \cdots, x_n)$ is not true of every distributive lattice, there is some distributive lattice $(K, \times, +)$, where K contains elements $a_1, \cdots, a_n$ such that $\alpha(a_1, \cdots, a_n)$ is false. By Lemma 2, there is a subclass K_1 of K which contains all the elements $a_1, \cdots, a_n$, and only 2^{2^n} elements altogether, such that $(K, \times, +)$ is a distributive lattice. It is then clear that $\alpha(a_1, \cdots, a_n)$ has the same truth-value in $(K_1, \times, +)$ as in $(K, \times, +)$. Hence $\alpha(x_1, \cdots, x_n)$ is not true of the finite distributive lattice $(K_1, \times, +)$.

THEOREM 7. There is a decision method for the class A of all open sentences corresponding to the class $\mathfrak{L}_2$ of all distributive lattices.

Proof. By Theorems 3 and 6.

We shall now show how Lemma 2 and Theorems 6 and 7 can be somewhat strengthened. To show this, let us think of $\mathfrak{L}$ as an arbitrary but fixed axiomatizable class of algebras which can be obtained from $\mathfrak{L}_2$ by taking as an additional axiom some open sentence. Then it is immediately evident that the proof of Lemma 2 gives us also the new lemma:

LEMMA 3. Let $(K, \times, +)$ be an algebra of the class $\mathfrak{L}$, and let F be a finite subclass of K containing just n elements. Then there exists a finite subclass K_1 of K such that:

(1) $(K_1, \times, +)$ is in $\mathfrak{L}$,
(2) F is a subclass of K_1,
(3) K_1 contains at most 2^{2^n} elements.

From this new lemma we can in turn prove:

THEOREM 8. Let α be any open sentence which is not true of every member of $\mathfrak{L}$; then there is a finite member of $\mathfrak{L}$, with at most 2^{2^n} elements, for which α is not true, where n is the number of different variables occurring in α.

THEOREM 9. There is a decision method for the class A of all open sentences corresponding to the class $\mathfrak{L}$ of algebras.

By means of this last theorem we can also somewhat extend the class of sentences of distributive lattices for which there is a decision method.

THEOREM 10. Let B be the class of all sentences of distributive lattices of the form

$$\alpha \rightarrow \beta$$

where α and β are the closures of open sentences. Then there is a decision method for B.

Proof. Let

$$\alpha \rightarrow \beta$$

be a sentence such that α and β are the closures, respectively, of the open sentences α' and β'. Let $\mathfrak{L}$ be the class of algebras which is obtained by adding α' to the axioms for distributive lattices. By Theorem 9 there is a decision method for the class of open sentences of $\mathfrak{L}$. Hence, in particular, we can decide whether β' is true of $\mathfrak{L}$. But it is clear that β' is true of $\mathfrak{L}$ if and only if

$$\alpha \rightarrow \beta$$

is true of all distributive lattices. Thus we can decide whether

$$\alpha \rightarrow \beta$$

is true of all distributive lattices, as was to be shown.

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