Chapter II

LAMBDA-CONVERSION

5. PRIMITIVE SYMBOLS, AND FORMULAS. We turn now to the development of a formal system, which we shall call the calculus of λ-conversion, and which shall have as a possible interpretation or application the system of ideas about functions described in Chapter I.

The primitive symbols of this calculus are three symbols,

$$\lambda,\ (,\ ),$$

which we shall call improper symbols, and an infinite list of symbols,

$$a,\ b,\ c,\ \ldots\ ,\ x,\ y,\ z,\ \bar{a},\ \bar{b},\ \ldots\ ,\ \bar{z},\ \bar{\bar{a}},\ \ldots\ ,$$

which we shall call variables. The order in which the variables appear in this originally given infinite list shall be called their alphabetical order.

A formula is any finite sequence of primitive symbols. Certain formulas are distinguished as well-formed formulas, and each occurrence of a variable in a well-formed formula is distinguished as free or bound, in accordance with the following rules (1-4), which constitute a definition of these terms by recursion:

1. A variable x is a well-formed formula, and the occurrence of the variable x in this formula is free.

2. If F and A are well-formed, (FA) is well-formed, and an occurrence of a variable y in F is free or bound in (FA) according as it is free or bound in F, and an occurrence of a variable y in A is free or bound in (FA) according as it is free or bound in A.

3. If M is well-formed and contains at least one free occurrence of x, then $(\lambda x M)$ is well-formed, and an occurrence

of a variable $\boldsymbol{y}$, other than $\boldsymbol{x}$, in $(\lambda \boldsymbol{x} \boldsymbol{M})$ is free or bound in $(\lambda \boldsymbol{x} \boldsymbol{M})$ according as it is free or bound in $\boldsymbol{M}$. All occurrences of $\boldsymbol{x}$ in $(\lambda \boldsymbol{x} \boldsymbol{M})$ are bound.

4. A formula is well-formed, and an occurrence of a variable in it is free, or is bound, only when this follows from 1-3.

The free variables of a formula are the variables which have at least one free occurrence in the formula. The bound variables of a formula are the variables which have at least one bound occurrence in the formula.

Hereafter (as was just done in the statement of the rules 1-4) we shall use bold capital letters to stand for variable or undetermined formulas, and bold small letters to stand for variable or undetermined variables. Unless otherwise indicated in a particular case, it is to be understood that formulas represented by bold capital letters are well-formed formulas. Bold letters are thus not part of the calculus which we are developing but are a device for use in talking about the calculus: they belong, not to the system itself, but to the metamathematics or syntax of the system.

Another syntactical notation which we shall use is the notation,

$$\mathrm{S}^{\boldsymbol{x}}_{\boldsymbol{N}} \boldsymbol{M}|$$

which shall stand for the formula which results by substitution of $\boldsymbol{N}$ for $\boldsymbol{x}$ throughout $\boldsymbol{M}$. This formula is well-formed, except in the case that $\boldsymbol{x}$ is a bound variable of $\boldsymbol{M}$ and $\boldsymbol{N}$ is other than a single variable -- see §7. (In the special case that $\boldsymbol{x}$ does not occur in $\boldsymbol{M}$, it is the same formula as $\boldsymbol{M}$.)

For brevity and perspicuity in dealing with particular well-formed formulas, we often do not write them in full but employ various abbreviations.

One method of abbreviation is by means of a nominal definition, which introduces a particular new symbol to replace or stand for a particular well-formed formula. We indicate such a nominal definition by an arrow, pointing from the new symbol which is being introduced to the well-formed formula which it is to replace (the arrow may be read "stands for"). As an example we make at once the nominal definition:

$$I \longrightarrow (\lambda a a).$$

This means that I will be used as an abbreviation for $(\lambda a a)$ -- and consequently that (II) will be used as an abbreviation for $((\lambda a a)(\lambda a a))$, $(\lambda a(aI))$ as an abbreviation for $(\lambda a(a(\lambda a a)))$, etc.

Another method of abbreviation is by means of a <u>schematic definition</u>, which introduces a class of new expressions of a certain form, specifying a scheme according to which each of the new expressions stands for a corresponding well-formed formula. Such a schematic definition is indicated in a similar fashion by an arrow, but the expressions on each side of the arrow contain bold letters. When a bold small letter -- one or several -- occurs in the expression following the arrow (the definiens) but not in the expression preceding the arrow (the definiendum), the following convention is to be understood:

> $\boldsymbol{a}$ stands for the first variable in alphabetical order not otherwise appearing in the definiens, $\boldsymbol{b}$ stands for the second such variable in alphabetical order, $\boldsymbol{c}$ the third, and so on.

As examples, we make at once the following schematic definitions:

$$[\boldsymbol{M}+\boldsymbol{N}] \longrightarrow (\lambda \boldsymbol{a}(\lambda \boldsymbol{b}((\boldsymbol{M}\boldsymbol{a})((\boldsymbol{N}\boldsymbol{a})\boldsymbol{b})))).$$
$$[\boldsymbol{M}\times\boldsymbol{N}] \longrightarrow (\lambda \boldsymbol{a}(\boldsymbol{M}(\boldsymbol{N}\boldsymbol{a}))).$$
$$[\boldsymbol{M}^{\boldsymbol{N}}] \longrightarrow (\boldsymbol{N}\boldsymbol{M}).$$

The first of these definitions means that, for instance, $[x+y]$ will be used as an abbreviation for $(\lambda a(\lambda b((xa)((ya)b))))$, and $[a+c]$ will be used as an abbreviation for $(\lambda b(\lambda d((ab)((cb)d))))$, and $[I+I]$ as an abbreviation for $(\lambda b(\lambda c((Ib)((Ib)c))))$, etc.

As a further device of abbreviation, we shall allow the omission of the parentheses ( ) in $(\boldsymbol{F}\boldsymbol{A})$ when this may be done without ambiguity, whether $(\boldsymbol{F}\boldsymbol{A})$ is the entire formula being written or merely some part of it. In restoring such omitted parentheses, the convention is to be followed that association is to the left (cf. Schönfinkel [49], Curry [17]). For example, fxy is an abbreviation of $((fx)y)$, $f(xy)$ is an abbreviation of $(f(xy))$, $fxyz$ is an abbreviation of $(((fx)y)z)$, $f(xy)z$ is an abbreviation of $((f(xy))z)$, $f(\lambda xx)y$ is an abbreviation of $((f(\lambda xx))y)$, etc.

In expressions which (in consequence of schematic definitions) contain brackets [ ], we allow a similar omission of brackets, subject to a similar convention of association to the left; thus $x+y+z$ is an abbreviation for $[[x+y]+z]$, which expression is in turn an abbreviation for a certain well-formed formula in accordance with the schematic definition already introduced. Moreover we allow, as an abbreviation, omitting a pair of brackets and at the same time putting a dot or period in the place of the initial bracket [; in this case the convention, instead of association to the left, is that the omitted bracket extends from the bold period as far to the right as possible, consistently with the formula's being well-formed -- so that, for instance, $x+.y+z$ is an abbreviation for $[x+[y+z]]$, and $x+.y+.z+t$ is an abbreviation for $[x+[y+[z+t]]]$, and $(\lambda x.x+x)$ is an abbreviation for $(\lambda x[x+x])$.

We also introduce the following schematic definitions:

$$(\lambda \mathbf{x}.\boldsymbol{FA}) \rightarrow (\lambda \mathbf{x}(\boldsymbol{FA})),$$
$$(\lambda \mathbf{xy}.\boldsymbol{FA}) \rightarrow (\lambda \mathbf{x}(\lambda \mathbf{y}(\boldsymbol{FA}))),$$
$$(\lambda \mathbf{xyz}.\boldsymbol{FA}) \rightarrow (\lambda \mathbf{x}(\lambda \mathbf{y}(\lambda \mathbf{z}(\boldsymbol{FA})))),$$

and so on for any number of variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... (which must be all different). And we allow similar omissions of λ's, preceding a bold period which represents an omitted bracket in the way described in the preceding paragraph -- using, e.g., $\lambda xyz.x+y+z$ as an abbreviation for $(\lambda x(\lambda y(\lambda z[[x+y]+z])))$.

Finally, we allow omission of the outside parentheses in $(\lambda \mathbf{x}\boldsymbol{M})$, or in $(\lambda \mathbf{x}.\boldsymbol{FA})$, or $(\lambda \mathbf{xy}.\boldsymbol{FA})$, or $(\lambda \mathbf{xyz}.\boldsymbol{FA})$, etc., when this is the entire formula being written -- but not when one of these expressions appears as a proper part of a formula.

Hereafter, in writing definitions, we shall abbreviate the definiens in accordance with previously introduced abbreviations and definitions. Thus the definition of $[\boldsymbol{M}+\boldsymbol{N}]$ would now be written:

$$[M+N] \longrightarrow \lambda ab . Ma(Nab) .$$

Definitions and other abbreviations are introduced merely as matters of convenience and are not properly part of the formal system at all. When we speak of the free variables of a formula, the bound variables of a formula, the length (number of symbols) of a formula, the occurrences of one formula as a part of another, etc., the reference is always to the unabbreviated form of the formulas in question.

The introduction and use of definitions and other abbreviations is, of course, subject to the restriction that there shall never be any ambiguity as to what formula a given abbreviated form stands for. In practice certain further restrictions are also desirable, e.g., that all free variables of the definiens be represented explicitly in the definiendum. Exact formulation of these restrictions is unnecessary for our present purpose, since all definitions and abbreviations are extraneous to the formal system, as just explained, and in principle dispensable.

6. CONVERSION. We introduce now the three following operations, or transformation rules, on well-formed formulas:

I. To replace any part M of a formula by $S_{y}^{x}M|$, provided that x is not a free variable of M and y does not occur in M.

II. To replace any part $((\lambda x M)N)$ of a formula by $S_{N}^{x}M|$, provided that the bound variables of M are distinct both from x and from the free variables of N.

III. To replace any part $S_{N}^{x}M|$ of a formula by $((\lambda x M)N)$, provided that $((\lambda x M)N)$ is well-formed and the bound variables of M are distinct both from x and from the free variables of N.

In the statement of these rules -- and hereafter generally -- it is to be understood that the word <u>part</u> (of a formula) means <u>consecutive well-formed part not immediately following an occurrence of the symbol λ</u>.

When the same formula occurs several times as such a part of another formula, each occurrence is to be counted as a different part. Thus, for instance, Rule I may be used to transform $ab(\lambda aa)(\lambda aa)$ into $ab(\lambda bb)(\lambda aa)$. Rule III may be used to transform λaa into $\lambda a.(\lambda aa)a$. But Rule III may not be used to transform (λaa) into $(\lambda((\lambda aa)a)a)$ -- the latter formula is, in fact, not even well-formed.

Rules I-III have the important property that they are effective or "definite," i.e., there is a means of always determining of any two formulas $\boldsymbol{A}$ and $\boldsymbol{B}$ whether $\boldsymbol{A}$ can be transformed into $\boldsymbol{B}$ by an application of one of the rules (and, if so, of which one).

If $\boldsymbol{A}$ can be transformed into $\boldsymbol{B}$ by an application of one of the Rules I-III, we shall say that $\boldsymbol{A}$ is <u>immediately convertible</u> into $\boldsymbol{B}$ (abbreviation, "$\boldsymbol{A}$ imc $\boldsymbol{B}$"). If there is a finite sequence of formulas, in which $\boldsymbol{A}$ is the first formula and $\boldsymbol{B}$ the last, and in which each formula except the last is immediately convertible into the next one, we shall say that $\boldsymbol{A}$ is <u>convertible</u> into $\boldsymbol{B}$ (abbreviation, "$\boldsymbol{A}$ conv $\boldsymbol{B}$"); and the process of obtaining $\boldsymbol{B}$ from $\boldsymbol{A}$ by a particular finite sequence of applications of Rules I-III will be called a <u>conversion</u> of $\boldsymbol{A}$ into $\boldsymbol{B}$ (no reference is intended to conversion in the sense of forming the converse -- for the corresponding noun we use, not "converse," but "convert"). It is not excluded that the number of applications of Rules I-III in a conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$ should be zero, $\boldsymbol{B}$ being then the same formula as $\boldsymbol{A}$.

The relation which holds between $\boldsymbol{A}$ and $\boldsymbol{B}$ when $\boldsymbol{A}$ conv $\boldsymbol{B}$ will be called <u>interconvertibility</u>, and we shall use the expression "$\boldsymbol{A}$ and $\boldsymbol{B}$ are interconvertible" as synonymous with "$\boldsymbol{A}$ conv $\boldsymbol{B}$." The relation of interconvertibility is transitive, symmetric, and reflexive -- symmetric because Rules II and III are inverses of each other and Rule I is its own inverse.

If there is a conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$ which contains no application of Rule II or Rule III, we shall say that $\boldsymbol{A}$ is <u>convertible-I</u> into $\boldsymbol{B}$ ($\boldsymbol{A}$ conv-I $\boldsymbol{B}$). Similarly we define "$\boldsymbol{A}$ conv-I-II $\boldsymbol{B}$" and "$\boldsymbol{A}$ conv-I-III $\boldsymbol{B}$."

A conversion which contains no application of Rule II and exactly one application of Rule III will be called an expansion. A conversion which contains no application of Rule III and exactly one application of Rule II will be called a reduction. If there is a reduction of **A** into **B**, we shall say that **A** is immediately reducible to **B** (**A** imr **B**). If there is a conversion of **A** into **B** which consists of one or more successive reductions, we shall say that **A** is reducible to **B** (**A** red **B**). (The meaning of "**A** red **B**" thus differs from that of "**A** conv-I-II **B**" only in that the former implies the presence of at least one application of Rule II in the conversion of **A** into **B**.)

An application of Rule II to a formula will be called a contraction of the part $((\lambda x\mathbf{M})\mathbf{N})$ which is affected.

A well-formed formula will be said to be in normal form if it contains no part of the form $((\lambda x\mathbf{M})\mathbf{N})$. We shall call **B** a normal form of **A** if **B** is in normal form and **A** conv **B**. We shall say that **A** has a normal form if there is a formula **B** which is a normal form of **A**.

A well-formed formula will be said to be in principal normal form if it is in normal form, and no variable is both a bound variable and free variable of it, and the first bound variable occurring in it (in the left-to-right order of the symbols which compose the formula) is the same as the first variable in alphabetical order which is not a free variable of it, and the variables which occur in it immediately following the symbol λ are, when taken in the order in which they occur in the formula, in alphabetical order, without repetitions, and without omissions except of variables which are free variables of the formula. For example, $\lambda ab.ba$, and $\lambda a.a(\lambda c.bc)$, and $\lambda b.ba$ are in principal normal form; and $\lambda ac.ca$, and $\lambda bc.cb$, and $\lambda a.a(\lambda a.ba)$ are in normal form but not in principal normal form.

We shall call **B** a principal normal form of **A** if **B** is in principal normal form and **A** conv **B**. A formula in normal form is always convertible-I into a corresponding formula in principal normal form, and hence every formula which has a normal form has a principal normal form. We shall show in the next section that the principal normal form of a formula, if it exists, is unique.

An example of a formula which has no normal form (and therefore no principal normal form) is $(\lambda x.xxx)(\lambda x.xxx)$.

It is intended that, in any interpretation of the formal calculus, only those well-formed formulas which have a normal form shall be meaningful, and, among these, interconvertible formulas shall have the same meaning. The condition of being well-formed is thus a necessary condition for meaningfulness but not a sufficient condition.

It is important that the condition of being well-formed is effective in the sense explained at the beginning of this section, whereas the condition of being well-formed and having a normal form is not effective.

7. FUNDAMENTAL THEOREMS ON WELL-FORMED FORMULAS AND ON THE NORMAL FORM. The following theorems are taken from Kleene [34] (with non-essential changes to adapt them to the present modified notation). Their proof is left to the reader; or an outline of the proof may be found in Kleene, loc. cit.

7 I. In a well-formed formula $\boldsymbol{K}$ there exists a unique pairing of the occurrences of the symbol (, each with a corresponding occurrence of the symbol), in such a way that two portions of $\boldsymbol{K}$, each lying between an occurrence of (and the corresponding occurrence of) inclusively, either are non-overlapping or else are contained one entirely within the other. Moreover, if such a pairing exists in the portion of $\boldsymbol{K}$ lying between the nth and the (n+r)th symbol of $\boldsymbol{K}$ inclusively, it is a part of the pairing in $\boldsymbol{K}$. .

7 II. A necessary and sufficient condition that the portion $\boldsymbol{N}$ of a well-formed formula $\boldsymbol{K}$ which lies between a given occurrence of (in $\boldsymbol{K}$ and a given occurrence of) in $\boldsymbol{K}$ inclusively be well-formed is that the given occurrence of (and the given occurrence of) correspond.

7 III. Every well-formed formula has one of the three forms, $\boldsymbol{x}$, where $\boldsymbol{x}$ is a variable, or $(\boldsymbol{FA})$, where $\boldsymbol{F}$ and $\boldsymbol{A}$ are well-formed, or $(\lambda\boldsymbol{xM})$, where $\boldsymbol{M}$ is well-formed and $\boldsymbol{x}$ is a free variable of $\boldsymbol{M}$.

7 IV. If $(\boldsymbol{FA})$ and either $\boldsymbol{F}$ or $\boldsymbol{A}$ is well-formed, then both $\boldsymbol{F}$ and $\boldsymbol{A}$ are well-formed.

7 V. If $(\lambda x \boldsymbol{M})$ is well-formed, x being a variable, then $\boldsymbol{M}$ is well-formed and x is a free variable of $\boldsymbol{M}$.

7 VI. A well-formed formula can be of the form $(\boldsymbol{F}\boldsymbol{A})$, where $\boldsymbol{F}$ (or $\boldsymbol{A}$) is well-formed, in only one way.

7 VII. A well-formed formula can be of the form $(\lambda x \boldsymbol{M})$, where x is a variable, in only one way.

7 VIII. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are well-formed parts of a well-formed formula $\boldsymbol{K}$, then either $\boldsymbol{P}$ is a part of $\boldsymbol{Q}$, or $\boldsymbol{Q}$ is a part of $\boldsymbol{P}$, or $\boldsymbol{P}$ and $\boldsymbol{Q}$ are non-overlapping.

7 IX. Two distinct occurrences of the same well-formed formula $\boldsymbol{P}$ as a part of a well-formed formula $\boldsymbol{K}$ must be non-overlapping.

7 X. If $\boldsymbol{P}$, $\boldsymbol{F}$, and $\boldsymbol{A}$ are well-formed and $\boldsymbol{P}$ is a part of $(\boldsymbol{F}\boldsymbol{A})$, then $\boldsymbol{P}$ is $(\boldsymbol{F}\boldsymbol{A})$ or $\boldsymbol{P}$ is a part of $\boldsymbol{F}$ or $\boldsymbol{P}$ is a part of $\boldsymbol{A}$.

7 XI. If $\boldsymbol{P}$ and $\boldsymbol{M}$ are well-formed and x is a variable and $\boldsymbol{P}$ is a part of $(\lambda x \boldsymbol{M})$, then $\boldsymbol{P}$ is $(\lambda x \boldsymbol{M})$ [or $\boldsymbol{P}$ is x] or $\boldsymbol{P}$ is a part of $\boldsymbol{M}$. (The clause in brackets is superfluous because of the meaning we give to the word <u>part</u> of a formula -- see §6).

7 XII. An occurrence of a variable x in a well-formed formula $\boldsymbol{K}$ is bound or free according as it is or is not an occurrence in a well-formed part of $\boldsymbol{K}$ of the form $(\lambda x \boldsymbol{M})$. (Hence, in particular, no occurrence of a variable in a well-formed formula is both bound and free.)

7 XIII. If $\boldsymbol{M}$ is well-formed and the variable x is not a free variable of $\boldsymbol{M}$ and the variable y does not occur in $\boldsymbol{M}$, then $S^{x}_{y}\boldsymbol{M}|$ is well-formed and has the same free variables as $\boldsymbol{M}$.

7 XIV. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are well-formed and the variable x occurs in $\boldsymbol{M}$ and the bound variables of $\boldsymbol{M}$ are distinct both from x and from the free variables of $\boldsymbol{N}$, then $S^{x}_{\boldsymbol{N}}\boldsymbol{M}|$ and $((\lambda x \boldsymbol{M})\boldsymbol{N})$ are well-formed and have the same free variables.

7 XV. If $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ are well-formed and all free variables of $\boldsymbol{P}$ are also free variables of $\boldsymbol{Q}$, the formula obtained

by substituting $\boldsymbol{Q}$ for a particular occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$, not immediately following an occurrence of λ, is well-formed.

7 XVI. If $\boldsymbol{A}$ is well-formed and $\boldsymbol{A}$ conv $\boldsymbol{B}$, then $\boldsymbol{B}$ is well-formed.

7 XVII. If $\boldsymbol{A}$ is well-formed and $\boldsymbol{A}$ conv $\boldsymbol{B}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same free variables.

7 XVIII. If $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ are well-formed, and $\boldsymbol{P}$ conv $\boldsymbol{Q}$, and $\boldsymbol{L}$ is obtained by substituting $\boldsymbol{Q}$ for a particular occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$, not immediately following an occurrence of λ, then $\boldsymbol{K}$ conv $\boldsymbol{L}$.

We shall call a well-formed part $\boldsymbol{P}$ of a well-formed formula $\boldsymbol{K}$ a <u>free</u> occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$ if every free occurrence of a variable in $\boldsymbol{P}$ is also a free occurrence of that variable in $\boldsymbol{K}$; in the contrary case (if some free occurrence of a variable in $\boldsymbol{P}$ is at the same time a bound occurrence of that variable in $\boldsymbol{K}$) we shall call the part $\boldsymbol{P}$ of $\boldsymbol{K}$ a <u>bound</u> occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$. If $\boldsymbol{P}$ is an occurrence of a variable in $\boldsymbol{K}$, not immediately following an occurrence of λ, this definition is in agreement with our previous definition of free and bound occurrences of variables.

Moreover we shall extend the notation $S_{\boldsymbol{N}}^{\boldsymbol{x}}\boldsymbol{M}|$ introduced in §5 by allowing $S_{\boldsymbol{N}}^{\boldsymbol{P}}\boldsymbol{M}|$ to stand for the result of substituting $\boldsymbol{N}$ for $\boldsymbol{P}$ throughout $\boldsymbol{M}$, where $\boldsymbol{N}$, $\boldsymbol{P}$, $\boldsymbol{M}$ are any well-formed formulas. This is possible without ambiguity, by 7 IX.

7 XIX. A well-formed part $\boldsymbol{P}$ of a well-formed formula $\boldsymbol{K}$ is a bound or free occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$ according as it is or is not an occurrence in a well-formed part of $\boldsymbol{K}$ of the form $(\lambda \boldsymbol{x}\boldsymbol{M})$ where $\boldsymbol{x}$ is a free variable of $\boldsymbol{P}$.

7 XX. If $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ are well-formed, the formula obtained by substituting $\boldsymbol{Q}$ for a particular free occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$ is well-formed.

7 XXI. If $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ are well-formed and there is no bound occurrence of $\boldsymbol{P}$ in $\boldsymbol{K}$, then $S_{\boldsymbol{Q}}^{\boldsymbol{P}}\boldsymbol{K}|$ is well-formed.

7 XXII. Let $\boldsymbol{x}$ be a free variable of the well-formed formula

$\mathbf{M}$ and let $\mathbf{P}$ be the formula obtained by substituting $\mathbf{N}$ for the free occurrences of x in $\mathbf{M}$. If the resulting occurrences of $\mathbf{N}$ in $\mathbf{P}$ are free, $((\lambda x\mathbf{M})\mathbf{N})$ conv $\mathbf{P}$.

In what follows we shall frequently make tacit assumption of these theorems.

In stating these theorems, it has been necessary to hold in abeyance the convention that formulas represented by bold capital letters are well-formed. Hereafter this convention will be restored, and formulas so represented are to be taken always as well-formed.

We turn now to a group of theorems on conversion taken from Church and Rosser [16]. In order to state these, it is necessary first to define the notion of the residuals of a set of parts $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$ of a formula $\mathbf{A}$ after a sequence of applications of Rules I and II to $\mathbf{A}$ (§6).

We assume that, if $p \neq q$, then $((\lambda x_p\mathbf{M}_p)\mathbf{N}_p)$ is not the same part of $\mathbf{A}$ as $((\lambda x_q\mathbf{M}_q)\mathbf{N}_q)$ -- though it may be the same formula. The parts $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$ of $\mathbf{A}$ need not be all the parts of $\mathbf{A}$ which have the form $((\lambda y\mathbf{P})\mathbf{Q})$. The residuals of the $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$ after a particular sequence of applications of Rules I and II to $\mathbf{A}$ are then certain parts, of the form $((\lambda y\mathbf{P})\mathbf{Q})$, of the formula into which $\mathbf{A}$ is converted by this sequence of applications of Rules I and II. They are defined as follows:

If the sequence of applications of Rules I and II in question is vacuous, each part $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$ is its own residual.

If the sequence consists of a single application of Rule I, each part $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$ is changed into a part $((\lambda y_j\mathbf{M}'_j)\mathbf{N}'_j)$ of the resulting formula, and this part $((\lambda y_j\mathbf{M}'_j)\mathbf{N}'_j)$ is the residual of $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$.

If the sequence consists of a single application of Rule II, let $((\lambda x\mathbf{M})\mathbf{N})$ be the part of $\mathbf{A}$ which is contracted (§6), and let $\mathbf{A}'$ be the resulting formula into which $\mathbf{A}$ is converted. Let $((\lambda x_p\mathbf{M}_p)\mathbf{N}_p)$ be a particular one of the $((\lambda x_j\mathbf{M}_j)\mathbf{N}_j)$, and distinguish the six following cases.

Case 1: $((\lambda x\mathbf{M})\mathbf{N})$ and $((\lambda x_p\mathbf{M}_p)\mathbf{N}_p)$ do not overlap. Under the reduction of $\mathbf{A}$ to $\mathbf{A}'$, $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ goes into a definite part of $\mathbf{A}'$, which is the same formula as $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$. This part of $\mathbf{A}'$ is the residual of $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$.

Case 2: $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$ is a part of $\mathbf{M}_p$. Under the reduction of $\mathbf{A}$ to $\mathbf{A}'$, $\mathbf{M}_p$ goes into a definite part $\mathbf{M}'_p$ of $\mathbf{A}$, which arises from $\mathbf{M}_p$ by contraction of $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$, and $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ goes into the part $((\lambda \mathbf{x}_p \mathbf{M}'_p)\mathbf{N}_p)$ of $\mathbf{A}'$. This part $((\lambda \mathbf{x}_p \mathbf{M}'_p)\mathbf{N}_p)$ of $\mathbf{A}'$ is the residual of $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$.

Case 3: $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$ is a part of $\mathbf{N}_p$. Under the reduction of $\mathbf{A}$ to $\mathbf{A}'$, $\mathbf{N}_p$ goes into a definite part $\mathbf{N}'_p$ of $\mathbf{A}'$, which arises from $\mathbf{N}_p$ by contraction of $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$, and $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ goes into the part $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}'_p)$ of $\mathbf{A}'$. This part $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}'_p)$ of $\mathbf{A}'$ is the residual of $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$.

Case 4: $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$ is $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$. In this case $((\lambda \mathbf{x}_p \mathbf{M}_p)$ $\mathbf{N}_p)$ has no residual in $\mathbf{A}'$.

Case 5: $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ is a part of $\mathbf{M}$. Let $\mathbf{M}'$ be the result of replacing all $\mathbf{x}$'s of $\mathbf{M}$ except those occurring in $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ by $\mathbf{N}$. Under these changes the part $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ of $\mathbf{M}$ goes into a definite part of $\mathbf{M}'$ which we shall denote also by $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$, since it is the same formula. If now we replace $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ in $\mathbf{M}'$ by $S^{\mathbf{x}}_{\mathbf{N}}((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)|$, $\mathbf{M}'$ becomes $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ and we denote by $S^{\mathbf{x}}_{\mathbf{N}}((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)|$ the particular occurrence of $S^{\mathbf{x}}_{\mathbf{N}}((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)|$ in $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ that resulted from replacing $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ in $\mathbf{M}'$ by the formula $S^{\mathbf{x}}_{\mathbf{N}}((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)|$. Then the residual in $\mathbf{A}'$ of $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ in $\mathbf{A}$ is defined to be the part $S^{\mathbf{x}}_{\mathbf{N}}((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)|$ in the particular occurrence of $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ in $\mathbf{A}'$ that resulted from replacing $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$ in $\mathbf{A}$ by $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$.

Case 6: $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ is a part of $\mathbf{N}$. Let $((\lambda \mathbf{y}_1 \mathbf{P}_1)\mathbf{Q}_1)$ respectively stand for the particular occurrences of the formula $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ in $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ which are the part $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ in each of those particular occurrences of the formula $\mathbf{N}$ in $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ that resulted from replacing the $\mathbf{x}$'s of $\mathbf{M}$ by $\mathbf{N}$. Then the residuals in $\mathbf{A}'$ of $((\lambda \mathbf{x}_p \mathbf{M}_p)\mathbf{N}_p)$ in $\mathbf{A}$ are the parts $((\lambda \mathbf{y}_1 \mathbf{P}_1)\mathbf{Q}_1)$ in the particular occurrence of the formula $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ in $\mathbf{A}'$ that resulted from replacing $((\lambda \mathbf{x}\mathbf{M})\mathbf{N})$ in $\mathbf{A}$ by $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$.

Finally, in the case of a sequence of two or more successive applications of Rules I, II to $\boldsymbol{A}$, say $\boldsymbol{A}$ imc $\boldsymbol{A}'$ imc $\boldsymbol{A}''$ imc ... , we define the residuals in $\boldsymbol{A}'$ of the parts $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ of $\boldsymbol{A}$ in the way just described, and we define the residuals in $\boldsymbol{A}''$ of the parts $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ of $\boldsymbol{A}$ to be the residuals of the residuals in $\boldsymbol{A}'$, and so on.

7 XXIII. After a sequence of applications of Rules I and II to $\boldsymbol{A}$, under which $\boldsymbol{A}$ is converted into $\boldsymbol{B}$, the residuals of the parts $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ of $\boldsymbol{A}$ are a set (possibly vacuous) of parts of $\boldsymbol{B}$ which each have the form $((\lambda \boldsymbol{y} \boldsymbol{P}) \boldsymbol{Q})$.

7 XXIV. After a sequence of applications of Rules I and II to $\boldsymbol{A}$, no residual of the part $((\lambda \boldsymbol{x} \boldsymbol{M}) \boldsymbol{N})$ of $\boldsymbol{A}$ can coincide with a residual of the part $((\lambda \boldsymbol{x}' \boldsymbol{M}') \boldsymbol{N}')$ of $\boldsymbol{A}$ unless $((\lambda \boldsymbol{x} \boldsymbol{M}) \boldsymbol{N})$ coincides with $((\lambda \boldsymbol{x}' \boldsymbol{M}') \boldsymbol{N}')$.

We say that a sequence of reductions on $\boldsymbol{A}_1$, say $\boldsymbol{A}_1$ imr $\boldsymbol{A}_2$ imr $\boldsymbol{A}_3$ ... imr $\boldsymbol{A}_{n+1}$, is a <u>sequence of contractions on the parts</u> $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ <u>of</u> $\boldsymbol{A}_1$ if the reduction from $\boldsymbol{A}_i$ to $\boldsymbol{A}_{i+1}$ ($i = 1, \ldots, n$) involves a contraction of a residual of the $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$. Moreover, if no residuals of the $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ occur in $\boldsymbol{A}_{n+1}$ we say that the sequence of contractions on the $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ <u>terminates</u> and that $\boldsymbol{A}_{n+1}$ is the result.

In some cases we wish to speak of a sequence of contractions on the parts $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ of $\boldsymbol{A}$ where the set $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ may be vacuous. To handle this we agree that, if the set $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ is vacuous, the sequence of contractions shall be a vacuous sequence of reductions.

7 XXV. If $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ are parts of $\boldsymbol{A}$, then a number m can be found such that any sequence of contractions on the $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$ will terminate after at most m contractions, and if $\boldsymbol{A}'$ and $\boldsymbol{A}''$ are two results of terminating sequences of contractions on the $((\lambda x_j \boldsymbol{M}_j) \boldsymbol{N}_j)$, then $\boldsymbol{A}'$ conv-I $\boldsymbol{A}''$.

This is proved by induction on the length of $\boldsymbol{A}$. It is trivially true if the length of $\boldsymbol{A}$ is 1 (i.e., if $\boldsymbol{A}$ consists of a single symbol), the number m being then 0. As hypothesis of induction, assume that the proposition is true of every formula $\boldsymbol{A}$ of length less than n. On this hypothesis we have to prove that the proposition is true of an arbitrary given formula $\boldsymbol{A}$ of length n. This we proceed to do, by means of a proof involving three cases.

Case 1: $\boldsymbol{A}$ has the form $\lambda \boldsymbol{x}\boldsymbol{M}$. All the parts $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$ must be parts of $\boldsymbol{M}$. Since $\boldsymbol{M}$ is of length less than n, we apply the hypothesis of induction to $\boldsymbol{M}$.

Case 2: $\boldsymbol{A}$ has the form $\boldsymbol{F}\boldsymbol{X}$, where $\boldsymbol{F}\boldsymbol{X}$ is not one of the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$. All the parts $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$ must be parts either of $\boldsymbol{F}$ or of $\boldsymbol{X}$. Since $\boldsymbol{F}$ and $\boldsymbol{X}$ are each of length less than n, we apply the hypothesis of induction.

Case 3: $\boldsymbol{A}$ is $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, where $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$ is one of the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$. By the hypothesis of induction, there is a number a such that any sequence of contractions on those $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$ terminates after at most a contractions, and there is a number b such that any sequence of contractions on those $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{N}_p$ terminates after at most b contractions; moreover, if we start with the formula $\boldsymbol{M}_p$ and perform a terminating sequence of contractions on those $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, the result is a formula $\boldsymbol{M}$, which is unique to within applications of Rule I, and which contains a certain number c, $\geqq 1$, of free occurrences of the variable $\boldsymbol{x}_p$.

Now one way of performing a terminating sequence of contractions on the parts $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$ is as follows. First perform a terminating sequence of contractions on those $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)$ $\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, so converting $\boldsymbol{A}$ into $((\lambda \boldsymbol{t}\boldsymbol{M})\boldsymbol{N}_p)$. Then there is one and only one residual of $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, namely the entire formula $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N}_p)$. Perform a contraction of this, so obtaining

$$S^{\boldsymbol{t}}_{\boldsymbol{N}_p} \boldsymbol{M}' |$$

where $\boldsymbol{M}'$ differs from $\boldsymbol{M}$ at most by applications of Rule I. Then in this formula there are c occurrences of $\boldsymbol{N}_p$ resulting from the substitution of $\boldsymbol{N}_p$ for $\boldsymbol{t}$. Take each of these occurrences of $\boldsymbol{N}_p$ in order and perform a terminating sequence of contractions on the residuals of the $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ occurring in it.

Let us call such a terminating sequence of contractions on the parts $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$ a special terminating sequence of contractions on the parts $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$. Clearly such a special terminating sequence of contractions contains at most $a+1+cb$ contractions.

Consider now any sequence of contractions, μ, on the parts $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$. The part $((\lambda \mathbf{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$ of $\boldsymbol{A}$ will have just one residual (which will always be the entire formula) up to the point that a contraction of its residual occurs, and thereafter will have no residual; moreover, if the sequence of contractions is continued, a contraction of the residual of $((\lambda \mathbf{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$ must occur within at most $a+b+1$ contractions. Hence we may suppose, without loss of generality, that μ consists of a sequence of contractions, ϕ, on the $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are different from $((\lambda \mathbf{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, followed by a contraction β_0 of the residual of $((\lambda \mathbf{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, followed by a sequence of contractions, ϑ, on the then remaining residuals of the $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$. Clearly, ϕ can be replaced by a sequence of contractions, α_0, on the $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, followed by a sequence of contractions, η, on the $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{N}_p$ -- in the sense that α_0 followed by η gives the same end formula as ϕ and the same set of residuals for each of the $((\lambda \mathbf{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$. Moreover, replacing ϕ by α_0 followed by η does not change the total number of contractions of residuals of parts of $\boldsymbol{M}_p$ or of residuals of parts of $\boldsymbol{N}_p$. Next, η followed by β_0 can be replaced by a contraction β' of the residual $((\lambda \boldsymbol{y} \boldsymbol{P})\boldsymbol{N}_p)$ of $((\lambda \mathbf{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$ followed by a set of applications of η on each of those occurrences of $\boldsymbol{N}_p$ in the resulting formula

$$\mathrm{S}^{\boldsymbol{y}}_{\boldsymbol{N}_p} \boldsymbol{P}'|$$

which arose by substituting $\boldsymbol{N}_p$ for $\boldsymbol{y}$ in $\boldsymbol{P}'$. (Here $\boldsymbol{P}'$ differs from $\boldsymbol{P}$ at most by applications of Rule I. Since η may be thought of as a transformation of the formula $\boldsymbol{N}_p$, the convention will be understood which we use when we speak of the sequence of reductions of a given formula which results from applying η to a particular occurrence of $\boldsymbol{N}_p$ in that formula.)

By this means the sequence of contractions, μ, is replaced by a sequence of contractions, μ', which consists of a sequence of contractions, α_0, on the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, followed by a contraction β' of the residual of $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, followed by further contractions on the then remaining residuals of the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$.

Consider now the part ζ of μ', consisting of β' and the contractions that follow it, up to and including the first contraction of a residual of a part of $\boldsymbol{M}_p$. Denoting the formula on which ζ acts by $((\lambda \boldsymbol{y} \boldsymbol{P})\boldsymbol{N}_p)$, we see that ζ can be considered as the act of first replacing the free $\boldsymbol{y}$'s of $\boldsymbol{P}$ by various formulas $\boldsymbol{N}_{pk}$, got from $\boldsymbol{N}_p$ by various sequences of reductions (which may be vacuous), and then (possibly after some applications of Rule I) contracting a residual $((\lambda \boldsymbol{z}\boldsymbol{R})\boldsymbol{S})$ of one of the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, say $((\lambda \boldsymbol{x}_q \boldsymbol{M}_q)\boldsymbol{N}_q)$. From this point of view, we see that none of the free $\boldsymbol{z}$'s of $\boldsymbol{R}$ are parts of any $\boldsymbol{N}_{pk}$, and hence ζ can be replaced by a contraction (possibly after some applications of Rule I) of that residual in $\boldsymbol{P}$ of $((\lambda \boldsymbol{x}_q \boldsymbol{M}_q)\boldsymbol{N}_q)$ of which $((\lambda \boldsymbol{z}\boldsymbol{R})\boldsymbol{S})$ is a residual, followed by a contraction (possibly after some applications of Rule I) of the residual of $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, followed by a sequence of contractions on residuals of parts of $\boldsymbol{N}_p$.

If μ' is altered by replacing ζ in this way, the result is a sequence of contractions, μ'', having the same form as μ', but having the property that after the contraction of the residual of $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$ one less contraction of residuals of parts of $\boldsymbol{M}_p$ occurs.

By repetitions of this process, μ is finally replaced by a sequence of contractions ν, which consists of a sequence of contractions, α, on the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, followed by a contraction β of the residual of $((\lambda \boldsymbol{x}_p \boldsymbol{M}_p)\boldsymbol{N}_p)$, followed by a sequence of contractions γ on residuals of the $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ which are parts of $\boldsymbol{N}_p$. Moreover, ν contains at least as many contractions as μ -- for in the process of obtaining ν from μ there is no step which can decrease the number of contractions. The sequence of contractions, α, contains at most a contractions, and γ contains at most cb contractions. Thus ν, and consequently μ, contains at most a+1+cb contractions.

Thus we have proved that any sequence of contractions on the parts $((\lambda \boldsymbol{x}_j \boldsymbol{M}_j)\boldsymbol{N}_j)$ of $\boldsymbol{A}$ will terminate after at most a+1+cb contractions.

Now suppose that μ is a <u>terminating</u> sequence of contractions. Then ν either is a <u>special</u> terminating sequence of contractions (see above) or can be made so by some evident changes in the order in which the contractions in γ are performed. By the hypothesis of induction, applied to $\boldsymbol{M}_p$ and $\boldsymbol{N}_p$, the result of a special terminating sequence of contractions is unique to within possible applications of Rule I. Therefore the result of any terminating sequence of contractions, μ, is unique to within possible applications of Rule I.

7 XXVI. If $\boldsymbol{A}$ imr $\boldsymbol{B}$ by a contraction of the part $((\lambda \boldsymbol{x M})\boldsymbol{N})$ of $\boldsymbol{A}$, and $\boldsymbol{A}_1$ is $\boldsymbol{A}$, and $\boldsymbol{A}_1$ imr $\boldsymbol{A}_2$, $\boldsymbol{A}_2$ imr $\boldsymbol{A}_3$, ..., and, for all k, $\boldsymbol{B}_k$ is the result of a terminating sequence of contractions on the residuals in $\boldsymbol{A}_k$ of $((\lambda \boldsymbol{x M})\boldsymbol{N})$, then:

(1) $\boldsymbol{B}_1$ is $\boldsymbol{B}$.

(2) For all k, $\boldsymbol{B}_k$ conv-I-II $\boldsymbol{B}_{k+1}$:

(3) Even if the sequence $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, ... can be continued to infinity, there is a number u_m, depending on the formula $\boldsymbol{A}$, the part $((\lambda \boldsymbol{x M})\boldsymbol{N})$ of $\boldsymbol{A}$, and the number m, such that, starting with $\boldsymbol{B}_m$, at most u_m consecutive $\boldsymbol{B}_k$'s occur for which it is not true that $\boldsymbol{B}_k$ red $\boldsymbol{B}_{k+1}$.

(1) is obvious.

To prove (2), let $((\lambda \boldsymbol{y}_i \boldsymbol{P}_i)\boldsymbol{Q}_i)$ be the residuals in $\boldsymbol{A}_k$ of $((\lambda \boldsymbol{x M})\boldsymbol{N})$ and let the reduction of $\boldsymbol{A}_k$ into $\boldsymbol{A}_{k+1}$ involve a contraction of (a residual of) the part $((\lambda \boldsymbol{z R})\boldsymbol{S})$ of $\boldsymbol{A}_k$. Then $\boldsymbol{B}_{k+1}$ is the result of a terminating sequence of contractions on $((\lambda \boldsymbol{z R})\boldsymbol{S})$ and the parts $((\lambda \boldsymbol{y}_i \boldsymbol{P}_i)\boldsymbol{Q}_i)$ of $\boldsymbol{A}_k$. If $((\lambda \boldsymbol{z R})\boldsymbol{S})$ is one of the $((\lambda \boldsymbol{y}_i \boldsymbol{P}_i)\boldsymbol{Q}_i)$, no residuals of $((\lambda \boldsymbol{z R})\boldsymbol{S})$ occur in $\boldsymbol{B}_k$, and $\boldsymbol{B}_k$ conv-I $\boldsymbol{B}_{k+1}$ by 7 XXV. If, however, $((\lambda \boldsymbol{z R})\boldsymbol{S})$ is not one of the $((\lambda \boldsymbol{y}_i \boldsymbol{P}_i)\boldsymbol{Q}_i)$, a set of residuals of $((\lambda \boldsymbol{z R})\boldsymbol{S})$

does occur in $\boldsymbol{B}_k$ and a terminating sequence of contractions on these residuals in $\boldsymbol{B}_k$ gives $\boldsymbol{B}_{k+1}$ by 7 XXV.

Thus $\boldsymbol{B}_k$ red $\boldsymbol{B}_{k+1}$ unless the reduction of $\boldsymbol{A}_k$ into $\boldsymbol{A}_{k+1}$ involves a contraction of a residual of $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N})$; but if we start with any particular $\boldsymbol{A}_k$ this can be the case only a finite number of successive times by 7 XXV. Hence (3) is proved, u_m being defined as follows:

Perform m successive reductions on $\boldsymbol{A}$ in all possible ways. This gives a finite set of formulas (since, for this purpose, we need not distinguish formulas differing only by applications of Rule I). In each formula find the largest number of reductions that can occur in a terminating sequence of contractions on the residuals of $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N})$. Then let u_m be the largest of these.

7 XXVII. If $\boldsymbol{A}$ conv $\boldsymbol{B}$, there is a conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$ in which no expansion precedes any reduction.

In the given conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$, let the last expansion which precedes any reduction be an expansion of $\boldsymbol{B}_1$ into $\boldsymbol{A}_1$. This expansion is followed by a sequence of one or more reductions, say $\boldsymbol{A}_1$ imr $\boldsymbol{A}_2$, $\boldsymbol{A}_2$imr $\boldsymbol{A}_3$, ..., $\boldsymbol{A}_{n-1}$ imr $\boldsymbol{A}_n$, and $\boldsymbol{A}_n$ conv-I-III $\boldsymbol{B}$. The inverse of the expansion of $\boldsymbol{B}_1$ into $\boldsymbol{A}_1$ is a reduction of $\boldsymbol{A}_1$ into $\boldsymbol{B}_1$; let $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N})$ be the part of $\boldsymbol{A}$ which is contracted in this reduction, and let $\boldsymbol{B}_k$ (k = 2, 3, ..., n) be the result of a terminating sequence of contractions on the residuals in $\boldsymbol{A}_k$ of $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N})$. By 7 XXVI, $\boldsymbol{B}_1$ conv-I-II $\boldsymbol{B}_2$, $\boldsymbol{B}_2$ conv-I-II $\boldsymbol{B}_3$, ..., $\boldsymbol{B}_{n-1}$ conv-I-II $\boldsymbol{B}_n$, $\boldsymbol{B}_n$ conv-I-III $\boldsymbol{A}_n$, $\boldsymbol{A}_n$ conv-I-III $\boldsymbol{B}$. This provides an alternative conversion of $\boldsymbol{B}_1$ into $\boldsymbol{B}$ in which no expansion precedes any reduction. The given conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$ may be altered by employing this alternative conversion of $\boldsymbol{B}_1$ into $\boldsymbol{B}$ instead of the one originally involved, with the result that the number of expansions which are out of place (precede reductions) in the conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$ is decreased by one. Repetitions of this process lead to a conversion of $\boldsymbol{A}$ into $\boldsymbol{B}$ in which no expansion precedes reductions.

7 XXVIII. If $\boldsymbol{B}$ is a normal form of $\boldsymbol{A}$, then $\boldsymbol{A}$ conv-I-II $\boldsymbol{B}$.

This is a corollary of 7 XXVII, since no reductions are possible of a formula in normal form.

7 XXIX. If $\boldsymbol{A}$ has a normal form, its normal form is unique to within applications of Rule I.

For if $\boldsymbol{B}$ and $\boldsymbol{B}'$ are both normal forms of $\boldsymbol{A}$, then $\boldsymbol{B}'$ is a normal form of $\boldsymbol{B}$. Hence $\boldsymbol{B}$ conv-I-II $\boldsymbol{B}'$. Hence $\boldsymbol{B}$ conv-I $\boldsymbol{B}$, since no reductions are possible of the normal form $\boldsymbol{B}$.

Note that 7 XXIX ensures a kind of consistency of the calculus of λ-conversion, in that certain formulas for which different interpretations are intended are shown not to be interconvertible.

7 XXX. If $\boldsymbol{A}$ has a normal form, it has a unique principal normal form.

7 XXXI. If $\boldsymbol{B}$ is a normal form of $\boldsymbol{A}$, then there is a number m such that any sequence of reductions starting from $\boldsymbol{A}$ will lead to $\boldsymbol{B}$ (to within applications of Rule I) after at most m reductions.

In order to prove 7 XXXI, we first prove the following lemma by induction on n:

> If $\boldsymbol{B}$ is a normal form of $\boldsymbol{A}$ and there is a sequence of n reductions leading from $\boldsymbol{A}$ to $\boldsymbol{B}$, then there is a number $v_{\boldsymbol{A},n}$ such that any sequence of reductions starting from $\boldsymbol{A}$ will lead to a normal form of $\boldsymbol{A}$ in at most $v_{\boldsymbol{A},n}$ reductions.

If $n = 0$, we take $v_{\boldsymbol{A},0}$ to be 0.

Assume, as hypothesis of induction, that the lemma is true when $n = k$. Suppose $\boldsymbol{A}$ imr $\boldsymbol{C}$, $\boldsymbol{C}$ imr $\boldsymbol{C}_1$, $\boldsymbol{C}_1$ imr $\boldsymbol{C}_2$, $\boldsymbol{C}_2$ imr $\boldsymbol{C}_3$, ..., $\boldsymbol{C}_{k-1}$ imr $\boldsymbol{B}$. Also, where $\boldsymbol{A}_1$ is the same as $\boldsymbol{A}$, suppose $\boldsymbol{A}_1$ imr $\boldsymbol{A}_2$, $\boldsymbol{A}_2$ imr $\boldsymbol{A}_3$, ... By 7 XXVI there is a sequence ($\boldsymbol{D}_1$ the same as $\boldsymbol{C}$), $\boldsymbol{D}_1$ conv-I-II $\boldsymbol{D}_2$, $\boldsymbol{D}_2$ conv-I-II $\boldsymbol{D}_3$, ..., such that $\boldsymbol{A}_j$ conv-I-II $\boldsymbol{D}_j$ for all j's for which $\boldsymbol{A}_j$ exists; and, if the reduction from $\boldsymbol{A}$ to $\boldsymbol{C}$ involves a contraction of $((\lambda x\boldsymbol{M})\boldsymbol{N})$, then, starting with $\boldsymbol{D}_m$, at most u_m consecutive

$\boldsymbol{D}_j$'s occur for which it is not true that $\boldsymbol{D}_j$ red $\boldsymbol{D}_{j+1}$.

Since the sequence $\boldsymbol{C}$ imr $\boldsymbol{C}_1$, $\boldsymbol{C}_1$ imr $\boldsymbol{C}_2$, ... leads to $\boldsymbol{B}$ in k reductions, there is, by hypothesis of induction, a number $v_{\boldsymbol{C},k}$ such that any sequence of reductions starting from $\boldsymbol{C}$ leads to a normal form (and thus terminates) after at most $v_{\boldsymbol{C},k}$ reductions. Hence there are at most $v_{\boldsymbol{C},k}$ reductions in the sequence $\boldsymbol{D}_1$ conv-I-II $\boldsymbol{D}_2$, $\boldsymbol{D}_2$ conv-I-II $\boldsymbol{D}_3$, ..., and this sequence must terminate after at most $f(v_{\boldsymbol{C},k})$ steps, f(x) being defined as follows:

$$f(0) = u_1,$$

$$f(x+1) = f(x)+M+1,$$

where M is the greatest of the numbers $u_1, u_2, \ldots, u_{f(x)+1}$. (Of course f(x) depends on the formula $\boldsymbol{A}$ and the part $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N})$ of $\boldsymbol{A}$, as well as on x, because u_m depends on $\boldsymbol{A}$ and $((\lambda \boldsymbol{x}\boldsymbol{M})\boldsymbol{N})$.

Since the sequence of $\boldsymbol{D}_j$'s continues as long as there are $\boldsymbol{A}_j$'s on which reductions can be performed, it follows that after at most $f(v_{\boldsymbol{C},k})$ reductions an $\boldsymbol{A}_j$ is reached on which no reductions are possible. But this is equivalent to saying that this $\boldsymbol{A}_j$ is in normal form. Thus any reductionsof $\boldsymbol{A}$ to a formula $\boldsymbol{C}$, such that there is a sequence of k reductions leading from $\boldsymbol{C}$ to a normal form of $\boldsymbol{A}$, determines an upper bound, $f(v_{\boldsymbol{C},k})$, which holds for all sequences of reductions starting from $\boldsymbol{A}$. Since the number of possible reductions of $\boldsymbol{A}$ to such formulas $\boldsymbol{C}$ is finite (reductions, or formulas $\boldsymbol{C}$, which differ only by applications of Rule I need not be distinguished as different), we take $v_{\boldsymbol{A},k+1}$ to be the least of the numbers $f(v_{\boldsymbol{C},k})$.

This completes the proof of the lemma. Hence 7 XXXI follows by 7 XXVIII.

7 XXXII. If $\boldsymbol{A}$ has a normal form, every [well-formed] part of $\boldsymbol{A}$ has a normal form.

This follows from 7 XXXI, since any sequence of reductions on a part of $\boldsymbol{A}$ implies a sequence of reductions on $\boldsymbol{A}$ and therefore must terminate.