I'm basically curious about the following set and associated size function:

$$S(n,k,z) = \{(x,y) \in (\mathbb{Z}_m^k)^2 \mid \langle x,y \rangle = z\}$$
$$f(n,k,z) = |S(n,k,z)|$$

It's clear that there is a recursive definition of the set $S(n,k,z)$, recurring on k:

$$S(n,k,z) = \oplus_{(x_1,y_1)\in\mathbb{Z}_m^2} append(x_1,y_1,S(n,k-1,z-x_1y_1))$$

Where *append* will place x_1, y_1 on the front of each x, y respectively. The reason the above is true is because $\sum_i x_iy_i = z$ which implies $x_1y_1+\sum_{i>1} x_iy_i = z$ which implies $\sum_{i>1} x_iy_i = z - x_1y_1$. Clearly this is a disjoint union, as each set of vectors will have "different heads". We thus can write the function f as:

$$f(n,k,z) = \sum_{(x_1,y_1)\in\mathbb{Z}_m^2} f(n,k-1,z-x_1y_1)$$

Using the fact that $ax \equiv b(mod\ n)$ has exactly (a,n) solutions if $(a,n) \mid (b,n)$ and has no solutions otherwise, we have that:

$$f(n,1,z) = \sum_{y\in\mathbb{Z}_n:(y,n)|(z,n)} (y,n) = \sum_{d|(z,n)} d\varphi(\frac{n}{d})$$
$$f(n,1,z) = n\sum_{d|(z,n)} \prod_{p|(n/d)} (1-\frac{1}{p})$$

For prime p, its easy to see that:

$$f(p,1,0) = 2p-1$$
$$f(p,1,z\neq 0) = p-1$$