

I'm basically curious about the following set and associated size function:

$$S(n, k, z) = \{(x, y) \in (\mathbb{Z}_m^k)^2 \mid \langle x, y \rangle = z\}$$

$$f(n, k, z) = |S(n, k, z)|$$

It's clear that there is a recursive definition of the set $S(n, k, z)$, recurring on k :

$$S(n, k, z) = \oplus_{(x_1, y_1) \in \mathbb{Z}_m^2} \text{append}(x_1, y_1, S(n, k-1, z - xy))$$

Where *append* will place x_1, y_1 on the front of each x, y respectively. The reason the above is true is because $\sum_i x_i y_i = z$ which implies $x_1 y_1 + \sum_{i>1} x_i y_i = z$ which implies $\sum_{i>1} x_i y_i = z - x_1 y_1$. Clearly this is a disjoint union, as each set of vectors will have "different heads". We thus can write the function f as:

$$f(n, k, z) = \sum_{(x_1, y_1) \in \mathbb{Z}_m^2} f(n, k-1, z - xy)$$

Using the fact that $ax \equiv b \pmod n$ has exactly (a, n) solutions if $(a, n) \mid (b, n)$ and has no solutions otherwise, we have that:

$$f(n, 1, z) = \sum_{y \in \mathbb{Z}_n : (y, n) \mid (z, n)} (y, n) = \sum_{d \mid (z, n)} d \varphi\left(\frac{n}{d}\right)$$

$$f(n, 1, z) = n \sum_{d \mid (z, n)} \prod_{p \mid (n/d)} \left(1 - \frac{1}{p}\right)$$

At this point, all I can come to is that this function is quite difficult to compute and essentially has no obvious and convenient closed form.